Title

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A note on random projections for preserving paths on a manifold

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Abstract

Random projections are typically used to study low distortion linear embeddings that approximately preserve Euclidean distances between pairs of points in a set $S \subset \mathbb{R}^D$. Of particular interest is when the set S is a low-dimensional submanifold of $\mathbb{R}^D$. Recent results by Baraniuk and Wakin [2007] and Clarkson [2007] shed light on how to pick the projection dimension to achieve low distortion of Euclidean distances between points on a manifold. While preserving ambient Euclidean distances on a manifold does imply preserving intrinsic path-lengths between pairs of points on a manifold, here we investigate how one can reason *directly* about preserving path-lengths without having to appeal to the ambient Euclidean distances between points. In doing so, we can improve upon Baraniuk and Wakin's result by removing the dependence on the ambient dimension D, and simplify Clarkson's result by using a single covering quantity and giving explicit dependence on constants.

1 Introduction

Random projections have turned out to be a powerful tool for linear dimensionality reduction that approximately preserve Euclidean distances between pairs of points in a set $S \subset \mathbb{R}^D$. Their simplicity and universality stems from the fact the target embedding space is picked *without* looking at the individual samples from the set S. Interestingly, recent results by Baraniuk and Wakin [2007] and Clarkson [2007] show that even if the underlying set is a non-linear manifold (say of intrinsic dimensionality n), a random projection into a subspace of dimension $O(n)$ suffices to preserve interpoint Euclidean distances between the pairs of points.

It turns out that requiring Euclidean distances to be approximately preserved between pairs of points in a manifold is in a sense the strictest condition one can pose. This condition suffices to imply that the random projection will also preserve several other useful properties on manifolds. For instance, if one has a random projection that can approximately preserve the Euclidean distances, it will also approximately preserve the lengths of arbitrary curves on the manifold, and the curvature of the manifold.

Here we are interested in analyzing whether one can use random projections to reason *directly* about preserving the lengths of arbitrary paths on a manifold, without having appeal to interpoint Euclidean distances. There is a two fold reason for doing this: i) one can possibly get a sharper bound on the dimension of target space by relaxing the Euclidean interpoint distance preservation requirement, and ii) since paths—unlike Euclidean distances—are inherently an intrinsic quantity, it should require a different technique to show path length preservation. Thus, giving us an alternate, direct proof.

In this manuscript, we make progress on both fronts. We can remove the dependence on ambient dimension from the bound provided by Baraniuk and Wakin [2007], as well as simplify the bound provided by Clarkson [2007] by giving an explicit bound for all settings of the isometry parameter (and not just asymptotically small values). Our key lemma (Lemma 6) uses an elegant chaining argument on the coverings of vectors in tangent spaces providing an alternate proof technique.

2 Random projections for preserving paths on a manifold

2.1 Notation and Preliminaries

Let $M \subset \mathbb{R}^D$ be a smooth compact n-dimensional submanifold of $\mathbb{R}^D$. For any two points p and q, we shall use $D_G(p, q)$ to denote the geodesic distance between points p and q when the underlying manifold is understood from the context.

Recall that the length of any given curve $\gamma : [a, b] \to M$ is given by $\int_a^b \|\gamma'(s)\| ds$ (that is, length of a curve is an infinitesimal sum of the lengths of vectors tangent to points along the path). It thus suffices to bound the distortion induced by a random projection to the lengths of arbitrary vectors tangent to M.

Since path lengths depend intimately on tangent vectors, we also need to know how the tangent vectors vary locally in the ambient space. This relationship between the local curvature of M in the ambient space $\mathbb{R}^D$ is captured formally by the notion of the *second fundamental form* (see e.g. Chapter 6 of do Carmo [1992]). It is a symmetric bilinear form $B_p : T_p \times T_p \to T_p^{\perp}$ (for any $p \in M$, tangent space T_p and normal space $T_p^{\perp}$). We shall assume that the norm of the second fundamental form of M is uniformly bounded by $1/\tau$. That is, for all $p \in M$, unit $u \in T_p$, and unit $\eta \in T_p^{\perp}$, we have $\langle \eta, B_p(u, u) \rangle \leq 1/\tau$.

As a final piece of notation, we require a notion of covering on our manifold M. We define the α-*geodesic covering number* of M as the size of the smallest set $S \subset M$, with the property: for all $p \in M$, there exists $p' \in S$ such that $D_G(p, p') \leq \alpha$.

2.2 Main Result

Theorem 1 *Let M be a smooth compact n-dimensional submanifold of $\mathbb{R}^D$ with the norm on its second fundamental form uniformly bounded by $1/\tau$. Let $G(M, \alpha)$ denote the α-geodesic covering number of M. Pick any $0 < \epsilon < 1$ and $0 < \delta < 1$. Let ϕ be a random projection matrix that maps points from $\mathbb{R}^D$ into a random subspace of dimension d ($d \leq D$) and define $\Phi := \sqrt{D/d}\phi$ as a scaled projection mapping.*

If $d \geq \left\{ \frac{64}{\epsilon^2} \ln \frac{4G(M, \tau\epsilon^2/2^{18})}{\delta} + \frac{64n}{\epsilon^2} \ln \frac{12}{\delta} \right\}$, then with probability at least $1 - \delta$ we have the following:

For any path-connected points p and q in $M \subset \mathbb{R}^D$ and any path γ from p to q in M, and their corresponding projections $\Phi(p)$, $\Phi(q)$ and $\Phi(\gamma)$ in $\Phi(M) \subset \mathbb{R}^d$,

$$(1 - \epsilon)L(\gamma) \leq L(\Phi(\gamma)) \leq (1 + \epsilon)L(\gamma),$$

where $L(\beta)$ denotes the length of the path β.

3 Proof

As discussed earlier, it suffices to uniformly bound the distortion induced by a random projection to the length of an arbitrary vector tangent to our manifold M. So we shall only focus on that. We start by stating a few useful lemmas that would help in our discussion.

Lemma 2 (random projection of a fixed vector – see e.g. Lemma 2.2 of Dasgupta and Gupta [1999]) *Fix a vector $v \in \mathbb{R}^D$. Let ϕ be a random projection map that maps points from $\mathbb{R}^D$ to a random subspace of dimension d. Then,*

i) For any $\beta \geq 1$,

$$\mathbf{P}\left[\|\phi(v)\|^2 \geq \beta \frac{d}{D}\|v\|^2\right] \leq e^{(\beta - 1 - \ln\beta)(-d/2)}.$$

ii) For any $0 < \epsilon < 1$, we have

$$\mathbf{P}\left[\|\phi(v)\|^2 \leq (1 - \epsilon)\frac{d}{D}\|v\|^2 \text{ or } \|\phi(v)\|^2 \geq (1 + \epsilon)\frac{d}{D}\|v\|^2\right] \leq 2e^{-d\epsilon^2/4}.$$

Lemma 3 (covering of a Euclidean unit-sphere – Chapter 13 of Lorentz et al. [1996]) *Let S^n be an n-dimensional Euclidean unit sphere. Then there exists a ϵ-cover of S^n of size at most $(12/\epsilon)^n$. That is, there exists a set $C \subset S^n$, of size at most $(12/\epsilon)^n$, with the property: for any $x \in S^n$, exists $c \in C$ such that $\|x - c\| \leq \epsilon$.*

Lemma 4 (covering of a section of a manifold – implicit in the proof of Theorem 22 of Dasgupta and Freund [2008]) *Let $M \subset \mathbb{R}^D$ be a smooth compact n-dimensional manifold with $1/\tau$ uniform bound on the norm of its second fundamental form. For any $x \in \mathbb{R}^D$ and $0 < r \leq \tau/2$, let $M' := M \cap B(x, r)$. Then, M' can be covered by 9^n balls of radius $r/2$. That is, there exists $C \subset M'$ with size at most 9^n, with the property: for any $p \in M'$, exists $c \in C$ such that $\|p - c\| \leq r/2$.*

Lemma 5 (relating closeby tangent vectors – implicit in the proof of Propositions 6.2 and 6.3 of Niyogi et al. [2006]) *Let $M \subset \mathbb{R}^D$ be a smooth compact n-dimensional manifold with $1/\tau$ uniform bound on the norm of its second fundamental form. Then,*

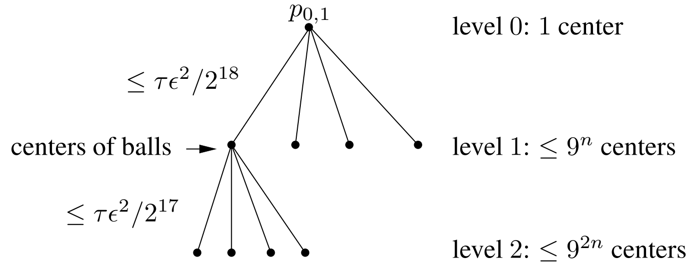

Figure 1: A hierarchy of covers of $S \subset B(p, \tau\epsilon^2/2^{18})$ for some point p in an n-manifold M with condition number $1/\tau$. Observe that at any level i, there are at most 9^{ni} points in the cover. Also note that the Euclidean distance between any point $p_{i,k}$ at level i and its parent $p_{i-1,j}$ in the hierarchy is at most $\tau\epsilon^2/2^{17+i}$.

i) Pick any two path-connected points $p, q \in M$. Let $u \in T_pM$ be a unit length tangent vector and $v \in T_qM$ be its parallel transport along the (shortest) geodesic path to q. Then, i) $u \cdot v \geq 1 - D_G(p,q)/\tau$, ii) $\|u - v\| \leq \sqrt{2D_G(p,q)/\tau}$.

ii) If $p, q \in M$ such that $\|p - q\| \leq \tau/2$, then $D_G(p,q) \leq \tau(1 - \sqrt{1 - 2\|p-q\|/\tau}) \leq 2\|p - q\|$.

Lemma 6 *Let $M \subset \mathbb{R}^D$ be a smooth compact n-dimensional manifold with $1/\tau$ uniform bound on the norm of its second fundamental form. Pick any $0 < \epsilon < 1$. Fix some p in M and let $S := \{p' \in M : D_G(p, p') \leq \tau\epsilon^2/2^{18}\}$. Let ϕ be a random orthoprojector from $\mathbb{R}^D$ to $\mathbb{R}^d$. Then, if $d > 30n \ln 9$,*

$$\mathbf{P}\left[\exists p' \in S : \exists v' \in T_{p'}M : \|\phi(v')\| \leq (1-\epsilon)\sqrt{\frac{d}{D}}\|v'\| \text{ or } \|\phi(v')\| \geq (1+\epsilon)\sqrt{\frac{d}{D}}\|v'\|\right]$$
$$\leq 2(e^{n\ln(108/\epsilon) - (d/30)} + e^{n\ln(12/\epsilon) - (d\epsilon^2/64)}).$$

Proof: Note that the set S is path-connected, and (see for instance Lemma 4) for any Euclidean balls $B(x, r)$, $S \cap B(x, r)$ can be covered by 9^n balls of half the radius. We will use this fact to create a hierarchy of covers of increasingly fine resolution. For each point in the hierarchy, we shall associate a covering of the tangent space at that point. We will inductively show that (with high probability) a random projection doesn't distort the lengths of the tangent vectors in the covering by too much. We will then conclude by showing that bounding the length distortion on tangent vectors in the covering implies a bound on the length distortion of all vectors in all the tangent spaces of all points in S. We now make this argument precise.

Constructing a hierarchical cover of S: Note that S is contained in a Euclidean ball $B(p, \tau\epsilon^2/2^{18})$. We create a hierarchy of covers as follows. Pick a cover of $S \subset B(p, \tau\epsilon^2/2^{18})$ by 9^n balls of radius $\tau\epsilon^2/2^{19}$ (see Lemma 4). WLOG, we can assume that the centers of these balls lie in S (see e.g. proof of Theorem 22 of Dasgupta and Freund [2008]). Each of these balls induces a subset of S, which in turn can then be covered by 9^n balls of radius $\tau\epsilon^2/2^{20}$. We can continue this process to get an increasingly fine resolution such that at the end, any point of S would have been arbitrarily well approximated by the center of some ball in the hierarchy. We will use the notation $p_{i,k}$ to denote the center of the k^{th} ball at level i of the hierarchy (note that with this notation $p_{0,1} = p$). (see Figure 1).

A tangent space cover associated with each point in the hierarchy: Associated with each $p_{i,k}$, we have a set $Q_{i,k} \subset T_{p_{i,k}}M$ of unit-length vectors tangent to M at $p_{i,k}$ that forms a $(\epsilon/6)$-cover of the unit-vectors in $T_{p_{i,k}}M$ (that is, for all unit $v \in T_{p_{i,k}}M$, there exists $q \in Q_{i,k}$ where $\|q\| = 1$ such that $\|q - v\| \leq \epsilon/6$). We will define the individual vectors in $Q_{i,k}$ as follows. The set $Q_{0,1}$ is any $(\epsilon/6)$-cover of the unit-sphere in $T_{p_{0,1}}M$. Note that, by Lemma 3, we can assume that $|Q_{0,1}| = L \leq e^{n\ln(12/\epsilon)}$. For levels $i = 1, 2, \ldots$, define $Q_{i,k}$ (associated with the point $p_{i,k}$) as the *parallel transport* (via the shortest geodesic path using the standard manifold connection, see Figure 2) of the vectors in $Q_{i-1,j}$ (associated with the point $p_{i-1,j}$) where $p_{i-1,j}$ is the parent of $p_{i,k}$ in the hierarchy. Note that parallel transporting a set of vectors preserves certain desirable properties – the dot product, for instance, between the vectors being transported is preserved (see, for instance, page 161 of Stoker [1969]). Thus, by construction, we have that $Q_{i,k}$ is a $(\epsilon/6)$-cover, since parallel transport doesn't change the lengths or the mutual angles between the vectors being transported.

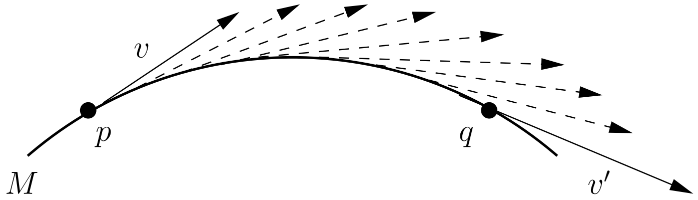

Figure 2: Parallel transport of the vector v at point $p \in M$ to the point $q \in M$. The resulting transported vector is v'. Parallel transport is the translation of a (tangent) vector from one point to another while remaining tangent to the manifold. As the vector is transported infinitesimally along a path, it is also required to be parallel. Note that the resulting vector v' is typically path-dependent: that is, for different paths from p to q, the transport of v is generally different. However, as expected, the transport does not change the length of the original vector. That is, $\|v\| = \|v'\|$.

A residual associated with each vector in the tangent space cover: For $i \geq 1$, let $q_l^{i,k}$ be the l^{th} vector in $Q_{i,k}$, which was formed by the transport of the vector $q_l^{i-1,j}$ in $Q_{i-1,j}$. We define the "residual" as $r_l^{i,k} := q_l^{i,k} - q_l^{i-1,j}$ (for $l = 1, \ldots, L$). Then we have that $\|r_l^{i,k}\|$ is bounded. In particular, since $\|q_l^{i-1,j}\| = \|q_l^{i,k}\| = 1$ (since the transport doesn't change vector lengths), and since $D_G(p_{i-1,j}, p_{i,k}) \leq 2\|p_{i-1,j} - p_{i,k}\| \leq \tau\epsilon^2/2^{16+i}$ (cf. Lemma 5)

$$\|r_l^{i,k}\|^2 \leq 2D_G(p_{i-1,j}, p_{i,k})/\tau \leq \epsilon^2 2^{-i-15}.$$

Effects of a random projection on the length of the residual: Note that for a fixed $r_l^{i,k}$ (corresponding to a fixed point $p_{i,k}$ at level i in the hierarchy and its parent $p_{i-1,j}$ in the hierarchy), using Lemma 2 (i), we have (for $\beta > 1$)

$$\mathbf{P}\left[\|\phi(r_l^{i,k})\|^2 \geq \beta \frac{d}{D}\|r_l^{i,k}\|^2\right] \leq e^{(\beta - 1 - \ln \beta)(-d/2)}. \tag{1}$$

By choosing $\beta = 2^{i/2}$ in Eq. (1), we have the following. For any fixed i and k, with probability at least $1 - e^{n \ln(12/\epsilon)} e^{(2^{i/2} - 1 - \ln 2^{i/2})(-d/2)} \geq 1 - e^{n \ln(12/\epsilon) - di/30}$, we have $\|\phi(r_l^{i,k})\|^2 \leq \epsilon^2 2^{i/2} \frac{d}{D} \|r_l^{i,k}\|^2 \leq \epsilon^2 2^{-(i/2)-15}(d/D)$ (for $l = 1, \ldots, L$). By taking a union bound over all edges in the hierarchy, (if $d > 30n \ln 9$)

$$\mathbf{P}\Big[\exists \text{ level } i : \exists \text{ ball } k \text{ at level } i \text{ with center } p_{i,k} : \exists \text{ residual } r_l^{i,k} : \|\phi(r_l^{i,k})\|^2 \geq \epsilon^2 2^{-(i/2)-15}(d/D)\Big]$$
$$\leq \sum_{i=1}^{\infty} e^{ni \ln 9} e^{n \ln(12/\epsilon)} e^{-di/30} \leq 2e^{n \ln(108/\epsilon) - (d/30)}.$$

Effects of a random projection on the vectors in the tangent space cover: We now use the (uniform) bound on $\|\phi(r_l^{i,k})\|^2$ to conclude inductively that ϕ doesn't distort the length of any vector $q_l^{i,k}$ too much (for any i, k, and l). In particular we will show that for all i, k and l, we will have $(1 - \frac{\epsilon}{2})\frac{d}{D} \leq \|\phi(q_l^{i,k})\|^2 \leq (1 + \frac{\epsilon}{2})\frac{d}{D}$.

Base case (level 0): Since $|Q_{0,1}| \leq e^{n \ln(12/\epsilon)}$ we can apply Lemma 2 (ii), to conclude with probability at least $1 - 2e^{-d\epsilon^2/64 + n \ln(12/\epsilon)}$, for all $q \in Q_{0,1}$, $(1 - \frac{\epsilon}{4})\frac{d}{D} \leq \|\phi(q)\|^2 \leq (1 + \frac{\epsilon}{4})\frac{d}{D}$.

Inductive hypothesis: We assume that for all vectors $q_l^{i,k} \in Q_{i,k}$ (for all k) at level i

$$\left(1 - \frac{\epsilon}{4} - \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right)\frac{d}{D} \leq \|\phi(q_l^{i,k})\|^2 \leq \left(1 + \frac{\epsilon}{4} + \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right)\frac{d}{D}. \tag{2}$$

Inductive case: Pick any $p_{i+1,k}$ at level $i+1$ in the hierarchy and let $p_{i,j}$ be its parent ($i \geq 0$). Then, for any $q_l^{i+1,k} \in Q_{i+1,k}$ (associated with the point $p_{i+1,k}$), let $q_l^{i,j} \in Q_{i,j}$ (associated with the point $p_{i,j}$) be the vector which after the parallel transport resulted in $q_l^{i+1,k}$. Then, we have:

$$\begin{aligned}
\|\phi(q_l^{i+1,k})\|^2 &= \|\phi(q_l^{i,j}) + \phi(r_l^{i+1,k})\|^2 \\
&= \|\phi(q_l^{i,j})\|^2 + \|\phi(r_l^{i+1,k})\|^2 + 2\phi(q_l^{i,j}) \cdot \phi(r_l^{i+1,k}) \\
&\geq \|\phi(q_l^{i,j})\|^2 + \|\phi(r_l^{i+1,k})\|^2 - 2\|\phi(q_l^{i,j})\|\|\phi(r_l^{i+1,k})\| \\
&\geq \frac{d}{D}\left(1 - \frac{\epsilon}{4} - \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right) - 2\sqrt{\left(1+\frac{\epsilon}{2}\right)\frac{d}{D}}\sqrt{\frac{\epsilon^2 2^{-(i/2)-15.5} d}{D}} \\
&\geq \frac{d}{D}\left[\left(1 - \frac{\epsilon}{4} - \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right) \underbrace{-\epsilon\sqrt{2^{-(i/2)-12.5}}}_{\geq -\epsilon 2^{-(i/4)-(1/4)-5}}\right] \\
&\geq \frac{d}{D}\left(1 - \frac{\epsilon}{4} - \frac{\epsilon}{32}\sum_{j=1}^{i+1} 2^{-j/4}\right).
\end{aligned}$$

Now, in the other direction we have

$$\begin{aligned}
\|\phi(q_l^{i+1,k})\|^2 &= \|\phi(q_l^{i,j})\|^2 + \|\phi(r_l^{i+1,k})\|^2 + 2\|\phi(q_l^{i,j})\|\|\phi(r_l^{i+1,k})\| \\
&\leq \frac{d}{D}\left(1 + \frac{\epsilon}{4} + \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right) + \frac{\epsilon^2 2^{-(i/2)-15.5} d}{D} + 2\sqrt{\left(1+\frac{\epsilon}{2}\right)\frac{d}{D}}\sqrt{\frac{\epsilon^2 2^{-(i/2)-15.5} d}{D}} \\
&\leq \frac{d}{D}\left[\left(1 + \frac{\epsilon}{4} + \frac{\epsilon}{32}\sum_{j=1}^{i} 2^{-j/4}\right) \underbrace{+\epsilon 2^{-(i/2)-15.5} + \epsilon\sqrt{2^{-(i/2)-12.5}}}_{\leq +\epsilon 2^{-(i/4)-(1/4)-5}}\right] \\
&\leq \frac{d}{D}\left(1 + \frac{\epsilon}{4} + \frac{\epsilon}{32}\sum_{j=1}^{i+1} 2^{-j/4}\right).
\end{aligned}$$

So far we have shown that by picking $d > 30n \ln 9$, with probability at least $1 - 2(e^{n\ln(108/\epsilon)-(d/30)} + e^{n\ln(12/\epsilon)-(d\epsilon^2/64)})$, for all i, k and l,

$$(1-\epsilon/2)(d/D) \leq \|\phi(q_l^{i,k})\|^2 \leq (1+\epsilon/2)(d/D).$$

Effects of a random projection on any tangent vector at any point in the hierarchy: Now, pick any point $p_{i,k}$ in the hierarchy and consider the corresponding set $Q_{i,k}$. We will show that for any unit vector $v \in T_{p_{i,k}}M$, $(1-\epsilon)\sqrt{d/D} \leq \|\phi(v)\| \leq (1+\epsilon)\sqrt{d/D}$.

Define $A := \max_{v \in T_{p_{i,k}}M, \|v\|=1} \|\phi(v)\|$ and let v_0 be a unit vector that attains this maximum. Let $q \in Q_{i,k}$ be such that $\|v_0 - q\| \leq \epsilon/6$. Now, if $\|v_0 - q\| = 0$, then we get that $A = \|\phi(v_0)\| = \|\phi(q)\| \leq (1+\epsilon)\sqrt{d/D}$. Otherwise,

$$A = \|\phi(v_0)\| \leq \|\phi(q)\| + \|\phi(v_0 - q)\| = \|\phi(q)\| + \|v_0 - q\|\left\|\phi\left(\frac{v_0 - q}{\|v_0 - q\|}\right)\right\| \leq (1+\epsilon/2)\sqrt{d/D} + (\epsilon/6)(A).$$

This yields that $A \leq (1+\epsilon)\sqrt{d/D}$, and thus for any unit $v \in T_{p_{i,k}}M$, $\|\phi(v)\| \leq \|\phi(v_0)\| = A \leq (1+\epsilon)\sqrt{d/D}$. Now, in the other direction, pick any unit $v \in T_{p_{i,k}}M$, and let $q \in Q_{i,k}$ be such that $\|v - q\| \leq \epsilon/6$. Again, if $\|v - q\| = 0$, then we have that $\|\phi(v)\| = \|\phi(q)\| \geq (1-\epsilon)\sqrt{d/D}$. Otherwise,

$$\begin{aligned}
\|\phi(v)\| &\geq \|\phi(q)\| - \|\phi(v-q)\| = \|\phi(q)\| - \|v - q\|\left\|\phi\left(\frac{v-q}{\|v-q\|}\right)\right\| \\
&\geq (1-\epsilon/2)\sqrt{d/D} - (\epsilon/6)(1+\epsilon)\sqrt{d/D} \geq (1-\epsilon)\sqrt{d/D}.
\end{aligned}$$

Since ϕ is linear, it immediately follows that for *all* $v \in T_{p_{i,k}}M$ (not just unit-length v) we have

$$(1-\epsilon)\sqrt{d/D}\|v\| \leq \|\phi(v)\| \leq (1+\epsilon)\sqrt{d/D}\|v\|.$$

Observe that since the choice of the point $p_{i,k}$ was arbitrary, this holds true for *any* point in the hierarchy.

Effects of a random projection on any tangent vector at any point in S: We can finally give a bound on any tangent vector v at any $p \in S$. Pick any v tangent to M at $p \in S$. Then, for any $\delta > 0$ such that $\delta \leq \tau/2$, we know that there exists some $p_{i,k}$ in the hierarchy such that $\|p - p_{i,k}\| \leq \delta$. Let u be the parallel transport (via the shortest geodesic path) of v from p to $p_{i,k}$. Then, we know that $\|u\| = \|v\|$ and (cf. Lemmas 5) $\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\| \leq \sqrt{4\delta/\tau}$. Thus,

$$\|\phi(v)\| \leq \|\phi(u)\| + \|\phi(v-u)\| \leq (1+\epsilon)\sqrt{d/D}\|u\| + \|(v-u)\| \leq (1+\epsilon)\sqrt{d/D}\|v\| + 2\sqrt{\delta/\tau}.$$

Similarly, in the other direction

$$\|\phi(v)\| \geq \|\phi(u)\| - \|\phi(v-u)\| \geq (1-\epsilon)\sqrt{d/D}\|u\| - \|(v-u)\| \geq (1-\epsilon)\sqrt{d/D}\|v\| - 2\sqrt{\delta/\tau}.$$

Note that since the choice of δ was arbitrary, by letting $\delta \to 0$ from above, we can conclude

$$(1-\epsilon)\sqrt{\frac{d}{D}}\|v\| \leq \|\phi(v)\| \leq (1+\epsilon)\sqrt{\frac{d}{D}}\|v\|.$$

∎

All the pieces are now in place to compute the distortion to tangent vectors induced by a random projection mapping. Let C be a $(\tau\epsilon^2/2^{18})$-geodesic cover of M. Noting that one can have C of size at most $G(M, \tau\epsilon^2/2^{18})$, we have (for $d > 30n \ln 9$)

$$\mathbf{P}\left[\exists c \in C : \exists p \text{ such that } D_G(c,p) \leq \tau\epsilon^2/2^{18} : \exists v \in T_pM : \|\phi(v)\| \leq (1-\epsilon)\sqrt{\frac{d}{D}}\|v\| \text{ or } \|\phi(v)\| \geq (1+\epsilon)\sqrt{\frac{d}{D}}\|v\|\right]$$
$$\leq 2G(M, \tau\epsilon^2/2^{18})(e^{n\ln(108/\epsilon)-(d/30)} + e^{n\ln(12/\epsilon)-(d\epsilon^2/64)}).$$

The theorem follows by bounding this quantity by δ.

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