STRUCTURAL RELIABILITY UNDER INCOMPLETE PROBABILITY INFORMATION

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A report on research sponsored by
the National Science Foundation
under Grant No. CEE-8205049

Report No. UCB/SESM-85/01
Division of Structural Engineering
and Structural Mechanics
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January 1985

Abstract

A comprehensive framework is set forth for first-order structural reliability analysis under incomplete probability information. Under stipulated requirements of consistency, invariance, operability, and simplicity, a method is developed to incorporate incomplete probability information on random variables beyond the second moments, including marginal distributions, partial joint distributions, bounds, and higher moments. The method complements existing first-order, second-moment and first-order, full-distribution structural reliability theories and is consistent with the philosophy of Ditlevsen's generalized reliability index.

New results are given for joint distribution models with known marginals, which are useful in many applications of probability and statistics.

Acknowledgement

The work presented in this report has been supported by the National Science Foundation under Grant No. CEE-8205049. This support is gratefully acknowledged. Opinions, findings, and conclusions expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Thanks are due Professor David R. Brillinger of the Department of Statistics and Dr. Takeru Igusa of the Department of Civil Engineering of the University of California, Berkeley, for their many valuable suggestions during the course of this study and for their critically reviewing the manuscript.

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1. Introduction

The structural reliability problem is often formulated in terms of a vector of basic random variables $\mathbf{X} = [X_1 \cdots X_n]^T$, representing uncertain quantities such as loads, environmental factors, material properties, structural dimensions, and variables introduced to account for modeling and prediction errors, and a performance function $g(\mathbf{X})$ describing the limiting state of the structure in terms of $\mathbf{X}$. By convention, the performance function is formulated such that for an outcome $\mathbf{X} = \mathbf{x}$ the structure fails if $g(\mathbf{x}) \leq 0$ and it survives if $g(\mathbf{x}) > 0$. The boundary between the failure and safe sets,

$$g(\mathbf{x}) = 0 \tag{1}$$

is known as the *limit-state surface*. This is illustrated in Fig. 1 for a case with $n = 2$.

Formally, the probability of failure is

$$p_f = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{2}$$

where $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint probability density function (PDF) of $\mathbf{X}$ and the integral is over the failure set. By definition, *reliability* is the probability of survival and is the complement of p_f.

For most problems of interest, direct evaluation of the multifold integral in Eq. 2 is impractical. Two distinct problems arise: (1) Usually $f_{\mathbf{X}}(\mathbf{x})$ is not known, since complete probability information on $\mathbf{X}$ is seldom available; (2) The multifold integral in Eq. 2 requires numerical evaluation, which is a formidable task for $n > 2$. Under incomplete probability information, strict evaluation of the failure probability is not possible. Instead, an alternative measure of reliability known as the *reliability index* is the object of the analysis.

In the well known, first-order reliability method, the reliability index is obtained by linearizing the performance function (at one or several points) on the limit-state surface [11, 5, 7, 21, 12]. Most existing literature on this

approach deal with second-moment methods where only the means, variances, and covariances of the basic variables are known [11, 5, 7]. In many applications, however, information beyond these moments but short of complete knowledge of $f_{\mathbf{X}}(\mathbf{x})$ is available. For example, marginal distributions, bounds, or higher moments of the variables might be known. Obviously, any such information should be properly incorporated in the reliability analysis. This problem has been previously recognized by Ditlevsen [5] and has been addressed by Veneziano [24] in relation to a reliability index based on upper Tchebycheff bounds. However, a methodology to properly incorporate such information within the context of the first-order reliability approach has not been developed.

In this work, a comprehensive framework for first-order structural reliability under incomplete probability information is developed. Under stipulated requirements of *consistency*, *invariance*, *operability*, and *simplicity*, a method is developed to incorporate information beyond the second moments, including marginal distributions, partial joint distributions, bounds, and higher moments. For the two extremes of probability information, which are second-moment and complete distributional knowledge, the method reduces to the existing first-order reliability methods[11, 5, 21, 12]. As the quality of available information improves from purely second moment to complete distributional knowledge, the complexity of the method progressively increases in accordance to the requirement of simplicity. Thus, the method presented herein complements existing first-order, second-moment [11, 5] and first-order, full-distribution [21, 12] structural reliability methods. The work also includes new results with regard to joint distribution models with known marginals which is of interest in other applications of probability and statistics.

2. The First-Order Reliability Method

In the first-order, second-moment reliability method [11,5] it is assumed that the available information on $\mathbf{X}$ is limited to the mean vector $\mathbf{M} = [m_1 \cdots m_n]^T$ and the covariance matrix $\mathbf{C}$. It is a common procedure to transform the basic variables into a set of standard variates with zero means and unit covariance matrix. For later developments in this paper, it is convenient to write this transformation in the form

$$\mathbf{Y} = \Gamma \mathbf{D}^{-1} (\mathbf{X} - \mathbf{M}) \tag{3}$$

in which $\mathbf{D} = diag[\sigma_i]$ is the diagonal matrix of the standard deviations, σ_i, and $\Gamma = \mathbf{L}^{-1}$, where $\mathbf{L}$ is a lower-triangular matrix obtained from Cholesky decomposition [25] of the correlation matrix $\mathbf{R} = \mathbf{D}^{-1}\,\mathbf{C}\,\mathbf{D}^{-1}$ such that

$$\mathbf{R} = \mathbf{L}\mathbf{L}^T \tag{4}$$

The above transformation exists if $\mathbf{R}$ is positive definite, which occurs when the variables are real and no linear relationship among the X_i exists. Note that $\mathbf{L}^{-1}$ is easily computed since $\mathbf{L}$ is triangular.

It is also necessary to transform the limit-state surface into the standard space. Using the inverse transformation,

$$\mathbf{X} = \mathbf{M} + \mathbf{D}\mathbf{L}\mathbf{Y} \tag{5}$$

the expression for the limit-state surface is

$$g(\mathbf{x}) = g(\mathbf{M} + \mathbf{D}\mathbf{L}\mathbf{y}) \equiv G(\mathbf{y}) = 0 \tag{6}$$

This is illustrated in Fig. 2.

An important property of the space of the standard variates is that it has rotational symmetry with respect to a second moment representation. To see this, consider the probability content p outside an arbitrary hyperplane of unit normal α and distance β from the origin, Fig. 2. By squeezing the probability density onto the line $u-u$ (normal to the hyperplane from the origin), p can be written in terms of the probability distribution of a single variable $U = -\alpha^T \mathbf{Y}$:

$$p = P(U \leq -\beta) \tag{7}$$

Since $\mathbf{Y}$ has zero mean and unit covariance and $\boldsymbol{\alpha}$ has unit length, it follows that U is a standard variate (i.e., it has zero mean and unit standard deviation). The symmetry property can now be stated as follows: *For all hyperplanes of equal distance from the origin the probability p is constant within a second-moment representation (i.e., upper Tchebycheff bound [19]) regardless of the orientation of the hyperplane (i.e., $\boldsymbol{\alpha}$).* Further, this probability is contained in the tail of the distribution of U located at β standard deviations from its mean. These properties (with a somewhat different interpretation) led to the introduction of the widely used *Hasofer-Lind* reliability index, β_{HL}, which is defined as the shortest distance from the origin to the limit-state surface in the standard space [11].

For nonflat limit-state surfaces, determination of the nearest point to the origin (the *Hasofer-Lind* point) requires an iterative procedure. Several such procedures are available in the literature. One due to Rackwitz and Fiessler [12] establishes a sequence of points $\mathbf{y}_1, \mathbf{y}_2, \cdots$ in the standard space according to the rule

$$\mathbf{y}_{i+1} = \left[\mathbf{y}_i^T \boldsymbol{\alpha}_i + \frac{G(\mathbf{y}_i)}{|\nabla G(\mathbf{y}_i)|}\right]\boldsymbol{\alpha}_i \tag{8}$$

where $\nabla G(\mathbf{y}) = [\frac{\partial G(\mathbf{y})}{\partial y_1} \cdots \frac{\partial G(\mathbf{y})}{\partial y_n}]^T$ is the gradient vector which, using Eqs. 5 and 6, is computed in terms of the gradient vector in the original space

$$\nabla G(\mathbf{y}) = \mathbf{L}^T \mathbf{D} \nabla g(\mathbf{x}) \tag{9}$$

and

$$\boldsymbol{\alpha}_i = -\frac{\nabla G(\mathbf{y}_i)}{|\nabla G(\mathbf{y}_i)|} \tag{10}$$

is the unit vector normal to the surface having its direction towards the unsafe set. The sequence normally converges to the Hasofer-Lind point, $\mathbf{y}^*$, in just few cycles. The reliability index then is computed from

$$\beta_{HL} = \sqrt{\mathbf{y}^{*T}\mathbf{y}^{*}} \tag{11}$$

and the corresponding point in the original space, $\mathbf{x}^*$, is obtained from Eq. 5.

In passing we note that the unit vector $\boldsymbol{\alpha}$ represents the gradient vector of β_{HL} with respect to $\mathbf{y}^*$. As such, it describes the sensitivities of the reliability index with respect to variations in each of the standard variates Y_i. A unit vector which measures the sensitivities with respect to standard variations in the basic variables, X_i, is [3]

$$\boldsymbol{\gamma} = \frac{\boldsymbol{\Gamma}^T\boldsymbol{\alpha}}{|\boldsymbol{\Gamma}^T\boldsymbol{\alpha}|} \tag{12}$$

Ditlevsen [5] has shown that for a nonflat limit-state surface β_{HL} lacks comparability: Ordering of β_{HL} values may not be consistent with the ordering of actual reliabilities. An example of this can be seen in Fig. 2 by considering the curved and flat surfaces as limit-state surfaces of two structures. Whereas the structure with the flat surface is clearly more reliable than the one with the curved surface, β_{HL} values are identical and suggest equal reliability. To overcome this inconsistency, Ditlevsen [5] introduced the ***generalized reliability index***, β_g, which is defined by

$$\beta_g = \Phi^{-1}\left[\int_{G(\mathbf{y})>0} \varphi(y_1)\varphi(y_2)\cdots\varphi(y_n)\,d\mathbf{y}\right] \tag{13}$$

in which $\Phi[.]$ is the standard normal cumulative probability, and $\varphi(.)$ is the standard normal probability density. The integral in the right-hand side is the probability volume of a standard, n-dimensional normal density over the safe set in the standard space. The use of this density is only a mathematical formalism and does not imply a distribution assumption for the basic variables. Its choice is based on the symmetry property described earlier and certain requirements of simplicity [5]. Since β_g includes the entire safe set in its definition, it provides a consistent ordering of reliability within a second-moment context.

For large n, the integral in Eq. 13 for the generalized reliability index

seemingly poses the same computational problem as direct evaluation of Eq. 2. However, two principles are used to resolve this difficulty [5]: One is that for a flat limit-state surface β_g is identical to β_{HL}, which is easily computed. The second is that, for a convex safe set, the nonflat limit-state surface can be approximated by a polyhedral surface consisting of tangent hyperplanes at selected points on the surface (e.g., the locally minimum-distance points). For small failure probabilities of interest in structural reliability, the probability content of the polyhedral surface closely approximates that of the actual surface. Thus, narrow bounds for the normal probability content of polyhedral surfaces, which are in terms of β_{HL} and $\boldsymbol{\alpha}$ values of the individual hyperplanes [4], are used to approximately evaluate the generalized reliability index from Eq. 13.

The symmetry property described earlier is valid only when the information on $\mathbf{X}$ is strictly limited to $\mathbf{M}$ and $\mathbf{C}$. Generally, any additional information disturbs this symmetry and both the β_{HL} and β_g definitions become inoperable. Consider, for example, the case $\mathbf{X}=[X_1,X_2]^T$, $\mathbf{M}=\mathbf{0}$, $\mathbf{C}=\mathbf{I}$, and assume it is known that $X_1 < 2$. In this case the original space is standard and no transformation is needed. Now observe that whereas it is certain that the probability content of the set $2-x_1<0$ is zero, the probability content of the set $2-x_2<0$ which is of equal distance from the origin is unknown and is conceivably nonzero. Thus, the rotational symmetry described earlier does not exist. For such cases, Ditlevsen [5] suggests introducing a nonlinear transformation

$$\mathbf{Y} = \mathbf{T}(\mathbf{X}) \tag{14}$$

such that the space of $\mathbf{Y}$ possesses rotational symmetry. In the context of the generalized reliability index, this essentially implies the formal selection of a probability distribution for $\mathbf{X}$ and a transformation $\mathbf{T}$ such that $\mathbf{Y}$ has the standard normal density.

When the probability information on **X** is incomplete, a multitude of distribution models consistent with the available information may exist and the reliability index may not be unique. To avoid this problem, Ditlevsen [5] suggests that "... it is necessary that *codes of practice* uniquely specify types of density functions (or transformations) to be used for various types of relevant uncertain quantities." As an example, he suggests a logarithmic transformation for an uncertain quantity which is known to be positive. Obviously, this corresponds to a formal selection of the lognormal distribution for the uncertain quantity. To the writers' knowledge, no other transformations have been proposed by Ditlevsen or others. In particular, it is not known what transformation should be used when the variables with incomplete information are statistically dependent.

The main contribution of the present work is in formulating distribution models for **X** and developing corresponding transformations **T** for use in structural reliability when the probability information on **X** is beyond the first and second moments but short of being complete. The approach is consistent with the philosophy of Ditlevsen's generalized reliability index.

Before closing this section it is relevant to cite the work of Veneziano [24], which is closely related to the present topic of study. After demonstrating the weakness of β_{HL} relative to the ordering property described earlier, he introduces an alternative reliability index which is in terms of the upper Tchebycheff bound of the failure probability. One advantage of this index, he points out, is that it can incorporate information other than the first and second moments. The paper includes a number of specific cases for which closed form solutions of the Tchebycheff bound are available. As pointed out by Ditlevsen [5], however, whereas the reliability index introduced by Veneziano is rational and satisfying from a philosophical point of view, its application in practice poses

severe difficulties since a general methodology for computing upper Tchebycheff bounds does not yet exist.

3. Fundamental Requirements

As ground rules for formulation of distribution models for **X** and corresponding transformations **T**, four requirements are stipulated as follows:

Consistency -- The distribution model for **X** shall satisfy the rules of probability and be consistent with the available information.

Invariance -- The reliability index, β_{HL} or β_g, shall be invariant to any arbitrariness in selection of the distribution or the transformation.

Operability -- The distribution model shall apply to an arbitrary number of basic variables and be capable of incorporating any and all available information.

Simplicity -- The computational effort needed for the transformation shall be commensurate with the quality of information available.

The last requirement above is included since sophisticated and extensive computations are not justified when the available information is of poor quality.

In the following section, the basic ideas of the paper are introduced for the special case where **M**, **C**, and the set of marginal distributions of the basic variables are known. In the subsequent sections, these ideas are extended to situations where other information beyond the first and second moments are available.

4. Marginal Distributions Known

Suppose the information available on **X** consists of **M**, **C**, and the set of marginal cumulative distribution functions (CDFs), $F_{X_i}(x_i)$, $i = 1, \cdots, n$. This situation may arise in practice when available observations of **X** are sufficient for

selecting appropriate marginal distributions but insufficient for determining joint statistics beyond the covariances. Also, in some situations, appropriate marginal distributions may be prescribed based on physical or mathematical considerations, e.g., the normal or lognormal distribution based on the central limit theorem, extreme-value distributions based on asymptotic theorems [2, 1].

Consider first two variables X_1 and X_2. In accordance with the first requirement above, we seek a bivariate distribution model which is consistent with the known marginals and the covariance between the two variables. A large number of such models are available in the literature and several examples are listed in Appendix I. With the consistency requirement alone, the choice between these models is entirely arbitrary. Thus, the choice should be guided by the requirements of invariance, operability, and simplicity. A review of Appendix I reveals that the models listed there fail to satisfy one or more of these requirements. In particular, most models are not applicable to more than two variables or are restricted to narrow ranges of correlation coefficients between the variables. The corresponding transformation to standard normal space requires at least the same amount of computation as would be required were the actual distributions known. Also, it can be shown that β_{HL} computed based on these models may lack invariance relative to the ordering of the variables.

Since the transformation in Eq. 14 is a mapping into the normal space, it is natural to seek a distribution model which itself is based on the normal distribution. Let Z_1 and Z_2 be standard normal variates obtained by marginal transformations of X_1 and X_2

$$Z_i = \Phi^{-1}[F_{X_i}(X_i)] \quad i = 1,2 \tag{15}$$

We assign a joint distribution to X_1 and X_2 such that Z_1 and Z_2 are jointly normal. Using the rules of probability transformation, the joint density of X_1 and

X_2 is

$$f_{X_1X_2}(x_1,x_2) = \varphi_2(z_1, z_2, \rho_{0,12}) \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{\varphi(z_1)\varphi(z_2)} \tag{16}$$

where $f_{X_i}(x_i) = \frac{d}{dx_i} F_{X_i}(x_i)$ is the marginal density of X_i, $z_i = \Phi^{-1}[F_{X_i}(x_i)]$, and $\varphi_2(z_1, z_2, \rho_{0,12})$ is the bivariate normal density of zero means, unit standard deviations, and correlation coefficient $\rho_{0,12}$

$$\varphi_2(z_1, z_2, \rho_{0,12}) = \frac{1}{2\pi\sqrt{1-\rho_{0,12}^2}} \exp\left[-\frac{z_1^2 - 2\rho_{0,12} z_1 z_2 + z_2^2}{2(1-\rho_{0,12}^2)}\right] \tag{17}$$

The correlation coefficient $\rho_{0,12}$ is expressed in terms of the correlation coefficient ρ_{12} of X_1 and X_2 through the integral relation

$$\begin{aligned}\rho_{12} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{x_1-m_1}{\sigma_1}\right]\left[\frac{x_2-m_2}{\sigma_2}\right]\varphi_2(z_1, z_2, \rho_{0,12}) \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{\varphi(z_1)\varphi(z_2)} dx_1 dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{x_1-m_1}{\sigma_1}\right]\left[\frac{x_2-m_2}{\sigma_2}\right]\varphi_2(z_1, z_2, \rho_{0,12})\, dz_1 dz_2\end{aligned} \tag{18}$$

The above bivariate distribution model, which was originally suggested by Nataf [13], is valid under two conditions: (1) The mappings in Eq. 15 are one to one, which is true if each marginal $F_{X_i}(x_i)$ is continuous and monotonically increasing; (2) The value of $\rho_{0,12}$ from Eq. 18 lies between -1 and 1. Since in general $|\rho_{0,12}| \geq |\rho_{12}|$ (see Appendix II), the latter condition is normally satisfied if X_1 and X_2 are not strongly correlated. For most marginal distributions, the difference between ρ_{12} and $\rho_{0,12}$ is small and the condition is usually satisfied.

For each pair of marginal distributions with known ρ_{12}, the preceding equation can be iteratively solved for $\rho_{0,12}$. Since this calculation can be tedious, a set of semi-empirical formulae relating $\rho_{0,12}$ to ρ_{12} for selected marginal distributions are developed in Appendix II. These are in the form $\rho_{0,12} = \rho_{12} F$, where, for two-parameter distributions, depending on the marginals of X_1 and X_2, F is either a constant or is a function of ρ_{12} and the coefficients of variation δ_i =

σ_i / m_i of X_1 and X_2. The formulae for F for five categories of distributions are shown in Tables 1-5. The detailed development of these formulae is described in Appendix II. Example plots of F for selected distributions may be seen in Figs. 3-5.

The above distribution model is easily generalized to n variables. Define standard normal variates $\mathbf{Z} = [\, Z_1 \ \cdots \ Z_n \,]^T$ according to Eq. 15. Now assume the joint distribution of $\mathbf{X}$ is such that $\mathbf{Z}$ is jointly normal. The distribution model then is the generalization of Eq. 16, i.e.,

$$f_{\mathbf{X}}(\mathbf{x}) = \varphi_n(\mathbf{z}, \mathbf{R}_0) \frac{f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)}{\varphi(z_1)\varphi(z_2) \cdots \varphi(z_n)} \tag{19}$$

where $\varphi_n(\mathbf{z}, \mathbf{R}_0)$ is the n-dimensional normal density of zero means, unit standard deviations, and correlation matrix $\mathbf{R}_0$. The elements $\rho_{0,ij}$ of this matrix are obtained from Eq. 18 (or Tables 1-5), according to the marginal distributions and the correlation coefficient ρ_{ij} of X_i and X_j. This distribution model is valid under two conditions: (1) Each marginal CDF $F_{X_i}(x_i)$ is continuous and monotonically increasing; (2) The correlation matrix $\mathbf{R}_0$ is positive definite. The second condition, of course, necessitates that $|\rho_{0,ij}| < 1$ for all $i \neq j$. Since the correlation matrix $\mathbf{R}$ is positive definite and the differences $|\rho_{0,ij} - \rho_{ij}|$ are small, the second condition is satisfied in nearly all cases of practical interest.

The transformation to the standard normal space for the above distribution model can be written in the form

$$\mathbf{Y} = \mathbf{T}(\mathbf{X}) = \Gamma_0 \mathbf{Z} = \Gamma_0 \begin{Bmatrix} \Phi^{-1}[F_{X_1}(X_1)] \\ \vdots \\ \Phi^{-1}[F_{X_n}(X_n)] \end{Bmatrix} \tag{20}$$

in which $\Gamma_0 = \mathbf{L}_0^{-1}$, where $\mathbf{L}_0$ is the lower triangular matrix obtained from Cholesky decomposition of $\mathbf{R}_0$. For non-normal marginals, the above transformation is nonlinear. In searching for the nearest point to the origin in the $\mathbf{Y}$ space, the

standard procedure is to linearize the transformation at each iteration point [6]. Following the idea of Rackwitz, et al. [21], the linearized transformation is written

$$\mathbf{Y} = \mathbf{T}_i(\mathbf{X}) = \Gamma_0 \mathbf{D}'^{-1}(\mathbf{X} - \mathbf{M}') \tag{21}$$

in which $\mathbf{M}' = [m'_1 \cdots m'_n]^T$ is an equivalent mean vector and $\mathbf{D}' = diag\ [\sigma'_i]$ is an equivalent standard deviation matrix. The elements of $\mathbf{M}'$ and $\mathbf{D}'$ are determined at each linearization point $\mathbf{x}$ by the requirement that the two transformations be identical up to their first derivatives. This leads to two sets of equations: $\mathbf{T}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})$, which simplifies to

$$\Phi^{-1}[F_{X_i}(x_i)] = \frac{x_i - m'_i}{\sigma'_i}, \quad i = 1, \cdots, n \tag{22}$$

and $\dfrac{\partial \mathbf{T}_i(\mathbf{x})}{\partial x_i} = \dfrac{\partial \mathbf{T}(\mathbf{x})}{\partial x_i}$, which simplifies to

$$\frac{d}{dx_i}\Phi^{-1}[F_{X_i}(x_i)] = \frac{1}{\sigma'_i}, \quad i = 1, \cdots, n \tag{23}$$

Solving for m'_i and σ'_i in Eqs. 22 and 23,

$$m'_i = x_i - \sigma'_i \Phi^{-1}[F_{X_i}(x_i)], \quad i = 1, \cdots, n \tag{24}$$

$$\sigma'_i = \frac{\varphi\{\Phi^{-1}[F_{X_i}(x_i)]\}}{f_{X_i}(x_i)}, \quad i = 1, \cdots, n \tag{25}$$

The similarity of Eq. 21 to Eq. 3 suggests that the Hasofer-Lind point can be determined by an algorithm very similar to that used for the second-moment case. The differences are that the correlation matrix $\mathbf{R}_0$ rather than $\mathbf{R}$ is used and that at each iteration point the equivalent means and standard deviations from Eqs. 24-25 are substituted for the actual means and standard deviations. Since $\mathbf{R}_0$ is independent of the linearization point, $\mathbf{L}_0$ and Γ_0 are computed only once at the beginning of the iteration. Also, explicit use of the distribution model in Eq. 19 is not required in determining the reliability index.

The above formulation satisfies the fundamental requirements set forth in the previous section, provided the conditions specified following Eq. 19 are met.

Specifically, the joint distribution model satisfies the rules of probability, incorporates and is consistent with all available information, and is uniquely defined for arbitrary number of variables with arbitrary marginals and covariances. The reliability index is invariant to arbitrariness in the transformation (e.g., ordering of the variables) and the amount of computation required for its determination is more than that required for a second-moment application, but, as shown subsequently, less than that required when the joint distribution is known. It is worth noting that for independent, non-normal variables with known marginals (a case of complete probability information) the formulation presented here reduces to the solution previously given by Rackwitz, et al. [21].

4.1 **Example** 1

As a simple illustration consider the performance function

$$g(\mathbf{X}) = X_1 - X_2 \tag{26}$$

and assume the available information consists of the uniform and exponential marginal PDF's

$$f_{X_1}(x_1) = 0.01 \qquad 0 \le x_1 \le 100 \tag{27}$$

$$f_{X_2}(x_2) = 0.08 \exp(-0.08 x_2) \qquad 0 \le x_2 \tag{28}$$

and the correlation coefficient $\rho_{12} = 0.5$. From this information, the mean, standard deviation, and correlation matrices are

$$\mathbf{M} = \begin{Bmatrix} 50 \\ 20 \end{Bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 28.87 & 0 \\ 0 & 12.5 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tag{29}$$

The solution for the Hasofer-Lind reliability index proceeds as follows:

(1) Construct the correlation matrix $\mathbf{R_0}$ by evaluating Eq. 18 or using Table 3:

$$\mathbf{R_0} = \begin{bmatrix} 1 & 0.57 \\ 0.57 & 1 \end{bmatrix} \tag{30}$$

(2) Determine $\mathbf{L_0}$ by performing Cholesky decomposition of $\mathbf{R_0}$ and compute its inverse Γ_0:

$$\mathbf{L}_0 = \begin{bmatrix} 1 & 0 \\ 0.57 & 0.82 \end{bmatrix} \qquad \Gamma_0 = \begin{bmatrix} 1 & 0 \\ -0.70 & 1.22 \end{bmatrix} \tag{31}$$

(3) Select an initial linearization point and compute the equivalent means and standard deviations from Eqs. 24-25. The transformation to the standard normal space, Eq. 21, is now complete and a new linearization point is determined according to the rule in Eq. 8. Repeat the process until convergence is achieved. These calculations, with the final result $\beta_{HL} = 1.653$, are summarized in Table 6. The solution in the standard space for this example is shown in Fig. 6. In order to make an approximation of the generalized reliability index, β_g, a second linearization is considered at $\mathbf{y} = [2.03\ 2.69]^T$ with distance $\beta = 3.370$. Using bounds for probability content of polyhedral surfaces [4], the generalized reliability index approximately is $\beta_g = 1.650$.

5. Marginal and Partial Joint Distributions Known

The available information on the basic variables $\mathbf{X}$ may include, in addition to the marginal distributions, the joint distribution for a subset of these variables. For notational convenience, $\mathbf{X}$ is partitioned into $\mathbf{X} = [\mathbf{X}_1\ \mathbf{X}_2]^T$, with $\mathbf{X}_1 = [X_1\ \cdots\ X_k]^T$ and $\mathbf{X}_2 = [X_{k+1}\ \cdots\ X_n]^T$, where the marginals $F_{X_i}(x_i)$, $i = 1, \cdots, k$, and the $(n-k)$-dimensional joint CDF $F_{\mathbf{X}_2}(\mathbf{x}_2)$ are all known. Assume that the mean, covariance, standard deviation, and correlation matrices are available and, using the above partition, are expressed in the form

$$\mathbf{M} = \begin{Bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{Bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^T & \mathbf{C}_{22} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}_2 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^T & \mathbf{R}_{22} \end{bmatrix} \tag{32}$$

First consider the special case where $\mathbf{X}_2$ is normal. A linear transformation of the form

$$\mathbf{Z}_2 = \mathbf{D}_2^{-1}(\mathbf{X}_2 - \mathbf{M}_2) \tag{33}$$

then leads to an $(n-k)$-vector of standard, but correlated normal variates $\mathbf{Z}_2$.

For $\mathbf{X}_1$, the transformation in Eq. 15 is used for each element to obtain a k-vector $\mathbf{Z}_1$ whose elements are correlated and have standard normal marginals. The joint distribution of $\mathbf{X}$ is now assumed to be such that the n-vector $\mathbf{Z} = [\mathbf{Z}_1\ \mathbf{Z}_2]^T$ is standard normal with correlation matrix

$$\mathbf{R}_0 = \begin{bmatrix} \mathbf{R}_{0,11} & \mathbf{R}_{0,12} \\ \mathbf{R}_{0,12}^T & \mathbf{R}_{22} \end{bmatrix} \tag{34}$$

where the elements $\rho_{0,ij}$ of $\mathbf{R}_{0,11}$ and $\mathbf{R}_{0,12}$ are obtained from Eq. 18 (or Tables 1-5), according to the marginal distributions and the correlation coefficients ρ_{ij} of X_i and X_j. Note that for $i > k$, the marginal distribution of X_i is normal.

The above distribution assumption for $\mathbf{Z}$ implies a distribution model for $\mathbf{X}$ of the form

$$f_{\mathbf{X}}(\mathbf{x}) = \varphi_n(\mathbf{z}, \mathbf{R}_0)\, \frac{f_{X_1}(x_1) \cdots f_{X_k}(x_k)}{\varphi(z_1) \cdots \varphi(z_k)\, \sigma_{k+1} \cdots \sigma_n} \tag{35}$$

The conditions for validity of this model are similar to those listed following Eq. 19. It is easy to show that the model is consistent with the available information, i.e., upon integration it produces the known marginals of $\mathbf{X}_1$, the joint normal density of $\mathbf{X}_2$, and the covariance matrix $\mathbf{C}$.

The linearized transformation to the standard normal space at a point $\mathbf{x}$ is written as in Eq. 21, where Γ_0 is obtained from Cholesky decomposition of the correlation matrix $\mathbf{R}_0$ in Eq. 34, and

$$\mathbf{M}' = \begin{Bmatrix} \mathbf{M}'_1 \\ \mathbf{M}_2 \end{Bmatrix} \qquad \mathbf{D}' = \begin{bmatrix} \mathbf{D}'_1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}_2 \end{bmatrix} \tag{36}$$

in which $\mathbf{M}'_1$ and $\mathbf{D}'_1$ are the equivalent mean and standard deviation matrices of $\mathbf{X}_1$ at the linearization point, whose elements m'_i and σ'_i, $i = 1, \cdots, k$, are obtained from Eqs. 24-25. Thus, the search algorithm for the Hasofer-Lind point remains the same as for the previous case, except that the equivalent means and standard deviations are computed only for the elements of $\mathbf{X}_1$. As before, the correlation matrix $\mathbf{R}_0$ remains independent of the linearization

point and $\mathbf{L}_0$ and Γ_0 are computed once.

Now consider the general case where the known joint distribution of $\mathbf{X}_2$ is non-normal. For arbitrary marginals of $\mathbf{X}_1$ and joint distribution of $\mathbf{X}_2$, it is not possible to construct a distribution model which is consistent with the available information. That is, it is not possible to formulate $f_{\mathbf{X}}(\mathbf{x})$ such that upon integration it produces the known marginals of $\mathbf{X}_1$, the known joint distribution of $\mathbf{X}_2$, and the covariance matrix $\mathbf{C}$. However, as subsequently shown, it is possible to construct a distribution model which is consistent with the available information in a first-order sense. This is entirely suitable, since the reliability formulation itself is based on a first-order approximation.

Consider the transformation

$$\mathbf{V}_2 = \mathbf{T}_2(\mathbf{X}_2) \tag{37}$$

such that $\mathbf{V}_2 = [V_{k+1} \cdots V_n]$ is an $(n-k)$-dimensional standard, uncorrelated normal vector. The linearized version of this transformation at a point $\mathbf{x} = [\mathbf{x}_1 \; \mathbf{x}_2]^T$ may be written

$$\mathbf{V}_2 = \mathbf{A}_2 + \mathbf{J}_2 \mathbf{X}_2 \tag{38}$$

where $\mathbf{J}_2$ is the Jacobian of the transformation at $\mathbf{x}_2$ with elements $J_{ij} = \partial v_i / \partial x_j$, $i,j = k+1, \cdots, n$, and

$$\mathbf{A}_2 = \mathbf{T}_2(\mathbf{x}_2) - \mathbf{J}_2 \mathbf{x}_2 \tag{39}$$

A convenient choice for the transformation in Eq. 37 is

$$V_i = \begin{cases} \Phi^{-1}[F_{X_i}(X_i)] & i = k+1 \\ \Phi^{-1}[H_i(X_i \mid \mathbf{X}_{i-1})] & i = k+2, \cdots, n \end{cases} \tag{40}$$

in which $\mathbf{X}_{i-1} = [X_{k+1} \cdots X_{i-1}]^T$ and

$$H_i(X_i \mid \mathbf{X}_{i-1}) = P(X_i \le x_i \mid \mathbf{X}_{i-1} = \mathbf{x}_{i-1}) \tag{41}$$

The Jacobian matrix then is lower triangular with elements

$$J_{ij} = \begin{cases} \dfrac{1}{\varphi(v_i)} \dfrac{\partial H_i(x_i \mid \mathbf{x}_{i-1})}{\partial x_j} & j \le i \\ 0 & j > i \end{cases} \tag{42}$$

In the structural reliability literature, this transformation is attributed to Rosenblatt [22] and was first utilized by Hohenbichler, et al. [12]. However, an earlier formulation of the transformation may be found in Segal [23]. The transformation in general is one-to-one, provided the functions H_i are continuous and monotonically increasing.

It has been shown [6] that the linearization in Eq. 38 of the transformation in Eq. 37 is equivalent to replacing $\mathbf{X}_2$ with a normal vector of mean and covariance

$$\mathbf{M}'_2 = -\mathbf{B}_2\mathbf{A}_2 \qquad \mathbf{C}'_2 = \mathbf{B}_2\mathbf{B}_2^T \tag{43}$$

where $\mathbf{B}_2 = \mathbf{J}_2^{-1}$. This result is easily obtained by taking expectations in Eq. 38. Let $\mathbf{D}'_2$ and $\mathbf{R}'_{22}$ denote the standard deviation and correlation matrices associated with $\mathbf{C}'_2$.

Now consider the n-vector $\mathbf{Z} = [\mathbf{Z}_1\ \mathbf{Z}_2]^T$, where $\mathbf{Z}_1$ is defined as before and $\mathbf{Z}_2 = \mathbf{D}'^{-1}_2(\mathbf{X}_2 - \mathbf{M}'_2)$. At the linearization point $\mathbf{x}$, $\mathbf{Z}_2$ is equivalent, in a first-order sense, to a standard normal vector of correlation matrix $\mathbf{R}'_{22}$. Based on the special case presented earlier in this section, the distribution of $\mathbf{X}$ at the linearization point is assumed to be such that $\mathbf{Z}$ has the n-dimensional standard normal distribution $\varphi_n(\mathbf{z},\mathbf{R}'_0)$ with

$$\mathbf{R}'_0 = \begin{bmatrix} \mathbf{R}_{0,11} & \mathbf{R}_{0,12} \\ \mathbf{R}_{0,12}^T & \mathbf{R}'_{22} \end{bmatrix} \tag{44}$$

in which the elements $\rho_{0,ij}$ of $\mathbf{R}_{0,11}$ and $\mathbf{R}_{0,12}$ are obtained from Eq. 18 (or Tables 1-5) according to the marginal distributions and the correlation coefficient ρ_{ij} of X_i and X_j. Note that for $i > k$, the actual marginal distribution of X_i rather than the equivalent normal should be used in computing $\rho_{0,ij}$.

The above distribution assumption for $\mathbf{Z}$ implies a distribution for $\mathbf{X}$ of the form

$$f_{\mathbf{X}}(\mathbf{x}) = \varphi_n(\mathbf{z},\mathbf{R}'_0)\,\frac{f_{X_1}(x_1)\cdots f_{X_k}(x_k)}{\varphi(z_1)\cdots\varphi(z_k)\,\sigma'_{k+1}\cdots\sigma'_n} \tag{45}$$

This is a valid model provided: (1) The transformation in Eq. 37 is one-to-one; (2) The known marginals of $\mathbf{X}_1$ are continuous and monotonically increasing; (3) the correlation matrix $\mathbf{R}'_0$ is positive definite. Obviously, the distribution depends on the linearization point. However, at each linearization point the distribution is consistent with the available information in a first-order sense, i.e., upon integration it produces the marginals of $\mathbf{X}_1$, the equivalent normal of $\mathbf{X}_2$, and the equivalent covariance matrix at the linearization point. Thus, the consistency requirement is satisfied in a first-order sense.

The linearized transformation to the standard normal space for the preceding distribution model may be written in the form

$$\mathbf{Y} = \Gamma''_0 \mathbf{D}'^{-1} (\mathbf{X} - \mathbf{M}') \tag{46}$$

where $\Gamma''_0 = \mathbf{L}'^{-1}_0$, where $\mathbf{L}'_0$ is the lower-triangular decomposition of the correlation matrix $\mathbf{R}'_0$ in Eq. 44, and

$$\mathbf{M}' = \begin{Bmatrix} \mathbf{M}'_1 \\ \mathbf{M}'_2 \end{Bmatrix} \qquad \mathbf{D}' = \begin{bmatrix} \mathbf{D}'_1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}'_2 \end{bmatrix} \tag{47}$$

in which the elements of $\mathbf{M}'_1$ and $\mathbf{D}'_1$ are obtained from Eqs. 24-25, as before, and $\mathbf{M}'_2$ and $\mathbf{D}'_2$ are obtained from Eq. 43. The algorithm for determining the Hasofer-Lind point remains the same, except that at each linearization point one needs to recompute, in addition to the equivalent means and standard deviations, the equivalent correlation matrix $\mathbf{R}'_{22}$ of $\mathbf{X}_2$ and the matrices $\mathbf{L}'_0$ and Γ''_0.

It is important to note that in the Cholesky decomposition of $\mathbf{R}'_0$ the lower-triangular matrices $\mathbf{L}'_0$ and Γ''_0 are computed rowwise from top to bottom. Since $\mathbf{R}_{0,11}$ remains independent of the linearization point, the first k rows of $\mathbf{L}'_0$ and Γ''_0 remain unchanged and need to be computed only once. Thus, at each linearization point, after computing $\mathbf{R}'_{22}$, one proceeds with computing rows $k+1$ to n of $\mathbf{L}'_0$ and Γ''_0. The amount of computation in excess of that required when marginal distributions are known is of order $(n-k)n^2$, which is proportional to the number of variables with known joint distribution. Thus, the

computational effort is commensurate with the quality of information, which satisfies the simplicity requirement. It is worth noting that when complete probability information is available (i.e., when $k = 0$), the method presented here becomes identical to the method previously developed by Hohenbichler, et al. [12] for that special case.

Unfortunately, the invariance of the reliability index cannot be guaranteed in the present case. This problem stems from the transformation of variables with known joint distribution (i.e., Eq. 37) and has been previously recognized [7]. In particular, with the transformation in Eq. 40, β_{HL} depends on the ordering of the variables. Under complete probability information (i.e., when $f_{\mathbf{X}}(\mathbf{x})$ is known), the generalized reliability index β_g theoretically remains invariant, since the transformation preserves the probability content of the safe set. However, the polyhedral approximation of β_g in general may depend on the ordering of the variables. With incomplete information, the transformation in Eq. 46 is not necessarily probability preserving and one may expect that the theoretical definition of β_g also lack invariance.

The preceding formulation is easily generalized to a situation where, besides the marginal distributions of $\mathbf{X}_1$, the joint distributions for $m-1$ disjoint subsets $\mathbf{X}_2, \cdots, \mathbf{X}_m$ of the basic variables are known. One only needs to write the mean, standard deviation, and correlation matrices in the form

$$\mathbf{M}' = \begin{Bmatrix} \mathbf{M}'_1 \\ \mathbf{M}'_2 \\ \cdot \\ \mathbf{M}'_m \end{Bmatrix} \quad \mathbf{D}' = \begin{bmatrix} \mathbf{D}'_1 & \mathbf{0} & \cdot & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}'_2 & \cdot & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{0}^T & \mathbf{0}^T & \cdot & \mathbf{D}'_m \end{bmatrix} \quad \mathbf{R}'_0 = \begin{bmatrix} \mathbf{R}_{0,11} & \mathbf{R}_{0,12} & \cdot & \mathbf{R}_{0,1m} \\ \mathbf{R}_{0,12}^T & \mathbf{R}'_{22} & \cdot & \mathbf{R}_{0,2m} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{R}_{0,1m} & \mathbf{R}_{0,2m} & \cdot & \mathbf{R}'_{mm} \end{bmatrix} \tag{48}$$

where the primed submatrices for $\mathbf{X}_2, \cdots, \mathbf{X}_m$ are evaluated from transformations similar to that in Eq. 37 and the elements of the submatrices $\mathbf{R}_{0,ij}$ are obtained from Eq. 18 (or Tables 1-5). The distribution model for $\mathbf{X}$ remains the same as in Eq. 45 and the search algorithm for the Hasofer-Lind point remains

unchanged. One further generalization that may be considered is the situation where joint distributions for non-disjoint subsets of the basic variables are known. In that case, some elements of the off-diagonal submatrices of $\mathbf{R}'_0$ also become primed quantities and must be computed at each linearization point. This situation, however, is not of practical interest and will not be further developed here.

The formulation presented here is seen to be applicable to arbitrary situations with known marginals and partial joint distributions, thus satisfying the operability requirement.

5.1 Example 2

Consider the performance function

$$g(\mathbf{X}) = X_1 - X_2 - X_3 \tag{49}$$

and assume that the available information consists of the following: X_1 is log-normally distributed

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\zeta x_1}\exp\left[-\frac{1}{2}\left(\frac{\ln x_1 - \lambda}{\zeta}\right)^2\right] \qquad 0 < x_1 \tag{50}$$

with parameters $\lambda = 1.590$ and $\zeta = 0.198$, X_2 and X_3 are bivariate exponentially distributed

$$f_{X_1X_2}(x_1, x_2) = (x_1 + x_2 + x_1x_2)\exp[-(x_1 + x_2 + x_1x_2)] \qquad 0 \le x_1, x_2 \tag{51}$$

and the correlation coefficients are $\rho_{12} = 0.3$ and $\rho_{13} = -0.2$. The means, standard deviations, and correlation coefficients based on this information are

$$\mathbf{M} = \begin{bmatrix} 5.0 \\ 1.0 \\ 1.0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1.0 & 0.3 & -0.2 \\ 0.3 & 1.0 & -0.4 \\ -0.2 & -0.4 & 1.0 \end{bmatrix} \tag{52}$$

The procedure for determining the Hasofer-Lind point proceeds as follows:

(1) Construct the unprimed submatrices $\mathbf{R}_{0,11}$ and $\mathbf{R}_{0,12}$ of $\mathbf{R}'_0$. From the joint distribution in Eq. 51, the marginals of X_2 and X_3 are both exponential and

using Table 4, one obtains

$$\mathbf{R}_{0,11} = [1] \qquad \mathbf{R}_{0,12} = [0.33 \ -0.23] \tag{53}$$

(2) Determine the rows of $\mathbf{L}'_0$ and Γ''_0 corresponding to the variables with known marginal distributions. For the present example, these are both simply [1 0 0].

(3) Transform $\mathbf{X}_2 = [X_1 \ X_2]^T$ into the standard normal space. Using the transformation in Eq. 40, one has

$$V_2 = \Phi^{-1}[1 - \exp(-X_2)] \tag{54}$$

$$V_3 = \Phi^{-1}\{1 - (1 + X_3)\exp[-(X_3 + X_2 X_3)]\} \tag{55}$$

(4) Select an initial linearization point $\mathbf{x} = [x_1 \ \mathbf{x}_2]^T$. Compute $\mathbf{J}_2$ (Eq. 42), $\mathbf{B}_2$, $\mathbf{A}_2$ (Eq. 39), $\mathbf{M}'_2$ and $\mathbf{C}'_2$ (Eq. 43), and $\mathbf{R}'_{22}$ in terms of $\mathbf{x}_2$. Also compute m'_1 and σ'_1 (Eqs. 24-25) in terms of x_1. Construct $\mathbf{R}'_0$, $\mathbf{M}'$, and $\mathbf{D}'$ according to Eqs. 44 and 47 and complete the calculation of the lower triangular matrices $\mathbf{L}'_0$ and Γ''_0. The transformation to the standard normal space is now complete and a new linearization point is determined according to the rule in Eq. 8. Repeat this process until convergence is achieved.

The calculations for this example are summarized in Table 7. The starting point is selected to be the mean point and convergence is achieved in 15 steps with $\beta_{HL} = 1.987$, $\mathbf{x}^* = [4.05 \ 0.198 \ 3.85]^T$, and $\boldsymbol{\alpha} = [0.486 \ 0.319 \ -0.814]^T$. The limit-state surface for the present example has a second local minimum-distance point which is obtained if one starts the iteration at $\mathbf{x} = [5.0 \ 4.0 \ 0.2]^T$. The results are $\beta = 2.507$, $\mathbf{x}^* = [4.77 \ 4.57 \ 0.206]^T$, and $\boldsymbol{\alpha} = [0.054 \ -0.995 \ -0.087]^T$. The limit-state surface in the standard space may now be approximated by planes tangent at the two linearization points. Using Ditlevsen's bounds for polyhedral surfaces [4], the generalized reliability index is approximately $\beta_g = 1.89$.

As stated earlier, the reliability index in general may depend on the ordering of the elements of $\mathbf{X}_2$. To examine this, the above analysis is repeated with reordered elements of $\mathbf{X}_2$. Since the joint distribution of X_2 and X_3 is symmetric, the only change is in the correlation matrix in Eq. 52 where the elements 0.30 and −0.20 are interchanged. The results for the two minimum-distance points now are $\beta_{HL} = 2.106$, $\mathbf{x}^* = [4.00\ 3.82\ 0.178]^T$, $\boldsymbol{\alpha} = [0.489\ -0.869\ -0.076]^T$ and $\beta = 2.361$, $\mathbf{x}^* = [4.79\ 0.206\ 4.58]^T$, $\boldsymbol{\alpha} = [0.051\ 0.399\ -0.915]^T$, which result in $\beta_g = 1.93$. Observe that whereas the variation in β_{HL} due to the reordering is quite significant, the variation in β_g, which is partly due to the polyhedral approximation, is relatively small.

6. Less than Marginal Distributions Known

The available information on a random variable might be more than the first two moments but less than the marginal distribution. For example, higher moments of a variable (e.g., coefficients of skewness or kurtosis) or bounds of its distribution might be known. A rational and practical approach in this case is to select for each such variable a marginal distribution which is consistent with the available information and then proceed in the manner described in the previous sections. The problem then arises as to how one selects the marginal distribution, as several candidates might appear to be equally consistent with the available information. Two possible approaches originally suggested by Ditlevsen [5] are explored here.

In the first approach one selects that marginal distribution from the set of candidate distributions which results in the lowest estimate of the structure reliability. The rationale behind this approach is that one should pay in terms of a lower reliability estimate when the available information is of poor quality. With improving quality of information, the set of acceptable candidates is reduced and the reliability estimate either remains unchanged or increases.

Thus, the approach provides incentive for the analyst to collect additional data in order to increase the reliability estimate. Obviously, when marginals for several variables are to be selected, one needs to consider all combinations of the candidate distributions and the method may require extensive computations. This aspect of the approach is not consistent with the simplicity requirement of this paper.

The second and more appealing approach is based on a Bayesian notion. The marginal distribution is assumed to be a weighted average of all candidate distributions with the weights representing subjective probabilities of each candidate being the true distribution. For a variable X_1 with k candidate distributions $F_{1i}(x_1)$, $i = 1, \cdots, k$, the Bayesian distribution is written in the form

$$F_{X_1}(x_1) = \sum_{i=1}^{k} p_{1i} F_{1i}(x_1) \tag{56}$$

where p_{1i}, satisfying $\sum_{i=1}^{k} p_{1i} = 1$, are the weights. Note that all candidate distributions have the same mean and variance, as these are assumed to be known quantities. This form of the distribution is used directly in Eqs. 15 and 24-25 to compute the transformed variable Z_1 and the equivalent mean m'_1 and standard deviation σ'_1. Also, for variables X_1 and X_2 with respectively k and l candidate distributions each, $\rho_{0,12}$ is evaluated from Eq. 18 by using $z_1 = \Phi^{-1}\left[\sum_{i=1}^{k} p_{1i} F_{1i}(x_1)\right]$ and $z_2 = \Phi^{-1}\left[\sum_{j=1}^{l} p_{2j} F_{2j}(x_2)\right]$. In general, this evaluation will require numerical integration. To avoid such tedious calculations, one may use the approximation

$$\rho_{0,12} \approx \sum_{i=1}^{k} \sum_{j=1}^{l} p_{1i} p_{2j} \rho_{0,12,ij} \tag{57}$$

in which $\rho_{0,12,ij}$ is the value of $\rho_{0,12}$ if X_1 had the distribution $F_{1i}(x_1)$ and X_2 had the distribution $F_{2j}(x_2)$. Values of $\rho_{0,12,ij}$ are obtained from Tables 1-5 according to the candidate distributions of X_1 and X_2 and the correlation coefficient

ρ_{12}. This approximation is based on the simple idea that when the weights p_{1i} and p_{2j} approach unity, $\rho_{0,12}$ approaches $\rho_{0,12,ij}$. Thus, a weighted average of the $\rho_{0,12,ij}$ values should provide a good approximation. The only situation where the approximation may not be good is when the dominant candidate distributions of a variable have dissimilar shapes. Such a situation, however, would rarely occur in practice, since, in a sense, it is inconsistent.

6.1 Example 3

Suppose, based on the available information, the distribution of X_1 is either lognormal or gamma with subjective probabilities 0.7 and 0.3, respectively, and the distribution of X_2 is either type-I or type-II (largest) extreme-value distribution with subjective probabilities 0.4 and 0.6, respectively. Also assume $\rho_{12} = 0.5$ and $\delta_1 = \delta_2 = 0.3$, where δ_i denotes the coefficient of variation of X_i. From Tables 4 and 5, one obtains $\rho_{0,12,11} = 0.513$, $\rho_{0,12,12} = 0.548$, $\rho_{0,12,21} = 0.511$, and $\rho_{0,12,22} = 0.550$. Using these values in Eq. 57 results in $\rho_{0,12} = 0.534$. The exact value obtained from numerical evaluation of Eq. 18 is $\rho_{0,12} = 0.534$, which coincides with the approximate result.

7. Summary and Conclusions

A comprehensive framework for first-order structural reliability under incomplete probability information is developed with stipulated requirements of *consistency*, *invariance*, *operability*, and *simplicity*. Under this framework, a method is developed to incorporate information on the basic variables beyond the second moments, including marginal distributions, partial joint distributions, bounds, and higher moments. The method is consistent with the philosophy of Ditlevsen's generalized reliability index [5] and complements the existing methods for first-order, second-moment [11] and first-order, full-distribution [12] reliability methods. An appealing feature of the method is that, while it

generally satisfies the requirements of consistency, invariance, and operability, its complexity is commensurate with the quality of available information.

The multivariate distribution models obtained by generalization of Nataf's bivariate model [13] and the formulae developed for $\rho_{0,ij}$ are useful in other applications of probability and statistics where joint distribution models with known marginals are of interest. Finally, the formulae in Table 8 for Morgenstern's [18] model are new results, which can be valuable in future applications of this model.

8. Appendix I -- Joint Distribution Models with Known Marginals

Since M. Frechet's original paper in 1951 [9], joint distribution models with known marginals have been extensively studied [13]. In this appendix, selected bivariate models from the literature are briefly reviewed and their utilization in structural reliability is discussed. It is assumed that, in addition to the marginals $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, the correlation coefficient ρ_{12} between the two variables is known. The bivariate CDF is denoted by $F_{X_1X_2}(x_1,x_2)$.

Frechet's [9] model is based on the fundamental relation

$$F_{X_1}(x_1) + F_{X_2}(x_2) - 1 \le F_{X_1X_2}(x_1,x_2) \le \min[\, F_{X_1}(x_1), F_{X_2}(x_2)\,] \tag{58}$$

and is expressed as

$$\begin{aligned} F_{X_1X_2}(x_1,x_2) = {} & \alpha \max[\, F_{X_1}(x_1) + F_{X_2}(x_2) - 1, 0\,] \\ & + (1-\alpha)\min[\, F_{X_1}(x_1), F_{X_2}(x_2)\,] \quad 0 \le \alpha \le 1 \end{aligned} \tag{59}$$

in which α must be determined in terms of the correlation coefficient ρ_{12}. The joint PDF for this model is a discontinuous function and a simple expression relating α to ρ_{12} does not exist. The range of correlation coefficients that can be modeled is limited and, in particular, the model does not include the special case of zero correlation. In addition, this model is restricted to two variables.

A model introduced by Gumbel [10] has the form

$$[-\log F_{X_1X_2}(x_1,x_2)]^\alpha = [-\log F_{X_1}(x_1)]^\alpha + [-\log F_{X_2}(x_2)]^\alpha \qquad \alpha \geq 1 \tag{60}$$

where α is related to the correlation coefficient. For $\alpha = 1$, the model reduces to the special case of statistically independent variables. Again, a simple expression between α and ρ_{12} does not exist.

A contingency-type distribution model by Plackett [20] is expressed in the form

$$\frac{F_{X_1X_2}(x_1,x_2)\,[\,1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1X_2}(x_1,x_2)\,]}{[\,F_{X_1}(x_1) - F_{X_1X_2}(x_1,x_2)\,][\,F_{X_2}(x_2) - F_{X_1X_2}(x_1,x_2)\,]} = \alpha \qquad 0 \leq \alpha \tag{61}$$

For a given α, the joint distribution is obtained by solving for $F_{X_1X_2}(x_1,x_2)$ in the above equation. For $\alpha = 1$, the variables are statistically independent. This model also lacks a simple relation between α and ρ_{12} and is limited to two variables.

A relatively simple model by Morgenstern [18] is

$$F_{X_1X_2}(x_1,x_2) = F_{X_1}(x_1)\,F_{X_2}(x_2)\,\{\,1 + \alpha[1 - F_{X_1}(x_1)][1 - F_{X_2}(x_2)]\,\} \qquad -1 \leq \alpha \leq 1 \tag{62}$$

It can be shown [17] that the relation between α and ρ_{12} is

$$\alpha = \rho_{12}\,\frac{\sigma_1\,\sigma_2}{4\,Q_1\,Q_2} \tag{63}$$

where

$$Q_i = \int_{-\infty}^{\infty} (x_i - m_i)\,f_{X_i}(x_i)\,F_{X_i}(x_i)\,dx_i \qquad i = 1,2 \tag{64}$$

Closed-form solutions of this integral for several useful distributions are listed in Table 8. Because of the bounds on α, the range of correlation coefficients that can be modeled is rather limited, as can be seen in Table 9 for several example distributions. This limitation becomes more severe if the model is generalized to more than two variables [14]

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) = \prod_i F_{X_i}(x_i)\,\{\,1 &+ \sum_{i<j} \alpha_{ij}\,[\,1 - F_{X_i}(x_i)\,][\,1 - F_{X_j}(x_j)\,] \\ &+ \sum_{i<j<k} \alpha_{ijk}\,[\,1 - F_{X_i}(x_i)\,][\,1 - F_{X_j}(x_j)\,][\,1 - F_{X_k}(x_k)\,] + \cdots\,\} \end{aligned} \tag{65}$$

Here, the parameters α_{ij}, α_{ijk}, etc., must satisfy certain conditions to ensure

the nondecreasing property of the joint CDF. These conditions severely restrict the possible range of correlation coefficients that can be described with this model.

Morgenstern's model has been generalized by Farlie [8] in the form

$$F_{X_1X_2}(x_1,x_2) = F_{X_1}(x_1)\,F_{X_2}(x_2)\{1+\alpha\,A[F_{X_1}(x_1)]\,B[F_{X_2}(x_2)]\} \quad -1\le\alpha\le 1 \qquad (66)$$

where A and B are functions of $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, respectively, defined such that the model is a valid bivariate distribution. For appropriate choices of A and B the model describes a wider range of correlation coefficients. However, simple relation between α and ρ_{12} for such general cases does not exist.

None of the preceding models are ideal for use in the context of first-order structural reliability. They are limited with regard to the range of correlation coefficients and their extension to more than two variables is cumbersome or impossible. In addition, these models violate the requirement of simplicity, since the corresponding transforming into the standard normal space requires at least the same amount of effort as would be required from any known non-normal distribution. Finally, it can be shown that these models in general violate the requirement of invariance.

A bivariate model due to Nataf [13] that possesses all the desired properties is described by Eqs. 16-18 and is extended to the multivariate case in the main body of this report.

9. Appendix II -- Formulae for $\rho_{0,12}$

In this appendix, several important properties of Eq. 18 are examined and semi-empirical formulae relating $\rho_{0,12}$ to ρ_{12} are developed. For many distributions, the difference between ρ_{12} and $\rho_{0,12}$ is small and can be entirely ignored without significantly affecting the reliability index. However, when the distributions are highly nonnormal, the difference can be important and accurate esti-

mation of $\rho_{0,12}$ is essential. For arbitrary marginals, the uniqueness of $\rho_{0,12}$ in Eq. 18 cannot be established, although numerical investigations indicate that it holds. Herein, a unique value for $\rho_{0,12}$ is defined as that root of Eq. 18 which is nearest to ρ_{12}. A precise definition of "nearness" will be given shortly. The following are important properties of the relation between $\rho_{0,12}$ and ρ_{12}:

Lemma 1. $\rho_{0,12} = 0$ *for* $\rho_{12} = 0$.

Proof. It is easy to see that $\rho_{0,12} = 0$ is a root of Eq. 18 when $\rho_{12} = 0$. Since it is a coinciding root, it is the nearest root by any definition of "nearness" and the lemma is proved.

Lemma 2. *For arbitrary marginals,* $|\rho_{12}| \leq |\rho_{0,12}|$, *where the equality stands when* $\rho_{12} = 0$ *or when both marginals are normal.* See Lancaster [15] for the proof.

Lemma 3. *The algebraic sign of* $\rho_{0,12}$ *is the same as that of* ρ_{12}. The proof can be found in Lehmann [16].

Based on Lemmas 1-3, the "nearest root" of Eq. 18 is defined as that root which minimizes $|\rho_{0,12} - \rho_{12}|$. Also, based on the above facts an expression for $\rho_{0,12}$ for two-parameter distributions may be written in the form

$$\rho_{0,12} = \rho_{12}\, F(\rho_{12}; m_1, \sigma_1; m_2, \sigma_2) \tag{67}$$

in which F, satisfying $F \geq 1$, is a function of ρ_{12} and the means and standard deviations of X_1 and X_2.

Lemma 4. F *is independent of* ρ_{12} *if one of the variables is normal.*

Proof. For normal X_1, $Z_1 = (X_1 - m_1)/\sigma_1$ and Eq. 18 reduces to

$$\begin{aligned}
\rho_{12} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_1 \left[\frac{x_2 - m_2}{\sigma_2}\right] \frac{1}{2\pi\sqrt{1-\rho_{0,12}^2}} \exp\left(-\frac{z_2^2}{2}\right) \exp\left[-\frac{1}{2}\left(\frac{z_1 - \rho_{0,12} z_2}{\sqrt{1-\rho_{0,12}^2}}\right)^2\right] dz_1\, dz_2 \\
&= \int_{-\infty}^{\infty} \left[\frac{x_2 - m_2}{\sigma_2}\right] \rho_{0,12} z_2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) dz_2 \\
&= \frac{1}{\sigma_2} E[X_2 Z_2]\, \rho_{0,12}
\end{aligned} \tag{68}$$

which shows that F is a function of the distribution of X_2 only.

Lemma 5. *F is invariant to linear transformations of X_1 and X_2.*

Proof. Let $\bar{X}_i = a_i + b_i X_i$ and $\bar{Z}_i = \Phi^{-1}[F_{\bar{X}_i}(\bar{X}_i)]$, $i = 1,2$, where a_i and b_i are arbitrary constants and $b_i > 0$. Using superposed bars to denote the properties of $\bar{X}_i$, $\bar{m}_i = a_i + b_i m_i$ and $\bar{\sigma}_i = b_i \sigma_i$. It is easy to see that $(\bar{x}_i - \bar{m}_i)/\bar{\sigma}_i = (x_i - m_i)/\sigma_i$. Also, since $F_{\bar{X}_i}(\bar{x}_i) = F_{X_i}(x_i)$, then $\bar{z}_i = z_i$. Now, Eq. 18 for $\bar{X}_1$ and $\bar{X}_2$ can be written

$$\bar{\rho}_{12} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{x_1 - m_1}{\sigma_1}\right]\left[\frac{x_2 - m_2}{\sigma_2}\right]\varphi_2(z_1, z_2, \bar{\rho}_{0,12})\, dz_1\, dz_2 \tag{69}$$

Since $\bar{\rho}_{12} = \rho_{12}$, it follows that $\bar{\rho}_{0,12} = \rho_{0,12}$, which proves that F is invariant.

Lemma 6. *For two-parameter distributions, F is independent of the mean and standard deviation, provided the distribution is reducible to a standard form through a linear transformation.*

Proof. This is a direct result of the preceding lemma, since the variable can be transformed to have a zero mean and unit standard deviation. Examples for such distributions are the uniform, shifted exponential, shifted Rayleigh, and type-I extreme-value distributions. This group of distributions are denoted herein as Group 1 Distributions.

Lemma 7. *For two-parameter distributions which are irreducible to standard forms through linear transformations, F can be expressed as a function of the coeffient of variation $\delta_i = \sigma_i / m_i$.*

Proof. This follows from Lemma 5, since the variable can be scaled to have a unit mean. Examples for such distributions are the lognormal, gamma, and type-II and type-III extreme-value distributions. This group of distributions are denoted herein as Group 2 Distributions.

Closed form solutions for F exist only for the normal and lognormal distri-

butions (see Tables 2 and 5). For other cases, numerical integration of Eq. 18 is required. To avoid such tedious calculations, a set of empirical formulae for selected marginal distributions are developed. These are obtained by least-square fitting of polynomial expressions to exact values computed by numerical integration of Eq. 18.

Based on Lemmas 4-7 above and depending on the distributions of X_1 and X_2, five categories of formulae are developed: (1) F = *constant* for normal X_1 and X_2 belonging to Group 1; (2) $F = F(\delta_2)$ for normal X_1 and X_2 belonging to Group 2; (3) $F = F(\rho_{12})$ for both X_1 and X_2 belonging to Group 1; (4) $F = F(\rho_{12}, \delta_2)$ for X_1 belonging to Group 1 and X_2 belonging to Group 2; and (5) $F = F(\rho_{12}, \delta_1, \delta_2)$ for both X_1 and X_2 belonging to Group 2. The results of these formulations are listed in Tables 1-5 together with the maximum errors resulting from the least-square fitting. The range of coefficients of variation used in generating the formulae in Tables 2, 4, and 5 is 0.1 to 0.5. For values outside this range, the errors in the formulae can be larger than those listed in these tables.

As demonstrations of the formulae in Tables 1-5, plots of F for several selected pairs of marginal distributions are shown in Figs. 3-5 together with the exact results obtained from numerical integration of Eq. 18. Figs. 3 and 4 represent typical cases, whereas Fig. 5 represents the case with poorest agreement with exact results (maximum error = 4.5%, as shown in Table 4). It is seen that for most distributions F is only slightly higher than unity. However, it can be as high as 1.5 or greater when X_1 and X_2 are negatively correlated and their marginals are skewed in the same direction (e.g., see curve in Fig. 4 for exponential X_1 and X_2).

References

1. A.H-S. Ang and W.H. Tang, *Probability Concepts in Engineering Planning and Design - Volume II: Decision, Risk, and Reliability*, J. Wiley & Sons, New York, N.Y., 1984.

2. J. R. Benjamin and C. A. Cornell, *Probability, Statistics, and Decision for Civil Engineers*, McGraw-Hill, New York, N.Y., 1970.

3. A. Der Kiureghian and J-B. Ke, "Finite-Element Based Reliability Analysis of Frame Structures," in *Proceedings*, Fourth International Conference on Structural Safety and Reliability, Kobe, Japan, May 1985.

4. O. Ditlevsen, "Narrow Reliability Bounds for Structural Systems," *Journal of Structural Mechanics*, vol. 7, no. 4, pp. 453-472, Dec. 1979.

5. O. Ditlevsen, "Generalized Second Moment Reliability Index," *Journal of Structural Mechanics*, vol. 7, no. 4, pp. 435-451, 1979.

6. O. Ditlevsen, "Principle of Normal Tail Approximation," *Journal of the Engineering Mechanics Division*, vol. 107, no. EM6, pp. 1191-1208, Dec. 1981.

7. O. Ditlevsen, "Basic Reliability Concepts," in *Reliability Theory and Its Application in Structural and Soil Mechanics*, ed. by P. Thoft-Christensen, pp. 1-55, NATO ASI Series, Martinus Nijhoff Pub., The Hague, 1983.

8. D.J.G. Farlie, "The Performance of Some Correlation Coefficients for a General Bivariate Distribution," *Biometrika*, vol. 47, pp. 307-323, 1960.

9. M. Fréchet, "Sur les Tableaux de Corrélation Dont les Marges Sont Donées," *Annals de l'Universite de Lyon*, vol. 13, pp. 53-77, Section A, Series 3, 1951.

10. E. J. Gumbel, "Bivariate Logistic Distribution," *Journal of American Statistical Association*, vol. 56, pp. 335-349, 1961.

11. A. M. Hasofer and N. C. Lind, "An Exact and Invariant First Order Reliability Format," *Journal of the Engineering Mechanics Division*, vol. 100, no. EM1,

pp. 111-121, Feb. 1974.

12. M. Hohenbichler and R. Rackwitz, "Non-Normal Dependent Vectors in Structural Safety," *Journal of the Engineering Mechanics Division*, vol. 107, no. EM6, pp. 1227-1238, Dec. 1981.

13. N. L. Johnson and S. Kotz, *Distributions in Statistics - Continuous Multivariate Distributions*, John Wiley & Sons, Inc., New York, N.Y., 1976.

14. S. Kotz, "Multivariate Distribution at A Cross Road," in *Statistical Distributions in Scientific Work*, ed. by G. P. Patil, et al., vol. 1, pp. 247-270, D. Reidel Publ. Co., Dordrecht, Holland, 1975.

15. H. O. Lancaster, "Some Properties of Bivariate Normal Distribution Considered in the Form of a Contingency Table," *Biometrika*, vol. 44, pp. 289-292, June 1957.

16. E. L. Lehmann, "Some Concept of Dependence," *Annals of Mathematical Statistics*, vol. 37, pp. 1137-1153, 1966.

17. P-L. Liu, "First-Order, Full-Distribution Structural Reliability Analysis Program," CE-249 Term Project Report, Department of Civil Engineering, University of California, Berkeley, Cal., May 1984.

18. D. Morgenstern, "Einfache Beispiele Zweidimensionaler Verteilungen," *Mitteilingsblatt für Mathematische Statistik*, vol. 8, pp. 234-235, 1956.

19. A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, New York, N.Y., 1965.

20. R. L. Plackett, "A Class of Bivariate Distributions," *Journal of American Statistical Association*, vol. 60, pp. 516-522, 1965.

21. R. Rackwitz and B. Fiessler, "Structural Reliability Under Combined Load Sequences," *Computers and Structures*, vol. 9, pp. 489-494, 1978.

22. M. Rosenblatt, "Remarks on a Multivariate Transformation," *Annals of Mathematical Statistics*, vol. 23, pp. 470-472, 1969.

23. I. E. Segal, "Fiducial Distribution of Several Parameters with Applications to a Normal System," *Proceedings of Cambridge Philosophical Society*, vol. 34, pp. 41-47, 1938.

24. D. Veneziano, "New Index of Reliability," *Journal of the Engineering Mechanics Division*, vol. 105, no. EM 5, pp. 277-296, ASCE, April 1979.

25. J. H. Wilkinson, *The Algebraic Eigenvalue Problem* , Oxford University Press, Oxford, England, 1965.

Table 1. Category 1 Formulae, $F = constant$, for Normal X_1 and X_2 belonging to Group 1 Distributions

X_2	$F = constant$	Max. Error
Uniform	1.023	0.0%
Shifted Exponential	1.107	0.0%
Shifted Rayleigh	1.014	0.0%
Type-I Largest Value	1.031	0.0%
Type-I Smallest Value	1.031	0.0%

Table 2. Category 2 Formulae, $F = F(\delta_2)$, for X_1 Normal and X_2 Belonging to Group 2 Distributions

X_2	$F = F(\delta_2)$	Max. Error
Lognormal	$\frac{\delta_2}{\sqrt{\ln(1+\delta_2^2)}}$	Exact
Gamma	$1.001-0.007\delta_2+0.118\delta_2^2$	0.0%
Type-II Largest Value	$1.030+0.238\delta_2+0.364\delta_2^2$	0.1%
Type-III Smallest Value	$1.031-0.195\delta_2+0.328\delta_2^2$	0.1%

Table 3. Category 3 Formulae, $F = F(\rho_{12})$, for X_1 and X_2 Both Belonging to Group 1 Distributions

X_2 \ X_1	1	2	3	4	5
1 (Max. Error)	$1.047-0.047\rho^2$ (0.0%)				
2 (Max. Error)	$1.133+0.029\rho^2$ (0.0%)	$1.229-0.367\rho+0.153\rho^2$ (1.5%)			
3 (Max. Error)	$1.038-0.008\rho^2$ (0.0%)	$1.123-0.100\rho+0.021\rho^2$ (0.1%)	$1.028-0.029\rho$ (0.0%)		
4 (Max. Error)	$1.055+0.015\rho^2$ (0.0%)	$1.142-0.154\rho+0.031\rho^2$ (0.2%)	$1.046-0.045\rho+0.006\rho^2$ (0.0%)	$1.064-0.069\rho+0.005\rho^2$ (0.0%)	
5 (Max. Error)	$1.055+0.015\rho^2$ (0.0%)	$1.142+0.154\rho+0.031\rho^2$ (0.2%)	$1.046+0.045\rho+0.006\rho^2$ (0.0%)	$1.064+0.069\rho+0.005\rho^2$ (0.0%)	$1.064-0.069\rho+0.005\rho^2$ (0.0%)

Note: $\rho = \rho_{12}$; distribution numbers are: 1 = uniform; 2 = shifted exponential; 3 = shifted Rayleigh; 4 = type-I largest value; 5 = type-I smallest value.

Table 4. Category 4 Formulae, $F = F(\rho_{12}, \delta_2)$, for X_1 Belonging to Group 1 Distributions and X_2 Belonging to Group 2 Distributions

X_2 \ X_1	1	2	3	4	5
6	$1.019+0.014\delta_2$ $+0.010\rho^2+0.249\delta_2^2$	$1.098+0.003\rho+0.019\delta_2$ $+0.025\rho^2+0.303\delta_2^2-0.437\rho\delta_2$	$1.011+0.001\rho+0.014\delta_2$ $+0.004\rho^2+0.231\delta_2^2-0.130\rho\delta_2$	$1.029+0.001\rho+0.014\delta_2$ $+0.004\rho^2+0.233\delta_2^2-0.197\rho\delta_2$	$1.029-0.001\rho+0.014\delta_2$ $+0.004\rho^2+0.233\delta_2^2+0.197\rho\delta_2$
(Max. Error)	(0.7%)	(1.6%)	(0.4%)	(0.3%)	(0.3%)
7	$1.023-0.007\delta_2$ $+0.002\rho^2+0.127\delta_2^2$	$1.104+0.003\rho-0.008\delta_2$ $+0.014\rho^2+0.173\delta_2^2-0.296\rho\delta_2$	$1.014+0.001\rho-0.007\delta_2$ $+0.002\rho^2+0.126\delta_2^2-0.090\rho\delta_2$	$1.031+0.001\rho-0.007\delta_2$ $+0.003\rho^2+0.131\delta_2^2-0.132\rho\delta_2$	$1.031-0.001\rho-0.007\delta_2$ $+0.003\rho^2+0.131\delta_2^2+0.132\rho\delta_2$
(Max. Error)	(0.1%)	(0.9%)	(0.9%)	(0.3%)	(0.3%)
8	$1.033+0.305\delta_2$ $+0.074\rho^2+0.405\delta_2^2$	$1.109-0.152\rho+0.361\delta_2$ $+0.130\rho^2+0.455\delta_2^2-0.726\rho\delta_2$	$1.036-0.038\rho+0.266\delta_2$ $+0.028\rho^2+0.383\delta_2^2-0.229\rho\delta_2$	$1.056-0.060\rho+0.263\delta_2$ $+0.020\rho^2+0.383\delta_2^2-0.332\rho\delta_2$	$1.056+0.060\rho+0.263\delta_2$ $+0.020\rho^2+0.383\delta_2^2+0.332\rho\delta_2$
(Max. Error)	(2.1%)	(4.5%)	(1.2%)	(1.0%)	(1.0%)
9	$1.061-0.237\delta_2$ $-0.005\rho^2+0.379\delta_2^2$	$1.147+0.145\rho-0.271\delta_2$ $+0.010\rho^2+0.459\delta_2^2-0.467\rho\delta_2$	$1.047+0.042\rho-0.212\delta_2$ $+0.353\delta_2^2-0.136\rho\delta_2$	$1.064+0.065\rho-0.210\delta_2$ $+0.003\rho^2+0.356\delta_2^2-0.211\rho\delta_2$	$1.064-0.065\rho-0.210\delta_2$ $+0.003\rho^2+0.356\delta_2^2+0.211\rho\delta_2$
(Max. Error)	(0.5%)	(0.4%)	(0.2%)	(0.2%)	(0.2%)

Note: $\rho = \rho_{12}$; range of coefficient of variation is δ_2 = 0.1 - 0.5; distribution numbers are: Group 1 distributions: 1 = uniform; 2 = shifted exponential; 3 = shifted Rayleigh; 4 = type-I largest value; 5 = type-I smallest value; and Group 2 distributions: 6 = lognormal; 7 = gamma ; 8 = type-II largest value; 9 = type-III smallest value.

Table 5. Category 5 Formulae, $F = F(\rho_{12}, \delta_1, \delta_2)$, for X_1 and X_2 Both Belonging to Group 2 Distributions

X_2 \ X_1	6	7	8	9
6	$\frac{\ln(1+\rho\delta_1\delta_2)}{\rho\sqrt{\ln(1+\delta_1^2)\ln(1+\delta_2^2)}}$			
(Max. Error)	(Exact)			
7	$1.001+0.033\rho+0.004\delta_1-0.016\delta_2$ $+0.002\rho^2+0.223\delta_1^2+0.130\delta_2^2$ $-0.104\rho\delta_1+0.029\delta_1\delta_2-0.119\rho\delta_2$	$1.002+0.022\rho-0.012(\delta_1+\delta_2)$ $+0.001\rho^2+0.125(\delta_1^2+\delta_2^2)$ $-0.077\rho(\delta_1+\delta_2)+0.014\delta_1\delta_2$		
(Max. Error)	(4.0%)	(4.0%)		
8	$1.026+0.082\rho-0.019\delta_1+0.222\delta_2$ $+0.018\rho^2+0.288\delta_1^2+0.379\delta_2^2$ $-0.441\rho\delta_1+0.126\delta_1\delta_2-0.277\rho\delta_2$	$1.029+0.056\rho-0.030\delta_1+0.225\delta_2$ $+0.012\rho^2+0.174\delta_1^2+0.379\delta_2^2$ $-0.313\rho\delta_1+0.075\delta_1\delta_2-0.182\rho\delta_2$	$1.086+0.054\rho+0.104(\delta_1+\delta_2)$ $-0.055\rho^2+0.662(\delta_1^2+\delta_2^2)$ $-0.570\rho(\delta_1+\delta_2)+0.203\delta_1\delta_2$ $-0.020\rho^3-0.218(\delta_1^3+\delta_2^3)$ $-0.371\rho(\delta_1^2+\delta_2^2)+0.257\rho^2(\delta_1+\delta_2)$ $+0.141\delta_1\delta_2(\delta_1+\delta_2)$	
(Max. Error)	(4.3%)	(4.2%)	(4.3%)	
9	$1.031+0.052\rho+0.011\delta_1-0.210\delta_2$ $+0.002\rho^2+0.220\delta_1^2+0.350\delta_2^2$ $+0.005\rho\delta_1+0.009\delta_1\delta_2-0.174\rho\delta_2$	$1.032+0.034\rho-0.007\delta_1-0.202\delta_2$ $+0.121\delta_1^2+0.339\delta_2^2$ $-0.006\rho\delta_1+0.003\delta_1\delta_2-0.111\rho\delta_2$	$1.065+0.146\rho+0.241\delta_1-0.259\delta_2$ $+0.013\rho^2+0.372\delta_1^2+0.435\delta_2^2$ $+0.005\rho\delta_1+0.034\delta_1\delta_2-0.481\rho\delta_2$	$1.063-0.004\rho-0.200(\delta_1+\delta_2)$ $-0.001\rho^2+0.337(\delta_1^2+\delta_2^2)$ $+0.007\rho(\delta_1+\delta_2)-0.007\delta_1\delta_2$
(Max. Error)	(2.4%)	(4.0%)	(3.8%)	(2.6%)

Note: $\rho = \rho_{12}$; ranges of coefficients of variation are $\delta_1, \delta_2 = 0.1$ - 0.5; distribution numbers are: 6 = lognormal; 7 = gamma ; 8 = type-II largest value; 9 = type-III smallest value.

Table 6. Summary Calculations of Example 1

Step	x	M'	*diag* D'	α	γ	β_{HL}
1	50.0 12.5	50.0 8.18	39.9 12.8	−0.952 0.307	−0.952 0.306	0.177
2	6.37 3.91	25.4 7.39	12.5 5.63	−0.895 0.447	−0.912 0.411	1.149
3	10.1 6.28	32.6 8.40	17.6 7.95	−0.895 0.447	−0.912 0.411	1.374
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
7	9.85 9.86	32.3 8.61	17.4 10.9	−0.780 0.626	−0.847 0.532	1.653
8	9.87 9.87	32.3 8.61	17.4 10.9	−0.780 0.626	−0.847 0.532	1.653

Table 7. Summary Calculations of Example 2

Step	$\mathbf{x}$	$\mathbf{M}'$	*diag* $\mathbf{D}'$	$\mathbf{R}'_0$			α	γ	β_{HL}
1	5.00	4.90	0.990	1.00			−0.557	0.714	0.698
	1.00	0.654	1.02	0.327	1.00		−0.262	0.576	
	1.00	0.732	1.06	−0.226	−0.642	1.00	−0.650	0.598	
2	2.92	4.44	0.579	1.00			0.458	−0.273	2.801
	0.694	0.693	0.799	0.327	1.00		0.283	0.377	
	2.23	0.252	1.88	−0.226	−0.685	1.00	−0.843	0.885	
3	3.38	4.64	0.670	1.00			0.462	−0.281	2.248
	0.321	0.596	0.459	0.327	1.00		0.361	0.193	
	3.13	−0.383	2.24	−0.226	−0.595	1.00	−0.810	0.940	
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
14	4.05	4.82	0.802	1.00			0.486	−0.310	1.987
	0.198	0.491	0.320	0.327	1.00		0.319	0.124	
	3.85	−0.773	2.44	−0.226	−0.510	1.00	−0.814	0.943	
15	4.05	4.82	0.802	1.00			0.486	−0.310	1.987
	0.198	0.491	0.320	0.327	1.00		0.319	0.124	
	3.85	−0.773	2.44	−0.226	−0.510	1.00	−0.814	0.943	

Table 8. $Q_i = \int_{-\infty}^{+\infty} (x-\mu)\, f_X(x)\, F_X(x)\, dx$ **For Selected Distributions**

Name	$F_X(x)$	Q_i
Normal	$\Phi\left[\frac{x-m}{\sigma}\right]$	$\frac{\sigma}{2\sqrt{\pi}}$
Uniform	$\frac{x-a}{b-a}$, $a \le x \le b$	$\frac{b-a}{12}$
Shifted Exponential	$1-\exp[-\lambda(x-x_0)]$, $x_0 \le x$	$\frac{1}{4\lambda}$
Shifted Rayleigh	$1-\exp\left[-\frac{1}{2}\left[\frac{x-x_0}{a}\right]^2\right]$, $x_0 \le x$	$\frac{a}{4}(\sqrt{2\pi}-\sqrt{\pi})$
Gamma	$\frac{\Gamma(k,\lambda x)}{\Gamma(k)}$, $0 \le x$	$\frac{\Gamma(2k)}{\lambda\Gamma^2(k)4^k}$
Type-I Largest Value	$\exp\{-\exp[-\alpha(x-u)]\}$	$\frac{\ln 2}{2\alpha}$
Type-I Smallest Value	$1-\exp\{-\exp[\alpha(x-u)]\}$	$\frac{\ln 2}{2\alpha}$
Type-II Largest Value	$\exp\left[-\left[\frac{u}{x}\right]^k\right]$, $0 < x$	$\frac{u(2^{\frac{1}{k}}-1)}{2}\Gamma(1-\frac{1}{k})$
Type-III Smallest Value	$1-\exp\left[-\left[\frac{x-e}{u-e}\right]^k\right]$, $e \le x$	$\frac{(u-e)(1-2^{-\frac{1}{k}})}{2}\Gamma(1+\frac{1}{k})$

Note: $\Gamma(k,u) = \int_0^u e^{-u} u^{k-1} du$ and $\Gamma(k) = \Gamma(k,\infty)$ are the incomplete and complete gamma functions, respectively.

Table 9. Maximum Permitted $|\rho_{12}|$ with Morgenstern's Model

	1	2	3	4	5	6
1	0.318	0.326	0.282	0.316	0.305	0.305
2	0.326	0.333	0.289	0.324	0.312	0.312
3	0.282	0.289	0.250	0.280	0.270	0.270
4	0.316	0.324	0.280	0.314	0.303	0.303
5	0.305	0.312	0.270	0.303	0.292	0.292
6	0.305	0.312	0.270	0.303	0.292	0.292

Note: Distribution numbers are: 1 = normal; 2 = uniform; 3 = shifted exponential; 4 = shifted Rayleigh; 5 = type-I largest value; 6 = type-I smallest value.

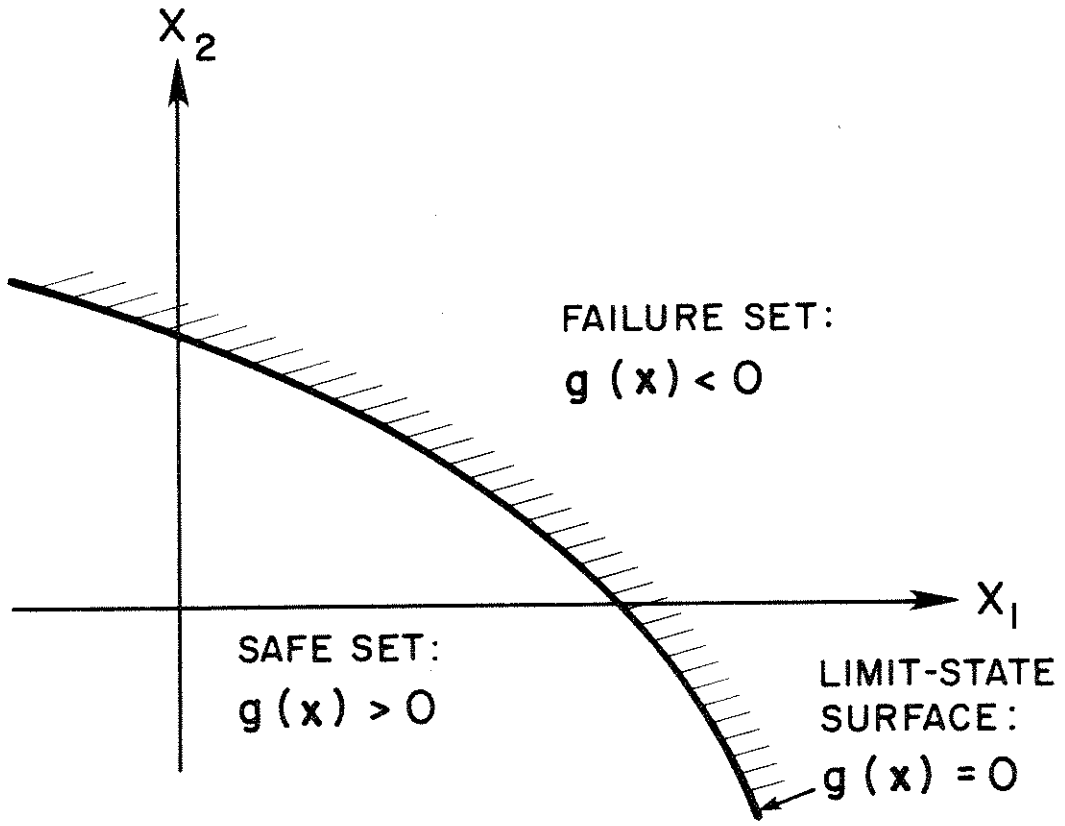

Figure 1. Basic Definitions of Structural Reliability

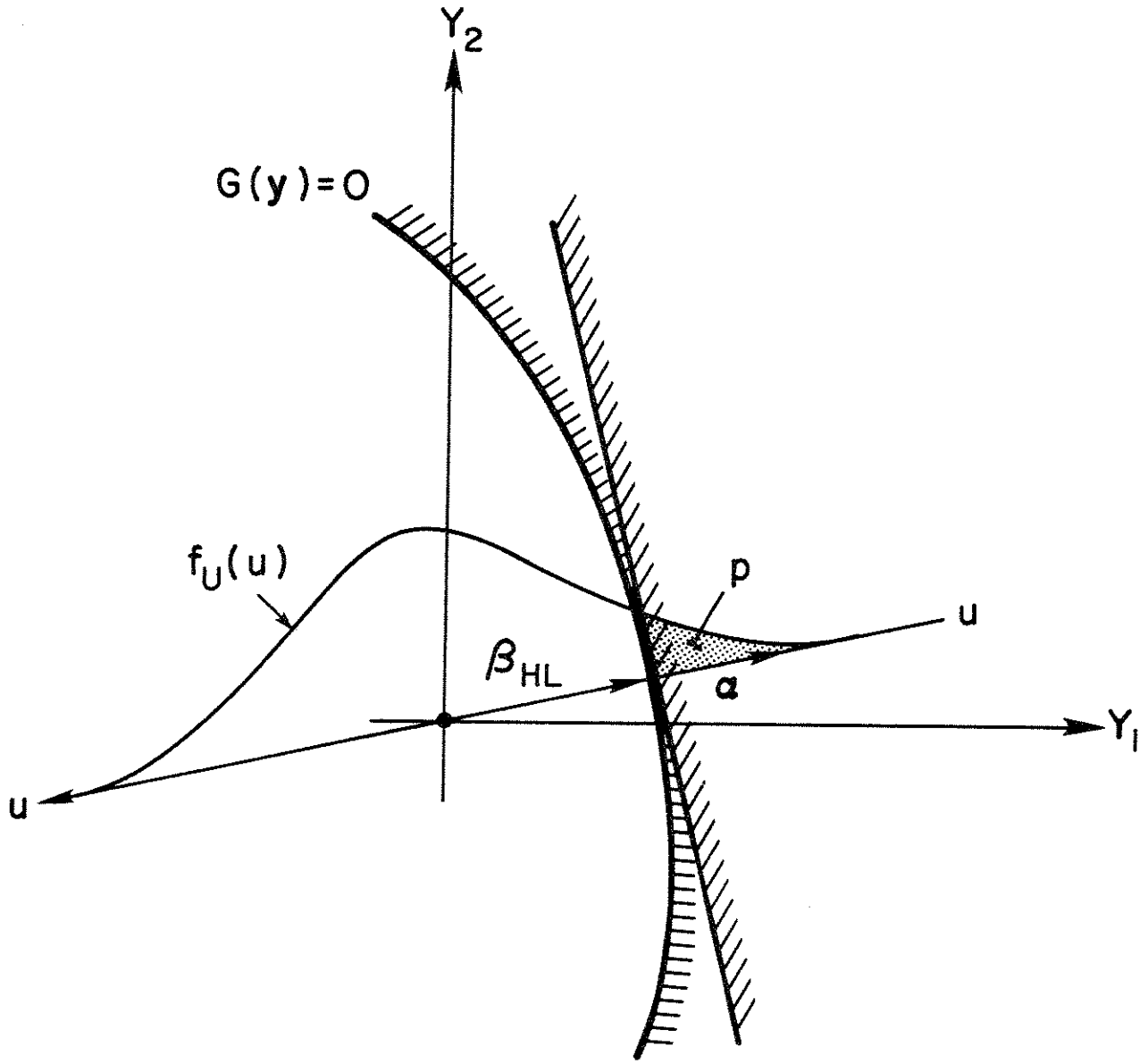

Figure 2. The Standard Space

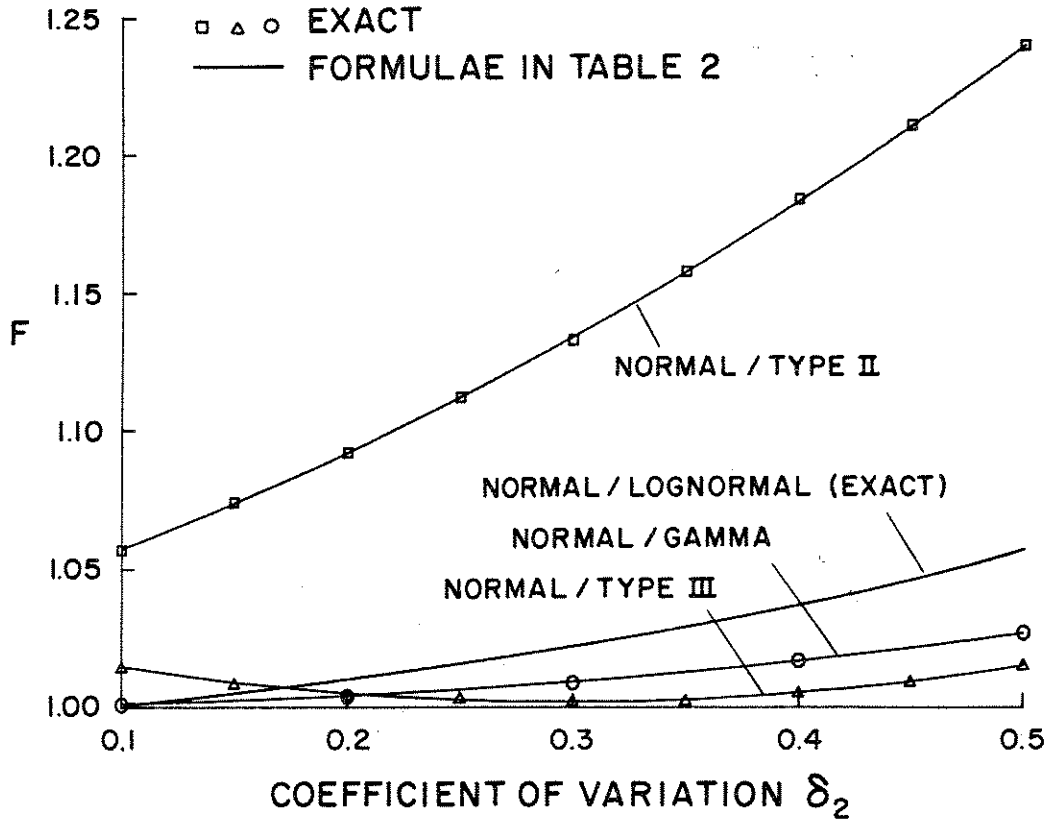

Figure 3. F **for** X_1 **Normal and** X_2 **Belonging to Group 2 Distributions**

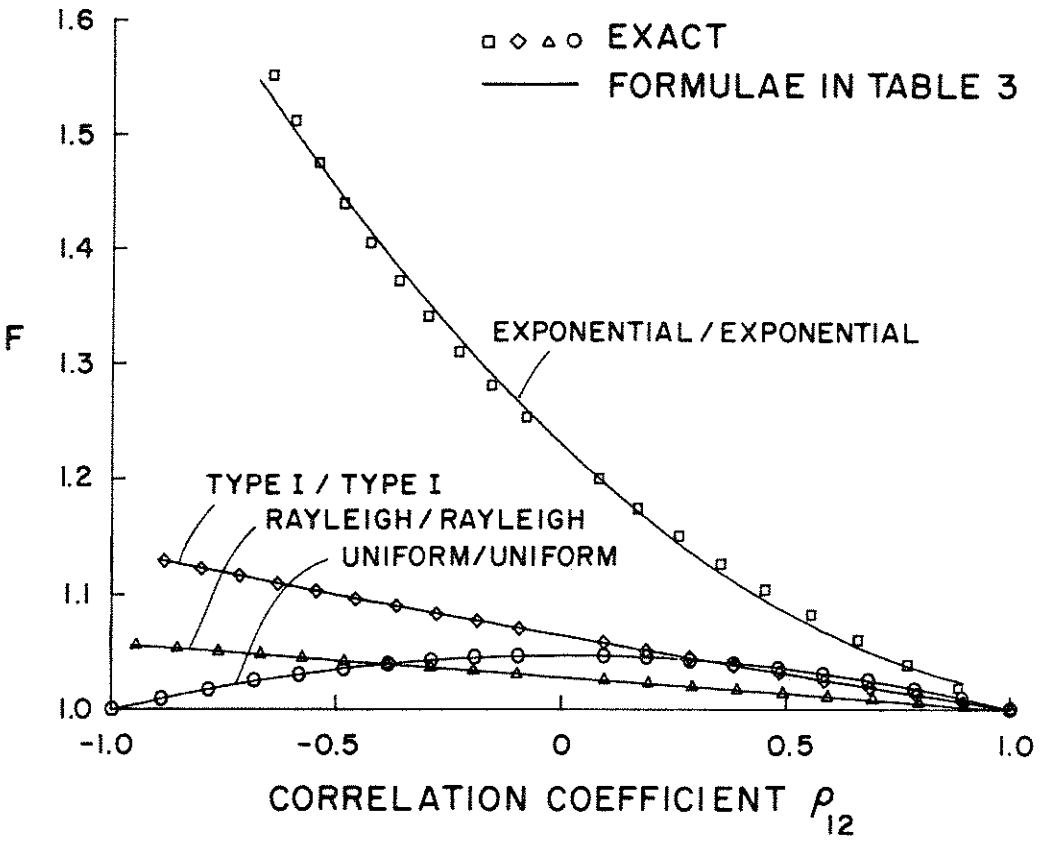

Figure 4. Selected Plots of F for X_1 and X_2 Both Belonging to Group 1 Distributions

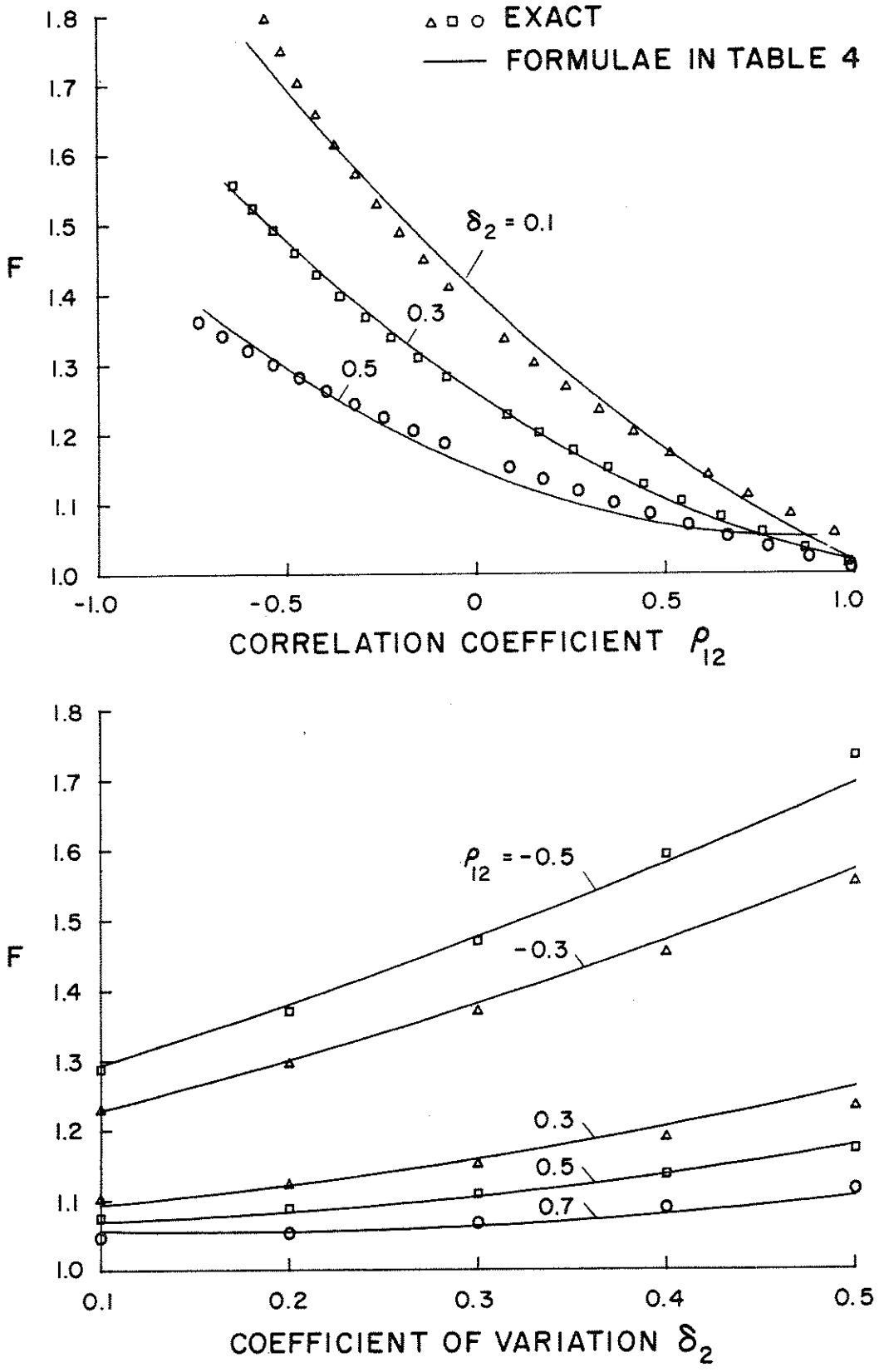

Figure 5. F **for** X_1 **Exponential and** X_2 **Type-II Largest Value**

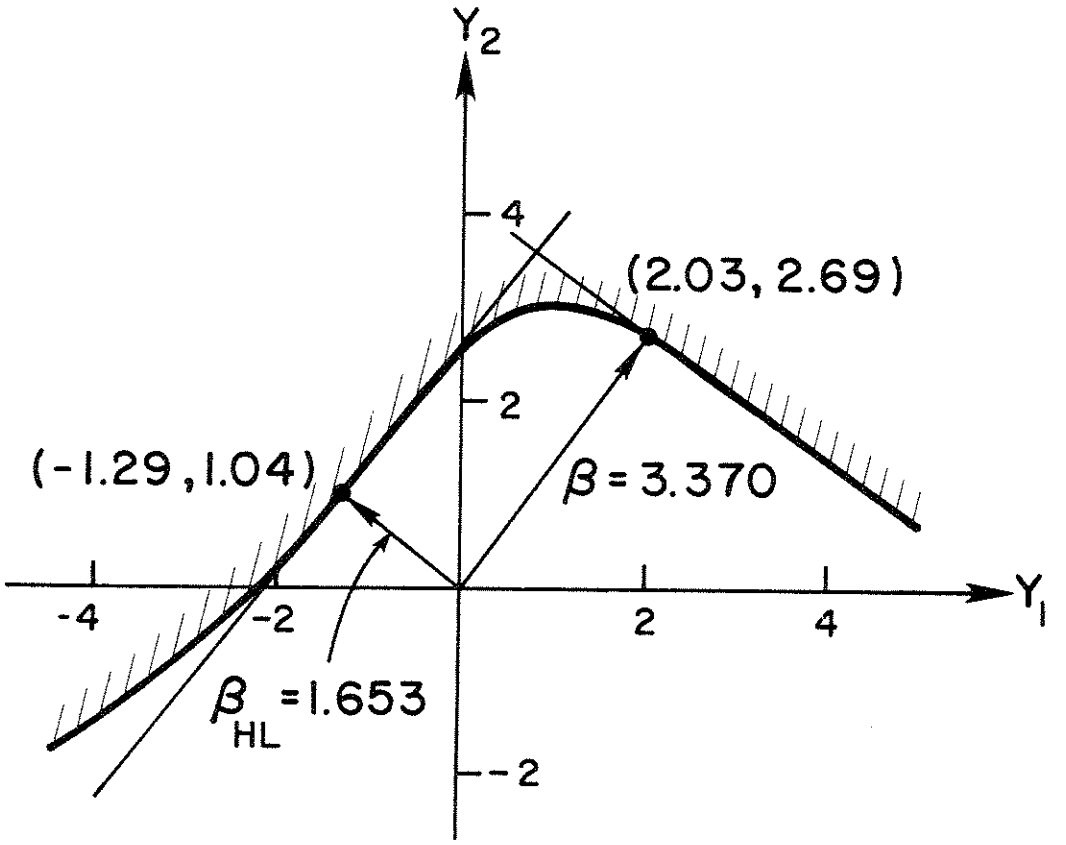

Figure 6. The Standard Space for Example 1