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BY

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ABSTRACT

In this report, the basic transfer functions and time response functions of linear phenomenological models are first revisited. The relation between the analyticity of a transfer function and the causality of the corresponding time response function is extended for the case of generalized transfer functions. Using the properties of the Hilbert transform and the associated Kramers-Kroning relations it is shown that transfer functions which have a singularity at $\omega = 0$ in their imaginary part, should be corrected by adding a delta function in their real part. This operation ensures that the resulting time response function is causal; and is consistent with the theory of generalized functions (Lighthill 1989). Accordingly, the transfer functions of classical viscoelastic models presented in standard vibration handbooks are revised. The addition of a delta function proposed by Crandall (1991) in the impedance of the non-causal ideal hysteretic damper is discussed.

Subsequently, the *causal hysteretic element* is constructed and analyzed. The dynamic stiffness of the proposed hysteretic model has the same imaginary part as the "ideal" hysteretic damper, but has the appropriate real part that makes the model causal. The proposed model is constructed by requiring that the real and imaginary parts of its transfer functions satisfy the Kramers-Kroning relations. This condition ensures that the corresponding time response functions of the proposed model are zero at negative times. The causal hysteretic element is physically realizable at finite frequencies; whereas at $\omega = 0$ is not defined. The behavior of the proposed model is analyzed both in frequency and time domain. It is shown that the causal hysteretic element is the limiting case of a linear viscoelastic model with nearly frequency-independent dissipation that was proposed by Biot (1958). Finally, the response of a mass supported by the causal hysteretic element is discussed with reference to the solutions presented by Caughey (1962).

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SECTION 1

DYNAMIC STIFFNESS, FLEXIBILITY, IMPEDANCE, MOBILITY AND THE HIDDEN DELTA FUNCTION

INTRODUCTION

When the vibration-amplitudes of a structure are small the response is usually nearly linear and the analysis can be computed using frequency-domain techniques. Frequency domain solutions are attractive because they are computational efficient since the frequency dependent behavior of structural components such as footings, pile-groups, isolators and dampers, is accounted for. When a frequency domain solution is used the causality of a model that describes the behavior of a structural component is not a concern; because, the model is expressed with a transfer function which can be directly incorporated in the dynamic stiffness or flexibility matrix of the structural system.

During strong ground motions structures are experiencing large deformations and their behavior is nonlinear Chopra (1995). In this case the structural response has to be computed in the time domain using time marching techniques; and therefore, a time-domain representation of the behavior of structural components is needed. This is possible either by following a state-space formulation (provided that a differential equation that relates force with displacement exists), or by computing the basic time response functions of the macroscopic models.

In most cases however, the response of structural components is available in terms of a transfer function obtained either from dynamic experiments or from numerical solutions of a continuum mechanics formulation. These transfer functions often depend on material behavior but also on the geometry of the configuration; and therefore, the formulation of time domain constitutive equations becomes cumbersome. On the other hand, the use of time response function is a practical alternative for time domain analysis (Veletsos and Verbic 1974). Time response functions can be directly computed by inverting in the time domain the transfer function. However, additional attention is needed when inverting a transfer function that has a singularity at $\omega = 0$, since a delta function has to be appended to its real part. This problem is addressed herein by examining the transfer functions of elementary viscoelastic models.

In this section the basic transfer functions of the linear spring, the viscous dashpot, the Maxwell model and the Kelvin model are examined. Each of these models has either an impedance of a flexibility transfer function with an imaginary part that has a singularity at $\omega = 0$. Using the properties of the Hilbert transform and the associated Kramers-Kroning relations it is shown that such transfer functions should be corrected by adding a delta function in their real part. This operation ensures causality of the resulting time response function and is consistent with the theory of generalized functions (Dirac 1958, Lighthill 1989).

THE BASIC TRANSFER FUNCTIONS OF MECHANICAL MODELS

Herein we are concerned with the integral representation of linear viscoelastic models with constant coefficients of the form:

$$\left[\sum_{m=0}^{M} a_m \frac{d^m}{dt^m}\right] P(t) = \left[\sum_{n=0}^{N} b_n \frac{d^n}{dt^n}\right] u(t), \tag{1}$$

where the coefficients a_m and b_n are restricted to real numbers and are the parameters of the constitutive model, and the order of differentiation, m and n, is restricted to integers. The linearity of (1) permits its transformation in the frequency domain using the Fourier transform

$$P(\omega) = [K_1(\omega) + iK_2(\omega)]\, u(\omega), \tag{2}$$

where $P(\omega) = \mathcal{F}\{P(t)\}$ and $u(\omega) = \mathcal{F}\{u(t)\}$ are the Fourier transforms of the force and displacement histories respectively and $K_1(\omega) + iK_2(\omega)$ is the dynamic stiffness of the model:

$$\mathcal{K}(\omega) = K_1(\omega) + iK_2(\omega) = \frac{\sum_{n=0}^{N} b_n (i\omega)^n}{\sum_{m=0}^{M} a_m (i\omega)^m}. \tag{3}$$

$\mathcal{K}(\omega)$ is the transfer function which relates a *displacement input* to a *force output*. Using the Laplace variable $s = i\omega$, the dynamic stiffness $\mathcal{K}(s/i)$ is a ratio of two polynomials. The numerator of the right-hand side of (3) is a polynomial of degree, n, and the denominator on degree, m; therefore, $\mathcal{K}(s/i)$, has n zeros and m poles.

A transfer function that has more poles than zeros (m>n) is called *strictly proper* and results a *strictly causal* impulse response function--later in this section the causality issue is discussed in detail. A strictly proper transfer function means that the output of the model can not react instantaneously to an input modulation; and this is what happens in reality with physical systems. The force P(t) in (1) can be computed in the time domain with the convolution integral

$$P(t) = \int_{-\infty}^{t} q(t-\tau)\, u(\tau)\, d\tau, \tag{4}$$

where q(t) is the *memory function* of the model (Bird et al. 1987), defined as the resulting force at the present time, t, due to an impulsive displacement input at time $\tau\,(\tau < t)$, and is the inverse Fourier transform of the dynamic stiffness,

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(\omega)\, e^{i\omega t} d\omega. \tag{5}$$

Herein the notation q(t) used for the memory function is the same to the notation used by Veletsos and Verbic (1974). The inverse Fourier transform given by (5) converges only when $\int_{-\infty}^{\infty} |\mathcal{K}(\omega)| d\omega < \infty$; and therefore, q(t) exists in the classical sense only when $\mathcal{K}(\omega)$ is a strictly proper function (m>n). Moreover, in some cases proper transfer functions have a pole at $\omega = 0$, and in this case further attention is needed. When the number of poles is equal to the number of zeros (m=n), the transfer function of the model is simply proper and results to an impulse response function, q(t), that has a singularity at the time origin because of the finite limiting value of the dynamic stiffness at high frequencies. This means that the model produces instantaneously an output at a given input. When the number of poles is less than the number of zeros (m<n), the model produces an output prior to the application of the input; and this corresponds to an unreal system. Such transfer functions are improper (Rohrs et *al.* 1993).

The inverse of the dynamic stiffness is the dynamic flexibility:

$$\mathcal{H}(\omega) = H_1(\omega) + iH_2(\omega) = \frac{1}{K_1(\omega) + iK_2(\omega)}, \tag{6}$$

which is the transfer function that relates a *force input* to a *displacement output*. From equation (3) and (6) it is clear that when a macroscopic model has a strictly proper dynamic stiffness it has an improper dynamic flexibility and vice versa. Accordingly, when the causality of a proposed model is a concern it is important to specify what is the input and what is the output.

When the dynamic flexibility, $\mathcal{H}(\omega)$, is a proper transfer function the displacement, u(t), in (1) can be computed in the time domain via the convolution integral

$$u(t) = \int_{-\infty}^{t} h(t-\tau) P(\tau)\, d\tau, \qquad (7)$$

where h(t) is the *impulse response function* defined as the resulting displacement at time, t, for an impulsive force input at time $\tau\,(\tau < t)$, and is the inverse Fourier transform of the dynamic flexibility,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{H}(\omega)\, e^{i\omega t} d\omega. \qquad (8)$$

The notation, h(t), for the impulse response function used by Veletsos and Verbic (1974) is also kept herein.

Another useful transfer function of a macroscopic model is the impedance, $\mathcal{Z}(\omega) = Z_1(\omega) + iZ_2(\omega)$, which relates a *velocity input* to a *force output,*

$$P(\omega) = [Z_1(\omega) + iZ_2(\omega)]\, v(\omega), \qquad (9)$$

where $v(\omega) = i\omega u(\omega)$ is the Fourier transform of the velocity time history. For the linear viscoelastic model given by (1) the impedance of the model is

$$\mathcal{Z}(\omega) = Z_1(\omega) + iZ_2(\omega) = \frac{\sum_{n=0}^{N} b_n (i\omega)^n}{\sum_{m=0}^{M} a_m (i\omega)^{m+1}}. \qquad (10)$$

In the geotechnical literature the term impedance is used occasionally for the dynamic stiffness. This usage is not recommended. The force P(t) in (1) can be computed in the time domain with an alternative convolution integral

$$P(t) = \int_{-\infty}^{t} k(t-\tau)\,\dot{u}(\tau)\,d\tau, \qquad (11)$$

where k(t) is the *relaxation stiffness* of the model defined as the resulting force at the present time t for a unit step displacement at time $\tau\,(\tau < t)$, and is the inverse Fourier transform of the impedance,

$$k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{Z}(\omega)\,e^{i\omega t}\,d\omega. \qquad (12)$$

Equation (10) indicates that if the dynamic stiffness of a model is a simple proper function, then the impedance of the model is a strictly proper function; and therefore, the relaxation stiffness of the model, k(t), is finite whereas the impulse response function, q(t), has a singularity at the time origin.

The inverse of the impedance is called *mobility* (Harris 1988), while in the electrical engineering literature the term *admittance* is used for the same quantity (Bode 1959). It is defined as

$$\mathcal{M}(\omega) = M_1(\omega) + iM_2(\omega) = \frac{1}{Z_1(\omega) + iZ_2(\omega)} \qquad (13)$$

The mobility (admittance) of a macroscopic model relates a *force input* to a *velocity output*. Herein the notation, $\mathcal{M}(\omega)$, proposed by Harris (1988) is preferred. The inverse Fourier transform of the mobility is defined as the step response function and is defined as the resulting displacement at time, t, for a step force input at time $\tau\,(\tau < t)$.

$$m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}(\omega)\,e^{i\omega t}\,d\omega \qquad (14)$$

REAL AND IMAGINARY PARTS OF PROPER TRANSFER FUNCTIONS

The Maxwell Model

Our discussion on the causality of transfer functions starts with the classical Maxwell model of viscoelasticity

$$P(t) + \lambda\frac{dP(t)}{dt} = C\frac{du(t)}{dt}, \tag{15}$$

which was found to describe satisfactorily the dynamic response of hydraulic fluid dampers (Constantinou and Symans 1993, Burton et *al.* 1996). Its dynamic stiffness and impedance are:

$$\mathcal{K}(\omega) = \frac{Ci\omega}{1+\lambda i\omega} = \frac{C\lambda\omega^2}{1+\lambda^2\omega^2} + i\frac{C\omega}{1+\lambda^2\omega^2} \tag{16}$$

$$\mathcal{Z}(\omega) = \frac{C}{1+\lambda i\omega} = \frac{C}{1+\lambda^2\omega^2} - i\frac{C\lambda\omega}{1+\lambda^2\omega^2} \tag{17}$$

After substituting of (17) into (12) and integrating,

$$k(t) = \frac{C}{\lambda}e^{-t/\lambda} \quad \text{for } t \geq 0 \quad \text{and} \quad k(t) = 0 \quad \text{for } t < 0. \tag{18}$$

Equation (18) shows that the relaxation stiffness of the Maxwell model is a decaying exponential for positive times and zero at negative times. It is therefore, a causal function with a finite value at the time origin $k(0) = C/\lambda$.

When $t < 0$, the relaxation stiffness, k(t)=0, and the corresponding impedance, $\mathcal{Z}(\omega)$, is an analytic function on the right-hand complex plane (Papoulis 1987). This condition on the analyticity of $\mathcal{Z}(\omega)$ dictates that its real part $Z_1(\omega)$ and its imaginary parts $Z_2(\omega)$ are related with the Hilbert transform:

$$Z_1(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Z_2(x)}{x-\omega}dx, \quad Z_2(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Z_1(x)}{x-\omega}dx. \tag{19}$$

The proof of the relations given by (19) can be found in textbooks (Bendat and Piersol 1986, Papoulis 1987). These relations between real and imaginary parts of a strictly proper transfer function are the necessary and sufficient conditions to ensure causality in the strict sense. They can be directly derived from the Cauchy integral theorem, and are known in the theories of optical processes and viscoelasticity as the Kramers-Kronig relations (Booij and Thoone, 1982; Bird et al., 1987). As an example the interested reader can show that the real and imaginary part of the impedance, $\mathcal{Z}(\omega)$, of the Maxwell model satisfy the relations given by (19). While the Kramers-Kronig relations given by (19) result from the analyticity of a complex valued function (Bracwell 1986), they can be used to relate generalized functions. For instance, with the help (19) it can be shown (Makris et al. 1996), that the real and imaginary parts of the conjugate of the complex-valued Dirac function, $\tilde{\mathcal{D}}(\omega)$, defined as

$$\tilde{\mathcal{D}}(\omega) = \delta(\omega) - i\frac{1}{\pi\omega}, \tag{20}$$

are Hilbert pairs. This particular combination of the delta function and the reciprocal function given by (20) has been used by Dirac in the study of collision processes in quantum theory (Dirac 1958, Feynman and Hibbs 1962).

When the transfer function is simply proper (m=n), the first relation given by (19) does not hold. This is because the transfer function maintains a finite value as frequency tends to infinity and the corresponding impulse response function has a singularity at the time origin.

In order to illustrate the presence of this singularity at the time origin, lets compute the memory function resulting from the dynamic stiffness of the Maxwell model, $\mathcal{K}(\omega)$, given by (16). Since $\int_{-\infty}^{\infty}|\mathcal{K}(\omega)|d\omega = \infty$ the integral given by (5) cannot be computed in the classical sense. However one can extract the limiting value of the dynamic stiffness at high frequencies, $\mathcal{K}(\infty) = C/\lambda$ and equation (16) can be rewritten as

$$\mathcal{K}(\omega) = \frac{C}{\lambda} - \frac{C}{\lambda}\frac{1}{1+\lambda i\omega}. \tag{21}$$

Substitution of (21) into (5) gives

$$q(t) = \frac{C}{\lambda}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\lambda i\omega} e^{i\omega t} d\omega\right], \tag{22}$$

and using the properties of the delta function: $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega = \delta(t-0)$,

$$q(t) = \frac{C}{\lambda}\left[\delta(t-0) - \frac{1}{\lambda}e^{-t/\lambda}\right], \tag{23}$$

which shows that in this case q(t) is not strictly causal since the delta function at the time origin is responsible for an instantaneous response.

The expression given by (23) can be generalized for any simply proper transfer function, $\mathcal{K}(\omega)$, which can be written in the form

$$\mathcal{K}(\omega) = A + S(\omega), \tag{24}$$

where, A=$\mathcal{K}(\infty)$, is a real-valued constant and, $S(\omega)$ is a strictly proper transfer function. In this case the memory function is given by

$$q(t) = A\delta(t-0) + f(t) = A\delta(t-0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega t} d\omega, \tag{25}$$

where f(t) is a strictly causal function with finite value at the time origin.

In order to derive the relation between the real and imaginary parts of a simply proper transfer function the causal function, f(t), in (25) is broken into the sum of an even function, $f_e(t)$, plus an odd function, $f_o(t)$, so that $f(t) = f_e(t) + f_o(t)$. In this case

$$f_e(t) = \frac{1}{2}f(t) \quad \text{and} \quad f_o(t) = \frac{1}{2}f(t) = f_e(t) \quad \text{for} \quad t \geq 0, \tag{26}$$

and

$$f_e(t) = \frac{1}{2}f(-t) \quad \text{and} \quad f_o(t) = -\frac{1}{2}f(-t) = -f_e(t) \quad \text{for } t < 0. \tag{27}$$

Now the Fourier transform of q(t) gives:

$$\mathcal{F}\{q(t)\} = A + \mathcal{F}\{f_e(t)\} + \mathcal{F}\{f_0(t)\} = K_1(\omega) + iK_2(\omega). \tag{28}$$

Since the Fourier transform of an even function is a real quantity and the Fourier transform of an odd function is an imaginary quantity, equation (28) decomposes to:

$$K_1(\omega) = A + \mathcal{F}\{f_e(t)\} \quad \text{and} \quad iK_2(\omega) = \mathcal{F}\{f_o(t)\}. \tag{29}$$

From (26) and (27) we have that

$$f_e(t) = f_o(t)\,\mathrm{sgn}(t), \tag{30}$$

and the substitution of (30) into the first relation in (29) gives

$$K_1(\omega) = A + \mathcal{F}\{f_o(t)\,\mathrm{sgn}(t)\}. \tag{31}$$

Using that

$$\mathrm{sgn}(t) = \mathcal{F}^{-1}\{-\frac{2i}{\omega}\}, \tag{32}$$

and that

$$f_o(t) = \mathcal{F}^{-1}\{iK_2(\omega)\} \tag{33}$$

the convolution integral theorem yields

$$K_1(\omega) = A + \frac{1}{2\pi}\int_{-\infty}^{\infty} iK_2(x)\left(\frac{-2i}{\omega - x}\right)dx = A - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{K_2(x)}{x-\omega}dx. \tag{34}$$

Consequently, in the case of a simple proper transfer function the Kramers-Kroning relations given by (19) becomes

$$K_1(\omega) = A - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{K_2(x)}{x-\omega}dx, \quad K_2(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{K_1(x)}{x-\omega}dx, \tag{35}$$

where the constant value, A, is now involved in the first relation.

The flexibility of the Maxwell model is

$$\mathcal{H}(\omega) = \frac{1+\lambda i\omega}{Ci\omega} = \frac{1}{C}(\lambda - i\frac{1}{\omega}). \tag{36}$$

The Fourier integral given by (8) exists in the generalized sense (Lighthill 1989), and with the help of (32) one can compute the corresponding impulse response function,

$$h(t) = \frac{1}{C}\left[\lambda\delta(t-0) + \frac{1}{2}\operatorname{sgn}(t)\right]. \tag{37}$$

The reader recognizes that although the flexibility expression given by (36) is a simply proper transfer function; the resulting impulse response function given by (37) is non-causal because of the function sgn(t), which has a finite value along the entire negative time axis. The origin of this causality violation is that although the flexibility expression given by (37) is a simply proper transfer function, the real and imaginary parts of $\mathcal{H}(\omega)$ given by (36) are not Hilbert pairs. The violation of causality shown by (37) can be cured by requiring the real and imaginary parts of (36) to be Hilbert pairs, or in other words to satisfy the Kramer-Kroning relations given by (35). The imaginary part of $\mathcal{H}(\omega)$, $-1/\omega$, is the Hilbert transform of $\lambda + \pi\delta(\omega - 0)$, where λ is any constant. This can be derived immediately from the second relation given by (35) using the change of variables $\xi = x - \omega$.

$$H_2(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\pi\delta(x)}{x-\omega}dx = \int_{-\infty}^{\infty}\frac{\delta(\xi+\omega)}{\xi}d\xi = -\frac{1}{\omega} \tag{38}$$

The presence of the constant λ, does not create any difficulty since the Hilbert transform of a constant is zero. Accordingly, the real part of $\mathcal{H}(\omega)$ is $\lambda + \pi\delta(\omega - 0)$ rather than λ, and the correct expression for the dynamic flexibility of the Maxwell model is

$$\mathcal{H}(\omega) = \frac{\pi}{C}\left(\frac{\lambda}{\pi} + \delta(\omega - 0) - i\frac{1}{\pi\omega}\right) = \frac{\pi}{C}\left[\frac{\lambda}{\pi} + \tilde{\mathcal{D}}(\omega - 0)\right] \tag{39}$$

in which, $\tilde{\mathcal{D}}(\omega - 0)$, is the complex conjugate Dirac functions given by (20).

By inverting in the time domain equation (39), the correct expression for the impulse response function of the Maxwell model is

$$h(t) = \frac{1}{C}\left[\lambda\delta(t - 0) + \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)\right]. \tag{40}$$

Equation (40) shows that h(t)=0 for t<0; and therefore, h(t) is a causal function.

The Hidden Delta Function

In the forgoing analysis it was shown that the delta function has to be appended to the flexibility of the Maxwell model so that its real and imaginary parts satisfy the Kramers-Kroning relations and the resulting impulse response function is causal. The addition of the delta function in the real part of a transfer function that has a pole at zero is not new. In a paper with outstanding insight, Crandall (1991) studied the time domain response of the viscous and hysteretic damping. Without referring to the Kramers-Kroning relations or the Hilbert transform, Professor Crandall explained in his own words that *the strong singularity* $1/\omega$ *so outshines the feeble singularity of* $\delta(\omega)$, *that an observer does not notice the presence of an arbitrary number of delta functions.* Accordingly, using this intuitive argument Crandall (1991), appended the delta function as a real part next to the imaginary reciprocal function. Herein this intuitive approach followed by Crandall is confirmed mathematically with the Hilbert transform and the associated Kramers-Kroning relations given by (35).

This intimate relation between the reciprocal function and the delta function was first noticed by Dirac (1958). In order to make the reciprocal function $1/x$ well defined in the neighbourhood of $x = 0$ (in the sense of a generalized function), Dirac imposed an extra condition, such that the integral of the reciprocal function from $-\varepsilon$ to $\varepsilon(\varepsilon > 0)$, vanishes. With reference to Figure 1,

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{x} dx = 0. \tag{41}$$

However, if one uses the standard expression from differential calculus, $\frac{d}{dx}\log x = \frac{1}{x}$, the result of the aforementioned integral equals to $-i\pi$, which is a contradiction with (41). Dirac explained, that as x passes through the value zero this pure imaginary term vanishes discontinuously. The differentiation of this pure imaginary term yields the result $-i\pi\delta(x)$, so that the correct expression for the derivative of the logarithm is: Dirac (1958),

$$\frac{d}{dx}\log x = \frac{1}{x} - i\pi\delta(x) = -i\pi\left[\delta(x) + i\frac{1}{\pi x}\right]. \tag{42}$$

The quantity within brackets in the right hand side of (42) is the complex Dirac function (its complex conjugate was introduced with Eqn. (20)), in which its real and imaginary parts are Hilbert pairs. The positive sign before the imaginary part, $1/(\pi x)$, is related to the direction of integration, which is counter-clockwise. The negative sign before the imaginary part of (20), $1/(\pi\omega)$, is present because impedances in general are analytic in the right-hand complex plane, and the contour integration is performed clockwise.

Using equation (42) one can construct the complex Heaviside function, $\Xi(x)$, which is defined as

$$\Xi(x) = \int_{-\varepsilon}^{x} \mathcal{D}(x)\, dx = \int_{-\varepsilon}^{x} \left(\delta(x) + i\frac{1}{\pi x}\right) dx, \tag{43}$$

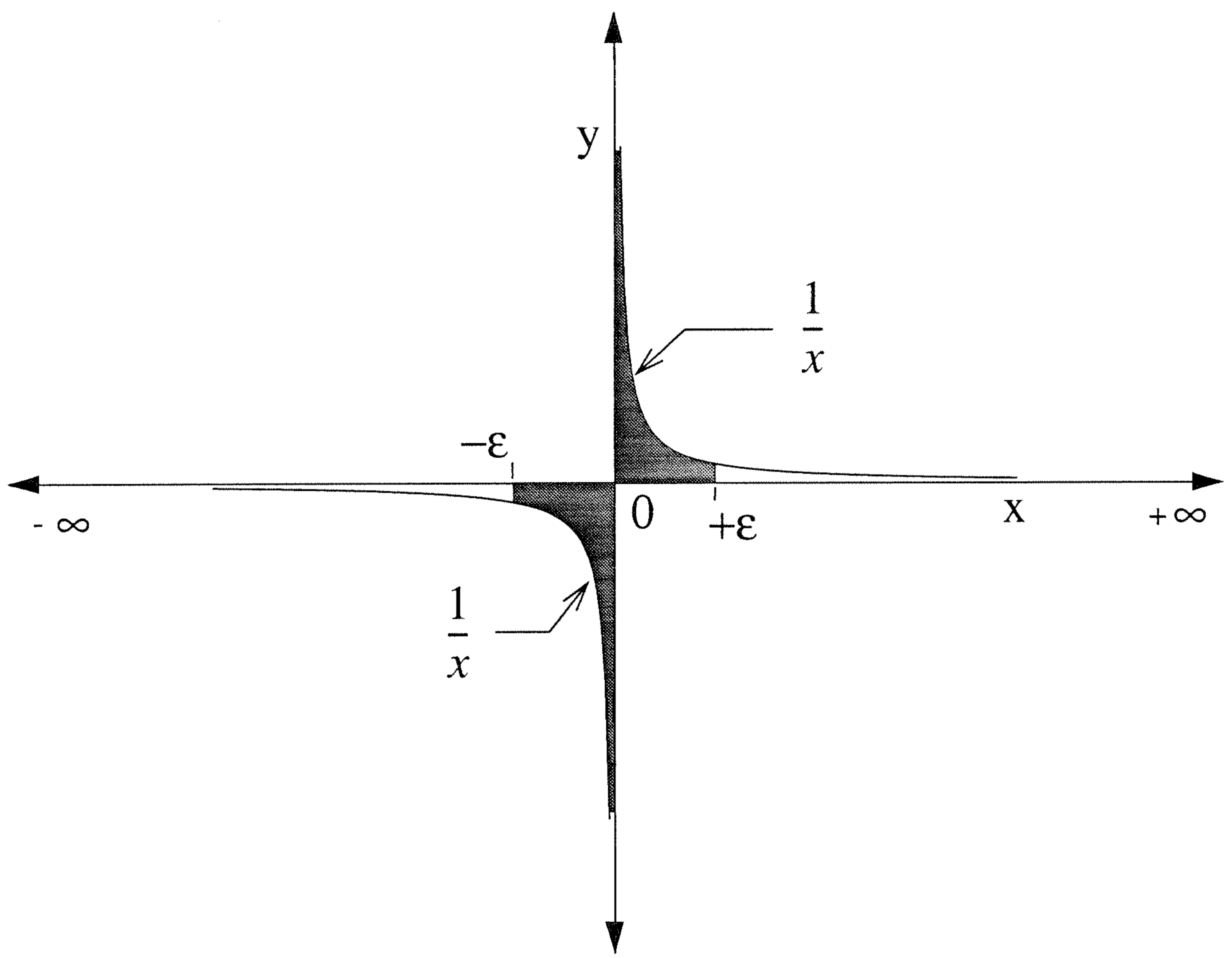

Figure 1. The reciprocal function.

where, ε, is an arbitrary positive real number. Substitution of (42) into (43) gives:

$$\Xi(x) = \frac{i}{\pi}\int_{-\varepsilon}^{x} \frac{d\log x}{dx}dx = \frac{i}{\pi}[\log x]^{x}_{-\varepsilon} \tag{44}$$

Distinguishing between the case where the upper limit is positive or negative, the integral of (44) gives:

$$\Xi(x) = \frac{i}{\pi}[ln|x| + i\pi - (ln|\varepsilon| + i\pi)] = \frac{i}{\pi}ln\left|\frac{x}{\varepsilon}\right| \text{, for } x \le 0 \tag{45}$$

and

$$\Xi(x) = \frac{i}{\pi}[ln|x| - (ln|\varepsilon| + i\pi)] = 1 + \frac{i}{\pi}ln\left|\frac{x}{\varepsilon}\right| \text{, for } x > 0 \tag{46}$$

Combining the results of (45) and (46) the final expression for the complex Heaviside function is

$$\Xi(x) = \xi(x) + i\frac{1}{\pi}ln\left|\frac{x}{\varepsilon}\right| \tag{47}$$

where, $\xi(x)$, is the classical real-valued Heaviside function and, ε, is an arbitrary positive real number. The symbol, $\xi(x)$, instead of the traditional symbol, h(x), is used herein for the real-valued Heaviside function, in order to avoid confusion with the impulse response function defined with (8). The interested reader can show that the real and imaginary parts of the complex Heaviside function are also Hilbert pairs. In the next section it is shown that the complex Heaviside function is of importance in structural mechanics since is intimately related with the dynamic stiffness of the causal hysteretic element.

The Linear Spring (Hook's Model)

With the forgoing study on the Maxwell model we have developed the necessary background to study the transfer functions and basic response functions of other classical macroscopic models. For the linear spring (Hook's model),

$$P(t) = Ku(t), \tag{48}$$

the dynamic stiffness, $\mathcal{K}(\omega) = K$, and flexibility, $\mathcal{H}(\omega) = 1/K$, are real-valued quantities. They are simply proper transfer functions with a constant real part and a zero imaginary part. Using the relations offered by (35) one can show that both the dynamic stiffness, $\mathcal{K}(\omega) = K$, and flexibility, $\mathcal{H}(\omega) = 1/K$, satisfy the Kramers-Kroning relations. However, the resulting impedance, $\mathcal{Z}(\omega) = \mathcal{K}(\omega)/(i\omega)$, has to be adjusted to satisfy the Kramers-Kroning relations in order to ensure causality. Using the result of (38), the correct expression for the impedance of the linear spring is

$$\mathcal{Z}(\omega) = \pi K\left[\delta(\omega - 0) - i\frac{1}{\pi\omega}\right] = \pi K\tilde{\mathcal{D}}(\omega - 0), \tag{49}$$

in which, $\tilde{\mathcal{D}}(\omega)$, is the conjugate complex Dirac function given by (20). The corresponding relaxation stiffness of the linear spring is

$$k(t) = \frac{K}{2}[1 + \operatorname{sgn}(t)] = K\xi(t), \tag{50}$$

in which, $\xi(t)$, is the real valued Heaviside function. Equation (50) shows that the relaxation stiffness is indeed a causal function. The rest of the transfer functions and the corresponding basic response functions of the linear spring can be computed using the calculus of generalized functions (Lighthill 1989), and are summarized in Table 1.

Transfer functions of classical phenomenological models, like these studied herein, are presented in tabulated form in standard shock and vibration handbooks. For instance, Table 10.2 in the shock and vibration handbook edited by Harris (1988) offers expressions for the impedances of classical viscoelastic models such as the linear spring, the viscous dashpot, the Kelvin and Maxwell model studied herein. The expression for the impedance

TABLE 1. Basic Transfer Functions and Time Response Functions of Standard Constitutive Models

	Hook (spring)	**Newton (dashpot)**	**Kelvin**	**Maxwell**
Constitutive Equation	$P(t) = Ku(t)$	$P(t) = C\frac{du(t)}{dt}$	$P(t) = Ku(t) + C\frac{du(t)}{dt}$	$P(t) + \lambda\frac{dP(t)}{dt} = C\frac{du(t)}{dt}$
Dynamic Stiffness $\mathcal{K}(\omega)$	$K + i0$	$0 + i\omega C$	$K + i\omega C$	$C\left[\frac{\lambda\omega^2}{1+\lambda^2\omega^2} + i\frac{\omega}{1+\lambda^2\omega^2}\right]$
Dynamic Flexibility $\mathcal{H}(\omega)$	$\frac{1}{K} + i0$	$\frac{\pi}{C}\left[\delta(\omega-0) - i\frac{1}{\pi\omega}\right]$	$\frac{1}{K}\left[\frac{1}{1+\lambda^2\omega^2} - i\frac{\omega\lambda}{1+\lambda^2\omega^2}\right]$	$\frac{\pi}{C}\left[\frac{\lambda}{\pi} + \delta(\omega-0) - i\frac{1}{\pi\omega}\right]$
Impedance $\mathcal{Z}(\omega)$	$\pi K\left[\delta(\omega-0) - i\frac{1}{\pi\omega}\right]$	$C + i0$	$\pi K\left[\frac{\lambda}{\pi} + \delta(\omega-0) - i\frac{1}{\pi\omega}\right]$	$C\left[\frac{1}{1+\lambda^2\omega^2} - i\frac{\omega\lambda}{1+\lambda^2\omega^2}\right]$
Mobility $\mathcal{M}(\omega)$	$0 + i\omega\frac{1}{K}$	$\frac{1}{C} + i0$	$\frac{1}{K}\left[\frac{\lambda\omega^2}{1+\lambda^2\omega^2} + i\frac{\omega}{1+\lambda^2\omega^2}\right]$	$\frac{1}{C}[1 + i\lambda\omega]$
Memory Function q(t)	$K\delta(t-0)$	$C\frac{d\delta(t-0)}{dt}$	$K\left[\delta(t-0) + \lambda\frac{d\delta(t-0)}{dt}\right]$	$\frac{C}{\lambda}\left[\delta(t-0) - \frac{1}{\lambda}e^{-t/\lambda}\right]$
Impulse Response Function h(t)	$\frac{1}{K}\delta(t-0)$	$\frac{1}{C}\left[\frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right]$	$\frac{1}{C}e^{-t/\lambda}$	$\frac{1}{C}\left[\lambda\delta(t-0) + \frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right]$
Relaxation Stiffness k(t)	$K\left[\frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right]$	$C\delta(t-0)$	$K\left[\lambda\delta(t-0) + \frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right]$	$\frac{C}{\lambda}e^{-t/\lambda}$
Step Response Function m(t)	$\frac{1}{K}\frac{d\delta(t-0)}{dt}$	$\frac{1}{C}\delta(t-0)$	$\frac{1}{C}\left[\delta(t-0) - \frac{1}{\lambda}e^{-t/\lambda}\right]$	$\frac{1}{C}\left[\delta(t-0) + \lambda\frac{d\delta(t-0)}{dt}\right]$

of the linear spring offered in table 10.2 (Harris 1988) should be corrected by adding a delta function in its real part as is shown in (49). Following the theory presented herein, this hidden delta function (after being multiplied with the appropriate constant) should be appended to all transfer functions offered in Table 10.2 (Harris 1988) that contain the term $1/(i\omega)$.

The Ideal Viscous Dashpot (Newton Model)

The four transfer functions and the corresponding basic response functions of the ideal viscous dashpot

$$P(t) = C\frac{du(t)}{dt} \tag{51}$$

have been presented and discussed in detail by Crandall (1991), and are summarized in Table 1. Following the same causality arguments that we used to construct the flexibility of the Maxwell model (Eqn (39)) and the impedance of the linear spring (Eqn (49), the flexibility of the ideal viscous dashpot is

$$\mathcal{H}(\omega) = \frac{\pi}{C}\left[\delta(\omega-0) - i\frac{1}{\pi\omega}\right] = \frac{\pi}{C}\tilde{\mathcal{D}}(\omega-0), \tag{52}$$

in which, $\tilde{\mathcal{D}}(\omega)$, is the conjugate complex Dirac function given by (20). Note that in all the afore-mentioned transfer functions, the pole at the frequency origin is at their *imaginary parts* and the delta function is appended as their *real part* so that real and imaginary parts are Hilbert pairs and causality of the model is ensured.

The Kelvin Model

The Kelvin model is a combination in parallel of the linear spring and the viscous dashpot and is expressed by

$$P(t) = Ku(t) + C\frac{du(t)}{dt}. \tag{53}$$

The four transfer functions and the corresponding basic response functions of the Kelvin model are summarized in Table 1.

THE "IDEAL" HYSTERETIC DASHPOT (STRUCTURAL DAMPING)

The attractive feature of the hysteretic dashpot is that its loss stiffness (dissipation) is independent of frequency. The hysteretic dashpot, also known as linear structural damping (Theodorsen and Garrick 1940), hysteretic damper (Bishop and Johnson 1960), or ideal hysteretic damper (Crandall 1991), is a pathological model since it is non-causal (Caughey 1962, Crandall 1963, 1970, 1991, Inaudi and Kelly 1995 among others). Because of its non-causality the ideal hysteretic dashpot can not be expressed with a real-valued time-domain constitutive law; and is introduced herein with its dynamic stiffness,

$$\mathcal{K}(\omega) = iK_2 \operatorname{sgn}(\omega), \tag{54}$$

where, K_2, is the loss factor and has units of stiffness $[M][T]^{-2}$. Application of (5) in conjunction with the theory of generalized functions (Lighthill 1989), yields the memory function of the ideal hysteretic dashpot:

$$q(t) = -\frac{K_2}{\pi}\frac{1}{t}. \tag{55}$$

This result, which was initially presented by Crandall (1970), shows that the memory function of the ideal hysteretic damper is non-causal since it does not vanish for $t<0$. This means that the ideal hysteretic dashpot responds prior to the application of the impulsive excitation. In fact, there is as much response before the impulse as there is afterwards. The impedance (velocity transfer function), of the ideal hysteretic dashpot can be derived from (54),

$$\mathcal{Z}(\omega) = \frac{\mathcal{K}(\omega)}{i\omega} = K_2\frac{\operatorname{sgn}(\omega)}{\omega}, \tag{56}$$

and is a real-valued transfer function with a pole at $\omega = 0$.

In the previous sections we studied transfer functions that their imaginary parts have a pole at zero, and it was found that a delta function has to be appended in their real parts, so that the resulting basic response functions are causal. In the case of the ideal hysteretic damping the argument of causality cannot be used since all time response functions of the model are non causal (Crandall 1970). This inherent non causality of the ideal hysteretic damping was the motivation for developing the causal hysteretic element which is presented in detail in the next section.

The transfer functions and basic response functions of the ideal hysteretic damper have been presented by Crandall (1991). Using the argument that the feeble singularity of a delta function is "unnoticeable" compared to the strong singularity, $\frac{\mathrm{sgn}(\omega)}{\omega}$, Crandall (1991) appended the term, $C\delta(\omega)$, to the real-valued expression given by (56) so that the impedance of the ideal hysteretic damper takes the form:

$$\mathcal{Z}(\omega) = C\delta(\omega - 0) + K_2 \frac{\mathrm{sgn}(\omega)}{\omega}. \tag{57}$$

Using the expression given by (57) and the properties of generalized functions, the resulting relaxation stiffness of the ideal hysteretic damping is

$$k(t) = C - \frac{1}{\pi} ln|t|, \tag{58}$$

which is clearly non-causal since it does not vanish for $t<0$.

The addition of the delta function in (57) proposed by Crandall has nothing to do with the Kramers-Kroning relations and the causality requirement. Professor Crandall appended the delta function in (57) in order to be consistent with the fact that for any positive integer, m, the function $\omega^{-m}\mathrm{sgn}\,\omega$, is indeterminate to the extent of an arbitrary multiple of $\delta^{(m-1)}(\omega)$ (Lighthill 1989).

CONCLUSIONS

In this section the basic transfer functions of the linear spring, the viscous dashpot, the Maxwell model and the Kelvin model are examined. Each of these models has either an impedance or a flexibility transfer function with an imaginary part that has a singularity at $\omega = 0$. Using the properties of the Hilbert transform and the associated Kramers-Kroning relations it is shown that such transfer functions should be corrected by adding a delta function in their real part. This operation ensures causality of the resulting time response function and is consistent with the theory of generalized functions. The transfer functions of classical viscoelastic models presented in vibration handbooks have been revised accordingly.

A singularity at $\omega = 0$ is also present in the impedance of the ideal hysteretic damping model which is non causal. The addition of the delta function in the impedance expression does not remove the causality violation of the ideal hysteretic damping model. Nevertheless, its presence can be justified from the calculus of generalized functions. The non causality problem of the ideal hysteretic damping is addressed in the next section-where the causal hysteretic element is introduced and analyzed.

SECTION 2

THE CAUSAL HYSTERETIC ELEMENT

INTRODUCTION

Early experimental studies on the dissipation properties of engineering materials indicated that the internal damping is nearly frequency independent, and the need of a mechanical model that generates frequency independent dissipation was apparent (see Caughey 1962, Crandall 1970 and references therein). In studying flutter, Theodorsen and Garrick (1940) introduced the *frequency independent structural damping* model, which was later given the label *hysteretic damping* by Bishop (1955) and more recently the label *ideal hysteretic damping* by Crandall (1991). The appeal of frequency independent dissipation generated by the hysteretic dashpot motivated a number of researchers to adapt hysteretic damping in time-domain vibration analysis. Both real-valued and complex-valued formulations were considered (Myklestad 1952, Bishop 1955, Reid 1956, Neumark 1957, among others), all being limited to a harmonic steady-state excitation. In 1963, Crandall (1963) raised the question of the physical unrealizability of the hysteretic dashpot, when negative frequencies are to be considered. Since then, this question remains with no satisfactory answer.

The hysteretic damping model is a pathological model since is not causal (Caughey 1962, Crandall 1963, 1970, Inaudi and Kelly 1995, among others). This non-physical behavior of the model is a flow, and further attempts have been made to cure this problem. For instance, Bishop and Price (1986) introduced the band-limited hysteretic damper and suggested that it might satisfy the causality requirement. However, Crandall (1991) showed that the band-limited hysteretic damper fails to satisfy causality.

Recently, Makris (1994a) showed that when a complex-valued formulation is used to analyze structures with hysteretic damping in the time domain, the excitation history should be complex valued, and he developed the procedure to compute the imaginary counterpart of real-valued transient records (Makris 1994b). About the same time, Inaudi and Kelly (1993, 1995), following a real valued formulation showed that in order to conduct a time domain analysis on a structure with hysteretic damping, the Hilbert transform of the excitation should be induced to the hysteretic dashpot. The procedures developed independently by Makris (1994a,b) and by Inaudi and Kelly (1993, 1995) are equivalent (see Makris et al. 1996), and consist a mathematically-consistent methodology to analyze in the time domain the transient response of structures with hysteretic damping. Nevertheless, this mathematically-consistent methodology does not remove the long-standing problem, that the "ideal" hysteretic dashpot is a non-causal element.

In this section, first the non-causality issue is addressed by investigating what is missing from the "ideal" hysteretic dashpot that makes it non-causal. Subsequently the causal hysteretic element is constructed using the Kramers-Kroning relations (Bird et al. 1987), and its behavior is analyzed both in frequency and time domain. It is found that the proposed causal hysteretic element is the limiting case of the Biot model of viscoelasticity. Finally, the response of a mass supported by the causal hysteretic element is examined.

DYNAMIC STIFFNESS OF CAUSAL HYSTERETIC ELEMENT

Since the "ideal" hysteretic dashpot is physically unrealizable, it can not be expressed with a real-valued constitutive law; and it is usually presented in terms of its dynamic stiffness,

$$\mathcal{K}(\omega) \;=\; 0 + iK_2 \operatorname{sgn}\left(\frac{\omega}{\varepsilon}\right) \tag{59}$$

where ω, is the frequency variable and ε is an arbitrate positive real number with units in rad/sec which is present to make the argument in the signum function dimensionless. The need to have dimensionless arguments becomes apparent later, where the logarithmic and exponential integral functions are involved.

The transfer function given by (59) has a zero real part and the signum function as imaginary part. It is this careless combination of real and imaginary parts that makes the "ideal" hysteretic dashpot physically unrealizable. Clearly, the real and imaginary parts of (59) do not satisfy the Kramers-Kroning relations (see equations (19) in the previous section); and this is why the "ideal" hysteretic dashpot is non-causal. In the aforementioned references several researchers investigated the *complex spring* where the zero real part in (59) is replaced with a constant, K_1. Again, in this case the same problem of non-causality arises, since the real and imaginary part fail to satisfy the Kramers-Kroning relations. In the previous section, it was shown that the fundamental relation between the analyticity of a transfer function and the causality of the corresponding time response function can be extended for the case of generalized transfer functions. This result is applied now in order to construct the causal hysteretic element.

The imaginary part of the dynamic stiffness (loss stiffness) of the causal hysteretic element is the same as the imaginary part of the "ideal" hysteretic dashpot given by (59), so that the energy dissipated by the model is frequency independent. With the imaginary part being established, the real part of the proposed model should be the Hilbert transform of, $K_2 \operatorname{sgn}(\omega/\varepsilon)$. This operation ensures that the resulting time response function of the proposed model is causal (Makris et al. 1996). Accordingly, if the dynamic stiffness of the causal hysteretic element is

$$\mathcal{K}(\omega) = K_1(\omega) + iK_2 \operatorname{sgn}\left(\frac{\omega}{\varepsilon}\right), \tag{60}$$

the storage stiffness (real part), $K_1(\omega)$, is given from equation (19)

$$K_1(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{K_2 \operatorname{sgn}(\phi)}{\frac{\omega}{\varepsilon} - \phi} d\phi . \tag{61}$$

From the theory of generalized functions (Dirac 1958, Lighthill 1989, Papoulis 1987), the evaluation of the convolution integral in (61) can be achieved using the Fourier transform. It can be shown that (61) can be expressed as (Makris 1994, Makris et al. 1996)

$$K_1(\omega) = i\int_{-\infty}^{\infty} \operatorname{sgn}(t)\, K_2(t)\, e^{-i\omega t} dt \tag{62}$$

where, $K_2(t)$, is the Fourier transform of $K_2(\omega)$,

$$K_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_2 \operatorname{sgn}(\omega)\, e^{i\omega t} d\omega = -K_2 \frac{1}{i\pi t}. \tag{63}$$

Substitution of (63) into (62) gives:

$$K_1(\omega) = -\frac{K_2}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(t)}{t} e^{-i\omega t} dt \tag{64}$$

The integral in (64) can be evaluated using the calculus of generalized function (Lighthill 1989),

$$K_1(\omega) = -\frac{K_2}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x)}{x} e^{-i2\pi xy} dx = \frac{2}{\pi} K_2 \left(ln\left|\frac{\omega}{\varepsilon}\right| + C\right) \tag{65}$$

in which, $x = t\nu$, $y = \frac{\omega}{\varepsilon}$, and $\nu = \frac{\varepsilon}{2\pi}$. C is an arbitrary constant since the function $\frac{\operatorname{sgn}(x)}{x}$, is indeterminate to the extent of an arbitrary multiple of $\delta(x)$. By setting, $K_1 = (2/\pi) K_2 C$, and after substituting (65) into (60), one obtains the dynamic stiffness of the causal hysteretic model,

$$\mathcal{K}(\omega) = K_1\left[1 + \frac{2}{\pi}\eta\, ln\left|\frac{\omega}{\varepsilon}\right| + i\eta \operatorname{sgn}\left(\frac{\omega}{\varepsilon}\right)\right], \tag{66}$$

where, $\eta = K_2/K_1$, is a positive real number named hysteretic damping coefficient. Equation (66) shows that while the loss stiffness (imaginary part) of the causal hysteretic element is frequency independent, its storage stiffness (real part) is frequency dependent, approaching negative infinity as frequency tends to zero. Nevertheless, the arbitrary constant, ε, does not depend on the physics of the problem; and can be set as small as desired so that the real part of (66) matches a realistic stiffness value measured at any finite frequency. Consequently, the causal hysteretic element is physically realizable at any finite frequency; whereas is not defined at $\omega = 0$.

The dynamic stiffness of the causal hysteretic element given by (66), is nothing more than the familiar non-causal *complex-stiffness element,* $K_1[1 + i\eta\,\mathrm{sgn}(\omega)]$, (Clough and Penzien 1991, Chopra 1994, Inaudi and Kelly 1995, among others), enhanced with the term $\frac{2}{\pi}\eta\, ln\left|\frac{\omega}{\varepsilon}\right|$ in its real part in order to satisfy causality. Figure 2 shows the real and imaginary parts of the dynamic stiffness given by (66) (solid line) for different values of $\eta = K_2/K_1$.

It is interesting to show that the dynamic stiffness of the causal hysteretic element given by (66) is related with the complex Heaviside function, $\Xi(\omega)$, introduced in the previous section

$$\Xi(\omega) = \xi(\omega) + i\frac{1}{\pi}ln\left|\frac{\omega}{\varepsilon}\right|. \tag{67}$$

With $\xi(\omega) = \frac{1}{2}\left[1 + \mathrm{sgn}\left(\frac{\omega}{\varepsilon}\right)\right]$, equation (66) gives

$$\mathcal{K}(\omega) = K_1\left\{1 + 2i\eta\left[\bar{\Xi}(\omega) - \frac{1}{2}\right]\right\} \tag{68}$$

where, $\bar{\Xi}(\omega)$, is the complex conjugate of (67). So equation (68) in conjunction with equation (41), show that the positive value, ε, in (66), is merely the value of the limits of the integral defined by Dirac (1958) in order to make the reciprocal function well defined in the neighborhood of zero.

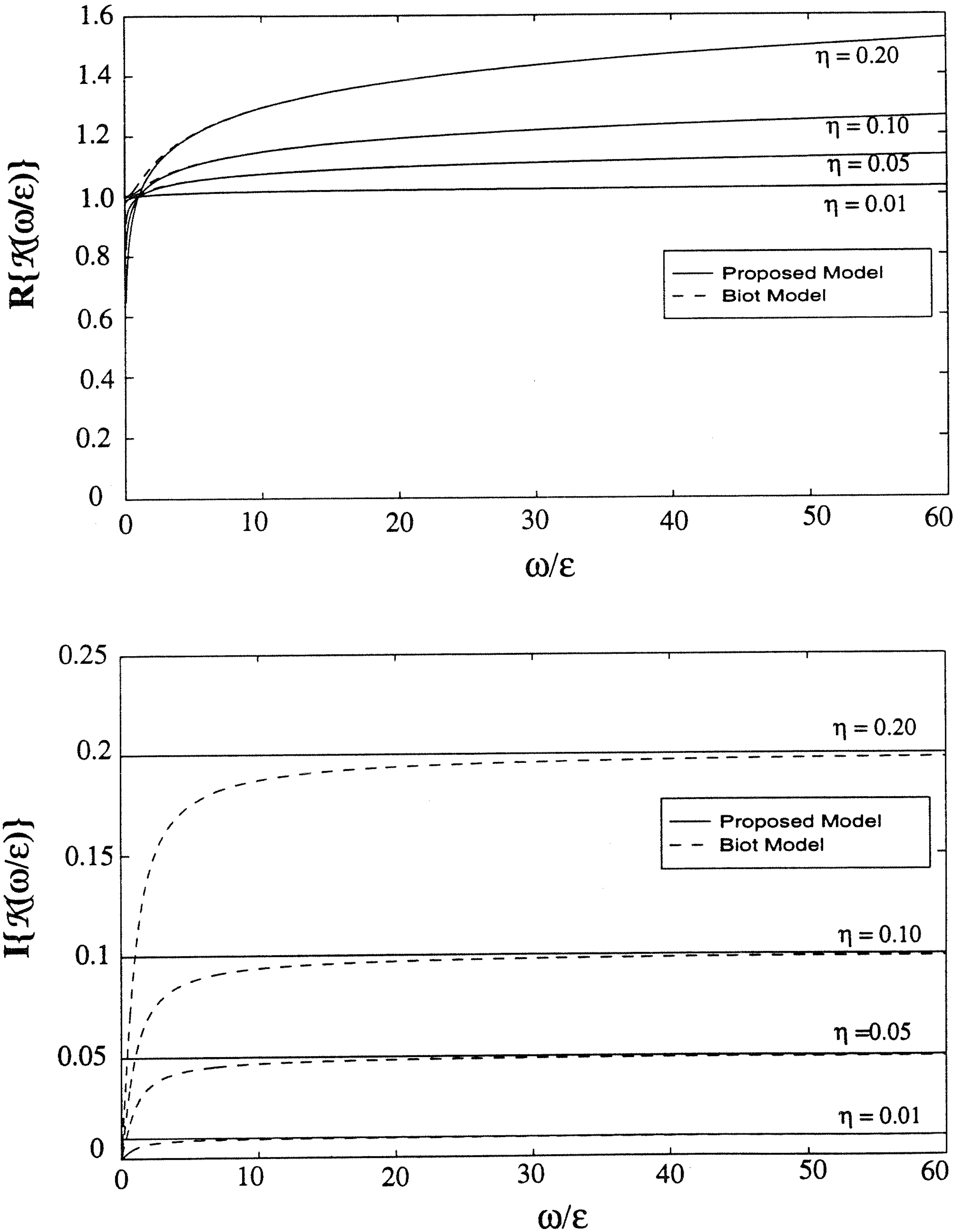

Figure 2. Real and imaginary parts of the dynamic stiffness of the causal hysteretic element (solid line) and of the Biot model (dashed line).

The proposed causal hysteretic element given by (66) or (68) was constructed by seeking a dynamic stiffness, $\mathcal{K}(\omega)$, that its real part is the Hilbert transform of its imaginary part, $K_2 \operatorname{sgn}(\omega/\varepsilon)$. The interested reader can show that the computed real part, $K_1 + \frac{2}{\pi} K_2 ln\left|\frac{\omega}{\varepsilon}\right|$, given by (65) is also the Hilbert transform of the imaginary part; and therefore the Kramers-Kroning relations are satisfied. This condition implies that the resulting time response function (memory function), q(t), should be zero at negative times.

MEMORY FUNCTION OF THE CAUSAL HYSTERETIC ELEMENT

The memory function of a constitutive model is defined as the resulting force at the present time, t, due to an impulsive displacement input at time $\tau\,(\tau < t)$, and is the inverse Fourier transform of the dynamic stiffness,

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(\omega)\, e^{i\omega t} d\omega = \nu \int_{-\infty}^{\infty} \mathcal{K}(y)\, e^{i2\pi xy} dy, \tag{69}$$

in which, $x = t\nu$, $y = \frac{\omega}{\varepsilon}$, and $\nu = \frac{\varepsilon}{2\pi}$. Substitution of (66) into (5) gives:

$$\frac{q(t)}{\nu} = K_1 \left[\delta(x-0) + \frac{2}{\pi}\eta \int_{-\infty}^{\infty} ln|y| e^{i2\pi xy} dy + i\eta \int_{-\infty}^{\infty} \operatorname{sgn}(y)\, e^{i2\pi xy} dy \right], \tag{70}$$

and from the calculus of generalized functions (Lighthill 1989),

$$\frac{q(t)}{\nu} = K_1 \left\{ \delta(x-0) + \frac{2}{\pi}\eta \left[-i\pi \frac{\operatorname{sgn}(x)}{2i\pi x} \right] + i\eta \left[\frac{-2}{2i\pi x} \right] \right\} \tag{71}$$

After replacing x/ν in (71) with t, the expression for the memory function of the causal hysteretic element becomes:

$$q(t) = K_1 \left[\delta(x-0) - \frac{\eta}{\pi} \frac{1}{t} (1 + \operatorname{sgn}(t)) \right]. \tag{72}$$

Equation (72) is zero for $t < 0$, showing that the proposed model is indeed causal.

The physically realizable response of the causal hysteretic element is compared to the non-causal response of the complex-stiffness model discussed in detail by Inaudi and Kelly (1995). Under a displacement induced excitation the force response is

$$P(t) = \int_{-\infty}^{t} q(t-\tau)\, u(\tau)\, d\tau. \tag{73}$$

Substitution of (72) into (73) yields a time domain force-displacement relation for the causal hysteretic element

$$P(t) = K_1 \left[u(t) - \frac{\eta}{\pi} \int_{-\infty}^{t} \frac{u(\tau)}{t-\tau} d\tau \right] \tag{74}$$

On the other hand the force response of the non-causal complex stiffness model is (Inaudi and Kelly 1995)

$$P(t) = K_1 \left[u(t) - \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau \right] \tag{75}$$

The only difference between equations (74) and (75) is that the upper limit in the convolution integral in (74) is, t, whereas in (75) is ∞. Because of this difference equation (74) is causal whereas (75) is not.

IMPEDANCE AND RELAXATION STIFFNESS OF THE CAUSAL HYSTERETIC ELEMENT

The impedance (velocity transfer function) of the causal hysteretic element results from the expression of its dynamic stiffness given by (66).

$$Z(\omega) = \frac{\mathcal{K}(\omega)}{i\omega} = \frac{1}{\varepsilon} K_1 \left[\eta \frac{\text{sgn}(y)}{y} - i \left(\frac{1}{y} + \frac{2}{\pi} \eta \frac{ln|y|}{y} \right) \right] \tag{76}$$

where $y = \frac{\omega}{\varepsilon}$. Again, since $\frac{\text{sgn}(y)}{y}$ is indeterminate to the extent of a delta function, the term $C\delta(\omega - 0)$ should be added in (76). The constant, C, is determined from the Kramers-Kroning relations. In Appendix II is shown that $C = \pi K_1 - 2K_2 [ln(2\pi) + \gamma]$, where γ= 0.5772 is the Euler constant. Accordingly the complete expression for the impedance of the causal hysteretic element is

$$\varepsilon Z\left(\frac{\omega}{\varepsilon}\right) = K_1 \{ [\pi - 2\eta [ln(2\pi) + \gamma]] \delta(y - 0) + \eta \frac{\text{sgn}(y)}{y} - i \left(\frac{1}{y} + \frac{2}{\pi} \eta \frac{ln|y|}{y} \right) \} \tag{77}$$

Figure 3 (solid line) shows the real and imaginary part of the impedance given by (77).

The relaxation stiffness, $k(t)$, is defined as the resulting force at the present time t, for a unit step displacement at time $\tau (\tau < t)$, and is the inverse Fourier transform of the impedance,

$$k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega = \nu \int_{-\infty}^{\infty} Z(y) e^{i2\pi xy} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon Z(y) e^{i2\pi xy} dy \tag{78}$$

where, $\varepsilon Z(y)$, in the last integral of (78) is given by (77). Following the integration rules of generalized functions the relaxation stiffness of the causal hysteretic element is:

$$k(t) = K_1 \{ \frac{1}{2} - \frac{\eta}{\pi} [ln|\varepsilon t| + \gamma] \} [1 + \text{sgn}(\varepsilon t)] , \tag{79}$$

which is also a causal function. Under a prescribed velocity excitation, the force response is

$$P(t) = \int_{-\infty}^{t} k(t - \tau) \dot{u}(\tau) d\tau, \tag{80}$$

and substitution of (79) into (80) gives:

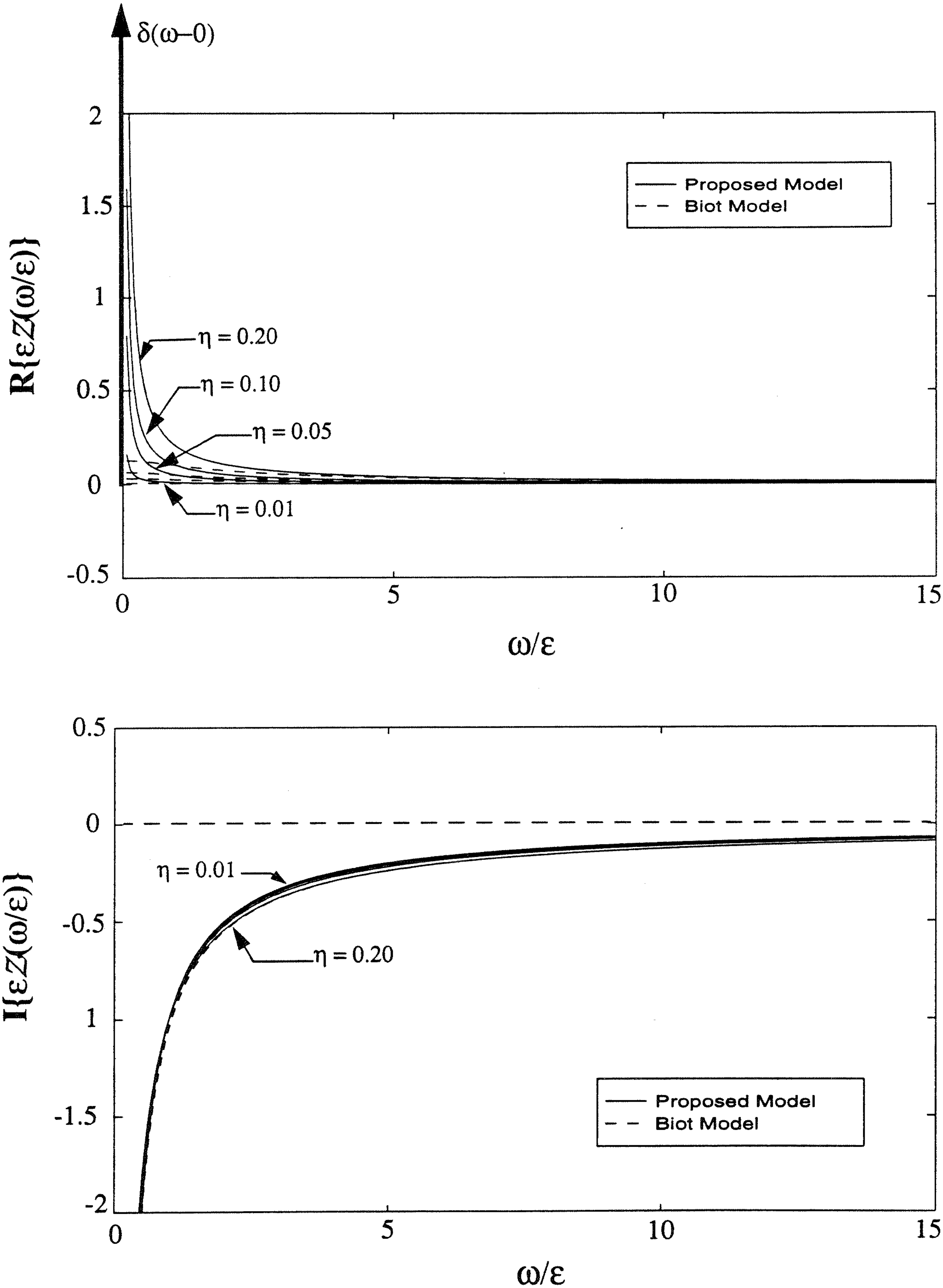

Figure 3. Real and Imaginary parts of the impedance of the causal hysteretic element (solid lines) and the Biot model (dashed lines).

$$P(t) = K_1 \left\{ u(t) - \frac{2}{\pi}\eta \int_{-\infty}^{t} (ln[\varepsilon(t-\tau)] + \gamma) \frac{du(\tau)}{d\tau} d\tau \right\}. \qquad (81)$$

Equation (81) is an alternative time domain force-displacement expression of the causal hysteretic element which is the limiting case of a linear viscoelastic model which was proposed nearly forty years ago by Biot (1958). This shows that there is a continues transition from a linear viscoelastic model to a model that is ideally hysteretic.

COMPARISON WITH THE BIOT'S MODEL OF VISCOELASTICITY

The first successful model exhibiting linear hysteretic damping was proposed by Biot (1958). In his paper entitled "*Linear thermodynamics and the mechanics of solids*," Biot, after discussing fundamental concepts and properties in continuum mechanics he presented a hysteretic model with nearly frequency independent dissipation with dynamic stiffness,

$$\mathcal{K}(\omega) = K_1 \left[1 + \frac{2}{\pi}\eta\, ln\sqrt{1 + \left(\frac{\omega}{\varepsilon}\right)^2} + i\frac{2}{\pi}\eta\, \mathrm{atan}\frac{\omega}{\varepsilon} \right]. \qquad (82)$$

At the limiting case where, $\varepsilon \ll \omega$ ($\sqrt{1+(\omega/\varepsilon)^2} \to |\omega/\varepsilon|$ and $\mathrm{atan}(\omega/\varepsilon) \to \pi/2$), the dynamic stiffness of the Biot model given by (82) reduces to the dynamic stiffness of the causal hysteretic element given by (66). Figure 2 compares the dynamic stiffness of the Biot mode (dashed lines) with the dynamic stiffness of the causal hysteretic element (solid lines).

While the two models are practically the same when ω/ε is sufficiently large for a given value of η, they behave differently as ω/ε approaches zero. At the zero limit the Biot model maintains a finite storage stiffness, but its loss stiffness depends linearly with frequency (standard Kelvin model). At the zero limit the causal hysteretic element maintains a frequency independent loss stiffness but the storage stiffness becomes strongly frequency dependent reaching negative values. Nevertheless, since ε does not depend on the physics of the problem, it can be arbitrary small, and the two models will yield the same answer for all practical purposes. One possible advantage of the causal hysteretic element is that integral transform techniques become a routine operation with the theory of generalized functions (Lighthill 1989).

The behavior of the two models is also compared in the time domain. Four years after Biot's (1958) paper appeared, Caughey (1962) presented and excellent paper where he studied the response of a mass supported by the Biot model. In his paper Caughey showed that the Biot model can be constructed using linear viscoelasticity by combining in series a linear spring, K_1, with an infinite number of Maxwell elements (Bird et al 1987). A time-domain representation of the Biot model is (Caughey 1962),

$$P(t) = K_1 \left\{ u(t) - \frac{2}{\pi}\eta \int_{-\infty}^{t} E_i[-\varepsilon(t-\tau)] \frac{du(\tau)}{d\tau} d\tau \right\} \tag{83}$$

where $E_i(x)$, is the exponential integral (Abramowitz and Stegun 1972),

$$E_i(x) = \int_{\infty}^{-x} \frac{e^{-\xi}}{\xi} d\xi. \tag{84}$$

The exponential integral kernel, $E_i[-\varepsilon t]$, appearing in (83) is directly related to the kernel of the causal hysteretic element, $ln(\varepsilon t) + \gamma$, appearing in (81) with the identity,

$$E_i(-\varepsilon t) = -E_1(\varepsilon t) = ln|\varepsilon t| + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n (\varepsilon t)^n}{nn!}. \tag{85}$$

Identity (85) shows that the kernels of the two models differ by the term, $\sum_{n=1}^{\infty} \frac{(-1)^n (\varepsilon t)^n}{nn!}$, which vanishes when εt is sufficiently small ("recent past"). However, since ε can be arbitrary small, the "recent past" can be as remote as desired in terms of real time, t; and the two models can yield the same answer for all practical purposes.

From equation (83) one can derive directly the relaxation stiffness of the Biot model

$$k(t) = K_1\left[1 - \frac{2}{\pi}\eta E_i(-\varepsilon t)\right]. \tag{86}$$

Figure 4 compares the relaxation stiffness resulting from the Biot model (dashed line) with the relaxation stiffness of the causal hysteretic element given by (79) (solid line).

RESPONSE OF MASS SUPPORTED BY THE CAUSAL HYSTERETIC SPRING

The solution of the equation,

$$m\frac{d^2u(t)}{dt} + P(t) = F(t), \tag{87}$$

where F(t) is the excitation history and P(t) is the reaction from the Biot model given by (83), has been investigated in depth by Caughey (1962). In his paper Caughey addressed the problem of free oscillations ($F(t) = 0$), impulse response ($F(t) = \delta(t-0)$), and forced oscillations; and he derived closed form solutions of (87) for the case where η and ε are small (in Caughey's paper $\mu = \varepsilon\sqrt{m/K_1}$).

In the case where the mass is supported by the causal hysteretic element, the reaction, P(t), in (87), is given by (81) rather than (83). Nevertheless, based on the forgoing analysis equations (81) and (83) behave the same when εt is sufficiently small; and therefore, the closed-form solutions presented by Caughey (1962) are perfectly valid when the proposed model is considered. Nevertheless, in the case of the causal hysteretic element Caughey's solutions can be simplified because in this limiting case, $\mu = \varepsilon\sqrt{m/K_1}$, is arbitrary small and the terms $exp(-\mu t)$ and $-E_i(-\mu t)$ in Caughey's solutions can be replaced with, 1 and $\gamma + ln(\varepsilon t)$, respectively.

Herein we only present the frequency response transfer function of a mass supported by the causal hysteretic element. The Fourier transform of (87) gives

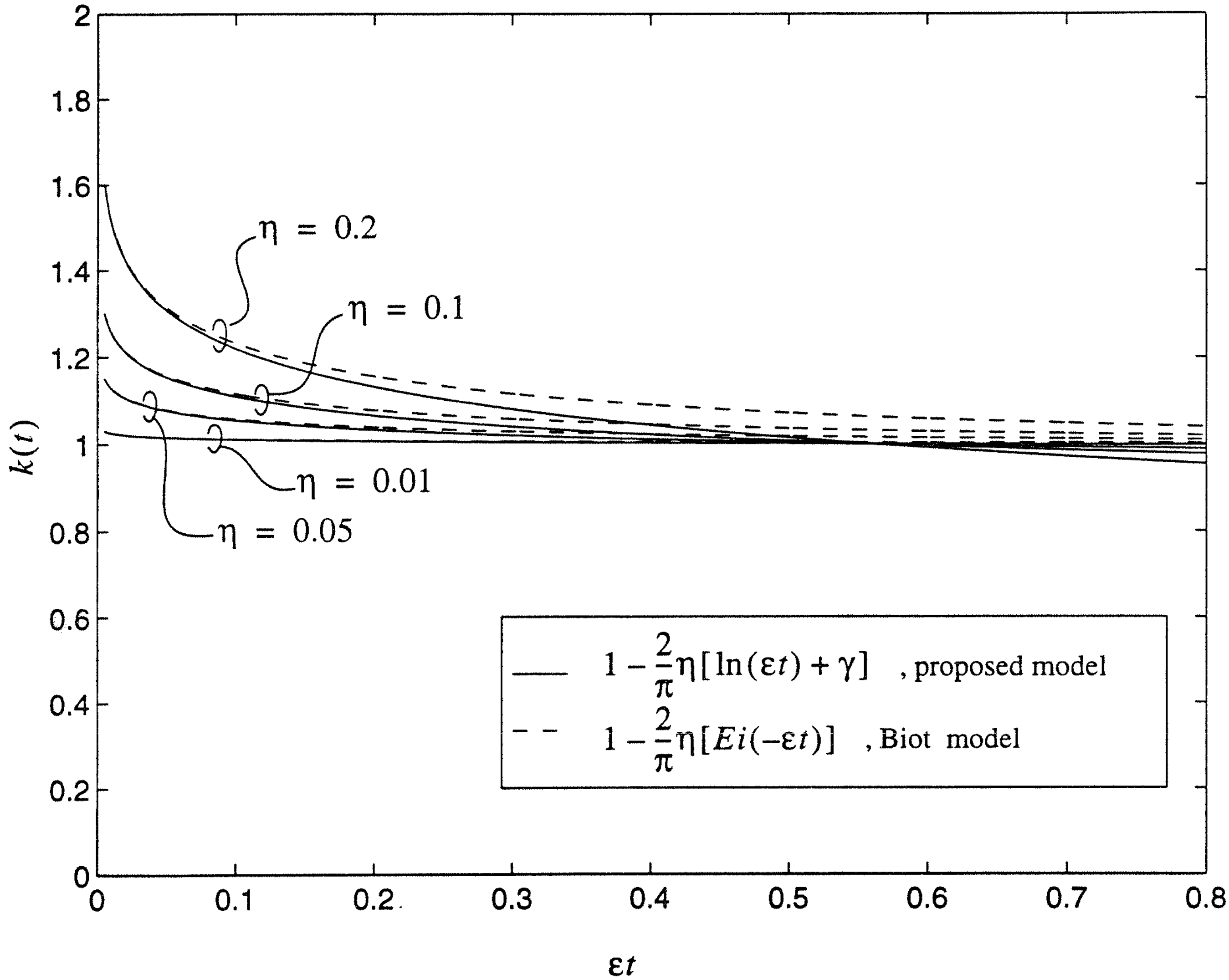

Figure 4. Relaxation stiffness of the causal hysteretic element (solid lines) and of the Biot model (dashed lines).

$$H(\omega) = \frac{F(\omega)}{u(\omega)} = \frac{1}{m\{-\omega^2 + \omega_0^2\left[1 + \frac{2}{\pi}\eta\, ln\left|\frac{\omega}{\varepsilon}\right| + i\eta\, \mathrm{sgn}(\omega)\right]\}} \tag{88}$$

where $\omega_0 = \sqrt{K_1/m}$. With the frequency ratio, $\beta = \omega/\omega_0$, equation (88) takes the form

$$H(\omega) = \frac{1}{K_1}\frac{1}{1 - \beta^2 + \frac{2}{\pi}\eta\, ln\left|\frac{\omega}{\varepsilon}\right| + i\eta\, \mathrm{sgn}(\omega)} \tag{89}$$

Figure 5 plots the amplitude of the frequency response function for $\varepsilon = 1$ (top) and $\varepsilon = 0.1$ (bottom) and different values of η. As ε becomes smaller the storage stiffness of the model increases and the resonant peaks move to the right.

CONCLUSIONS

In this section the issue of developing a causal macroscopic model that generates frequency independent energy dissipation was addressed in a rigorous and systematic manner. The *causal hysteretic element* was constructed by adopting the imaginary part of the "ideal" hysteretic damper, and computing the appropriate real part that makes the model causal. The proposed model was constructed by requiring that the real and imaginary parts of its transfer functions satisfy the Kramers-Kroning relations -- a condition that ensures that the corresponding time response functions are zero at negative times. The causal hysteretic element is physically realizable at finite frequencies; whereas at $\omega = 0$ is not defined. The behavior of the proposed model was analyzed both in frequency and time domain, and it was shown that the causal hysteretic element is the limiting case of a linear viscoelastic model with nearly frequency-independent dissipation that was proposed by Biot (1958). This finding demonstrates that there is a continues transition from a linear viscoelastic model to a model that is ideally hysteretic. Finally, the response of a mass supported by the causal hysteretic element was discussed with reference to the solutions presented by Caughey (1962).

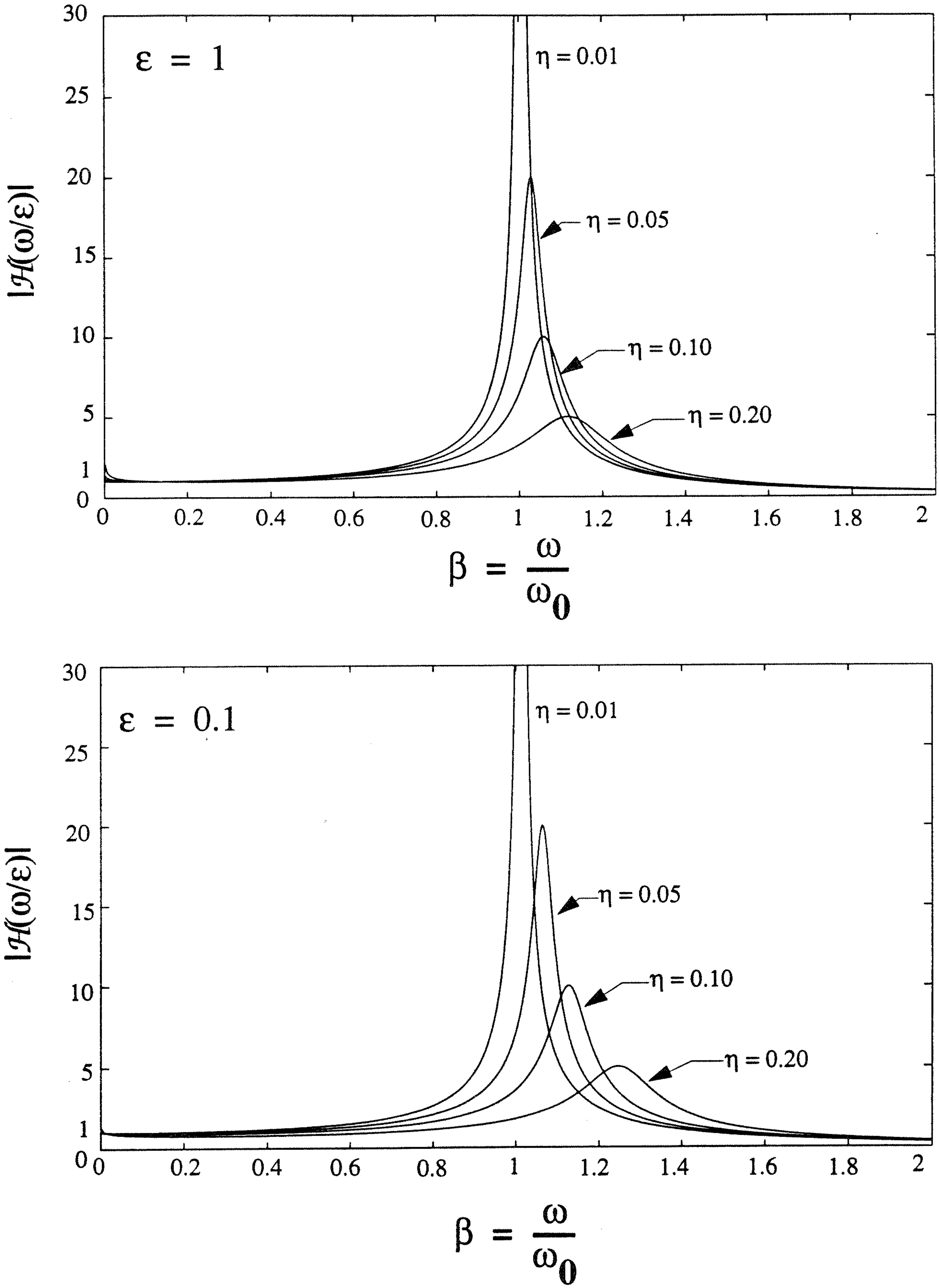

Figure 5. Frequency response function of a mass supported by the causal hysteretic element.

Appendix I

The real part of the impedance given in by (77) was obtained by computing the Hilbert transform of its imaginary part. This is possible by applying the convolution integral theorem in conjunction with the Fourier transform of generalized functions (Makris 1994, Makris et al 1996). Accordingly,

$$\varepsilon Z_1(y) = i\int_{-\infty}^{\infty} \operatorname{sgn}(x)\,\varepsilon Z_2(x)\,e^{-i2\pi xy}dx, \qquad (90)$$

in which, $\varepsilon Z_2(x)$, is the Fourier transform of $\varepsilon Z_2(y)$ appearing in (76).

$$\varepsilon Z_2(x) = K_1\int_{-\infty}^{\infty} -\left(\frac{1}{y} + \frac{2}{\pi}\eta\frac{ln|y|}{y}\right)e^{i2\pi xy}dy, \qquad (91)$$

and after evaluating the integrals in (91)

$$\varepsilon Z_2(x) = (-K_1)\,i\pi\operatorname{sgn}(x)\left[1 - \frac{2}{\pi}\eta\{\log(2\pi|x|) + \gamma\}\right]. \qquad (92)$$

Substitution of (92) into (62) gives:

$$\varepsilon Z_1(y) = K_1[\pi - 2\eta\{\log(2\pi) + \gamma\}]\,\delta(y-0) + K_2\frac{\operatorname{sgn}(y)}{y}, \qquad (93)$$

and therefore, the constant C is

$$C = \pi - 2\eta\{\log(2\pi) + \gamma\}. \qquad (94)$$

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