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ABSTRACT

A boundary value problem of quasi-static deformation of physically nonlinear viscoelastic solids is discussed in the context of providing a scheme for characterization of material response in a manner compatible with an algorithm for numerical solution of boundary value problems. An illustrative example involving axial deformation of a statically indeterminate nonlinear viscoelastic rod is included.

1. INTRODUCTION

Numerical solution of boundary value problems in the mechanics of nonlinear deformable solids depends intimately upon two factors: characterization of the mechanical properties of the solid (representation of the constitutive response functional) and development of an algorithm leading to a solution scheme for the nonlinear differential (or integral) mechanical field equations appropriate to the problem. Unfortunately, from an engineering viewpoint, these two areas have often been studied almost independently by workers in widely separated disciplines. As a result progress in the area of engineering analysis of nonlinear solids has been slower than either of the underlying areas of mechanical constitutive theory or numerical analysis. The present paper is concerned with this difficulty--the questions of mechanical characterization and numerical solution are considered simultaneously so that the experimentation required for mechanical characterization of a particular solid forms an integral part of the computational algorithm for a particular boundary value problem. This is accomplished by examining the response of the solid in the neighborhood of a (prescribed) reference state, a technique which has been discussed previously in a different context by Pipkin [1], among others.

The details of this method of characterization are presented in Section 2, following some preliminaries. In Section 3 is presented a solution algorithm for a class of boundary value problems associated with quasi-static deformation of physically nonlinear viscoelastic solids. The algorithm is deduced from a finite element approximation of a stationary value problem for a functional. In Section 4 a simple example is presented to illustrate the method. The problem considered is that of a statically indeterminate nonlinear viscoelastic rod subjected to time-dependent axial forces.

2. MECHANICAL CHARACTERIZATION AT A PRESCRIBED REFERENCE STATE

To place emphasis on the characterization problem and its subsequent interaction with a solution algorithm for boundary value problems, we limit discussion to nonlinear viscoelastic (or elastic) solids undergoing quasi-static deformations for which linear kinematic assumptions are adequate. We call these materials physically nonlinear and adopt as mechanical field variables the small strain tensor $\underset{\sim}{e}$ and stress tensor $\underset{\sim}{\sigma}$, both referred to a fixed curvilinear coordinate system.* We suppose that the temperature of the body, as well as all other non-mechanical fields, remains constant. We may then define a physically nonlinear viscoelastic solid by the constitutive equation

$$\underset{\sim}{\sigma}(\underset{\sim}{x},\, t) = \underset{s=-\infty}{\overset{s=t}{\underset{\sim}{f}}} \left[\underset{\sim}{e}(\underset{\sim}{x},\, s);\ \underset{\sim}{x},\, t\right] \tag{2.1}$$

The mechanical response functional $\underset{\sim}{f}$ assigns to each strain history at the place $\underset{\sim}{x}$ the value of the stress tensor at the current time. The appearance of the arguments $(\underset{\sim}{x},\, t)$ in the functional allows for the qualities of non-homogeneity and aging in the mechanical response of the solid. Although this symbolic statement is logically explicit, it is clear that a specific mathematical form must be ascribed to the representation if calculations are to be performed. This is precisely the point at which the crux of the nonlinear boundary value problem lies--the tendency has been to incorporate a highly over-simplified "mechanical model" of the solid, without regard to its limits of applicability, in order to obtain a tractable boundary value

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* Direct notation will be employed; i.e., symbols underlined with a tilde are tensors of order indicated by the context. See [2] for a discussion of this notation.

problem; on the other hand, in rational mechanics the response functional is often regarded as "defined" by expressions such as (2.1). We will adopt an intermediate position with the specific intention of providing a representation of the solid suitable for use in solving problems. To this end we suppose that a reference history $\bar{\underset{\sim}{e}}(s)$ is specified. The collection of reference histories for all points in the solid will be called a reference state. Assuming that the solid is initially quiescent, from (2.1) we have at any point*

$$\bar{\underset{\sim}{\sigma}}(t) = \underset{\sim}{f}\,{}_{s=0}^{s=t}\,[\bar{\underset{\sim}{e}}(s)] \tag{2.2}$$

We will further suppose that the value of the functional $\underset{\sim}{f}$ for the history $\bar{\underset{\sim}{e}}(s)$ can be determined experimentally. This means that the multiaxial stress state corresponding to a multiaxial deformation history can be measured. (Clearly, in some instances there may be limitations imposed by current experimental technology.) We now consider the response resulting from a small change in the input strain history; i.e., replace $\bar{\underset{\sim}{e}}$ by $\bar{\underset{\sim}{e}} + \underset{\sim}{\varepsilon}$ in (2.2), where the history $\underset{\sim}{\varepsilon}(s)$ is assumed to be small.** Using a Fréchet expansion of the functional [3], we can write

$$\underset{\sim}{f}\,{}_{s=0}^{s=t}\,[\bar{\underset{\sim}{e}}(s) + \underset{\sim}{\varepsilon}(s)] \triangleq \underset{\sim}{f}\,{}_{s=0}^{s=t}\,[\bar{\underset{\sim}{e}}(s)] + \int_0^t \underset{\sim}{h}\,{}_{s=0}^{s=t}\,[\bar{\underset{\sim}{e}}(s);\ t - \tau]: \frac{\partial \underset{\sim}{\varepsilon}}{\partial \tau}\, d\tau \tag{2.3}$$

where the colon denotes doubly contracted composition of the fourth rank

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* In the sequel we will assume that the solid is non-aging.

** This implies that the norm of the history is small; we prefer to proceed on an intuitive basis here.

tensor $\underset{\sim}{h}$ with the second rank tensor $\partial\underset{\sim}{\varepsilon}/\partial\tau$. Note that the kernel function $\underset{\sim}{h}$ in the linear functional appearing in (2.3) is itself a functional of the reference strain history; if the reference history is the zero history, (2.3) reduces to the usual hereditary integral representation of a linear functional. Note further that the linear functional in (2.3) resembles that of a linear, anisotropic, aging viscoelastic solid; i.e., characterization at a non-zero reference state produces an effect on the material equivalent to aging. Finally, we repeat that the accuracy of the approximation in (2.3) depends upon the magnitude of $\underset{\sim}{\varepsilon}(s)$ as measured by a suitable norm. In some neighborhood of the reference history $\bar{\underset{\sim}{e}}(s)$, (2.3) will provide a predictive characterization of the solid. The form of the equation suggests the pattern for an experimental program to characterize the solid in a narrow band surrounding the reference history $\bar{\underset{\sim}{e}}(s)$. First conduct an experiment using $\bar{\underset{\sim}{e}}(s)$ as the input, measuring $\bar{\underset{\sim}{g}}(t)$ appearing in (2.2) and (2.3). The experiment is then repeated for a strain history input $\underset{\sim}{e} = \bar{\underset{\sim}{e}} + \underset{\sim}{\varepsilon}_o H(t-\tau_k)$ where $\underset{\sim}{\varepsilon}_o$ is a constant tensor, and H is the Heaviside step function. Substituting this input into (2.3) there follows

$$\underset{\sim}{f}[\underset{\sim}{e}] = \underset{\sim}{f}[\bar{\underset{\sim}{e}}] + \underset{s=0}{\overset{s=t}{\underset{\sim}{h}}}\ [\bar{\underset{\sim}{e}}(s);\ t - \tau_k]:\ \underset{\sim}{\varepsilon}_o \tag{2.4}$$

Since the two output functions $\underset{\sim}{f}[\underset{\sim}{e}]$, $\underset{\sim}{f}[\bar{\underset{\sim}{e}}]$ are measured in the experiment and $\underset{\sim}{\varepsilon}_o$ is a prescribed constant tensor, (2.4) constitutes a set of simultaneous linear equations for a trace of the components of the tensor $\underset{\sim}{h}$ in a two-dimensional time space with coordinates $(t,\ t - \tau_k)$. Note that since $\bar{\underset{\sim}{e}}(s)$ is a prescribed history, the value of the kernel appearing in (2.4) can be written

$$\underset{\sim}{h} = \underset{\sim}{h}(t, t-\tau_k) \tag{2.5}$$

By repeating the experimental procedure for a sequence of values of τ_k the function $\underset{\sim}{h}$ (more properly the components of the tensor-valued function $\underset{\sim}{h}$) may be generated by the set of its experimentally determined traces in the time space. The number of experiments required to characterize the material at a prescribed reference history (reference state) clearly will depend on the complexity of the strain history and degree of symmetry of the solid in its initial state. Graphical representation of a typical component of $\underset{\sim}{h}$, and its experimental traces, is shown in Figure 1. Path (2) corresponding to $\tau_k = 0$ can be taken as a plane of symmetry without loss of generality. For values of $\tau_k > 0$, traces defined by curves such as path (1) will result. Note that all components of h vanish for $t - \tau_k < 0$ to satisfy the condition that future strain inputs cannot affect the current value of the stress.

To summarize this section on characterization, we have described a procedure for determining "incremental" linear viscoelastic material functions for the purpose of characterizing a physically nonlinear viscoelastic solid at a prescribed reference state. In order to guide the choice of reference states we must look to a solution scheme for nonlinear boundary value problems in order to set the experimentally chosen reference states in correspondence with deformation state iterates in a numerical solution algorithm for a boundary value problem. In this way only those experiments actually required to effect a solution of the problem will be called for.

3. SOLUTION ALGORITHM FOR BOUNDARY VALUE PROBLEMS

The characterization method described in the previous section has been introduced to provide a natural interface with iterative computational

methods for linear boundary value problems. We have previously defined a physically nonlinear viscoelastic solid in terms of a constitutive response functional, which in turn requires experimental determination. At the same time, the mechanical variables (stress, strain, and displacement) constitute fields which must satisfy certain principles of mechanics: mass conservation, continuity of deformation, and balance of momentum. Together with prescribed boundary data describing external actions on the solid, a particular physical system can be represented by a boundary value problem. For nonlinear solids a direct attack on the resulting differential or integro-differential equations is usually not fruitful. What we have emphasized in Section 2 is a method of characterization which lends itself to a method of numerical analysis of the field problem based upon global balance of linear momentum. The latter balance equation is equivalent to the condition that the virtual work vanish for arbitrary admissible virtual displacements associated with an equilibrium configuration of the solid. Accordingly, with ρ designating the mass density of the solid, and $\underset{\sim}{b}$ the body force vector per unit mass, we adopt as our global balance principle

$$\int_V [\underset{\sim}{\sigma}:\delta\underset{\sim}{e} - \rho\underset{\sim}{b}\cdot\delta\underset{\sim}{u}]\, dv - \int_{S_\sigma} \underset{\sim}{\bar{t}}\cdot\delta\underset{\sim}{u}\, ds = 0 \tag{3.1}$$

In (3.1) we assume that the symmetric stress tensor $\underset{\sim}{\sigma}$ is given by the constitutive equation (2.1) and that the strain tensor and displacement vector satisfy the set of strain-displacement equations.

$$2\underset{\sim}{e} = \nabla\underset{\sim}{u} + (\nabla\underset{\sim}{u})^T \tag{3.2}$$

Furthermore, the virtual displacement vector $\delta\underset{\sim}{u}$ vanishes on the part of the boundary S_u of the solid over which displacements are specified, and

the traction vector $\bar{\underset{\sim}{t}}$ is assigned on the part of the boundary S_σ, which is the complement of S_u. We now write the balance equation (3.1) for a neighboring equilibrium state obtained by replacing $\underset{\sim}{e}$ by $\bar{\underset{\sim}{e}} + \underset{\sim}{\varepsilon}$, $\underset{\sim}{u}$ by $\bar{\underset{\sim}{u}} + \underset{\sim}{u}$ and evaluating the stress $\underset{\sim}{\sigma}$ using the right hand side of (2.3). Substituting these replacements in (3.1) and noting that $\delta\bar{\underset{\sim}{e}} = \delta\bar{\underset{\sim}{u}} = 0$ since $\bar{\underset{\sim}{e}}$ and $\bar{\underset{\sim}{u}}$ are prescribed functions of space and time, we obtain

$$\int_V [\underset{\sim}{\sigma}:\delta\underset{\sim}{\varepsilon} - \rho\underset{\sim}{b}\cdot\delta\underset{\sim}{u}]\, dv - \int_{S_\sigma} \bar{\underset{\sim}{t}}\cdot\delta\underset{\sim}{u}\, ds = 0 \tag{3.3}$$

where the stress is given by

$$\underset{\sim}{\sigma} = \underset{s=0}{\overset{s=t}{\underset{\sim}{f}}}\,[\bar{\underset{\sim}{e}}(s)] + \int_0^t \underset{s=0}{\overset{s=t}{\underset{\sim}{h}}}\,[\bar{\underset{\sim}{e}}(s);\ t-\tau]:\frac{\partial\underset{\sim}{\varepsilon}}{\partial\tau}\, d\tau \tag{2.3}_1$$

In the sequel $\underset{\sim}{\sigma}$ is the stress associated with the actual state of deformation of the solid, while $\underset{\sim}{u}$ and $\underset{\sim}{\varepsilon}$ are respectively the incremental displacement and strain relative to a prescribed reference state defined by the displacement $\bar{\underset{\sim}{u}}$ and strain $\bar{\underset{\sim}{e}}$ relative to an undeformed initial state of the solid. At this point it will be more convenient to employ standard reduced notation [4] to represent the tensors appearing in the field equations as vectors in a six-space; the components of the vectors are physical components of the tensors referred to appropriate orthogonal curvilinear coordinates. Proceeding in this fashion, we define the vectors

$$\sigma_i = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})$$

$$\varepsilon_i = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12})$$

$$u_i = (u_1, u_2, u_3, 0, 0, 0) \tag{3.4}$$

$$b_i = (b_1, b_2, b_3, 0, 0, 0)$$

$$\bar{t}_i = (\bar{t}_1, \bar{t}_2, \bar{t}_3, 0, 0, 0)$$

corresponding, respectively, to the stress and incremental strain tensors, incremental displacement vector, body force and surface traction vectors. A common range $i = 1, 2, \ldots, 6$ is chosen to avoid complication of the notation. The constitutive equation $(2.3)_1$ can now be written

$$\sigma_i = f_i + k_{ij}\varepsilon_j \tag{3.5}$$

where summation convention on repeated indices is understood and

$$k_{ij} = \int_0^t h_{ij} \frac{\partial}{\partial \tau} (\ ) \, d\tau \tag{3.6}$$

h_{ij} is the symmetric* 6×6 matrix of the independent components of the fourth rank tensor $\underset{\sim}{h}$. In (3.5) and (3.6) the components f_i of the tensor-valued functional $\underset{\sim}{f}$ are determined experimentally for a prescribed reference history $\underset{\sim}{\bar{e}}(s)$. The same holds true for the matrix h_{ij}. Accordingly, the (symmetric) linear matrix operator k_{ij} is a functional of the reference history. We now express the balance principle (3.3) in reduced notation. Using $(2.3)_1$, (3.4) and (3.5) in (3.3):

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* In general, the linear anisotropic viscoelastic relaxation modulus is taken as symmetric [5].

$$\int_V [(f_i + k_{ij}\varepsilon_j)\delta\varepsilon_i - \rho b_i \delta u_i]\, dv - \int_{S_\sigma} \bar{t}_i \delta u_i ds = 0 \qquad (3.7)$$

Recalling that the functions appearing in (3.7) depends on both space and time, we treat first the spatial problem. The functional (3.7) can be discretized (in space) following the well-known Ritz or finite element method [6]. For a particular boundary value problem the domain V is divided into regular sub-domains V_n and (3.7) is replaced by the sum

$$\sum_{n=1}^{N} \left\{ \int_{V_n} [(f_i^n + k_{ij}^n \varepsilon_j^n)\delta\varepsilon_i^n - \rho^n b_i^n \delta u_i^n] dV_n - \int_{S_{\sigma_n}} \bar{t}_i^n \delta u_i^n dS_n \right\} = 0 \qquad (3.8)$$

The label n on each of the variables or prescribed functions indicates the support of the function and is not subject to the summation convention; V_n denotes the volume of the "nth" sub-domain and S_{σ_n} the part of the surface of V_n on which the traction vector $\bar{t}_i^n$ is prescribed.

Over each sub-domain the incremental displacement vector field is approximated by a polynomial in spatial coordinates with time-dependent coefficients;

$$u_i^n(\underset{\sim}{x}, t) = A_{ij}^n(\underset{\sim}{x})\alpha_j^n(t) \qquad (3.9)$$

The vector $\alpha_j^n(t)$ constitutes a set of generalized kinematic coordinates for the nth sub-domain. These generalized coordinates are converted to nodal point values by the transformation

$$\alpha_j^n(t) = \phi_{jk}^n q_k^n(t) \qquad (3.10)$$

The range of the index k on the nodal point displacement vector q_k^n for the nth element depends upon the number of nodal points per element as well as the kinematic conditions of a particular problem (i.e., plane strain, etc.). A similar statement applies to the transformation matrix ϕ_{jk}^n. Combining (3.9) and (3.10) gives the element displacement vector

$$u_i^n(\underset{\sim}{x}, t) = A_{ij}^n(\underset{\sim}{x})\phi_{jk}^n q_k^n(t) \tag{3.11}$$

Recalling (3.2) and carrying out the differentiation formally, the strain tensor can be written

$$\varepsilon_i^n(\underset{\sim}{x}, t) = B_{ij}^n(\underset{\sim}{x})\phi_{jk}^n q_k^n(t) \tag{3.12}$$

Before substituting these results into (3.8) it is necessary to relate the element nodal point displacements to the nodal point displacements for the entire domain; i.e., if we denote the incremental displacement vector for the entire domain by $v_p(t)$, where the range of p depends on the number of nodal points in the entire domain, the vector q_k^n can be made compatible with v_p through a localizing transformation

$$q_k^n(t) = L_{kp}^n v_p(t) \tag{3.13}$$

If all element displacements are referred to a common reference frame the localizing matrix L_{kp} is Boolean. Returning now to the equations for displacement and strain, we can write

$$u_i^n(\underset{\sim}{x}, t) = A_{iq}^n(\underset{\sim}{x})\phi_{qk}^n L_{kp}^n v_p(t) \tag{3.14}$$

and

$$\varepsilon_i^n(\underset{\sim}{x}, t) = B_{iq}^n(\underset{\sim}{x})\phi_{qk}^n L_{kp}^n v_p(t) \tag{3.15}$$

Now substituting (3.14) and (3.15) into (3.8)

$$\sum_{n=1}^{N}\Big\{\int_{V_n}\Big[f_i^n B_{iq}^n \phi_{qk}^n L_{kp}^n + k_{ij}^n B_{iq}^n \phi_{qk}^n L_{kp}^n B_{jr}^n \phi_{rl}^n L_{ls}^n v_s - \rho^n b_i^n A_{iq}^n \phi_{qk}^n L_{kp}^n\Big] dV_n - \int_{S_{\sigma_n}} \bar{t}_i^n A_{iq}^n \phi_{qk}^n L_{kp}^n dS_n\Big\} \delta v_p = 0 \tag{3.16}$$

Since δv_p is an arbitrary virtual displacement it follows that the expression $\sum_{n=1}^{N}\{\ \}$ must vanish for each value of p independently. Accordingly, (3.16) can be rewritten;

$$K_{ps} v_s - F_p + H_p - T_p = 0 \quad s, p = 1, 2, \ldots \tag{3.17}$$

where we have defined

$$\begin{aligned}
K_{ps} &= \sum_{n=1}^{N} \phi_{qk}^n L_{kp}^n \phi_{rl}^n L_{ls}^n \int_{V_n} B_{iq}^n B_{jr}^n k_{ij}^n dV_n \\
F_p &= \sum_{n=1}^{N} \phi_{qk}^n L_{kp}^n \int_{V_n} \rho^n b_i^n A_{iq}^n dV_n \\
H_p &= \sum_{n=1}^{N} \phi_{qk}^n L_{kp}^n \int_{V_n} f_i^n B_{iq}^n dV_n \\
T_p &= \sum_{n=1}^{N} \phi_{qk}^n L_{kp}^n \int_{S_{\sigma_n}} \bar{t}_i^n A_{iq}^n dS_n
\end{aligned} \tag{3.18}$$

Equation (3.17) constitutes the basis for a discretized solution algorithm for the nonlinear boundary value problem. Techniques for the automatic formation of the matrices (3.18) and the system of equations (3.17) are

well-known [6]. Since K_{ps} is a linear integral matrix operator for a prescribed reference state, the solution of (3.17) for nodal point displacements $v_s(t)$ may be obtained by step-forward integration in time, solving at each step a linear boundary value problem in space. Numerical solution of problems of this type, in the context of linear thermoviscoelasticity, has been discussed by Taylor, et al. [7].

It remains to discuss the details of iteration in the solution algorithm. In (3.17) the vectors T_p and F_p are prescribed functions of space and time, once the details of the sub-division of the domain V and selection of the approximating functions $A^n_{ij}(\underset{\sim}{x})$ have been accomplished. The vector H_p and operator K_{ps} depend upon the coordinate functions $A^n_{ij}(\underset{\sim}{x})$ in addition to the reference history $\underset{\sim}{\bar{e}}(x, t)$ selected. We may proceed as follows: a trial solution of the boundary value problem is assumed; for example, it can be assumed that the trial solution corresponds to a linear constitutive law, in which case K_{ps} is associated with the zero reference history and H_p vanishes. Solution of (3.17) for $v_s = v^{(1)}_s(t)$ constitutes the first approximation to the discretized problem. (Actual displacements, strains, stresses may be obtained from nodal point values through the appropriate transformation equations.) Corresponding to the displacements $v^{(1)}_s(t)$ is a new reference strain history $\underset{\sim}{\bar{e}}^{(1)}(\underset{\sim}{x}, t)$. Using this history f_i, h_{ij}, k_{ij} and hence $K^{(1)}_{ps}$, $H^{(1)}_p$ can be evaluated experimentally. Using the generalized Newton's Method [8] the second approximation for $v^{(2)}_s(t)$ can be obtained by solving the system of equations.

$$K^{(1)}_{ps} v^{(2)}_s + H^{(1)}_p = 0 \tag{3.19}$$

Repeating this process, we have the recurrence equation

$$K_{ps}^{(i-1)} v_s^{(i)} + H_p^{(i-1)} = 0 \qquad i > 2 \tag{3.20}$$

When the "ith" iteration produces $H_p^{(i)} \cong 0$ to the accuracy desired, the solution has been obtained, since $v_s^{(i)}$ is zero for the homogeneous boundary value problem that appears as a result of solving a nonhomogeneous problem in the first cycle of iteration. Note that after the first trial solution the displacements $v_s^{(i)}$ correspond to incremental displacements. Accordingly, the total nodal point displacement is obtained by addition of the iterate and the reference displacement. Other field variables are obtained in a similar fashion.

In view of the substantial amount of experimental and computational effort involved in a typical problem, it should be remarked that considerable savings is effected if the simplified, or chord-type, Newton's Method [8] can be employed. Using this basis for solution, the operator $K_{ps}^{(1)}$ is obtained once and for all, and subsequent iterates $v_s^{(i)}$ depend only upon the changes in pseudo body force $H_p^{(i)}$.

At this point it is well to acknowledge the purely formal nature of the algorithm. No claims are made for either uniqueness or convergence of the iterated solution. In view of the complexity of these questions for nonlinear operators [8] it seems best to rest these questions on intuition at the present time. Finally, in passing, it should be noticed that all of the previous remarks also apply to a nonlinear elastic boundary value problem, for which the constitutive equation for stress is a tensor-valued function of strain rather than a functional of the strain history. In this case the matrix K_{ps} is a nonlinear function of the reference strain rather than a functional of the reference history. Thus, the solution algorithm involves construction of a sequence of solutions for linear time-independent boundary value problems.

We turn now to a simple example to illustrate the procedure described in the previous sections.

4. EXAMPLE: AXIAL DEFORMATION OF A STATICALLY INDETERMINATE VISCOELASTIC ROD

To illustrate the solution algorithm for a nonlinear boundary value problem, a simple one-dimensional example has been chosen. The system is a nonlinear viscoelastic rod of unit cross-sectional area clamped at each end, as shown in the inset of Figure 3. The rod undergoes axial deformation produced by two forces

$$\sigma(t) = 400[H(t) - H(t-s)], \quad s = 1, 2, \ldots \tag{4.1}$$

applied as shown in Figure 3. Although the example is one-dimensional in space, the same technique is applicable to two- and three-dimensional states of deformation at the expense of computational effort. In the absence of experimental data for a particular material, we substitute for our experimentally determined response functional (2.2) the equation*

$$\bar{\sigma}(t) = \underset{s=0}{\overset{s=t}{f}} [\bar{e}(s)] = \int_0^t \left\{G_0 + E_0 \exp\,[-k_0(t - s)]\right\} \frac{\partial \bar{e}}{\partial s}\, ds$$

$$\int_0^t\int_0^t\int_0^t \left\{ G_1 + E_1 \exp[-k_1(t-s_1)]\exp[-k_1(t-s_2)]\exp[-k_1(t-s_3)]\right\}\frac{\partial \bar{e}}{\partial s_1}\frac{\partial \bar{e}}{\partial s_2}\frac{\partial \bar{e}}{\partial s_3}\, ds_1 ds_2 ds_3 \tag{4.2}$$

corresponding to a "third order" nonlinear viscoelastic solid. For this constitutive equation the kernel function h appearing in (2.3) and $(2.3)_1$

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* Tensor notation is dropped for the one-dimensional example.

is readily found to be

$$\overset{s=t}{\underset{s=0}{h}}\ [\bar{e}(s),\ t - \tau] = G_0 + E_0 \exp\ [-k_0(t - \tau)] + 3G_1\bar{e}_G^2(t) + 3E_1\bar{e}_E^2(t) \exp\ [-k_1(t - \tau)] \tag{4.3}$$

where

$$\bar{e}_G(t) = \int_0^t \frac{\partial \bar{e}}{\partial s}\, ds$$

$$\bar{e}_E(t) = \int_0^t \exp\ [-k_1(t - s)]\, \frac{\partial \bar{e}}{\partial s}\, ds \tag{4.4}$$

Proceeding with the discretization required by the solution algorithm, the rod is divided into three elements (four nodal points); the coordinate functions $A_{ij}^n(x)$ in (3.9) are taken to be linear in x and the generalized coordinates $\alpha_j^n(t)$ linear in time.

The constants describing the material in (4.2) are selected as

$$k_0 = 1, \quad k_1 = 0.7$$

$$G_0 = E_0 = 1000 \tag{4.5}$$

and for the nonlinear moduli

$$G_1 = E_1 = 0 \qquad (0/0)$$

$$G_1 = E_1 = 20{,}000 \qquad (20/20)$$

$$G_1 = 10{,}000, \quad E_1 = 30{,}000 \qquad (20/10) \tag{4.6}$$

$$G_1 = 25{,}000, \quad E_1 = 75{,}000 \qquad (50/25)$$

$$G_1 = 0, \qquad E_1 = 100{,}000 \qquad (50/0)$$

The symbols (I/F) measure nonlinearity in the constitutive equation (4.2) for the limiting values of time $t = 0$, $t \to \infty$. They are defined by

$$I \equiv \frac{G_1 + E_1}{G_0 + E_0}, \quad F \equiv \frac{G_1}{G_0} \tag{4.7}$$

A plot of the strain-dependent relaxation modulus associated with a step function strain input is shown in Figure 2 for the material constants given by (4.5) and (4.6).

The formation of the matrices (3.18) using the above data and approximation functions is straightforward. The system of equations (3.17) was solved using the simplified (chord-type) Newton's Method employing the matrix operator K_{ps} evaluated for a zero reference history. For the cases studied here iteration was completed in approximately 10-15 cycles, depending upon the degree of nonlinearity. Further computational details may be found in [9].

Figure 3 presents the strain history in segment 1 of the rod for different values of material constants. From a physical standpoint it is interesting to note that, in addition to the effect of the ratio of initial to final nonlinearity (I/F), the influence of "aging" due to nonlinearity is pronounced. That is, the instantaneous elastic change in strain due to removal of the load depends upon the time at which unloading occurs, a phenomenon observed in many engineering materials.

Figure 4 shows the stress history in segment 1 for the case of loading only. Note that in contrast to a linear material stress is a function of time, although the highly constrained support conditions of the example preclude any large changes with time.

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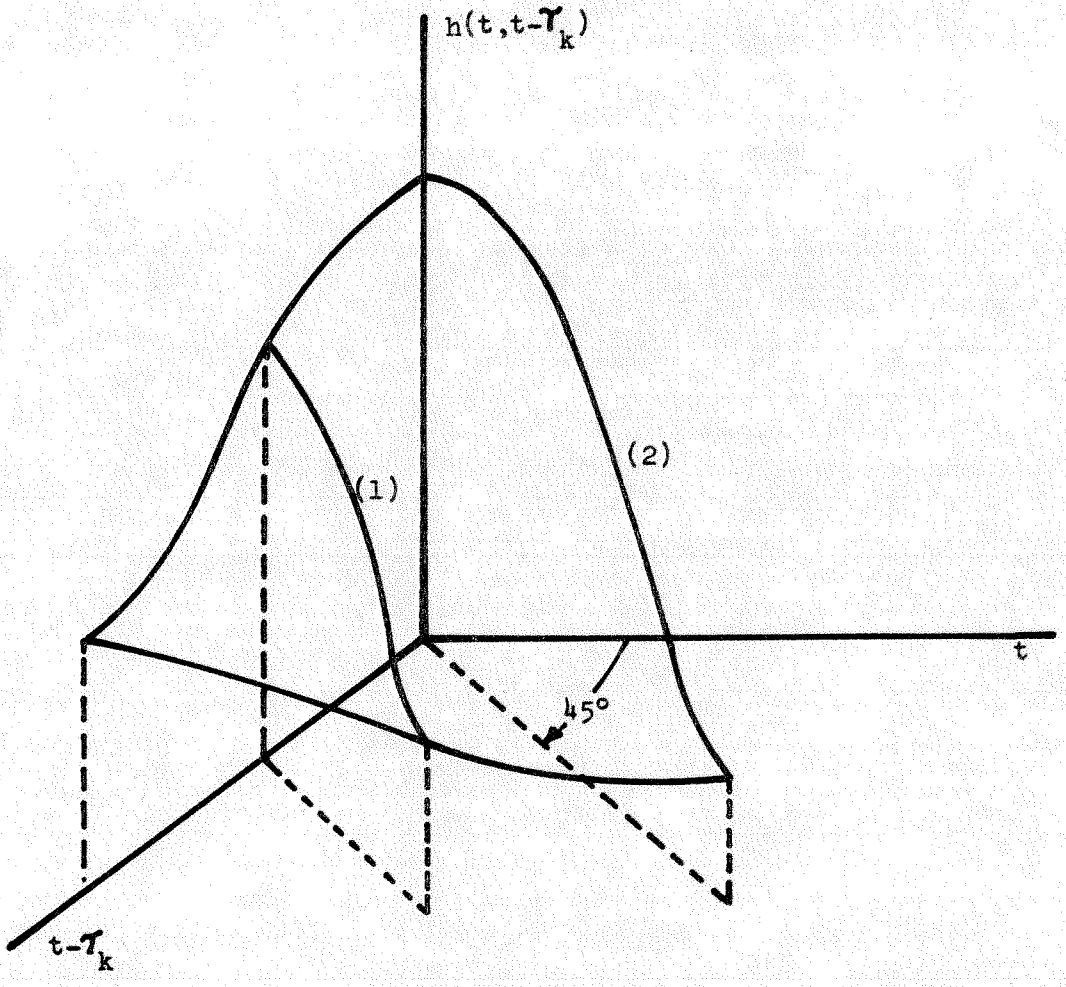


FIGURE 1

FIGURE 2

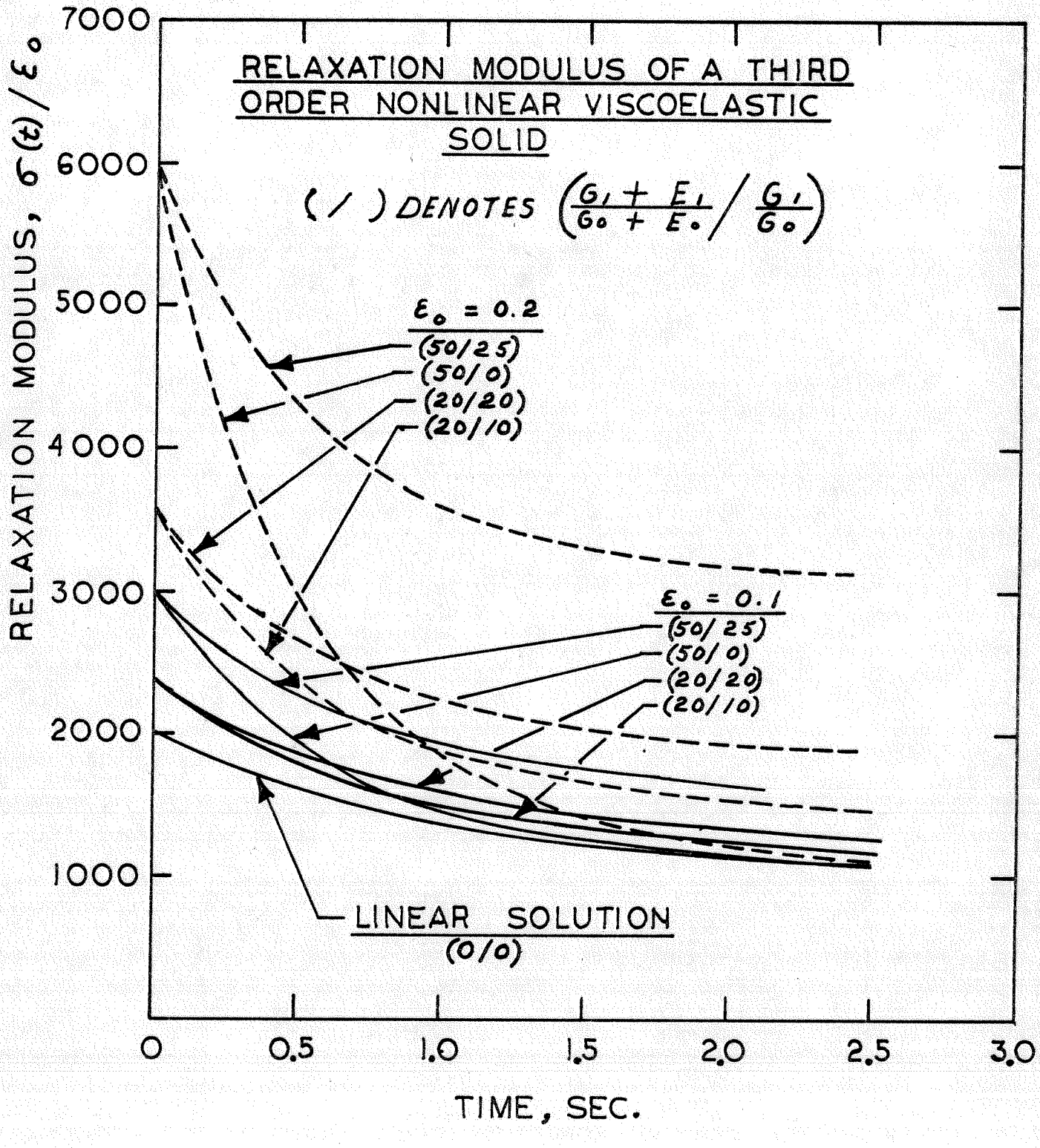
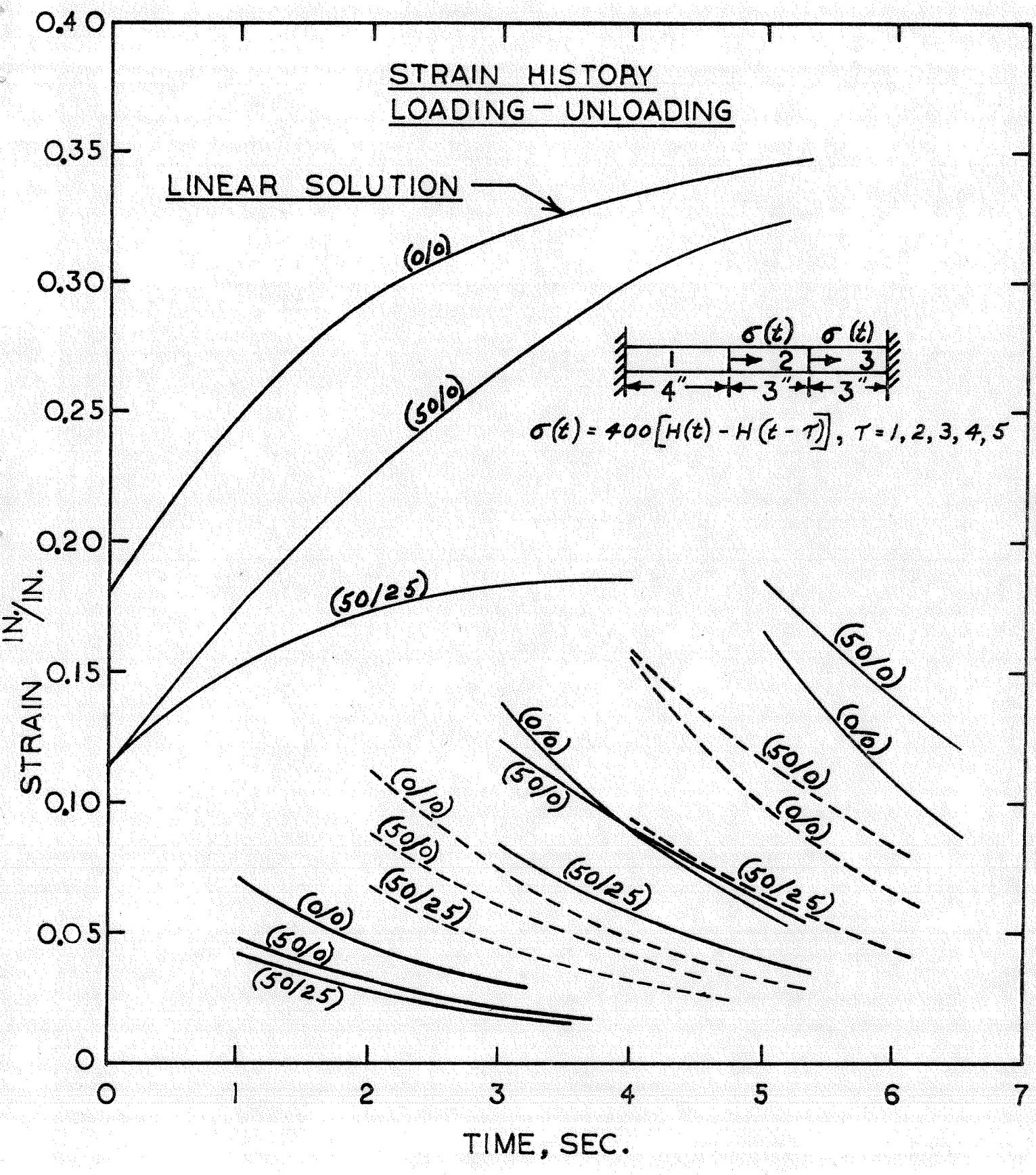

RELAXATION MODULUS OF A THIRD ORDER NONLINEAR VISCOELASTIC SOLID
(/) DENOTES $\left(\frac{G_1 + E_1}{G_0 + E_0} / \frac{G_1}{G_0}\right)$
RELAXATION MODULUS, $\sigma(t)/\varepsilon_0$
TIME, SEC.
7000
6000
5000
4000
3000
2000
1000
0
0.5
1.0
1.5
2.0
2.5
3.0
$\varepsilon_0 = 0.2$
(50/25)
(50/0)
(20/20)
(20/10)
$\varepsilon_0 = 0.1$
(50/25)
(50/0)
(20/20)
(20/10)
LINEAR SOLUTION
(0/0)

FIGURE 3


STRAIN HISTORY
LOADING – UNLOADING
LINEAR SOLUTION
(0/0)
(50/0)
(50/25)
σ(t) σ(t)
1 2 3
4" 3" 3"
σ(t) = 400[H(t) - H(t - τ)], τ = 1, 2, 3, 4, 5
(0/10)
(50/0)
(50/25)
STRAIN IN./IN.
TIME, SEC.
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
0 1 2 3 4 5 6 7

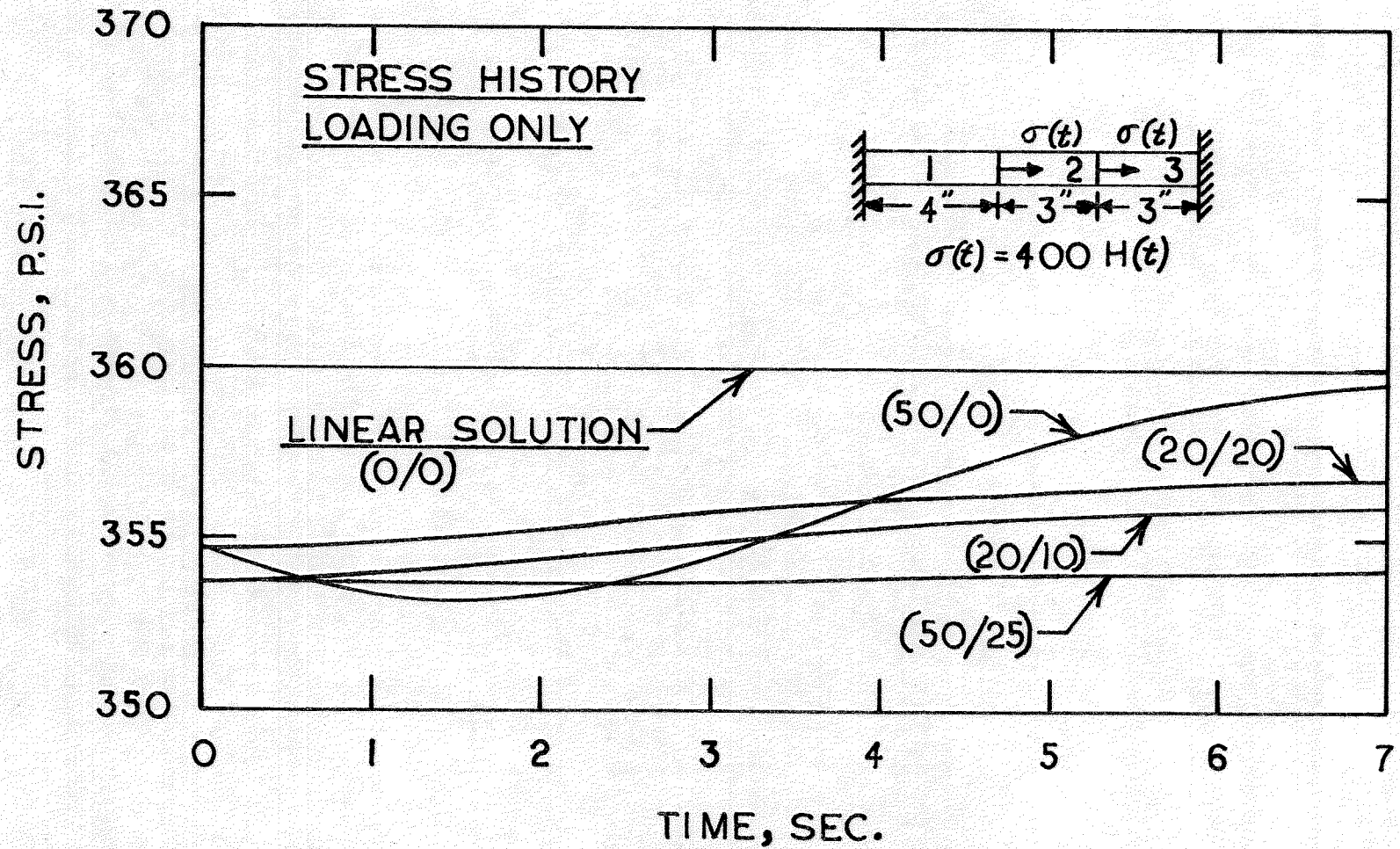

370
365
360
355
350
STRESS, P.S.I.
STRESS HISTORY
LOADING ONLY
σ(t) σ(t)
1
2
3
4"
3"
3"
σ(t) = 400 H(t)
LINEAR SOLUTION
(0/0)
(50/0)
(20/20)
(20/10)
(50/25)
0
1
2
3
4
5
6
7
TIME, SEC.

FIGURE 4