Title
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Journal
Journal of Experimental and Theoretical Physics, 83(12)

ISSN
0021-3640

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Publication Date
2006

DOI
10.1134/S0021364006120137

Peer reviewed

Statistical Properties of Quantum Graph Spectra

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(August 15, 2006)

A general analytical approach to the statistical description of quantum graph spectra based on the exact periodic orbit expansions of quantum levels is discussed. The exact and approximate expressions obtained in [5] for the probability distribution functions using the spectral hierarchy method are analyzed. In addition, the mechanism of appearance of the universal statistical properties of spectral fluctuations of quantum-chaotic systems is considered in terms of the semiclassical theory of periodic orbits.

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I. INTRODUCTION

A quantum graph system consists of a quantum particle moving along the bonds of an arbitrary finite graph G [1]. In the classical limit, this system generates a simple stochastic dynamics, which is specified by the translational motion along the bonds of the graph and stochastic scattering at its vertices with preset scattering probabilities. This dynamics has many common features with the dynamics of usual chaotic systems [2]. For example, periodic trajectories in such a system are isolated and their number increases exponentially with the period. At the same time, the statistical behavior of various spectral characteristics of sufficiently complex quantum graphs, e.g. the probability distribution of spacings $s_n = k_n - k_{n-1}$ between the nearest levels of the momentum was numerically shown [1] to follow the predictions of the Random Matrix Theory (RMT) [3,4], as it is usually the case for classically nonintegrable systems. It also turns out that a great number of problems of classical and quantum dynamics on the graph allow exact solutions, which makes these systems convenient models in the context of the analytical theory of "quantum chaos". In particular, for these systems there exist the exact periodic orbit expansions of the quantum density of states (Gutzwiller formula) [1] along with a similar expansion for the spectral staircase:

$$N(k) \equiv \sum_{j=1}^{\infty} \Theta\left(k - k_j\right) = \bar{N}(k) + \frac{1}{\pi} \operatorname{Im} \sum_{p} A_p e^{iL_p^{(0)} k}, \tag{1}$$

Here $\bar{N}(k)$ is the average number of levels in the range $[0, k]$, $L_p^{(0)}$ is the optical length of the periodic trajectory with the index p, and A_p is a certain weight factor explicitly defined in terms of the scattering coefficients at the graph vertices. It should be emphasized that the existence of the explicit expansions of the global characteristics such as (1) is not equivalent to the ultimate solution of the spectral problem, which should provide local information about the individual levels in the form of an explicit dependence $k_n = k(n)$. An approach for determining the quantities k_n explicitly was proposed in [5], which is based on using a finite system of $r + 2$ auxiliary "separators" $\hat{k}_n^{(0)}$, $\hat{k}_{n+1}^{(1)}$, ..., $\hat{k}_n^{(r+1)}$, the first of which is the physical spectral sequence $k_n = \hat{k}_n^{(0)}$, and the last one is a globally defined explicit function of n:

$$\hat{k}_n^{(r+1)} = \frac{\pi}{L_0}\left(n + \frac{1}{2}\right). \qquad (2)$$

The key property of these sequences is that they must satisfy the "bootstrapping" conditions

$$\hat{k}_n^{(j)} < \hat{k}_n^{(j-1)} < \hat{k}_{n+1}^{(j)}, \qquad (3)$$

which guarantee that between every pair of the neighboring points $\hat{k}_n^{(j)}$ and $\hat{k}_{n+1}^{(j)}$ (see Fig. 1) there exists a single point $\hat{k}_n^{(j-1)}$. In [5] it was also pointed out that due to certain analytical properties of the spectral determinant $\Delta(k) = 1 + \sum_i a_i e^{ikL_{(i)}}$, where $L_{(i)}$, are different linear combinations for the bond lengths l_1, l_2, ..., l_{N_B}, the set $\hat{k}_n^{(j)}$ can be provided by the sequence of zeros of the j-th derivative of the function $\Delta(k)$ [5,8]. In this case, the quantity r characterizing the degree of spectral irregularity is defined as the minimal number for which the condition $\sum_i \left|a_i \left(L_{(i)}/L_0\right)^r\right| < 1$ is satisfied [5].

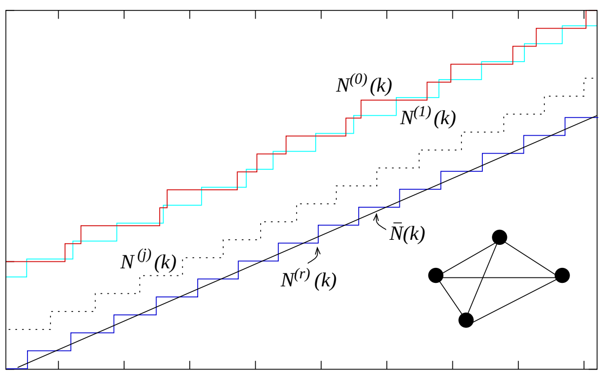

FIG. 1. Bootstrapping of spectral staircases for separating sequences $\hat{k}_n^{(j)}$ of the completely connected four-vertex graph with $r = 7$. The plots $N^{(j)}(k)$ are vertically shifted for the sake of clarity. It is clear that the physical spectral staircase $N^{(0)}(k)$ is interlaced by the staircase $N^{(1)}(k)$, etc. The last staircase $N^{(r)}(k)$ is intersected by the Weyl average $\bar{N}(k)$

In the simplest case of regular graphs when $r = 0$ [6,7], only one auxiliary sequence (2) is required and various spectral characteristics can be calculated using the formula

$$f\left(k_n\right)=\int_{\hat{k}_{n-1}^{(1)}}^{\hat{k}_n^{(1)}} f(k)\rho(k)dk. \tag{4}$$

As pointed out in [6,5], this case corresponds to the situation in which the straight line with the slope L_0/π, representing the Weyl average $\bar{N}(k)$, "pierces" the physical spectral staircase $N(k)$, i.e., $\bar{N}(k)$ intersects every stair step of $N(k)$ at the points $\hat{k}_n^{(1)}$.

II. STATISTICAL PROPERTIES OF THE SPECTRA OF REGULAR GRAPHS

Using the Gutzwiller formula in Eq. (4), one can derive the explicit expansions for various spectral characteristics $f_n^{(0)}$, for example, for fluctuations $\delta_n^{(0)}=\frac{L_0}{\pi}\left(k_n-\bar{k}_n\right)$, of the eigenvalues k_n around the Weyl average or for the distances between levels $s_{n,m}=k_{n+m}-k_n$. Such expansions have the form [6,7]

$$f_n^{(0)}=\bar{f}^{(0)}-\sum_p C_p^{(0)}\cos\left(\omega_p^{(0)}n+\varphi_p^{(0)}\right), \tag{5}$$

where the frequencies $\omega_p^{(0)}$ are defined via the periodic orbit lengths as, $\omega_p^{(0)}=\pi L_p^{(0)}/L_0$. The first term of expansion (5) determines the average value of the quantity $f_n^{(0)}$, whereas the following sum describes fluctuations around the average. Each frequency $\omega_p^{(0)}$ is an integer combination $\omega_p^{(0)}=m_{p,1}^{(0)}\Omega_1+m_{p,2}^{(0)}\Omega_2+...+m_{p,N_B}^{(0)}\Omega_{N_B}$, of the quantities Ω_i, which are expressed in terms of the lengths of the graph bonds as $\Omega_i=l_i/L_0$, and the coefficients $m_{p,i}^{(0)}$ indicate how many times the orbit passes along the bond l_i. The sum $\left|m_p^{(0)}\right|=m_{p,1}^{(0)}+m_{p,2}^{(0)}+...+m_{p,N_B-1}^{(0)}$ specifies the total number of scattering events that the particle moving along the trajectory p undergoes at the vertices. If Eq. (5) includes only the orbits for which $|m_p^{(0)}|<m$, we arrive at the m-th approximation to the exact value $f_n^{(0)}$ [1,6].

Since the numbers Ω_i satisfy the condition $\Omega_1+\Omega_2+...+\Omega_{N_B}=1$, only N_B-1 of these numbers are independent. Expressing one of them, e.g., Ω_{N_B}, in terms of the others, let us consider the (generic) case when the numbers $\tilde{\Omega}_i=\Omega_i-\Omega_{N_B}$ are irrational and algebraically independent. Let us call the orbit p algebraically simple (with the notation p') if the integer coefficients $\tilde{m}_{p,i}^{(0)}=m_{p,i}^{(0)}-m_{p,N_B}^{(0)}$ have no common divisors. Such orbits in general differ from the *dynamically* simple orbits that correspond to single traversals along closed sequences of bonds during the particles motion along the graph [1,2,5–7].

The expansion (5) enables one to pass immediately to the statistical description of the sequence $f_n^{(0)}$. Indeed, it is well known that the sequence of the remainders $x_n = [\alpha n]_{\mathrm{mod}\,1}$ for any irrational number α and $n = 1, 2, ...$, is uniformly distributed in the interval $[0, 1]$ [14]. Since the arguments of the trigonometric functions appearing in series (5) are defined modulo 2π, parsing through the values $f_n^{(0)}$ yields a sequence which is statistically equivalent to the series

$$f_x^{(0)} = \bar{f}^{(0)} - \sum_p \tilde{C}_p^{(0)} \sin\left(\tilde{m}_p^{(0)} x + \varphi_p^{(0)}\right), \tag{6}$$

Here, $\tilde{C}_p^{(0)}$ and $\tilde{m}_p^{(0)}$ correspond to the coefficients of Eq. (5) in which the condition $\sum_i \Omega_i = 1$ is taken into account, and x is a set of $N_B - 1$ independent, uniformly distributed random variables. The distribution of the quantities $\delta f_x^{(0)}$ in this case is obtained from the expression $P_f^{(0)} = \langle \delta\left(f^{(0)} - f_x^{(0)}\right)\rangle$:

$$P_f^{(0)} = \int dk e^{ik\left(f^{(0)} - \bar{f}^{(0)}\right)} \int_0^{2\pi} \prod_p \Lambda_p(x) \frac{dx}{2\pi} \tag{7}$$

where every factor $\Lambda_p(\vec{x}) = e^{ik\tilde{C}_p^{(0)} \cos\left(\tilde{m}_p^{(0)} x + \varphi_p^{(0)}\right)}$ determines the contribution to the integral from the corresponding periodic orbit p. Thus, Eq. (7) gives the exact expression for the distribution $P_f^{(0)}$ in terms of the periodic orbit theory. It is important to point out that the properties of the asymptotic distributions of trigonometric sums of form (6) are one of the traditional areas of research of mathematical statistics (see, e.g., [9,10] and references therein). In particular, it is known that separate terms (or groups of terms) of lacunary trigonometric series of form (6) can be considered as weakly dependent random variables, for which one can be establish a generalization of the central limit theorem, and consequently their sum is asymptotically Gauss distributed according to

$$P_f^{(0)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(\delta f^{(0)}\right)^2}{2\sigma^2}}, \tag{8}$$

with the variance

$$\sigma^2 = \frac{1}{2}\sum_p \tilde{C}_p^{(0)2} = \left\langle \left(\delta f_x^{(0)}\right)^2 \right\rangle . \tag{9}$$

The conclusion about the Gaussian form of the distribution of the fluctuations also appears to be applicable to this kind of spectral characteristics expansions of most of the regular quantum graphs (and other scaling systems), which are described by series of form (6) with constant coefficients. A hypothesis about the Gaussian nature of the distribution the spectral staircase fluctuations $\delta N(k) = N(k) - \bar{N}(k)$, confirmed by extensive numerical investigations, was previously proposed in [11] as the universal "central limit theorem for spectral fluctuations" applicable to general quantum chaotic systems. Owing to the existence of additional explicit expansions (5) this hypothesis, corroborated by the relation with the theory of weakly dependent random variables (trigonometric sums) can actually be extended to a much wider set of spectral characteristics.

III. APPROXIMATE DESCRIPTION OF THE DISTRIBUTION FUNCTIONS

Since the contributions of individual orbits to the series $\delta f_x^{(0)}$ behave as weakly dependent random variables, some physical simplifications are possible in Eq. (7). Expanding the exponentials $\Lambda_p(\vec{x})$, one can note that because expansion (5) is made in orthogonal harmonics, most integrals of the cross terms appearing from the product of the expansions $\Lambda_p(\vec{x})$ in Eq. (7). Contributions come only from the "resonant" terms for which one of the algebraic sums of the frequencies vanishes. The amplitude of these contributions decreases rapidly in the orders of the corresponding degrees of $C_p^{(0)}$, that are proportional to the product of the corresponding number of scattering coefficients at graph vertices [6,7]. This argumentation can be used to simplify the integral for $P_f^{(0)}$. For example, in a simple approximation the contributions from resonances between different algebraically simple orbits can be disregarded. This is equivalent to untangling of the factors $\Lambda_{p'}(x)$ corresponding to different algebraically simple orbits, i.e., to the introducing an independent set of variables $x_{p'}$ for every algebraically simple orbit. In this case, the distribution probability is represented in the form

$$P_f^{(0)} = \int dk e^{ik\left(f^{(0)}-\bar{f}^{(0)}\right)} \prod_{p'} Q_{p'}\left(k\tilde{C}_{p'}^{(0)}\right), \tag{10}$$

where every factor

$$Q_{p'} = \int_0^{2\pi} e^{ik\sum_\nu \tilde{C}_{p'\nu}^{(0)} \cos\left(\nu\tilde{m}_{p'}x_{p'}+\varphi_p\right)} dx_{p'}, \tag{11}$$

corresponds to the algebraically simple orbit p' and the sum with respect to ν in Eq. (11) is calculated over orbits whose indices are multiples of $\tilde{m}_p$. For a more crude description of the probability distribution profile, one can disregard the resonances between any distinct orbits, which is equivalent to the introduction of an independent phase x_p for every orbit. Under this assumption, the integral in Eq. (7) is separated into independent integrals and, as a result, we arrive at the simple expression

$$P_f^{(0)} = \int dk e^{ik\left(f^{(0)}-\bar{f}^{(0)}\right)} \prod_{p} J_0\left(k\tilde{C}_{p}^{(0)}\right), \tag{12}$$

where $J_0(x)$ is the zeroth Bessel function. Distributions of form (12) appear in communication theory, for example, when analyzing the intensity of interfering telecommunication channels, the theory of wave propagation in random media, and other fields where stochastic signal models are used [12,13]. It is also worth noting that, in the approximation of independent random contributions, the conditions of the Lindeberg–Feller theorem and central limit theorem are satisfied, which establish the normal distribution law for the sum of independent random variables. For spectral expansions (6) these conditions on the variances $\sigma_p^2 = \left(\tilde{C}_p^{(0)}\right)^2/2$ of individual contributions are satisfied due to the exponential increase in the number of periodic orbits and the uniform exponential decrease of the magnitude of the coefficients $\tilde{C}_p^{(0)}$. As a result, in the approximation of independent random contributions, distribution (12) has the same Gaussian form (8), with the variance $\sigma^2 = \sum_p \tilde{C}_p^{(0)2}/2 < \infty$ as that predicted in [9,10] and [11] for the case of weakly dependent variables. Such description is applicable to the statistical properties of various spectral characteristics of the regular graphs beginning with their harmonic expansions [5–7]. For example one can consider the fluctuations $\delta_n^{(0)} = \frac{L_0}{\pi}\left(k_n - \bar{k}_n\right)$, of levels around the average value, which have form (5) with $\bar{\delta}^{(0)} = 0$, $\varphi_p^{(0)} = -\frac{\pi}{2}$, and the coefficients

$$C_p^{(0)} = -\frac{2}{\pi}\frac{A_p^{(0)}}{\omega_p}\sin\left(\frac{\omega_p}{2}\right), \tag{13}$$

or the difference $s_{m,n}^{(0)} = k_{n+m} - k_n$ with $\bar{s}_{m,n}^{(0)} = \frac{\pi}{L_0}m$, $\varphi_p^{(0)} = \frac{\omega_p m}{2}$ and the coefficients

$$D_{p,m}^{(0)} = \frac{4}{L_0}\frac{A_p^{(0)}}{\omega_p}\sin\left(\frac{\omega_p}{2}\right)\sin\left(\frac{\omega_p m}{2}\right). \tag{14}$$

Knowing the distributions of these quantities, one can describe more complex objects such as the correlation function of fluctuations $\left\langle \delta_n^{(0)}\delta_{n+m}^{(0)}\right\rangle$, autocorrelation function $R_2(x)$, and the form factor $K_2(\tau)$, given by the expression

$$K_2 = \frac{\pi}{L_0}\sum_m \left\langle e^{-is_{mn}\tau}\right\rangle = \frac{\pi}{L_0}\sum_m e^{-i\frac{\pi m}{L_0}\tau}F_{s_m}^{(0)}(k), \tag{15}$$

where $F_{s_m}^{(0)}(k)$ is the characteristic function of distributions of form (7), (10) or (12), which are obtained from expansion (5) for $s_{m,n}$ with coefficients (14), and thus,

$$R_2(x) = \frac{\pi}{L_0}\sum_{m=1}^{\infty} P_{s_m}^{(0)}(x). \tag{16}$$

It is important that all above distributions are closed expressions consistently describing the spectral characteristics in terms of periodic orbit theory.

IV. SPECTRAL HIERARCHY

As mentioned above, in general quantum graphs are not regular and so for them the spectral expansions of form (5) cannot be obtained directly. A generalization to the irregular case can be obtained by using the relationship between the two neighboring separator systems $\hat{k}_n^{(j)}$ and $\hat{k}_n^{(j-1)}$ and by applying Eq. (4) to $f(k) = k$ at the $(j-1)$th level of the hierarchy:

$$\hat{k}_n^{(j-1)} = \int_{\hat{k}_{n-1}^{(j)}}^{\hat{k}_n^{(j)}} k dN^{(j-1)}. \tag{17}$$

Here, $N^{(j)}(k)$ corresponds to the spectral staircase of the sequence $\hat{k}_n^{(j)}$. Bootstrapping of the sequences $\hat{k}_n^{(j-1)}$ by $\hat{k}_n^{(j)}$ (or $N^{(j-1)}(k)$ by $N^{(j)}(k)$, see Fig. 1) means that $N^{(j-1)}\left(\hat{k}_n^{(j)}\right) = n$. Substituting expansion (1) for $N^{(j-1)}\left(\hat{k}_n^{(j)}\right)$ into Eq. (17), and using $\hat{k}_n^{(j)}$ in the form

$$\hat{k}_n^{(j)} = \frac{\pi}{L_0}\left(n + \delta_n^{(j)}\right), \tag{18}$$

we obtain the oscillating part of $\hat{k}_n^{(j-1)}$ in the form

$$\delta_n^{(j-1)} = f_\delta^{(j-1)} - \sum_p C_p^{(j-1)} \sin\left(\omega_p^{(j-1)} n + \varphi_p^{(j-1)}\right), \tag{19}$$

Here, the zeroth term

$$f_\delta^{(j-1)} = \frac{1}{2}\left(\delta_n^{(j)} - \delta_{n-1}^{(j)}\right) - \frac{1}{2}\left((\delta_n^{(j)})^2 - (\delta_{n-1}^{(j)})^2\right), \tag{20}$$

the amplitudes,

$$C_p^{(j-1)} = \frac{2}{L_0}\frac{A_p^{(j-1)}}{\omega_p^{(j-1)}} \sin\frac{\omega_p^{(j-1)}}{2}\left(\delta_n^{(j)} - \delta_{n-1}^{(j)} + 1\right), \tag{21}$$

and phases $\varphi_p^{(j-1)} = \omega_p^{(j-1)}\left(\delta_n^{(j)} + \delta_{n-1}^{(j)} - 1\right)/2$ for every level j are functions of the fluctuations $\delta_n^{(j)}$ and $\delta_{n-1}^{(j)}$ at the preceding hierarchy level.

Similar expansions are easily obtained for other spectral characteristics, for example, for $s_{n,m}^{(j-1)} = \hat{k}_{n+m}^{(j-1)} - \hat{k}_n^{(j-1)}$:

$$s_{n,m}^{(j-1)} = f_s^{(j-1)} + \frac{2}{L_0}\sum_p D_{p,m}^{(j-1)} \cos\omega_p^{(j-1)}\left(n - \frac{m}{2}\varphi_p^{(j-1)}\right), \tag{22}$$

with the zeroth term

$$\begin{aligned} f_s^{(j-1)} = s_{n,m}^{(j)} + \left(s_{n,m}^{(j)} - s_{n,m-1}^{(j)}\right) \times \\ \left(\pi m/L_0 - (s_{n,m}^{(j)} + s_{n-1,m}^{(j)})/2\right) - \xi_n^{(j)}\left(s_{n,m}^{(j)} - s_{n-1,m}^{(j)}\right), \end{aligned} \tag{23}$$

where $\xi_n^{(j)} = (\delta_n^{(j)} + \delta_{n-1}^{(j)})/2$ and the expansion coefficients $\tilde{D}_{p,m}^{(j-1)}$ are obtained from the corresponding expansion for $s_{n,m}^{(j)}$. The equations relating the neighboring sequences can also be considered as describing the transition of a single separating sequence $f_n^{(j)}$ from one hierarchy level to another.

V. STATISTICAL DESCRIPTION OF SPECTRAL HIERARCHY

As in the case of the regular graphs, the description of the stochastic properties of sequences such as $\delta_n^{(j)}$ or $s_{n,m}^{(j)}$ is based on the observation that parsing through the indices

n in the arguments of harmonic functions (19) and (22) leads to the appearance of random variables x. The idea of finding the distribution functions for various spectral characteristics is based on using the structural relations between the separating sequences obtained above in order to relate the probability distributions $P_f^{(j)}$ at different hierarchy levels. Beginning with the distribution $P_f^{(r)}$ at the regular level, one can determine the distribution $P_f^{(r-1)}$ at the next level and so on, ending with the last, physical level.

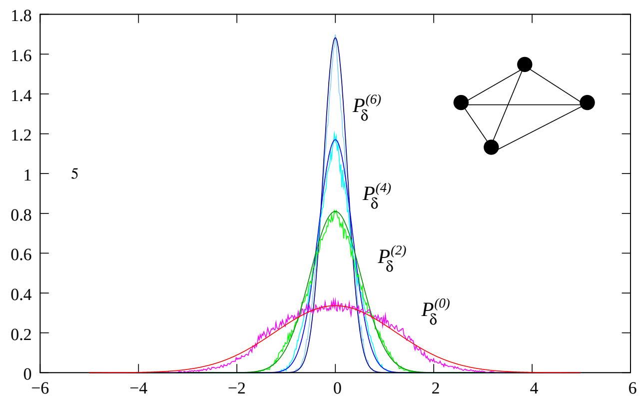

FIG. 2. Distribution of variances at the odd levels of the spectral hierarchy for the four-vertex quantum graph with $r = 7$. The solid lines are the Gaussian approximations of the numerically calculated histograms.

As an example, let us consider the behavior of the sequences $\delta_n^{(j)}$. For simplicity, we treat the fluctuations $\delta_n^{(j)}$ and $\delta_{n-1}^{(j)}$ as independent random variables δ_1 and δ_2 distributed according to $P_\delta^{(j)}$. Correspondingly, one can write for the density $P_\delta^{(j-1)}(\delta)$

$$P_\delta^{(j-1)} = \int \delta\left(\delta - \delta_x^{(j-1)}\right) P_{\delta_1}^{(j)} P_{\delta_2}^{(j)} d\delta_1 d\delta_2 dx. \tag{24}$$

Using Eq. (19) and representing the delta functional in exponential form, we obtain

$$P_\delta^{(j-1)}(\delta) = \int dk e^{ik\delta} \left\langle \prod_p \Lambda_p^{(j-1)}\left(x, \delta_1, \delta_2\right) dx \right\rangle_{\Omega^{(j-1)}}, \tag{25}$$

Here, the factors $\Lambda_p^{(j)}\left(x, \delta_1, \delta_2\right)$ correspond to the terms of expansion (19), which are now explicit functions of fluctuations at preceding hierarchy levels, and $\langle * \rangle_{\Omega^{(j)}}$ denotes averaging over these fluctuations with the weight

$$\Omega^{(j-1)}\left(\delta_1, \delta_2, k\right) = e^{-ikf_\delta^{(j-1)}(\delta_1,\delta_2)} P_\delta^{(j)}\left(\delta_1\right) P_\delta^{(j)}\left(\delta_2\right). \tag{26}$$

The expression (25) generalizes regular expansions (7), (10) and (12) for the single-level hierarchy to the general expressions for $j > 0$, averaged over the disorder at the preceding levels.

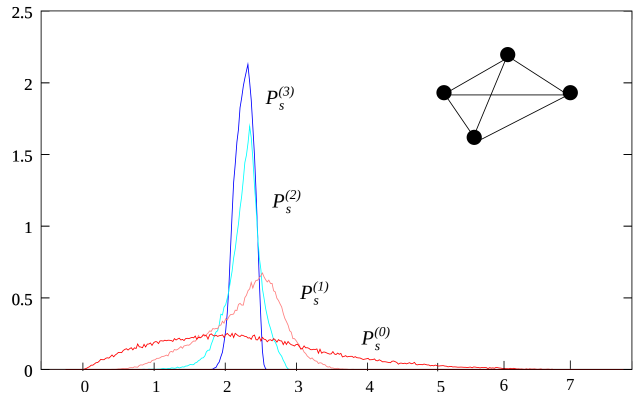

FIG. 3. Development of the probability distributions for the distances between the nearest neighbors $s_n^{(j)} = \hat{k}_n^{(j)} - \hat{k}_{n-1}^{(j)}$, $r = 3$. The maximum distance between the nearest neighbors at the $j = 0$ level in this case is $s_{max} = 8.68$, for the regular cell size $\pi/S_0 = 2.28$.

We note that the argumentation concerning the Gaussian distribution form in Section 2 [9,10] can be directly applied to the distribution of $\delta_x^{(r)}$ at the regular level. However, as shown in Fig. 2, the distribution of $\delta_x^{(j)}$ at higher levels $j > 0$ is also Gaussian-like. For other spectral characteristics, for example, $s_n^{(j)}$ (see Fig. 3), the sequence of transitions of form (25) can lead to asymmetric (non-Gaussian) distributions.

VI. DISCUSSION

The method proposed in [5] for solving the spectral problem is based on establishing the structural relationships between the sequence of physical levels k_n and the regular sequence $\hat{k}_n^{(r+1)}$ specified as an explicit function $\hat{k}_n^{(r+1)} = \hat{k}^{(r+1)}(n)$. For quantum graphs, the regular sequence is given by (2) and relation to k_n is established through the system of auxiliary sequences $\hat{k}_n^{(j)}$, bootstrapping k_n with $\hat{k}^{(r+1)}(n)$. The spectral hierarchy thus obtained consists of the system of sequences $\hat{k}_n^{(j)}$ and transition equations (17) from $\hat{k}_n^{(j)}$ to $\hat{k}_n^{(j-1)}$.

This approach allows not only the description of the evolution of base sequences $\hat{k}_n^{(j)}$ from low to high hierarchy levels, but also the complete probability description of spectral characteristics in the framework of periodic orbit theory including those that are not directly described by the Gutzwiller formula. In this case, it is possible to follow the development of the scales of spectral fluctuations, distributing disorder over the intermediate hierarchy levels, gradually passing from less to more disordered sequences. While the base sequence is maximally ordered, the amplitude of fluctuations in each next sequence $\hat{k}_n^{(j)}$ increases as the index j decreases, i.e. with the approach to the physical spectrum [5]. The minimum number of auxiliary sequences $\hat{k}_n^{(j)}$ necessary for bootstrapping $\hat{k}_n^{(r+1)}$ with $\hat{k}_n^{(0)}$ defines to the complexity of the spectral problem with respect to the given bootstrapping method.

The above relation between the properties of the series of expansions (19) and the properties of weakly dependent random variables [9,10] reveals the physical origins of the universality of the distributions of different spectral characteristics following from the limiting properties of the sums of such quantities. The existence of a sufficient number of transitions between hierarchy levels of irregular systems and, correspondingly, of averaging processes over random phases and disordered sequences $\hat{k}_n^{(j)}$ in Eq. (25) leads not only to the Gaussian shape of the distribution of probabilities $P_f^{(0)}$ [as, e.g., for $\delta N(k)$ and, correspondingly for $\delta_n^{(0)}$, see [11] and Fig. 2), but also to the appearance of more complex (e.g., Wignerian, see [3] and Fig. 3) distributions.

It is also important that determining the fluctuation probabilities in form (25) makes it possible not only to follow the appearance of general, universal statistical relations, but also to describe in detail the specific features of distributions $P_f^{(j)}$, which present the individual properties of each particular system.

Work supported in part by the Sloan–Swartz Foundation.

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