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# Dynamic Portfolio Choice and Risk Aversion \*

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## **Abstract**

This paper explicitly solves a dynamic portfolio choice problem in which an investor allocates his wealth between a riskless and a risky asset. The solution shows that insights gained from studying static portfolio choice problems do not necessarily carry over to dynamic choice settings. For example, even though the risk premium of the risky asset in the problem presented here is strictly positive, holdings of that risky asset might increase with risk aversion. More surprisingly, a risk-averse investor might take a short position in the risky asset. The findings suggest that using stock holdings as a proxy for risk aversion may be inappropriate. Finally, I show that volatility might not prevent a risk averse investor from holding an infinite amount of a risky asset, contrary to Harry Markowitz's insights on the static portfolio choice.

# 1 Introduction

Many studies have been done on dynamic portfolio choice problems since the seminal work of Merton (1969, 1971). However, with the exception of Kim and Omberg (1996), most solutions to these problems are merely numerical, and those explicit solutions that are available are static in nature [for example, Merton (1969, 1971)]. Therefore, these studies do not provide much insight into dynamic choice.

The goal of this paper is twofold. First, I explicitly solve a problem of dynamic choice between a riskless and risky asset. I study the comparative statics of the optimal portfolio weight of the risky asset.<sup>1</sup> Second, using the derived portfolio weight, I show that qualitative differences exist between dynamic and static portfolio choice. For example, two well-known fundamental theorems provide intuition on static portfolio choice: Namely, a risk averse agent (that is, investor) will hold a strictly positive amount of a risky asset and a more-risk-averse agent will hold less of a risky asset, provided that the risk premium is strictly positive. I show that, in the dynamic setting, both theorems are violated.

This paper uses the framework developed by Merton (1969, 1971). The agent chooses the proportion of his wealth to invest between two assets, a riskless asset and a risky one. The riskless asset has a constant return. Both the risky asset return and the instantaneous variance of the risky asset return follow diffusion processes. The risk premium is a power function

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<sup>1</sup> Because there are only two assets in the problem, the portfolio weight of the riskless asset is known once the portfolio weight of the risky asset is known. Therefore, in the remainder of this paper, I refer to the portfolio weight of the risky asset as the “portfolio weight”.

of the volatility.<sup>2</sup> The agent rebalances his portfolio position continuously to maximize his utility which is a power function of end-of-period wealth.<sup>3</sup> The optimal portfolio weight is derived explicitly as a function of the instantaneous volatility, the constant interest rate, the investment horizon, the constant relative risk aversion coefficient, and the parameters of the volatility process.

The optimal portfolio weight in general depends on the volatility of the risky asset return; therefore, the agent must time the volatility. The dependence of the portfolio weight on the volatility is determined by the instantaneous market price of risk (IMPR), which is the ratio of the instantaneous risk premium over the instantaneous variance. If the IMPR is increasing in the volatility, the return for bearing the risk is high at high volatilities; therefore, the agent holds more risky assets when the volatility is high. Conversely, if the IMPR is decreasing in volatility, then the return for bearing the risk is low at high volatilities and the agent holds less of the risky asset when the volatility is low.

The optimal portfolio weight is a monotonic function of the investment horizon. For example, when the correlation between the instantaneous Sharpe ratio (ISR) (which is the ratio of instantaneous risk premium over the instantaneous volatility) and the risky asset return is

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<sup>2</sup> Merton (1980) pointed out that the risk premium is crucial to portfolio choice and is difficult to estimate empirically. He proposed that the risk premium might depend on powers of volatility, with the power being either 0, 1, or 2.

<sup>3</sup> Because the interest rate is constant, one can also get explicit solutions for the more general hyperbolic risk averse (HARA) utility.

negative,<sup>4</sup> a conservative agent will hold more risky assets over a longer investment horizon, whereas an aggressive agent will hold fewer risky assets.<sup>5</sup> On the other hand, when the instantaneous correlation between the ISR and the risky asset return is positive, a conservative agent will hold fewer risky assets whereas an aggressive agent will hold more.

The most striking feature of the optimal portfolio weight is its dependence on risk aversion. If the instantaneous correlation between the ISR and the risky asset return is negative, the graph of the portfolio weight as a function of risk aversion may display a humped shape—starting from negative infinity at a non-zero minimum risk aversion, increasing to a maximum at a finite value of risk aversion, and then decreasing to zero as the risk aversion tends to infinity. In other words, agents who are close to being risk-neutral will *short* an infinite amount of risky assets, whereas some other agents will short a finite amount, and still others will hold a positive amount. This is so despite the fact that the risk premium is strictly positive! This finding is in direct contrast with the risk aversion dependence of the static portfolio weight, which is positive and decreasing for all levels of risk aversion if the risk premium is positive. For all other cases, the optimal dynamic portfolio weight decreases with the risk aversion dependence; however, the optimal portfolio weight can be infinitely large and positive for a nonzero minimum risk aversion.

Although many researchers have studied dynamic portfolio choice since Mossin (1968), Samuelson (1969), and Merton (1969, 1971), it seems that few have attempted to understand

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<sup>4</sup> In other studies, the terms instantaneous market price of risk (IMPR) and instantaneous Sharpe ratio (ISR) are used interchangeably. In this paper,  $IMPR = \frac{ISR}{\sqrt{V_t}}$ .

<sup>5</sup> Throughout the paper, I refer to agents with a constant relative risk aversion (CRRA) larger than 1 as “conservative” agents and those with a constant relative risk aversion less than 1 but larger than 0 as “aggressive” agents. Note that aggressive agents are nevertheless risk averse.

the qualitative differences between dynamic and static choice that arise from rebalancing. One notable exception is the work of Kim and Omberg (1996), who studied the dynamic choice problem in which the risk premium is time-varying. However, Kim and Omberg's counter-intuitive results might be attributed to the negative risk premium in their model. It is well known that risk averse agents will short risky assets and that the portfolio weight of a risky asset increases with risk aversion in static choice problems if the risk premium of the risky asset is negative. But, in the stochastic volatility model studied in this paper, the risk premium is always positive and the short rate is constant, so the counter-intuitive result can come only from the effects of dynamic rebalancing.

Recently, there have been several studies on portfolio choice problems in which the risky asset return displays stochastic volatility, including Liu (1998) and Chacko and Viceira (1999). Liu assumes that the risk premium is proportional to the variance of the risky asset return whereas Chacko and Viceira (1999) assume that the risk premium is independent of the variance. In this paper, I consider a more general risk premium. More important, neither Liu (1998) nor Chacko and Viceira (1999) studies the differences of the dynamic and static choices, which is the focus of this paper. Longstaff (2000) studies the effect of liquidity on portfolio choice. In his paper, the risky asset return displays stochastic volatility and the agent has logarithmic utility. Liu, Longstaff, and Pan (2000) study the effects of jumps in both stock return and volatility. Finally, Ang and Bekaert (1998) and Das and Uppal (2000) study portfolio problems that can be viewed as alternative specifications of the stochastic volatility.

The remainder of this paper is organized as follows. In section 2, I specify the asset price dynamics and the utility function of the agent. In section 3, I derive the optimal portfolio

weight, study its comparative statics, and discuss its dependence on the volatility and the investment horizon. In section 4, I review two important properties of static portfolio choice. I then show that striking differences exist between dynamic portfolio choice and static portfolio choice. Section 5 gives conclusion remarks and the Appendix provides further details on my calculations.

## 2 Model

In this section, I specify the stochastic volatility model and the utility function of the agent.

### 2.1 Asset Price Dynamics

The price of the risky asset satisfies the following equation:

$$dP_t = P_t(r + \lambda V_t^{\frac{1+\beta}{2}}) + P_t\sqrt{V_t}dB_t. \quad (1)$$

To focus on stochastic volatility, I assume that the short rate  $r$  is constant.<sup>6</sup> The risk premium  $\lambda V_t^{\frac{1+\beta}{2}}$ , where  $\lambda$  and  $\beta$  are both constants, has CEV (constant elasticity of volatility or variance)).<sup>7</sup> Without loss of generality, I assume that  $\lambda \geq 0$ . This means that the risk premium is

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<sup>6</sup>Liu (1998) studies a portfolio choice problem in which both the short rate and the volatility of stock returns are stochastic.

<sup>7</sup>There is an important difference between predictability and stochastic volatility models. In the case of predictability models, the volatility is implicitly assumed to be constant while predictability specifies the risk premium. In stochastic volatility models, the focus is often on volatility, and the risk premium is often left unspecified when markets are complete and the researchers are interested only in derivative pricing. As Merton (1980) pointed out, for portfolio choice problems, the risk premium plays a critical role and it is difficult to estimate the risk



positive, which is economically sensible. The solution also applies for  $\lambda < 0$ . In this specification of the asset price dynamics, the instantaneous market price of risk IMPR is  $\lambda V^{\frac{\beta-1}{2}}$  and the instantaneous Sharpe ratio ISR is  $\lambda V^{\frac{\beta}{2}}$ . As I show later in this paper, the IMPR determines the volatility dependence of the portfolio weight whereas the ISR determines the risk aversion and horizon dependence.

## 2.2 Volatility Process and Instantaneous Sharpe Ratio

I assume an instantaneous variance process

$$V_t = X_t^{\frac{1}{\beta}} \quad (2)$$

where  $X_t$  is a square-root process,

$$dX_t = (k - KX_t)dt + \sigma\sqrt{X_t}dB_t^V. \quad (3)$$

Using Ito's lemma, one can show that the assumption for  $V_t$  is equivalent to directly assuming that the variance process  $V_t$  satisfies the following equation:

$$dV_t = \frac{1}{\beta}V_t^{1-\beta} \left\{ \left( k + \frac{\sigma^2(1-\beta)}{2\beta} - KV_t^\beta \right) dt + \sigma V_t^{\frac{\beta}{2}} dB_t^V \right\}, \beta \neq 0. \quad (4)$$

Because  $V_t$  is a deterministic function of a square-root process with the long-term mean  $\frac{k}{K}$ ,  $V_t$  is well-defined provided  $\frac{k}{K} \geq 0$ . I assume that  $\sigma > 0$  for definiteness.<sup>8</sup> I assume that the premium. He proposes three possible forms for the risk premium,  $\lambda$ ,  $\lambda V_t^{\frac{1}{2}}$ , and  $\lambda V_t$ , which are special cases of my specification ( $\beta = -1$ ,  $\beta = 0$ , and  $\beta = 1$  in equation (1) respectively). Subsequent empirical studies have yielded conflicting results on the relation between volatility and risk premium; some studies have found positive relationships, some have found negative relationships, while still others have found no relationship.

<sup>8</sup> There is no loss of generality because  $-\sigma$  should lead to the same process for the risky asset and volatility when one changes  $B_t^V$  to  $-B_t^V$  and  $\rho$  to  $-\rho$ .

correlation between the Brownian motions  $dB_t$  and  $dB_t^V$  is a constant value  $\rho$ . Note that the instantaneous correlation between asset return shocks  $\sqrt{V_t}dB_t$  and variance shocks  $\frac{\sigma}{\beta}V_t^{1-\beta/2}dB_t^V$  is  $\text{sign}(\beta)\rho$ : The instantaneous correlation between the two shocks is  $\rho$  when  $\beta > 0$  and  $-\rho$  when  $\beta < 0$ .

Using equation (2), the instantaneous Sharpe ratio can be expressed as

$$\text{ISR} = \lambda V_t^{\frac{\beta}{2}} = \lambda X^{\frac{1}{2}}.$$

Therefore, up to a proportional constant  $\lambda^2$ , the state variable  $X$  is the squared instantaneous Sharpe ratio (SISR). As noted in Liu (1998), the indirect utility function is determined by the ISR (instead of by the instantaneous risk premium or the instantaneous volatility separately). I am able to solve explicitly for the indirect utility function, and thus explicitly for the portfolio weight because of the simple specification of the SISR in equation (3). I should point out that the dynamics of the volatility process in equation (4) are not as simple as that of the SISR. Note that the instantaneous correlation between the ISR or SISR and the risky asset return is  $\rho$  and does not depend on  $\beta$ , unlike the instantaneous correlation between the volatility and the risky asset return.

A couple comments are in order: First, when  $\beta = 1$ , the return dynamics specified in equations (1) and (2) is the same as that of the Heston model (1993). Heston did not explicitly specify the risk premium of the stock return because his primary interest was option pricing. The risk premium of the form  $\lambda V_t$  was proposed by Bates (1997) and used in Bakshi, Cao, and Chen (1997). This form of risk premium was motivated by the capital asset pricing model (CAPM) and was originally suggested for any stochastic volatility model (and not just the Heston model)

by Merton (1980). The portfolio selection in the Heston model is studied in Liu (1998).

Second, when  $\beta = -1$ , the return dynamics specified by equations (1) and (2) are the same as proposed by Chacko and Viceira (1999). For those dynamics, they study the portfolio choice problem when the agent has recursive utility defined over intermediate consumption.

### 2.3 Utility of the Agent

The agent's objective is to allocate his wealth between a riskless asset with constant return  $r$  and the risky asset with dynamics specified in equations (1) and (2) to maximize his utility, which is a power function of end-of-period wealth:

$$\max_{\phi_t, 0 \leq t \leq T} E_0 \left[ \frac{W_T^{1-\gamma}}{1-\gamma} \right], \quad (5)$$

where  $W_T$  is end-of-period wealth of a self-financing trading strategy  $\phi_t$ :

$$dW_t = W_t \left( r + \lambda \phi_t V^{\frac{1+\beta}{2}} \right) dt + \phi_t \sqrt{V_t} dB_t. \quad (6)$$

The utility over end-of-period wealth (instead of over intermediate consumption) is chosen for three reasons. First, maximizing end-of-period wealth while controlling for risk is the objective of many investors, such as fund managers. Second, I would like to study the effects of the investment horizon on portfolio choice. The concept of an investment horizon is blurred when there is intermediate consumption. Third, and most important, the main focus of this paper is the effects of agents' risk aversion on portfolio choice. With power utility over intermediate consumption, it is difficult to distinguish risk aversion from elasticity of intertemporal substitution.

### 3 Optimal Dynamic Strategy

The optimization problem (5) in the previous section can be solved as was done in Liu (1998). The problem is first reduced to a Hamilton-Jacobi-Bellman (HJB) partial differential equation by using the principle of optimality, following Merton (1969, 1971). The partial differential equation is then solved by reducing it to an ordinary differential equation. This is possible because of the specification of the asset dynamics and utility function in section 2. The optimal portfolio weight is given in the following theorem:

**Theorem 1 (Optimal Portfolio Weight)** *The optimal portfolio weight is given by*

$$\phi_t^* = V_t^{\frac{\beta-1}{2}} \frac{\lambda}{\gamma} \left( 1 + \frac{1}{K - \frac{1-\gamma}{\gamma} \lambda \rho \sigma + \xi \coth(\xi \tau / 2)} \frac{(1-\gamma) \lambda \rho \sigma}{\gamma} \right) \quad (7)$$

$$= V_t^{\frac{\beta-1}{2}} \frac{\lambda(K + \xi \coth(\xi \tau / 2))}{\gamma(K + \xi \coth(\xi \tau / 2)) - (1-\gamma) \lambda \rho \sigma} \quad (8)$$

$$= V_t^{\frac{\beta-1}{2}} \frac{\lambda}{\gamma} \left( 1 + \frac{1}{K - \frac{1-\gamma}{\gamma} \lambda \rho \sigma + \eta \cot(\eta \tau / 2)} \frac{(1-\gamma) \lambda \rho \sigma}{\gamma} \right) \quad (9)$$

$$= V_t^{\frac{\beta-1}{2}} \frac{\lambda(K + \eta \cot(\eta \tau / 2))}{\gamma(K + \eta \cot(\eta \tau / 2)) - (1-\gamma) \lambda \rho \sigma}, \quad (10)$$

with  $\tau = T - t$ ,  $\xi = \sqrt{K^2 - \frac{1-\gamma}{\gamma} (2K \lambda \rho \sigma + \lambda^2 \sigma^2)}$ , and  $\eta = -i\xi$ .

The proof of this theorem is in the Appendix. In equations (7) and (9), the first term is what Merton called the “myopic component” and the second term is the “intertemporal hedging component”. When  $\gamma \geq 1$ , the parameter  $\xi$  is real for all  $\tau \geq 0$ . When  $\gamma < \gamma_{\min} \equiv 1 - \frac{K^2}{(K + \lambda \rho \sigma)^2 + \lambda^2 \sigma^2 (1 - \rho^2)}$ ,  $\xi$  is purely imaginary. In this case,  $K^2 - \frac{1-\gamma}{\gamma} (2K \lambda \rho \sigma + \lambda^2 \sigma^2) < 0$  and  $\eta = -i\xi$  is real. The above expressions are valid even when  $\xi$  or  $\eta$  are purely imaginary. The portfolio weight is given in both  $\xi$  and  $\eta$  because at least one of them is real and it is con-

venient to work with real variables. Note also that the optimal dynamic portfolio weight does not depend on the long-run mean  $\frac{k}{K}$  of the variance process  $V$ .

### 3.1 Comparative Statics

I now turn to the dependence of the portfolio weight on the various parameters that specify the asset dynamics. Because the myopic component depends on the parameters in a simple way, I focus mainly on the intertemporal hedging component.

First, as noted earlier, the portfolio weight does not depend on the parameter  $k$ , which specifies the long-term mean of the state variable. This is fairly obvious: The long-term mean does not characterize the changes in the opportunity set, and therefore does not affect the intertemporal hedging component.

The following proposition describes the dependence of the magnitude of the intertemporal hedging component on the other parameters: the mean-reversion  $K$ , the volatility of the volatility  $\sigma$ , the correlation  $\rho$ , and the parameter  $\lambda$  that describes the risk premium  $\lambda V^{\frac{1+\beta}{2}}$ .

**Proposition 1 (Comparative Statics)** *The magnitude of the intertemporal hedging component is decreasing in  $K$ , increasing in  $\sigma$  for  $\gamma < 1$ , increasing in the magnitude of  $\rho$  for  $\gamma > 1$ , and increasing in  $\lambda$  when  $(1 - \gamma)\rho > 0$ .*

The proof of this proposition is in the Appendix. Although proposition 1 is proved for the various restrictions, I believe it is also true without those restriction. The results are intuitive but not easy to prove, even with explicit solution given in equations (7-10). To the best of my knowledge, no other paper in the literature provides proof of similar results.

Note that the intertemporal hedging component has the same sign as  $(1 - \gamma)\rho$  and does not change signs as a function of the investment horizon. Thus, one needs to study only the magnitude of the intertemporal hedging component; the magnitude is larger when the dynamic hedging effect is larger.

Consider the effect of  $K$ . Because  $K$  is the mean-reverting parameter of the state variable [equation (3)], the larger the value of  $K$ , the faster  $X$  reverts back to its mean  $\frac{k}{K}$  and the faster changes in the state variable are damped out, which should lead to a smaller hedging effect. In fact, when  $K \rightarrow +\infty$  while keeping  $\frac{k}{K}$  fixed,  $X_t$  is equal to its mean  $\frac{k}{K}$  and the intertemporal hedging effect is zero. Therefore, the magnitude of the intertemporal hedging component is decreasing in  $K$ . Note that when  $K$  is negative, the state variable becomes explosive (it is no longer stationary). And the more negative  $K$  is, the more volatile the state variable is and the larger the dynamic hedging effect will be, which leads to a larger intertemporal hedging component.

On the other hand, the state variable is more volatile when  $\sigma$  is large. Therefore, the dynamic hedging effect should be large and the magnitude of the intertemporal hedging component increases with  $\sigma$ .

The sign of the intertemporal hedging component depends on the sign of the correlation  $\rho$  (as well as  $\gamma$ ). When the correlation is zero, the risky asset cannot be used to hedge the changes in the opportunity set and the intertemporal hedging component is zero. Because more changes in the opportunity set can be hedged with a larger magnitude of correlation, the magnitude of the intertemporal hedging component will be larger.

The larger the risk premium parameter  $\lambda$ , the larger the position of the risky asset in the

myopic component, and therefore the larger position to be hedged, so that the magnitude of the intertemporal hedging component is increasing in  $\lambda$ .

### 3.2 Volatility Timing

**Proposition 2 (Volatility Timing)** *The optimal portfolio weight is decreasing in volatility if IMPR is decreasing in volatility ( $\beta < 1$ ); and increasing if IMPR is increasing in volatility ( $\beta > 1$ ). The ratio of myopic component and intertemporal hedging component is independent of the volatility.*

The proof is obvious using Theorem 1. Proposition 2 implies that the agent times the volatility as long as  $\beta \neq 1$ . The intuition is clear. When the IMPR is increasing in volatility ( $\beta > 1$ ), the return for bearing risk also increases with volatility; therefore, the agent will hold more risky assets when the volatility is high. When the IMPR is decreasing in volatility ( $\beta < 1$ ), the return for bearing risk also decreases with volatility; therefore, the agent will hold less of risky asset when the volatility is high. The constant ratio of the myopic component to the intertemporal hedging component is due to the specification of the CEV risk premium and variance processes. In general, this feature of the constant ratio is not true.

Proposition 2 provides some perspective on the so-called flight-to-quality phenomenon, which refers to investors moving capital from stock markets to government bond markets when the stock markets are more volatile than usual. According to proposition 2, investors should invest less in the risky asset if the IMPR decreases with volatility, assuming everything else (including the wealth of the agent) constant. On the other hand, if the IMPR increases with

volatility, investors should do the opposite and invest more in the risky asset when the volatility is high.

### 3.3 Investment Horizon

When asset returns display stochastic volatility, the optimal portfolio weight depends on the investment horizon, contrary to the case when asset returns are distributed independently over time.

**Definition 1 (Aggressive, Logarithmic, and Conservative Agents)** *An aggressive agent is a risk averse agent with constant relative risk aversion  $\gamma$  smaller than 1; a logarithmic agent is a risk averse agent with constant relative risk aversion  $\gamma$  equal to 1; a conservative agent is a risk averse agent with constant relative risk aversion  $\gamma$  greater than 1;*

The portfolio choice of conservative agents is very different from that of aggressive agents because their utility functions are qualitatively different, as shown in figure 1.

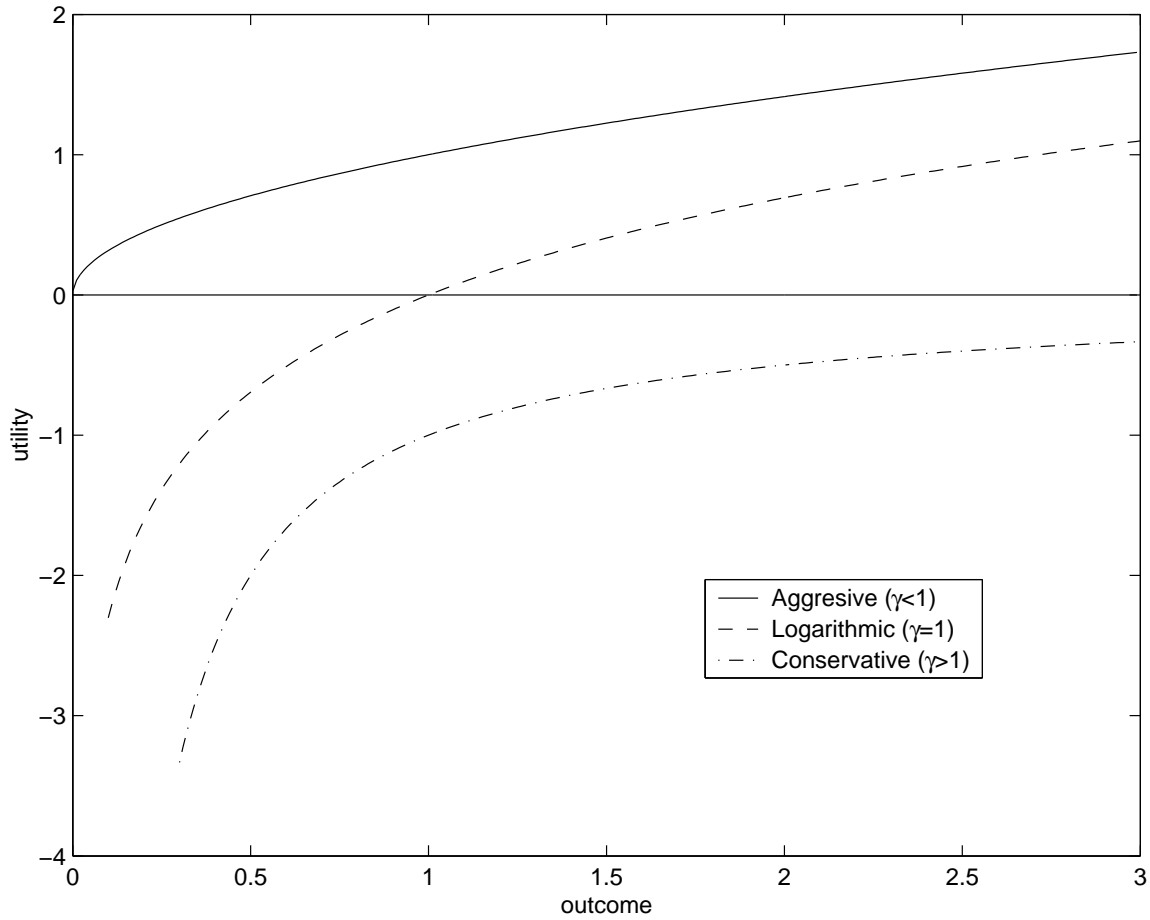
The horizon dependence of the portfolio weight is summarized by the following proposition.

**Proposition 3 (Horizon Dependence)**

*For conservative agents ( $\gamma > 1$ ), the optimal portfolio weight is smaller than the myopic component and is decreasing as a function of the investment horizon if  $\rho > 0$ ; the optimal portfolio weight is greater than the myopic component and is increasing if  $\rho < 0$ .*

*For aggressive agents ( $0 < \gamma < 1$ ), the optimal portfolio weight is greater than the myopic component and is increasing as a function of the investment horizon if  $\rho > 0$  and always goes to infinity when  $\gamma$  is close to 0; the optimal portfolio weight is smaller than the myopic component*





This figure characterizes the utility function of various agents. The utility function of an aggressive agent ( $\gamma < 1$ ) is bounded from below by zero and unbounded from above; the utility function of a logarithmic agent ( $\gamma = 1$ ) is unbounded from below and above; the utility function of a conservative agent ( $\gamma > 1$ ) is unbounded from below and bounded from above by zero.

Figure 1: Utility Functions of Various Agents

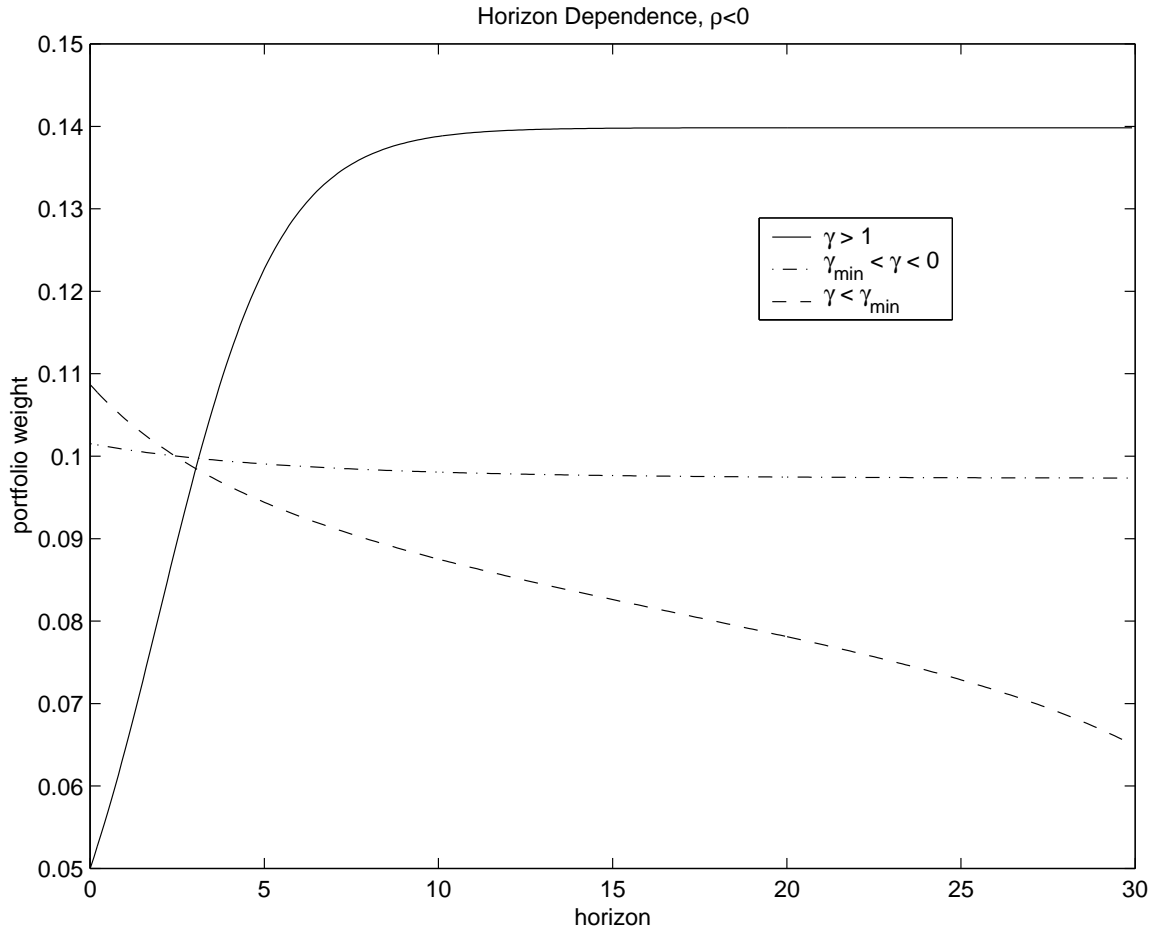
and is decreasing if  $\rho < 0$  and could go to positive infinity at a finite horizon if  $\gamma$  is close to 0 and  $2K\rho + \lambda\sigma > 0$ .

*In all cases, the effect (that is, the magnitude) of intertemporal hedging demand is increasing with the investment horizon.*

Using theorem 1 the proof is straightforward and is given in the Appendix. It is not an accident that the parameter  $\beta$  has no influence on the investment horizon dependence of the optimal portfolio weight. The dependence on the investment horizon is determined by the intertemporal hedging component, and thus by the indirect utility function, which in turn is determined by the ISR or SISR, whose dynamics [as specified in equation (3)] does not depend on  $\beta$ . Figures 2 and 3 graph the cases of  $(\beta = 1, \rho < 0)$  and  $(\beta = 1, \rho > 0)$ , respectively.

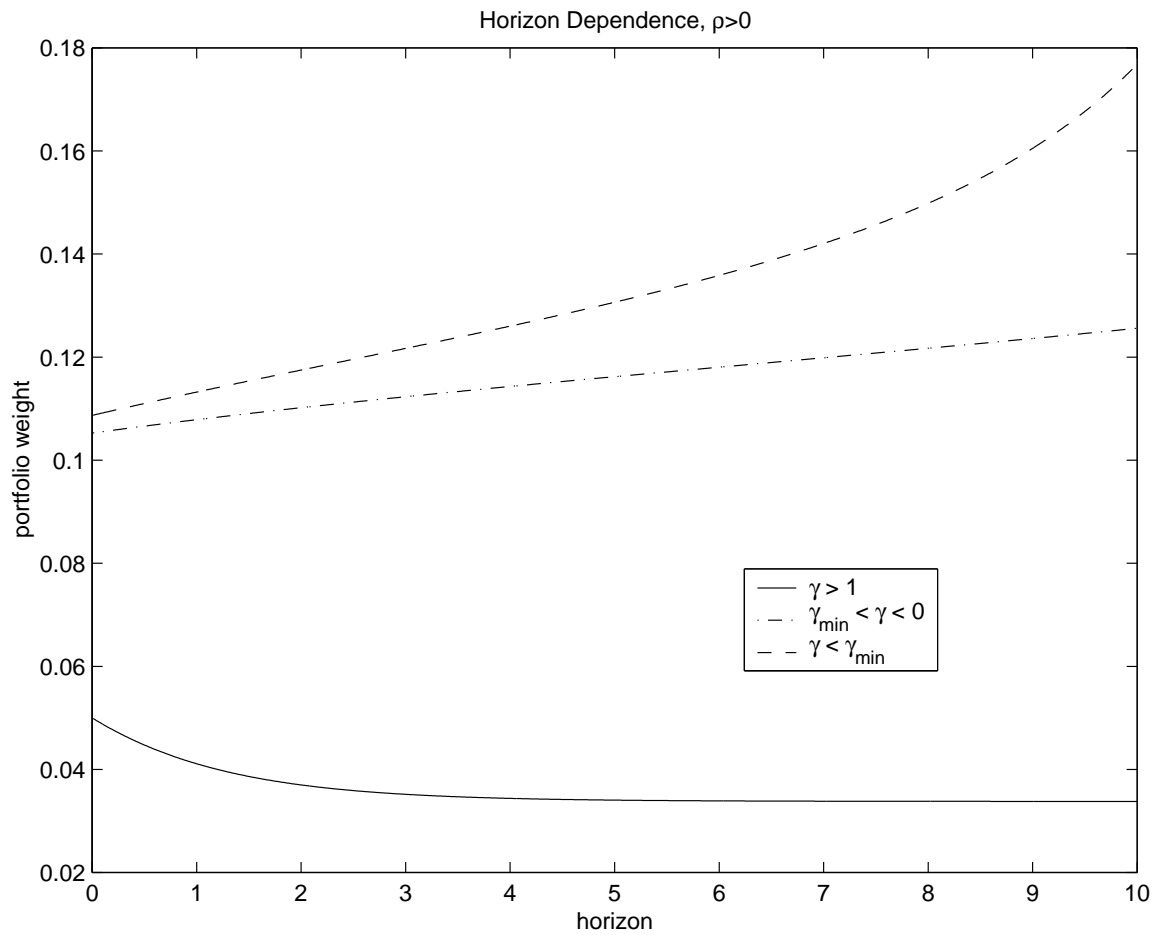
To understand proposition 3 intuitively, consider the case of  $\rho < 0$ . In this case, the ISR and the risky asset return are negatively correlated. Therefore, a low return is likely to be accompanied by a high ISR, that is, a better future risk-return trade-off, as if the returns of the risky asset are mean-reverting.

For a conservative agent, the utility function is bounded from above by zero and is unbounded from below, as shown in figure 1. Therefore the agent suffers huge losses in utility from large losses in returns but does not have large gains in utility from large gains in returns, and the agent will focus on avoiding losses or will prefer an asset with an mean-reverting return. As pointed out earlier, in the case  $\rho < 0$ , a low return is likely to be accompanied by a better risk-return trade-off, so that the agent is less likely to incur losses in holdings of the risky asset if  $\rho < 0$  than if  $\rho = 0$ . Therefore, the agent will hold more risky assets if  $\rho < 0$  than if  $\rho = 0$ . Since the portfolio weight when  $\rho = 0$  is the myopic portfolio weight, this means



This graph summarizes the possible investment horizon dependence of the optimal portfolio weight when  $\beta = 1$  and the correlation coefficient  $\rho < 0$ . For a conservative agent ( $\gamma > 1$ ), the portfolio weight increases monotonically to a finite value; for an aggressive agent ( $\gamma < 1$ ) with  $\gamma \geq \gamma_{\min}$ , the portfolio weight decreases monotonically to a finite value; for an aggressive agent ( $\gamma < 1$ ) with  $\gamma < \gamma_{\min}$ , the portfolio weight decreases monotonically and reaches  $-\infty$  at a finite horizon.

Figure 2: The Optimal Portfolio Weight as a Function of Investment Horizon



This graph summarizes the possible investment horizon dependence of the optimal portfolio weight when  $\beta = 1$  and the correlation coefficient  $\rho > 0$ . For a conservative agent ( $\gamma > 1$ ), the portfolio weight decreases monotonically to a finite value; for an aggressive agent ( $\gamma < 1$ ) with  $\gamma \geq \gamma_{\min}$ , the portfolio weight increases monotonically to a finite value; for an aggressive agent ( $\gamma < 1$ ) with  $\gamma < \gamma_{\min}$ , the portfolio weight increases monotonically and reaches  $+\infty$  at a finite horizon.

Figure 3: The Optimal Portfolio Weight as a Function of Investment Horizon

that the dynamic portfolio weight will be larger than the myopic portfolio weight. Furthermore, the longer the horizon, the bigger the effect. Therefore, the portfolio weight for a conservative agent increases with the investment horizon.

For an aggressive agent on the other hand, the utility function is bounded from below by zero and is unbounded from above, as shown in figure 1. Therefore the agent does not suffer huge losses in utility from large losses in returns but does enjoy large gains in utility from large gains in returns; thus, an aggressive agent will focus more on gains or will prefer an asset with momentum in its return. Because a high return is less likely to be accompanied by a better risk-return trade-off in the case  $\rho < 0$ , the risky asset is less likely to compile large gains if  $\rho < 0$  than if  $\rho = 0$ . Therefore, the conservative agent will hold less of the risky asset if  $\rho < 0$  than if  $\rho = 0$ . Hence, the dynamic asset will be smaller than the myopic component. Furthermore, the longer the horizon, the bigger the effect. Therefore, the portfolio weight for an aggressive agent decreases with the investment horizon.

Finally, for a logarithmic agent, the utility is unbounded from below as well as from above and the agent is indifferent between  $\rho < 0$  and  $\rho = 0$ ; therefore, for a logarithmic agent, the dynamic portfolio weight is equal to the myopic component.

Here are some special cases in which  $\beta = 1$ . At a short horizon ( $\tau \rightarrow 0$ ), the portfolio weight is given by

$$\phi = \frac{\lambda}{\gamma} \left( 1 - \frac{1}{2} \tau \rho \sigma \left( 1 - \frac{1}{\gamma} \right) \lambda \right). \quad (11)$$

In particular, when  $\tau = 0$ , one gets  $\phi = \frac{\lambda}{\gamma}$ , which is the myopic component. For  $\gamma \geq \gamma_{\min}$ , the parameter  $\xi$  in equation (7) is real for all  $\tau$  and the portfolio weight is finite for all horizons. For

a long horizon ( $\tau = \infty$ ), the portfolio weight is given by

$$\phi = \frac{1}{1 + \gamma \left( \frac{1}{\rho^2} - 1 \right)} \left( \frac{\lambda}{\rho^2} + \frac{K - \sqrt{K^2 + \left(1 - \frac{1}{\gamma}\right) (2K\rho + \sigma\lambda)\sigma\lambda}}{\rho\sigma} \right). \quad (12)$$

For  $\gamma < \gamma_{\min}$ , the parameter  $\xi$  in equation (7) is imaginary and  $\eta$  in equation (9) is real. The function  $\cot(\eta\tau/2)$  in equation (9) varies monotonically from  $+\infty$  to  $-\infty$  when  $\tau$  changes from 0 to  $2\pi/\eta$ ; therefore, the denominator in equation (10) approaches 0 when  $\tau \rightarrow \tau_{\max} \equiv \frac{2}{\eta} \arctan \left( -\frac{\eta}{K - \frac{1-\gamma}{\gamma} \lambda \rho \sigma} \right)$ .<sup>9</sup> In this case, the portfolio weight reaches infinity at finite  $\tau$ , and the agent will take an infinite position (short if  $\rho < 0$  or long if  $\rho > 0$ ) in the risky asset.

When  $\beta = 1$ , the risk premium  $\lambda V_t$  is proportional to the variance  $V_t$ , and the result in proposition 3 reduces to the result of Liu (1998). When  $\beta = -1$ , the risk premium  $\lambda V_t$  is independent of the variance  $V_t$ , and the result in proposition 3 is similar to the result of Chacko and Viceira (1999). The portfolio behavior is qualitatively different in the two papers because the ISR is increasing with variance in Liu whereas it is decreasing in Chacko and Viceira.

## 4 Portfolio Choice and Risk Aversion

In this section, I show that the dependence of the dynamic portfolio weight is qualitatively different from that of static portfolio weights. I first state two fundamental theorems on static

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<sup>9</sup> The inverse tangent function  $\arctan(x)$  is defined to be valued between 0 and  $\pi$ ; this is different from the standard definition, which is valued between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  and is denoted as  $\text{atan}(x)$  in matlab. The relation between  $\arctan(x)$  defined here and  $\text{atan}(x)$  is  $\arctan(x) = \text{atan}(x) + \pi\theta(-x)$ , where  $\theta(x) = 1$  if  $x > 0$  and  $\theta(x) = 0$  if  $x < 0$ .

choice; then I show that dynamic choice violates both theorems. Finally, I argue that the violation is the consequence of rebalancing.

## 4.1 Static Portfolio Choice and Risk Aversion

Two fundamental theorems provide basic intuitions on static choice. I refer to them as the “participation theorem” and the “calibration theorem”, respectively.

**Theorem 2 (Participation Theorem)** *If the risk premium  $E[r_e]$  of an asset is positive, then a risk-averse agent will hold a positive amount of the asset.*

The theorem and the proof are given in Huang and Litzenberger (1988). The proof is simple. Suppose  $E[r_e] > 0$ . The marginal utility at  $\phi = 0$  is  $E[U'(r_f + \phi r_e)r_e]|_{\phi=0} = U'(r_f)E[r_e] > 0$ . This implies that  $\phi^* > 0$  because the utility function  $U$  is concave in wealth and therefore the expected utility function is concave in  $\phi$ .

According to the participation theorem, agents should always take advantage of the excess returns of risky assets. As applied to the U.S. stock markets, this theorem implies that all investors should hold a positive amount of stock because the equity risk premium is positive. The fact that a significant proportion of the U.S. population does not hold stocks is the so-called non-participation puzzle, which is being actively studied in the literature [see for example, Mankiw and Zeldes (1991), Heaton and Lucas (1997), and Barsak and Cuoco (1998)].

**Theorem 3 (Calibration Theorem)** *If the risk premium  $E[r_e]$  of an asset is positive, the optimal portfolio weight decreases with the risk aversion of the agent.*

The calibration theorem is a variation of Arrow's theorem on insurance premia. The proof is given in the Appendix. The theorem is very intuitive: It implies that a less risk-averse agent holds more risky assets. This theorem has both theoretical as well as practical importance. The optimal portfolio weight as well as many other economic variables such as the pricing kernel in the consumption-based asset pricing model, depends on the risk aversion  $\gamma$  of agents. But the risk aversion of agents is not easily measured. One common method to calibrate  $\gamma$ , which is used in academic studies as well as in practice, is to infer it from the agents' stock portfolio weight. The calibration theorem provides the theoretical foundation for this method.

Both the participation and the calibration theorems provide basic intuition for expected utility theory as a theory of risk and have economic importance. However, I show that both theorems are violated in dynamic choice models.

I should emphasize that both theorems require that the risk premium is positive. If the risk premium is negative, then *all* risk averse agents will short the risky asset and holdings of the risky asset *increase* with risk aversion.

## 4.2 Dynamic Portfolio Choice and Risk Aversion

The dependence of the dynamic portfolio weight on risk aversion is summarized in the following proposition.

### **Proposition 4 (Risk Aversion Dependence)**

*The optimal portfolio weight decreases in  $\gamma$  and (if  $\tau > 0$ ) always reaches  $+\infty$  at a non-zero risk aversion when the correlation  $\rho > 0$ .*

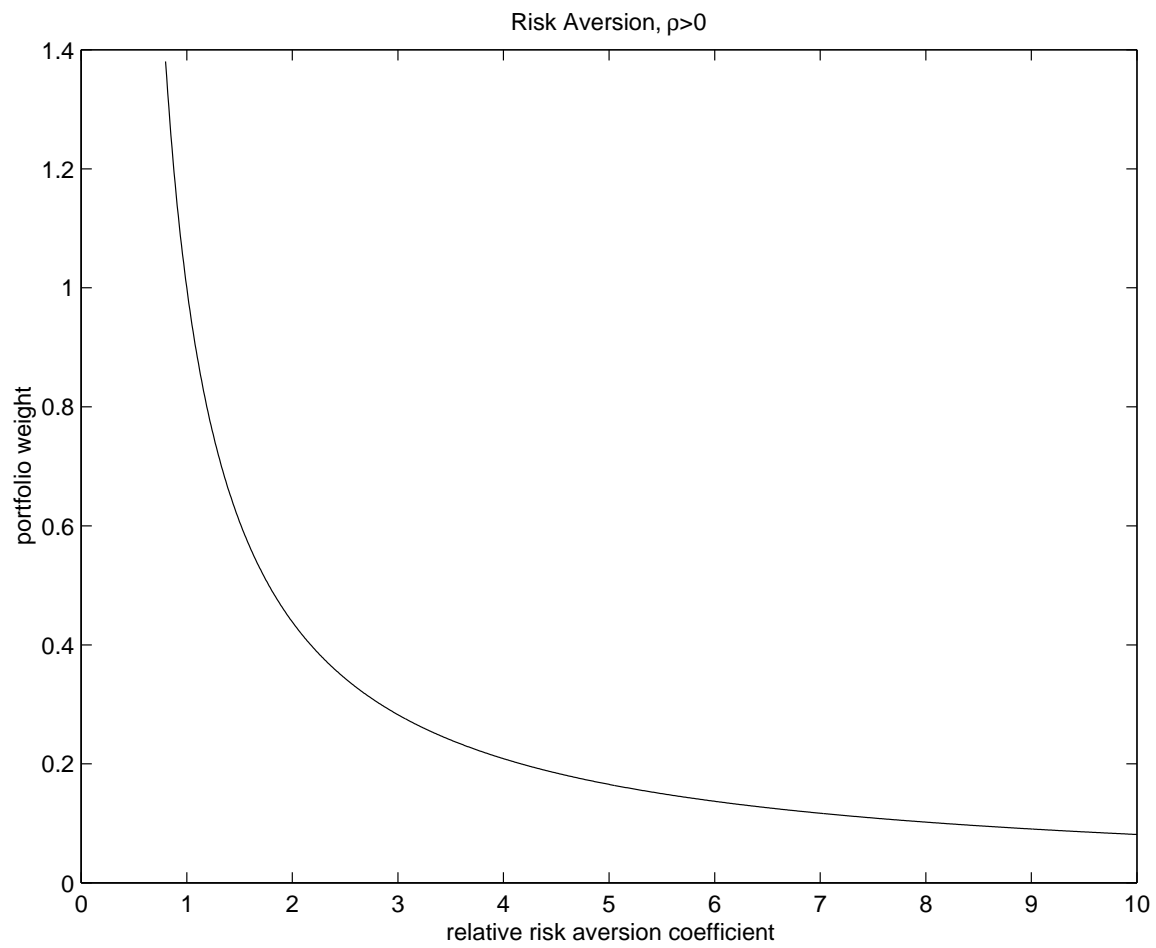


*The optimal portfolio weight decreases in  $\gamma$  and reaches  $+\infty$  at zero risk aversion when the correlation  $\rho < 0$  and  $2K\rho + \lambda\sigma \leq 0$ ; the graph of the portfolio weight as a function of risk aversion displays a hump shape (if  $\tau > 0$ ), starting from  $-\infty$  at a non-zero risk aversion, increasing for small  $\gamma$ , and decreasing for large  $\gamma$ , when the correlation  $\rho < 0$  and  $2K\rho + \lambda\sigma > 0$ . The portfolio weight is non-monotonic in  $\gamma$  in the last case.*

The proof is given in the appendix. Figures 4 and 5 graph the portfolio weight as a function of risk aversion for  $\beta = 1$ . Note the hump shape of the solid line in figure 5, which demonstrates the violation of the participation and calibration theorems.

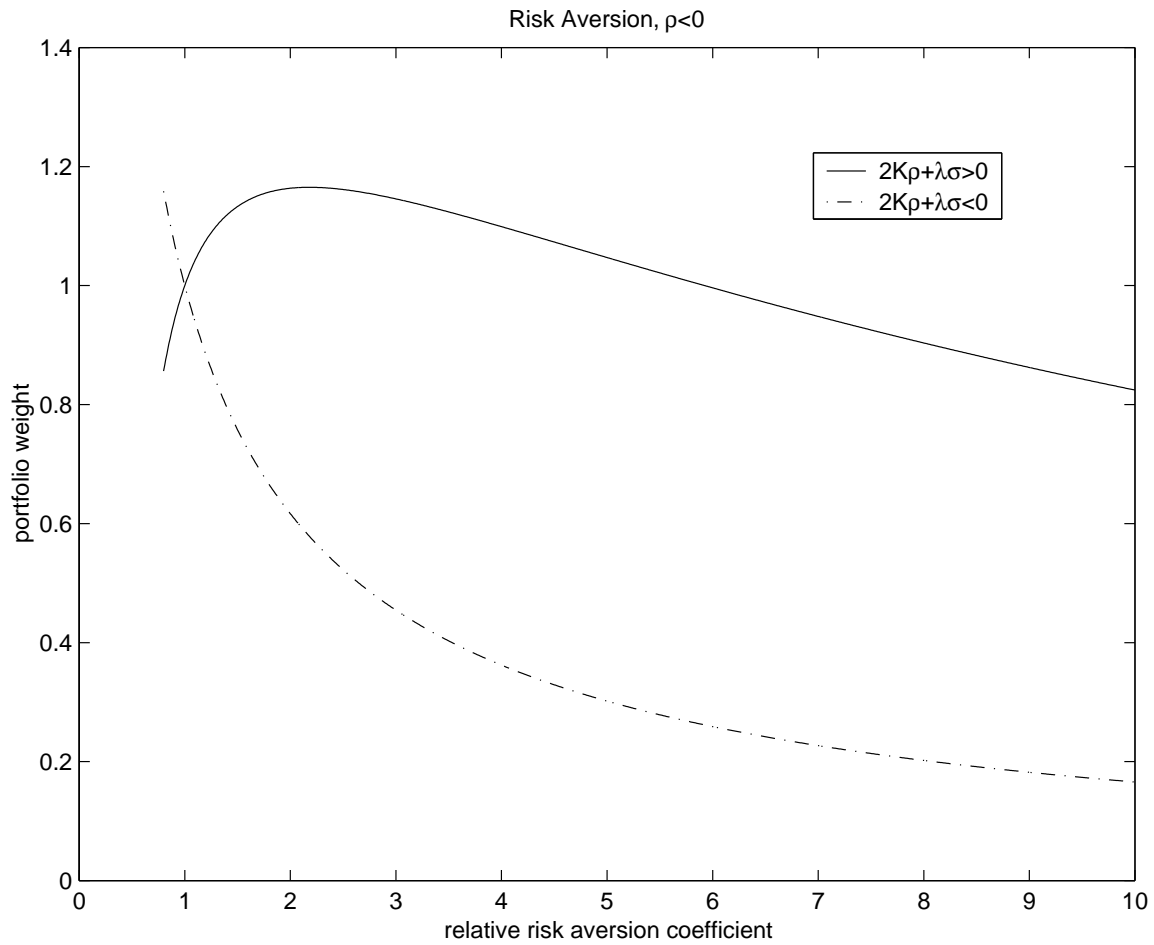
The participation and calibration theorems are violated when  $\rho < 0$  and  $2K\rho + \lambda\sigma > 0$ . To understand why this might happen, again consider the case of  $\rho < 0$ . Because the myopic component is the portfolio weight of a static problem, it decreases with risk aversion, which is the result of the calibration theorem. However, the effect of increasing risk aversion on the intertemporal hedging component [the second term in equation (7) or (8)] is not clear. On the one hand, the smaller myopic amount implies a smaller amount to be hedged and therefore a smaller intertemporal hedging component; on the other hand, a more conservative agent values more greatly the mean-reversion effect of the risky asset, which will lead to a larger intertemporal hedging component. If the latter effect is big enough, the portfolio weight will increase with risk aversion and a risk-averse agent may want to short the risky asset with a positive risk premium.

The reason that an aggressive agent may short the risky asset can also be understood as follows. Because an aggressive agent prefers momentum returns but the return of the risky asset is mean-reverting with  $\rho < 0$ , the aggressive agent can create momentum returns from



This graph shows that the optimal portfolio weight is always decreasing in risk aversion when  $\beta = 1$  and  $\rho > 0$ .

Figure 4: The Optimal Portfolio Weight as a Function of Risk Aversion



This graph summarizes the possible risk aversion dependence of the optimal portfolio weight when  $\beta = 1$  and  $\rho < 0$  and when the investment horizon  $\tau$  is finite but non-zero. The solid line shows that the optimal portfolio weight is increasing for low risk aversion  $\gamma$  if the dynamic hedging effect is large ( $2K\rho + \lambda\sigma > 0$ ); the dashed-dotted line shows that the optimal portfolio weight is always decreasing if the dynamic hedging effect is small ( $2K\rho + \lambda\sigma < 0$ ).

Figure 5: The Optimal Portfolio Weight as a Function of Risk Aversion

mean-reverting returns. Note that when the investment horizon is short the dynamic portfolio choice problem reduces to a static portfolio choice problem. Thus, according to the participation theorem, the aggressive agent will hold a positive amount of the risky asset in the future when the investment horizon becomes short. By shorting the risky assets now and holding them later, the agent effectively creates a trading strategy with momentum returns (which is what an aggressive agent prefers) from an asset with mean-reverting returns!

Another interesting feature of dynamic choice is that a risk averse agent can hold an infinite amount of the risky asset. I believe that this is a norm of dynamic choice rather than an exception. In static choice problems, a risk-averse agent will hold a finite amount of a risky asset. This is exactly Harry Markowitz's insight on static portfolio choice: Even though a stock might have higher return than T-bills, the stock is risky and because of this risk-return trade-off an risk-averse investor will hold a finite amount of stock. However, in the dynamic setting, the ability to exploit returns intertemporally gives agents an extra dimension of opportunity. Furthermore, there may not be a counterbalancing force for such exploits and the agent may hold an infinite amount of the risky asset when the opportunity is good enough.

One question still remains: How does dynamic choice evade the participation and calibration theorems of static choice? After all, the dynamic choice problems are usually reduced to a static problem by the principle of optimality. The key is that the distribution of the risky asset in the reduced static choice problem is different from the true distribution, and the reduced distribution depends on risk aversion. It is this dependence that invalidates the assumption of the two theorems.

The violation of the participation theorem means that investors may not always want to invest in the stock market and this may help to explain market non-participation. Because the risk aversion of an economic agent is arguably difficult to measure, stock holdings are often used as a proxy for risk aversion. But the violation of the calibration theorems implies that stock holdings may not be a good proxy. My results suggest that one should use stock holdings with short investment horizons instead of long investment horizons, because in the limit of the short investment horizon dynamic portfolio choice problems reduce to static portfolio choice problems for which the calibration theorem holds.

Kim and Omberg (1996) also find that a risk averse agent may short a risky asset and the portfolio weight of the risky asset may increase with risk aversion. However, the risk premium in their paper and in other recent papers on dynamic choice problems with predictability [Brennan, Schwartz, and Lagnado (1997); Barberis (1999); Brennan and Xia (1999); Campbell and Viceira (1996)] is an AR(1) random process and thus can be negative. As pointed out earlier, an agent will short a risky asset and his holdings of risky assets may increase with risk aversion if the risk premium is negative. So, it is at least not conclusive from these studies that the violation of the two theorems is the result of dynamic rebalancing. On the other hand, the risk premium considered in this paper,  $\lambda V^{\frac{1+\beta}{2}}$  [equation (1)], is always positive, so that the violation can only be the result of dynamic choice. Thus, fundamental differences exist between dynamic and static portfolio choice.

## 5 Conclusion

This paper provides one of the few explicitly solved dynamic portfolio choice problems. I study dynamic portfolio choice between a riskless and a risky asset. The optimal portfolio weight is derived in closed form for the risky asset whose return displays stochastic volatility. Various results on comparative statics are proved; in all cases, the magnitude of the dynamic hedging effect increases with changes in parameters that lead to a more volatile opportunity set. To the best of my knowledge, no other papers give proof of similar results. The magnitude of the optimal portfolio weight increases with the investment horizon.

The optimal portfolio weight decreases with volatility, a phenomenon called “flight-to-quality” when applied to stock markets, if the IMPR decreases with volatility. However, the optimal portfolio weight might increase with volatility, if the IMPR increases with volatility. Therefore, it is important to investigate empirically for the U.S. stock market whether IMPR decreases with volatility.

I show that a risk-averse agent may short a risky asset with a positive risk premium and a more risk averse agent may hold more of risky assets. Both results go against the basic intuitions on static portfolio choice and are a consequence of agents rebalancing optimally. One implication of these results is that stock holdings over short horizons (instead of over long horizons) should be used as a proxy for risk aversion.

In static portfolio choice problems, the volatility of risky asset returns prevents agents from taking an infinite position in risky assets, which is the insight of Harry Markowitz. However, in a dynamic setting, the opportunity to exploit the intertemporal relation of asset returns can

be so great that volatility alone might not be the counterbalancing force to prevent a risk averse agent from holding an infinite amount of risky assets. The optimal dynamic portfolio weight often becomes infinite when the investment horizon is long enough. In practice one does not observe investors with an infinite position on stocks. Presumably, this may be due to transaction costs, among many other reasons.

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## 6 Appendix

### 6.1 Proof of Theorem 1

The price process can be expressed in terms of the state variable  $X_t$ :

$$dP_t = P_t(r + \lambda X_t^{\frac{1}{2\beta} + \frac{1}{2}}) + P_t X_t^{\frac{1}{2\beta}} dB_t. \quad (13)$$

The wealth process  $W_t$  satisfies the following equation:

$$dW_t = W_t(r + \phi_t \lambda X_t^{\frac{1}{2\beta} + \frac{1}{2}}) dt + W_t \phi_t X_t^{\frac{1}{2\beta}} dB_t.$$

The derived utility function is the utility of the agent when the optimal allocation  $\phi_t^*$  is followed

$$J(W, V, t) \equiv E_0 \left[ \frac{W_T^{*1-\gamma}}{1-\gamma} \right].$$

The HJB equation implies that  $J$  satisfies the following equation,

$$\begin{aligned} \max_{\phi} \left[ j + \frac{1}{2} W^2 \phi^2 X^{\frac{1}{\beta}} J_{WW} + W [r + \phi \lambda X^{\frac{1}{2\beta} + \frac{1}{2}}] J_W \right. \\ \left. + W \phi \sigma \rho X^{\frac{1}{2\beta} + \frac{1}{2}} J_{WX} + \frac{1}{2} \sigma^2 X J_{XX} + (k - KX) J_X \right] = 0, \\ J(T, W, X) = \frac{W^{1-\gamma}}{1-\gamma}, \end{aligned} \quad (14)$$

where  $j$ ,  $J_W$ , and  $J_X$  denote the derivatives of  $J$  with respect to  $t$ ,  $W$ , and  $X$ , respectively.

Substituting the first order condition

$$\phi^* = -\frac{J_W}{W J_{WW}} X^{-\frac{1}{2\beta} + \frac{1}{2}} \left( \lambda + \sigma \rho \frac{\partial \ln J_W}{\partial X} \right)$$

into the above equation, one obtains

$$j - \frac{1}{2} X \frac{J_W^2}{J_{WW}} \left( \lambda + \sigma \rho \frac{\partial \ln J_W}{\partial X} \right)^2 + r W J_W + \frac{1}{2} \sigma^2 X J_{XX} + (k - KX) J_X = 0.$$

Guessing that the value function  $J$  has the form

$$J(W, X, t) = \frac{W^{1-\gamma}}{1-\gamma} e^{\gamma(c(t)+d(t)X)},$$

the above partial differential equation reduces to

$$\dot{c} + \dot{d}X + \frac{1-\gamma}{2\gamma^2}X(\lambda + \gamma\sigma\rho d(t))^2 + \frac{1-\gamma}{\gamma}r + \frac{\gamma}{2}\sigma^2Xd^2 + (k - KX)d = 0.$$

The functions  $c(t)$  and  $d(t)$  satisfy the following ordinary differential equation (ODE):

$$\begin{aligned} \dot{c} + kd + \frac{1-\gamma}{\gamma}r &= 0, \\ \dot{d} + \left(-K + \frac{1-\gamma}{\gamma}\lambda\sigma\rho\right)d + \frac{\sigma^2}{2}(1 + (\gamma-1)(1-\rho^2))d^2 + \frac{1-\gamma}{2\gamma^2}\lambda^2 &= 0, \\ c(T) = d(T) &= 0. \end{aligned}$$

Solving the above ODE produces the following:

$$\begin{aligned} c &= \frac{2k}{\sigma^2(\rho^2 + \gamma(1-\rho^2))} \ln \left( \frac{2\tilde{\xi}e^{(\tilde{\xi}+\tilde{K})\tau/2}}{((K - (1-\gamma)/\gamma\lambda\rho\sigma) + \tilde{\xi})(\exp(\tilde{\xi}\tau) - 1) + 2\tilde{\xi}} \right) + \frac{1-\gamma}{\gamma}r\tau \\ d &= -\frac{2(\exp(\tilde{\xi}\tau) - 1)}{((K - (1-\gamma)/\gamma\lambda\rho\sigma) + \tilde{\xi})(\exp(\tilde{\xi}\tau) - 1) + 2\tilde{\xi}} \delta \end{aligned} \quad (15)$$

with  $\tau = T - t$ ,  $\delta = -\frac{1-\gamma}{2\gamma^2}\lambda^2$ , and

$$\xi = \sqrt{(K - (1-\gamma)/\gamma\lambda\rho\sigma)^2 + 2\delta(\rho^2 + \gamma(1-\rho^2))\sigma^2} = \sqrt{K^2 - \frac{1-\gamma}{\gamma}(2K\lambda\rho\sigma + \lambda^2\sigma^2)}.$$

See Liu (1998) for further calculation details. The optimal weight  $\phi^*$  is given by

$$\begin{aligned} \phi_t^* &= X_t^{-\frac{1}{2\beta} + \frac{1}{2}} \left( \frac{1}{\gamma}\lambda + \rho\sigma d(t) \right) \\ &= V_t^{-\frac{1}{2} + \frac{\beta}{2}} \left( \frac{1}{\gamma}\lambda + \rho\sigma d(t) \right). \end{aligned} \quad (16)$$

Note that the function  $c(t)$  is not used in  $\phi^*$ . When  $\gamma \geq 1$ , the parameter  $\xi$  is real and it is easy to verify that the functions  $c(t)$  and  $d(t)$  are well defined over  $[0, T]$ . When  $\gamma < 1$ , it is possible that  $\xi$  will be imaginary. In this case,  $d(t)$  is still real but becomes unbounded for finite  $t < T$ . When  $(K - (1 - \gamma)/\gamma\lambda\rho\sigma)^2 < -2\delta(\rho^2 + \gamma(1 - \rho^2))\sigma^2$ ,  $\xi = i\eta$  is purely imaginary, and therefore  $\eta$  is real. In this case, one can easily show that

$$d = -\frac{2}{K - \frac{1-\gamma}{\gamma}\lambda\rho\sigma + \eta \cot(\eta\tau/2)} \delta. \quad (17)$$

One still needs to verify that the above solution of the HJB equation is the optimal solution to the original problem. For  $\gamma \leq 1$ , one can use the positivity of  $J(W_t)$  to show that

$$M_t = J(W_0, X_0, 0) + \int_0^t J_W W_s \phi_s X_s^{\frac{1}{2\beta}} dB_s + J_X \sigma \sqrt{X_s} dB_s^v$$

is a supermartingale. We can then use the method of Duffie [(1996), p. 200], to establish that  $J(W_0, V_0, 0)$  is the upper bound for utility of all admissible trading strategies. For  $\gamma > 1$ , the previous method does not apply and the proof of  $J(W_0, V_0, 0)$  being the upper bound might be much more involved.

Given that  $J(W_0, V_0, 0)$  is the upper bound, and the additional technical condition that ensures that  $J(W_t^*, V_t, t)$  is a martingale, one can then prove that  $\phi_t^*$  is the optimal portfolio weight, following Duffie [(1996), p. 200].

## 6.2 Proof of Theorem 3

The proof is a variation of the proof of Arrow's theorem on insurance premia. Suppose that the utility function  $U$  is more concave than the utility function  $V$  and both are increasing. There

then exists a concave function  $G$ , such that  $U = G(V)$ . Suppose the optimal portfolio weight for  $V$  is  $\phi^*$ , and the risk premium is positive, so that  $\phi^* > 0$ . Then  $E[V'(r_f + \phi^* r_e)r_e] = 0$ .

Therefore:

$$\begin{aligned}
E[U'(r_f + \phi^* r_e)r_e] &= E[G'(r_f + \phi^* r_e)V'(r_f + \phi^* r_e)r_e] \\
&= E[(G'(r_f + \phi^* r_e) - G'(r_f))V'(r_f + \phi^* r_e)r_e] \\
&= E[(G'(r_f + \phi^* r_e) - G'(r_f))V'(r_f + \phi^* r_e)r_e 1_{\{r_e \geq 0\}}] \\
&\quad + E[(G'(r_f + \phi^* r_e) - G'(r_f))V'(r_f + \phi^* r_e)r_e 1_{\{r_e < 0\}}] \\
&\geq E[(G'(r_f) - G'(r_f))V'(r_f + \phi^* r_e)r_e 1_{\{r_e \geq 0\}}] \\
&\quad + E[(G'(r_f) - G'(r_f))V'(r_f + \phi^* r_e)r_e 1_{\{r_e < 0\}}] = 0.
\end{aligned}$$

This implies that the optimal portfolio weight for  $U$  is smaller than  $\phi^*$ .

For the case of CRRA preferences, I explicitly show that the derivative of the optimal portfolio weight with respect to the relative risk aversion coefficient  $\gamma$  is negative. The first order condition for  $\phi^*$  is

$$E[(r_f + \phi^* r_e)^{-\gamma} r_e] = 0.$$

Taking the derivative with respect to  $\gamma$  of the above equation, one gets

$$\begin{aligned}
\frac{\partial \phi^*}{\partial \gamma} &= -\frac{E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(r_f + \phi^* r_e)]}{\gamma E[(r_f + \phi^* r_e)^{-\gamma-1} r_e^2]} \\
&= -\frac{E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(1 + \phi^* \frac{r_e}{r_f})]}{\gamma E[(r_f + \phi^* r_e)^{-\gamma-1} r_e^2]} \\
&= -\frac{\text{cov}((r_f + \phi^* r_e)^{-\gamma} r_e, \ln(1 + \phi^* \frac{r_e}{r_f}))}{\gamma E[(r_f + \phi^* r_e)^{-\gamma-1} r_e^2]},
\end{aligned}$$

where I use the first order condition for the second and third equality. Because  $\ln(r_f + \phi^* r_e)$  is

strictly increasing if  $\phi^* > 0$ , it follows that

$$\begin{aligned}
& \mathbb{E}\left[(r_f + \phi^* r_e)^{-\gamma} r_e \ln\left(1 + \phi^* \frac{r_e}{r_f}\right)\right] \\
&= \mathbb{E}\left[(r_f + \phi^* r_e)^{-\gamma} r_e \ln\left(1 + \phi^* \frac{r_e}{r_f}\right) 1_{\{r_e \geq 0\}}\right] + \mathbb{E}\left[(r_f + \phi^* r_e)^{-\gamma} r_e \ln\left(1 + \phi^* \frac{r_e}{r_f}\right) 1_{\{r_e < 0\}}\right] \\
&\geq \mathbb{E}\left[(r_f + \phi^* r_e)^{-\gamma} r_e (\ln 1) 1_{\{r_e \geq 0\}}\right] + \mathbb{E}\left[(r_f + \phi^* r_e)^{-\gamma} r_e (\ln 1) 1_{\{r_e < 0\}}\right] = 0.
\end{aligned}$$

Therefore,  $\frac{\partial \phi^*}{\partial \gamma} < 0$ . Note that, if  $\phi^* < 0$ ,  $\ln(r_f + \phi^* r_e)$  is strictly decreasing and  $\frac{\partial \phi^*}{\partial \gamma} > 0$ .

### 6.3 Proof of Proposition 1

Let  $u$  denote the intertemporal hedging component,  $u = \rho \sigma d$ . Using equation (15), one can show that the function  $u$  satisfies the following equation:

$$\dot{u} + \left(-K + \frac{1-\gamma}{\gamma} \lambda \sigma \rho\right) u + \frac{\sigma}{2\rho} (\gamma + (1-\gamma)\rho^2) u^2 + \frac{1-\gamma}{2\gamma^2} \rho \sigma \lambda^2 = 0. \quad (18)$$

Differentiating equation (18) with respect to  $K$ , one obtains the following equation for the derivative  $u_K$  of  $u$  with respect to  $K$ :

$$\dot{u}_K + \left(-K + \frac{1-\gamma}{\gamma} \lambda \sigma \rho\right) u_K + (1 + (\gamma - 1)(1 - \rho^2)) u u_K - u = 0,$$

which can be solved to give

$$u_K = -e^{\int_t^T h ds} \int_t^T e^{-\int_s^T h dv} u ds < 0,$$

with the function  $h$  defined by

$$h = \left(-K + \frac{1-\gamma}{\gamma} \lambda \sigma \rho\right) + (1 + (\gamma - 1)(1 - \rho^2)) u.$$

Because the function  $u$  does not change signs as a function of  $t$ , the above equation implies that  $u$  is decreasing in  $K$  if  $u > 0$  and  $u$  is increasing in  $K$  if  $u < 0$ . In other words, the magnitude of  $u$  is always decreasing in  $K$ .

Differentiating equation (18) with respect to  $\sigma$ , one obtains the following equation for the derivative  $u_\sigma$  of  $u$  with respect to  $\sigma$ :

$$\dot{u}_\sigma + hu_\sigma + \frac{1-\gamma}{\gamma}\lambda\rho u + \frac{1}{2\rho}(1+(\gamma-1)(1-\rho^2))u^2 + \frac{1-\gamma}{\gamma^2}\lambda^2\rho = 0,$$

which can be written as

$$\dot{u}_\sigma + hu_\sigma + \frac{\gamma}{2\rho}u^2 + \frac{1-\gamma}{\gamma}\rho\left(u + \frac{\lambda}{\gamma}\right)^2 = 0.$$

The above equation can be expressed as

$$u_\sigma = e^{\int_t^T hds} \int_t^T e^{-\int_s^T hdv} \left( \frac{\gamma}{2\rho}u^2 + \frac{1-\gamma}{\gamma}\rho\left(u + \frac{\lambda}{\gamma}\right)^2 \right),$$

for  $\gamma < 1$ ,  $u_\sigma > 0$  if  $\rho > 0$ ; and  $u_\sigma < 0$  if  $\rho < 0$ . Therefore, the magnitude of the function  $u$  (the intertemporal hedging component) is increasing in  $\sigma$ .

Differentiating equation (18) with respect to  $\rho$ , one obtains the following equations for the derivative  $u_\rho$  of  $u$  with respect to  $\rho$ :

$$\dot{u}_\rho + hu_\rho + \frac{1-\gamma}{\gamma}\lambda\sigma u + \frac{\sigma}{2}\left(-\frac{\gamma}{\rho^2} + (1-\gamma)\right)u^2 + \frac{1-\gamma}{2\gamma^2}\lambda^2\sigma = 0,$$

or

$$\dot{u}_\rho + hu_\rho - \frac{\gamma\sigma}{2\rho^2}u^2 + \frac{\sigma}{2}(1-\gamma)\left(\frac{\lambda}{\gamma} + u\right)^2 = 0.$$

Therefore,

$$u_\rho = \frac{\sigma}{2}e^{\int_t^T hds} \int_t^T e^{-\int_s^T hdv} \left( -\frac{\gamma}{\rho^2}u^2 + (1-\gamma)\left(\frac{\lambda}{\gamma} + u\right)^2 \right).$$



From this equation, one concludes that for  $\gamma > 1$ ,  $u_\rho < 0$ . Because  $u$  has the opposite sign as  $\rho$  for  $\gamma > 1$ ,  $u_\rho > 0$  implies that the magnitude of  $u$ , and thus the magnitude of the intertemporal hedging component, is increasing in the magnitude of  $\rho$ .

Differentiating equation (18) with respect to  $\lambda$ , one obtains the following equation for the derivative  $u_\lambda$  of  $u$  with respect to  $\lambda$ :

$$\dot{u}_\lambda + hu_\lambda + \frac{1-\gamma}{\gamma}\sigma\rho u + \frac{1-\gamma}{\gamma^2}\lambda\rho\sigma = 0,$$

so

$$u_\lambda = \frac{1-\gamma}{\gamma}\sigma\rho e^{\int_t^T hds} \int_t^T e^{-\int_s^T hdv} \left(u + \frac{\lambda}{\gamma}\right) ds.$$

Noting that  $u > 0$  if  $\rho(1-\gamma) > 0$ , this equation implies that  $u$  is increasing in  $\lambda$  when  $\rho(1-\gamma) > 0$  or when  $u > 0$ .

## 6.4 Proof of Proposition 3

When  $\xi$  is real, the proposition is obvious by using equation (7). When  $\gamma < \gamma_{\min}$ ,  $\eta$  is real and the proposition can be readily proved by using equation (9).

## 6.5 Proof of Proposition 4

Using the equation (8), I define function  $g$ :

$$\begin{aligned} \phi_t^* &= V_t^{\frac{\beta-1}{2}} \frac{\lambda(K + \xi \coth(\xi\tau/2))}{\gamma(K + \xi \coth(\xi\tau/2)) - (1-\gamma)\lambda\rho\sigma} \\ &= V_t^{\frac{\beta-1}{2}} g. \end{aligned}$$

First consider  $\rho > 0$ . Let  $\Delta = 2K\rho + \lambda\sigma$ ,  $a = K/\sqrt{\Delta\lambda\sigma}$ , and  $b = \frac{\lambda\rho\sigma}{\sqrt{\Delta}}$ . Then  $\xi = \sqrt{\Delta}\sqrt{a^2 + 1} - \frac{1}{\gamma}$  and  $\gamma = \frac{1}{1+a^2-\frac{\xi^2}{\Delta}}$ . Letting  $z = \frac{\xi}{\sqrt{\Delta\lambda\sigma}}$  and  $\nu = \sqrt{\Delta\lambda\sigma}\tau$ , one obtains

$$\begin{aligned} g &= \frac{K + \xi \coth(\xi\tau/2)}{\gamma(K + \xi \coth(\xi\tau/2)) - (1 - \gamma)\lambda\rho\sigma} \\ &= \frac{(a + z \coth(z\nu/2))(1 + (a^2 - z^2))}{a + z \coth(z\nu/2) - b(a^2 - z^2)} = \frac{(a + f)(1 + a^2 - z^2)}{h}, \end{aligned}$$

where  $h = a + z \coth(z\nu/2) - b(a^2 - z^2)$  and  $f = z \coth(z\nu/2)$ . The following, then, is straightforward:

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{(f'(1 + a^2 - z^2) + (a + f)(-2z))h - (f' + 2bz)(a + f)(1 + a^2 - z^2)}{h^2} \\ &= \frac{-f'(a^2 - z^2)^2b - f'ba^2 - 2z(a + f)^2 - 2azb - fzb - zb(f - f'z)}{h^2} < 0. \end{aligned} \quad (19)$$

Now, consider the case  $\rho < 0$  and  $2K\rho + \lambda\sigma < 0$ . It is easy to verify that  $\phi^*$  is decreasing in  $\gamma$  when  $\gamma > 1$  by using equation (7):

$$\phi_t^* = V_t^{\frac{\beta-1}{2}} \frac{\lambda}{\gamma} \left( 1 + \frac{1}{K - \frac{1-\gamma}{\gamma}\lambda\rho\sigma + \xi \coth(\xi\tau/2)} \frac{(1-\gamma)\lambda\rho\sigma}{\gamma} \right). \quad (20)$$

Note that

$$\frac{\partial \xi}{\partial \gamma} = \frac{(2K\rho + \lambda\sigma)\lambda\sigma}{2\xi\gamma^2} < 0$$

and  $\xi \coth(\xi\tau/2)$  is increasing in  $\xi$  and therefore decreasing in  $\gamma$ . Hence,  $\frac{1}{K - \frac{1-\gamma}{\gamma}\lambda\rho\sigma + \xi \coth(\xi\tau/2)}$  is increasing in  $\gamma$  (noting that  $\lambda\rho\sigma < 0$ ). Because  $\frac{1-\gamma}{\gamma}$  is decreasing in  $\gamma$  and is negative when  $\gamma > 1$ , it follows that the term in parentheses in the above equation is decreasing in  $\gamma$ . Because  $\frac{1}{\gamma}$  and the term in parentheses are both positive and decreasing in  $\gamma$ , their product is also decreasing, which implies that  $\phi^*$  is decreasing in  $\gamma$ .

Now consider the case of  $\gamma < 1$ ,  $\rho < 0$ , and  $2K\rho + \lambda\sigma < 0$ . The function  $g$  can be written

as

$$\begin{aligned}
g &= \frac{K + \xi \coth(\xi\tau/2)}{\gamma(K + \xi \coth(\xi\tau/2)) - (1 - \gamma)\lambda\rho\sigma} \\
&= \frac{1 + x \coth(x\mu/2)}{\gamma(1 + x \coth(x\mu/2)) + (1 - \gamma)C} = \frac{1 + f}{\gamma(1 + f - C) + C}
\end{aligned} \tag{21}$$

where  $x = \frac{\xi}{K}$ , and  $C = \frac{-\lambda\rho\sigma}{K}$ , and  $f(x) = x \coth(x\mu/2)$ . The derivative of  $g$  with respect to  $x$

is given by:

$$\begin{aligned}
\frac{\partial}{\partial x}g &= \frac{f'(\gamma(1 + f - C) + C) - (1 + f)(\gamma f' + (1 + f - C)\gamma_x)}{(\gamma(1 + f - C) + C)^2} \\
&= \frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2} = \frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2}.
\end{aligned}$$

Note that

$$x = \sqrt{1 + (1/\gamma - 1)C(2 - C/\rho^2)}$$

and

$$\gamma_x^{-1} = \frac{1}{2x} \frac{-1}{\gamma^2} C(2 - C/\rho^2),$$

therefore,

$$\frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2} = \frac{f'(x^2 - 1)(C + \frac{x^2-1}{2-\frac{C}{\rho^2}}) + (1 + f)(1 + f - C)2x}{(\gamma(1 + f - C) + C)^2}.$$

For  $x \geq 1$ , then  $f'(x^2 - 1)(C + \frac{x^2-1}{2-\frac{C}{\rho^2}}) = f'(C(x^2 - 1) + \frac{(x^2-1)^2}{2-\frac{C}{\rho^2}}) > 0$  (noting that  $f'(x) > 0 \forall x$ ).

Because  $f(x) \geq x$ , it follows that  $(1 + f(x))(1 + f(x) - C)2x \geq (1 + x)(1 + x - C)2x \geq 0$ .

I used the fact that  $x \geq x_{\min} = \sqrt{(C - 1)^2 + (1/\rho^2 - 1)C^2}$  so that  $x \geq C - 1$ .

Now consider  $x_{\min} < x < 0$ . Because  $1 - x^2 - C(2 - C/\rho^2)$  for all  $x \geq x_{\min}$ , it follows that  $f'(x^2 - 1)(C + \frac{x^2-1}{2-\frac{C}{\rho^2}}) \geq 0$ . Note that  $f'(x) \geq 1$ , we know that  $f'(x^2 - 1)(C + \frac{x^2-1}{2-\frac{C}{\rho^2}}) \geq (x^2 - 1)(C + \frac{x^2-1}{2-\frac{C}{\rho^2}})$ .

Therefore, one obtains

$$\begin{aligned}
& f'(x^2 - 1)\left(C\left(2 - \frac{C}{\rho^2}\right) + x^2 - 1\right) + (1 + f)(1 + f - C)2x\left(2 - \frac{C}{\rho^2}\right) \\
& \geq (x^2 - 1)\left(C\left(2 - \frac{C}{\rho^2}\right) + x^2 - 1\right) + (1 + x)(1 + x - C)2x\left(2 - \frac{C}{\rho^2}\right) \\
& = (1 - x^2)^2 - (1 - x^2)C\left(2 - \frac{C}{\rho^2}\right) + 2x((1 + x)^2 - (1 + x)C)\left(2 - \frac{C}{\rho^2}\right) \\
& = (1 - x^2)^2 + \left(2 - \frac{C}{\rho^2}\right)(1 + x^2)(2x - C) \\
& = (1 + x)^2\left((1 - x)^2 + \left(2 - \frac{C}{\rho^2}\right)(2x - C)\right) \\
& = (1 + x)^2\left(2 - 2x + x^2 - 1 + \left(2 - \frac{C}{\rho^2}\right)(2x - C)\right) \\
& = (1 + x)^2\left(2(1 - x) + \left(2 - \frac{C}{\rho^2}\right)2x\right) \geq 0.
\end{aligned}$$

When  $\rho < 0$  and  $2K\rho + \lambda\sigma > 0$ , one can easily show that the portfolio weight also reaches  $-\infty$  when  $\gamma$  is small enough, as long as  $\tau > 0$ . Therefore, the portfolio weight cannot be decreasing in  $\gamma$  for any strictly positive  $\tau$ .