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S. Roberts
(Ph.D. Thesis)

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# CONVERGENCE OF A RANDOM WALK METHOD FOR THE BURGERS EQUATION[1]

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# Convergence of a Random Walk Method foe the Burgers Equation

**Stephen Gwyn Roberts**

**Abstract**

In this paper we consider a random walk algorithm for the solution of Burgers' equation. The algorithm uses the method of fractional steps. The non-linear advection term of the equation is solved by advecting 'fluid' particles in a velocity field induced by the particles. The diffusion term of the equation is approximated by adding an appropriate random perturbation to the positions of the particles. Though the algorithm is inefficient as a method for solving Burgers' equation, it does model a similar method, the random vortex method, which has been used extensively to solve the incompressible Navier-Stokes equations.

The purpose of this paper is to demonstrate the strong convergence of our random walk method and so provide a model for the proof of convergence for more complex random walk algorithms; for instance, the random vortex method without boundaries. We are able to show that the expected value of the $L^1$ norm of the error of our method is of order $\frac{1}{m^{1/4}}(\ln(m))^2$, where $m$ is the number of particles that generate the solution, provided the time step of the method is proportional to $\frac{1}{m^{1/4}}$. In addition, we show that the probability of the $L^1$ error being greater than a constant multiple of the expected value of the error, decreases exponentially as the constant tends to infinity. Consequently, the expected value of the error provides a reliable estimate for the error expected in any particular numerical run of our method.

Finally, we remark that this work provides the first proof of convergence in a strong sense, for a random walk method for a problem in which the related advection equation is non-linear.

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# Contents

## Introduction

In this paper we will study a stochastic numerical method for solving the Burgers equation

$$u_t + uu_x = \nu u_{xx} \quad , u(x,0) = u_0(x). \tag{0.1}$$

This equation was advanced by Burgers [8] as a one dimensional model for the Navier-Stokes equations. An interesting survey of many of the properties of this equation is provided by Burgers [9].

The numerical method that we present, though impractical as a method for solving Burgers' equation, does model a similar method, the random vortex method (Chorin [11]), which has been used extensively to solve the incompressible Navier-Stokes equations.

Our numerical method will be a fractional step method. The method of fractional steps is discussed in Richtmyer and Morton [41], § 8.9, and in the fine survey paper, Chorin et. al. [15]. For a more theoretical account of fractional step methods and product formulas in general, we refer to Chernoff [10].

The first step of our method approximates the solution of the inviscid Burgers equation

$$u_t + uu_x = 0 \quad , u(x,0) = u_0(x). \tag{0.2}$$

We suppose that the gradient of the solution is approximated by a collection of particles so that

$$u_x \approx \sum_{i=1}^{m} \delta_{x_i} \varsigma_i$$

where $x_i$ signifies the position of each particle, $\varsigma_i$ denotes the strength of each particle and $\delta_{x_i}$ denotes the delta function concentrated at the point $x_i$. The solution of (0.2) is obtained by allowing the positions of the particles to move with a velocity induced by the step function solution generated by the particles.

In the second fractional step we solve the diffusion equation

$$u_t = \nu u_{xx} \quad , u(x,0) = u_0(x) \tag{0.3}$$

by utilizing the correspondence between the probability distribution of the position of a particle undergoing a random walk and the solution of the diffusion equation as discussed in Einstein [21], Feller [23], Chorin and Marsden [14], and Chorin [11]. In essence, the diffusion is simulated by randomly perturbing the positions of the particles that generate the numerical solution. We notice that the statistical errors of our method are greatly reduced since our numerical solution is obtained by integrating the function approximated by the particles. In random walk methods it is advantageous to move particles which approximate the gradient of the solution instead of particles which approximate the solution itself.

The random vortex method, [11], is also a fractional step method; the first step involves advecting a collection of 'vortex particles' using an approximation of the Euler equations; the second step diffuses the particles as in our method. If boundaries are present, it is necessary to add an additional fractional step in which particles are created on the boundary so as to satisfy the boundary conditions. This method has proved to be a practical tool in the study of incompressible fluid flow; see for instance Laitone [34], Stansby and Dixon [45], Sung et. al. [46], Teng [47] and Van der Vegt and Huijsmans [48].

Similar 'random walk' methods have been developed to solve other problems which contain diffusion (see Ghoniem and Sherman [26]). A random vortex sheet method has been developed to solve the Prandtl boundary layer equation (Chorin [12]). A combination of the random vortex method and the the random vortex sheet method has been used to study turbulent combustion (see Ghoniem et. al. [25] and Oppenheim and Ghoniem [40]). In addition, random walk methods have been developed for the solution of scalar reaction advection diffusion equations (see Brenier [6], [7], Chorin [13] and Hald [27]). In all of these methods, the diffusive part of the equation is solved by applying a random walk technique to a set of particle positions (Brenier [7] uses a pseudo random walk technique).

The usefulness of these random walk methods depend on the following facts:

(1) If the Reynolds number for the equation is large ($\nu$ small), then it may be computationally too expensive to use a standard finite difference scheme to solve the equation. Random walk methods produce little, if any numerical diffusion. Consequently, the computational labour for these methods are essentially independent of the Reynolds number.

(2) The analogy between a random walk method and the underlying physical process usually justifies the good qualitative behaviour of these methods.

The convergence of these methods have still to be proved in a completely satisfactory sense. Marchioro and Pulvirenti [36] have shown that the two dimensional random vortex method converges weakly to a solution of the Navier-Stokes equations. In Benefatto and Pulvirenti [4], it is shown that a similar result holds for the random walk solution of the Prandtl boundary layer equation. From a numerical standpoint we would prefer the convergence to occur in one of the standard norms and we also need to have some idea on the rate of convergence of the method.

Hald [28] has proved the strong convergence of a random walk method for a coupled system of diffusion equations with boundary. This is the first proof of convergence of a random walk method in which particles are created at the boundary to satisfy the boundary conditions. Hald [27] has also proved the strong convergence of a method for solving a reaction diffusion equation. Unfortunately his method does not readily generalize to equations with advection. Brenier [7] has generalized Chorin's [13] reaction diffusion method to the case of scalar reaction advection diffusion equations. His method is very similar to our method for the Burgers equation, in that particles are moved via the action of the velocity field generated by the particles and the diffusion is simulated by adding random perturbations to the particle positions. The approximation of the reaction step of his equation is undertaken by changing the strengths of the particles in an appropriate way. Brenier has been able to prove the $L^1$ convergence of a modified version of his method in which the diffusion algorithm is solved using a deterministic 'random walk' algorithm.

In this paper we prove the strong convergence of our random walk method, in an appropriate probabilistic sense, and provide an estimate of the rate of convergence. This is the first proof of convergence of a random walk method, in a strong sense, for an advection diffusion equation in which the advection equation is non-linear.

In the rest of this chapter we will give an outline of our proof of convergence. To facilitate this discussion we will first introduce some notation and at the same time define more precisely our numerical method.

We will use the notation $F_t$, $A_t$ and $D_t$ to denote the solution operators for the equations (0.1), (0.2) and (0.3) respectively, i.e., $F_t$ is the solution operator for the full equation, $A_t$ is the operator associated with advection, and $D_t$ is associated with diffusion. We will denote the operators that approximate the advection and the diffusion operators by $\widetilde{A}_t$ and $\widetilde{D}_t$, where we understand that these operators depend on the spatial parameter $h$, which denotes the maximum absolute strength of the particles that generate the numerical solution.

For initial data $u_0$, time step $k$ and spatial parameter $h$, the numerical approximation of $F_{nk}\, u_0$ is obtained as follows:

**Step 1.** In the initial step of the algorithm one approximates the smooth data $u_0$ with a step function $S^0 u_0$ (see Section 1.1) which is generated by $m$ particles with positions $X_i^{\,0}$ ( $X_i^{\,j}$, $j=0$ ) and strengths $\varsigma_i$, $i = 1,..,m$, such that $|\varsigma_i| = h$. The initial step function approximation is then given by

$$S^0 u_0(y) = u_0^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i^{\,0})\, \varsigma_i$$

where $u_0^L$ is the limit of $u_0$ at minus infinity and H denotes the Heaviside function (defined in Section 1.1).

**Step 2.** Given particles at positions $X_i^{\,j}$, we evolve the positions of the particles so that their motion approximates the flow of particles in the exact velocity field. Specifically, the new particle positions $X_i^{\,j+1/2}$ are given by

$$X_i^{\,j+1/2} = x_i(k)$$

where $x_i(t)$ solves the evolution equation

$$\frac{dx_i(t)}{dt} = s_i(x_1(t),..,x_m(t))$$

with initial conditions $x_i(0) = X_i^{\,j}$. The function $s_i(x_1,..,x_m)$ gives the value of the velocity field generated by the $m$ particles at position $x_i$. The definition of $s_i$ is chosen to approximate the behaviour of the exact entropy solution. If two or more particles form a discontinuity which would correspond to a shock satisfying the entropy condition for the exact equation, then $s_i$ is defined so all of those particles will be moved with a common velocity given by the correct shock velocity. On the other hand, if the particles are part of a discontinuity which would naturally form into a rarefaction wave, then the particles are made to fan out in such a way as to approximate the exact rarefaction wave.

To denote that the approximate advection operator actually operates on particle positions, we will use the notation

$$\widetilde{A}_k(X_1^{\,j},.., X_m^{\,j}) = (X_1^{\,j+1/2},.., X_m^{\,j+1/2}).$$

In addition, we will also use the notation

$$S^{j+1/2}u_0(y) = u_0^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i^{\,j+1/2})\,\varsigma_i = \widetilde{A}_k[\widetilde{D}_k\widetilde{A}_k]^j S^0 u_0(y).$$

to denote the step function generated by the random variables $X_i^{\,j+1/2}$. Here we are using the notation $\widetilde{A}$ to denote an operator on particle positions or alternatively an operator on step functions. For both cases we implicitly require that we have a collection of generating particles.

**Step 3.** We must now solve the diffusion step. To the position of the particles we add a random component. Let $B_i^{\,j+1}$, $i = 1,..,m$ be an independent collection of normally distributed random variables such that E $[B_i^{\,j+1}] = 0$ and $\mathrm{Var}[B_i^{\,j+1}] = 2\nu k$ . Then the new positions of the particles are given by

$$X_i^{\,j+1} = X_i^{\,j+1/2} + B_i^{\,j+1}$$

and the new numerical approximation is given by

$$S^{j+1}u_0(y) = u_0^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i^{\,j+1})\,\varsigma_i = [\widetilde{D}_k \widetilde{A}_k]^{j+1} S^0 u_0(y).$$

**Step 4.** Finally, we set $j = j+1$ and go back to Step 2 if $j < n$ .

Our numerical approximation for $F_{nk}\, u_0$ is then given by

$$S^n u_0(y) = u_0^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i^{\,n})\,\varsigma_i = [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0(y).$$

Our convergence results, as contained in Theorems 5.4.2 and 5.4.3, can now be stated explicitly. If the time step $k$ satisfies $k = \nu^{1/2} h^{1/4}$, then for any $\alpha > 1$,

$$\mathrm{P}(\, \| F_{nk}\, u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{\mathrm{L}^1} > M_5 \alpha h^{1/4} (\ln(1/h))^2) \qquad (0.4)$$

$$\leq M_6 h^{\frac{1}{2}\alpha \ln(1/h) - 2}$$

and

$$\mathrm{E}\,[\, \| F_{nk}\, u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{\mathrm{L}^1}] \leq \mathrm{C}_{10} h^{1/4} (\ln(1/h))^2. \qquad (0.5)$$

The constants that appear in these results and the constants that will be introduced in the rest of this chapter will depend on $\nu$, $u_0$ and $T = nk$ , unless stated otherwise.

The choice of the $\mathrm{L}^1$ norm as a measure of the error is determined by the existence of stability results in this norm, for all of the operators under consideration, and lack of such results in any other Lebesgue norm.

The derivation of statements (0.4) and (0.5) is accomplished in five stages, the final one being a synthesis of the preceding stages. Three of the stages consist of analyzing the accuracy of steps 1, 2 and 3 of the numerical method. That is, analyzing the accuracy of the operators $S^0$, $\widetilde{A}_t$ and $\widetilde{D}_t$.

The first stage involves studying the accuracy of the exact fractional step algorithm, namely, the fractional step algorithm in which $[D_k A_k]^n$ is used to approximate $F_{nk}$. Note that we are using the exact operators $D_k$ and $A_k$. It would be foolhardy to hope that a fractional step algorithm using random walks would converge if the corresponding exact fractional step algorithm did not convergence. In Theorem 2.2.1 we show that the use of the method is justified, when we show that

$$\| F_{nk} u_0 - [D_k A_k]^n u_0 \|_{L^1} \leq C_2 n k^2. \tag{0.6}$$

This result is not only a justification, it is also an integral part of the proof of convergence for our random walk algorithm. By considering the exact operators, we are able to use standard tools of analysis to obtain an estimate on the interaction of the advection and diffusion parts of the fractional step algorithm.

The derivation of expression (0.6) is based on the one step estimate

$$\| F_k v_0 - [D_k A_k] v_0 \|_{L^1} \leq C_2 k^2 \tag{0.7}$$

where $v_0 = F_{jk} u_0$ for $j = 0,..,n-1$. Expression (0.6) follows from (0.7) by using a simple triangle inequality argument.

It is interesting to note that the constant $C_2$ has a $\nu^{-1}$ dependence, which can be verified by letting $v_0$ be a travelling wave solution of the viscous Burgers equation (see section 2.3). On the other hand, numerical experiments indicate that

$$\| F_{nk} v_0 - [D_k A_k]^n v_0 \|_{L^1} \leq C_2' k^2$$

where $C_2' \to 0$ as $\nu \to 0$. Hence, the interaction of the errors over a number of time steps produce an error which is smaller than that predicted by our simple analysis, based on one time step. Beale and Majda [3] have studied the corresponding exact fractional step algorithm for the incompressible Navier-Stokes equation. They have shown that the constant $C_2$, for the one step error of their problem, tends to zero as $\nu \to 0$. This difference in behaviour between the fractonal step algorithm for the Burgers equation and the corresponding algorithm for the Navier-Stokes equation can be attributed to the existence of a priori bounds on the derivatives of the solution of Euler's equations (see Beale and Majda [3], McGarth [37]) compared to the existence of a priori bounds on the derivative of the solution of Burgers' equation which depend on $\nu^{-1}$.

Having completed the analysis of the exact fractional step algorithm, it is then necessary to study the accuracy of the operators $S^0$, $\widetilde{A}_k$ and $\widetilde{D}_k$. In Section 1.1 we define the operator $S^0$ and show that

$$\| S^0 u_0 - u_0 \|_{L^1} \leq C_1 h. \tag{0.8}$$

The advection operator is studied in Chapter 3, where it is shown that

$$\| A_k S^j u_0 - \widetilde{A}_k S^j u_0 \|_{L^1} < C_9 hk. \tag{0.9}$$

We note that Brenier [6] uses a simplier method to move his particles during the advection step. Even though his method converges and, as $k \to 0$, produces numerical results similar to those obtain with our method, we found it necessary, for the proof of convergence of our random walk method, to use an approximate advection operator which was more accurate during each time step.

In Chapter 4 we develop the tools to show that for $\alpha > 1$,

$$\begin{aligned} P( \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} &> M_3 \alpha h^{1/2} (\ln(1/h))^2 ) \\ &\leq M_4 h^{\frac{1}{2}\alpha \ln(1/h) - 1} \end{aligned} \tag{0.10}$$

and

$$E [ \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} ] \leq C_8 h^{1/2} (\ln(1/h))^2. \tag{0.11}$$

The proof of these results is based on the observation that $\widetilde{D}_k S^{j-1/2} u_0 (y)$ can be represented as a sum of bounded random variables. We use a result due to Hoeffding ([30], Theorem 1) to show that for any fixed $y \in \mathbb{R}$ the following strong result (Theorem 4.3.4) holds.

$$P(\ |D_k S^{j-1/2}(y) - \widetilde{D}_k S^{j-1/2}(y)| > mh\,\delta\ ) \leq 2\exp(-2\delta^2 m).$$

This estimate can be extended to obtain $L^\infty$ estimates in any bounded interval. The $L^1$ result, (0.10), then follows by noting that there exists a bounded interval, outside of which the $L^1$ error is very small, with large probability.

In Chapter 5 we show that the random variable

$$\| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1}$$

is less than or equal to

$$\| F_{nk} u_0 - [D_k A_k]^n u_0 \|_{L^1} \tag{0.12}$$

$$+ \| u_0 - S^0 u_0 \|_{L^1} \tag{0.13}$$

$$+ \sum_{j=1}^{n} \| A_k S^{j-1} u_0 - \widetilde{A}_k S^{j-1} u_0 \|_{L^1} \tag{0.14}$$

$$+ \sum_{j=1}^{n} \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}. \tag{0.15}$$

To obtain this estimate we need to make extensive use of the stability of the operators $D_k$ and $A_k$. The convergence proof is concluded by comparing expressions (0.6) and (0.8)-(0.11) with expressions (0.12)-(0.15). With appropriate restrictions placed on $h$, the convergence results (0.4) and (0.5) then follow.

We conclude this introduction with a review of the organization of this paper. In Chapter 1 we introduce some notation and results that are used in the rest of the paper. Chapter 2 contains our analysis of the exact fractional step algorithm which shows that the use of a fractional step algorithm for Burgers equation in justified. In Chapter 3 we define and study our approximate advection operator and in Chapter 4 we prove the $L^1$ norm convergence of the random walk approximation to the solution of the diffusion equation. Finally, in the fifth chapter we bring together the results of the preceding chapters to complete the proof of convergence of our method.

# Chapter 1

## Preliminary Results

In this chapter we will introduce some of the notation and general results that will be used in the rest of the paper. The first section will concentrate on general definitions and results pertaining to the space of step functions on which our approximation operators will be defined. Section two will contain the properties of the solution of the viscous Burgers equation that will be needed. Section 3 contains similar results pertaining to the inviscid Burgers equation. Finally, in Section four we state some results from probability theory that are of use.

**1.1. Definitions and Notation.** The Heaviside function is defined to be the function

$$
\begin{aligned}
H(x) &= 1 \quad \text{if } x > 0, \\
&= 1/2 \quad \text{if } x = 0, \\
&= 0 \quad \text{if } x < 0.
\end{aligned}
$$

Related to the Heaviside function is the sgn function, which is defined by the following relationship.

$$
\begin{aligned}
\operatorname{sgn}(x) &= 1 \quad \text{if } x > 0, \\
&= 0 \quad \text{if } x = 0, \\
&= -1 \quad \text{if } x < 0.
\end{aligned}
$$

We will say that a function $u$ is constant in a neighbourhood of infinity if there exist constants $R > 0$, $u^L$ and $u^R$ such that

$$
\begin{aligned}
u(x) &= u^L \quad \text{for } x < -R, \\
&= u^R \quad \text{for } x > R.
\end{aligned}
$$

**1.1.1. Pseudo-norms.** We denote the set of functions with continuous $j^{th}$ derivatives by $C^j(\mathbb{R})$. For $u \in C^j(\mathbb{R})$ we let

$$\| u \|_{BD^j} = \| \partial_x^j u \|_{L^\infty}.$$

For a locally integrable function $u$, the variation of $u$ is defined to be

$$\| u \|_{BV} = \sup_{z \neq 0} |z|^{-1} \int_{-\infty}^{\infty} | u(x+z) - u(x) | \, dx.$$

If $u \in C^1(\mathbb{R})$ and constant in a neighbourhood of infinity then

$$\| u \|_{BV} = \| \partial_x u \|_{L^1}.$$

This follows from the fact that

$$\int_{-\infty}^{\infty} \left| \frac{u(x+z) - u(x)}{z} \right| dx \leq \int_{-\infty}^{\infty} \int_0^1 | u_x(x + \theta z) | \, d\theta dx = \| u_x \|_{L^1}.$$

**1.1.2. The Set of Step Functions S.** For a given $h > 0$, we let S denote the step functions with a finite number of discontinuities, each with a step size a multiple of $h$. We will regard an element of S as being generated by a set of particles. For each $s \in S$ we can find two finite sequences $\{x_i\}_{i=1}^m$ and $\{\varsigma_i\}_{i=1}^m$ such that $|\varsigma_i| = h$, so that $s$ can be written,

$$s(y) = s^L + \sum_{i=1}^m H(y - x_i) \varsigma_i .$$

For convenience, we think of the function $s \in S$ as being generated by the set of $m$ particles with positions $x_i$ and strengths $\varsigma_i$. It should be noted that this description is not unique, since it is possible that $x_i = x_j$, $i \neq j$.

We observe that the variation norm of an element of S can be estimated by the following sum:

$$\| s \|_{BV} \leq \sum_{i=1}^m |\varsigma_i| ,$$

with equality if $x_i = x_j$ implies $\text{sgn}(\varsigma_i) = \text{sgn}(\varsigma_j)$ for all $i, j = 1,..,m$.

**1.1.3. Approximation of Smooth Functions by Step Functions.** In Chapters 3 and 4 we will define our approximate advection and diffusion operators. They will be defined to operate on the positions of a collection of particles which generate elements of **S**. Hence the first step of our algorithm is to approximate smooth data, $u_0$, by step functions $S^0 u_0$ in **S** and consequently define a corresponding set of generating particles.

**Definition of $S^0$.** Let the initial data $u_0 \in C^1(\mathbf{R})$ be constant in a neighbourhood of infinity. Let $h > 0$ be given so that $h$ divides $u_0^L - u_0^R$. We will define the step function $S^0 u_0$ in the following manner. Consider the set

$$E = \{ y \in \mathbf{R} : u_0(y) - u_0(-\infty) \notin h\mathbb{Z} \}$$

where $\mathbb{Z}$ is the set of integers. We will suppose that $u_0$ is not a constant and so $E$ is non-empty. Since $u_0$ is a continuous function which is constant in a neighbourhood of infinity and $h$ divides $u_0^L - u_0^R$, it follows that the set $E$ is a union of open intervals contained in some compact set. Hence there exist two sequences of distinct points $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$ ($I$ countable, possibly infinite) such that $a_i < b_i$ for all $i \in I$ and

$$E = \bigcup_{i \in I} (a_i, b_i).$$

In addition, we may suppose that $a_i < a_j$ implies that $b_i \leq a_j$ (This implies that the sets $(a_i, b_i)$ are disjoint; see figure 1.). We define the step function $S^0 u_0$ by specifying a set of generating particles. The positions of the particles are given by

$$\{x_i\}_{i=1}^{m} = \{\frac{1}{2}(a_j + b_j) : u_0(b_j) \neq u_0(a_j)\}.$$

The corresponding strength of the $i^{th}$ particle is given by

$$\varsigma_i = u_0(b_j) - u_0(a_j)$$

where $j$ satisfies $x_i = \frac{1}{2}(a_j + b_j)$ and we note that $| u_0(b_j) - u_0(a_j) | = h$. The step function $S^0 u_0$ is then given by

$$S^0 u_0(y) = u_0^L + \sum_{i=1}^{m} \mathrm{H}(y - x_i)\, \varsigma_i .$$

Since the function $u_0$ is of bounded variation, it follows that there exist only a finite number of intervals $(a_j, b_j)$ such that $u_0(b_j) \neq u_0(a_j)$. This implies that $S^0 u_0$ is well defined. Hence the function $S^0 u_0$ is generated by $m$ particles at distinct positions $x_i$, $i = 1,..,m$ each with strength of absolute size $h$. We observe that the above discussion holds even if the set $\{j : u_0(b_j) \neq u_0(a_j)\}$ is empty. In that case

$$S^0 u_0(y) = u_0^L.$$

The error introduced by approximating a function $u_0$ by $S^0 u_0$ is estimated in the following theorem.

**1.1.4. Theorem.** *Let $u_0 \in \mathrm{C}^1(\mathbf{R})$ be constant in a neighbourhood of infinity such that $u_0(y) = u_0^L$, if $y < -R$ and $u_0(y) = u_0^R$, if $y > R$. Let $h$ divide $u_0^L - u_0^R$. Then $S^0 u_0$ is generated by a collection of $m$ particles with strengths $\varsigma_i$ such that $|\varsigma_i| = h$, $mh \leq \| u_0 \|_{\mathrm{BV}}$ and*

$$\| S^0 u_0 - u_0 \|_{\mathrm{L}^1} \leq \mathrm{C}_1 h ,$$

*where $\mathrm{C}_1 = 2R$.*

**Proof.** Since $S^0 u_0$ is a step function with a finite number of discontinuities, we have, by section 1.1.2,

$$\| S^0 u_0 \|_{\mathrm{BV}} = \sum_{j : u_0(b_j) \neq u_0(a_j)} | u_0(b_j) - u_0(a_j) |$$

$$= \sum_j | u_0(b_j) - u_0(a_j) | = \sum_j \left| \int_{a_j}^{b_j} \partial_x u_0(x) dx \right|$$

$$\leq \sum_j \int_{a_j}^{b_j} | \partial_x u_0(x) | \, dx \leq \int_{-\infty}^{\infty} | \partial_x u_0(x) | \, dx = \| u_0 \|_{\mathrm{BV}} .$$

Hence the first part of the theorem holds, since $mh = \| S^0 u_0 \|_{BV}$. To prove the second statement we observe that by definition $| S^0 u_0(y) - u_0(y) | < h$, for all $y \in E$. On the other hand, if $y \notin E$, then $S^0 u_0(y) = u_0(y)$. The theorem now follows since $E \subseteq [-R, R]$.

**1.2 Viscous Burgers Equation.** We denote the solution operator for the viscous Burgers equation,

$$u_t + u u_x = \nu u_{xx} \ , \ u(x,0) = u_0(x), \tag{1.1}$$

by $F_t$. The existence and uniqueness of a classical solution of this equation is guaranteed by standard results from the theory of second order quasilinear parabolic equations (see Friedman [24] or Oleinik and Kruzkov [39]). An explicit formula for the solution of (1.1) also exists:

**1.2.1. Theorem (Hopf [31], Cole [17]).** *If the initial data* $u_0$ *for equation* (1.1) *satisfies* $\int_0^x u_0(\xi) d\xi = o(x^2)$, *for large* $|x|$, *then the solution of the viscous Burgers equation is given by the explicit formula,*

$$u(x,t) = -2\nu \frac{\phi_x(x,t)}{\phi(x,t)},$$

*where*

$$\phi(x,t) = \frac{1}{(4\pi\nu t)^{1/2}} \int_{-\infty}^{\infty} \phi_0(x-y) \exp\left(-\frac{y^2}{4\nu t}\right) dy$$

*and*

$$\phi_0(y) = \exp\left(-\frac{1}{2\nu} \int_0^y u_0(\xi) d\xi\right).$$

Using this explicit form it is possible to obtain special solutions of the viscous Burgers equation given specific initial data. Chapter 4 of Whitham [50] gives a good survey of such specific solutions. We will use *Theorem* 1.2.1 to obtain estimates for the $BD^j$ norms of the solution of the viscous Burgers equation in terms of the initial data.

**1.2.2. Theorem (A priori Bounds).** *Let $u_0 \in C^2(\mathbf{R})$ be constant in a neighbourhood of infinity. Then*

$$\| F_t u_0 \|_{L^\infty} \leq \| u_0 \|_{L^\infty},$$

$$\| F_t u_0 \|_{BD^1} \leq \frac{1}{\nu}[\, \nu \| u_0 \|_{BD^1} + \| u_0 \|^2_{L^\infty} ],$$

$$\| F_t u_0 \|_{BD^2} \leq \frac{4}{\nu^2}[\, \nu^2 \| u_0 \|_{BD^2} + \nu \| u_0 \|_{L^\infty} \| u_0 \|_{BD^1} + \| u_0 \|^3_{L^\infty} ].$$

**Sketch of Proof.** We will use the explicit formula for $F_t u_0$ given in Theorem 1.2.1. Since,

$$\phi_0(y) = \exp(-\frac{1}{2\nu}\int_0^y u_0(\xi)d\xi)$$

it follows that

$$\phi_0'(y) = -\frac{1}{2\nu} u_0(y)\phi_0(y),$$

$$\phi_0''(y) = [\frac{1}{4\nu^2} u_0^2(y) - \frac{1}{2\nu} u_0'(y)]\, \phi_0(y)$$

and

$$\phi_0'''(y) = [-\frac{1}{8\nu^3} u_0^3(y) + \frac{3}{4\nu^2} u_0(y) u_0'(y) - \frac{1}{2\nu} u_0''(y)\,]\, \phi_0(y).$$

We substitute these expressions for $\phi_0'$, $\phi_0''$ and $\phi_0'''$ into the integral expressions for $\phi$ and its derivatives to obtain

$$| \phi_x(y,t) | \;\leq \frac{1}{2\nu} \| u_0 \|_{L^\infty} \phi_0(y),$$

$$| \phi_{xx}(y,t) | \;\leq [\frac{1}{4\nu^2} \| u_0 \|^2_{L^\infty} + \frac{1}{2\nu} \| u_0 \|_{BD^1}]\, \phi_0(y)$$

and

$$| \phi_{xxx}(y,t) | \;\leq [\frac{1}{8\nu^3} \| u_0 \|^3_{L^\infty} + \frac{3}{4\nu^2} \| u_0 \|_{L^\infty} \| u_0 \|_{BD^1} + \frac{1}{2\nu} \| u_0 \|_{BD^2}]\, \phi_0(y).$$

The normed estimates are now easily obtained. As an example we will derive the bound on the quantity $\| F_t u_0 \|_{BD^1}$.

$$|\partial_x F_t u_0(y)| \leq 2\nu \frac{|\phi_{xx}(y,t)\phi(y,t) - \phi_x^2(y,t)|}{\phi^2(y,t)}$$

$$\leq 2\nu \left[\frac{1}{4\nu^2}\| u_0 \|_{L^\infty}^2 + \frac{1}{2\nu}\| u_0 \|_{BD^1}\right] + 2\nu\left[\frac{1}{4\nu^2}\| u_0 \|_{L^\infty}^2\right]$$

$$\leq \frac{1}{\nu}[\nu\| u_0 \|_{BD^1} + \| u_0 \|_{L^\infty}^2].$$

Hence

$$\| F_t u_0 \|_{BD^1} \leq \frac{1}{\nu}[\nu\| u_0 \|_{BD^1} + \| u_0 \|_{L^\infty}^2].$$

The other bounds can be derived in a similar manner.

**1.2.3. Stability in $L^1$ Norm.** We note that the $F_t$ operator is stable in the $L^1$ norm. That is, if the $L^1$ norm of the difference between two solutions of the viscous Burgers equation is small at initial time, then the difference at later times will be no larger. It can be shown that this stability does not hold in any other $L^p$ space ($p > 1$). The statement of the stability result follows. We refer the reader to the the paper of Kruzkov [33] for a proof of the result.

**1.2.4. Stability Theorem (Kruzkov [33] p. 239).** *Let $u_0$ and $v_0$ be bounded measurable functions such that $\| u_0 - v_0 \|_{L^1} < \infty$. Then*

$$\| F_t u_0 - F_t v_0 \|_{L^1} \leq \| u_0 - v_0 \|_{L^1}.$$

**1.2.5. Bound on the Variation of $F_t u_0$.** As a corollary of the stability result we observe that if the initial data $u_0$ is of bounded variation, then the solution $F_t u_0$ at time $t$ must also be of bounded variation. This can be seen by substituting $v_0(y) = u_0(y+z)$ into Theorem 1.2.4. Hence we have the result:

**1.2.6. Theorem (Kruzkov [33] pp. 231-236).** *Let $u_0$ be a bounded and measurable function such that $\| u_0 \|_{BV} < \infty$. Then,*

$$\| F_t u_0 \|_{BV} \leq \| u_0 \|_{BV}.$$

We have indicated that Kruzkov proves Theorem 1.2.6 as a corollary of Theorem 1.2.4. In fact Kruzkov first proves Theorem 1.2.6 in a more general setting and then comments that the proof of Theorem 1.2.4 can be obtained in an analogous manner.

**1.3. Inviscid Burgers Equation.** The inviscid Burgers equation is

$$u_t + uu_x = 0 , \ u(x,t) = u_0(x). \tag{1.2}$$

This equation is a one dimensional model for inviscid fluid flow. The velocity of a fluid particle at position $x$ and time $t$ is given by $u(x,t)$. The trajectory, $x(\alpha,t)$, of a fluid particle at position $\alpha$ at $t = 0$, satisfies the equation,

$$\dot{x}(\alpha,t) = u(x(\alpha,t),t).$$

where $\dot{}$ denotes differentiation with respect to time. The trajectories are commonly known as characteristic curves (see John [32], Bardos [2]). For smooth initial data there exists a classical solution of equation (1.2) for a short time.

**1.3.1 Theorem (John [32] p. 18).** *Let the initial data $u_0 \in C^2(\mathbb{R})$ be given such that $\| u_0 \|_{BD^1} < \infty$ and $\| u_0 \|_{BD^2} < \infty$. Then for $t < \| u_0 \|_{BD^1}^{-1}$, the solution $u(x,t)$ of equation (1.2), is given by the implicit formula*

$$u(x,t) = u_0(x - tu(x,t)) \tag{1.3}$$

*and satisfies*

$$\| u(.,t) \|_{L^\infty} \leq \| u_0 \|_{L^\infty},$$

$$\| u(.,t) \|_{BD^1} \leq \frac{\| u_0 \|_{BD^1}}{1 - t \| u_0 \|_{BD^1}},$$

$$\| u(\cdot,t) \|_{BD^2} \le \frac{\| u_0 \|_{BD^2}}{[1 - t \, \| u_0 \|_{BD^1}]^3}.$$

**Proof.** If $u_0$ is a smooth function, then the implicit function theorem implies that the above implicit equation has a smooth solution $u(x,t)$, for $|t| < \| u_0 \|_{BD^1}^{-1}$. It is then a simple matter to show that $u$ satisfies (1.2). If we differentiate equation (1.3) with respect to $x$ we find that

$$u_x(x,t) = \frac{u_0'(x - tu(x,t))}{1 + tu_0'(x - tu(x,t))}.$$

Differentiating again we find that

$$u_{xx}(x,t) = \frac{u_0''(x - tu(x,t))}{[1 + tu_0'(x - tu(x,t))]^3}.$$

The proof of the norm estimates now follows.

**1.3.2. The Weak Entropy Solution.** We observe that the derivative of the solution given in Theorem 1.3.1 blows up in finite time if $u_0$ has any point with negative slope. It is possible to define a generalized notion of solution for equation (1.2) which includes solutions with a blow up in the derivative (solutions with discontinuities). We say that a bounded measurable function $u(x,t)$, is a weak entropy solution of the inviscid Burgers equation for $0 \le t \le T$, if for any constant $a$ and any smooth function $f(x,t) \ge 0$ which is supported in $\mathbf{R} \times [0,T]$,

$$\int_0^T \int_{-\infty}^{\infty} \{ | u - a | f_t + sgn(u - a)[\tfrac{1}{2}u^2 - \tfrac{1}{2}a^2] f_x \} \, dxdt \ge 0 \tag{1.4}$$

and

$$\lim_{t \to 0} \int_{-r}^{r} | u(x,t) - u_0(x) | \, dx = 0$$

for all $r > 0$.

The first expression above implies that a weak entropy solution of the inviscid Burgers equation satisfies the weak version of the differential equation (1.2). That is, if $u$ satisfies

(1.4), then it must also satisfy

$$\int_0^r \int_{-\infty}^{\infty} [u f_t + \frac{1}{2} u^2 f_x] \, dx dt = 0. \tag{1.5}$$

This can be show by substituting $a = sup(u)$ and $a = inf(u)$ into expression (1.4).

Equation (1.4) also contains an entropy condition which specifies the appropriate physically realistic solution of the weak inviscid Burgers equation. For a survey of equivalent formulations of a general entropy condition see Bardos [2]. Here we have used the formulation used by Volpert [49], § 6.3, and Kruzkov [33], p. 220.

We will now quote two results from Kruzkov's [33] paper. The first concerns the stability in the $L^1$ norm of the weak entropy solutions with respect to changes in the initial data.

**1.3.3. Stability Theorem (Kruzkov [33], p. 223).** *Let $u(x,t)$ and $v(x,t)$ be weak entropy solutions satisfying equation (1.4), with initial data $u_0$ and $v_0$ respectively. Suppose that $|u(x,t)| \leq M$ and $|v(x,t)| \leq M$ almost everywhere in $\mathbb{R} \times [0,T]$. Then, for all $r > 0$ and $t \leq T$,*

$$\int_{-r}^{r} |u(x,t) - v(x,t)| \, dx \leq \int_{-r-tM}^{r+tM} |u_0(x) - v_0(x)| \, dx. \tag{1.6}$$

*In addition, if $\| u_0 - v_0 \|_{L^1} < \infty$, then $\| u(\cdot,t) - v(\cdot,t) \|_{L^1} < \infty$ and*

$$\| u(\cdot,t) - v(\cdot,t) \|_{L^1} \leq \| u_0 - v_0 \|_{L^1}.$$

Just as the stability of the $F_t$ operator implies the boundedness of the variation of the function $F_t u_0$, so the stability of the solution operator of the inviscid Burgers equation implies the following result:

**1.3.4. Theorem.** *Let $u_0$ be a bounded and measurable function such that $\| u_0 \|_{BV} < \infty$. If $u$ is a weak entropy solution of the inviscid Burgers equation with $u_0$ as initial data, then*

$$\| u(\cdot,t) \|_{BV} \leq \| u_0 \|_{BV}.$$

Theorem 1.3.3 also implies that the weak entropy solution of the inviscid Burgers equation is unique. In addition, it follows that the domain of dependence of the solution is bounded. On the other hand Theorem 1.3.3 does not imply the existence of a solution to problem (1.4). The existence of a solution is guaranteed by the next result.

**1.3.5. Existence Theorem (Kruzkov [33], p. 237).** *The solutions $u^{\nu}(x,t)$ of the viscous Burgers equation (equation (1.1)), with viscosity constant $\nu$ and initial data $u_0$, converge as $\nu \to 0$, almost everywhere in $\mathbf{R} \times [0,T]$ to a function $u(x,t)$, which is a weak entropy solution of the inviscid Burgers equation with the same initial data. The convergence also exists in the $L^1$ sense on any compact set in $\mathbf{R} \times [0,T]$.*

**1.3.6. Solution Operator.** Theorems 1.3.3 and 1.3.5 together imply that for any bounded measurable initial data $u_0$, there exists a unique weak entropy solution for the inviscid Burgers equation. We define $A_t u_0$ to be that measurable and bounded solution of equation (1.4) with initial data $u_0$.

As $A_t u_0$ is the point-wise limit of a sequence of bounded functions, we conclude from Theorem 1.2.2 that $A_t u_0$ must be bounded and satisfy the following result:

**1.3.7. Theorem.** *Let $u_0$ be a bounded measurable function. Then for $t \geq 0$,*

$$\| A_t u_0 \|_{L^\infty} \leq \| u_0 \|_{L^\infty}.$$

**1.3.8. Piecewise Smooth Solutions and Riemann's Problem.** Suppose that $u$ is a weak entropy solution of the inviscid Burgers equation, such that $u$ is smooth except on a curve,

$$\Gamma = \{ (s(t),t) : t \in \mathbf{R}^+ \}.$$

Away from the curve $\Gamma$, the function $u$ satisfies the differential equation $u_t + uu_x = 0$. On the curve $\Gamma$ the function must satisfy the Rankine-Hugoniot condition,

$$\dot{s}(t) = \frac{1}{2}[u_L + u_R] \tag{1.11}$$

and the entropy condition,

$$u_R \leq \dot{s}(t) \leq u_L \tag{1.12}$$

where $u_L$ is the left hand limit of $u$ as we approach the point $(s(t), t)$ and $u_R$ is the corresponding right hand limit.

An important example of a piece-wise smooth solution of the inviscid Burgers equation is given by the solution of the Riemann problem for the inviscid Burgers equation. This problem consists of finding the unique weak entropy solution $v(x,t)$ for initial data $v_0(x)$ of the form,

$$\begin{aligned} v_0(x) &= v_0^L \quad \text{if } x < a, \\ &= v_0^R \quad \text{if } x > a \end{aligned}$$

where $v_0^L$ and $v_0^R$ are constants. The solution to the Riemann problem has two forms depending on the values of $v_0^L$ and $v_0^R$.

**1.3.9. The Shock Solution of Riemann's Problem.** Suppose that $v_0^L > v_0^R$. Then the solution of the corresponding Riemann problem is given by,

$$\begin{aligned} v(x,t) &= v_0^L \quad \text{if } x < a + \frac{1}{2}[v_0^L + v_0^R]t, \\ &= v_0^R \quad \text{if } x > a + \frac{1}{2}[v_0^L + v_0^R]t. \end{aligned}$$

The line $s(t) = a + \frac{1}{2}[v_0^L + v_0^R]t$ is called a shock.

**1.3.10. Rarefaction Wave Solution of Riemann's Problem.** For the case of $v_0^L < v_0^R$, the solution of the Riemann problem is given by the function,

$$\begin{aligned} v(x,t) &= v_0^L \quad \text{if } x < a + tv_0^L, \\ &= \frac{x-a}{t} \quad \text{if } a + tv_0^L < x < a + tv_0^R, \end{aligned}$$

$$= \quad v_0^R \qquad \text{if } x > a + tv_0^R .$$

The solutions of the Riemann problem will be of particular importance when we describe our approximate advection operator in Chapter 3. In that chapter we will be studying the solution of the inviscid Burgers equation with initial data given by a step function in S . For that case the solution for small times can be constructed from the solutions of the individual Riemann problems associated with each discontinuity of the initial function.

**1.4. Probability Results.** We will let ( $\Omega$, $\Sigma$, P ) denote a probability space consisting of a point set $\Omega$, a $\sigma$-algebra contained in the power set of $\Omega$ denoted $\Sigma$, and a probability measure defined on $\Sigma$ given by P. A generic element of $\Omega$ will be denoted by $\omega$. The probability of an event $E \in \Sigma$ will be denoted by P($E$). The expected value of a random variable $X$ (measurable function on measure space ($\Omega$, $\Sigma$)) will be given by E $[X]$. For a given $\omega \in \Omega$, a representative of a random variable $X$ will be denoted by $X|_\omega$.

We present two probability results that will be used in the sequel.

**1.4.1 Theorem.** *Let $X_1$ ,.., $X_m$ be random variables and $a_1$ ,.., $a_m$ be constants. Then,*

$$\mathrm{P}\Big(\sum_{i=1}^{m} X_i > \sum_{i=1}^{m} a_i\Big) \le \sum_{i=1}^{m} \mathrm{P}(X_i > a_i).$$

**Proof.** Let the random variables $X_i$ be defined on a common probability space ( $\Omega$, $\Sigma$, P ). For any $\omega \in \Omega$,

$$\sum_{i=1}^{m} X_i \,|_\omega > \sum_{i=1}^{m} a_i$$

implies that $X_i|_\omega > a_i$ for some $i$ . Hence

$$\{ \omega \in \Omega : \sum_{i=1}^{m} X|_{\omega} > \sum_{i=1}^{m} a_i \} \subseteq \bigcup_{i=1}^{m} \{ \omega \in \Omega : X|_{\omega} > a_i \}.$$

Consequently,

$$\mathrm{P}( \sum_{i=1}^{m} X_i > \sum_{i=1}^{m} a_i ) \leq \sum_{i=1}^{m} \mathrm{P}( X_i > a_i ).$$

**1.4.2. Theorem (Chung [16] p. 42).** *Let $Y \geq 0$ be a random variable. Suppose $a > 0$. Then,*

$$\mathrm{E}[Y] \leq a \sum_{j=0}^{\infty} \mathrm{P}( Y > ja ).$$

**Proof.** This result follows from an explicit calculation:

$$\mathrm{E}[Y] \leq \sum_{j=0}^{\infty} a(j+1)\, \mathrm{P}(ja < Y \leq (j+1)a)$$

$$\leq \sum_{j=0}^{\infty} a(j+1)\,[\, \mathrm{P}( Y > ja ) - \mathrm{P}(Y > (j+1)a )\,]$$

$$\leq a \sum_{j=0}^{\infty} \mathrm{P}( Y > ja ).$$

# Chapter 2

## Fractional Step Algorithm with Exact Operators

As stated in the introduction, we plan to prove the convergence of our random walk method by studying the behaviour of the random variable

$$\| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} \tag{2.1}$$

where $\widetilde{D}_k$ and $\widetilde{A}_k$ denote the approximate diffusion and advection operators and $S^0 u_0$ is a step function approximation of the smooth data $u_0$.

In this chapter we will study the behaviour of the related 'exact' fractional step algorithm. Namely, we will study the quantity

$$\| F_{nk} u_0 - [D_k A_k]^n u_0 \|_{L^1}. \tag{2.2}$$

It is intuitively obvious that it would be impossible for the quantity (2.1) to converge to zero as $n \to \infty$ ($nk \leq T$), if expression (2.2) did not converge to zero. Indeed, the proof of convergence for our random walk method, in Chapter 5, uses the fact that expression (2.2) converges to zero.

We obtain a bound on (2.2) by using a technique very similar to that used by Kruzkov, [33] p. 239, to show that the operator $F_k$ is stable in the $L^1$ norm. We will use this technique to prove a general result (Lemma 2.1.1) which is then used to show (Theorem 2.1.3) that the error arising from one time step of the exact fractional step algorithm is of order $k^2$, where $k$ is the time step. In Theorem 2.2.1 we then prove that the error for $n$ time steps can be estimated by summing the estimates for each of the $n$ individual fractional steps. Hence we show that the fractional step algorithm with $n$ steps will have an error which is at most order $nk^2$. For $nk \leq T$, the error will be of order $k$, which implies that the exact fractional step algorithm converges.

Finally in the third section we investigate the behaviour of the algorithm as $\nu \to 0$.

**2.1. Accuracy of Fractional Step Algorithm for One Time Step.** We proceed with the estimation of the accuracy of one time step of the exact fractional step method by first proving the following $L^1$ norm estimate for the solution of a second order parabolic differential equation. The proof is essentially identical to Kruzkov's ([33], p. 239) proof of stability for the operator $F_k$.

**2.1.1. Lemma.** *Let $\nu$ and $T$ be positive, $a, b \in C^3(\mathbb{R} \times [0,T])$ and $w_0 \in C^3(\mathbb{R})$ such that $\| w_0 \|_{L^\infty} < \infty$, $\| w_0 \|_{L^1} < \infty$, $\| w_0 \|_{BD^1} < \infty$, $\| a \|_{L^\infty} < \infty$ and $\| b \|_{L^1} < \infty$. Then the solution, in the region $\mathbf{R} \times [0,T]$, of the equation*

$$w_t + (aw)_x + b = \nu w_{xx} \quad , \quad w(x,0) = w_0(x) \tag{2.3}$$

*satisfies*

$$\int_{-\infty}^{\infty} | w(x,t) | \, dx \le \int_{-\infty}^{\infty} | w_0(x) | \, dx + \int_0^t \int_{-\infty}^{\infty} | b(x,s) | \, dx ds$$

*for $0 \le t \le T$.*

**Proof.** First note that the conditions on $a, b$ and $w_0$ guarantee the existence of a unique solution of equation (2.3), such that $\| w \|_{L^\infty}$ and $\| w_x \|_{L^\infty}$ are bounded (see Oleinik and Kruzkov [39] Theorem 14 together with Kruzkov [33] p. 237).

Let us choose an arbitrary smooth bounded function $g(x,s)$ on $\mathbb{R} \times [0,T]$, which has compact support. If we multiple equation (2.3) by $g$ and integrate over the region $\mathbf{R} \times [0,t]$, then integration by parts implies that

$$\int_{-\infty}^{\infty} w(x,t) g(x,t) \, dx - \int_0^t \int_{-\infty}^{\infty} Lg(x,s) \, w(x,s) \, dx ds \tag{2.4}$$

$$= \int_{-\infty}^{\infty} w_0(x) g(x,0) \, dx - \int_0^t \int_{-\infty}^{\infty} g(x,s) b(x,s) \, dx ds$$

where $Lg = g_t + a g_x + \nu g_{xx}$. We want to extend this result to the case of smooth functions $g$ which decrease exponentially to zero as $|x|$ tends to infinity. Specifically, we want to show that (2.4) holds for functions $g$ which satisfy the condition that there exist constants $c_1, c_2, c_3 > 0$ such that $|g(x,s)| \le c_1 \exp(-c_2 |x|)$ for $(x,s) \in \mathbf{R} \times [0,t]$ such that $|x| > c_3$. Let us suppose that a given function $g$ satisfies such a relation. For a given parameter $R > 0$, let $\eta$ denote a smoothed version of the characteristic function for the region $[-R, R]$. That is, $\eta = \psi * \chi_{[-R,R]}$, where $\psi$ is a smooth positive function with support in [-1,1] whose integral equals one, $\chi_{[-R,R]}$ is the characteristic function of the region $[-R, R]$, and $*$ denotes convolution. Expression (2.4) implies that

$$\int_{-\infty}^{\infty} wg\,\eta\Big|_{s=t} dx - \int_0^t \int_{-\infty}^{\infty} Lg\,\eta w \;dxds \tag{2.5}$$

$$- \int_0^t \int_{-\infty}^{\infty} [wa\,\eta_x - 2\nu w_x \eta_x - \nu w\,\eta_{xx}] g \;dxds$$

$$= \int_{-\infty}^{\infty} w_0\, g\,\eta\Big|_{s=0} dx - \int_0^t \int_{-\infty}^{\infty} g\,\eta b \;dxds.$$

We note that the term

$$[wa\,\eta_x - 2\nu w_x \eta_x + \nu w\,\eta_{xx}]$$

is bounded and has support contained in two compact regions centered about the points $-R$ and $R$. Here we have used the fact that $\| w \|_{L^\infty}$ and $\| w_x \|_{L^\infty}$ are bounded (see the remark at the beginning of the proof). Since $g$ is assumed to decrease to zero as $|x|$ approaches infinity, it follows that

$$\lim_{R \to \infty} \int_0^t \int_{-\infty}^{\infty} [wa\,\eta_x - 2\nu w_x \eta_x + \nu w\,\eta_{xx}] g \;dxds = 0.$$

Obviously $g\,\eta$ convergences pointwise to the integrable function $g$, as $R \to \infty$. Hence the Lebesque dominated convergence theorem can be applied to expression (2.5) to show that $g$ satisfies (2.4) provided $Lg$ is integrable.

For $\alpha > 0$, let $\psi^\alpha$ be a smooth approximate delta function with compact support. Specifically, for a smooth positive function, $\psi$, with support contained in [-1,1], we define $\psi^\alpha$ by the relation

$$\psi^\alpha(x) = \frac{1}{\alpha}\psi(\frac{x}{\alpha}).$$

Now choose an $r > 0$ and let $\beta(x) = \text{sgn}(w(x,t))$ for $x \in [-r,r]$ and 0 otherwise. Define $\beta^\alpha = \psi^\alpha * \beta$. Note that $\beta^\alpha$ has compact support. Let $g^\alpha(x,t)$ be the unique bounded solution of linear parabolic equation $Lg^\alpha = 0$ in $\mathbf{R} \times [0,t]$, with initial conditions $g^\alpha(x,t) = \beta^\alpha(x)$, where we note that we are solving backwards in time. The uniqueness and existence of the solution $g^\alpha$ is given by Friedman [24] (p. 25 and p. 29). Using the compactness of the support of the initial data $g^\alpha(x,t)$ it is possible to show that $g^\alpha$ decreases to zero exponentially as $|x| \to \infty$ (see Kruzkov [33], lemma 4). Hence expression (2.4) is satisfied with $g$ replaced by $g^\alpha$.

Now the maximum principle (Friedman [24] p. 34) implies that for $0 \le s \le t$,

$$| g^\alpha(x,s) | \le \sup_{x \in \mathbf{R}} | g^\alpha(x,t) | \le \| \beta^\alpha \|_{L^\infty} \le 1. \tag{2.6}$$

Hence equations (2.4) and (2.6), together with the assumptions that $\| w_0 \|_{L^1}$ and $\| b \|_{L^1}$ are finite, imply that

$$| \int_{-\infty}^{\infty} w(x,t) g^\alpha(x,t)\, dx | \le \int_{-\infty}^{\infty} | w_0(x) |\, dx + \int_0^t \int_{-\infty}^{\infty} | b(x,s) |\, dx ds.$$

The function $\beta$ is a bounded measurable function and so it follows that $\beta^\alpha(x)$ converges to $\beta(x)$ for each Lebesque point, $x$, of the function $\beta$ (Kruzkov [33], p. 221). The set of Lebesgue points of $\beta$ has zero measure and so $\beta^\alpha$ converges to $\beta$ pointwise almost everywhere on $\mathbf{R}$. That is, $g^\alpha(x,t)$ converges to $\beta(x)$ pointwise almost everywhere on $\mathbb{R}$. Hence, it follows from the Lebesque dominated convergence theorem that

$$\int_{-r}^{r} | w(x,t) |\, dx \le \int_{-\infty}^{\infty} | w_0(x) |\, dx + \int_0^t \int_{-\infty}^{\infty} | b(x,s) |\, dx ds.$$

Finally the lemma is concluded by applying the monotone convergence theorem as $r \to \infty$.

In the next lemma we will prove a technical result that will be of use in the proof of Theorem 2.1.3.

**2.1.2. Lemma.** *Let* $f, g \in C^1(\mathbf{R})$.

*(i) If* $\|f\|_{L^\infty} < \infty$ *and* $\|g\|_{BV} < \infty$, *then*

$$\|D_t f D_t g - D_t f g\|_{L^1} \leq 2(\nu t)^{1/2} \|f\|_{L^\infty} \|g\|_{BV}.$$

*(ii) If* $\|f\|_{L^1} < \infty$ *and* $\|g\|_{BD^1} < \infty$, *then*

$$\|D_t f D_t g - D_t f g\|_{L^1} \leq 2(\nu t)^{1/2} \|f\|_{L^1} \|g\|_{BD^1}.$$

**Proof.** Let us introduce the notation

$$G_{2\nu t}(x) = \frac{1}{(4\pi\nu t)^{1/2}} \exp\left(-\frac{x^2}{4\nu t}\right)$$

and observe that $D_t = G_{2\nu t} *$, that is, $D_t$ is equivalent to convolution by the function $G_{2\nu t}$. In addition, observe that

$$\int_{-\infty}^{\infty} G_{2\nu t}(x - y_2)\, dy_2 = 1.$$

Hence,

$$\|D_t f D_t g - D_t f g\|_{L^1} = \|G_{2\nu t} * f\, G_{2\nu t} * g - G_{2\nu t} * (f g)\|_{L^1}$$

$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(x - y_1) G_{2\nu t}(x - y_2) f(y_1) g(y_2) \right.$$

$$\left. - G_{2\nu t}(x - y_1) G_{2\nu t}(x - y_2) f(y_1) g(y_1)\, dy_1\, dy_2 \right| dx$$

$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(x - y_1) G_{2\nu t}(x - y_2) f(y_1) [g(y_2) - g(y_1)]\, dy_1\, dy_2 \right| dx.$$

That is, we have

$$\|D_t f D_t g - D_t f g\|_{L^1} \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(x - y_1) G_{2\nu t}(x - y_2) f(y_1) [g(y_2) - g(y_1)]\, dy_1\, dy_2 \right| dx.$$

Let us suppose that the condition in $(i)$ holds. Then we can use the integral form of the mean value theorem to bound the second term in expression (2.7) by the quantity

$$\|f\|_{L^\infty} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(x-y_1) G_{2\nu t}(x-y_2) \mid y_2-y_1 \mid \ \mid g_x(y_1+\theta(y_2-y_1)) \mid dy_1 dy_2 dx d\theta$$

$$\leq \|f\|_{L^\infty} \| g \|_{BV} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(y_1) G_{2\nu t}(y_2) \mid y_2-y_1 \mid dy_1 dy_2$$

$$= \frac{(8\nu t)^{1/2}}{\pi^{1/2}} \|f\|_{L^\infty} \| g \|_{BV} \leq 2(\nu t)^{1/2} \|f\|_{L^\infty} \| g \|_{BV}$$

which proves the first part of the lemma. Note that we have used the result

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(y_1) G_{2\nu t}(y_2) \mid y_2-y_1 \mid dy_1 dy_2 = \frac{(8\nu t)^{1/2}}{\pi^{1/2}}$$

which is obtained by an explicit calculation in which we make a change of variables and interchange the order of integration.

Let us suppose that the condition in (*ii*) holds. The differential mean value theorem shows that (2.7) is bounded by

$$\| g \|_{BD^1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(x-y_1) G_{2\nu t}(x-y_2) \mid f(y_1) \mid \ \mid y_2-y_1 \mid dy_1 dy_2 dx$$

$$\leq \|f\|_{L^1} \| g \|_{BD^1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{2\nu t}(y_1) G_{2\nu t}(y_2) \mid y_2-y_1 \mid dy_1 dy_2$$

$$\leq 2(\nu t)^{1/2} \|f\|_{L^1} \| g \|_{BD^1}.$$

This completes the proof of Lemma 2.1.2.

**2.1.3. Theorem.** *Let* $v_0 \in C^2(\mathbf{R})$ *such that* $\| v_0 \|_{BD^j} < \infty$, $j = 0,1,2$ *and* $\| v_0 \|_{BV} < \infty$. *Let* $0 \leq k \leq \frac{1}{2} \| v_0 \|_{BD^1}^{-1}$ *and* $\nu > 0$. *Then*

$$\| F_k v_0 - D_k A_k v_0 \|_{L^1} \leq B_1 k^2 + B_2 k^{5/2}$$

*where*

$$B_1 = 8\nu \| v_0 \|_{BV} \| v_0 \|_{BD^2}$$

*and*

$$B_2 = 5\nu^{1/2}\| v_0 \|_{BV} [\, 4\| v_0 \|_{BD^2}\| v_0 \|_{L^\infty} + \| v_0 \|^2_{BD^1} ].$$

**Proof.** We will show that the function

$$w(x,t) = F_t v_0(x) - D_t A_t v_0(x)$$

satisfies an equation of form (2.3). This will allow us to apply Lemma 2.1.1 and so obtain an estimate of the $L^1$ error for one step of the fractional step algorithm. The main part of the proof involves estimating the $L^1$ norm of the function $b(x,t)$ (see equation (2.8)). Lemma 2.1.1 will be used to obtain an estimate of the size of $b$. This leads to a problem of estimating quantities of the form

$$\| D_t f D_t g - D_t f g \|_{L^1},$$

which can be estimated using Lemma 2.1.2.

We will first introduce some notation. Let $v^\nu(x,t) = F_t v_0(x)$ and $v(x,t) = A_t v_0(x)$. Using the notation introduced in the last lemma, we write the function $D_t A_t v_0$ as $G_{2\nu t} * v$. Then

$$w(x,t) = v^\nu(x,t) - G_{2\nu t} * v(x,t).$$

Since $v(x,t)$ is a solution of the invicsid Burgers equation, it follows from Theorem 1.3.1 that the solution is smooth for $0 \le t \le \frac{1}{2}\| v_0 \|^{-1}_{BD^1}$. Since $v^\nu$ satisfies the viscous Burgers equation and $G_{2\nu t}(x)$, as a function of $x$ and $t$, satisfies the diffusion equation, it follows that for $0 < t < \frac{1}{2}\| v_0 \|^{-1}_{BD^1}$, the function $w$ satisfies

$$w_t + (aw)_x + b = \nu w_{xx} \quad , \quad w(x,0) = 0 \tag{2.8}$$

where

$$a = \frac{1}{2}[v^\nu + G_{2\nu t} * v]$$

and

$$b = [(G_{2\nu t} * v)(G_{2\nu t} * v_x) - G_{2\nu t} * (vv_x)].$$

Theorems 1.2.2 and 1.3.1, together with the smoothness of the diffusion operator imply that $a, b \in C^3(\mathbb{R})$ and that $\| a \|_{L^\infty} \leq \infty$. Hence, Lemma 2.1.1 implies that

$$\| w(\cdot,k) \|_{L^1} \leq \int_0^k \int_{-\infty}^{\infty} | b(x,t) | \, dxdt \tag{2.9}$$

provided the right hand side is finite.

It follows from Lemma 2.1.2 ($i$) (with $f = v_x$ and $g = v$) that

$$\int_{-\infty}^{\infty} | b(x,t) | \, dx \leq 2(\nu t)^{1/2} \| v \|_{BD^2} \| v \|_{BV} .$$

Since $0 < t < \frac{1}{2} \| v_0 \|_{BD^1}^{-1}$, it follows from Theorem 1.3.1 that the functions $v$, $v_x$ and $v_{xx}$ are defined and satisfy

$$\| v \|_{L^\infty} \leq \| v_0 \|_{L^\infty} , \| v \|_{BD^1} \leq 2 \| v_0 \|_{BD^1} , \| v \|_{BD^2} \leq 8 \| v_0 \|_{BD^2} . \tag{2.10}$$

In addition by Theorem 1.3.4 we have that

$$\| v \|_{BV} \leq \| v_0 \|_{BV} . \tag{2.11}$$

Hence we conclude that

$$\int_{-\infty}^{\infty} | b(x,t) | \, dx \leq 16(\nu t)^{1/2} \| v_0 \|_{BD^2} \| v_0 \|_{BV}$$

and so by expression (2.9) we have that

$$\| w(\cdot,k) \|_{L^1} \leq 11 \nu^{1/2} k^{3/2} \| v_0 \|_{BD^2} \| v_0 \|_{BV}$$

While this result is sufficient to show that the exact fractional step algorithm converges, it is possible to obtain a sharper result by a closer examination of $b$. We will use another application of Lemma 2.1.1 to obtain a better estimate for the size of $\int_{-\infty}^{\infty} | b(x,t) | \, dx$ for $0 \leq t \leq k$. We first observe that in the region $\mathbb{R} \times [0,k]$ the function $b$ satisfies the equation

$$b_t + (ab)_x + c = \nu b_{xx} \quad , \quad b(x,0) = 0 \tag{2.12}$$

with

$$a = -G_{2\nu t} * v \quad , \quad c = c_1 + c_2 + c_3$$

where

$$c_1 = 2\nu (G_{2\nu t} * v_x)(G_{2\nu t} * v_{xx})$$

$$c_2 = 2[(G_{2\nu t} * v)(G_{2\nu t} * v_x)^2 - G_{2\nu t} * v\,(v_x)^2]$$

$$c_3 = [(G_{2\nu t} * v)^2 (G_{2\nu t} * v_{xx}) - G_{2\nu t} * v^2 v_{xx}].$$

Expression (2.10) implies that the $L^1$ norm of the quantity $c_1$ satisfies the following estimate:

$$\int_{-\infty}^{\infty} |\, c_1(x,t) \,| \;\; dx \;\leq 2\nu \| G_{2\nu t} * v_x \|_{L^1} \| G_{2\nu t} * v_{xx} \|_{L^\infty} \tag{2.13}$$

$$\leq 16\nu \| v_0 \|_{BV} \| v_0 \|_{BD^2}.$$

Estimates for the $L^1$ norms of $c_2$ and $c_3$ can be obtained by using Lemma 2.1.2.

$$\int_{-\infty}^{\infty} |\, c_2(x,t) \,| \;\; dx$$

$$\leq 2 \| (G_{2\nu t} * v)(G_{2\nu t} * v_x)^2 - (G_{2\nu t} * v_x)(G_{2\nu t} * v v_x) \|_{L^1}$$

$$+ 2 \| (G_{2\nu t} * v_x)(G_{2\nu t} * v v_x) - G_{2\nu t} * v\,(v_x)^2 \|_{L^1}$$

The first normed quantity can be estimated as follows:

$$\| (G_{2\nu t} * v)(G_{2\nu t} * v_x)^2 - (G_{2\nu t} * v_x)(G_{2\nu t} * v v_x) \|_{L^1}$$

$$\leq \| G_{2\nu t} * v_x \|_{L^\infty} \| (G_{2\nu t} * v)(G_{2\nu t} * v_x) - G_{2\nu t} * v v_x \|_{L^1}$$

$$\leq 2(\nu t)^{1/2} \| v_x \|_{L^\infty}^2 \| v_x \|_{L^1} \leq 8 (\nu t)^{1/2} \| v_0 \|_{BD^1}^2 \| v_0 \|_{BV}.$$

Here we have used Lemma 2.1.2 ($i$), with $f = v_x$ and $g = v$, to bound the quantity

$$\| (G_{2\nu t} * v)(G_{2\nu t} * v_x) - G_{2\nu t} * v v_x \|_{L^1}.$$

In addition, we have have used expressions (2.10) and (2.11) to bound the norm quantities at time $t$, in terms of the initial data $v_0$.

If we use condition ($ii$) of Lemma 2.1.2 with $f = v v_x$ and $g = v_x$, then we conclude that

$$\| (G_{2\nu t} * v_x)(G_{2\nu t} * v v_x) - G_{2\nu t} * v\,(v_x)^2 \|_{L^1}$$

$$\leq 2(\nu t)^{1/2} \| v_{xx} \|_{L^\infty} \| v v_x \|_{L^1} \leq 2(\nu t)^{1/2} \| v_{xx} \|_{L^\infty} \| v \|_{L^\infty} \| v_x \|_{L^1}$$

$$\leq 16(\nu t)^{1/2} \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} \| v_0 \|_{BV} .$$

Consequently

$$\int_{-\infty}^{\infty} | c_2(x,t) | \; dx \tag{2.14}$$

$$\leq 16(\nu t)^{1/2} \| v_0 \|_{BV} [2 \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} + \| v_0 \|_{BD^1}^2 ].$$

Similarly, we can use condition ($i$) of Lemma 2.1.2, with $f = v v_{xx}$ and $g = v$, to show that

$$\int_{-\infty}^{\infty} | c_3(x,t) | \; dx \leq 32(\nu t)^{1/2} \| v_0 \|_{BV} \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} . \tag{2.15}$$

Results (2.13), (2.14) and (2.15) imply that for $0 \leq s \leq \frac{1}{2} \| v_0 \|_{BD^1}^{-1}$

$$\int_{-\infty}^{\infty} | c(x,s) | \; dx$$

$$\leq 16\nu \| v_0 \|_{BV} \| v_0 \|_{BD^2} + 16(\nu s)^{1/2} \| v_0 \|_{BV} [4 \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} + \| v_0 \|_{BD^1}^2 ].$$

Lemma 2.1.1, together with the fact that $b$ satisfies equation (2.12) implies that

$$\int_{-\infty}^{\infty} | b(x,t) | \; dx \leq \int_0^t \int_{-\infty}^{\infty} | c(x,s) | \; dx ds$$

$$\leq 16\nu t \| v_0 \|_{BV} \| v_0 \|_{BD^2} + 11\nu^{1/2} t^{3/2} \| v_0 \|_{BV} [ 4 \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} + \| v_0 \|_{BD^1}^2 ].$$

If we substitute this result into expression (2.9) then we conclude that

$$\| w(\cdot,k) \|_{L^1}$$

$$\leq 8\nu k^2 \| v_0 \|_{BV} \| v_0 \|_{BD^2} + 5\nu^{1/2} k^{5/2} \| v_0 \|_{BV} [ 4 \| v_0 \|_{BD^2} \| v_0 \|_{L^\infty} + \| v_0 \|_{BD^1}^2 ]$$

which concludes the proof of the theorem.

**2.2. Convergence of the Fractional Step Algorithm.** The preceding result estimates the error arising from one time step. For $n$ time steps, the error can be estimated by summing the errors from each individual time step.

**2.2.1. Theorem.** *Let* $u_0 \in C^2(\mathbb{R})$ *such that* $\| u_0 \|_{BD^j} < \infty$, *for* $j = 0,1,2$ *and* $\| u_0 \|_{BV} < \infty$. *Then for* $n \in \mathbb{Z}^+$ *and* $0 < k < \frac{\nu}{2}(\| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1})^{-1}$,

$$\| F_{nk} u_0 - [D_k A_k]^n u_0 \|_{L^1} \leq C_2 n k^2$$

*where*

$$C_2 = \frac{90}{\nu} \| u_0 \|_{BV} [\nu^2 \| u_0 \|_{BD^2} + \nu \| u_0 \|_{L^\infty} \| u_0 \|_{BD^1} + \| u_0 \|^3_{L^\infty}]$$

$$+ \frac{4}{\nu} \| u_0 \|_{BV} [\, \| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1}]^{3/2}.$$

**Proof.** We will prove this result using induction. Let us assume that

$$\| F_{jk} u_0 - [D_k A_k]^j u_0 \|_{L^1} \leq C_2 j k^2. \tag{2.16}$$

We must then show that

$$\| F_{(j+1)k} u_0 - [D_k A_k]^{j+1} u_0 \|_{L^1} \leq C_2 (j+1) k^2.$$

The triangle inequality and the $L^1$ stability of the operators $D_t$ and $A_t$ (Theorem 1.3.3) imply that

$$\| F_{(j+1)k} u_0 - [D_k A_k]^{j+1} u_0 \|_{L^1} \tag{2.17}$$

$$\leq \| F_{(j+1)k} u_0 - [D_k A_k] F_{jk} u_0 \|_{L^1} \tag{2.18}$$

$$+ \| F_{jk} u_0 - [D_k A_k]^j u_0 \|_{L^1}. \tag{2.19}$$

If we let $v_0 = F_{jk} u_0$, then expression (2.18) can be written as

$$\| F_k v_0 - [D_k A_k] v_0 \|_{L^1}.$$

We can use Theorem 2.1.3 to estimate this quantity, since by Theorem 1.2.2 we have

$$0 \leq k \leq \frac{\nu}{2}(\| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1})^{-1} \leq \frac{1}{2} \| v_0 \|^{-1}_{BD^1}.$$

Consequently, for $0 \leq k \leq \frac{\nu}{2}(\| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1})^{-1}$

$$\| F_k v_0 - [D_k A_k] v_0 \|_{L^1}$$

$$\leq [8\nu\| v_0 \|_{BD^2} + 20(\nu k)^{1/2}\| v_0 \|_{BD^2}\| v_0 \|_{L^\infty} + 5(\nu k)^{1/2}\| v_0 \|^2_{BD^1}] \; \| v_0 \|_{BV}\, k^2.$$

Using the apriori bounds obtained in Theorem 1.2.2 we conclude that

$$8\nu\| v_0 \|_{BD^2} \leq \frac{32}{\nu}[\nu^2\| u_0 \|_{BD^2} + \nu\| u_0 \|_{L^\infty}\| u_0 \|_{BD^1} + \| u_0 \|^3_{L^\infty}].$$

Since $k \leq \frac{\nu}{2}\| v_0 \|^{-2}_{L^\infty}$, we have that

$$20(\nu k)^{1/2}\| v_0 \|_{BD^2}\| v_0 \|_{L^\infty} \leq \frac{58}{\nu}[\nu^2\| u_0 \|_{BD^2} + \nu\| u_0 \|_{L^\infty}\| u_0 \|_{BD^1} + \| u_0 \|^3_{L^\infty}].$$

Finally, using the fact that $k \leq \frac{1}{2}\| v_0 \|^{-1}_{BD^1}$, we find that

$$5(\nu k)^{1/2}\| v_0 \|^2_{BD^1} \leq 4\nu^{1/2}\| v_0 \|^{3/2}_{BD^1} \leq \frac{4}{\nu}[\| u_0 \|^2_{L^\infty} + \nu\| u_0 \|_{BD^1}]^{3/2}.$$

Combining these estimates we conclude that

$$\| F_k v_0 - [D_k A_k] v_0 \|_{L^1} \leq C_2 k^2. \tag{2.20}$$

Note that if $j = 0$, then $v_0 = u_0$, and we have

$$\| F_k u_0 - [D_k A_k] u_0 \|_{L^1} \leq C_2 k^2,$$

which shows that the initial proposition of our induction hypothesis is true.

Now, if we compare (2.18) with (2.20) and (2.16) with (2.19), we conclude that if

$$\| F_{jk} u_0 - [D_k A_k]^j u_0 \|_{L^1} \leq C_2 j \; k^2.$$

then

$$\| F_{(j+1)k} u_0 - [D_k A_k]^{j+1} u_0 \|_{L^1} \leq C_2 (j+1) k^2.$$

Hence our induction proof is complete.

**2.3. Behaviour of the Algorithm for small $\nu$.** In this section we will study the behaviour of the exact fractional step algorithm for one time step as $\nu \to 0$. Specifically, we will investigate the actual behaviour of the error for one time step when the initial data is given by a steady state solution of the viscous Burgers equation. With such a choice of initial data we are able to give an explicit formulation for the error produced in one time step. Let,

$$f_\nu(x) = \frac{1 - e^{x/\nu}}{1 + e^{x/\nu}}.$$

The function $f_\nu$ is a steady state solution of the viscous Burgers equation with diffusion constant $\nu$ and so $F_t f_\nu = f_\nu$ ($F$ depends on $\nu$). The error produced by one time step of the exact fractional step algorithm is given by

$$e_{\nu,t} = \| F_t f_\nu - D_t A_t f_\nu \|_{L^1}.$$

Let $g_\nu(x,t) = A_t f_\nu(x)$. Now for $t < 1$, $g_1(x,t)$ satisfies the implicit equation for the solution of the inviscid Burgers equation with initial data $f_1$ (see Theorem 1.3.1). That is,

$$g_1(x,t) = f_1(x - t g_1(x,t)).$$

Hence, for $t < \nu$

$$g_1(\frac{x}{\nu}, \frac{t}{\nu}) = f_1(\frac{x}{\nu} - \frac{t}{\nu} g_1(\frac{x}{\nu}, \frac{t}{\nu}))$$

$$= f_\nu(x - t g_1(\frac{x}{\nu}, \frac{t}{\nu})).$$

The function $g_1(\frac{x}{\nu}, \frac{t}{\nu})$ solves the implicit equation for the inviscid Burgers equation with initial data $f_\nu(x)$ and so $g_\nu(x,t) = g_1(\frac{x}{\nu}, \frac{t}{\nu})$.

If we substitute this result into the equation for $e_{\nu,t}$ and make a change of variables, then we obtain

$$e_{\nu,t} = \| f_1(\frac{\cdot}{\nu}) - G_{2\nu t} * g_1(\frac{\cdot}{\nu}, \frac{t}{\nu}) \|_{L^1}$$

$$= \nu \int_{-\infty}^{\infty} | f_1(x) - G_{2t/\nu} * g_1(x, \frac{t}{\nu}) | \, dx = \nu h(\frac{t}{\nu})$$

where

$$h(a) = \int_{-\infty}^{\infty} | f_1(x) - G_{2a} * g_1(x,a) | \, dx.$$

From Theorem 2.1.3. we conclude $e_{\nu,t}$ is of order $t^2$ as $t \to 0$. Consequently, $h(a)$ must be of order $a^2$ as $a \to 0$ and so $e_{\nu,t} \leq Ct^2$, where $C \to \infty$ as $\nu \to 0$, provided $t < \nu$. Hence, the small $\nu$ behaviour of the one step error as obtained in Theorem 2.1.3 is consistent with the actual error obtained with the specific initial data $f_\nu$.

On the other hand, for a fixed time step $k$ the error of the fractional step algorithm tends to zero as $\nu \rightarrow 0$. To see this, observe that there exists a constant $B$ such that,

$$\| D_k v_0 - v_0 \|_{L^1} \leq B \; \| v_0 \|_{BV} (2\nu k)^{1/2}.$$

Theorem 1.3.5 implies that

$$\lim_{\nu \rightarrow 0} \| F_{nk} u_0 - A_{nk} u_0 \|_{L^1} = 0.$$

The stability of the diffusion and transport operators can then be used to show that

$$\| F_{nk} u_0 - [D_k A_k]^n u_0 \|_{L^1}$$
$$\leq \| F_{nk} u_0 - A_{nk} u_0 \|_{L^1} + nB \; \| u_0 \|_{BV} (2\nu k)^{1/2}$$

Hence the error of the fractional step algorithm tends to zero as $\nu \rightarrow 0$ (at least) if the size of the time step is fixed.

## Chapter 3

## The Approximate Advection Operator

The purpose of this chapter is to describe our approximation advection operator. That is, the operator with which we approximate the solution of the inviscid Burgers equation. The operator will be defined in terms of the evolution of the positions of a set of particles. In Section 3.1 we will define an evolution equation for these particles which approximates the way in which particles would flow under the influence of the exact velocity field. The accuracy of the method is then discussed in Section 3.2 where it is shown that the error in the $L^1$ norm is bounded by a quantity of order $h$, where $h$ is the absolute strength of each particle. Finally in Section 3.3 we generalize our definition to encompass initial data which is stochastic in nature. Specifically we define our approximate advection operator for particles whose initial positions are described by random variables.

**3.1. Definition of Method.** We assume that our initial data $s \in S$ is generated by $m$ basic particles with initial positions $\{x_i\}_{i=1}^{m}$ and strengths $\{\varsigma_i\}_{i=1}^{m}$ with $|\varsigma_i| = h$, such that

$$s(y) = s^L + \sum_{i=1}^{m} H(y - x_i)\, \varsigma_i .$$

The definition of the numerical method is obtained by specifying the evolution of the positions of these particles.

We first observe that for small times, the weak entropy solution of the inviscid Burgers equation with step functions as initial data can be obtained by splicing together the Riemann problem solutions (see sections 1.3.8-1.3.10) associated with each discontinuity considered separately.

We define the strength of the discontinuity of the step function $s \in S$ at the particle position $x_i$ by

$$[s]_i = s(x_i+0) - s(x_i-0) = \sum_{j:x_j=x_i} \varsigma_j .$$

Hence $[s]_i$ is the total strength of all the particles positioned at $x_i$.

The solution to the Riemann problem takes two distinct forms depending on the sign of the discontinuity, one being a shock solution, the other a rarefaction wave. Hence the way in which the $i^{th}$ particle is transported by our approximate operator will be determined by the sign of the quantity $[s]_i$.

For particles that form a negative discontinuity (particles with $[s]_i$ negative), we want the trajectories of the particles to coincide with the shock that occurs in the exact solution. This is accomplished by moving all of the particles that generate a specific negative discontinuity with a common velocity given by the Rankine-Hugoniot condition for that discontinuity. Consequently, if $[s]_i \leq 0$, we define the velocity of the $i^{th}$ particle to be,

$$s_i = s^L + \sum_{j:x_j<x_i} \varsigma_j + \frac{1}{2} \sum_{j:x_j=x_i} \varsigma_j . \tag{3.1}$$

This implies that

$$s_i = \frac{s(x_i-0) + s(x_i+0)}{2}$$

(see figure 2) which is the velocity given by the Rankine-Hugoniot condition.

Suppose that a positive discontinuity is generated by $q+p$ particles with positive strength, and $p$ particles with negative strength, where $qh$ is the strength of the discontinuity and $p$ is some non-negative integer. We will allow the trajectories of the first $q$ particles with positive strength to fan out in such a way as to approximate the exact rarefaction solution. The other $2p$ particles will be given a common velocity and allowed to evolve together. Hence, if $[s]_i > 0$, we define the velocity of the $i^{th}$ particle as follows:

If $\varsigma_i > 0$ and

$$\sum_{j\leq i:x_j=x_i,\varsigma_j>0} \varsigma_j \leq [s]_i \tag{3.2}$$

(condition for being in the group of particles that will approximate the rarefaction fan) then

$$s_i = s^L + \sum_{j : x_j < x_i} \varsigma_j + \sum_{j < i : x_j = x_i, \varsigma_j > 0} \varsigma_j + \frac{1}{2} \varsigma_i . \qquad (3.3)$$

If on the other hand $\varsigma_i < 0$ or $\varsigma_i > 0$ and condition (3.2) does not hold, then the velocity of the $i^{th}$ particle is given by

$$s_i = s^L + \sum_{j : x_j < x_i} \varsigma_j + [s]_i = s(x_i + 0) \qquad (3.4)$$

(see figure 3).

Note that $s_i$ depends both on the ordering of the particles (relative to the index $i$) and on the ordering of the particle positions.

The positions of the particles at a later time $t$, denoted $x_i(t)$, are then given by the equation

$$x_i(t) = x_i + t s_i . \qquad (3.5)$$

(see figures 2 and 3) provided $t$ satisfies the condition that for all $i$ and $j$ with $x_i < x_j$, we have that $x_i(t) \leq x_j(t)$. That is, we use equation (3.5) to evolve the particle positions until such time as the trajectories of two particles, initially at distinct positions, have intersected (see figure 4). Let us denote the time of first intersection by $t^1$. We conclude, from the definition of the velocities $s_i$, that the step function generated by the particles at time $t$ ($0 < t < t^1$) is composed of positive discontinuities generated by distinct particles with positive strength and by groups of particles with common position and accumulated strength which is non-positive. We will refer to these groups of particles as numerical shocks. The strength of a numerical shock is given by the quantity

$$\sum_{j : x_j(t) = x_i(t)} \varsigma_j$$

where the $i^{th}$ particle is any one of the particles forming the numerical shock at time $t$. We note that the strength of each numerical shock is non-positive. In addition, if the $i^{th}$ particle satisfies the condition

$$\sum_{j : x_j(t) = x_i(t)} \varsigma_j \leq 0$$

then that particle is contained in a numerical shock.

For $0 < t < t^1$, let us observe the following facts;

(1) The distance between two adjacent discontinuities of positive strength increases as $t$ increases (see figure 5).

(2) The distance between a positive discontinuitity and an adjacent numerical shock will only decrease with time if the strength of the numerical shock is less than or equal to $-2h$ (see figure 6).

(3) We conclude from (1) and (2) that at time $t^1$, the accumulated strength of the particles intersecting at a point must be non-positive. Hence at time $t^1$, at each point of intersection, numerical shocks and positive discontinuities join to form one numerical shock. So at time $t^1$ the sum of the number of numerical shocks and the number of discontinuities of size $h$ decreases.

We will extend our definition past time $t^1$ by using an inductive argument. Suppose that the positions, $x_i(t)$, of our particles are given for $0 \leq t \leq t^l$, where $l$ is an integer greater than or equal to one. In addition suppose that for $0 < t \leq t^1$

$$\sum_{j : x_j(t) = x_i(t)} \varsigma_j > 0,$$

can only hold if $x_j(t) = x_i(t)$ implies that $i = j$. That is, we suppose that the positive discontinuities of the step functions generated by the particles at any time $0 < t \leq t^l$ are of strength $h$ and are generated by distinct particles.

Let $x_i^l = x_i(t^l)$ and let $s^l$ be the correponding step function

$$s^l(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - x_i^l)\, \varsigma_i \,.$$

Let the velocity of the particle with position $x_i^l$ be defined as follows:

$$s_i^l = s^L + \sum_{j : x_j^l < x_i^l} \varsigma_j + \frac{1}{2} \sum_{j : x_j^l = x_i^l} \varsigma_j . \tag{3.6}$$

Note that equations (3.3) and (3.4) reduce to equation (3.6) since we are assuming that the positive discontinuities of the function $s^l$ are generated by individual particles.

Let

$$t^{l+1} = \min_{i,j} \{ t^l + \frac{x_j^l - x_i^l}{s_i^l - s_j^l} : x_j^l > x_i^l \text{ and } s_i^l > s_j^l \}. \tag{3.7}$$

If there does not exist an $i$ and $j$ such that $x_j^l > x_i^l$ and $s_i^l > s_j^l$ then we let $t^{l+1} = \infty$. We observe that $t^{l+1} > t^l$ (provided $t^l$ is finite). This follows from the fact that there exists a bound on the magnitude of the velocities $s_i^l$.

Now for $t^l < t \leq t^{l+1}$, we define $x_i(t)$ to be

$$x_i(t) = x_i^l + (t - t^l) s_i^l. \tag{3.8}$$

Observe that for any $i$ and $j$ such that $x_i^l < x_j^l$, we have that $x_i(t) \leq x_j(t)$ for $t^l < t \leq t^{l+1}$. In addition, if $t^{l+1}$ is finite, then there exist an $i$ and $j$ such that $x_i^l < x_j^l$ and $x_i(t^{l+1}) = x_j(t^{l+1})$. Hence $t^{l+1}$ gives the time of first intersection of the trajectories $x_i(t)$ for $t > t^l$.

If $t^l < t \leq t^{l+1}$, then equation (3.8) can be rewritten in the form,

$$x_i(t) = x_i + \sum_{j=1}^{l-1} (t^{j+1} - t^j) s_i^j + (t - t^l) s_i^l \tag{3.9}$$

This reformulation allows us to estimate the maximum displacement that a particle can receive via this algorithm. In particular, equation (3.9) implies that for $t^l < t \leq t^{l+1}$,

$$| x_i(t) - x_i | \leq t \sup_l |s_i^l| \leq t ( |s^L| + mh ). \tag{3.10}$$

Notice that the maximum displacement of a particle depends only on the initial data and on the time $t$.

The description of the evolution of the particles will be complete provided we can show that $\sup_l (t^l)$ is infinite. In fact we can show that there exists an $l$ such that $t^{l+1} = \infty$. Fact

(3) can be shown to hold for all intersection times $t^l$. Hence, at each time $t^l$, the sum of the number of numerical shocks and the number of discontinuities of strength $h$ decreases. As there are only a finite number of particles, these events can only occur a finite number of times. So there must exist a time $t^l$ after which no more interactions occur. Hence $t^{l+1} = \infty$ for some $l$, as required. Note that since $\sup_l (t^l)$ is infinite, we conclude that expression (3.10) holds for any $t \geq 0$.

We now use the above evolution operator to define the approximate advection operator as follows:

$$\widetilde{A}_t (x_1, .., x_m) = (x_1(t), .., x_m(t)). \tag{3.11a}$$

We will also denote the step function generated by the particles with positions $x_i(t)$ by

$$\widetilde{A}_t s(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - x_i(t)) \varsigma_i. \tag{3.11b}$$

Hence, we will regard $\widetilde{A}_t$ as either an operator on step functions, or as an operator on particle positions, but we will always assume that there is an underlying set of generating particles on which $\widetilde{A}_t$ is defined.

We remark that the operator $\widetilde{A}_t$ satisfies the semi-group property relative to the variable $t$. That is, if $\tau_1, \tau_2 > 0$, then for any step function $s \in \mathrm{S}$

$$\widetilde{A}_{\tau_1} \widetilde{A}_{\tau_2} s = \widetilde{A}_{\tau_1 + \tau_2} s. \tag{3.12}$$

**Remark.** Brenier [6] has discussed a method which is similar to our method, for the case of reaction advection equations. His method applied to the inviscid Burgers equation involves moving positions of a set of generating particles by associating to each particle the velocity

$$s_i = s^L + \sum_{j : x_j < x_i} \varsigma_j + \sum_{j < i : x_j = x_i} \varsigma_j + \frac{1}{2} \varsigma_i$$

for all $i$. Let us denote his approximation operator by $\widetilde{B}_t$. It is an easy exercise to show that if $s$ is a step function with a single discontinuity of negative strength then there exists a constant C, which is independent of the size of the strengths of the particles, such that

$$\| A_t s - \widetilde{B}_t s \|_{L^1} \geq Ct. \tag{3.13}$$

On the other hand, using a compactness argument similar to the argument found in Crandall and Majda [20], Brenier [6] is able to show that for $nk = T$

$$\lim_{n \to \infty} \| A_{nk} s - \widetilde{B}_k^n s \|_{L^1} = 0. \tag{3.14}$$

There is no conflict between (3.13) and (3.14) as the operator $\widetilde{B}_t$ does not satisfy the semi-group property.

We do not use Brenier method to solve the advection equation since our proof of convergence for the full random method needs an approximate advection operator which is accurate enough to allow us to add the $n$ errors from each time step and still obtain a quantity which converges to zero as $n \to \infty$ ($nk = T$). Obviously equation (3.14) implies that this is not possible using Brenier's method. It should be noted though, that both methods produce similar numerical results as $n \to \infty$.

**3.2. Accuracy of the Method.** We will now study the accuracy of the operator $\widetilde{A}_t$.

**3.2.1. Theorem.** *Suppose we have $m$ particles with initial positions $\{x_i\}_{i=1}^m$ and strengths $\{\varsigma_i\}_{i=1}^m$ such that $|\varsigma_i| = h$. Let $s \in S$ be given by*

$$s(y) = s^L + \sum_{i=1}^m H(y - x_i)\varsigma_i .$$

*Then for $t \geq 0$,*

$$\| A_t s - \widetilde{A}_t s \|_{L^1} \leq C_3 ht \tag{3.15}$$

*where $C_3 = \frac{1}{4} hm$ .*

The operators $A_t$ and $\widetilde{A}_t$ propagate information at a finite velocity (see equation (3.10) and the remarks following Theorem 1.3.4.). Since $s$ is constant in a neighbourhood of infinity, it follows that for any given $t>0$, there exists $R>0$ such that $A_t s(y)=\widetilde{A}_t s(y)$ for $|y|>R$. Hence the quantity $\|A_t s - \widetilde{A}_t s\|_{L^1}$ is defined and bounded.

We will prove the theorem by first proving the following lemma.

**3.2.2. Lemma.** *As in Theorem 3.2.1, suppose that $m$ particles are given with initial positions $\{x_i\}_{i=1}^m$ and strengths $\{\varsigma_i\}_{i=1}^m$ such that $|\varsigma_i|=h$. Then for any step function, $s$, generated by these particles, we have*

$$\|A_t s - \widetilde{A}_t s\|_{L^1} \leq \frac{1}{4}h^2 mt$$

*for $0 \leq t \leq t^1$ where*

$$t^1 = \min_{i,j}\{\frac{x_j - x_i}{s_i - s_j} : x_j > x_i \text{ and } s_i > s_j\} \tag{3.16}$$

*where the quantities $s_i$ defined in expressions (3.1), (3.3) and (3.4) and the quantity $t^1$ is equated to $\infty$ if the conditioning in (3.16) is vacuous.*

Note that $t^1$ is the time of first intersection of the trajectories $x_i(t)$ as discussed in Section 3.1.

**Proof of Lemma.** To prove the lemma we will make use of the following facts.

$$\lim_{\tau\to t}\|A_\tau s - A_t s\|_{L^1}=0$$

$$\lim_{\tau\to t}\|\widetilde{A}_\tau s - \widetilde{A}_t s\|_{L^1}=0$$

for $\tau,t\geq 0$. Theorem 1.3.3 allows us to bound the quantity $\|A_\tau s - A_t s\|_{L^1}$ by $\|s - A_{|\tau-t|}s\|_{L^1}$. The weak entropy solution of the inviscid Burgers equation converges in $L^1_{loc}$ to its initial data as $t\to 0$ (see Section 1.3.2) and so it follows that

$$\lim_{|\tau-t|\to 0}\|s - A_{|\tau-t|}s\|_{L^1}=0.$$

so we conclude that the first limit is correct. The second limit follows directly from the construction of the approximate operator. It follows that

$$\| A_t s - \widetilde{A}_t s \|_{L^1} \tag{3.17}$$

is continuous in $t$.

The lemma will be proved if we can show that the set

$$E = \{ t \in [0,t^1] : \| A_\tau s - \widetilde{A}_\tau s \|_{L^1} \leq \frac{1}{4} h^2 m \tau \text{ for all } 0 \leq \tau \leq t \}$$

is nonempty, closed and open in $[0,t^1]$. It is evident that $E$ is non-empty ($0 \in E$) and closure follows from the continuity of the function given in expression (3.17).

It remains to show that $E$ is open. Without loss of generality we may suppose that $t \in E$ and $t < t^1$. We want to show that there exists a $\delta > 0$ such that $\tau \in E$ for all $|\tau - t| < \delta$. Without loss of generality we may suppose that $t < \tau < t + \delta$, since $\tau \in E$ for $\tau \leq t$. Now,

$$\| A_\tau s - \widetilde{A}_\tau s \|_{L^1}$$

$$\leq \| A_{\tau-t} A_t s - A_{\tau-t} \widetilde{A}_t s \|_{L^1} \tag{3.18}$$

$$+ \| A_{\tau-t} \widetilde{A}_t s - \widetilde{A}_{\tau-t} \widetilde{A}_t s \|_{L^1}. \tag{3.19}$$

The operator stability of $A_{\tau-t}$ (Theorem 1.3.3), together with the assumption that $t \in E$ and the fact that $\| A_t s - \widetilde{A}_t s \|_{L^1}$ is a continuous function of $t$, implies that

$$\| A_\tau A_t s - A_\tau \widetilde{A}_t s \|_{L^1}$$

$$\leq \| A_t s - \widetilde{A}_t s \|_{L^1} \leq \frac{1}{4} h^2 m t.$$

The normed quantity (3.19) can be estimated by observing two facts.

(1) Using the nomenclature of Section 3.1 we have that the step function $\widetilde{A}_t s$ is generated by numerical shocks and positive discontinuities of size $h$ which are generated by distinct particles.

(2) Given the step function $\widetilde{A}_t s$, there exists a $\delta > 0$ so that for $0 < \tau - t < \delta$, the function $A_{\tau-t} \widetilde{A}_t s$ has an explicit formulation in terms of shock waves and rarefaction waves (see figure 7). In other words, $\delta$ can be chosen so that for $t \leq \tau < t + \delta$ the solution of the

inviscid Burgers equation with initial data $\widetilde{A}_t s$ at 'initial time' $t$, is composed of non-interacting elementary waves (shock and rarefaction waves) (see sections 1.3.8-1.3.10).

The difference between the function $A_{\tau-t}\widetilde{A}_t s$ and $\widetilde{A}_{\tau-t}\widetilde{A}_t s$ is simply the difference between representing a number of small rarefaction waves of height $h$ with positive discontinuities of the same height, situated at the average position of the rarefaction wave (see figure 8). These small rarefaction waves have width $(\tau - t)h$ at time $\tau$. Hence the $L^1$ norm error due to each discontinuity with positive strength will be $\frac{1}{4}h^2(\tau - t)$. If $N$ is the number of discontinuities with positive strength then we have

$$\| A_{\tau-t}\widetilde{A}_t s - \widetilde{A}_{\tau-t}\widetilde{A}_t s \|_{L^1}$$
$$\leq \frac{1}{4}h^2 N(\tau-t) \leq \frac{1}{4}h^2 m(\tau-t).$$

Hence we conclude that there exist a $\delta > 0$ such that

$$\| A_\tau s - \widetilde{A}_\tau s \|_{L^1} \leq \frac{1}{4}h^2 m\tau$$

for $t < \tau < t + \delta$. This shows that $E$ is open and that the lemma is true.

**Proof of Theorem 3.2.1.** We will use Lemma 3.2.2 together with an induction argument to show that expression (3.15) is true for all $t > 0$. This will of course imply that the theorem is true.

Our induction hypothesis is that

$$\| A_{t^l} s - \widetilde{A}_{t^l} s \|_{L^1} \leq \frac{1}{4}h^2 m t^l \tag{3.20}$$

where $t^l$ is one of the intersection times defined inductively in Section 3.1 (see equation (3.7)).

Let us suppose that $t \in [t^l, t^{l+1}]$. The lemma implies that

$$\| A_{t-t^l}\widetilde{A}_{t^l} s - \widetilde{A}_{t-t^l}\widetilde{A}_{t^l} s \|_{L^1} \leq \frac{1}{4}h^2 m(t - t^l). \tag{3.21}$$

Here we have applied the lemma to the step function $\widetilde{A}_{t^l} s$. In addition we have made the observation that the time $t^1$ defined in equation (3.16) for particles initially generating the function $\widetilde{A}_{t^l} s$ is equal to $t^{l+1} - t^l$, where $t^{l+1}$ and $t^l$ are defined in equation (3.7) for particles generating $s$. Consequently,

$$\| A_t s - \widetilde{A}_t s \|_{L^1}$$

$$\leq \| A_{t-t^l} A_{t^l} s - A_{t-t^l} \widetilde{A}_{t^l} s \|_{L^1} + \| A_{t-t^l} \widetilde{A}_{t^l} s - \widetilde{A}_{t-t^l} \widetilde{A}_{t^l} s \|_{L^1}$$

$$\leq \frac{1}{4} h^2 m t$$

where the first normed quantity is estimated using the stability of $A$ (see Theorem 1.3.3) together with assumption (3.20); the second quantity is estimated using (3.21). The proof is completed by observing that Lemma 3.2.2 implies that the initial induction hypothesis, (3.20) for $l = 1$, is true.

**3.3. Definition for Random Initial Data.** In general the initial data for the approximate advection operator will consist of step functions generated by a set of particles whose positions have been obtained as representatives of random variables. Consequently the approximate advection operator must be defined for random initial data. Suppose we have $m$ random variables, $X_i$, $i = 1,\ldots,m$ defined on a probability space $(\Omega, \Sigma, P)$. We will consider a collection of $m$ particles with strengths $\varsigma_i$ such that $|\varsigma_i| = h$ and with initial positions given by the random variables $X_i$. For a given constant $s^L$, the step function generated by these particles is given by

$$S(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i)\,\varsigma_i \tag{3.22}$$

where we have used upper case $S$ to emphasize that $S(y)$ is a random variable. For any random step function $S$, defined by a relationship of the form (3.22), we will denote a representative of $S$, for a given $\omega \in \Omega$ by $S|_\omega$. That is, we define

$$S\big|_\omega(y) = s^L + \sum_{i=1}^{m} H(y - X_i\big|_\omega)\,\varsigma_i$$

We will define the exact and approximate transport operators for initial data $S$ by requiring that for any $\omega \in \Omega$

$$\widetilde{A}_t S\big|_\omega = \widetilde{A}_t(S\big|_\omega),$$

and

$$A_t S\big|_\omega = A_t(S\big|_\omega).$$

In other words, we define our operators, for random data $S$, by using our previous definitions for each representative of $S$. To emphasize that $\widetilde{A}_t$ actually maps particle positions, we will also use the dual notation

$$\widetilde{A}_t(X_1, ..., X_m)\big|_\omega = \widetilde{A}_t(X_1\big|_\omega, ..., X_m\big|_\omega)$$

where $\widetilde{A}_t(x_1, ..., x_m)$ is defined by equation (3.11a).

Since the estimate of accuracy of the operator $\widetilde{A}_t$ obtained in Theorem 3.2.1 depends only on the number and the strength of the particles and not on their positions, we conclude that a similar estimate of accuracy holds for the case of random initial data. In particular, we have the following:

**3.3.1. Theorem.** *Suppose we have $m$ particles with random initial positions $\{X_i\}_{i=1}^{m}$ and strengths $\{\varsigma_i\}_{i=1}^{m}$ such that $|\varsigma_i| = h$. Let $S$ be random initial data generated by these particles. Then for $t \geq 0$,*

$$\|\widetilde{A}_t S - A_t S\|_{L^1} \leq C_3 ht$$

*where* $C_3 = \frac{1}{4}hm$ .

## Chapter 4

## The Random Walk Operator

To approximate the diffusion in Burgers equation we will use the well known correspondence between the probability distribution of the position of a particle undergoing a random walk and the solution of the diffusion equation (see Lamperti [35], Ch. 4, Arnold [1]). To utilize this correspondence we recall that a step function $s \in \mathbf{S}$ can be considered as being generated by a collection of particles with positions $x_i$ and strengths $\varsigma_i$, such that

$$s(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - x_i)\, \varsigma_i .$$

If the positions $x_i$ of the particles are perturbed by appropriate gaussian random variables, $\eta_i$, then the above correspondence allows us conclude that the perturbed function

$$s^L + \sum_{i=1}^{m} \mathrm{H}(y - x_i - \eta_i)\, \varsigma_i$$

approximates the solution of the diffusion equation in some probabilistic sense. The exact sense is described in section 4.3 and in Theorems 4.5.1 and 4.6.1.

It should be noted that the standard random walk method for solving the diffusion equation involves first approximating the initial data by a measure of the form

$$\sum_{i=1}^{m} \xi_i\, \delta_{z_i}$$

where $\delta_{z_i}$ denotes a delta function supported at the point $z_i$ and $\xi_i \in \mathbf{R}$. The positions $z_i$ are then perturbed by the random variables, $\eta_i$, so that the above correspondence can be used to show that the measure

$$\sum_{i=1}^{m} \xi_i\, \delta_{z_i + \eta_i}$$

approximates the solution to the diffusion equation in some weak sense. Unfortunately this method introduces large statistical errors. Our method of solution is equivalent to integrating the solution obtained using the standard method with the initial data given by the measure

$$\sum_{i=1}^{m} \delta_{x_i} \varsigma_i .$$

It is plausible that the solution obtained by our method has less statistical error, since our solution is obtained by integration, which is an averaging process (see Courant, et. al. [18], Hammersley and Handscomb [29]). A similar improvement in the accuracy of a random walk method is obtained in the random vortex method, in which case the vorticity is described by a measure and the velocity is obtained by integrating the vorticity with a smooth weight function (see Chorin [11], Roberts [42]). The moral is that the statistical error is reduced if we randomly walk particles which carry strength corresponding to the gradient of the function instead of the function itself.

**4.1. Definition of the Random Walk Operator.** We will now give the precise definition of the random walk operator. We note that in general the random walk operator will need to operate on data which has been derived from the application of our numerical method over an arbitrary number of time steps. Consequently the random walk operator must be defined for random initial data. As in Section 3.3, we consider the initial positions of the particles to be given by $m$ random variables, $X_i$ , $i = 1,...,m$ defined on the probability space ( $\Omega_1$ , $\Sigma_1$ , $P_1$ ). We will suppose that the strengths of the particles satisfy $|\varsigma_i| = h$ . Let $B_i$ , $i=1,..,m$ , be a collection of $m$ independent normally distributed random variables, defined on a probability space ( $\Omega_2$ , $\Sigma_2$ , $P_2$ ), such that $E[B_i] = 0$ and $\mathrm{Var}[B_i] = 2\nu k$ . We will regard the random variables $X_i$ and $B_i$ , $i=1,..,m$ as being defined on the product probability space ( $\Omega_1 \times \Omega_2$ , $\Sigma_1 \times \Sigma_2$ , $P_1 \times P_2$ ), and hence regard the $B_i$ random variables as being independent of the $X_i$ random variables. We will denote a generic element of $\Omega_1$ by $\omega_1$, an element of $\Omega_2$ by $\omega_2$ and an element of $\Omega_1 \times \Omega_2$ by $(\omega_1,\omega_2)$.

For a given constant $s^L$ , the initial step functions generated by these particles are given by

$$S(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - X_i)\, \varsigma_i . \tag{4.1}$$

The position of each particle after one time step will be given by the random variables

$$Y_i = X_i + B_i \tag{4.2}$$

where $Y_i$ is a random variable on the product space $( \Omega_1 \times \Omega_2 , \Sigma_1 \times \Sigma_2 , \mathrm{P}_1 \times \mathrm{P}_2 )$. The random walk operator, defined with respect to the random variables $\mathrm{B}_i$, is given by the mapping

$$\widetilde{D}_k ( X_1 ,..., X_m ) = ( Y_1 ,..., Y_m ). \tag{4.3a}$$

As in the case of the $A_t$ operator, we will also use the notation

$$\widetilde{D}_k\, S(y) = s^L + \sum_{i=1}^{m} \mathrm{H}(y - Y_i)\, \varsigma_i \tag{4.3b}$$

to denote the step function generated by the random variables $Y_i$.

The numerical approximation is given by one representative of the random step function $\widetilde{D}_t\, S$, which is equivalently given by one representative of the random variables $Y_i$. Hence the numerical approximation is given by

$$s^L + \sum_{i=1}^{m} \mathrm{H}(y - Y_i |_{(\omega_1,\omega_2)})\, \varsigma_i$$

for some $(\omega_1,\omega_2) \in \Omega_1 \times \Omega_2$. Computationally the numbers $Y_i |_{(\omega_1,\omega_2)}$ are given by $x_i + (2\nu k)^{1/2}\, \xi_i$, where the numbers $\xi_i$ are derived from a standard normal random number generator and the numbers $x_i$ are representatives of the random variables $X_i$, provided by the numerical approximation at the previous time step of the scheme.

**4.2. Conditional Expectations and Probabilities.** To simplify the analysis of the method we find it convenient to introduce the following notation. Let $\Psi$ be a random variable on the probability space $( \Omega_1 \times \Omega_2 , \Sigma_1 \times \Sigma_2 , \mathrm{P}_1 \times \mathrm{P}_2 )$, such that $\mathrm{E}[\,|\Psi|\,] < \infty$. Fubini's Theorem (Rudin [44], p. 150) implies that

$$\mathrm{E}[\Psi] = \int_{\Omega_1} [ \int_{\Omega_2} \Psi |_{(\omega_1,\omega_2)} \mathrm{P}_2( d\omega_2) ]\, \mathrm{P}_1( d\omega_1) \tag{4.4}$$

$$= \int_{\Omega_2} [ \int_{\Omega_1} \Psi|_{(\omega_1,\omega_2)} P_1( d\omega_1) ] P_2( d\omega_2).$$

If we let

$$E_2[\Psi]|_{\omega_1} = \int_{\Omega_2} \Psi|_{(\omega_1,\omega_2)} P_2( d\omega_2) \tag{4.5}$$

and

$$E_1[\Psi]|_{\omega_2} = \int_{\Omega_1} \Psi|_{(\omega_1,\omega_2)} P_1( d\omega_1)$$

then Fubini's Theorem also implies that $E_1$ and $E_2$ are random variables on $\Omega_2$ and $\Omega_1$ respectively. We note that the random variables $E_2$ and $E_1$ are closely to the conditional expectations $E [\Psi \mid \Sigma_1 \times \Omega_2]$ and $E [\Psi \mid \Omega_1 \times \Sigma_2]$. For the definition of conditional expectations see Arnold [1] or Chung [16]. It is easy to see that

$$E_2[\Psi]|_{\omega_1} = E [\Psi \mid \Sigma_1 \times \Omega_2]|_{(\omega_1,\omega_2)},$$
$$E_1[\Psi]|_{\omega_2} = E [\Psi \mid \Omega_1 \times \Sigma_2]|_{(\omega_1,\omega_2)}$$

almost everywhere on the spaces $\Omega_1$ and $\Omega_2$.

Equations (4.4) and (4.5) imply that

$$E [\Psi] = E [ E_1[\Psi] ] = E [ E_2[\Psi] ]. \tag{4.6}$$

We can use the preceding results to obtain an alternative formulation for quantities of the form $P(\Psi > \delta)$, where $\delta \in \mathbb{R}$ . Let us introduce the notation

$$P_2(\Psi > \delta) = E_2[ H (\Psi - \delta ) ].$$

Here $P_2(\Psi > \delta)$ denotes a random variable on $\Omega_1$. Equation (4.6) then implies that

$$P(\Psi > \delta) = E [ P_2(\Psi > \delta) ]. \tag{4.7}$$

Finally the quantity $P_2(\Psi > \delta)$ can be calculated using relationship (4.5). Namely,

$$P_2(\Psi > \delta)|_{\omega_1} = P_2( \omega_2 : \Psi|_{(\omega_1,\omega_2)} > \delta ). \tag{4.8}$$

**4.3. Point-wise Accuracy.** Suppose that we have a random step function $S$. The solution of the exact diffusion equation with $S$ as initial data, denoted $D_k S$, is defined by the relation

$$D_k S\big|_{\omega_1} = D_k(S\big|_{\omega_1})$$

for $\omega_1 \in \Omega_1$. That is, the solution of the diffusion equation for random initial data is defined in terms of the standard definition for each representative.

We will show that $E_2[\tilde{D}_k S(y)] = D_k S(y)$. This can be interpreted to mean that on the average the random walk algorithm approximates the solution of the diffusion equation with initial data provided by the numerical method at the previous time step.

To calculate $E_2[\tilde{D}_k S]$, we first observe that

$$E_2[\,H(y - Y_i)\,] = \int_{\Omega_2} H(y - X_i - B_i\big|_{\omega_2})\,P_2(d\,\omega_2)$$

$$= \frac{1}{(4\pi\nu k)^{1/2}} \int_{-\infty}^{\infty} H(y - X_i - z)\exp(-\frac{z^2}{4\nu k})\,dz = D_k H(y - X_i).$$

Here we have been able to equate the integral over $\Omega_2$ to an integral over $\mathbb{R}$, since we know that the random variables $B_i$ are normally distributed. We conclude that

$$E_2[\,H(y - Y_i)\,] = D_k H(y - X_i). \tag{4.9}$$

Consequently,

$$E_2[\,\tilde{D}_k S(y)\,] = s^L + \sum_{i=1}^{m} E_2[\,H(y - Y_i)\,]\,\varsigma_i$$

$$= s^L + \sum_{i=1}^{m} D_k H(y - X_i)\,\varsigma_i = D_k S(y)$$

as required.

It is still important to estimate the accuracy of the random walk algorithm. We will first use Chebyshev's inequality to estimate the error in approximating $D_k S(y)$ by a representative of $\tilde{D}_k S(y)$. In Theorem 4.3.4 we use a result due to Hoeffding [30] to show that the term $1/\alpha^2$ in Theorem 4.3.1 can be replaced by a term of the form $\exp(-\alpha^2)$.

**4.3.1. Theorem.** *Suppose that the initial data $S$ for the random walk operator is generated by particles all with strengths of absolute size $h$. Then for a fixed $y \in \mathbb{R}$*

$$P(\,|\,\widetilde{D}_k\,S(y) - D_k\,S(y)\,|\, > C_4\,h^{1/2}\alpha\,) \leq 1/\alpha^2.$$

*for any $\alpha > 1$, where $C_4 = \frac{1}{2}(mh)^{1/2}$.*

**Proof.** Chebyshev's inequality (Feller [23] p. 151) states that for $\alpha > 1$

$$P(\,|\,\Theta - E[\Theta]\,\,|\, > \alpha \mathrm{Var}[\Theta]^{1/2}\,) \leq 1/\alpha^2.$$

where $\Theta$ is a random variable on a probability space $(\Omega\,,\Sigma\,,P)$. Using the results of section 4.2, it can be shown that Chebyshev's inequality implies that if $\Psi$ is a random variable on $\Omega_1 \times \Omega_2$, then

$$P_2(\,|\,\Psi - E_2[\Psi]\,\,|\, > \alpha \mathrm{Var}_2[\Psi]^{1/2}\,) \leq 1/\alpha^2. \tag{4.10}$$

This result is obtained by substituting into Chebyshev's inequality, the random variable $\Theta$ on $\Omega_2$, given by $\Theta\big|_{\omega_2} = \Psi\big|_{(\omega_1,\omega_2)}$ where $\omega_1$ is held fixed.

We will apply result (4.10) to the random variable $\Psi = \widetilde{D}_k\,S(y)$. We recall from the previous discussion that

$$E_2[\Psi] = D_k\,S(y). \tag{4.11}$$

To estimate the variance of $\Psi$ we observe that for a fixed $\omega_1$

$$\sum_{i=1}^{m} H(y - X_i - B_i)\,\varsigma_i$$

is a sum of independent random variables on the space $\Omega_2$. Consequently

$$\mathrm{Var}_2[\Psi] = \sum_{i=1}^{m} \mathrm{Var}_2[\,H(y - X_i - B_i)\,]\,\varsigma_i^2. \tag{4.12}$$

The equality follows from the fact that the variance of a sum of independent random variables is equal to the sum of the variances of the individual random variables. The individual terms in the sum can be estimated in the following manner.

$$\mathrm{Var}_2[\,\mathrm{H}\,(y - X_i - B_i\,)\,] \tag{4.13}$$

$$= \mathrm{P}_2(\,B_i < y - X_i\,)\,[\,1 - \mathrm{P}_2(\,B_i < y - X_i\,)\,] \leq \frac{1}{4}.$$

Equations (4.12) and (4.13) together imply that

$$\mathrm{Var}_2[\Psi] \leq \frac{1}{4}\sum_{i=1}^{m} \varsigma_i^2 = \frac{1}{4} m h^2 = \mathrm{C}_4\, h. \tag{4.14}$$

By applying equation (4.10) to $\Psi$, together with the estimates for the expectation and variance obtained in expressions (4.11) and (4.14), we conclude that

$$\mathrm{P}_2(\;|\,\tilde{D}_k\,S\,(y\,) - D_k\,S\,(y\,)\,| \;>\; \mathrm{C}_4\, h^{1/2}\alpha) \leq 1/\alpha^2$$

Equation (4.7) can then be used to show that

$$\mathrm{P}(\,|\,\tilde{D}_k\,S\,(y\,) - D_k\,S\,(y\,)\,| \;>\; \mathrm{C}_4\, h^{1/2}\alpha) \leq 1/\alpha^2.$$

which completes the proof.

The result just obtained depends on the use of Chebyshev's inequality. A much stronger result can be obtained by using the following result due to Hoeffding and its simple consequence, Corollary 4.3.3.

**4.3.2. Theorem (Hoeffding [30], Theorem 1.).** *If $Z_1, \ldots, Z_m$ are $m$ independent random variables such that $0 \leq Z_i \leq 1$, $i = 1,..,m$, then for $\delta > 0$,*

$$\mathrm{P}(\,\frac{1}{m}\sum_{i=1}^{m} Z_i - \frac{1}{m}\sum_{i=1}^{m}\mathrm{E}\,[Z_i\,] > \delta\,) \leq \exp(-2\delta^2 m\,).$$

**4.3.3. Corollary.** *Let the random variables $Z_i$ $i = 1,\ldots,m$ be defined as in Theorem 4.3.2. Then,*

$$\mathrm{P}(\,|\,\frac{1}{m}\sum_{i=1}^{m} Z_i - \frac{1}{m}\sum_{i=1}^{m}\mathrm{E}\,[Z_i\,]\;|\;>\delta\,) \leq 2\,\exp(-2\delta^2 m\,).$$

To prove this corollary we apply Hoeffding's result to the random variables $Z_i$ and $1 - Z_i$, $i = 1,..,m$.

We will now use Corollary 4.3.3 to obtain a strengthening of Theorem 4.3.1.

**4.3.4. Theorem.** *Suppose that the initial data $S$ for the random walk operator is generated by particles with strengths $\varsigma_i$ such that $\varsigma_i = h$. Then for any fixed $y \in \mathbb{R}$ and $\delta > 0$,*

$$P_2(\ |\tilde{D}_k S(y) - D_k S(y)|\ > mh\,\delta\ ) \leq 2\exp(-2\delta^2 m)$$

*and so*

$$P(\ |\tilde{D}_k S(y) - D_k S(y)|\ > mh\,\delta\ ) \leq 2\exp(-2\delta^2 m).$$

If we let $\delta = \alpha\, m^{-1/2}$, *then this theorem implies that for any step function generated by particles with stengths equal to $h$,*

$$P(\ |\tilde{D}_k S(y) - D_k S(y)|\ > C_5 \alpha h^{1/2}) \leq 2\exp(-2\alpha^2)$$

where $C_5 = (mh)^{1/2}$. This estimate will also be true if the strengths of the particles are equal to $-h$. Now, if the strengths of the particles satisfy $|\varsigma_i| = h$, then we can consider the initial data as being the sum of two functions, one with particles all of negative strength, the other with particles all of positive strength. The triangle inequality together with Theorem 4.3.4 applied to these two functions then implies that

$$P(\ |\tilde{D}_k S(y) - D_k S(y)|\ > 2C_5 \alpha h^{1/2}) \leq 4\exp(-2\alpha^2).$$

Obviously this bound is superior to the bound obtained using Chebyshev's inequality.

**Proof of Theorem.** Without loss of generality we may assume that the limit at minus infinity of $S$ is zero ($s^L = 0$). For a fixed $\omega_1 \in \Omega_1$, let us define $m$ independent random variables on $\Omega_2$ by

$$Z_i = H(y - X_i|_{\omega_1} - B_i).$$

Note that $Z_i$ depends on $\omega_1$ and $y$ and satisfies $0 \leq Z_i \leq 1$. In addition we observe that

$$\frac{1}{mh}\tilde{D}_k S(y) = \frac{1}{m}\sum_{i=1}^{m} Z_i$$

and

$$\frac{1}{mh} D_k S(y) = \frac{1}{m} \sum_{i=1}^{m} \mathrm{E}_2[Z_i]$$

where we have used the fact that the strengths of the particles are all equal to $h$. If we apply Corollary 4.3.3 to the random variables $Z_i$ then by equation (4.8) we have that

$$\mathrm{P}_2( \mid \widetilde{D}_k S(y) - D_k S(y) \mid \; > mh\,\delta ) \leq 2 \exp(-2\delta^2 m ).$$

Equation (4.7) then implies that

$$\mathrm{P}( \mid \widetilde{D}_k S(y) - D_k S(y) \mid \; > mh\,\delta ) \leq 2 \exp(-2\delta^2 m ).$$

**4.4. A Condition on the Initial Distribution of the Particles.** We have obtained results regarding the point-wise behaviour of the random walk operator. To obtain a convergence proof for the full numerical method it is necessary to obtain an estimate of the $\mathrm{L}^1$ behaviour of the random walk operator. This will be possible if the following assumption is made about the probability distribution of the random variables $X_i$.

**4.4.1. Assumption.** *Let the time step $k$, the diffusion constant $\nu$ and the the random variables $X_i$ , $i=1,..,m$ be given. Let $B_i$ , $i=1,..,m$ be any collection of $m$ independent normally distributed random variables satisfying $\mathrm{E}\,[B_i] = 0$ and $\mathrm{Var}[B_i] = 2\nu k$, such that $\{X_i \, , B_i\}_{i=1}^{m}$ is an independent set of random variables. We assume that the random variables $X_i$ satisfy the condition that there exist constants $K > 0$ and $T > 0$ (independent of the $B_i$ random variables ), such that for all $a > 0$,*

$$\mathrm{P}(X_i > K + a ) \leq \Phi(-\frac{a}{(2\nu T)^{1/2}}) , \qquad \mathrm{P}(X_i < -K - a ) \leq \Phi(-\frac{a}{(2\nu T)^{1/2}})$$

and

$$\mathrm{P}(X_i + B_i > K + a ) \leq \Phi(-\frac{a}{(2\nu T)^{1/2}}) , \qquad \mathrm{P}(X_i + B_i < -K - a ) \leq \Phi(-\frac{a}{(2\nu T)^{1/2}})$$

*where,*

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} \exp(-\frac{y^2}{2})\, dy.$$

**Remark.** Since Assumption 4.4.1 deals only with the probability distribution of the random variables $X_i$ and $X_i + B_i$, it follows that the assumption will hold if conditions (4.15) and (4.16) are true for any one particular choice of random variables $B_i$. Consequently, the random variables $X_i$ introduced in Section 4.1 will satisfy Assumption 4.4.1 if there exist constants $K$ and $T$ such that condition (4.15) holds for the $X_i$ random variables and condition (4.16) holds for the random variables $Y_i = X_i + B_i$ defined in equation (4.2).

Assumption 4.4.1 is intended to capture the behaviour of random variables $X_i^{j-1/2}$, which represent the positions of the particles that generate our numerical approximation $\widetilde{A}_k\,[\widetilde{D}_k\,\widetilde{A}_k\,]^{j-1}$. In Theorem 5.2.1 we show that the random variables $X_i^{j-1/2}$ satisfy Assumption 4.4.1 with $K = R + [\,\|u_0\|_{L^\infty} + mk\,]jk$ and $T = jk$, where $R$ is the size of the support of the set containing the particles at time zero.

**4.5. $L^1$ Error Analysis.** We can now study the probability distribution of the $L^1$ norm of the error of the random walk operator.

**4.5.1. Theorem.** *Let $k$ be the time step. Let the initial data $S$ be generated by $m$ particles with initial positions given by random variables $X_i$, $i = 1,\dots,m$ which satisfy Assumption 4.4.1, with constants $K > 0$ and $T > 0$. Let the strengths of the particles have absolute size $h < 1/3$. Then for $\alpha > 1$,*

$$P(\,\|\widetilde{D}_k S - D_k S\|_{L^1} > M_1 \alpha h^{1/2} (\ln(1/h))^2)$$

$$\leq M_2 h^{\frac{1}{2}\alpha \ln(1/h) - 1}$$

*where the constants $M_1$ and $M_2$ are defined as,*

$$M_1 = 8\,(mh)^{1/2}[K + (8\nu T)^{1/2}] + (2\nu T)^{1/2}\,mh\,,$$

$$M_2 = 8\,(mh)^{1/2} + 16mh.$$

The exponential bound on the probability distribution of the $L^1$ norm of the error is of crucial importance in the subsequent proof of convergence of our numerical method. It allows us to sum the errors at each time step and still obtain a bound on the probability distribution of the overall error which is exponentially small for large errors.

The proof of the theorem will be based on the following lemma. Notice that in this lemma the strengths of the particles generating the solution are assumed to be equal to $h$ .

**4.5.2. Lemma.** *Let the initial data $S$ be generated by $m$ particles with strengths $\varsigma_i$ that satisfy $\varsigma_i = h$ , and with random initial positions $X_i$ which satisfy Assumption 4.4.1 with constants $K > 0$ and $T > 0$. For $r > K$ and $\delta > 0$,*

(*i*) $$P(\, \| \widetilde{D}_k S - D_k S \|_{L^1(-r,r)} > 2rmh\,[\delta + m^{-1/2}]) \leq 4m^{1/2}\exp(-2\delta^2 m\,),$$

(*ii*$_-$) $$P(\, \| \widetilde{D}_k S - D_k S \|_{L^1(-\infty,-r)} > \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \, \exp(-\frac{(r-K)^2}{16\nu T})) \leq 4m\frac{(2\nu T)^{1/2}}{r-K}\exp(-\frac{(r-K)^2}{16\nu T}),$$

*and*

(*ii*$_+$) $$P(\, \| \widetilde{D}_k S - D_k S \|_{L^1(r,\infty)} > \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \, \exp(-\frac{(r-K)^2}{16\nu T})) \leq 4m\frac{(2\nu T)^{1/2}}{r-K}\exp(-\frac{(r-K)^2}{16\nu T}).$$

**Proof of the Lemma.** We will first prove statement ($i$) of the lemma. Let us define

$$\epsilon(y) = |\, \widetilde{D}_k S(y) - D_k S(y) \,| .$$

The function $\epsilon(y)$ defines a random variable on $\Omega_1 \times \Omega_2$ which gives a measure of the pointwise error at the point $y$ . Since all of the particles are assumed to have equal strength, $h$ , it follows that $\widetilde{D}_k S(y)|_{(\omega_1,\omega_2)}$ and $D_k S(y)|_{\omega_1}$ are monotonically increasing functions of $y$ . Hence for $y \in [a_1, a_2]$,

$$\widetilde{D}_k S(y) - D_k S(y) \leq \widetilde{D}_k S(a_2) - D_k S(a_1)$$
$$\leq \widetilde{D}_k S(a_2) - D_k S(a_2) + D_k S(a_2) - D_k S(a_1)$$

$$\leq \epsilon(a_2) + D_k S(a_2) - D_k S(a_1).$$

(see figure 9). Similarly

$$\widetilde{D}_k S(y) - D_k S(y) \geq -\epsilon(a_1) + D_k S(a_1) - D_k S(a_2).$$

Consequently, for $y \in [a_1, a_2]$,

$$| \widetilde{D}_k S(y) - D_k S(y) | \leq \max(\epsilon(a_1), \epsilon(a_2)) + D_k S(a_2) - D_k S(a_1).$$

We conclude that

$$\| \widetilde{D}_k S - D_k S \|_{L^1[a_1, a_2]} \tag{4.17}$$

$$\leq (a_2 - a_1)[\max(\epsilon(a_1), \epsilon(a_2)) + D_k S(a_2) - D_k S(a_1)].$$

As in the proof of Theorem 4.3.4 we now consider the random variables in expression (4.17) as random variables on $\Omega_2$ for each fixed $\omega_1 \in \Omega_1$. For each such $\omega_1 \in \Omega_1$ we can choose a sequence of points $a_i$, $i = 1, \ldots, N_0$, where $N_0 \leq 2m^{1/2}$, such that

$$-r = a_1 < a_2 < \ldots < a_{N_0} = r,$$

and

$$D_k S(a_{i+1}) - D_k S(a_i) \leq hm^{1/2}.$$

This follows from the fact that the range of $D_k S$ is contained in the interval $[s^L, s^L + mh]$ and that $D_k S(y)|_{\omega_1}$, for each fixed $\omega_1 \in \Omega_1$, is a smooth monotonic function of $y$ (see figure 10).

Consider the function on $\Omega_1 \times \Omega_2$ given by

$$\xi = \max_{i=1,\ldots,N_0} \frac{1}{mh} | \widetilde{D}_k S(a_i) - D_k S(a_i) | = \max_{i=1,\ldots,N_0} \frac{\epsilon(a_i)}{mh}.$$

For each $\omega_1 \in \Omega_1$, and so each choice of the $a_i$'s, it is obvious that $\xi$ is a random variable on $\Omega_2$. On the other hand it is not at all obvious whether it is possible to choose the points $a_i$, as functions of $\omega_1$, such that $\xi$ is a random variable on $\Omega_1 \times \Omega_2$. Fortunately we only need $\xi$ to be a random variable on $\Omega_2$ for each fixed $\omega_1$.

We observe that equation (4.17) can be used to show that

$$\| \widetilde{D}_k S - D_k S \|_{L^1[-r,r]}$$

$$= \sum_{i=1}^{N_0-1} \| \widetilde{D}_k S - D_k S \|_{L^1[a_i, a_{i+1}]}$$

$$\leq \sum_{i=1}^{N_0-1} (a_{i+1} - a_i)[mh\, \xi + D_k S(a_{i+1}) - D_k S(a_i)].$$

$$\leq 2rmh\, [\xi + m^{-1/2}].$$

Since, for each fixed $\omega_1 \in \Omega_1$, $\xi$ is a random variable on $\Omega_2$, it follows that

$$P_2( \| \widetilde{D}_k S - \widetilde{D}_k S \|_{L^1[-r,r]} > 2rmh\, [\delta + m^{-1/2}] )$$

$$\leq P_2( 2rmh\, [\xi + m^{-1/2}] > 2rmh\, [\delta + m^{-1/2}] )$$

$$= P_2( \xi > \delta ).$$

We will use Theorem 4.3.4 to estimate this last expression. Observe that for a fixed $\omega_1 \in \Omega_1$,

$$P_2( \xi > \delta )$$

$$= P_2( \mid \widetilde{D}_k S(a_i) - D_k S(a_i) \mid \; > mh\, \delta \text{ for some } a_i )$$

$$\leq \sum_{i=1}^{N_0} P_2( \mid \widetilde{D}_k S(a_i) - D_k S(a_i) \mid \; > mh\, \delta )$$

$$\leq 2N_0 \exp(-2\delta^2 m) \leq 4m^{1/2}\exp(-2\delta^2 m).$$

Combining the last two series of calculations, we see that

$$P_2( \| \widetilde{D}_k S - D_k S \|_{L^1[-r,r]} > 2rmh\, [\delta + m^{-1/2}]) \tag{4.18}$$

$$\leq 4m^{1/2}\exp(-2\delta^2 m).$$

Notice that the function $\xi$ was only used as an intermediate to obtain estimate (4.18). Now we have a conditional estimate on the random variable $\| \widetilde{D}_k S - D_k S \|_{L^1(-r,r)}$, and so we can use equations (4.7) and (4.8) to show that

$$P( \| \widetilde{D}_k S - D_k S \|_{L^1(-r,r)} > 2rmh\, [\delta + m^{-1/2}] )$$

$$\leq 4m^{1/2}\exp(-2\delta^2 m).$$

This completes the proof of the first part of the lemma

**Proof of Statements** $(ii_-)$ **and** $(ii_+)$. We will only prove statement $(ii_-)$ and note that the proof for statement $(ii_+)$ follows in a similar way. We recall that $Y_i = X_i + B_i$ for appropriate normally distributed random variables $B_i$. For a given $r > K$ we will consider the set $E \subseteq \Omega_1 \times \Omega_2$ given by

$$E = \bigcap_{i=1}^{m} X_i^{-1}(J) \cap Y_i^{-1}(J)$$

$$= \{(\omega_1,\omega_2) : X_i\big|_{(\omega_1,\omega_2)} \in J \ , \ Y_i\big|_{(\omega_1,\omega_2)} \in J \ \text{ for all } i = 1,..,m\}$$

where $J$ is the interval $[-\frac{1}{2}(r+K), \frac{1}{2}(r+K)]$.

Let

$$L = \sup_E \| D_k S - s^L \|_{L^1(-\infty,-r)}.$$

We want to show that

$$\{ \| \widetilde{D}_k S - D_k S \|_{L^1(-\infty,-r)} > L \} \subseteq E^c \tag{4.19}$$

where $E^c$ denotes the complement of $E$, and the notation $\{\Psi \in I\}$ represents the set $\{ (\omega_1,\omega_2) : \Psi\big|_{(\omega_1,\omega_2)} \in I \}$, where $\Psi$ is a random variable on $\Omega_1 \times \Omega_2$ and $I$ is an interval contained in $\mathbf{R}$.

Expression (4.19) will hold if we can show that

$$E \subseteq \{ \| \widetilde{D}_k S - D_k S \|_{L^1(-\infty,-r)} \leq L \}. \tag{4.20}$$

For $(\omega_1,\omega_2) \in E$, and $y \leq -r$

$$\mathrm{H}(y - Y_i\big|_{(\omega_1,\omega_2)}) = 0$$

for $i = 1,..,m$ which in turn implies that

$$\widetilde{D}_k S(y)\big|_{(\omega_1,\omega_2)} = s^L.$$

Consequently, for $(\omega_1,\omega_2) \in E$,

$$\| \tilde{D}_k S - D_k S \|_{L^1(-\infty,-r)}$$

$$\leq \| \tilde{D}_k S - s^L \|_{L^1(-\infty,-r)} + \| D_k S - s^L \|_{L^1(-\infty,-r)}$$

$$\leq \sup_E \| D_k S - s^L \|_{L^1(-\infty,-r)} = L.$$

This shows that relation (4.20) holds, and so it follows that expression (4.19) is true.

Expression (4.19) implies that

$$P( \| \tilde{D}_k S - D_k S \|_{L^1(-\infty,-r)} > L ) \leq P(E^c). \tag{4.21}$$

The proof of the lemma will be completed once we show that

$$L \leq \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \ \exp(-\frac{(r-K)^2}{16\nu T})), \tag{4.22}$$

and

$$P(E^c) \leq 4m \frac{(2\nu T)^{1/2}}{r-K} \exp(-\frac{(r-K)^2}{16\nu T}). \tag{4.23}$$

Statement (4.22) depends on the observation that for $(\omega_1,\omega_2) \in E$, we have $-\frac{1}{2}(r+K) \leq X_i$ and so the monotonicity of $D_k H(y)$ as a function $y$ implies that

$$D_k H(y - X_i) \leq D_k H(y + \frac{1}{2}(r+K)).$$

Consequently, for $(\omega_1,\omega_2) \in E$

$$0 \leq D_k S(y) - s^L \leq mhD_k H(y + \frac{1}{2}(r+K))$$

and so

$$\| D_k S - s^L \|_{L^1(-\infty,-r)} \tag{4.24}$$

$$\leq mh \| D_k H(\cdot + \frac{1}{2}(r+K)) \|_{L^1(-\infty,-r)} = mh \| D_k H \|_{L^1(-\infty,\frac{1}{2}(K-r))}.$$

Here we have made a simple translational change of variables to the last normed quantity.

We observe that $D_k H(y) = \Phi(\frac{y}{(2\nu T)^{1/2}})$ and so

$$\| D_k \mathrm{H} \|_{L^1(-\infty,\frac{1}{2}(K-r))} = (2\nu T)^{1/2} \int_{-\infty}^{\frac{K-r}{(8\nu T)^{1/2}}} \Phi(y)\, dy. \tag{4.25}$$

We will now estimate this integral quantity. Suppose $x < 0$, then

$$\int_{-\infty}^{x} \Phi(y)\, dy = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} (x - z)\exp(\frac{z^2}{2})\, dz$$

$$\leq \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} -z \exp(-\frac{z^2}{2})\, dz \leq \frac{1}{(2\pi)^{1/2}} \exp(-\frac{x^2}{2}).$$

If we set $x = \dfrac{K-r}{(8\nu T)^{1/2}}$ and apply this estimate to equation (4.25), we conclude that

$$\| D_k \mathrm{H} \|_{L^1(-\infty,\frac{1}{2}(K-r))} \leq \frac{(\nu T)^{1/2}}{\pi^{1/2}} \exp(-\frac{(r-K)^2}{16\nu T})$$

and so by equation (4.24) we have that

$$L \leq \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \exp(-\frac{(r-K)^2}{16\nu T})$$

as required.

Finally we must estimate the probability measure of the set $E^c$. Let $I_1$ be the interval $(-\infty, -\frac{1}{2}(r+K))$ and let $I_2$ be the interval $(\frac{1}{2}(r+K), \infty)$. The random variables $X_i$ satisfy Assumption 4.4.1, with constants $K > 0$ and $T > 0$, and so for $a > 0$, the random variables $X_i$ satisfy condition (4,15) and the random variables $Y_i = X_i + B_i$ satisfy condition (4.16). If we set $a = \frac{1}{2}(r+K)$, then we have that

$$\mathrm{P}(E^c) = \mathrm{P}( \bigcup_{i=1}^{m} X_i^{-1}(I_1) \cup X_i^{-1}(I_2) \cup Y_i^{-1}(I_1) \cup Y_i^{-1}(I_2) )$$

$$\leq \sum_{i=1}^{m} [\mathrm{P}(X_i < -\frac{1}{2}(r+K)) + \mathrm{P}(X_i > \frac{1}{2}(r+K))]$$

$$+ \sum_{i=1}^{m} [\mathrm{P}(Y_i < -\frac{1}{2}(r+K)) + \mathrm{P}(Y_i > -\frac{1}{2}(r+K))]$$

$$\leq 4m\, \Phi(\frac{\frac{1}{2}(K-r)}{(2\nu T)^{1/2}}) \leq 4m \frac{(2\nu T)^{1/2}}{r-K} \exp(-\frac{(r-K)^2}{16\nu T})$$

where the last inequality follows from the standard estimate (see Feller [22] p. 175)

$$\Phi(x) \leq \frac{-1}{x}\frac{1}{(2\pi)^{1/2}}\exp(-\frac{x^2}{2})$$

for $x < 0$.

We have now shown that estimates (4.22) and (4.23) hold. The conclusion of the proof is obtained once these estimates are substituted into equation (4.21) to produce

$$P(\| \tilde{D}_k S - D_k S \|_{L^1(-\infty,-r)} > \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \, \exp(-\frac{(r-K)^2}{16\nu T}))$$

$$\leq 4m\frac{(2\nu T)^{1/2}}{r-K}\exp(-\frac{(r-K)^2}{16\nu T}).$$

**Proof of the Theorem 4.5.1.** We will now use Lemma 4.5.2 with an appropriate choice of $r$ and $\delta$ to prove Theorem 4.5.1 for the case of monotonic initial data; that is, data generated by particles with equal strengths. Specifically, let us assume that the strengths of the particles satisfy $\varsigma_i = h$. Let us introduce the following notation.

$$a_1 = 4(mh)^{1/2}[K + (8\nu T)^{1/2}] \; \alpha \; h^{1/2}(\ln(1/h))^2,$$

$$a_2 = \frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \; \alpha \; h^{1/2}(\ln(1/h))^2,$$

$$b_1 = 4(mh)^{1/2} h^{\frac{1}{2}\alpha\ln(1/h)-1},$$

$$b_2 = 4(mh) h^{\frac{1}{2}\alpha\ln(1/h)-1}.$$

Now let $r = (8\nu T)^{1/2}\alpha^{1/2}\ln(1/h) + K$, and $\delta = \alpha^{1/2} m^{-1/2}\ln(1/h)$ and observe that

$$2rmh\,[\delta + m^{-1/2}] \leq a_1, \qquad (4.26)$$

$$\frac{(\nu T)^{1/2}}{\pi^{1/2}} mh \exp(-\frac{(r-K)^2}{16\nu T}) \leq a_2, \qquad (4.27)$$

$$4m^{1/2}\exp(-2\delta^2 m) \leq b_1, \qquad (4.28)$$

$$4m\frac{(2\nu T)^{1/2}}{r-K}\exp(-\frac{(r-K)^2}{16\nu T}) \leq b_2. \qquad (4.29)$$

In all of these estimates we make use of the fact that $\alpha > 1$, $\ln(1/h) > 1$ and $h < 1$. Expression (4.26) is a simple estimate which uses the elementary fact that $cd+e < c[d+e]$ if $d, e > 0$ and $c > 1$. Expressions (4.27) and (4.29) follow from the calculation

$$\exp(-\frac{(r-K)^2}{16\nu T}) \leq \exp(\ [\frac{1}{2}\alpha\ln(1/h)]\ln(h)\ ) = h^{\frac{1}{2}\alpha\ln(1/h)} \leq h^{1/2};$$

the last inequality depending on the fact that $\alpha\ \ln(1/h) > 1$ and $h < 1$. Expression (4.28) follows from the calculation

$$\exp(-2\delta^2 m\ ) = \exp([2\alpha\ln(1/h)]\ln(h)) = h^{2\alpha\ln(1/h)} \leq h^{\frac{1}{2}\alpha\ln(1/h)}.$$

where we use the fact that $\frac{1}{2}\alpha\ \ln(1/h) < 2\alpha\ \ln(1/h)$ and that $h < 1$.

Estimates (4.26)-(4.29) and the results of Lemma 4.5.2 imply that

$$P(\ \|\tilde{D}_k S - D_k S\|_{L^1(-r,r)} > a_1) \leq b_1,$$

$$P(\ \|\tilde{D}_k S - D_k S\|_{L^1(-\infty,-r)} > a_2) \leq b_2,$$

$$P(\ \|\tilde{D}_k S - D_k S\|_{L^1(r,\infty)} > a_2) \leq b_2.$$

An application of Theorem 1.4.1 and the triangle inequality then implies that

$$\begin{aligned} &P(\ \|\tilde{D}_k S - D_k S\|_{L^1} > a_1 + 2a_2) \\ &\leq P(\ \|\tilde{D}_k S - D_k S\|_{L^1(-r,r)} > a_1\ ) \\ &+ P(\ \|\tilde{D}_k S - D_k S\|_{L^1(-\infty,-r)} > a_2\ ) \\ &+ P(\ \|\tilde{D}_k S - D_k S\|_{L^1(r,\infty)} > a_2\ ) \\ &\leq b_1 + 2b_2. \end{aligned}$$

The constants $M_1$ and $M_2$ defined in the statement of the theorem satisfy the condition that $a_1 + 2a_2 \leq \frac{1}{2}M_1\alpha\ h^{1/2}(\ln(1/h))^2$ and $b_1 + 2b_2 \leq \frac{1}{2}M_2 h^{\frac{1}{2}\alpha\ln(1/h)-1}$. We conclude that if all of the particles have strength equal to $h$, then

$$P(\ \|\tilde{D}_k S - D_k S\|_{L^1} > \frac{1}{2}M_1\alpha h^{1/2}(\ln(1/h))^2) \tag{4.30}$$

$$\leq \frac{1}{2} M_2 h^{\frac{1}{2}\alpha \ln(1/h) - 1} .$$

It is easy to see that equation (4.30) also holds if the numerical solution is generated by particles, all with strength equal to $-h$ .

Let us suppose that $S$ is generated by particles which have strength of absolute size $h$ , but may be of different sign. For this case we can write $S = S_1 + S_2$ where

$$S_1(y) = s^L + \sum_{\varsigma_i = h} \mathrm{H}(y - X_i)\, \varsigma_i ,$$

and

$$S_2(y) = \sum_{\varsigma_i = -h} \mathrm{H}(y - X_i)\, \varsigma_i .$$

We observe that

$$\widetilde{D}_k S = \widetilde{D}_k S_1 + \widetilde{D}_k S_2$$

$$D_k S = D_k S_1 + D_k S_2 .$$

Let $M_1^i$ and $M_2^i$ denote the constants defined in the statement of the theorem for the step functions $S_i$ , $i=1,2$, and let $M_1$ and $M_2$ be the corresponding constants for $S$ . Since the number of particles generating $S_1$ and $S_2$ are less than or equal to $m$ , it follows that $M_j^i \leq M_j$ for $i,j=1,2$. We can complete the proof of Theorem 4.5.1 by applying Theorem 1.4.1 together with equation (4.30) to the monotonic initial data $S_1$ and $S_2$ to show that

$$\mathrm{P}(\, \| D_k S - \widetilde{D}_k S \|_{L^1} > M_1 \alpha h^{1/2} (\ln(1/h))^2 \,)$$

$$\leq \mathrm{P}(\, \sum_{i=1}^{2} \| D_k Si - \widetilde{D}_k Si \|_{L^1} > \sum_{i=1}^{2} \frac{1}{2} M_1^i\, \alpha h^{1/2} (\ln(1/h))^2 \,)$$

$$\leq \sum_{i=1}^{2} \mathrm{P}(\, \| D_k Si - \widetilde{D}_k Si \|_{L^1} > \frac{1}{2} M_1^i\, \alpha h^{1/2} (\ln(1/h))^2 \,)$$

$$\leq \sum_{i=1}^{2} \frac{1}{2} M_2^i h^{\frac{1}{2}\alpha \ln(1/h) - 1} \leq M_2 h^{\frac{1}{2}\alpha \ln(1/h) - 1} .$$

**4.6. Expected Value of the Error.** We can now use Theorems 1.4.2 and 4.5.1 to estimate the expected value of the $L^1$ error of the random walk algorithm. The following result implies that the random walk algorithm produces an error in $L^1$ which has an expected value of order at most $h^{1/2}(\ln(1/h))^2$, provided the initial particle positions satisfy Assumption 4.4.1.

**4.6.1. Theorem.** *Let $k$ be the time step. Let $S$ be generated by particles with initial positions given by random variables $X_i$, $i=1,\ldots,m$ which satisfy Assumption 4.4.1, with constants $K>0$ and $T>0$. Let the strengths of the particles have absolute size $h$, where $h < 1/10$. Then,*

$$E\left[\, \| \tilde{D}_k S - D_k S \|_{L^1} \right] \leq C_6 h^{1/2}(\ln(1/h))^2$$

*where*

$$C_6 = [8\,(mh)^{1/2}[K + (8\nu T)^{1/2}] + (2\nu T)^{1/2}\, mh\,]\;[1 + 8\,(mh)^{1/2} + 16mh\,].$$

**Proof.** We will apply Theorem 1.4.2 with

$$Y = \| D_k S - \tilde{D}_k S \|_{L^1}$$

and

$$a = M_1 h^{1/2}(\ln(1/h))^2.$$

Using the estimates obtained in Theorem 4.5.1, with $j = \alpha$, we conclude that,

$$E\left( \| D_k s - \tilde{D}_k s \|_{L^1} \right)$$

$$\leq M_1 h^{1/2}(\ln(1/h))^2 \sum_{j=0}^{\infty} P\left( \| D_k S - \tilde{D}_k S \|_{L^1} > j M_1 h^{1/2}(\ln(1/h))^2 \right)$$

$$\leq M_1 h^{1/2}(\ln(1/h))^2 \;\left[1 + M_2 \sum_{j=1}^{\infty} h^{\frac{1}{2} j \ln(1/h) - 1}\right] \tag{4.31}$$

where we have used the fact that

$$P\left( \| D_k S - \tilde{D}_k S \|_{L^1} > 0 \right) \leq 1.$$

Since $h < 1/10$, we can easily show that

$$h^{\frac{1}{2}\ln(1/h)-1} < \frac{4}{5}$$

and

$$h^{\frac{1}{2}\ln(1/h)} < \frac{2}{25}.$$

Consequently, we conclude that

$$\sum_{j=1}^{\infty} h^{\frac{1}{2}j\ln(1/h)-1} = \frac{h^{\frac{1}{2}\ln(1/h)-1}}{1 - h^{\frac{1}{2}\ln(1/h)}} \leq \frac{25}{23} h^{\frac{1}{2}\ln(1/h)-1} \leq 1.$$

We can use this estimate to obtain a bound for the quantity (4.31) which in turn implies that

$$\mathrm{E}\left( \| D_k s - \tilde{D}_k s \|_{L^1} \right) \leq M_1 h^{1/2} (\ln(1/h))^2 \, [1 + M_2].$$

The theorem is concluded by observing that

$$C_6 = M_1 [1 + M_2].$$

## Chapter 5

## Convergence of the Numerical Method

In this chapter we will bring together the results of the last 4 chapters to produce a proof of convergence for our overall numerical method. We will show that the expected value of the $L^1$ norm of the error tends to zero, for an appropriate choice of dependence between time-step $k$ and spatial discretation parameter, $h$. If $k = \nu^{1/2}h^{1/4}$, then Theorem 5.4.3 allows us to assert that

$$E\left[\,\| F_{nk}u_0 - [\widetilde{D}_k\widetilde{A}_k]^n S^0 u_0 \|_{L^1}\right] \leq C_{10}h^{1/4}(\ln(1/h))^2 \tag{5.1}$$

where $n$ is any positive integer satisfying $nk \leq T$ and $C_{10}$ is a constant that depends only on the diffusion constant $\nu$, the final time $T$ and the initial data $u_0$. In Theorem 5.4.2, under the same conditions on $h$, $k$ and $n$, we prove that for any $\alpha > 1$,

$$P\left(\,\| F_{nk}u_0 - [\widetilde{D}_k\widetilde{A}_k]^n S^0 u_0 \|_{L^1} > M_5\alpha h^{1/4}(\ln(1/h))^2\right) \tag{5.2}$$
$$\leq M_6 h^{\frac{1}{2}\alpha\ln(1/h)-2}$$

where the constants $M_3$ and $M_4$ depend on $\nu$, T and $u_0$. From estimate (5.2) we conclude that as the parameter $h$ decreases, the probability that the $L^1$ error is greater than a constant multiple of $h^{1/4}(\ln(1/h))^2$ becomes exponentially small as the constant tends to infinity.. This is an important result from a numerical stand point.

**5.1. Description of the Numerical Method.** We will now give a complete description of our method in terms of the random variables $X_i^{\,j}$ and $X_i^{\,j+1/2}$ and the corresponding random step functions $S^{j-1}u_0(y)$ and $S^{j+1/2}u_0(y)$, introduced in the introduction.

The first step of our algorithm consists of producing an approximation of the smooth data $u_0$ by a step function $S^0u_0$ generated by $m$ (say) particles with initial positions $x_i$, $i = 1,..m$, and strengths $\varsigma_i$ of absolute size $h$, as defined in section 1.1.3.

The algorithm is then defined in terms of the random variables which describe the positions of these particles after each step of the fractional step algorithm. We will define these random variables inductively. Let us denote the random variables which describe the initial positions of the particles by

$$X_i^{\,0} = x_i\,,$$

for $i = 1,..,m$, where we suppose that these random variables are defined on a probability space ( $\Omega_0$ , $\Sigma_0$ , $P_0$ ). Of course the 'random variables' $X_i^{\,0}$ are just constants.

Suppose that the positions of the particles at the end of the $j^{th}$ fractional step are described by random variables $X_i^{\,j}$, which are defined on the product probability space

$$( \prod_{l=0}^{j} \Omega_l \ , \ \prod_{l=0}^{j} \Sigma_l \ , \ \prod_{l=0}^{j} P_l \ )$$

The first half of the next fractional step is associated with solving the advection equation. The random variables $X_i^{\,j+1/2}$, which denote the positions of the particles after this half step are given by the equation

$$( X_1^{j+1/2} ,..., X_m^{\,j+1/2} ) = \widetilde{A}_k ( X_1^j ,..., X_m^{\,j} )$$

(see section 3.3 for the definition of $\widetilde{A}_t$ ). Notice that the $X_i^{\,j+1/2}$ random variables are defined on the same probability space as the $X_i^{\,j}$ random variables.

Let us now chose a set of $m$ independent normally distributed random variables, $B_i^{\,j+1}$, $i = 1,..,m$, defined on the probability space ( $\Omega_{j+1}$ , $\Sigma_{j+1}$ , $P_{j+1}$ ), such that $E\,[B_i^{\,j+1}] = 0$ and $\mathrm{Var}[B_i^{\,j+1}] = 2\nu k$. The random variables $X_i^{\,j+1}$ which describe the particle positions at the end of the $j+1$ fractional step are obtained by applying the random walk operator to the $X_i^{\,j+1/2}$ random variables. Specifically, the random variables $X_i^{\,j+1}$ are defined on the product space

$$( \prod_{l=1}^{j+1} \Omega_l \ , \ \prod_{l=1}^{j+1} \Sigma_l \ , \ \prod_{l=1}^{j+1} P_l \ ),$$

and are given by

$$( X_1^{j+1} ,.., X_m^{\ j+1} ) = \widetilde{D}_k ( X_1^{j+1/2} ,.., X_m^{\ j+1/2} )$$

where the operator $\widetilde{D}_k$ is defined with respect to the random variables $B_i^{\ j+1}$ (see section 4.1 for definition of $\widetilde{D}_k$ ). Equivalently,

$$X_i^{\ j+1} = X_i^{\ j+1/2} + B_i^{\ j+1} \quad .$$

for $i = 1,..,m$ .

The random step functions which describe the numerical approximation at each step of the algorithm are defined as follows:

$$S^j u_0 (y) = [\widetilde{D}_k \widetilde{A}_k]^j S^0 u_0 (y) = u_0^L + \sum_{i=1}^{m} \mathrm{H} (y - X_i^{\ j}) \varsigma_i$$

and

$$S^{j+1/2} u_0 (y) = \widetilde{A}_k [\widetilde{D}_k \widetilde{A}_k]^j S^0 u_0 (y) = u_0^L + \sum_{i=1}^{m} \mathrm{H} (y - X_i^{\ j+1/2}) \varsigma_i$$

for $j$ a non-negative integer.

**5.2. A Property of the Distribution of the Particles.** We will now show that the random variables $X_i^{\ j-1/2}$ satisfy Assumption 4.4.1.

**5.2.1. Theorem.** *Let $u_0 \in C^2(\mathbb{R})$ be constant in a neighbourhood of infinity, with variation contained in the set $[-R, R]$. For $h > 0$ such that $h$ divides $u_0^L - u_0^R$, suppose that $m$ particles generate the step function $S^0 u_0$. Then, for a time step $k$ and for a positive integer, $j$, the random variables $X_i^{\ j-1/2}$, satisfy Assumption 4.4.1, with constants*

$$K_j = R + [\| u_0 \|_{L\infty} + mh ] jk$$

*and $T_j = jk$.*

**Proof.** The proof of this theorem involves showing that the displacement of a particle can be broken into two sums, one a pure random walk, the other bounded by a displacement due to the operator $T$.

We observe that the random variables $X_i^{\,j-1/2}$ and $X_i^{\,j}$ can be written in the form

$$X_i^{\,j-1/2} = \sum_{l=1}^{j-1} B_i^{\,l} + \sum_{l=0}^{j-1} ( X_i^{\,l+1/2} - X_i^{\,l} ) + X_i^{\,0} \tag{5.3}$$

and

$$X_i^{\,j} = \sum_{l=1}^{j} B_i^{\,l} + \sum_{l=0}^{j-1} ( X_i^{\,l+1/2} - X_i^{\,l} ) + X_i^{\,0}. \tag{5.4}$$

Using the nomenclature of Sections 3.1 and 3.3, we recall that for a fixed $\omega \in \prod_{p=0}^{l} \Omega_p$

$$X_i^{\,l+1/2}\big|_\omega = x_i(k),$$

for $i = 1,..,m$, where $x_i(k)$ satisfies equation (3.9) with the initial conditions $x_i(0) = X_i^{\,l}\big|_\omega$. From expression (3.10) we conclude that

$$| x_i(k) - x_i(0) | \le k [ | u_0^L | + mh ] \le k [ \| u_0 \|_{L\infty} + mh ]. \tag{5.5}$$

Since this is true for all choices of initial data $x_i(0) = X_i^{\,l}\big|_\omega$, we conclude that

$$| X_i^{\,l+1/2} - X_i^{\,l} | \le k [ \| u_0 \|_{L\infty} + mh ]. \tag{5.6}$$

The assumption on $u_0$, together with the properties of the $S^0$ operator (see Section 1.1), implies that the initial positions $X_i^{\,0}$ satisfy $| X_i^{\,0} | \le R$. If we substitute the estimate given in (5.6) into equations (5.3) and (5.4), we then see that

$$X_i^{\,j-1/2} \le \sum_{l=1}^{j-1} B_i^{\,l} + jk [ \| u_0 \|_{L\infty} + mh ] + R = \sum_{l=1}^{j-1} B_i^{\,l} + K_j$$

and

$$X_i^{\,j} \le \sum_{l=1}^{j} B_i^{\,l} + jk [ \| u_0 \|_{L\infty} + mh ] + R = \sum_{l=1}^{j} B_i^{\,l} + K_j$$

where $K_j = jk [ \| u_0 \|_{L\infty} + mh ] + R$.

The quantities $\sum_{l=1}^{j-1} B_i^{\,l}$ and $\sum_{l=1}^{j} B_i^{\,l}$ are sums of independent normally distributed random variables with zero mean and variance $2\nu k$ and so are themselves normally distributed with mean zero and with variances of $2\nu k (j-1)$ and $2\nu kj$ respectively. Hence for $a > 0$

$$P( X_i^{\,j-1/2} > a + K_j ) \le P( \sum_{l=1}^{j-1} B_i^{\,l} + K_j > a + K_j )$$

$$= P( \sum_{l=1}^{j-1} B_i^{\,l} > a ) = \Phi(-\frac{a}{(2\nu(j-1)k)^{1/2}})$$

$$\le \Phi(-\frac{a}{(2\nu jk)^{1/2}})$$

and

$$P( X_i^{\,j} > a + K ) \le P( \sum_{l=1}^{j} B_i^{\,l} > a ) \le \Phi(-\frac{a}{(2\nu jk)^{1/2}}).$$

Similarly, we have that

$$X_i^{\,j-1/2} \ge \sum_{l=1}^{j-1} B_i^{\,l} - K_j$$

and

$$X_i^{\,j} \ge \sum_{l=1}^{j} B_i^{\,l} - K_j$$

and so for $a > 0$

$$P( X_i^{\,j-1/2} < -a - K_j ) \le \Phi(-\frac{a}{(2\nu jk)^{1/2}})$$

and

$$P( X_i^{\,j} < -a - K_j ) \le \Phi(-\frac{a}{(2\nu jk)^{1/2}}).$$

Hence the random variables $X_i^{\,j-1/2}$ satisfy Assumption 4.4.1 with constants $K_j$ and $T_j = jk$.

**5.3. Accuracy of the Random Walk Operator.** Theorem 5.2.1, together with the results of Theorems 4.5.1 and 4.6.1 allow us to study the accuracy of the diffusive step of our method for each fractional time step. Specifically we have the following result:

**5.3.1. Theorem.** *Let $u_0 \in C^2(\mathbb{R})$ be constant in a neighbourhood of infinity, with variation contained in the set $[-R, R]$. Let $k$ be the time step, $T$ be the final time, and let $h$ be the spatial parameter, such that $h < 1/10$. Then, for positive integers, $j$, such that $jk \leq T$ and for $\alpha > 1$*

$$P(\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} > M_3 \alpha h^{1/2} (\ln(1/h))^2) \tag{5.7}$$

$$\leq M_4 h^{\frac{1}{2}\alpha \ln(1/h) - 1},$$

*and*

$$E[\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}] \leq C_8 h^{1/2} (\ln(1/h))^2, \tag{5.8}$$

*where $S^{j-1/2} u_0$ is the random step function generated by the random variables $X_i^{j-1/2}$ (see Section 5.1) and the constants $M_3$, $M_4$ and $C_8$ are defined as follows:*

$$M_3 = 8 \| u_0 \|_{BV}^{1/2} [R + T \| u_0 \|_{L^\infty} + T \| u_0 \|_{BV} + (8\nu T)^{1/2}] + (2\nu T)^{1/2} \| u_0 \|_{BV},$$

$$M_4 = 8 \| u_0 \|_{BV}^{1/2} + 16 \| u_0 \|_{BV},$$

$$C_8 = M_3 [1 + M_4].$$

**Proof.** Theorem 5.2.1 implies that the random variables $X_i^{j-1/2}$, satisfy Assumption 4.4.1, with constants $K_j$ and $T_j$, where

$$K_j = jk [\| u_0 \|_{L^\infty} + mh] + R$$

and $T_j = jk$. Since $h < 1/10$, it follows from Theorem 4.5.1 that

$$P(\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} > \tag{5.9}$$

$$[8 (mh)^{1/2} [K_j + (8\nu T_j)^{1/2}] + (2\nu T_j)^{1/2} mh] \alpha h^{1/2} (\ln(1/h))^2)$$

$$\leq [8 (mh)^{1/2} + 16 mh] h^{\frac{1}{2}\alpha \ln(1/h) - 1}.$$

Similarly, we find from Theorem 4.6.1 that

$$E[\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}] \tag{5.10}$$

$$\leq [8 (mh)^{1/2} [K_j + (8\nu T_j)^{1/2}] + (2\nu T_j)^{1/2} mh] [1 + 8 (mh)^{1/2} + 16 mh] h^{1/2} (\ln(1/h))^2.$$

It is now a matter of simplifying the constants in expressions (5.9) and (5.10). This will be achieved by making use of the following observations:

(1) $mh \leq \| u_0 \|_{BV}$ (from Theorem 1.1.4).

(2) $T_j = jk \leq T$, where $T$ is the final time.

(3) $K_j \leq R + [\, \| u_0 \|_{L^\infty} + \| u_0 \|_{BV} \,] T$.

Hence we conclude that

$$8\,(mh)^{1/2}[K_j + (8\nu T_j)^{1/2}] + (2\nu T_j)^{1/2}\, mh \leq M_3,$$

$$8\,(mh)^{1/2} + 16mh \leq M_4,$$

and

$$8\,(mh)^{1/2}[K_j + (8\nu T_j)^{1/2}] + (2\nu T_j)^{1/2}\, mh\,]\,[1 + 8\,(mh)^{1/2} + 16mh\,] \leq C_8.$$

The proof of the theorem follows once these estimates are substituted into expressions (5.9) and (5.10).

**5.4. Convergence of the Numerical Method.** In this section we will prove the convergence results discussed in the introduction of this chapter. We will first rigorously show that the $L^1$ error of the method is bounded by a sum of terms which can be analyzed using the results of the previous 4 chapters.

**5.4.1. Theorem.** *Let $u_0 \in C^2(\mathbb{R})$ be given initial data, constant in a neighbourhood of infinity, such that the variation of $u_0$ is supported in the set $[-R, R]$. Suppose the time step $k \leq \frac{\nu}{2}(\| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1})^{-1}$ and that the spatial parameter $h > 0$ divides $u_0^L - u_0^R$. Let $T$ be the final time. Then for a positive integer, $n$, such that $nk \leq T$,*

$$\| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1}$$

$$\leq C_1 h + C_2 n k^2 + C_9 n k h + \sum_{j=1}^{n} \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}$$

*where the random step functions* $S^{j-1/2}u_0$ *are defined in Section* 5.1 *and the constants are defined as follows:*

$$C_1 = 2R\,,$$

$$C_2 = \frac{90}{\nu}\,\|u_0\|_{BV}\,[\nu^2\|u_0\|_{BD^2} + \nu\|u_0\|_{L^\infty}\|u_0\|_{BD^1} + \|u_0\|^3_{L^\infty}]$$

$$+\frac{4}{\nu}\,\|u_0\|_{BV}\,[\,\|u_0\|^2_{L^\infty} + \nu\|u_0\|_{BD^1}]^{3/2},$$

$$C_3 = \frac{1}{4}\|u_0\|_{BV}\,.$$

**Proof.** As we indicated in the introduction of this paper, the triangle inequality implies that the $L^1$ error

$$\|F_{nk}u_0 - [\widetilde{D}_k\widetilde{A}_k]^n S^0 u_0\|_{L^1}$$

is less than or equal to

$$\|F_{nk}u_0 - [D_k A_k]^n u_0\|_{L^1} \tag{5.11}$$

$$+\,\|[D_k A_k]^n u_0 - [D_k A_k]^n S^0 u_0\|_{L^1} \tag{5.12}$$

$$+\,\|[D_k A_k]^n S^0 u_0 - [\widetilde{D}_k\widetilde{A}_k]^n S^0 u_0\|_{L^1}. \tag{5.13}$$

**Step 1.** Expression (5.11) is just the error of the exact fractional step algorithm, as discussed in Chapter 2. An estimate for this quantity can be obtain via Theorem 2.2.1. Namely

$$\|F_{nk}u_0 - [D_k A_k]^n u_0\|_{L^1} \le C_2 n k^2.$$

**Step 2.** A simple induction argument using the stability of the advection (see Theorem 1.3.3) and diffusion operators, together with the accuracy estimate of the $S^0$ operator contained in Theorem 1.1.4, shows that

$$\|[D_k A_k]^n u_0 - [D_k A_k]^n S^0 u_0\|_{L^1} \le \|u_0 - S^0 u_0\|_{L^1} \le C_1 h.$$

**Step 3.** We notice that expression (5.13) is bounded by the sum of terms

$$\| [D_k A_k]^{n-j} D_k A_k S^{j-1} u_0 - [D_k A_k]^{n-j} D_k \widetilde{A}_k S^{j-1} u_0 \|_{L^1} \tag{5.14}$$

and

$$\| [D_k A_k]^{n-j} D_k S^{j-1/2} u_0 - [D_k A_k]^{n-j} \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} \tag{5.15}$$

for $j = 1,..,n$ . The stability of the transport and diffusion operators can again be utilized to show that the terms (5.14) are less than or equal to

$$\| A_k S^{j-1} u_0 - \widetilde{A}_k S^{j-1} u_0 \|_{L^1}.$$

This estimate together with the accuracy of the transport operator $\widetilde{A}_k$ as estimated in Theorem 3.3.1, implies that the terms given in (5.14) are bounded by $\frac{1}{4} h^2 mk$ . Since $mh \leq \| u_0 \|_{BV}$ (see Theorem 1.1.4) we conclude that

$$\| [D_k A_k]^{n-j} D_k A_k S^{j-1} u_0 - [D_k A_k]^{n-j} D_k \widetilde{A}_k S^{j-1} u_0 \|_{L^1} \tag{5.16}$$

$$\leq \frac{1}{4} \| u_0 \|_{BV} hk.$$

The stability results also imply that the terms given in expression (5.15) are bounded by the random variables

$$\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}. \tag{5.17}$$

Hence expression (5.13) is bounded by the sum of terms given in (5.16) and (5.17) for $j = 1,..,n$ ; that is

$$\| [D_k A_k]^n S^0 u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1}$$

$$\leq \frac{1}{4} \| u_0 \|_{BV} nkh + \sum_{j=1}^{n} \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1}.$$

**Step 4.** Pooling the results of the last three steps we see that the proof of the theorem is complete.

**Remark.** In the following convergence result we need to have the time step $k$ and the spatial parameter $h$ satisfy the relation $k = \nu^{1/2} h^{1/4}$. This requirement seems to be of a technical nature. The actual requirement, as observed in numerical experiments, seems to be of the form $k < h^{1/2}$.

**5.4.2. Theorem.** *Let $u_0 \in C^2(\mathbb{R})$ be given initial data, constant in a neighbourhood of infinity, such that the variation of $u_0$ is supported in the set $[-R, R]$. Suppose that $h > 0$ divides $u_0^L - u_0^R$ and that the time step $k = \nu^{1/2} h^{1/4}$. Let $T$ be the final time. In addition, suppose that*

$$h \leq \min\left(\frac{1}{10}, \frac{\nu^2}{16}(\| u_0 \|^2_{L^\infty} + \nu \| u_0 \|_{BD^1})^{-4}\right)$$

*and that $n$ is a positive integer such that $nk \leq T$. Then for $\alpha > 1$*

$$P( \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} > M_5 \alpha h^{1/4} (\ln(1/h))^2 )$$
$$\leq M_6 h^{\frac{1}{2}\alpha \ln(1/h) - \frac{5}{4}}$$

*with the constants $M_5$ and $M_6$ defined as follows:*

$$M_5 = C_1 + T\nu^{-1/2}[\nu C_2 + \nu^{1/2} C_9 + M_3]$$
$$M_6 = T\nu^{-1/2} M_4$$

*where the constants $M_3$ and $M_4$ are defined in Theorem 5.3.1, and the constants $C_1$, $C_2$ and $C_9$ are defined in Theorem 5.4.1.*

**Proof.** We first note that the conditions on $k$ and $h$ and the initial data $u_0$ imply that Theorems 5.3.1 and 5.4.1 are applicable. Theorem 5.4.1, together with Theorem 1.4.1, implies that

$$P( \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} >$$
$$C_1 h + C_2 n k^2 + C_9 n k h + M_3 n \alpha h^{1/2} (\ln(1/h))^2 )$$

$$\leq P\left(\sum_{j=1}^{n} \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} > M_3 n \, \alpha h^{1/2} (\ln(1/h))^2 \right)$$

$$\leq \sum_{j=1}^{n} P(\| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} > M_3 \alpha h^{1/2} (\ln(1/h))^2 ). \tag{5.18}$$

The terms of the sum in expression (5.18) can be estimated by using Theorem 5.3.1. Namely

$$P( \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} > M_3 \alpha h^{1/2} (\ln(1/h))^2 )$$

$$\leq M_4 h^{\frac{1}{2}\alpha \ln(1/h) - 1}.$$

Hence we conclude that

$$P( \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} >$$

$$C_1 h + C_2 n k^2 + C_9 n k h + M_3 n \, \alpha h^{1/2} (\ln(1/h))^2 )$$

$$\leq M_4 n h^{\frac{1}{2}\alpha \ln(1/h) - 1}.$$

Since $k = \nu^{1/2} h^{1/4}$ and $h < 1/10$, we have

$$C_1 h + C_2 n k^2 + C_9 n k h + M_3 n \, \alpha h^{1/2} (\ln(1/h))^2 \leq M_5 \alpha h^{1/4} (\ln(1/h))^2,$$

and

$$M_4 n h^{\frac{1}{2}\alpha \ln(1/h) - 1} \leq M_6 h^{\frac{1}{2}\alpha \ln(1/h) - \frac{5}{4}}.$$

Our result now follows. Specifically

$$P( \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} > M_5 \alpha h^{1/4} (\ln(1/h))^2 )$$

$$\leq P( \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} > C_1 h + C_2 n \; k^2 + C_9 n k h + M_3 n \, \alpha h^{1/2} (\ln(1/h))^2 )$$

$$\leq M_4 n h^{\frac{1}{2}\alpha \ln(1/h) - 1} \leq M_6 h^{\frac{1}{2}\alpha \ln(1/h) - \frac{5}{4}}.$$

**Remark.** In a similar manner we can now estimate the $L^1$ norm error of our numerical method.

**5.4.3. Convergence Theorem.** *Let $u_0 \in C^2(\mathbb{R})$ be given initial data, constant in a neighbourhood of infinity, such that the variation of $u_0$ is supported in the set $[-R, R]$. Suppose that $h > 0$ divides $u_0^L - u_0^R$ and that the time step $k = \nu^{1/2} h^{1/4}$. Let $T$ be the final time. If*

$$h \leq \min\left(\frac{1}{10}, \frac{\nu^2}{16}(\| u_0 \|_{L^\infty}^2 + \nu \| u_0 \|_{BD^1})^{-4}\right)$$

*and $n$ is a positive integer such that $nk \leq T$, then*

$$E[\, \| F(nk) u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} ] \leq C_{10} h^{1/4} (\ln(1/h))^2$$

*where*

$$C_{10} = C_1 + T \nu^{-1/2} [\nu C_2 + \nu^{1/2} C_9 + C_8] \;;$$

*the constant $C_8$ being defined in Theorem 5.3.1, and the constants $C_1$, $C_2$ and $C_9$ being defined in Theorem 5.4.1.*

**Proof.** As in the previous theorem, the conditions on $k$, $h$ and $u_0$ guarantee that we can apply Theorems 5.3.1 and 5.4.1. Theorem 5.4.1 implies that

$$E[\, \| F_{nk} u_0 - [\widetilde{D}_k \widetilde{A}_k]^n S^0 u_0 \|_{L^1} ]$$

$$\leq C_1 h + C_2 n k^2 + C_9 n k h + \sum_{j=1}^{n} E[\, \| D_k S^{j-1/2} u_0 - \widetilde{D}_k S^{j-1/2} u_0 \|_{L^1} ],$$

which, by Theorem 5.3.1, is less than or equal to

$$C_1 h + C_2 n k^2 + C_9 n k h + C_8 n h^{1/2} (\ln(1/h))^2.$$

Since $k = \nu^{1/2} h^{1/4}$, $nk \leq T$, and $\ln(1/h) > 1$ ($h < 1$), we conclude that

$$C_1 h + C_2 n k^2 + C_9 n k h + C_8 n h^{1/2} (\ln(1/h))^2$$

$$\leq C_1 h + C_2 T \nu^{1/2} h^{1/4} + C_9 T h + C_8 T \nu^{-1/2} h^{1/4} (\ln(1/h))^2$$

$$\leq [C_1 + C_2 T \nu^{1/2} + C_9 T + C_8 T \nu^{-1/2}]\, h^{1/4} (\ln(1/h))^2 = C_{10} h^{1/4} (\ln(1/h))^2$$

Hence

$$C_1 h + C_2 n k^2 + C_9 n k h + C_8 n h^{1/2} (\ln(1/h))^2 \leq C_{10} h^{1/4} (\ln(1/h))^2$$

and so

$$\mathrm{E}\,[\,\|\,F_{nk}\,u_0\; - [\widetilde{D}_k\,\widetilde{A}_k\,]^n\,S^0 u_0\,\|_{L^1}\,] \le \mathrm{C}_{10} h^{1/4} (\ln(1/h\,))^2.$$

This concludes the proof of the convergence of our numerical method.

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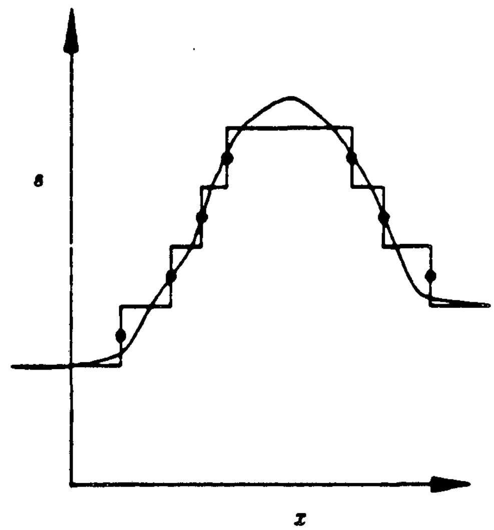


figure 1

Approximation of smooth data with a step function

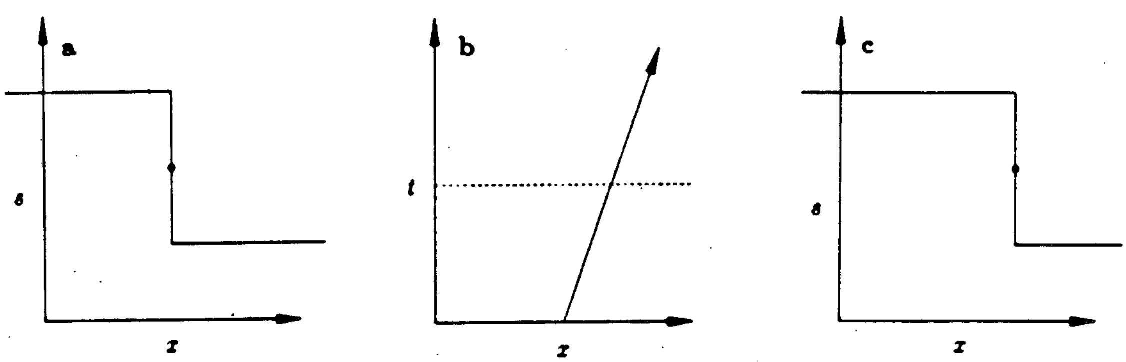


figure 2

Evolution of particles which form a negative discontinuity. (a) Initial discontinuity. (b) Particle trajectories. (c) Step function solution at time $t$.

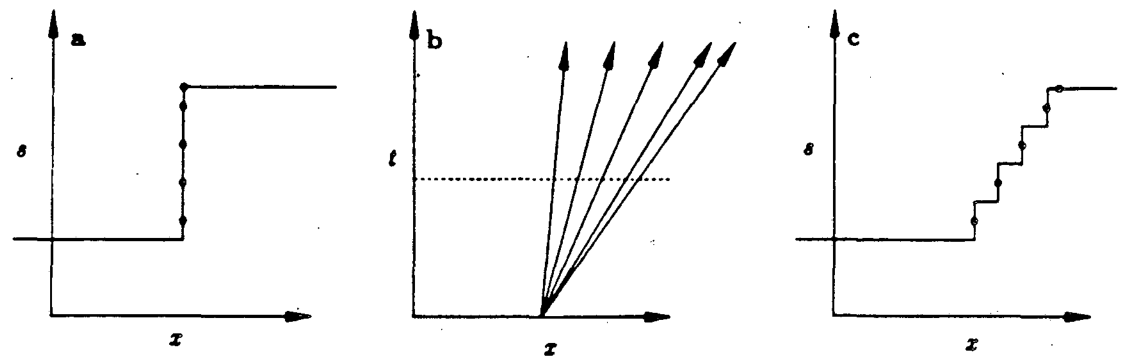


figure 3

Evolution of 6 particles which form a positive discontinuity. (a) Initial discontinuity. (b) Particle trajectories (Note that two particles are contained in the right most trajectory). (c) Step function solution at time $t$.

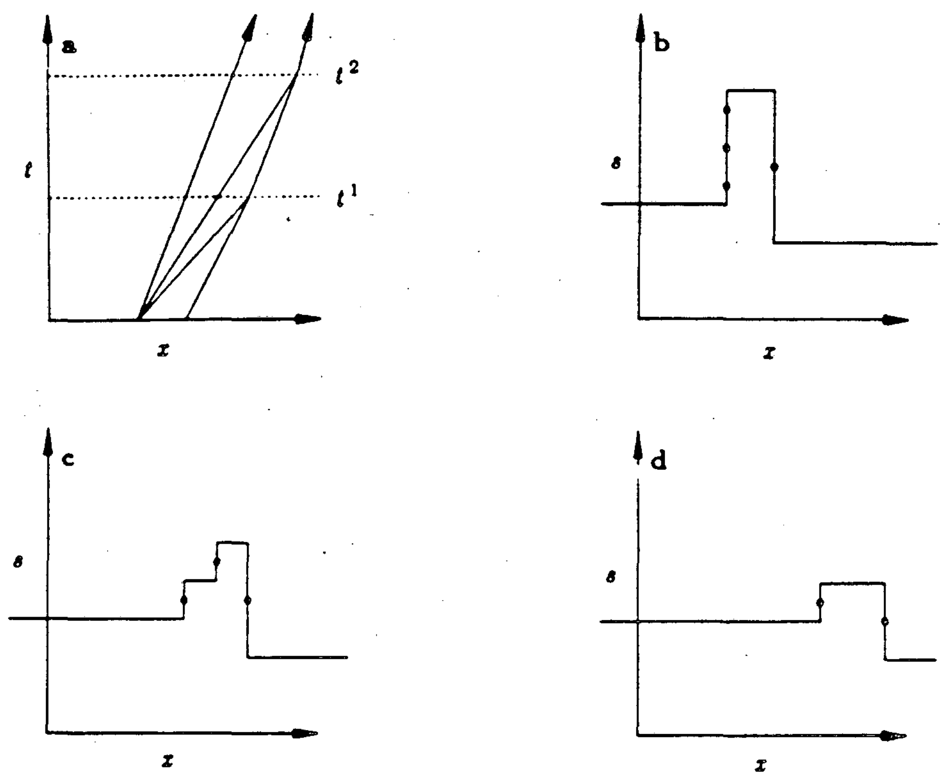


figure 4

Intersection of particle trajectories. (a) Trajectories of particles. (b) Initial step function generated by particles. (c) Solution at time $t^1$. (d) Solution at time $t^2$.

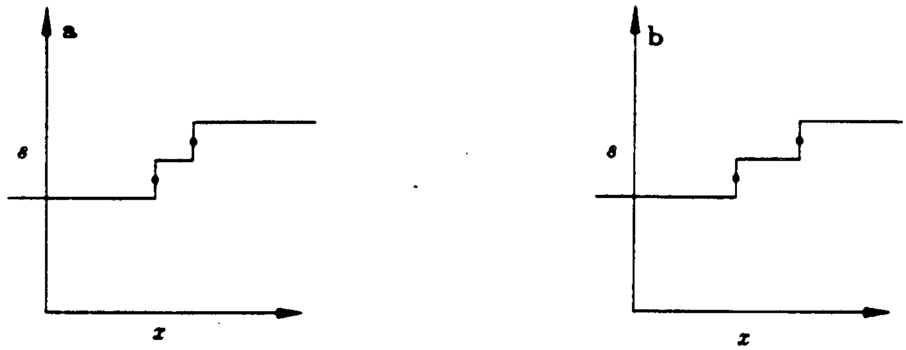


figure 5

Distance between two particles with positive strength increases with time.

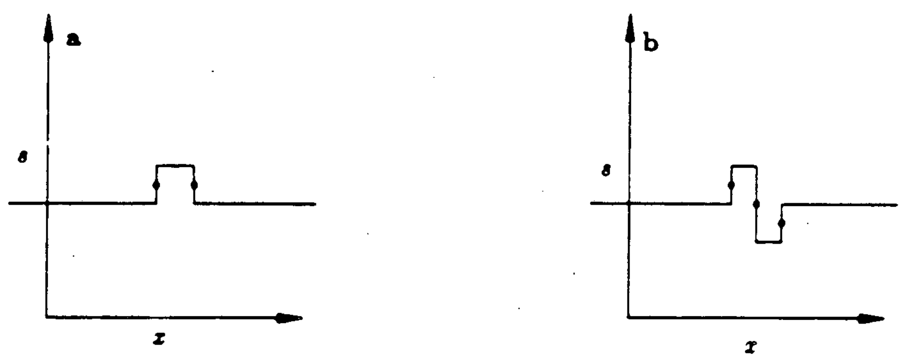


figure 6

(a) No interaction between particle with positive strength and adjacent numerical shock of strength greater than $-2h$ . (b) Interaction possible between particles with positive strength and numerical shock of strength less than $-h$ .

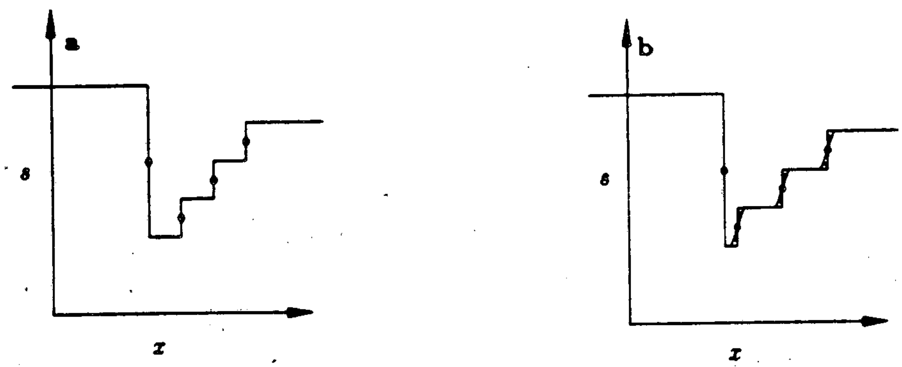


figure 7

(a) Step function $\widetilde{A}_t\, s$ . (b) Step functions $\widetilde{A}_{\tau-t}\, \widetilde{A}_t\, s$ and $A_{\tau-t}\, \widetilde{A}_t\, s$ .

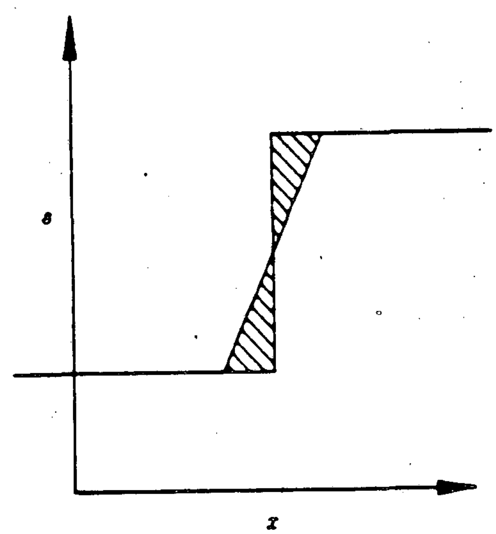


figure 8

Enlargement of one positive step and rarefaction wave shown in figure 7. Height of jump is $h$ and width of rarefaction wave is $(\tau - t)h$ .

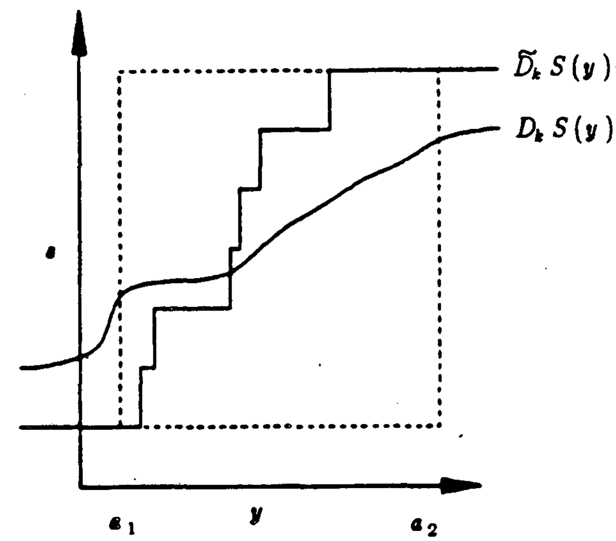


figure 9

Difference between smooth function $D_k S(y)$ and the step function $\tilde{D}_k S(y)$ in the range $[a_1, a_2]$.

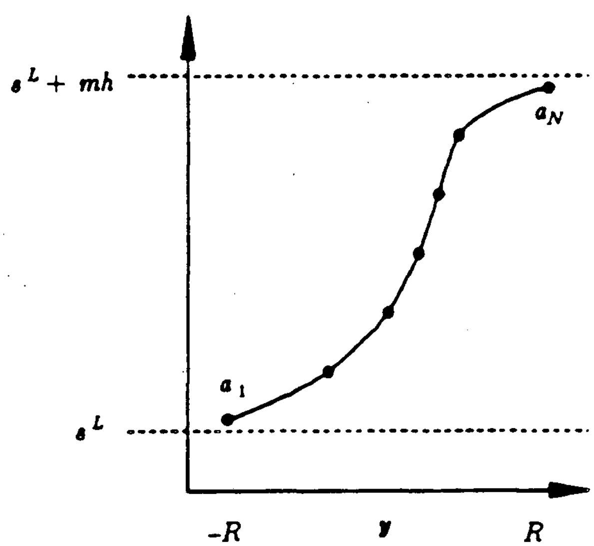


figure 10

Points $a_i$ chosen so that $D_k S(a_{i+1}) - D_k S(a_i) \leq hm^{1/2}$.

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