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Contract No. W-7405-eng-48

π^+ - p ELASTIC SCATTERING AT 310 Mev: PHASE-SHIFT ANALYSIS

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ABSTRACT

A comprehensive phase-shift analysis of π^+-p elastic-scattering data at 310-Mev incident-pion laboratory kinetic energy has been performed. The experimental data utilized include measurements of the differential and total cross sections and of the recoil-proton polarization. The D-wave phase shifts were found to be definitely needed in order to attain an adequate fit to the data. A general search for phase-shift solutions was carried out, using S-, P-, and D-wave phase shifts. One solution--of the Fermi type--was found that fits the data significantly better than any of the other solutions obtained. The calculated errors in the phase shifts of this set vary from 0.4 to 0.6 deg. Because it was felt that these errors might be deceivingly restrictive, the effects of small nuclear F-wave phase shifts on the results of the analysis were investigated and were found to be large: not only are the uncertainties in the original Fermi-type solution increased, but additional sets of phase shifts arise that fit the data well. One of these new solutions is similar to the original Fermi set except that the magnitudes of the phase shifts in this new fit are in general larger than those in the initial solution, and the signs of the D-wave phase shifts are reversed. The nuclear phase shifts in the original Fermi solution and their rms errors are (when

F-wave phase shifts are allowed): $S_{3,1} = -17.2 \pm 2.6$ deg, $P_{3,1} = -2.9 \pm 4.0$ deg, $P_{3,3} = 135.0 \pm 0.6$ deg, $D_{3,3} = 3.1 \pm 2.6$ deg, $D_{3,5} = -4.9 \pm 2.1$ deg, $F_{3,5} = 0.5 \pm 0.6$ deg, $F_{3,7} = -0.6 \pm 1.4$ deg. Although theory appears to favor this set, further theoretical and experimental evidence is desirable. The values given here for the first five phase shifts approximate the corresponding values obtained when the F-wave phase shifts were assumed negligible. However, all except $P_{3,3}$ fall outside the limits set by the small original errors. Inelastic-scattering processes were neglected during the phase-shift analysis. Calculations indicate that, if these processes could properly be taken into account, any changes in the quoted values of the phase shifts would probably be well within the corresponding errors given here. Extension of the phase-shift inquiries to include G waves was attempted, but it was observed that the available data and theory do not allow the G-wave interaction to be significantly incorporated into the analysis.

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I. INTRODUCTION

A series of experimental measurements on π^+-p scattering at an incident-pion laboratory kinetic energy of 310 Mev has been completed. Data obtained include values of the recoil-proton polarization at four angles of observation,[1] differential-cross-section (DCS) measurements at 23 distinct angles,[2] and total-cross-section values.[2] The polarization and cross-section data are noteworthy because of the relatively high accuracy that has been attained.

Scattering data such as these can be analyzed in terms of phase shifts, by using the method of partial waves. The amount of success with which a phase-shift analysis can be performed is a measure of the completeness of the experimental data at the energy being considered. A satisfactory comprehensive theory must predict the behavior and magnitude of the phase shifts. These parameters therefore provide a meeting place for theory and experiment. The more accurately the phase shifts are known, the more severely is an acceptable theory limited.

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*This work was done under the auspices of the U. S. Atomic Energy Commission.

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Many phase-shift analyses of $\pi^{\pm}$-p cross-section data have been performed in the past. At pion laboratory kinetic energies below about 200 Mev, the experimental data have been fitted satisfactorily by using only the first two terms of the partial-wave expansion--that is, S and P waves. Above the 200-Mev energy region, the possible participation of D waves in the pion-proton interaction has made the results of the data analyses uncertain. It has been difficult to determine the values of the D-wave phase shifts because of the insensitive manner in which these parameters enter into the cross-section equations and the relatively large errors in many of the cross-section measurements. The indefiniteness of the D-wave phase shifts has introduced uncertainties in other phase shifts. In these earlier analyses, not only have the values and signs of some of the phase shifts in a solution been uncertain, but also several different types of solution have been obtained. These dissimilar sets of phase shifts are all good fits to the data.

We have performed a phase-shift analysis, employing the experimental data now available at 310 Mev. The phase-shift uncertainties just mentioned have been investigated. Not only has the role of D waves in the π^{+}-p interaction been examined, but the available data also have enabled us to extend the phase-shift investigations to include F waves.

The equations used in our analysis are discussed in Section II. The different types of phase-shift ambiguities that have arisen in the past are briefly mentioned there. In Section III, we describe our phase-shift investigations, and present the results obtained. A discussion of these results follows in Section IV.[3]

II. BASIC EQUATIONS AND RELATED DISCUSSION

In this section, we present the equations used in our phase-shift analysis. General expressions are given for the non-spin-flip and spin-flip elastic-scattering amplitudes as derived through the use of the method of partial waves. These equations apply to π^+-p scattering and take into account both nuclear and Coulomb effects. First-order relativistic corrections to the Coulomb-scattering amplitudes will be incorporated into these equations. We include in this section the expressions, in terms of the scattering amplitudes, for the DCS and recoil-proton polarization in pion-proton elastic scattering. Finally, the various phase-shift ambiguities are noted, and our notation for the phase shifts is given.

It is convenient to discuss the pion-proton scattering in the center-of-mass (c.m.) system. One generally investigates the scattering that takes place in the horizontal plane, which is experimentally the simplest plane to treat. Consider a right-handed x-y-z Cartesian coordinate system, with the pion and proton moving along the z axis before the collision. Let the scattering occur at the origin and allow the +y direction to be up, perpendicular to the plane of the scattering. We will use the symbol $\theta_{c.m.}$ to represent the angle in the c.m. system between the direction of scattering and the initial direction of motion of either particle. This angle will be referred to as the c.m. scattering angle.

A. Scattering Amplitudes

The non-spin-flip and spin-flip scattering amplitudes in π^+-p elastic scattering can be written

$$g(\theta) = -\frac{k\eta}{2\sin^2(\theta/2)} \exp\left\{-i\eta \ln\left[\sin^2(\theta/2)\right]\right\}$$

$$+ k \sum_{L=0}^{\infty} \left[(L+1)\left(\frac{b_L^+ \exp(2i\delta_L^+) - \exp(2i\bar{\phi}_L)}{2i}\right) + L\left(\frac{b_L^- \exp(2i\delta_L^-) - \exp(2i\bar{\phi}_L)}{2i}\right)\right] P_L(\cos\theta) \tag{1}$$

and

$$h(\theta, \phi) = k \sum_{L=1}^{\infty} \left[\frac{b_L^+ \exp(2i\delta_L^+) - b_L^- \exp(2i\delta_L^-)}{2i}\right] D_L Y_L^{\pm 1}(\theta, \phi) . \tag{2}$$

The term "non-spin-flip" refers to the type of scattering in which the component of the proton spin in the direction of the incident beam is unchanged; "spin-flip" refers to the scattering in which the z component of the proton spin is reversed. In Eqs. (1) and (2), $g(\theta)$ is the non-spin-flip scattering amplitude, $h(\theta, \phi)$ is the spin-flip scattering amplitude, L is the orbital-angular-momentum quantum number, θ and ϕ are the spherical angular coordinates defining the direction of scattering of the particle (either pion or proton) considered to move in the $+z$ direction before the collision,[4] k is the wavelength of either particle, divided by 2π, in the c.m. system, $\delta_L^{\pm}$ are the phase shifts describing the total (nuclear plus Coulomb) interaction and relating to states with a specified L and with $J = L \pm 1/2$, where J is the total-angular-momentum quantum number (these phase shifts are real quantities), $b_L^{\pm}$ are the "inelastic parameters" (these are real numbers with magnitudes less than or equal to unity, and take into account inelastic reactions; they are all equal to unity only if no inelastic scattering occurs), and $P_L(\cos\theta)$ is the Legendre polynomial. In addition, we have $D_L = [4\pi L(L+1)/(2L+1)]^{1/2}$, and

$$Y_L^{\pm 1}(\theta, \phi) = \text{spherical harmonics}$$

$$= \mp \left(\frac{2L+1}{4\pi L(L+1)}\right)^{1/2} \sin\theta \frac{d}{d(\cos\theta)} [P_L(\cos\theta)] e^{\pm i\phi} . \tag{3}$$

For $L = 0$, the quantity $\overline{\Phi}_L$ is zero; for $L \geq 1$,

$$\overline{\Phi}_L = \sum_{x=1}^{L} \tan^{-1}(\eta/x), \tag{4}$$

with $\eta = e^2/\hbar v$ (positive for π^+-p scattering), where v is the laboratory velocity of the incident pion.

Equations (1) and (2), in a slightly different form and with the inelastic parameters set equal to unity, can be found in Critchfield and Dodder.[5] These equations take into account both Coulomb and nuclear scattering. Although we will refer to $\overline{\Phi}_L$ as the nonrelativistic Coulomb phase shift of order L, it is actually the difference between the nonrelativistic Coulomb phase shifts of order L and of order zero.[6] The upper signs in the expression for the spherical harmonics are to be used when the proton spin is pointing in the $+z$ direction before the collision; the lower signs, when the proton spin is initially pointing in the $-z$ direction.

The first term in Eq. (1) is the nonrelativistic Coulomb-scattering amplitude, which approaches infinity as the scattering angle approaches 0 deg. Because of this singular behavior, we will find the form of Eq. (1) advantageous. The summation in this expression for $g(\theta)$ contains just the difference between the total and the nonrelativistic Coulomb-scattering amplitudes, and is expected to converge more rapidly than an expansion in which the nonrelativistic Coulomb-scattering amplitude has not been separated out.

The phase shifts always enter into the equations in the form $2\delta_L^{\pm}$. Thus multiples of 180 deg can be added to or subtracted from the phase shifts without changing any function of these parameters. Before quoting phase-shift values, we will frequently make changes of 180 deg in order to reach a desired angular region.

Let us divide the phase shifts describing the total interaction into a pure Coulomb part and an additional portion that arises only when the nuclear interaction is added to the Coulomb interaction. We then can write the total phase shifts as $\delta_L^{\pm} = \Phi_L^{\pm} + \delta_{L,N}^{\pm}$, where the symbols $\Phi_L^{\pm}$ represent the relativistic Coulomb phase shifts of order L and are set equal to $\Phi_L + \Delta\Phi_L^{\pm}$.[7] The quantities $\Delta\Phi_L^{\pm}$ are corrections to Φ_L (the nonrelativistic Coulomb phase shift) due to modifications of the nonrelativistic Coulomb scattering. The modifications that we will discuss are the relativistic corrections given by Solmitz.[8] The quantities $\delta_{L,N}^{\pm}$ approximate the pion-proton nuclear phase shifts of order L. By nuclear phase shifts, we mean those that would describe the interaction if no Coulomb effects existed. It is to be stressed that the $\delta_{L,N}^{\pm}$ are only approximations to the nuclear phase shifts; the quantities obtained when the pure Coulomb phase shifts are subtracted from the total phase shifts still contain remnants of the Coulomb interaction. We assume that the additional corrections needed to obtain the true nuclear phase shifts are small.

B. Inclusion of First-Order Relativistic Coulomb Corrections

First-order relativistic corrections to the nonrelativistic Coulomb-scattering amplitudes can be written

$$\Delta g_C = \lambda \eta A \qquad \text{(non-spin-flip correction)}, \tag{5}$$

and

$$\Delta h_C = \mp \frac{\lambda \eta B \sin\theta}{2 \sin^2(\theta/2)} e^{\pm i\phi} \quad \text{(spin-flip correction)}, \tag{6}$$

where

$$A = \frac{(\beta_\pi \beta_P)/2 + (2\mu_P - 1)\beta_P^2/4}{1 + \beta_\pi \beta_P} ,$$

and

$$B = \frac{(\mu_P \beta_\pi \beta_P)/2 + (2\mu_P - 1)\beta_P^2/4}{1 + \beta_\pi \beta_P} .$$

Here β_P and β_π are the c.m. velocities of the proton and pion, respectively, divided by the velocity of light, and μ_P is the magnetic moment of the proton in nuclear magnetons. The other quantities in Eqs. (5) and (6) have been previously defined. These formulas were obtained from Eqs. (2) and (3) of Solmitz;[8] we used the relationship $v/c = (\beta_\pi + \beta_P)/(1 + \beta_\pi \beta_P)$, where (as in the expression for η) v is the laboratory velocity of the incident pion. The effect of the magnetic moment of the proton is included in these corrections. The double sign before the expression for Δh_C, and the $e^{\pm i\phi}$ factor after, are necessary to account for the two possible initial spin states. The double-sign convention is the same as in Eqs. (2) and (3) of this report. The order of these signs has been chosen so that the relative phase of the nuclear and Coulomb spin-flip scattering amplitudes in Eq. (1) of reference 8 agrees with the corresponding relative phase in our Eq. (8).

To incorporate these corrections into our analysis, we decompose them into partial waves. This allows them to be separated into two parts--one corresponding to states with $L \leqslant L_{MAX}$, and the second containing the remainder. The quantity L_{MAX} is the maximum value of the quantum number L whose related partial wave is affected by the nuclear interaction. For $L \leqslant L_{MAX}$, unitarity is maintained by employing the usual partial-wave expressions but now interpreting part of each phase shift as arising from the correction terms. These phase-shift corrections are estimated by comparing the first-order Solmitz corrections with Eqs. (1) and (2) taken to lowest order. Our basic assumption is that these corrections to the Coulomb phase shifts are not altered by the other interactions. We subtract them, along with the nonrelativistic Coulomb phase shifts, from the total phase shifts, to obtain estimates of the nuclear phase shifts. In contrast to the method for $L \leqslant L_{MAX}$, the part of the correction Δh_C for $L > L_{MAX}$ is simply added to the rest of the spin-flip scattering amplitude, with no attempt to preserve unitarity in the higher-order states. Because Δg_C is independent of angle, it is entirely taken into account by the correction to the S-wave phase shift.

The procedure just described yields the following expressions for the corrections to the nonrelativistic Coulomb phase shifts:

$$\Delta\bar{\Phi}_0 \; (= \Delta\bar{\Phi}_0^+) \approx \Delta g_C/\lambda \, ,$$

$$\Delta\bar{\Phi}_L^+ \approx \frac{\eta B}{L+1} \quad \text{for} \quad L \geqslant 1 \, ,$$

$$\Delta\bar{\Phi}_L^- \approx -\frac{\eta B}{L} \quad \text{for} \quad L \geqslant 1 \, .$$

Using these results and Eq. (4), we can compute the numbers presented in Table I. It is observed that the quantities $\Delta\bar{\Phi}^{\pm}_{L}$ are small and, for low L, $\bar{\Phi}_{L}$ is also small. Thus, for low L and θ not too near 0 deg, the approximations made in expanding Eqs. (1) and (2) to first order (with only the Coulomb interaction allowed) are justified.

Handling the Solmitz corrections as discussed, we can write the non-spin-flip and spin-flip elastic-scattering amplitudes as

$$g(\theta) = -\frac{k\eta}{2\sin^2(\theta/2)}\left\{\exp\left[-i\eta\ln\left[\sin^2(\theta/2)\right]\right]\right\}$$

$$+ k\sum_{L=0}^{L_{MAX}}\left[(L+1)\left(\frac{b^{+}_{L}\exp(2i\delta^{+}_{L}) - \exp(2i\bar{\Phi}_{L})}{2i}\right)\right.$$

$$\left. + L\left(\frac{b^{-}_{L}\exp(2i\delta^{-}_{L}) - \exp(2i\bar{\Phi}_{L})}{2i}\right)\right] \times P_{L}(\cos\theta), \qquad (7)$$

and

$$h(\theta,\phi) = \mp\frac{k\eta B\sin\theta}{2\sin^2(\theta/2)}e^{\pm i\phi}$$

$$+ k\sum_{L=1}^{L_{MAX}}\left[\frac{b^{+}_{L}\exp(2i\delta^{+}_{L}) - b^{-}_{L}\exp(2i\delta^{-}_{L})}{2i} - \eta B\left(\frac{2L+1}{L(L+1)}\right)\right] D_{L}Y_{L}^{\pm 1}(\theta,\phi). \qquad (8)$$

The part of the correction Δh_{C} for $L > L_{MAX}$ has been included in $h(\theta,\phi)$ by adding the entire Δh_{C} and then subtracting off the $L \leq L_{MAX}$ portion. We summarize the sign conventions employed in Eqs. (7) and (8):

(a) In each place where double signs occur in the expression for $h(\theta, \phi)$, the upper sign is to be used when the proton spin is pointing in the $+z$ direction before the collision; the lower sign, when the proton spin is initially pointing in the $-z$ direction.

(b) The $\pm$ superscripts on δ_L and b_L refer to states with $J = L \pm 1/2$.

Equations (7) and (8) are similar to expressions that are obtained if one simply adds the nuclear and Coulomb scattering amplitudes. However, differences exist because the method presented here adds nuclear and Coulomb phase shifts rather than amplitudes for $L \leqslant L_{MAX}$. Except for the modifications due to the Solmitz corrections, our approach is essentially that used by Stapp, Ypsilantis, and Metropolis.[9]

C. Cross-Section and Polarization Expressions

To obtain expressions for the DCS and recoil-proton polarization in elastic π^+-p scattering in terms of phase shifts, when both nuclear and Coulomb effects are present, we use the equations

$$I(\theta_{c.m.}) = \left|g_{aa}\right|^2 + \left|h_{\beta a}\right|^2 , \tag{9}$$

and

$$P(\theta_{c.m.}) = \frac{2\ \mathrm{Im}(g^*_{aa}\, h_{\beta a})}{I(\theta_{c.m.})} . \tag{10}$$

Here the quantity g_{aa} is given directly by Eq. (7), and $h_{\beta a}$ is given by Eq. (8) when one sets $\phi = 0$ or 180 deg and employs the upper sign in each place where double signs occur.

Equation (10) follows from the results of Fermi's article,[10] and Eq. (9) can be found, in a somewhat different form, in Bethe and de Hoffmann.[11] In obtaining Eq. (10), we have used the polarization definition $P = (N_U - N_D)/(N_U + N_D)$, where N_U and N_D are the intensities of recoiling protons with their spin vectors pointing in the $+y$ (assumed up) and $-y$ (assumed down) directions, respectively. The subscripts α and β denote the proton spin states in which the spin points in the $+z$ and $-z$ directions, respectively. The first subscript on g and h refers to the spin state after the collision, and the second to the spin state before the collision (the reverse of Fermi's subscript notation). In obtaining Eq. (10), we have used $h_{\beta\alpha} = - h_{\alpha\beta}$, a relationship that can be seen from Eq. (8) to be valid for $\phi = 0$ and 180 deg. This specification of the ϕ value is actually no restriction because one may choose the x-z plane, which contains $\phi = 0$ and 180 deg, to coincide with any scattering plane of interest. With ϕ specified, $g_{\alpha\alpha}$ and $h_{\beta\alpha}$ depend only on the one angular coordinate θ. Because θ can refer to the angle between the direction of scattering and the initial direction of motion of either particle, we have used the symbol $\theta_{c.m.}$ in Eqs. (9) and (10), following the definition at the beginning of Section II.

D. Ambiguities and Phase-Shift Notation

Owing to the nature of the equations, more than one set of phase shifts have arisen in the analysis of pion-proton scattering data. Each set has distinct characteristics and, within certain limitations, yields a satisfactory fit to the experimental data. It is important to determine which of the several possible solutions corresponds to the true solution. The various uncertainties in the π^+-p phase shifts may be classed as the Fermi-Yang-Minami ambiguity,[12-14] the D-wave phase-shift ambiguity,[15] and the uncertainty in the absolute sign of a given set of phase shifts.[12] We shall let the term "Minami-Yang" refer to the set of phase shifts obtained when the Minami transformation is applied to the Yang set,[13] as opposed to the "Minami" set, which is similarly obtained from the Fermi-type solution.

The phase-shift notation that we will employ is given in Table II. The conventional symbols for the S-, P-, and D-wave phase shifts have been modified to present a consistent notation when F waves are included in the analysis. As before, the first subscript is twice the total isotopic spin, and the second is twice the total angular momentum. Because we are dealing with π^+-p scattering, only the state with isotopic spin of 3/2 enters into the interaction.

III. PHASE-SHIFT ANALYSIS

Our phase-shift analysis and the results obtained will now be discussed.[16] We first examine the general method used in these investigations. Then, we describe the analysis involving S, P, and D waves and the evidence that the D-wave phase shifts are needed in order to attain an adequate fit to the data. The ambiguity in the D-wave phase shifts is mentioned. Finally, the inclusion of F waves in the analysis is discussed, and also described is the attempt to add G waves.

A. General Method

In the analysis of our experimental cross-section and polarization data, we used an IBM-704 electronic computer and the formulas presented in Section II. The grid search procedure was employed, in which the phase shifts are varied in cycles.[17] When varying a phase shift by the increment Δ, our computer program makes use of the equality $\exp[2i(\delta+\Delta)] = \exp(2i\delta) \times \exp(2i\Delta)$. This equation, when separated into real and imaginary parts, contains the sine and cosine of 2δ and 2Δ on the right-hand side. After these four trigonometric functions have been initially calculated, variations of the size Δ can be made in δ without the computation of any new trigonometric functions. Because only relatively simple arithmetic operations are involved, this method reduces the computational time.[18]

Our program is arranged so that, in the search for a fit to the data, the computer varies the phase shifts but not the inelastic parameters. In the major portion of our phase-shift investigations, and unless otherwise stated, the inelastic parameters were assumed to be unity; that is, only elastic scattering was allowed. This assumption is reasonable owing to the apparently small amount of inelastic scattering at 310 Mev (see Section IV-A). If there

were substantial inelastic scattering, the inelastic parameters could be considerably less than unity. We might then have had to vary both the inelastic parameters and the phase shifts in the search for the true solution, and the analysis would have become more complicated.

Although we generally disregarded inelastic scattering, we eventually wanted to investigate its influence on the results of the phase-shift analysis. Our program enables the computer to accept selected values of the inelastic parameters and employ these initial values throughout the search procedure. Various combinations of these parameters can be chosen, the solution of interest can be redetermined, and the resultant phase-shift changes can be examined. In this way, one is able to obtain estimates of the errors introduced into the analysis by the assumption that all the inelastic parameters are unity.

The predictions of a given set of phase shifts are compared with the available experimental data by computing the quantity M, where

$$M = \sum_i \frac{X_i^{(c)} - X_i^{(e)}}{E_i}^2 .$$

Here $X_i^{(e)}$ is the quantity X_i as obtained from experiment, E_i is the experimental error (standard deviation) in $X_i^{(e)}$, and $X_i^{(c)}$ is the quantity X_i as calculated by the computer from a given set of phase shifts. We sum over all the experimental measurements.

Expressing M in terms of quantities for which we have experimental data, we write

$$M = \sum_j \left[\frac{P_j^{(c)} - P_j^{(e)}}{E_j}\right]^2 + \sum_k \left[\frac{I_k^{(c)} - (1 \pm \epsilon) I_k^{(e)}}{E_k^{(I)}}\right]^2 + \left[\frac{\epsilon}{E^{(\epsilon)}}\right]^2 \left[\frac{I_T^{(c)} - I_T^{(e)}}{E^{(T)}}\right]^2 ,$$

where P_j is the polarization of the recoil protons at the c. m. scattering angle $\theta^{(j)}_{c.m.}$, $E^{(P)}_j$ is the experimental error in $P^{(e)}_j$, I_k is the elastic DCS for scattering at the c. m. angle $\theta^{(k)}_{c.m.}$, $E^{(I)}_k$ is the experimental error in $I^{(e)}_k$, ϵ is the variable normalization parameter for the DCS, $E^{(\epsilon)}$ is the experimental error in ϵ (the experimental value of ϵ is $0 \pm E^{(\epsilon)}$), I_T is the total cross section (elastic plus inelastic) between the cutoff angles $\theta^{(1)}_{c.m.}$ and $\theta^{(2)}_{c.m.}$, and $E^{(T)}$ is the experimental error in $I^{(e)}_T$. The quantities $I^{(c)}_k$ and $P^{(c)}_j$ are calculated by using Eqs. (9) and (10). The program computes $I^{(c)}_T$ by integrating the elastic DCS over the angular region between $\theta^{(1)}_{c.m.}$ and $\theta^{(2)}_{c.m.}$, and by adding on the total inelastic cross section when it is assumed to be nonnegligible. The first summation in the expression for M extends over all angles for which polarization data exist; the second summation, over all angles for which elastic DCS data were obtained. We assume that the experimental errors entering into M are independent, normally distributed, and realistically estimated.

The search program requires the computer to find a set of phase shifts for which M has a minimum value, beginning at a given set of phase shifts. In this way, a least-squares fit to the data is attained. Such a fit corresponds to a minimum point in the sense that a change of $\pm\Delta_{FINAL}$ in any one of the phase shifts gives a larger value of M than the value calculated at the minimum. Here Δ_{FINAL} is the smallest increment employed when the phase shifts are varied. The resulting value of M may not have the absolute minimum magnitude obtainable, because the computer stops at the first relative minimum that it notices. Different initial sets of phase shifts can lead to different minima, some of which may have even lower M values.

During the search procedure, the computer varies ϵ in the same manner that it varies the phase shifts. Thus the computer is able to modify the absolute scale of the DCS in order to improve the fit to the data. The experimental error in ϵ, $E^{(\epsilon)}$, is comprised of the uncertainties in the DCS absolute scale. Errors of this type include uncertainties in the intensity and contamination of the incident pi-meson beam and in the thickness of the liquid-hydrogen target. Independent errors, such as statistical counting uncertainties, are attached to each DCS measurement individually and are denoted $E_k^{(I)}$. These independent errors indicate the accuracy with which the various measurements are known with respect to one another (effects of systematic uncertainties in the shape of the DCS are discussed in Section III-B). The use of the variable ϵ enables the phase-shift analysis to keep the independent errors in the individual DCS measurements separate from the uncertainties in the absolute scale, thus allowing an optimum amount of information to be obtained from the DCS data and permitting independent errors in the expression for M. Although we will generally disregard ϵ in our further discussion of the program and when quoting results, it was always present in our analysis.

Owing to the influence of the small relative error in the value of $I_T^{(e)}$ used, the principal effect of ϵ in our analysis was to enable the elastic DCS curve to be normalized to the total-cross-section measurement. In performing this normalization, we usually assumed that we could neglect the inelastic-scattering contribution to the total cross section. Because the amount of inelastic scattering at 310 Mev is apparently not appreciable, the error introduced by its disregard in the normalization procedure appears to be small compared with the error in the total-cross-section measurement.

It is illuminating to visualize the hypersurface that would be obtained if M could be plotted as a function of the phase shifts. The region around a point where M has a minimum value corresponds to a depression in the hypersurface. In the phase-shift discussions to follow, we will sometimes refer to this visual representation.

The usefulness of any possibly acceptable phase-shift fit is increased if one can ascertain the accuracy with which the experimental data determine the individual phase shifts. We employed the customary method of error calculation, which involves the error matrix. Although the details of our calculation differ somewhat from those described by Anderson et al.,[19] the general method is the same. The square roots of the diagonal elements of the error matrix give the rms errors in the phase shifts. Each off-diagonal element is the product of a correlation coefficient and the two related rms errors.

As a check on the results obtained from the error matrix, the rms errors in the phase shifts were also calculated by a second method. In this method, one phase shift is changed from its value at the minimum and then held fixed while all the other phase shifts are varied until M can be decreased no further. If we let the resulting value of M be denoted M'_0 and let M_0 be the value of M at the minimum point corresponding to the solution under consideration, the change required in the fixed phase shift to give a difference of unity between M'_0 and M_0 is the rms error in that phase shift. Errors in all the phase shifts can be calculated in this way, but at the expense of considerably more computer time than when the error-matrix method is used. We obtained satisfactory agreement between the results of the two methods of error determination.

B. The SPD Random Search[20]

The phase-shift investigations were begun with a random search involving S-, P-, and D-wave phase shifts. In order to find every minimum that might lie in the neighborhood of the true solution, the computer was asked to begin searching at a large number of random points scattered over the M hypersurface. A total of 244 random sets of phase shifts were fed into the computer. The values of all five phase shifts ($S_{3,1}$, $P_{3,1}$, $P_{3,3}$, $D_{3,3}$, $D_{3,5}$) in every set were randomly selected. The initial value of ϵ was always zero. From these 244 random positions on the hypersurface, the computer searched and found 27 distinct clusters of solutions (phase-shift fits). The solutions in each cluster agree with one another to within a few tenths of a degree in every phase shift. The different clusters apparently correspond to various relative minima. Each of the ten relative minima in the group with the lowest values of M was detected by the computer at least five times. If one assumes that the relative minima are randomly spaced on the M hypersurface and can be entered with equal ease, then the probability of having overlooked a set of phase shifts with a low M value is less than 1%.

Since the completion of our SPD random search, both the computer program and the input data have been revised and extended. The most important changes were the addition of a total-cross-section measurement and the inclusion of DCS data at angles sufficiently small so that Coulomb-nuclear interference effects are noticeable. It is assumed that no new minima with low values of M were created by the changes made. (The validity of this assumption is supported by the results of the SPDF random search to be described in Section III-D.) In general, the changes in the data

and program produced only small alterations in the phase-shift values related to each minimum. The presence of the DCS data at small angles caused the M values of several of the original minima to increase considerably. These minima correspond to sets of phase shifts that give the incorrect sign for the Coulomb-nuclear interference effects.

In all results to follow, we employ the revised and extended data and program. The data used include four recoil-proton polarization measurements,[1] values of the elastic DCS at 23 anglesof observation,[2] and a total-cross-section measurement of 56.4 ± 1.4 mb (between the c. m. cutoff angles 14.7 and 158.0 deg).[2] The polarization data are given in Table V of reference 1, and the DCS data are listed in Table III of this report. These experimental measurements are plotted in Figs. 1 and 2.

Of the 27 distinct sets of phase shifts found in the SPD random search, all but three have negligible probabilities of lying in the vicinity of the true solution. We base this statement on the χ^2 distribution of statistical theory, which can be applied at least approximately to our results.[21] The χ^2 distribution for 23 degrees of freedom is used here because we are endeavoring to fit 29 pieces of experimental information (including $\epsilon = 0.00 \pm 0.06$) with five phase shifts and the parameter ϵ. The 24 solutions that were discarded on the basis of statistical theory have values of M in the range 86 to 1100, and are therefore highly improbable (the mean M value expected is equal to the number of degrees of freedom). If the polarization data had not been present in the analysis, some of these improbable sets of phase shifts would have had low M values and therefore could not have been discarded on the statistical basis alone.

Our three possibly acceptable solutions are presented in Table IV. The phase shifts given there are of the nuclear type. They were acquired by subtracting the Coulomb phase shifts $\mp\Phi^{\pm}_{L}$, which are listed in Table I,

from the total phase shifts obtained by the search program. The three solutions in Table IV are of the Fermi type, Minami type, and Yang type, in order of increasing M. The connections between these sets of phase shifts are not precisely the relationships one might expect because of the additional constraints created by the polarization data. However, the features that characterize these solutions can be noted.

Two other sets of phase shifts are good fits to all but the DCS data at small angles. These solutions are similar to the Fermi and Yang fits in Table IV except that the signs of most of the phase shifts are opposite to the signs of the corresponding quantities in the table. Because these two solutions give destructive Coulomb-nuclear interference in the forward direction of scattering, we can definitely exclude them by using the DCS data at small angles (see Fig. 2).

Figures 1 and 2 show the manner in which the SPD solutions in Table IV fit the data. The DCS curves calculated from the Minami and Yang sets of phase shifts are not shown; they closely resemble the Fermi plot. All three phase-shift sets give values for the total cross section that are in good agreement with the experimental measurement.

We present in Table V the error matrix that is associated with our SPD Fermi solution. The phase-shift uncertainties obtained from this matrix are based on the errors in the experimental data. In order to make the problem manageable, we have neglected the systematic uncertainties in the shape of the DCS and have used only the independent uncertainties referred to in Section III-A. It is these independent errors that are given in Table III and shown in Fig. 2. We investigated the influence on the phase shifts of the systematic uncertainties just mentioned, and found the effects to be small compared with the rms errors obtained from the error matrix for the SPD Fermi solution.

In the remainder of this section, our attention will often be concentrated on the Fermi solution given in Table IV. The reasons for disregarding the Minami and Yang sets of phase shifts will be briefly discussed in Section IV-A.

C. Inadequate SP Fit; Ambiguity in the D-Wave Phase Shifts

Besides our SPD analysis, we have also analyzed the data by assuming that the pion-nucleon nuclear interaction affects only the S and P waves. The best SP fit that we obtained is given in Table VI; the corresponding polarization and DCS curves are shown in Figs. 1 and 2. This solution is of the Fermi type and is obviously an inadequate fit to the experimental data. The poor fit is shown numerically in the large M value of 92.5. Although the D-wave nuclear phase shifts are small in our SPD Fermi set, they are definitely needed in order to obtain a satisfactory fit.[22]

By comparing the SP and SPD Fermi solutions, we observe that the inclusion of D waves in the analysis has a noticeable effect on $S_{3,1}$ and $P_{3,1}$. Each is reduced in absolute magnitude when the D-wave nuclear phase shifts are allowed to have values other than zero. Only the phase shift $P_{3,3}$ is rather insensitive to the number of partial waves included in the analysis.

When our four polarization measurements are excluded from the SPD analysis, an uncertainty appears in the D-wave phase shifts. This ambiguity was mentioned in Section II-D. It gives rise to two Fermi-type solutions yielding low values of M, instead of just the one previously discussed. The two Fermi phase-shift sets, obtained when only the cross-section data are utilized, are given in Table VI. (They possess lower M values than the Fermi solution in Table IV because there are fewer experimental measurements to fit.) A principal difference between these two solutions is that the

D-wave phase shifts in one set have signs reversed compared with those in the other set. The usefulness of the polarization data in differentiating between these two SPD phase-shift solutions is demonstrated in Fig. 3.

D. Inclusion of F Waves

Because of the relatively high accuracy with which the phase shifts in our SPD Fermi fit are determined, we felt it necessary to extend the analysis to include F waves. It appeared quite possible that the addition of small F-wave phase shifts might cause changes in the other phase shifts larger than the quoted errors. This indeed turned out to be true. We found that the inclusion of a small F-wave nuclear interaction not only alters the values of almost all the S-, P-, and D-wave phase shifts but also causes their errors to increase considerably. Also, new solutions appear that fit the data well.

With the F-wave nuclear phase shifts allowed to be different from zero, another random search for solutions was conducted. New random initial values were picked for the phase shifts related to the S, P, and D waves. The initial F-wave phase shifts were also chosen at random, but were restricted to the interval 0±9 deg because we assumed these parameters to be small. The number of random sets used was 260, and about twice as many minima were found as in the SPD random search. Every solution with an M value of less than 40 was obtained at least five times. According to the χ^2 distribution, now for 21 degrees of freedom, the probability is less than 1% that the M value of the true solution is greater than 40.

As a check on the SPD random-search results, we made SPD fits to the data using as starting points the first five phase shifts in the various SPDF solutions. All the original SPD solutions appeared. In addition,

only two new minima were found and these possess extremely high M values. Therefore, we had apparently obtained all the existing SPD solutions with low M values in our original random search.

Every SPDF solution discovered, with a value of M less then 40, is listed in Table VII. The Fermi-I, Minami-I, and Yang-I solutions correspond to the three SPD fits given in Table IV. The designation "Minami-Yang" refers to the type of fit of that name mentioned in Section II-D. Many of the phase-shift values in the various solutions denoted "I" in Table VII are approximately connected by the ambiguity interrelationships discussed in the references cited in Section II-D. Similarly interrelated are the three fits denoted "II". We will disregard solution 6 because of its excessively large $F_{3,7}$. When SPD fits to the cross-section data only are obtained, the SPDF Fermi-I and -II solutions reduce to the solutions of the same names given in Table VI and therefore appear to be manifestations of the ambiguity in the D-wave phase shifts. The error matrices for these two sets of phase shifts are presented in Tables VIII and IX.

The Fermi-II solution and the two Minami-Yang fits were also found in the SPD random search but then had improbably large M values because of their inability to fit the polarization data. The presence of small F-wave phase shifts has enabled these three previously unacceptable solutions to become good fits to the polarization measurements. We present in Fig. 4 the variation of the polarization with c.m. scattering angle predicted by the first four SPDF solutions in Table VII. The analogous curve for the solution Minami-Yang II is intermediate between those for Fermi II and Minami-Yang I. The polarization plots for the SPDF Minami-I and Yang-I sets are essentially the same as the corresponding curves in Fig. 1.

E. Addition of G Waves

An attempt was made to observe the effects of G waves on the SPDF analysis, again with the aid of the IBM-704 computer. When no restrictions are placed on the size of the G-wave phase shifts, we found that our former solutions become poorly defined, and additional sets of phase shifts appear that fit the data well. The SPDF Fermi-I and Fermi-II solutions are altered in character considerably when the nuclear G-wave interaction is allowed because the computer is best able to fit the data by changing some of the phase shifts in these solutions by as much as 10 to 20 deg (the M values dropping to about 10 and 16, respectively). Even if the magnitudes of the nuclear G-wave phase shifts are held to within the arbitrary limit of 0.2 deg, the uncertainties in many of the other phase shifts in the two Fermi solutions increase to one and one-half to two times their former values. With the nuclear G-wave interaction allowed, we re-investigated all the minima obtained in the SPDF random search. The magnitudes of the nuclear G-wave phase shifts in a given fit were arbitrarily restricted to be less than one-fifth the magnitude of the larger nuclear F-wave phase shift in the same fit. Even this constraint did not prevent new solutions with low M values from arising. With our present data and the limited amount of available theoretical information concerning the phase shifts related to angular-momentum states of higher order, we conclude that we cannot meaningfully include G waves in the analysis.

IV. DISCUSSION OF RESULTS

A. Phase-Shift Analysis

A comprehensive phase-shift analysis has been performed, utilizing the polarization and cross-section data now available on π^+-p scattering at 310 Mev. The D-wave phase shifts were found to be definitely needed in order to attain an adequate fit to the data. We investigated the influence on the analysis of the presence of small F-wave phase shifts; not only are the errors in our original Fermi-type solution increased, but additional solutions arise that fit the data well. Although the introduction of a small F-wave interaction does not greatly improve the best obtainable fit to the data, no justification can be found for completely neglecting $F_{3,5}$ and $F_{3,7}$. We attempted to extend the phase-shift inquiries to include G waves but found that the available data and theory do not allow the G-wave interaction to be significantly incorporated into the analysis. Evidently the region of angles over which polarization data exist is not large enough to enable us to satisfactorily define the phase shifts when G waves are also assumed affected by the nuclear interaction.

Our investigations indicate that it is difficult to obtain a completely meaningful set of phase shifts from pion-nucleon experimental data by using the partial-wave treatment alone. Further assistance from theory may be required before one can handle with confidence all the angular-momentum states measurably affected by the interaction. The discussions to follow will principally be limited to the results of our SPDF investigation.

Let us begin the discussion of the various phase-shift solutions by discarding all those that are of the Yang, Minami, or Minami-Yang type. A principal reason for rejecting these sets of phase shifts is that they appear

to disagreewiththe requirements of the dispersion relations for the spin-flip amplitude of the pion-nucleon scattering in the forward direction.[13, 23, 24] The Minami-type solution is also unreasonable because of its large $D_{3,3}$ and the implausible behavior of its phase shifts at low energy.[13, 25]

Of the phase-shift solutions listed in Table VII, only the Fermi-I and Fermi-II sets remain to be considered (we earlier rejected set 6 because of its excessively large $F_{3,7}$). In Table X, we summarize the characteristics of these two SPDF Fermi-type fits. The SPD Fermi set is also included for comparison. In comparing the closely related SPD Fermi and SPDF Fermi-I solutions, we notice that only $P_{3,3}$ is essentially unaffected by the addition of the F-wave interaction (owing to the strong dependence of this phase shift on only the total cross section). Although $F_{3,5}$ and $F_{3,7}$ in the SPDF Fermi-I solution are small and their errors overlap 0 deg, the effect of their presence is considerable.

Table X shows the drastic increases in the phase-shift errors that occur when F waves are added to the SPD Fermi solution and the SPDF Fermi-I set is thereby obtained. This would seem, at first glance, to indicate that much less information can be derived from this type of solution now that F waves are allowed. Actually this is not true because many of the correlation coefficients are large in the SPDF Fermi-I solution. Large correlation coefficients signify strong relationships between the phase shifts, and thus information about one phase shift will, in general, give useful information about other phase shifts. In any comparison of theory with the SPDF Fermi-I set, it will be important to use the entire error matrix (Table VIII).

To facilitate the phase-shift analysis, we neglected inelastic scattering. Additional uncertainties in the solutions of Table X exist because of this disregard of all but the elastic-scattering reaction. There is little experimental information available on inelastic processes in π^+-p scattering at 310 Mev. However, estimates can be made of the magnitude of the total inelastic cross section at this energy by combining the experimental measurements of Willis[26] at 500 Mev with theories such as those by Rodberg,[27] Franklin,[28] and Kazes.[29] The results indicate that the π^+-p total inelastic cross section is less than 1 mb at 310 Mev.

The inclusion in our analysis of even this small amount of inelastic scattering can cause changes in the phase shifts. We have observed the alterations in the solutions given in Table X when a total inelastic cross section of 1 mb is allowed. Various extreme assumptions were made about the manner in which this amount of inelastic scattering might be distributed among the different angular-momentum states of the interaction. Each inelastic parameter was assumed, in turn, to have a value sufficiently less than unity so as to account for the entire 1-mb cross section (all the other inelastic parameters remaining at unity). Equation (7) of Willis[26] was used in order to calculate these values. For each assumed set of inelastic parameters and for each solution considered, the computer redetermined the values of the phase shifts yielding the minimum magnitude of M (this general procedure was discussed briefly in Section III-A). We conclude from the results of this investigation that, if inelastic-scattering processes could properly be taken into account, any changes in the quoted values of the phase shifts would probably be well within the corresponding errors given in Table X.

B. Comparison of the SPDF Fermi-Type Solutions

Let us examine more closely the two SPDF Fermi-type solutions, both of which are excellent fits to the data. Both sets are reasonable from the point of view that the F-wave phase shifts are small compared with those related to the D wave. We are unwilling to discard the Fermi-II solution on the basis of lack of continuity with results of phase-shift analyses at other energies because we believe these other analyses may suffer the same uncertainties as our SPD results. In the remainder of this section, comparisons between the two SPDF Fermi solutions will be made in an attempt to eliminate one of these two sets of phase shifts.

Both solutions give $\mathrm{Re}[f(0^\circ)] = -0.686 \pm 0.012$ in units of $\hbar/\mu c$ (μ denotes the pi-meson rest mass) where $\mathrm{Re}[f(0^\circ)]$ is the real part of the forward-scattering amplitude, for π^+-p nuclear elastic scattering, in the c.m. system. The result, -0.686, was calculated by inserting the nuclear phase shifts of Table X into Eq. (12) of Anderson and Davidon.[30] (The value computed for $\mathrm{Re}[f(0^\circ)]$ is almost independent of the number of partial waves assumed to be affected by the nuclear interaction.) We obtained the error by using the error matrices in Tables VIII and IX. The sign of $\mathrm{Re}[f(0^\circ)]$ is determined by the absolute sign of the set of phase shifts used, which in turn is determined by the sign of the Coulomb-nuclear interference contribution to the DCS. We neglect a small correction (apparently less than 1%) to $\mathrm{Re}[f(0^\circ)]$ arising from the disregard of possible inelastic contributions to the total cross section when the computer normalizes the experimental elastic DCS to the experimental value of the total cross section. If inelastic scattering takes place but is neglected in the phase-shift analysis, DCS values calculated from the resulting sets of phase shifts will be too large. Because of the close relationship

between $\mathrm{Re}[f(0^\circ)]$ and the value of the DCS for nuclear scattering at $\theta_{c.m.} = 0$ deg, the disregard of inelastic scattering causes the magnitude quoted for $\mathrm{Re}[f(0^\circ)]$ to be slightly too great.

Our result for $\mathrm{Re}[f(0^\circ)]$ agrees well with values predicted by the dispersion relations and based on other experimental data.[31] The curve calculated by Spearman gives $\mathrm{Re}[f(0^\circ)] \approx -0.70$ for $f^2 = 0.08$, where f^2 is the renormalized, unrationalized, pion-nucleon coupling constant.[32] Another recent analysis is that by Cronin, who predicts -1.35×10^{-13} cm at 310 Mev for the real part of the forward-scattering amplitude in the laboratory system (for $f^2 = 0.08$).[33] When transformed to the laboratory system, our result becomes $(-1.36 \pm 0.02)\times10^{-13}$ cm, again in good agreement with the dispersion relations.

When the two SPDF Fermi-type solutions are compared with the predictions of the phase-shift formulas of Chew, Goldberger, Low, and Nambu,[34] we find that Fermi I is in better agreement. The P-wave phase shifts of Fermi I are more in accord with the effective-range formulas of Chew et al. than are the corresponding phase shifts of Fermi II. The effective-range equations predict approximately -5 deg for $P_{3,1}$ and 127 deg for $P_{3,3}$ at 310 Mev. We obtained these results by assuming $f^2 = 0.08$ and $\omega_r = 2.1$. The quantity ω_r is the value of ω at the 3,3 resonance, where ω denotes the total energy in the c.m. system, exclusive of the nucleon rest energy, in units of μc^2. The effective-range formulas are expected to be valid only at low energies. Therefore the fact that the Fermi-II set disagrees more noticeably with these equations than does the Fermi-I solution is not sufficient reason by itself for discarding the former set of phase shifts.

One often compares experimentally obtained values of $P_{3,3}$ with the effective-range theory by means of the Chew-Low plot[35] [i.e. $(q^3 \cot P_{3,3})/\omega$ versus ω, where q is the momentum of the pi meson in the c.m. system, in units of μc]. The values of $P_{3,3}$ in both Fermi I and Fermi II give results that fall below the straight line passing through the low-energy points on this type of plot, in accord with the results of other experiments at energies near or above 300 Mev. The D-wave phase shifts in the SPDF Fermi-I solution agree in sign and reasonably well in magnitude with the theoretical formulas of Chew et al., which predict $D_{3,3} = +0.3$ deg and $D_{3,5} = -2.5$ deg at 310 Mev; the D-wave phase shifts in Fermi II disagree in both sign and magnitude. However, these formulas do not include the effects of the pion-pion interaction and thus may not give accurate predictions.

The straight-line plot[36] at low energies of $S_{3,1}$ as a function of q can be linearly extrapolated to 310 Mev and compared with the values of this phase shift in our two SPDF Fermi solutions. The extrapolated value obtained is near -13 deg, and therefore the comparison yields the better agreement for Fermi I. Once again, this alone is not adequate evidence against Fermi II because the linear relationship between $S_{3,1}$ and q probably does not extend to energies as high as 310 Mev.

Although both the SPDF Fermi-I and Fermi-II solutions give results that agree with the dispersion relations predicting $\mathrm{Re}[f(0^\circ)]$, these two sets of phase shifts yield contrasting results when compared with the dispersion relations for the spin-flip forward-scattering amplitude, following the method of Davidon and Goldberger.[23,37] Dispersion-relation theory predicts that $y = f^2 + Cx$, where f^2 is again the pion-nucleon coupling constant, C is a constant, x is a given function of the energy, and y depends in a stated way on the phase shifts and the energy. As shown in reference 23, Fermi-

type phase shifts that are based on SP analyses over a range of energies lower than 310 Mev exhibit approximately the predicted y-x linear behavior and extrapolate to a reasonable value of f^2. (At sufficiently low energies, we would expect the SP-type analysis to be adequate.) Strictly speaking, the function y depends on the phase shifts at all energies. However, for Fermi-type solutions and for the region of energies considered in the Davidon and Goldberger article, y depends principally on the values of the phase shifts at the energy at which it is being evaluated and on the behavior of $P_{3,3}$ at other energies, about which reasonable assumptions can be made when necessary. Approximate calculations using the Fermi-I solution give $y \approx +0.03 \pm 0.08$; when Fermi II is considered, $y \approx +0.33 \pm 0.02$. We have included in the errors quoted only the error arising from the term $\mathrm{Re}(a_3)$ in Eq. (2.6) of reference 23. The entire error matrices (Tables VIII and IX) were used when calculating these errors. Assuming that the other uncertainties in the calculation do not greatly change the general features of these results for y, we find that the Fermi-I solution is in moderately good agreement with the straight line of reference 23 (which yields about 0.15 for y at 310 Mev) but that Fermi II disagrees. Relying on the Davidon and Goldberger analysis, then, we apparently may say that only the Fermi-I solution is admissible.

C. Concluding Remarks

Although theory appears to favor the Fermi-I set over the Fermi-II, further theoretical evidence and, in addition, experimental justification are desirable. Useful experimental information could probably be obtained by performing supplemental polarization measurements at sufficiently small angles. We note in Fig. 4 that appreciably different values of the polarization are predicted by the two Fermi solutions at c. m. scattering angles in the vicinity of 60 deg. If a practicable method could be developed for determining the polarization of protons with energies approximating 50 Mev, one could perform recoil-proton polarization measurements that might distinguish between the two SPDF Fermi solutions. The same data might also provide experimental evidence against the SPDF Minami, Yang, and Minami-Yang solutions.

In conclusion, the success of the SPD analysis was so striking that an investigation of the effects of F waves was in order. The inclusion of F waves has given a good fit to the data, but not an appreciably better fit than in the SPD analysis. The errors in the phase shifts of the Fermi-I type have become very much larger than they were before the F waves were added, but because many of the correlation coefficients are quite large there is still a great deal of information contained in the SPDF analysis. It is hoped that this work constitutes a significant step in the quantitative study of pion-nucleon scattering.

ACKNOWLEDGMENTS

The analysis described herein would not have been possible without the facilities of the computer centers of the Lawrence Radiation Laboratory at both the Berkeley and Livermore sites, which are supervised by Mr. Kent K. Curtis and Dr. Sidney Fernbach, respectively. Necessary hand calculations were competently performed by Messrs. Stephan J. Krieger, Michael D. Scadron, and William F. Tivol. We are indebted to Dr. Henry P. Stapp for discussions clarifying the Coulomb-scattering problem. It is also a pleasure to acknowledge the support and counsel of Professor Emilio Segrè.

FOOTNOTES

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Table I. Nonrelativistic Coulomb phase shifts, first-order relativistic corrections, and corrected Coulomb phase shifts (all in degrees) at an incident pion laboratory kinetic energy of 310 Mev. The signs given here apply to π^+-p scattering.

L	$\bar{\Phi}_L$	$\Delta\bar{\Phi}^+_L$	$\Delta\bar{\Phi}^-_L$	$\bar{\Phi}^+_L$	$\bar{\Phi}^-_L$
0	0.00	0.09	—	0.09	—
1	0.44	0.09	-0.17	0.53	0.27
2	0.66	0.06	-0.09	0.72	0.57
3	0.81	0.04	-0.06	0.85	0.75
4	0.92	0.03	-0.04	0.95	0.88

Table II. Phase-shift notation for π^+-p scattering

L	J	Phase-shift symbol
0	1/2	$S_{3,1}$
1	1/2	$P_{3,1}$
1	3/2	$P_{3,3}$
2	3/2	$D_{3,3}$
2	5/2	$D_{3,5}$
3	5/2	$F_{3,5}$
3	7/2	$F_{3,7}$

Table III. Experimental DCS measurements (in the c. m. system) used in the phase-shift analysis.[2] The errors given are standard deviations and are independent. Not included is an rms error of approximately ±6% in the absolute DCS scale.

C. m. scattering angle (deg)	$I(\theta_{c.m.})$ (mb/sterad)
14.0	18.71±0.60
19.6	16.05±0.46
25.2	13.82±0.31
30.6	12.99±0.25
34.6	12.28±0.27
36.2	11.65±0.27
44.0	9.82±0.15
51.8	8.59±0.26
56.8	7.54±0.28
60.0	6.58±0.22
69.6	4.73±0.10
75.3	3.62±0.09
81.6	2.77±0.08
97.8	1.66±0.07
105.0	1.51±0.06
108.1	1.62±0.07
120.9	2.08±0.08
135.2	2.93±0.14

Table III. Continued

C. m. scattering angle (deg)	$I(\theta_{c.m.})$ (mb/sterad)
140.6	3.36±0.12
144.7	3.76±0.15
152.2	4.10±0.21
156.4	4.51±0.17
165.0	4.88±0.12

Table IV. Solutions found in the SPD random search that best fit the experimental data. The mean M value expected is 23.

Type of solution	M	Nuclear phase shift(deg)				
		$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
Fermi	15.8	-18.5	- 4.7	134.8	1.9	-4.0
Minami	32.0	- 7.1	-22.3	-1.9	135.6	0.8
Yang	37.7	-23.2	126.2	159.0	7.5	-4.6

Table V. Error matrix for the SPD Fermi solution. The matrix elements are in $(deg)^2$.

$S_{3,1}$	0.41	0.26	0.17	0.11	-0.20
$P_{3,1}$		0.32	0.05	0.11	-0.18
$P_{3,3}$			0.42	-0.01	0.05
$D_{3,3}$				0.13	-0.10
$D_{3,5}$					0.19

Table VI. The "SP Fermi" solution is our best SP fit to the experimental data. "Fermi I" and "Fermi II" are the two SPD Fermi solutions with low M values that are obtained when the computer is required to fit only the cross-section data (these solutions exhibit the ambiguity in the D-wave phase shifts).

Type of solution	Mean M expected	Computed M	Nuclear phase shift (deg)				
			$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
SP Fermi	25	92.5	-22.3	-8.1	136.1	0	0
Fermi I	19	13.9	-16.8	-4.0	134.8	3.3	-5.4
Fermi II	19	14.1	-24.0	-8.8	137.3	-3.5	2.4

Table VII. Solutions found in the SPDF random search that possess values of M less than 40. The mean M value expected is 21.

No.	Type of Solution	M	Nuclear phase shift (deg)						
			$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
1	Fermi I	14.1	-17.2	- 2.9	135.0	3.1	-4.9	0.5	-0.6
2	Minami-Yang I	17.6	123.1	-22.4	3.1	158.6	0.2	-2.8	-0.1
3	Fermi II	18.3	-35.5	-16.1	151.4	-11.4	13.1	-1.1	-1.8
4	Yang II	26.6	-32.0	142.2	160.4	17.8	-6.4	-1.7	-1.3
5	Minami-Yang II	26.9	139.9	-39.0	13.1	164.0	-4.9	-5.7	2.0
6	————	27.8	-19.2	-7.6	153.8	2.0	-21.1	-2.7	13.0
7	Minami I	31.7	-7.2	-22.4	-2.0	136.8	0.8	0.2	0.1
8	Yang I	34.2	-23.6	124.7	159.5	5.8	-4.1	-1.5	0.7

Table VIII. Error matrix for the SPDF Fermi-I solution. The matrix elements are in $(\text{deg})^2$.

	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
$S_{3,1}$	6.93	10.38	-0.08	6.65	-5.56	1.27	-3.61
$P_{3,1}$		16.14	-0.36	10.34	-8.54	1.96	-5.66
$P_{3,3}$			0.42	-0.28	0.27	-0.05	0.16
$D_{3,3}$				6.76	-5.51	1.28	-3.67
$D_{3,5}$					4.61	-1.04	3.00
$F_{3,5}$						0.31	-0.70
$F_{3,7}$							2.03

Table IX. Error matrix for the SPDF Fermi-II solution. The matrix elements are in $(\text{deg})^2$.

	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
$S_{3,1}$	0.50	-0.11	0.30	-0.08	0.08	-0.08	0.13
$P_{3,1}$		0.43	-0.37	0.24	-0.30	0.13	-0.11
$P_{3,3}$			0.70	-0.25	0.26	-0.13	0.12
$D_{3,3}$				0.22	-0.22	0.08	-0.08
$D_{3,5}$					0.29	-0.11	0.11
$F_{3,5}$						0.08	-0.06
$F_{3,7}$							0.09

Table X. Phase shifts for solutions of the Fermi type arising in the SPD and SPDF analyses of π^+-p scattering data at 310 Mev. The units are degrees. The errors are standard deviations and are the square roots of the diagonal elements of the error matrices presented in Tables V, VIII, and IX.

	Solution		
Nuclear phase shift	SPD (M = 15.8	SPDF Fermi I 14.1	SPDF Fermi II 18.3)
$S_{3,1}$	-18.5±0.6	-17.2±2.6	-35.5±0.7
$P_{3,1}$	- 4.7±0.6	- 2.9±4.0	-16.1±0.7
$P_{3,3}$	134.8±0.6	135.0±0.6	151.4±0.8
$D_{3,3}$	1.9±0.4	3.1±2.6	-11.4±0.5
$D_{3,5}$	- 4.0±0.4	- 4.9±2.1	13.1±0.5
$F_{3,5}$	———	0.5±0.6	- 1.1±0.3
$F_{3,7}$	———	- 0.6±1.4	- 1.8±0.3

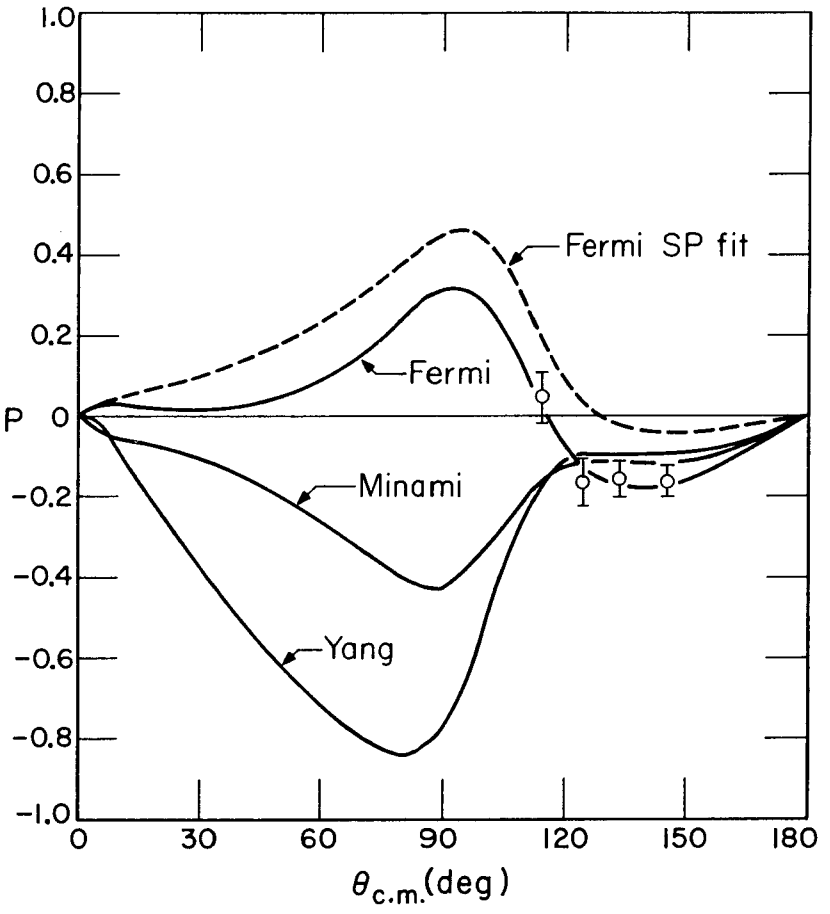
FIGURE LEGENDS

Fig. 1. Experimental recoil-proton polarization measurements given in Table V of reference 1. The solid curves represent the fits to the data predicted by the SPD solutions in Table IV of this work. The SP fit, which is discussed in Section III-C, is indicated by the dashed curve.

Fig. 2. The experimental c. m. DCS measurements given in Table III have been multiplied by $1 + \epsilon$ to normalize them to the total cross section. The value of ϵ used (-0.018) is that giving the minimum magnitude of M for both the SPD and SP Fermi-type solutions. Independent errors only are shown. The solid curve, which represents the Fermi SPD solution, fits the data well. The dot-dash curve at small angles shows the behavior of the SPD Fermi and Yang solutions that possess phase-shift signs opposite to those given in Table IV. The curve with short dashes, shown only at large angles, is the Fermi SP fit discussed in Section III-C. It is given only where it deviates sufficiently from the SPD fit to be easily drawn.

Fig. 3. Variation of polarization with angle predicted by the two SPD Fermi solutions with low M values that are obtained when the computer fits only the cross-section data. These solutions exhibit the ambiguity in the D-wave phase shifts. The values of the phase shifts for these fits are given in Table VI. When the four polarization measurements (shown above) are included in the SPD analysis, the Fermi-I curve can be easily altered to fit the polarization data but the Fermi-II curve cannot.

Fig. 4. Variation of polarization with c. m. scattering angle predicted by the first four SPDF solutions in Table VII. For reasons of clarity, the large-angle behavior of two of the curves is not shown. All curves satisfactorily fit the three negative polarization measurements.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0.4
0.2
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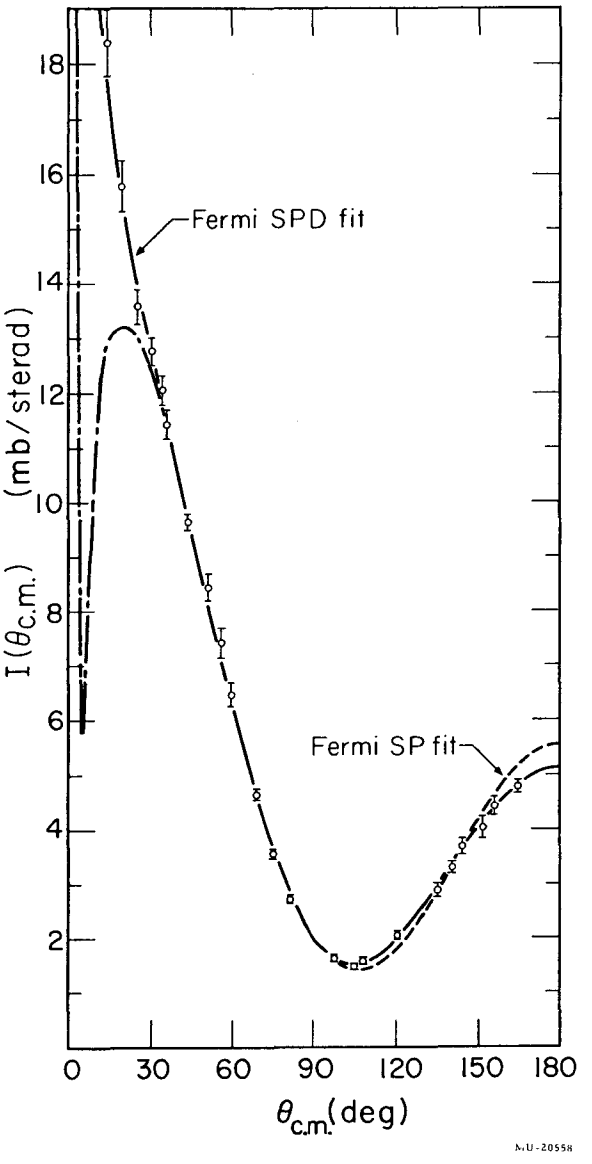
-0.2
-0.4
-0.6
-0.8
-1.0
0
30
60
90
120
150
180
θc.m.(deg)
Fermi SP fit
Fermi
Minami
Yang
MU-20552

Fig. 1.


Fermi SPD fit
Fermi SP fit
I(θc.m.) (mb/sterad)
θc.m.(deg)
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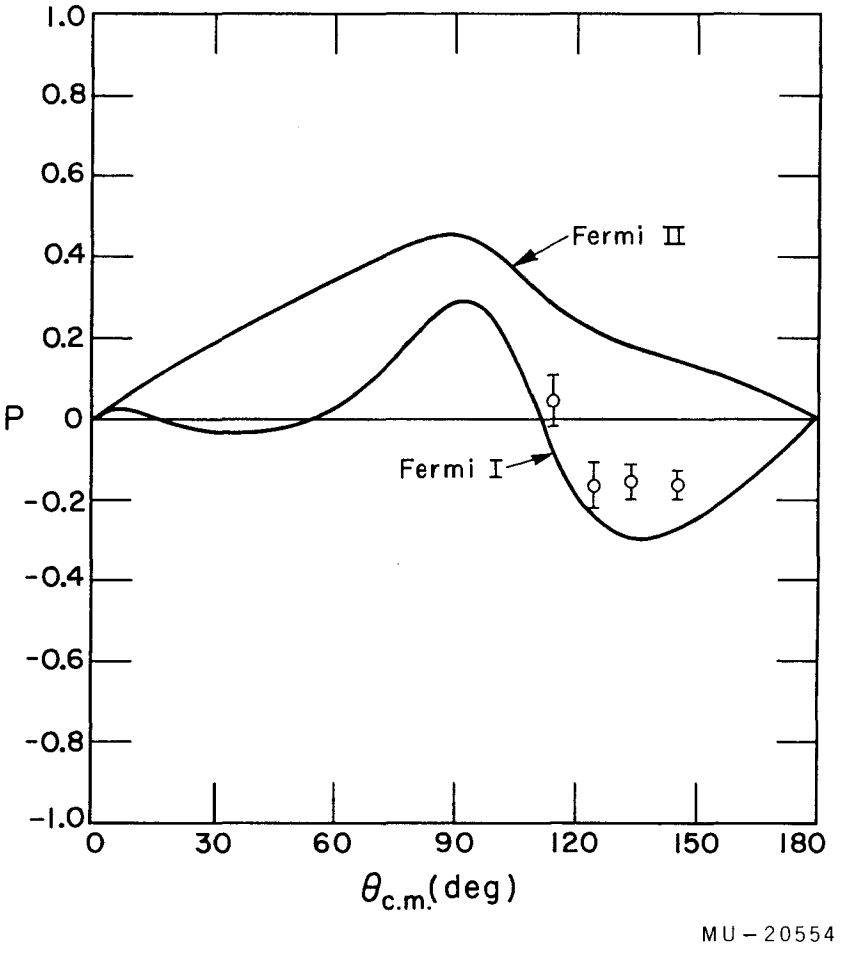
60
90
120
150
180
2
4
6
8
10
12
14
16
18

Fig. 2.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0.6
0.4
0.2
P 0
-0.2
-0.4
-0.6
-0.8
-1.0
0
30
60
90
120
150
180
θc.m.(deg)
Fermi II
Fermi I
MU-20554

Fig. 3.

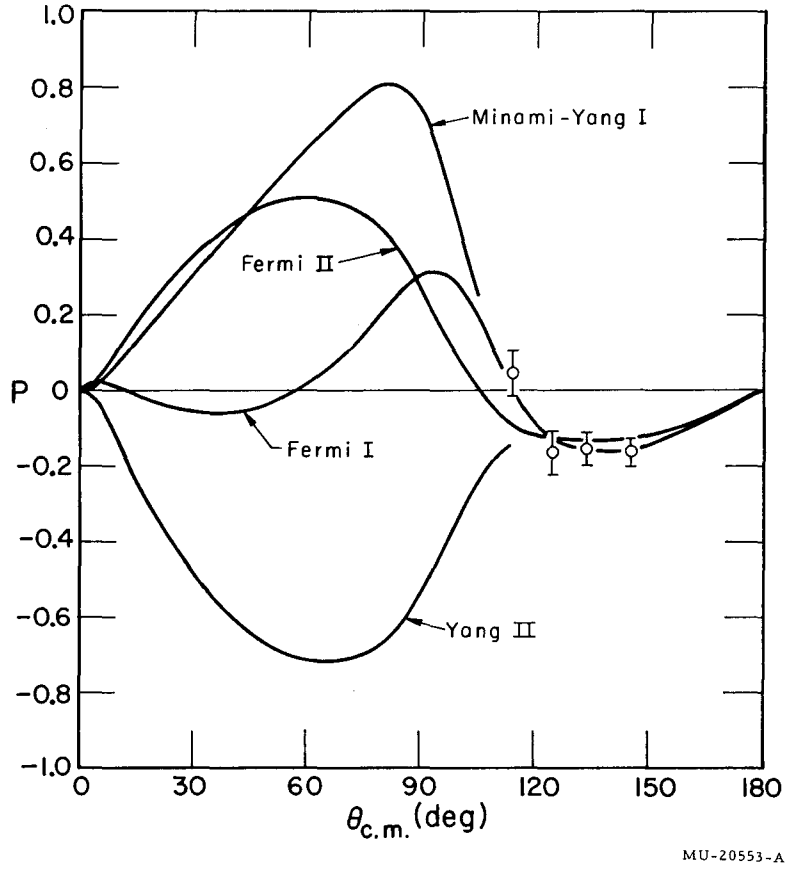
1.0
0.8
0.6
0.4
0.2
P 0
-0.2
-0.4
-0.6
-0.8
-1.0
0
30
60
90
120
150
180
θc.m. (deg)
Minami-Yang I
Fermi II
Fermi I
Yang II
MU-20553-A

Fig. 4.