UNIVERSITY OF CALIFORNIA
RIVERSIDE

Generalized Fractal Strings, Complex Dimensions and a Spectral Reformulation of The Riemann hypothesis

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

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August 2011

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Acknowledgments

I would like to thank my advisor for making me work on this research project, for all his help, advices and support. I would like also to thank all my colleagues at the Fractal Research Group and Mathematical Physics and Dynamical Systems seminar at the University of California, Riverside for their encouragements. Finally, I am very grateful to my family and all my good friends support during the period of my doctoral studies.

To my family and all my friends who supported me throughout this modest journey of passion for learning and research.

"Nature has its perfections showing it is an image of god but also has its faultiness showing it is nothing more than an image..."

-Blaise Pascal: Thoughts-

ABSTRACT OF THE DISSERTATION

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The spectral operator was introduced for the first time by M. L. Lapidus and his collaborator M. van Frankenhuijsen in their theory of complex dimensions in fractal geometry [La-vF1, La-vF2]. The corresponding inverse spectral problem was first considered by M. L. Lapidus and H. Maier in their work on a spectral reformulation of the Riemann hypothesis in connection with the question "Can One Hear The Shape of a Fractal String?" [LaMa2]. The spectral operator is defined on a suitable Hilbert space as the operator mapping the counting function of a generalized fractal string η to the counting function of its associated spectral measure. It relates the spectrum of a fractal string with its geometry. The inverse spectral problem for vibrating fractal strings studied by M. L. Lapidus and H. Maier in [LaMa2] has a positive answer if and only if the Riemann zeta function $\zeta(s)$ has no zeros on $Re(s) = D$, where $D \in (0, 1)$ is the dimension of the fractal string. In this work, we provide a functional analytic framework allowing us to study the spectral operator. In particular, by determining the spectrum of the spectral operator, we give a necessary and sufficient condition providing its invertibility in the critical strip. We show that such a condition is related to the location of the critical zeroes of the Riemann zeta function or equivalently that the spectral operator is invertible

if and only if the Riemann hypothesis is true [HerLa1]. As a result, the spectral operator is invertible for any $D \in (0,1) - \{\frac{1}{2}\}$ if and only if the Riemann hypothesis is true [HerLa1]. The latter results provides a spectral reformulation of the Riemann hypotesis in terms of a rigorously defined map (the spectral operator). Hence, it sheds new light to the earlier work obtained by M. L. Lapidus and H. Maier in [LaMa1, LaMa2] and later revisited by M. L. Lapidus and M. van Frankenhuijsen in [La-vF1, La-vF2, La-vF3].

Contents

Introduction

The spectral operator was introduced by M. L. Lapidus and M. van Frankenhuijsen in their theory of complex dimensions in fractal geometry and number theory [La-vF3]. The corresponding inverse spectral problem was first considered by M. L. Lapidus and H. Maier in their work [LaMa2] on a spectral reformulation of the Riemann hypothesis in connection with the question "Can One Hear The Shape of a Fractal String?" (See also [LaMa1].) The spectral operator is defined on a suitable Hilbert space. Such a space is equipped with boundary conditions that are suggested by the work of M. L. Lapidus and C. Pomerance [LaPo2] on the Riemann zeta function and the Weyl-Berry conjecture and on a characterization of the Minkowski measurability of fractal strings in terms of certain asymptotic bounds of their counting functions. (See also [LapPo1].)

The spectral operator (henceforth denoted by a) is the operator mapping the counting function of a generalized fractal string η to the counting function of its associated spectral measure, $\nu = \eta * h$, where $*$ denotes the multiplicative convolution of measures and h is the generalized harmonic string,

$$N_\eta(x) \longmapsto N_\nu(x)$$

or equivalently, and under the change of variable $x = e^t$

$$f(t) \longmapsto a(f)(t) = \sum_{k=0}^{\infty} f(t - logk)\,,$$

where f is the counting function of the underlying generalized fractal string. It relates the spectrum of a fractal string with its geometry. The spectral operator also has an operator-valued Euler product representation, which provides a counterpart to the usual

Euler product expansion for the Riemann zeta function $\zeta = \zeta(s)$, but is conjectured in [La-vF1] to be also convergent in the critical strip of the complex plane:

$$a(f)(t) = \zeta(\partial)(f)(t) = \prod_{p \in \mathcal{P}} (1 - p^{-\partial})^{-1}(f)(t) \quad \text{with } \partial := \tfrac{d}{dt},$$

where $\mathcal{P}$ denotes the set of prime numbers and for each $p \in \mathcal{P}$, the operator-valued pth Euler factor is defined by

$$a_p(f)(t) = \zeta_p(\partial)(f)(t) = (1 - p^{-\partial})^{-1}(f)(t)$$

with $\zeta_p(s) := (1-p^{-s})^{-1}$. We note that the last two equalities are understood in the sense of the functional calculus for unbounded normal operators. More specifically, ∂ denotes here a suitable realization of the differential operator $\frac{d}{dt}$, acting on an appropriate Hilbert space H_c, depending on a parameter $c \geq 0$. We will show, in particular, that ∂ (to be also sometimes denoted by ∂_c later on) is an unbounded normal operator: $\partial^* \partial = \partial \partial^*$, where ∂^* denotes the adjoint of ∂ and is given by $\partial^* = 2c - \partial$.

The goal of this research project is to study the spectral operator $a = a_c$. Such a study will be focused on describing its spectrum and on obtaining a necessary and sufficient condition for its invertibility, thereby enabling us to answer the following question in this context: "What kind of geometric information about a fractal string of dimension $D \in (0, 1)$ can be derived from its vibrational spectrum ?".

The thesis is organized as follows. In Chapter 1, we define our object of study, the class of ordinary fractal strings as introduced by M. L. lapidus and M. van Frankenhuijsen throughout their development of the theory of complex dimensions in fractal geometry [La-vF1, La-vF2]. An ordinary fractal string is a bounded open subset of the real line. Such a set is a disjoint union of open intervals, the lengths of which form a sequence $\mathcal{L} = l_1, l_2, l_3, ...$ which we assume to be infinite. Important geometric information about $\mathcal{L}$ is encoded in its geometric zeta function $\zeta_{\mathcal{L}}$ which will be assumed to have a suitable meromorphic extension. The complex dimensions of $\mathcal{L}$ are defined as the poles of the meromorphic extension of $\zeta_{\mathcal{L}}$. The complex dimensions carry in their real and imaginary parts important information about the oscillations in the geometry of the set $\mathcal{L}$. A standard example of such a class of strings is the Cantor string which is studied in Section 1.3. The notion of fractality within the theory of complex dimensions is defined as follows: "A set is a fractal if and only if its geometric zeta function has at least one complex dimension whose real part is strictly positive"(see [La-vF1].)

Chapter 2 is devoted to the study of the class of generalized fractal strings. These can be viewed as natural generalizations of the measures associated to ordinary fractal strings. Furthermore, in Section 2.5 we present some fundamental tools that are needed in our study of the spectral operator. We also recall the explicit formulas for generalized fractal strings; such formulas were given for the first time, in the context of fractal geometry, by M. L. Lapidus and M. van. Frankenhuijsen [La-vF1, La-vF2, La-vF3]. They give a representation of the kth distributional primitive (or antiderivative) of a generalized fractal string as a sum over its complex dimensions (defined as the poles of an associated geometric zeta function).

We refer the reader to [HerLa1] for all the results obtained in Chapters 3-6. In Chapter 3, we define the spectral operator $a = a_c$, using the explicit formulas at levels $k = 0$ and $k = 1$. We also define its operator-valued Euler factors a_p (where $p \in \mathcal{P}$ is a prime number) and Euler product $\prod_{p\in\mathcal{P}} a_p$, the detailed study of which will be the object of the sequel to the present research project [HerLa2].

In Chapter 4, we will define a suitable Hilbert space H_c, which is equipped with some boundary conditions naturally satisfied by the class of counting functions of generalized fractal strings. Based on the work of M. L. Lapidus and C. Pomerance [LaPo1, LaPo2], such a space can be viewed as a suitable realization in our context of the space of fractal strings of dimension $D \in (0,1)$. In Chapter 5, We give a description of the spectra of the operators ∂_c (in Section 5.1). We also exhibit the dependence of these spectra on the asymptotic behaviour of the counting functions of fractal strings at infinity. More explicitly, we show in Section 5.2 that the spectrum of ∂_c is exactly the vertical line $\{Re(s) = D\}$, which (since $D \in (0,1)$) is a subset of the critical strip $\{0 < Re(s) < 1\}$ (see [HerLa1].)

In Chapter 6, we provide a precise definition of the spectral operator. Furthermore, in Section 6.4 we deduce from the functional calculus form of the spectral theorem for unbounded normal operators [Ru2] that since ∂_c is an unbounded normal operator and $a_c = \zeta(\partial_c)$, as proved in Sections 4.3 and 5.2, respectively, the spectrum of a_c is given by $\{\zeta(s) : Re(s) = D\}$, the range of the values of the Riemann zeta function along the vertical line $\{Re(s) = D\}$ [HerLa1]. In fact, it turns out in our present situation that $D = c$ (although different choices of boundary conditions could yield a different conclusion, as well as a different spectrum for ∂_c and hence also for a_c), and that D is therefore allowed to be any nonnegative real number. Hence, we may have $D = 0$, $D = 1$ or $D > 1$, in addition to the present case when $D \in (0,1)$.

Finally, in Section 6.4, we explain how the above result concerning the spectrum of a_c can be used to provide necessary and sufficient conditions for the invertibility of the spectral operator on the critical strip. More specifically, we show that the spectral operator is invertible on the space of counting functions of generalized fractal strings whose dimension is $D \in (0,1)$ if and only if $\zeta(s)$ does not have any zeroes on the vertical line $Re(s) = D$. In particular, in Section 6.5 we deduce that the spectral operator is not invertible in the "midfractal" case when $D = \frac{1}{2}$, and that it is invertible everywhere else if and only if the Riemann hypothesis is true [HerLa1]. The last result recaptures the work of M. L. Lapidus and H. Maier [LaMa1, LaMa2] on the inverse spectral problem for the vibrations of fractal strings which provided a spectral reformulation of the Riemann hypothesis. See also the later reformulation of the results of [LaMa1, LaMa2] given in [La-vF2, La-vF3] in terms of explicit formulas and rigorously defined complex dimensions. (See, especially, [La-vF3], Sections 6.3.1, 6.3.2 and Chapter 7.) Our work provides, in particular, a precise and rigorous functional analytic justification of this latter reformulation (see, especially, [La-vF2; Corollary 9.6]), as well as a precise determination of the spectrum of a_c (thereby answering a key open problem in [La-vF3], Section 6.3.2, see [La-vF2], Problem 6.10). It also shows that the question of the invertibility of the spectral operator (now properly formulated as a precise mathematical problem) is intimately connected with the location of the critical zeroes of the Riemann zeta function, and therefore with the Riemann hypothesis.

We close this introduction by providing some general background references. For the theory of unbounded self-adjoint operators and its applications to physics, especially quantum mechanics, we refer to [Br, CouHi, JoLa, Kat, ReSi 1-2]. For the more general, but less well-known, theory of unbounded normal operators (including the powerful spectral theorem) and some of its applications, we refer to [Ru2], [Kat] and [Jola]. For distribution theory, we mention [Sch1], [Sch2] and [GeSh], while for other notions of measure theory, real and functional analysis (including the theory of absolutely continuous functions) used in this paper, see for example [Coh, Fo, Ru2]. For general references on fractal geometry, we point out [Fa, Man, Mat]. Finally, for the theory of the Riemann zeta function, see, for example [Edw, Ing, Ivi, Pat, Tit].

Chapter 1

Ordinary Fractal Strings and Their Complex Dimensions

In this chapter, we introduce our basic object of study: the class of *ordinary fractal strings* as defined and studied by M. L. Lapidus and M. van Frankenhuijsen throughout their development of the theory of complex dimensions in fractal geometry and number theory [La-vF1]. Such an object is a bounded open subset of the real line. The theory of complex dimensions studies the class of ordinary fractal strings by viewing them as the complement of rough subsets of the real line. Important information about the geometry of a fractal string is located in the sequence of lengths associated to the connected components of the set and is encoded in its geometric zeta function. Information about the oscillations in the geometry of the fractal string is encoded in its complex dimensions, which are the poles of the meromorphic continuation of the geometric zeta function associated to the fractal string. Given an ordinary fractal string, one can listen to its sound. Information about the spectrum of an ordinary fractal string is located in its spectral zeta function. We start providing the necessary backgound material allowing us to study the class of ordinary fractal strings (we refer the reader to [La-vF1, La-vF2, La-vF3] for more details on this topic.)

1.1 Ordinary fractal strings

Definition 1 *An* ordinary fractal string $\mathcal{L}$, *called also a* standard fractal string, *is a bounded open subset* Ω *of* R. *Such a set consists of countably many disjoint open intervals*

$$\Omega = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

We will denote the lengh of (a_j, b_j) *by* l_j, *for* $i \geq 1$ *and consider* $\Omega = \{l_j\}_{j=1}^{\infty}$.

Remark 2 *We will assume without loss of generality that*

$$l_1 \geq l_2 \geq l_3 \geq ... \geq 0$$

where each l_i, $i \geq 1$ *will be counted according to its multiplicty* w_{l_i}, $i \geq 1$. *We will allow for* Ω *to be a finite union of intervals, in which case the sequence of lengths is finite. Note that in the case where such a union is finite we have*

$$\sum_{j=1}^{\infty} l_j < \infty$$

and this last quantity is equal to the Lebesgue measure of Ω.

An ordinary fractal string can be thought of as a one-dimensional drum with fractal boundary. In fact, one can also call Ω *a* fractal harp, *and each connected interval of* Ω *could be called a string of the harp.*

Definition 3 *The* counting function *of the reciprocal lengths, also called the* geometric counting *function of $\mathcal{L}$, is the function*

$$N_{\mathcal{L}}(x) = \#\{\, j \geq 1 : l_j^{-1} \leq x \,\} = \sum_{j \geq 1\,,\, l_j^{-1} \leq x} 1.$$

Remark 4 *We will use the convention that the integers j such that $l_j^{-1} = x$ must be counted with weight $\frac{1}{2}$. A similar convention will be assumed for the spectral counting functions; with such convention, the pointwise explicit formulas without error term, as defined in [La-vF1], will hold at the jumps of the counting functions (see [La-vF1] section 5.3 for further details.)*

Proposition 5 *Let $\mathcal{L}$ be an ordinary fractal string with sequence of lengths $l_1, l_2, l_3, \ldots$. Then, $N_{\mathcal{L}}(x) = O(x^D)$ as $x \to +\infty$ if and only if $l_j = O(j^{-\frac{1}{D}})$ as $j \to +\infty$.*

Definition 6 *We will denote the boundary of an ordinary fractal string $\mathcal{L}$ by $\partial\mathcal{L}$. For a given positive ϵ, let $V(\epsilon)$ be the* volume of the inner tubular neighborhood *of $\partial\mathcal{L}$ with radius ϵ, defined as follows ;*

$$V(\epsilon) = vol_1\{\, x \in \Omega : d(x, \partial\Omega) < \epsilon \,\},$$

where vol_1 is the one-dimensional Lebesgue measure on R and $d(x, A)$ is defined as $d(x, A) = \inf\{\, d(x, y)\,,\, y \in A \subset R \,\}$.

1.2 Minkowski measurability and the tubular neighborhood of $\mathcal{L}$

Definition 7 *The* dimension *of a fractal string* $\mathcal{L}$*, is defined as the* inner Minkowski dimension *of* $\partial\mathcal{L}$*;*

$$D = D_{\mathcal{L}} = \inf\{\, \alpha \geq 0 : V(\epsilon) = O(\epsilon^{1-\alpha}) \, as \, \epsilon \to 0^+ \,\}.$$

$\mathcal{L}$ *is said to be* Minkowski measurable*, with* Minkowski content

$$\mathcal{M} = \mathcal{M}(D, \mathcal{L}) = \lim_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)}$$

if this limit exists in $(0, +\infty)$*.*

Remark 8 *Clearly* $\mathcal{L}$ *is Minkowski measurable if and only if* $\mathcal{M}^* = \mathcal{M}_* = \mathcal{M} \in (0, +\infty)$*, where*

$$\mathcal{M}^*(D, \mathcal{L}) = \limsup_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)}$$

and

$$\mathcal{M}_*(D, \mathcal{L}) = \liminf_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)}.$$

Note that we always have

$$0 \leq \mathcal{M}_* \leq \mathcal{M}^* \leq +\infty.$$

Remark 9 *Even though the Hausdorff dimension H is used in many works on fractals, throughout this work, the Minkowski dimension will be used instead, because it* *is invariant under displacement of intervals of which a fractal string is composed (see [LaPo1, LaPo2, LaPo3]). From now we will denote such a string by* $\mathcal{L} = \{l_j\}_{j=1}^{+\infty}$, *where each length* l_j *is counted according to multiplicity* w_{l_j} *for any* $j \geq 1$.

Remark 10 *The more irregular the boundary of* $\mathcal{L}$, $\partial\mathcal{L}$, *the larger* D *is, the Minkowski dimension of the string. Moreover, if* Ω *is a bounded open set in* R^d, *we always have*

$$d \leq H \leq D \leq d\,,$$

where H *and* D *denote respectively the* Hausdorff dimension *and the* Minkowski dimension *of the boundary* $\partial\Omega$ *of* Ω. *As a result, and in the case of an ordinary fractal string, we always have*

$$0 \leq H \leq D \leq 1\,.$$

Remark 11 *In the case of ordinary fractal strings, both the Minkowski dimension and content depend only on the lengths* l_j, *and hence are invariant under arbitrary rearrangements of the intervals* l_j, *the connected components of* Ω.

1.3 The Cantor string as an example of an ordinary fractal string

We will consider the ordinary string $\Omega = CS$, defined as the complement in $[0, 1]$ of the usual ternary Cantor set.

Hence, $CS = (\frac{1}{3}, \frac{2}{3}) \bigcup (\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9}) \bigcup (\frac{1}{27}, \frac{2}{27}) \bigcup \frac{7}{27}, \frac{8}{27}) \bigcup (\frac{19}{27}, \frac{20}{27}) \bigcup (\frac{25}{27}, \frac{26}{27}) \bigcup ...$

It turns out that the CS has Minkowski dimension $D = \log_3 2,$ and that it is not

Minkowski measurable, because its upper and lower Minkowski content do not coincide. In fact, we have

$$\mathcal{M}^* = 2^{2-D} = 2.5830 \neq 2.495 = 2^{2-D} D^{-(1-D)} = \mathcal{M}_*.$$

The volume of the tubular neighborhood of the boundary of $\mathcal{L}$ (see [La-vF1], [La-vF2]) is given by the formula

$$V(\epsilon) = \sum_{j:l_j \geq 2\epsilon} 2\epsilon + \sum_{j:l_j < 2\epsilon} l_j = 2\epsilon N_{\mathcal{L}}(\frac{1}{2\epsilon}) + \sum_{j:l_j < 2\epsilon} l_j.$$

Applying this to the Cantor string, we get

$$V(\epsilon) = 2\epsilon(2^n - 1) + \sum_{k=n}^{\infty} 2^k 3^{-k-1} = 2\epsilon 2^n + (\frac{2}{3})^n - 2\epsilon$$

where n is the positive integer such that $3^{-n} \geq 2\epsilon \geq 3^{-n-1}$, that is $n = [-\log_3(2\epsilon)]$. One can show that

$$V_{CS}(\epsilon) = (2\epsilon)^{1-D} \left((\frac{1}{2})^{\{-\log_3(\epsilon)\}} + (\frac{3}{2})^{\{-\log_3(2\epsilon)\}} \right) - 2\epsilon\,.$$

The function between parentheses is bounded, non constant and multiplicatively periodic, it takes the same values at ϵ and $\frac{\epsilon}{3}$, and has no limit as $\epsilon \to 0^+$. By using the Fourier series for the periodic function

$$u \longmapsto b^{-\{u\}},\, b > 0,\, b \neq 1\,,$$

we get the following series expansion

$$b^{-\{u\}} = \frac{b-1}{b} \sum_{n \in Z} \frac{e^{2\pi i n u}}{\log b + 2\pi n i}.$$

Combining this equation with the previous one, we get for $0 < \epsilon \leq \frac{1}{2}$

$$V_{CS}(\epsilon) = \frac{1}{2\log 3} \sum_{n=-\infty}^{\infty} \frac{(2\epsilon)^{1-D-in\mathbf{p}}}{(D+in\mathbf{p})(1-D-in\mathbf{p})} - 2\epsilon\,,$$

where $D = \log_3 2 := \frac{\log 2}{\log 3}$ and $\mathbf{p} = \frac{2\pi}{\log 3}$ is called the oscillatory period of the Cantor string. (For further details on how to derive the last equation, we encourage the reader to see [La-vF1] pages 13-15.)

Remark 12 *Note that the previous sum expresses $V_{CS}(\epsilon)$, for any given $0 < \epsilon \leq \frac{1}{2}$, as an infinite sum of complex numbers. This sum is in fact real-valued, as can be seen by combining the terms for n and -n into one, for $n \geq 1$. Indeed, these two terms are the complex conjugates of one another and hence the sum is real-valued. As a result, we get the following expression for $V_{CS}(\epsilon)$;*

$$V_{CS}(\epsilon) = \frac{2^{-D}\epsilon^{1-D}}{D(1-D)\log 3} + \frac{1}{\log 3}\sum_{n=1}^{\infty} Re(\frac{(2\epsilon)^{1-D-inp}}{D+inp}) - 2\epsilon\,.$$

Note that in this formula, we did manage to express $V_{CS}(\epsilon)$ as a sum of a term at $D = \log_3 2$ and an infinite sum of real-valued terms.

1.4 The geometric zeta function of an ordinary fractal string

Definition 13 *Let $\mathcal{L} = \{l_j\}_{j=1}^{\infty}$ be an ordinary fractal string. We recall that $\sum_{j=1}^{\infty} l_j^{\sigma}$ converges for $\sigma = 1$. It follows that the Dirichlet series $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$ defines a holomorphic function for $Re(s) > 1$. In fact, this series converges in the open right half-plane*

$Re(s) > D$ where D is the Minkowski dimension of the fractal string. The geometric zeta function *of $\mathcal{L}$ is defined as*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_{l} w_l l^s \ , \ for \ Re(s) > \sigma = D_{\mathcal{L}}.$$

Remark 14 *Some values of the geometric zeta function of an ordinary fractal string $\mathcal{L}$ do have a special interpretation. For instance, if $\mathcal{L}$ does have only finitely many lengths l_j, then $\zeta_{\mathcal{L}}(0)$ equals the number of lengths of the string, and we have $\zeta_{\mathcal{L}}(0) = \sum_{l_j\,,1\leq j\leq N} w_{l_j}$, where $N \geq 1$. One can also observe that the total length L of the fractal string $\mathcal{L}$ is $L := \zeta_{\mathcal{L}}(1) = vol_1(\Omega) = \sum_{j=1}^{\infty} l_j$, where vol_1 is the one-dimensional Lebesgue measure of Ω.*

Definition 15 *The abscissa of convergence of the Dirichlet series $\sum_{j=1}^{\infty} l_j^s$ is defined by*

$$\sigma = \inf\{\, \alpha \in R : \sum_{j=1}^{\infty} l_j^{\alpha} < +\infty \,\}.$$

As a consequence, $\{\, s \in C : Re(s) > \sigma \,\}$ is the largest open half-plane on which this series converges absolutely. Note that the function

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$$

is holomorphic in this half-plane.

Theorem 16 *Suppose $\mathcal{L}$ has infinitely many lengths. Then the abscissa of convergence of the geometric zeta function associated to $\mathcal{L}$ coincides with D, the Minkowski dimension of $\partial\mathcal{L}$, (see [La2, La-vF1].)*

Remark 17 *Using the previous theorem and the result of Remark 2.3, we deduce that the dimension D of an ordinary fractal string $\mathcal{L}$ satisfies the following combined equality:* $0 \leq D \leq 1$.

Remark 18 *In general ,$\zeta_{\mathcal{L}}$ may not have an analytic continuation to all of C. As a consequence, we introduce the screen $\mathcal{S}$ as the contour $\mathcal{S} : S(t) + it\,, t \in R$, where $S : R \longmapsto [-\infty, D_{\mathcal{L}}]$ is a continuous function. We will call the set*

$$\mathcal{W} = \{\, s \in C : Re(s) \geq S(Im(s)) \,\}$$

the window. We will also assume that $\zeta_{\mathcal{L}}$ has a suitable meromorphic extension to a neighborhood of $\mathcal{W}$; we will continue to denote by $\zeta_{\mathcal{L}}$ the meromorphic extension of the Dirichlet series given in Definition 4.1.

Let $D_{\mathcal{L}} = D_{\mathcal{L}}(\mathcal{W}) = \{\omega \in \mathcal{W}, \omega$ is a pole of $\zeta_{\mathcal{L}}(s)\}$; then we will call the set of poles $D_{\mathcal{L}} = D_{\mathcal{L}}(\mathcal{W}) \subset \mathcal{W}$ the set of visible complex dimensions of $\mathcal{L}$. We will require that $\zeta_{\mathcal{L}}$ does not have any poles on the screen.

Definition 19 *The set of* visible complex dimensions *of the fractal string $\mathcal{L}$ is defined as $D_{\mathcal{L}} = D_{\mathcal{L}}(\mathcal{W}) = \{\, \omega \in C : \zeta_{\mathcal{L}}$has a pole at $\omega\}$.*

Remark 20 *If $\mathcal{W} = C$, that is, if $\zeta_{\mathcal{L}}$ has a meromorphic extension to all of C, we call $D_{\mathcal{L}} = D_{\mathcal{L}}(C) = \{\, \omega \in C : \zeta_{\mathcal{L}}$ has a pole at ω the set of* complex dimensions *of $\mathcal{L}$.*

Firstable, note that $D_{\mathcal{L}}(\mathcal{W})$ is a discrete subset of C because it is the set of poles of a meromorphic function. Secondly, if $\mathcal{L}$ consists of finitely many lengths, we have $D_{\mathcal{L}} = D_{\mathcal{L}}(\mathcal{W}) =$, since $\zeta_{\mathcal{L}}(s)$ is in this case an entire function. Assume that $\mathcal{W}$ is symmetric with respect to the real axis. Then since $\zeta_{\mathcal{L}}(\overline{s}) = \overline{\zeta_{\mathcal{L}}(s)}$, the non-real complex *dimensions of an ordinary fractal string always come in complex conjugate pairs, i.e. if* $\omega \in D_{\mathcal{L}}$*, then* $\overline{\omega} \in D_{\mathcal{L}}$*.*

Theorem 21 *Let* $\mathcal{L}$ *be a fractal string of dimension D, and assume that* $\zeta_{\mathcal{L}}$ *has a meromorphic extension to a neighborhood of D. If* $N_{\mathcal{L}} = O(x^D)$ *as*

$x \to +\infty$ *or if the volume of the tubular neighborhood satisfies*

$$V(\epsilon) = O(\epsilon^{1-D}) \text{ as } \epsilon \to 0^+,$$

then $\zeta_{\mathcal{L}}$ *has a simple pole at D.*

Proof. See Section 1.2 pages 20 and 21 in [La-vF1]. ■

Example 22 *We recall that the Cantor string is denoted by* $\mathcal{L} = CS$ *and that its geometric zeta function is*

$$\zeta_{\mathcal{L}}(s) = \sum_{l} w_l l^s = \sum_{n=0}^{\infty} 2^n 3^{-(n+1)s} = \frac{3^{-s}}{1 - 2.3^{-s}}.$$

By choosing $W = C$*, the complex dimensions of the CS are found by solving the equation* $1 - 2.3^{-\omega} = 0$*, where* $\omega \in C$*. These complex dimensions are of the form* $D + in\boldsymbol{p}$*, where* $n \in Z, D = \log_3 2$ *and* $\boldsymbol{p} = \frac{2\pi}{\log 3}$ *is the CS oscillatory period ; they are all simple poles of* $\zeta_{\mathcal{L}}$*, and we have*

$$D_{CS} = \{ D + in\boldsymbol{p}, \, n \in Z \}.$$

We recall that we expressed $V(\epsilon)$ *as a sum over these complex dimensions, which encode in their real and imaginary parts important information about the oscillations in the geometry of the Cantor string.*

1.5 Fractality in the light of the theory of complex dimensions

The classical definition of fractality, as stated by Mandelbrot [Ma1,p.15], is the following;

> *"A fractal is by definition a set for which the Hausdorff-Besicovitch dimension exceeds the topological dimension".*

In other words, and as quoted in [La-vF1]

> *"If H denotes the Hausdorff dimension of a set $F \subset R^d$ and T denotes its topological dimension, then F is fractal, according to Mandelbrot, if and only if $H > T$ ".*

In view of the expression found for the volume of the tubular neighborhood $V(\epsilon)$, the complex dimensions describe *oscillations in the geometry* of fractal strings; we will show shortly see that they describe oscillations for the spectrum as well. As a consequence, the proposed notion of fractality by M. L. Lapidus and M. v. Frankenhuijsen will only depend on the geometry of a given object. Throughout this work, and as proposed by M. L. Lapidus and M. v. Frankenhuijsen in [La-vF1],

> *"A set will be called fractal if and only if its geometric zeta function has at least a non-real complex dimension with positive real part ".*

Note that, in the light of this new definition of fractality, the Cantor set, viewed as the boundary of the Cantor string, constitutes a fractal set because the complex dimensions of its geometric zeta function are of the form $D + in\mathbf{p}$, where $D = \log_3 2 > 0$, $\mathbf{p} = \frac{2\pi}{\log 3}$ is its oscillatory period, and $n \in Z$.

Next, we will provide an example of an object which is not fractal (see Section 6.5.1 in [La-vF1]).

Example 23 *Let a be a positive real number. We define the a-string to be the complement in (0,1) of the sequence* $\{j^{-a}\}_j$ *where j is a positive integer. As a consequence, the boundary of the a-string is*

$$F = \{1, 2^{-a}, 3^{-a}, 4^{-a}, 5^{-a}, ..., 0\}.$$

It is shown in Section 6.5.1 in [La-vF1] that all the complex dimensions associated to the geometric zeta function of the a-string are real. Hence, and in view of the above new definition of fractality, the a-string is a non-fractal object.

1.6 A reformulation of the Minkowski measurability of the boundary of an ordinary fractal string

We recall that an ordinary fractal string $\mathcal{L}$ is Minkowski measurable if $\lim V(\epsilon)\epsilon^{D-1}$ as $x \to +\infty$ exists and lies in $(0, +\infty)$, and that D, the dimension of an ordinary fractal string, does coincide with its Minkowski dimension.

Next, we present a criterion for the Minkowski measurability of the boundary of a fractal string given in [La-vF1] which completes and extends the earlier criterion obtained by M. L. Lapidus and C. Pomerance in [LaPo1] and [LaPo2].

Theorem 24 *Let $\mathcal{L}$ be an ordinary fractal string that is languid (i.e. $\mathcal{L}$ satisfies some polynomial growth conditions along the screen $\mathcal{S}$ passing between the vertical line $Re(s) = D$ and all the complex dimensions of $\mathcal{L}$ with real part strictly less than D, and not passing through zero, see [La-vF1].)*
Then, the following are equivalent:

1. *D is the only complex dimension with real part D and it is simple.*

2. $N_{\mathcal{L}}(x) = E.x^D + o(x^D)$, *as* $x \to +\infty$, *for some positive constant E.*

3. *The boundary of $\mathcal{L}$ is Minkowski measurable.*

Moreover, if any of these conditions is satisfied, then

$$\mathcal{M} = 2^{1-D}.\frac{E}{1-D} = 2^{1-D}.\frac{res(\zeta_{\mathcal{L}}(s); D)}{D(1-D)},$$

where $\mathcal{M}$ is its Minkowski content, [La-vF1].

Remark 25 *Let $\{l_j\}_{j=1}^{\infty}$ denote the sequence of lengths of the fractal string $\mathcal{L}$, then condition (ii) of the previous theorem is equivalent to the following condition ;*

$$(ii')\, l_j M.j^{\frac{-1}{D}}; \quad \text{that is, } j^{\frac{1}{D}}.l_j \to M \, as \, j \to +\infty$$

for some positive constant $M > 0$, where $E = M^D$ and E is the constant in condition (ii).

The criterion for Minkowski measurability that was obtained by M. L. Lapidus and C. Pomerance in [LaPo1] and [LaPo2] is the following;

"Let $\mathcal{L} = \{l_j\}_j$ be an ordinary fractal string of Minkowski dimension $D \in (0,1)$. Then the boundary of $\mathcal{L}$ is Minkowski measurable if and only if there exists $E > 0$ such that $N_{\mathcal{L}}(x) = E.x^D + o(x^D)$, as $x \to +\infty$ ".

The Minkowski measurability of $\mathcal{L}$ in condition (3) of the previous theorem, means that the leading term of the volume of small tubular neighborhoods does not oscillate. Hence the absence of geometric oscillations of order D in $\mathcal{L}$ is equivalent to the absence of non-real complex dimensions of $\mathcal{L}$ above D. Note that $\mathcal{L}$ could still have oscillations of lower order.

1.7 Frequencies, spectral zeta function of an ordinary fractal string and application to the Cantor string

Given an ordinary fractal string $\mathcal{L}$, we can listen to its sound. For that, we consider the bounded open set $\Omega \subset R$, with the positive Dirichlet Laplacian $\Delta = -\frac{d^2}{dx^2}$ on Ω. An eigenvalue λ of Δ will correspond to the normalized frequency $f = \frac{\sqrt{\lambda}}{\pi}$ of the fractal string. As a consequence, the frequencies of the unit interval are 1,2,3,... each counted with multiplicity one and the frequencies of an interval of length l will be $1l^{-1}, 2l^{-1}, 3l^{-1}, 4l^{-1}, ...$ also counted with multiplicity one. Hence, the frequencies of $\mathcal{L}$ are the numbers $f = k.l_j^{-1}$ where $j, k = 1, 2, 3, ...$ The total multiplicity of the frequency f is equal to

$$w_f^{(\nu)} = \sum_{j:f.l_j \in N^*} 1 = \sum_{l:f.l \in N^*} w_l.$$

To study the frequencies, we introduce the spectral counting function N_ν and spectral zeta function ζ_ν associated to $\mathcal{L}$.

Definition 26 *The* counting function *of the frequencies of $\mathcal{L}$ is defined for $x > 0$ as* $N_\nu(x) = \#\{\, f \leq x : \quad$ *frequency of $\mathcal{L}$, counted with multiplicity* $\}$ $= \sum_{f \leq x} w_f^{(\nu)}$, *where* $w_f^{(\nu)} = \sum_{j: f.l_j \in N} 1 = \sum_{l: f.l \in N} w_l$. *The* spectral zeta function *of $\mathcal{L}$ will be defined as* $\zeta_\nu(s) = \sum_{k.j=1}^{\infty} (k.l_j^{-1})^{-s} = \sum_f w_f^{(\nu)} f^{-s}$, *which converges for Re(s) sufficiently large.*

Next, we present a theorem (see [La3] and [La-vF1], Thm 1.19) relating the spectrum of an ordinary fractal string with its geometry, it will be of importance during our study of the direct spectral problem and in understanding the work of M. L. Lapidus and his collaborator H. Mayer on the inverse spectral problem studied in [LaMa1] and [LaMa2].

Theorem 27 *Let $\mathcal{L}$ be an ordinary fractal string. The spectral counting function of $\mathcal{L}$ is given by*

$$N_\nu(x) = \sum_{k=1}^{\infty} N_{\mathcal{L}}(\frac{x}{k}) = \sum_{j=1}^{\infty} [l_j x]$$

and the spectral zeta function of $\mathcal{L}$ is given by

$$\zeta_\nu(s) = \zeta_{\mathcal{L}}(s)\zeta(s).$$

As a consequence, $\zeta_\nu(s)$ is holomorphic for Re(s) > 1, it has a pole at $s = 1$ with residue L, the total length of $\mathcal{L}$; moreover, it has a meromorphic extension to a neighborhood of *the window $\mathcal{W}$ of $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be an ordinary fractal string and let $x > 0$; then

$$N_\nu(x) = \sum_{k=1}^{\infty} \sum_{j:kl_j^{-1} \leq x} 1 = \sum_{k=1}^{\infty} \#\{\, j : l_j^{-1} \leq \frac{x}{k} \,\} = \sum_{k=1}^{\infty} N_{\mathcal{L}}(\frac{x}{k}),$$

which is a finite sum provided that $N_{\mathcal{L}}(y) = 0,$ for $y < l_1^{-1}$.

We also have

$$N_\nu(x) = \sum_{j=1}^{\infty} \sum_{k \leq l_j x} 1 = \sum_{j=1}^{\infty} [l_j x]$$

for $x > 0.$ Note that for Re(s) large enough, we have

$$\zeta_\nu(s) = \sum_{k,j=1}^{\infty} k^{-s} l_j^s = \sum_{j=1}^{\infty} l_j^s \sum_{k=1}^{\infty} k^{-s} = \zeta_{\mathcal{L}}(s).\zeta(s).$$

■

Remark 28 *The last result on the previous theorem provides a connection between the spectrum and the geometry of an ordinary fractal string; it was first obtained in [La2] and [La3].*

Chapter 2

Generalized Fractal Strings and The Explicit Formulas

As introduced in the work of M. L. Lapidus and M. v. Frankenhuijsen in [La-vF1] throughout their development of the theory of fractal geometry and complex dimensions, given an ordinary fractal string $\mathcal{L} = \{l_j\}_{j=1}$ where each $l_j > 0$ is given according to multiplicity w_{l_j} for $j \geq 1$, one can construct a local complex or positive measure which we will denote by η. Such a measure is the measure associated to $\mathcal{L}$; it is defined in terms of $\mathcal{L}$ as

$$\eta = \sum_{l_j,\, j\geq 1} w_j \delta_{\{l_j^{-1}\}}\,,$$

where $\delta_{\{.\}}$ is the Dirac measure at $\{.\}$. We point out that η does have certain properties that will be studied in details in this chapter and it will represent an example of a generalized fractal string. Note that w_{l_j} is a positive integer for the case of an ordinary fractal string. Following the definition that was proposed by M. L. Lapidus and M. van Frankenhuijsen for η, we are not going to restrict w_j for $j \geq 1$ to be necessarily an integer, hence it can take non-integer values and as a consequence, this will explain to the reader why the world *generalized* would be associated to such a class of fractal strings. Again, and as in the case of ordinary fractal strings, the notion of fractality of such objects will be described in terms of tools of the theory of complex dimensions. Important information about the geometry of η is encoded in its geometric zeta function which is the Mellin transform of η. Spectral information about η is also encoded in its spectral zeta function.

In their theory of complex dimensions, M. L. Lapidus and M. van Frankenhuijsen have developed pointwise and distributional explicit formulas for this class of fractal strings. Throughout this research project, we will be only using the distributional explicit formulas in our framework. The latter ones express the kth distributional primitive of the counting function of η, when the geometric zeta function associated to η statisfies certain polynomial growth conditions, as a sum over its complex dimensions. The original explicit formulas were first developed by Riemann in 1858 as an analytic tool to study and give a better understanding of the distribution of the primes. The distributional explicit formulas obtained by M. L. Lapidus and M. van Frankenhuijsen will play an important role in defining the spectral operator.

Generalized fractal strings are fundamental objects used, for example, to reformulate (as was done in [La-vF1] and [La-vF2]) the study of the direct spectral problem and the inverse spectral problem, which was conducted by M. L. Lapidus and H. Mayer in [LaMa1] and [LaMa2]. Next, we start recalling the definition of this class of fractal strings and also provide the background material needed to study them. (We refer the reader to [La-vF1] for further details.)

2.1 Generalized fractal strings

Definition 29 *A generalized fractal string is a local (complex or positive) measure* η *on* $(0, +\infty)$ *such that* $|\eta|(0, x_0) = 0$ *for some positive* x_0*.*

Here, $|\eta|$ *is the total variation measure associated to* η*, and is defined as*

$$|\eta|(A) = \sup\{\sum_{k=1}^{\infty} |\eta(A_k)|\},$$

where $\{A_k\}_{k=1}^{\infty}$ *ranges over all finite partitions of* A *into disjoint measurable subsets of* $(0, +\infty)$.

Example 30 *Given an ordinary fractal string* $\mathcal{L}$ *consisting of distincts lengths* l_j *each counted with multiplicity* w_j *for* $j \geq 1$. *Then, the measure associated to* $\mathcal{L}$

$$\eta = \sum_{j=1}^{\infty} w_j \delta_{\{l_j^{-1}\}};$$

is a generalized fractal string.

2.2 Counting function, geometric and spectral zeta functions

Definition 31 *The* counting function *of the reciprocal lengths, also called the geometric counting function of* η, *is denoted by* N_η *and is defined for any positive real number* x *as*

$$N_\eta(x) = \int_0^x d\eta = \eta(0, x).$$

Remark 32 *If* η *has pure points, it is necessary to specify how the endpoints are counted. We will adopt the convention that* x *is counted half. That is,*

$$N_\eta(x) = \int_0^x d\eta := \eta(0, x) + \frac{1}{2}\eta(\{x\}).$$

With this convention, the pointwise explicit formula without error term (see [La-vF1]), will be well defined and make sense at the jumps of the counting function.

Definition 33 *Let* η *be a generalized fractal string. Then, its dimension will be denoted by* $D = D_\eta$; *it is the abscissa of convergence of the Dirichlet integral*

$$\zeta_{|\eta|}(\sigma) = \int_0^{\infty} x^{-\sigma} |\eta|(dx)\,,$$

in other words ,

$$D = D_\eta = \inf\{\sigma \in R : \int_0^{+\infty} x^{-\sigma} |\eta|(dx) < \infty \}.$$

Definition 34 *Let η be a generalized fractal string. Then, its* geometric zeta function, *denoted as ζ_η, is the* Mellin transform *of η*

$$\zeta_\eta(s) = \int_0^{+\infty} x^{-s} \eta(dx) \quad \textit{for } Re(s) > D_\eta \,.$$

Remark 35 *We will use the convention that $D_\eta = +\infty$ stands for having $x \longmapsto x^{-\sigma}$ not $|\eta|$ integrable for any σ and that $D_\eta = -\infty$ means that $x \longmapsto x^{-\sigma}$ is $|\eta|$ integrable for all $\sigma \in R$; in this latter case, ζ_η is a holomorphic function, defined by its Dirichlet integral on the whole complex plane.*

Remark 36 *As in the case of the geometric zeta function, $\zeta_{\mathcal{L}}$, of an ordinary fractal string, we will be interested in the meromorphic continuation of ζ_η. For that, we will define the screen $\mathcal{S}$ and window $\mathcal{W}$. We will assume that ζ_η has a meromorphic continuation to an open neighborhood of $\mathcal{W}$, and we will require that ζ_η does not have any poles on $\mathcal{S}$. We will call the set of poles of ζ_η located inside $\mathcal{W}$, the set of visible complex dimensions; such a set will be denoted by $D_\eta(\mathcal{W})$. Hence, $D_\eta = D_\eta(\mathcal{W}) = \{ w \in \mathcal{W} : \zeta_\eta$ has a pole at $w \}$. Since $D_\eta(\mathcal{W})$ is a discrete subset of C, its intersection with any compact subset of C is finite.*

Note also that, since ζ_η is holomorphic for $\Re(s) > D = D_\eta$, then $D_\eta \subset \{ s \in C, Re(s) \leq D \}$. If $\mathcal{W} = C$, then $D_\eta(C)$ will be called the set of complex dimensions of η; in this case, we set $\mathcal{S}(t) = -\infty$ and we will not define the screen.

2.3 The generalized Cantor string and generalized prime string

Next, we provide *the generalized Cantor string* and *the generalized prime string* as examples of generalized fractal strings. We also discuss some of their basic properties. For more details in the discussion below, we refer the reader to Section 4.1.1 in [La-vF1].

The generalized Cantor string is constructed by considering a string consisting of a sequence of lengths a^{-n} with multiplicity b^n, where $1 < b < a$ and $n \in Z_+$. The measure associated to this string is

$$\eta_{CS} = \sum_{n=0}^{\infty} b^n \delta_{\{a^n\}};$$

it is an example of a discrete non-geometric string. Note that if we let b to be integral, then η_{CS} will be the measure associated to an ordinary fractal string since, in this case, such a string will have integers multiplicities.

The geometric zeta function associated to η_{CS} is

$$\zeta_{\eta_{CS}}(s) = \int_0^{+\infty} x^{-s} \eta(dx), \quad \text{for } Re(s) > D_{\eta_{CS}}\,.$$

As a consequence,

$$
\begin{aligned}
\zeta_{\eta_{CS}}(s) &= \int_0^{+\infty} x^{-s} \sum_{n=0}^{\infty} b^n \delta_{\{a^n\}}(dx) \quad (\text{for } Re(s) > D_{\eta_{CS}}) \\
&= \sum_{n=0}^{\infty} b^n \int_0^{+\infty} x^{-s} \delta_{\{a^n\}} \\
&= \sum_{n=0}^{\infty} b^n a^{-ns} \\
&= \sum_{n=0}^{\infty} (ba^{-s})^n \quad (\text{which is a convergent geometric series}) \\
&= \frac{1}{1-ba^{-s}} \quad , \text{where } 1 < b < a\,.
\end{aligned}
$$

The set of complex dimensions of this generalized fractal string is

$$
D_{\eta_{CS}}(\mathcal{W}) = \{\, D + in\mathbf{p},\ where\, D = \log_a b,\ n \in Z\, and\, \mathbf{p} = \frac{2\pi}{\log a}\,\};
$$

it is obtained by solving the equation $1 - ba^{-s} = 0$, where $s \in C$.

Clearly, all these complex dimensions lie on a single vertical line, the line $Re(s) = D = \log_a b$.

The poles of the geometric zeta function associated to this string are simple, with residue $\frac{1}{\log a}$. The geometric counting function is computed as follows. There are $1+b+b^2+b^3+ \ldots + b^n$ lengths less than x satisfiying $n = [\log_a x]$. Thus

$$
N_{\eta_{CS}}(x) = \frac{b^{n+1}-1}{b-1} = \frac{b}{b-1} b^{[\log_a x]} - \frac{1}{b-1}.
$$

Now, using the Fourier series expansion of the periodic function $u \longmapsto b^{-\{u\}}$, we get

$$N_{\eta_{CS}}(x) = \frac{x^D}{\log a}.\sum_{n\in Z}\frac{x^{in\mathbf{p}}}{D+in\mathbf{p}} - \frac{1}{b-1}.$$

This result coincides with the explicit distributional formulas, obtained by M. L. Lapidus and his collaborator M. van Frankenhuijsen in [La-vF1], for the class of generalized fractal strings while applied to the generalized Cantor string.

The prime string is defined as the positive measure

$$B = \sum_{m\geq 1,p} \log p\, \delta_{\{p^m\}},$$

where $p \in \mathcal{P}$:=the set of all prime numbers. Note that B is not a measure associated with any ordinary fractal string since its reciprocal lengths p^m, $m \geq 1$ do have a non-integer multiplicity $\log p$, where $p \in \mathcal{P}$.

Next, we wish to find the geometric zeta function associated to B and then find its complex dimensions; for that, we recall that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}$ has an Euler product expansion, for $Re(s) > 1$, as

$$\zeta(s) = \prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}}.$$

Hence, by a logarithmic differentiation of both sides of the previous equality, we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{m\geq 1,p}(\log p)p^{-ms},$$

and due to the fact that we have

$$
\begin{aligned}
\zeta_B(s) &= \int_0^{+\infty} x^{-s} B(dx) \quad (\text{ for } Re(s) > 1\,) \\
&= \int_0^{+\infty} x^{-s} \sum_{m\geq 1, m} (\log p)\delta_{\{p^m\}}(dx) \\
&= \sum_{m\geq 1, p} (\log p) \int_0^{+\infty} x^{-s}\delta_{\{p^m\}}(dx) \\
&= \sum_{m\geq 1, p} (\log p) p^{-ms} \quad (\text{ for } Re(s) > 1\,),
\end{aligned}
$$

we get

$$
\zeta_B(s) = -\frac{\zeta'(s)}{\zeta(s)}\,,\ s \in C.
$$

As a consequence,

$$
D_B(\mathcal{W}) = \{\, s \in C\,,\ \zeta(s) = 0\,\}.
$$

We recall that such a set consists of the non-trivial zeroes, also called the critical zeros, which are located in the critical strip $0 \leq Re(s) \leq 1$, and the trivial zeroes which are simple, located on the negative real line, and taking even negative integers values $..., -2n, ..., -8, -6, -4, -2.$ We also recall that ζ does not have any zeroes on the vertical line $Re(s) = 1$ as was shown by Hadamard and de la Vallée Poussin. Moreover, Riemann showed how to continue zeta analytically in s and established the functional equation

$$
\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \Lambda(1-s),
$$

where Γ is the gamma function. The symmetry in this equation implies that ζ does not have any zeroes on $Re(s) = 0$. The Riemann hypothesis states that the critical zeroes of ζ are all located on the critical line $Re(s) = \frac{1}{2}$.

2.4 The harmonic string and spectral measure associated to a generalized fractal string

First, we start introducing the multiplicative operation of convolution of measures, since it plays an important role in the definition of the spectral measure associated to a generalized fractal string.

Let η and η' be two complex Borel measures on R^n_+. Then the map

$$f \longmapsto \int_{R^n_+} \int_{R^n_+} f(xy)\eta(dx)\eta(dy).$$

is a bounded linear functional on $C_0(R^n_+)$, the space of all continuous functions on R^n_+ that vanish at infinity. By the Riesz representation theorem for measures, there exists a unique Borel measure $\eta * \eta'$ on R^n_+ satisfying

$$\int_{R^n_+} f(x)\eta * \eta'(.) = \int_{R^n_+} \int_{R^n_+} f(x+y)\eta(dx)\eta(dy)\ ,\ f \in C_0(R^n_+).$$

A standard approximation argument shows that this also holds for every bounded Borel function f, and as a consequence

$$|\eta * \eta'| \leq |\eta|.|\eta'|,$$

where $|.|$ represents the total variation of the measure which we have previously defined. Since generalized fractal strings are positive or complex measures on $(0, +\infty)$, the convolution of measures from now on, and for the rest of this work, will be defined for the case where n=1.

The harmonic string is defined as the positive measure

$$h = \sum_{k=1}^{\infty} \delta_{\{k\}} \,,$$

where

$$\delta_{\{k\}}(x) = \begin{cases} 1\,, & \text{if}\, x = k\,, \\ 0\,, & \text{if}\, x \neq k\,. \end{cases}$$

Note that this generalized fractal string is the measure associated to the ordinary fractal string whose sequence of lengths is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ with multiplicity one. Its geometric zeta function is

$$\begin{aligned} \zeta_h(s) &= \int_0^{+\infty} x^{-s} h(dx) \quad (\text{for } Re(s) > 1\,) \\ &= \int_0^{+\infty} x^{-s} \sum_{k=1}^{\infty} \delta_{\{k\}}(dx) \\ &= \sum_{k=1}^{\infty} \int_0^{+\infty} x^{-s} \delta_{\{k\}}(dx) \\ &= \sum_{k=1}^{\infty} k^{-s} \\ &= \zeta(s) \quad , \text{for } Re(s) > 1\,. \end{aligned}$$

Hence, $\zeta_h(s) = \zeta(s)$, the Riemann zeta function. This is valid for $Re(s) > 1$ but then after analytic continuation, remains true for all $s \in C$.

Next, we define *the prime harmonic string* as the positive measure

$$h_p = \sum_{k=1}^{\infty} \delta_{\{p^k\}} \,,$$

where $p \in \mathcal{P}$. As a consequence,

$$h = \underset{p \in \mathcal{P}}{*} h_p.$$

Note that the geometric zeta function associated to h_p is

$$\begin{aligned}
\zeta_{h_p}(s) &= \int_0^{+\infty} x^{-s} h_p(dx) \quad (\text{for } Re(s) > 1) \\
&= \int_0^{+\infty} \sum_{k=0}^{\infty} \delta_{\{p^k\}}(dx) \\
&= \sum_{k=1}^{\infty} \int_0^{+\infty} x^{-s} \delta_{\{p^k\}}(dx) \\
&= \sum_{k=1}^{\infty} p^{-ks} \quad (\text{a convergent geometric series}) \\
&= \frac{1}{1-p^{-s}} \quad (\text{the } p^{th} \text{ Euler factor of } \zeta(s).)
\end{aligned}$$

Hence, we have for $Re(s) > 1$,

$$\zeta_h(s) = \zeta_{\underset{p \in \mathcal{P}}{*} h_p}(s) = \zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1-p^{-s}} = \prod_{p \in \mathcal{P}} \zeta_{h_p}(s).$$

Motivated by the definition adopted from the case of an ordinary fractal string, the *spectral measure* associated to a generlized fractal string η will be denoted by ν; it is defined as follows

Definition 37 *Let η be a generalized fractal string. Then, the spectral measure associated to η is*

$$\nu(A) = \sum_{k=1}^{\infty} \eta(\frac{A}{k}),$$

for each bounded Borel subset $A \subset (0, +\infty)$. The spectral zeta function *associated to η is defined as the geometric zeta function of ν.*

Remark 38 *Note that the sum defining $\nu(A)$ is finite for any bounded Borel set $A \subset (0, +\infty)$ since A is bounded and $|\eta|$ is assumed to have no mass near 0. In fact, we can choose k large enough so that $k^{-1}A \subset (0, x_0)$, where $|\eta|(0, x_0) = 0$, then $\eta(k^{-1}A) = 0$.*

Lemma 39 *Let η and η' be two generalized fractal strings. Then,*

$$\zeta_{\eta*\eta'}(s) = \zeta_\eta(s)\zeta_{\eta'}(s),$$

for $Re(s) > \max\{D_\eta, D_{\eta'}\}$.

Proof. Let η and η' be two generalized fractal strings whose dimensions are respectively D_η and $D_{\eta'}$, then

$$\begin{aligned}
\zeta_{\eta*\eta'}(s) &= \int_0^{+\infty} x^{-s}\eta * \eta'(dx) \qquad (\text{for } Re(s) > \max\{D_\eta, D_{\eta'}\}) \\
&= \int_0^{+\infty}\int_0^{+\infty} (xy)^{-s}\eta(dx)\eta'(dy) \\
&= \int_0^{+\infty} x^{-s}\eta(dx)\int_0^{+\infty} y^{-s}\eta'(dy) \\
&= \zeta_\eta(s)\zeta_{\eta'}(s).
\end{aligned}$$

■ Next, we show that the spectral measure associated to a given generalized fractal string is related to the harmonic string via convolution of measures.

Lemma 40 *Let η be a generalized fractal string. Then, the spectral measure associated to η is the convolution of h, the harmonic string associated to η, with η. That is*

$$\nu = \eta * h.$$

Proof. Let η be a generalized fractal string whose dimension is D_η, and let $A \subset (0, +\infty)$ be a bounded Borel subset. Then

$$\begin{aligned}
\eta * h(A) &= (\eta * \sum_{k=1}^{\infty} \delta_{\{k\}})(A) \\
&= \sum_{k=1}^{\infty} \eta * \delta_{\{k\}}(A) \\
&= \sum_{k=1}^{\infty} \eta * \delta_{\{k\}}(\frac{kA}{k}) \\
&= \sum_{k=1}^{\infty} \eta(\frac{A}{k}) \\
&= \nu(A).
\end{aligned}$$

■

Remark 41 *When $\eta = \sum_{j=1}^{\infty} \delta_{\{l_j^{-1}\}}$ is the measure associated with an ordinary fractal string $\mathcal{L} = l_1, l_2, l_3, ...$, then its spectral zeta function ζ_ν coincides with the spectral zeta function of $\mathcal{L}$. Indeed we have*

$$\zeta_\nu(s) = \int_0^\infty x^{-s}\nu(dx) = \zeta_\eta(s).\zeta(s),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Also, N_ν coincides with the spectral counting function of $\mathcal{L}$. Next, we will show that the spectral measure associated to a generalized fractal string encodes information about its frequencies. In fact, the spectral measure ν of an ordinary fractal string $\mathcal{L} = \{l_j\}_{j=1}^{\infty}$ with multiplicities w_{l_j}, viewed as the measure $\eta = \sum_{j=1}^{\infty} w_{l_j}\delta_{\{l_j^{-1}\}}$, is given by

$$\nu = \sum_{f} w_f^{(\nu)} \delta_f$$
$$= \eta * h$$
$$= \sum_{n,j=1} w_{n,l_j} \delta_{\{n.l_j^{-1}\}} \, ,$$

where, f runs over the (distinct) frequencies of $\mathcal{L}$ and $w_f^{(\nu)}$ denotes the multiplicity of f.

Example 42 *We will use the result derived on Remark 41 to compute the frequencies of the prime string B. We will first evaluate the spectral zeta function of B. We have*

$$\zeta_{\nu,B}(s) = \zeta_B(s).\zeta(s) = -\frac{\zeta'(s)}{\zeta(s)}.\zeta(s) = -\zeta'(s) \, ,$$

where $\zeta'(s)$ is the derivative of the Riemann zeta function. Since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $Re(s) > 1$, we obtain (for $Re(s) > 1$) the following expression for the spectral zeta function,

$$\zeta_\nu(s) = -\zeta'(s) = \sum_{n=1}^{\infty} (\log n) n^{-s}.$$

As a result, we deduce that

$$\nu = \sum_{n=1}^{\infty} (\log n) \delta_{\{n\}}.$$

Thus, the frequencies of the prime string B are all the positive integers $1, 2, ..., n, ...$ and the frequency n has a non-integer multiplicity $\log n$. Note that this result is in contrast with the case of the class of ordinary fractal strings whose lengths and frequencies always have integer multiplicities. More details on the discussion we just provided are given in Example 4.8 of Section 4.2 in [La-vF1].

Remark 43 *We could have recovered the result we just obtained by determining directly* *the spectral measure ν associated with B. We recall that we always have*

$$\delta_{\{x\}} * \delta_{\{y\}} = \delta_{\{xy\}} \,.$$

Hence, using the fact that

$$\nu = \eta * h \,,$$

we get

$$\begin{aligned}
\nu &= \sum_{m\geq 1, p} (\log p)\delta_{\{p^m\}} * \sum_{k\geq 1} \delta_k \\
&= \sum_{m,k\geq 1, p} (\log p)\delta_{\{p^m . k\}} \\
&= \sum_{n\geq 1} \delta_{\{n\}} \sum_{m\geq 1, p: p^m|n} \log p \,,
\end{aligned}$$

where p runs over all the primes. Since we have $\sum_{m\geq 1, p^m|n} \log p = \log p^m$, so that $\sum_{m\geq 1,\, p:p^m|n} \log p \; = \; \log n$ by the unique factorization of integers into prime powers, we conclude that

$$\nu = \sum_{n=1}^{\infty} (\log n)\delta_{\{n\}} \,,$$

which is coinciding with the result previously obtained.

2.5 The distributional explicit formulas for Generalized Fractal Strings

Throughout their development of the theory of fractal geometry, number theory and complex dimensions (see [La-vF1], [La-vF2] and [La-vF3]), M. L. Lapidus and M. van Frankenhuijsen obtained pointwise and distributional explicit formulas for the lengths and frequencies of fractal strings. In this section, we will omit giving pointwise explicit formulas for generalized fractal strings, in fact we refer the reader to [La-vF1] and [La-vF2] for more details. We will restrict our exposition to presenting the distributional explicit formulas which will be used while defining the spectral operator. For that we make a few preliminary comments.

The original explicit formula was given by Riemann in 1858 as an analytic tool to understand the distribution of the primes. It was later extended by by Von Mangoldt and led in 1896 to the first rigorous proof of the Prime Number Theorem, independently by Hadamard and de la Vallée Poussin. If we let $f(x) = \sum_{p^n \leq x} \frac{1}{n}$ be the function that counts prime powers $p^n \leq x$ with a weight $\frac{1}{n}$, the explicit formula for Riemann is

$$f(x) = Li(x) - \sum_{z} Li(x^z) + \int_x^\infty \frac{1}{t^2-1} \frac{dt}{t \log t} - \log 2\,,$$

where the sum is taken over all zeroes z of the Riemann zeta function, taken in order of increasing absolute value, and $Li(x)$ is the logarithmic integral $\int_0^x \frac{dt}{\log t}$.

The distributional explicit formulas of [La-vF1, La-vF2] express the counting function of the lengths or of the frequencies as a sum over the visible complex dimensions w of the fractal string. Moreover, they will usually contain an error term, which will be in general given by an integral over the screen $\mathcal{S}$.

The distributional formula at a given level $k \in Z$ allows to express η, in the distributional sense, as a sum over its complex dimensions. The space on which these distributions will act is Schwartz space. The distribution η acts by

$$< \eta, \varphi >= \int_0^\infty \varphi(x)\eta(dx).$$

The k-th *primitive* of this distribution will be denoted by $\mathcal{P}_\eta^{[k]}$. More specifically, $\mathcal{P}_\eta^{[k]}$ is the distribution given for all test functions ϕ by the formula

$$< \mathcal{P}_\eta^{[k]}, \varphi >= (-1)^k < \eta, \mathcal{P}_\eta^{[k]}\varphi > ,$$

where $\mathcal{P}_\eta^{[k]}\varphi$ is the $k-th$ primitive of φ that vanishes at infinity together with its derivatives. Hence, for a test function φ, we have

$$< \mathcal{P}_\eta^{[k]}, \varphi >= \int_0^\infty \int_0^\infty \frac{(x-y)^{k-1}}{(k-1)!}\varphi(x)dx\, \eta(dy),$$

where $\mathcal{P}_\eta^{[0]} = \eta$. We recall from the general theory of distributions in the sense of Laurent Schwartz that any locally integrable function f on $(0, +\infty)$

defines a distribution in the obvious manner. More precisely, for any measurable function on $(0, +\infty)$ such that $\int_a^b |f(x)|dx$ is finite and for every $[a, b] \subset (0, \infty)$, we have

$$< f, \varphi >= \int_0^\infty f(x)\varphi(x)dx\,,$$

where ϕ is a test function with compact support contained in $(0, \infty)$. This applies in particular, for each fixed $k \geq 1$, to the k-th integrated counting function, $f(x) = N_\eta^{[k]}(x)$, associated with an arbitrary generalized fractal string. The k-th primitive $\mathcal{P}_\eta^{[k]}$ of η can be extended for $k \leq 0$ by differentiating $|k| + 1$ times the distribution $\mathcal{P}_\eta^{[1]}$. Thus in particular, $\mathcal{P}_\eta^{[0]} = \eta$.

In the following,

$$(s)_k := \frac{\gamma(s+k)}{\gamma(s)},$$

for $k \in Z$. This will extend the definition of Poachhammer symbol for $k \leq 0$. As a consequence, $(s)_0 = 1$ and for $k \geq 1$, $(s)_k = s(s+1)(s+2)...(s+k-1)$. We will denote by

$\tilde{\varphi}$ The Mellin transform of a test function ϕ on $(0, +\infty)$

$$\tilde{\varphi}(s) = \int_0^\infty \phi(x) x^{s-1} dx.$$

We denote by $res(g(s); \omega)$ the residue of a meromorphic function $g(s)$ at ω.

It vanishes unless ω is a pole of g. Since $res(g(s), \omega)$ is linear in g, we have that $\int_0^\infty \phi(x) res(x^{s+k-1} g(s); \omega) dx = res(\tilde{\varphi}(s+k) g(s); \omega)$, for any test function ϕ we will be considering throughout this work.

Remark 44 *A generalized fractal string η is called* languid *if its geometric zeta function satisfies a suitable polynomial growth condition along the vertical direction of the screen and some polynomial growth conditions along a sequence of horizontal lines necessarily avoiding the poles of ζ_η; see* [La-vF3,§5.3] *for the precise definition.*

Theorem 45 (See [La-vF3, Ch.5]). *Let η be a languid generalized fractal string. Then, for any $k \in Z$, its kth distributional primitive is given by*

$$\mathcal{P}_\eta^{[k]}(x) = \sum_{\omega \in D_\eta(\mathcal{W})} res(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega) + \frac{1}{(k-1)!} \sum_{j=0}^{k-1} C_j^{k-1} (-1)^j x^{k-1-j}$$

$$.\zeta_\eta(-j) + \mathcal{R}_\eta^{[k]}(x), \tag{2.1}$$

where $\mathcal{R}_\eta^{[k]}(x) = \frac{1}{2\pi i} \int_{\mathcal{S}} x^{s+k-1} \zeta_\eta(s) \frac{ds}{(s)_k}$ is the error term and can be suitably estimated as $x \mapsto +\infty$.[1] *In addition, if η is strongly languid, then we may choose $\mathcal{W} = C$ and $\mathcal{R}_\eta(x) \equiv 0$.*

The action of $\mathcal{P}_\eta^{[k]}$ on a test function φ is given by

[1]Here, $(s)_k := \frac{\Gamma(s+k)}{\Gamma(s)}$, where Γ is the usual Gamma function, $k \in Z$ and $Re(s) > D_\eta$. This extends the usual definition of the Pochhammer symbol to $k \leq 0$. We also recall that $C_j^{k-1} := \frac{(k-1)!}{j!(k-j-1)!}$, where $0 \leq j \leq k-1$.

$$< \mathcal{P}_\eta^{[k]}, \varphi >= \sum_{\omega \in D_\eta(\mathcal{W})} res(\frac{\zeta_\eta(s)\widetilde{\varphi}(s+k)}{(s)_k}; \omega) + \frac{1}{(k-1)!} \sum_{j=0}^{k-1} C_j^{k-1}(-1)^j \zeta_\eta(-j)$$

$$.\widetilde{\varphi}(k-j) + \mathcal{R}_\eta^{[k]}(x). \tag{2.2}$$

When we apply the distributional explicit formulas at level $k = 0$, assuming that η is a languid generalized fractal string whose complex dimensions are simple and satisfies certain mild additional conditions, we obtain that, as a distribution, the measure η is given by the following *density of geometric states* (or *density of lengths*) formula (see [La-vF3,§6.3.1]):

$$\eta = \sum_{\omega \in \mathcal{D}_\eta(\mathcal{W})} res(\zeta_\eta(s); \omega) x^{\omega - 1}.$$

We also obtain for the spectral measure, by applying the previous theorem to $\nu = \mathcal{P}_\nu^{[0]}$, the following *density of spectral states* (or *density of frequencies*) formula:

$$\nu = \zeta_\eta(1) + \sum_{\omega \in \mathcal{D}_\eta(\mathcal{W})} res(\zeta_\eta(s)\zeta(s)x^{s-1}; \omega) x^{w-1}.$$

Note that if $\omega \in D_\eta(\mathcal{W}) - \{1\}$ is a simple pole of ζ_η, then

$$res(\zeta_\eta(s)\zeta(s)x^{s-1}; \omega) = \zeta(\omega) res(\zeta_\eta(s); \omega) x^{\omega - 1}.$$

Chapter 3

The Spectral Operator a

The spectral operator (henceforth denoted by a) was introduced by M. L. Lapidus and M. van Frankenhuijsen in their development of the theory of complex fractal dimensions; see [La-vF3,§6.3.2]. It maps the density of geometric states to the density of spectral states. It therefore plays an important role in relating the spectrum of a generalized fractal string to its geometry; see [La-vF3,§6.3.1].

The discussion of this topic provided in [La-vF3] is, in some sense, of a poetic and non-rigorous nature. In the present work (see [HerLa1]), we will provide a rigorous functional analytic framework within which to define this spectral operator. We will also establish new properties of a and answer several open questions partaining to the spectrum and the invertibility of a. For now, we begin by motivating the definition of a in a more informal and intuitive manner, in the spirit of [La-vF3,§6.3.2].

3.1 The multiplicative and additive spectral operators.

Using the distributional explicit formula as a motivation, the spectral operator will be defined at level $k = 0$ as the map

$$\eta \longmapsto \nu \tag{3.1}$$

(see Definition 37 and the end of Section §2.5) and at level $k = 1$ (see Definition 3) as the map

$$N_\eta(x) \longmapsto \nu(N_\eta)(x) = N_\nu(x) = \sum_{k=1}^{\infty} N(\frac{x}{k}). \tag{3.2}$$

Here, η ranges through the class of *continuous* generalized fractal strings.(By "continuous", we mean that η, restricted to any Borel bounded subset of $(0, +\infty)$, is a continuous measure; namely, $|\eta|(\{x\}) = 0$, for every $x > 0$.) Note that the previous sum has finitely many nonzero terms, since $N(x) = 0$ for all x close to zero, a condition naturally satisfied by the class of continuous generalized fractal strings.

For every prime number p, we also define the p-factor of ν by

$$N_\eta(x) \longmapsto \nu_p(N_\eta)(x) = N_{\nu_p}(x) = N_{\eta * h_p}(x) = \sum_{k=0}^{\infty} N(xp^{-k}), \tag{3.3}$$

where the terms in the sum necessarily vanish when $p^k \geq x$. The operators ν_p commute with each other and their composition gives the Euler product for ν :

$$N_\eta(x) \longmapsto \nu(N_\eta)(x) = (\prod_{p \in \mathcal{P}} \nu_p)(N_\eta). \tag{3.4}$$

Making the change of variable $x = e^t$ $(x > 0)$ or equivalently, $t = \log x$ (and hence, $t \in R$), and writing $f(t) = N(x)$, we obtain the additive version of the spectral operator

$$f(t) \mapsto a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k), \tag{3.5}$$

and of its operator-valued Euler factors

$$f(t) \mapsto a_p(f)(t) = \sum_{k=0}^{\infty} f(t - k \log p). \tag{3.6}$$

From now on, we will only work with the additive (rather than with the multiplicative) version of these operators. Note that these last two sums have finitely many nonzero terms for each $t \in R$ provided that f is supported on a right half-line $[b, +\infty)$, for some positive real number b.

Remark 46 *We will assume that* b *is independent of* f *and that, in fact,* $b = 0$. *Hence, in the additive t-variable, all the functions* $f = f(t)$ *on* $R = (-\infty, +\infty)$ *are supported on* $[0, +\infty)$. *Of course, this means that, in the original multiplicative x-variable, "the geometric counting function"* $N_\eta = N_\eta(x)$ *are assumed to be supported on* $[1, +\infty)$. *This normalizing hypothesis amounts to assuming that all the continuous generalized fractal strings considered here are supported on* $[1, +\infty)$*; i.e.,* $x_0 = 1$ *in Remark 32. For later development, a crucial issue to be addressed in Section 4.2 is the appropriate asymptotic behavior of* $f(t)$ *as* $t \to +\infty$ *(or equivalently, of* $N_\eta(x)$ *as* $x \to +\infty$*).*

3.2 The spectral operator and its Euler product

The spectral operator a and its Euler factors a_p are also related by an operator-valued Euler product

$$f(t) \longmapsto a(f)(t) = (\prod_{p \in \mathcal{P}} a_p)(f)(t),$$

where the product is given in the sense of the composition of operator.

If we denote by $\partial := \frac{d}{dt}$ the first order differential operator with respect to t, the Taylor series associated to f, a smooth function, can be written as

$$\begin{aligned} f(t+h) &= f(t) + \frac{hf'(t)}{1!} + \frac{h^2 f''(t)}{2!} + \ldots \\ &= e^{h\frac{d}{dt}}(f)(t) = e^{h\partial}(f)(t); \end{aligned}$$

that is, $\partial = \frac{d}{dt}$ is the infinitesimal generator of the (one-parameter) group of shifts on the real line. Note that this gives a new representation for the spectral operator and its prime factors:

$$\begin{aligned} a(f)(t) &= \sum_{k=1}^{\infty} e^{-(\log k)\partial}(f)(t) = \sum_{k=1}^{\infty} (\frac{1}{k^{\partial}})(f)(t) \\ &= \zeta(\partial)(f)(t) = \prod_{p \in \mathcal{P}} (1 - p^{-\partial})^{-1}(f)(t) \end{aligned}$$

and for any prime p,

$$\begin{aligned} a_p(f)(t) &= \sum_{k=0}^{\infty} f(t - k\log p) = \sum_{k=0}^{\infty} e^{-k(\log p)\partial}(f)(t) = \sum_{k=0}^{\infty} (\frac{1}{p^k})\partial(f)(t) \\ &= (\frac{1}{1-p^{-\partial}})(f)(t) = (1-p^{-\partial})^{-1}(f)(t) = \zeta_{h_p}(\partial)(t). \end{aligned}$$

These representations of a and a_p were given without proof in [La-vF3, §6.3.2] and will be rigorously justified in this work and in [HerLa1] and [HerLa2]. More precisely, in the present work, we will specify boundary conditions that make ∂ a (possibly unbounded) normal operator and call A the resulting operator. Furthermore, we will show that $a = \zeta(\partial)$. (i.e, $a = \zeta(A)$). Moreover, in [HerLa2], we will prove the convergence of the operator-valued Euler product for a (viewed as a possibly unbounded operator) and study each of the operator-valued prime factors a_p. It will follow from the work in this paper that the spectral operator is unbounded for any value of $c \geq 0$, where c is the parameter indexing the underlying Hilbert space H_c.

Chapter 4

The Differentiation Operator ∂_c

We begin by defining (in terms of a suitable weight) the Hilbert space H_c on which both the differentiation operator $A = \partial_c$ and the spectral operator a will act. This Hilbert space is dependent on a parameter $c \geq 0$. We will also define the domain of the first order differential operator ∂_c which we will denote by $D(\partial_c)$ or also sometimes by $D(A)$. Moreover, we will show that ∂_c is an unbounded normal linear operator, while defined on $D(\partial_c)$, whose adjoint is given by $\partial_c^* = 2c - \partial_c$ and that $D(\partial_c) = D(\partial_c^*)$ (see [HerLa1]).

4.1 The weighted Hilbert space H_c.

For $c \geq 0$, let

$$\mathcal{H}_c := \{f \in C^{\infty}(R) : supp(f) \subset (0, +\infty) \, and \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt < +\infty\}, \tag{4.1}$$

where all the functions f involved are complex-valued and $supp(f)$ denotes the support of f (so that $f \in \mathcal{H}_c$ implies that $f(t) = 0$ for all $t \leq 0$). Then, $\mathcal{H}_c$ is a pre-Hilbert space for the natural inner product indicated below. Its completion is a Hilbert space and is denoted by H_c. It is equipped with the following inner product

$$< f, g >_c = \int_0^\infty f(t)\overline{g(t)} e^{-2ct} dt$$

and the associated Hilbert norm $||.||_c = \sqrt{< . , . >_c}$ (so that $||f||_c^2 = \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt$). We usually write $< ., . >_c$ or $||.||_c$ in order to stress the dependence on the parameter c. More specifically, (see Lemma 48 below), let $supp(f)$ denote the support of a Borel measurable function f with respect to the standard Lebesgue measure on R and consider the absolutely continuous and positive (Borel) measure $\mu_c(dt) := e^{-2ct}dt$. Then,

$$H_c = \{f \in L^2(R, \mu_c(dt))| : supp_{\mu_c}(f) \subset [0, +\infty)\}. \tag{4.2}$$

In other words H_c is the space of all square-integrable functions on R with respect to the measure μ_c, which vanish Lebesgue almost everywhere (a.e.) on $(-\infty, 0]$; see Lemma 48 and Remark 47. (Note for later use that if $f \in H_c$ is continuous, then it must vanish everywhere on $(-\infty, 0]$; in particular, $f(0) = 0$.)

Remark 47 *It is useful to observe that, since $supp(f) \subset [0, +\infty)$ for $f \in H_c$, we may replace $[0, +\infty)$ by R in the integral defining the inner product $< ., . >_c$ or the norm $||.||_c$ on H_c. In particular, we have $||f||_c^2 = \int_R |f(t)|^2 e^{-2ct} dt$, for $f \in H_c$.*

The following lemma is key to our above discussion of the Hilbert space H_c (as will be seen in the proof of Proposition 50 and of Remark 49 and will enable us to simplify many arguments in the remainder of the paper.

Lemma 48 *The measure μ_c and ordinary Lebesgue measure Leb have exactly the same sets of measure zero. Namely, for any Borel set $N \subset R$, $\mu_c(N) = 0$ if and only if $Leb(N) = 0$. (See Remark 49 below.)*

Proof. *Step1:* Suppose first that $N \subset [0, +\infty)$ and is bounded. Then, since $c \geq 0$, we have $e^{-2c\beta} \leq e^{-2ct} \leq 1$, for all $t \in N$ (where $\beta := \sup N < \infty$). Hence,

$$e^{-2c\beta} Leb(N) \leq \mu_c = \int_N e^{-2ct} dt \leq Leb(N),$$

from which it follows that $\mu_c(N) = 0$ if and only if $Leb(N) = 0$.

Step2: Next, if $N \subset [0, +\infty)$ is unbounded, we write N as a countable disjoint union: $N = \bigcup_{n=1}^{\infty} N_n$, where $N_n := N \cap [n-1, n)$ for each $n \geq 1$, and since $\mu_c(N) = \sum_{n=1}^{\infty} \mu_c(N_n)$ and $Leb(N) = \sum_{n=1}^{\infty} Leb(N)$ we deduce the same conclusion in this case as well. Note that when $N \subset (-\infty, 0]$, exactly the same reasoning (with $\alpha := \inf N$ instead of $\beta := \sup N$, when N is bounded) allows one to conclude the desired result. Indeed, if $\mu_c(N) = 0$, then $\mu_c(N_n) = 0$ for all $n \geq 1$, and hence, by Step 1, $Leb(N_n) = 0$ for all $n \geq 1$, from which it follows that $Leb(N) = 0$. [The converse is deduced from Step 1, since in light of the inequality $e^{-2ct} \leq 1$ for $t \geq 0$ we have $\mu_c(N) \leq Leb(N)$ (even when N is bounded) and hence $Leb(N) = 0$ implies that $\mu_c(N) = 0$.]

Step3: Finally, if $N \subset R$ is an arbitrary Borel set, we write $N = N^+ \cup N^-$ (disjoint union) with $N^+ \subset [0, +\infty)$ and $N^- \subset (-\infty, 0)$, and use the above results to complete the proof of the lemma. ∎

Remark 49 (Equality almost everywhere.) *It follows from Lemma 48 that given any two measurable functions f,g on R (or some subinterval of R), f=g* μ_c*–a.e. if and only if f=g Leb a.e. As a result, most of the time, we will simply write f=g a.e. in order to denote the equality of f and g almost everywhere (indifferently, with respect to* μ_c *or Leb). The same convention will be used for sequences of measurable pointwise converging* *functions almost everywhere (a.e.), except in the proof of Proposition 50.*

Proposition 50 *The space $\mathcal{H}_c$ (as defined by the right-hand side of equation 4.1 is a closed subspace of H_c. Moreover, $\mathcal{H}_c$ is dense in H_c. Therefore, H_c is a complex Hilbert space and is the completion of $\mathcal{H}_c$.*

Proof. First, to see that H_c is a closed subspace of $L^2(R, \mu_c)$, we use the fact that every limit f of a $||.||_c$-convergent sequence is the μ_c-a.e. limit of a convergent subsequence $\{f_j\}_{j=1}^{\infty}$. Hence, if $f_j = 0$, μ_c-a.e. on $(-\infty, 0]$, it follows that $f = 0$, $\mu_c - a.e.$ on $(-\infty, 0]$ (and thus, by Lemma 48, Leb a.e.) $f \in H_c$.)

Next, let $C_0^{\infty}(0, +\infty)$ denote the space of all C^{∞} functions with compact support contained in $(0, +\infty)$. Then, $C^{\infty}(0, +\infty)$ is dense in $L^2((0, +\infty),$

$\mu_c)$. Furthermore, it can be viewed as (continuously) embedded in $\mathcal{H}_c$ and hence, in H_c by simply identically extending every function $f \in C^{\infty}(0, +\infty)$ by 0 on $(-\infty, 0]$. (Note that clearly, $\mathcal{H}_c \subset H_c$). ■

Remark 51 *In the above proof of Proposition 50, we have distinguished, for clarity between convergence pointwise μ_c–almost everywhere and Leb–almost everywhere. However, due to Lemma 48, these two notions are equivalent. This is the reason why, in the definition of H_c given previously, we did not have to distinguish between the support of a function relative to Lebesgue measure or to the measure μ_c.*

4.2 The domain of the differentiation operator ∂_c

We now precisely define the differentiation operator A. This will enable us later on to study the spectral operator a. *Given $c \geq 0$, we define $A := \partial = \partial_c = \frac{d}{dt}$ as the unbounded linear operator from H_c to itself with domain $D(A)$ consisting of all the functions $f \in H_c$ that are (locally) absolutely continuous on R* (i.e., $f \in C_{abs}(R)$),[1] *and such that $f' \in H_c$ (where f' denotes the pointwise derivative of f, which exists Lebesgue almost everywhere on R). Furthermore, for $f \in D(A)$, we let $Af = \partial f := f'$, for all $f \in D(A)$.*

Remark 52 *Note that if $f \in D(A)$, then since $f \in C_{abs}(R) \cap H_c$, we have $f(t) = 0$ for all $t \leq 0$, and hence also $f'(t) = 0$ for all $t \leq 0$. Therefore, to require that f' (the almost everywhere derivative of f) belongs to H_c (when $f \in D(A)$) merely amounts to requiring that $\int_0^{+\infty} |f'(t)|^2 e^{-2ct} dt < +\infty$.*

It follows from the definition of $H_c, D(A)$ and a well-known lemma (about absolutely continuous functions) that every $f \in D(A)$, which is clearly a Borel measurable function, naturally satisfies the following boundary conditions at $-\infty$ and $+\infty$, respectively:

(1) (Boundary condition at -∞) $f(t) = 0$ for all $t \leq 0$; in particular, we have $f(0) = 0$.

[1] Here and thereafter, given an interval $J \subset R$, we say that g is (locally) absolutely continuous on J (and write $g \in C_{abs}(J)$) if the restriction of g to any compact subinterval of J is absolutely continuous (see [Ru1, Fo].)

(2)(Boundary condition at $+\infty$) $\lim_{t\to+\infty} f(t)e^{-tc} = 0$.

Indeed, the second boundary condition follows from the fact (applied to $g(t) := f(t)e^{-ct}$) according to which g is a Lebesgue square-integrable and locally absolutely continuous function on $[0, +\infty)$, then $\lim_{t\to+\infty} g(t) = 0$.(See Appendix A for a detailed proof.)

Proposition 53 *For any $c \geq 0$, $D(A)$ is dense in H_c.*

Proof. This follows immediately from Lemma 48. Indeed, it clearly follows from the definition of $D(A)$ and of $\mathcal{H}_c$ that $\mathcal{H}_c \subset D(A)$. (Note that a C^1 function is absolutely continuous since it satisfies the fundamental theorem of calculus for Riemann integrable functions). Furthermore, by Proposition 50, $\mathcal{H}_c$ is dense in H_c. Hence, $D(A)$ is also dense in H_c, as required. ■

Remark 54 *Note that H_c is a separable, infinite dimensional Hilbert space. Indeed, $H_c \subset L^2(R, \mu_c)$ and $C_0^\infty(0, +\infty) \subset H_c$.*

4.3 Normality of the unbounded operator ∂_c

Clearly, $A = \partial_c$ as defined in Section 4.2, is an unbounded linear operator (from H_c to itself). We will prove next that A is in fact normal, which means that it commutes with its adjoint. (See [Ru2] for the theory of unbounded normal operators, along with [Kat] and [Jola] for its applications.)

Recall that A is densily defined (by Proposition 53), the (Hilbert) adjoint of A, denoted by A^*, exists (as an unbounded linear operator from the complex Hilbert space H_c to itself). (This follows from the Riesz's representation theorem.) Furthermore, its domain $D(A^*)$ consists of the set of functions $g \in H_c$ for which there is a $\psi \in H_c$ (necessarily unique, by the density of D(A)) such that

$$< Af, g >=< f, \psi >, \quad \text{for all } f \in D(A).$$

Then, by definition, $A^*g = \psi$. Hence,

$$< Af, g >=< f, A^*g >, \quad \text{for all } f \in D(A), g \in D(A^*)$$

Definition 55 *The (densily defined, unbounded) operator A is said to be* normal *if $AA^* = A^*A$ and, in particular,* self-adjoint *if $A^* = A$ (which implies that $D(A^*) = D(A)$).*

Remark 56 *In the present case, when $A = \partial_c$ it will follow from Theorem 63 that A is* skew-adjoint *(i.e., $A^* = -A$) or equivalently, that $\frac{A}{i}$ is self-adjoint if and only if $c = 0$. That is $i^{-1}A$ is self-adjoint, where $i := \sqrt{-1}$ if and only if $c = 0$.*

Definition 57 *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear transformation. The graph of A is the set of pairs $\{< \phi, A\phi > \mid \phi \in D(A)\}$. The graph of A, denoted by $G(A)$, is a subset of $\mathcal{H} \times \mathcal{H}$ which is a Hilbert space with innner product $(< \phi_1, \psi_1 >, < \phi_2, \psi_2 >) = (\phi_1, \phi_2) + (\psi_1, \psi_2)$. A is called a* closed *operator if $G(A)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$.*

Definition 58 *Let B and A be operators on $\mathcal{H}$. If $G(A) \subset G(B)$, then B is said to be an* extension *of A and we write $A \subset B$. Equivalently, $A \subset B$ if and only if $D(A) \subset D(B)$ and if $A\phi = B\phi$ for all $\phi \in D(A)$.*

Definition 59 *An operator A is* closable *if it has a closed extension. Every closable operator has a smallest closed extension, called its* closure*, which we denote by $\overline{A}$.*

Proposition 60 *If $A : \mathcal{H} \to \mathcal{H}$ is a closable operator, then $G(\overline{A}) = \overline{G(A)}$.*

Theorem 61 *Let A be a densily defined operator on a Hilbert space $\mathcal{H}$. Then:*

1. *A^* is closed.*

2. *A is closable if and only if $D(A^*)$ is dense in which case $\overline{A} = A^{**}$.*

3. *If A is closable, then $(\bar{(A)})^* = A^*$.*

Proof. The proof of this theorem is omitted, we refer the reader to [ReSi1] for a rigorous proof. ■

Definition 62 *If A is a linear operator whose domain is $D(A)$, a subset $D \subset D(A)$ is called a* core *for A if $\overline{A/D} = A$.*

Theorem 63 *$A = \partial_c$ is an unbounded normal linear operator on H_c. Moreover, its adjoint A^* is given by $A^* = 2c - A$, with $D(A^*) = D(A)$.*

Proof. To prove the normality of A, we must show that A is closed densely defined and that $AA^* = A^*A$. By Proposition 53, A is an unbounded linear operator whose domain is dense in H_c. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $D(A)$ such that $\lim\limits_{n\to\infty} f_n = f$ and $\lim\limits_{n\to\infty} Af_n = g$. We want to show that $f \in D(A)$ and that $Af = g$. (This will imply that A is a closed operator).

Next, we will show that f also satisfies the boundary condition at the origin. Note that if $\phi_n \to 0$ in H_c, as $n \to \infty$, then $\int_0^t \phi_n(t)dt \to 0$, uniformly in t on compact subsets of $(0, +\infty)$.

Indeed, for any $t \in I$, a compact subinterval of $(0, +\infty)$, we have

$$\left|\int_0^t \phi_n(t)dt\right|^2 \leq \left(\int_0^t \left(|\phi_n(\tau)|e^{-c\tau}\right) e^{c\tau} d\tau\right)^2$$
$$\leq \left(\int_0^t |\phi_n(\tau)|^2 e^{-2c\tau} d\tau\right)\left(\int_0^t e^{2c\tau} d\tau\right)$$
$$\leq M||\phi_n||_c \to 0 \quad \text{as } n \to \infty.$$

(Here, the positive constant M is finite and depends only on I.) Consequently,

If $\phi_n \to \phi$ in H_c, then $\int_0^t \phi_n(\tau)d\tau \to \int_0^t \phi(\tau)d\tau$ uniformly on any compact subset of $(0, +\infty)$. Since $Af_n \to g$ in H_c and $Af_n = f_n'$, then $\int_0^t f_n'(\tau)d\tau \to \int_0^t g(\tau)d\tau$.

Now, since $f_n \in C_{abs}(R)$ (the set of absolutely continuous functions on R), then $f_n(t) = f_n(0) + \int_0^t f_n'(t)dt$; since $f_n(0) = 0$ (because $f_n \in H_c \cap C(R)$), it follows that (in H_c, and hence, a.e.) $f_n(t) = \int_0^t f_n'(t)dt \to \int_0^t g(t)dt$ as $n \to \infty$. Therefore,

$$f_n(t) \to \int_0^t g(t)dt \quad \text{as } n \to \infty \tag{4.3}$$

But since $f_n \to f$, in H_c, as $n \to \infty$, then there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $f_{n_j} \to f$ for a.e $t \in R$. (Here, "a.e" means almost everywhere with respect to Lebesgue measure. Note that we use throughout this part of the proof the fact that μ_c–null sets and Leb–null sets are the same; see Lemma 48.) Thus,

$$\lim_{j \to \infty} f_{n_j} = f(t). \tag{4.4}$$

Therefore, by 4.3 and 4.4, we have

$$f(t) = \int_0^t g(t)dt \quad \text{a.e.} \tag{4.5}$$

Note that since $g \in H_c$, it is the case that $g \in L^1([0, t])$ and in fact that $g \in L^1_{loc}(R)$.

Indeed, we have

$$\int_0^t |g(t)|dt \leq \left(\int_0^t |g(\tau)|^2 e^{-2c\tau} d\tau\right)^{\frac{1}{2}} \left(\int_0^t e^{-2c\tau} d\tau\right) < +\infty$$

Also if I is a bounded interval and since $g = 0$ a.e. on $(-\infty, 0]$, it follows that $g \in L^1_{loc}(I)$. (Here, $L^1(I)$ denotes the space of Lebesgue integrable functions on I, with respect to Lebesgue measure. Similarly, $L^1_{loc}(R)$ is the space of locally Lebesgue integrable functions on R.) Hence, it follows from (4.5) that f(0)=0 and that $f \in C_{abs}(R)$ with $f'(t) = g(t)$ a.e. on R; and as a consequence, (since $g \in H_c$), we have $\partial f = g$ and thus $f \in D(A)$ and $Af = g$, as desired. (Note that it follows from (4.4) that since $f_n = 0$ a.e. on $(-\infty, 0]$, the same is true for f and hence also for f'.) Therefore, A is a closed operator.

Next, we will prove that $D(A^*) = D(A)$ and that in fact, $A^* = 2c - A$, where A^*, is the adjoint of A. We begin by showing that $D(A) \subset D(A^*)$ and that $A^*f = 2cf - Af$, for $f \in D(A)$. For that, we will use the characterization of $D(A^*)$ recalled in Equation (4.6) and the discussion surrounding it.

Let $f \in D(A)$ and $g \in D(A)$; so that, in particular $f(0) = g(0) = 0$ and both f and g are absolutely continuous on R and belong to $L^2(H^1, \mu_c)$. [Note that it follows from the method used in the proof of Lemma A.2 in Appendix A that $h(t) := e^{-2ct}g(t)$ is also absolutely continuous on R.] Then, we have successively :

$$\begin{aligned}
< Af, g >_c &= \int_0^\infty Af(t)\overline{g(t)}e^{-2ct}dt = \int_0^\infty f'(t)\overline{g(t)}e^{-2ct}dt \\
&= \left[f(t)\overline{g(t)}e^{-2ct}\right]_0^\infty - \int_0^\infty f(t)\left[\overline{g'(t)}e^{-2ct} - 2c\overline{g(t)}e^{-2ct}\right]dt \\
&= -\int_0^\infty f(t)\overline{g'(t)}e^{-2ct}dt + 2c\int_0^\infty f(t)\overline{g(t)}e^{-2ct}dt \\
&= 2c < f, g >_c - < f, Ag >_c \\
&= < f, (2c - A)g >_c .
\end{aligned}$$

We briefly justify some of the steps above. To obtain the third equality, we have used the integration by parts formula for absolutely continuous functions (see, e.g., [Fo, Thm.3.36].) Furthermore, the fourth equality follows from the fact that the boundary term $[f(t)\overline{g(t)}e^{-2ct}]_0^\infty$ vanishes. Indeed, $f(0) = g(0) = 0$ and by Theorem A.1 in Appendix A (which can be applied since $f, g \in D(A)$ and $|g| = |\overline{g}|$), we have $\lim f(t)e^{-ct} = \lim \overline{g(t)}e^{-ct}$ as $t \to +\infty$ and hence also, $\lim f(t)\overline{g(t)}e^{-2ct} = 0$ as $t \to \infty$.

We have shown therefore that for a given $f \in D(A)$

$$< Af, g >=< f, 2cg - Ag >, \text{for all } f \in D(A).$$

Since $\psi := 2cg - Ag \in H_c$ (as the sum of two vectors in H_c), it follows from the discussion surrounding Equation 4.6 that $g \in D(A^*)$ and $A^*g = (2c - A)g$. Hence, $D(A) \subset D(A^*)$ and A^* is equal to $2c - A$ on $\mathcal{K} := C_{abs}(R) \cap H^1((0, +\infty), e^{-2ct})$. That is , any function $g \in C_{abs}(R)$ such that g and g' are in $L^2((0, +\infty), e^{-2ct})$ belongs to $D(A^*)$ and satisfies $A^*g(t) = 2cg(t) - Ag(t)$.

Now, we will show that $D(A^*) \subset D(A)$. Let $\psi \in D(A^*)$ and $A^*\psi = \psi^*$. Then, for each $\phi \in D(A)$

$$
\begin{aligned}
< A\phi, \psi > = < \phi, \psi^* > = & \int_0^{+\infty} \phi(t)\overline{\psi^*(t)}e^{-2ct}dt \\
= & \int_0^{+\infty} \phi(t)\frac{d}{dt}\overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt \\
= & \left[\phi(t)e^{-2ct}\overline{\left(\int_0^t \psi^*(s)ds\right)}\right]_0^{+\infty} + 2c\int_0^{+\infty} \phi(t). \\
& \overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt - \int_0^{+\infty} \phi'(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}. \\
& e^{-2ct}dt \\
= & 2c\int_0^{+\infty} \phi(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt - \int_0^{+\infty} \phi'(t). \\
& \overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_0^{+\infty} \phi'(t)\overline{\psi(t)}e^{-2ct}dt = & \int_0^{+\infty} 2c\phi(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt \\
& - \int_0^{+\infty} \phi'(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt
\end{aligned}
$$

which is equivalent to

$$
\int_0^{+\infty} \phi'(t)\overline{\left(\psi(t) + \int_0^t \psi^* ds\right)}e^{-2ct}dt = \int_0^{+\infty} 2c\phi(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}. \quad e^{-2ct}dt
$$

We claim that

$$
\int_0^{+\infty} \phi'(t)\overline{\left(\int_0^t \psi(s)ds\right)}e^{-2ct}dt = \int_0^{+\infty} \phi(t)\overline{\left(\int_0^t \psi^*(s)ds\right)}e^{-2ct}dt,
$$

Indeed, we have successively:

$$\begin{aligned}
&\int_0^{+\infty} \phi'(t) \left(\overline{\int_0^t \psi(s)ds}\right) e^{-2ct}dt \\
&= \left[\phi(t)e^{-2ct} \left(\overline{\int_0^t \psi(s)ds}\right)\right]_0^{+\infty} - \int_0^{+\infty} \frac{d}{dt} \left(\overline{\int_0^t \psi(s)ds}\right) e^{-2ct}\phi(t)dt \\
&\quad + 2c\int_0^{+\infty} \left(\overline{\int_0^t \psi(s)ds}\right) \phi(t)e^{-2ct}dt \\
&= \int_0^{+\infty} \left[2c\left(\overline{\int_0^t \psi(s)ds}\right) - \frac{d}{dt}\left(\overline{\int_0^t \psi(s)ds}\right)\right] \phi(t)e^{-2ct}dt \\
&= \int_0^{+\infty} \left(2c - \frac{d}{dt}\right) \left(\overline{\int_0^t \psi(s)ds}\right) \phi(t)e^{-2ct}dt \\
&= \int_0^{+\infty} A^* \left(\overline{\int_0^t \psi(s)ds}\right) \phi(t)e^{-2ct}dt \\
&= \int_0^{+\infty} \left(\overline{\int_0^t \psi^*(s)ds}\right) \phi(t)e^{-2ct}dt.
\end{aligned}$$

As a consequence, we obtain

$$\int_0^{+\infty} \phi'(t)\overline{\left(\psi(t) + \int_0^t \psi^*(s)ds\right)}e^{-2ct}dt = \int_0^{+\infty} 2c\phi'(t)\overline{\left(\int_0^t \psi(s)ds\right)}. \\ e^{-2ct}dt$$

or equivalently that

$$\int_0^{+\infty} \phi'(t) \left(-2c\int_0^t \psi(s)ds + \psi(t) + \int_0^t \psi^*(s)ds + \gamma\right) e^{-2ct}dt = 0 \tag{4.6}$$

where we define the real constant γ by the equation

$$\int_0^{+\infty} \left(-2c\int_0^t \psi(s)ds + \psi(t) + \int_0^t \psi^*(s)ds + \gamma\right) e^{-2ct}dt = 0$$

and substitute for $\phi(t)$ the function

$$\phi_0(t) = \int_0^t \left(-2c\int_0^\tau \psi(s)ds + \psi(t) + \int_0^\tau \psi^*(s)ds + \gamma\right) e^{-2c\tau}d\tau.$$

We claim that $\phi_0 \in D(A)$. (See Appendix C for a proof of this claim.) Then Equation (4.6) assumes the form

$$\int_0^{+\infty} \left| -2c \int_0^{\tau} \psi(s)ds + \psi(t) + \int_0^{\tau} \psi^*(s)ds + \gamma \right|^2 e^{-2c\tau} d\tau = 0.$$

which implies that

$$N(\tau) := -2c \int_0^{\tau} \psi(s)ds + \psi(\tau) + \int_0^{\tau} \psi^*(s)ds + \gamma = 0 \quad \text{(for a.e. } \tau \geq 0.)$$

Therefore $N'(\tau) = 0$ for a.e. $\tau \geq 0$, that is $-2c\psi(\tau) + \psi'(\tau) + \psi^*(\tau) = 0$, or equivalently that

$$\psi^*(\tau) = 2c\psi(\tau) - \psi'(\tau) = (2c - \partial)\psi(\tau) = (2c - A)\psi(\tau)$$

which implies that $\psi \in D(A)$ and henceforth that $D(A^*) \subset D(A)$.

As a consequence $D(A) = D(A^*)$.

Next, we will show that $AA^* = A^*A$. Note that $D(AA^*) = D(A^*A)$, since $D(A) = D(A^*)$ and $A^* = 2c - A$. Let $f \in D(AA^*)$, then we have $AA^*f = A(2c - A)f = (2c - A^2)f = (2c - A)Af = A^*Af$ which implies that $AA^* = A^*A$. Thus normality of A is proved. ■

Chapter 5

The Spectrum of the Differential Operator ∂_c

5.1 Characterization of the spectrum of an unbounded normal operator.

We begin by recalling the definitions of the resolvent, spectrum and resolvent set of an unbounded, normal, linear operator.

Definition 64 *A complex number z is in the* resolvent set $\rho(A)$ *of an unbounded, normal, linear operator A on a Hilbert space $\mathcal{H}$ if there is a bounded operator B on $\mathcal{H}$ such that*

$$(z-A)Bu = u, u \in \mathcal{H}, \text{ and } B(z-A)v = v, v \in D(A) \subset \mathcal{H} \text{ (the domain of } A).$$

The operator B depends on z and is called the resolvent operator *of A. It is sometimes denoted by $R(z)$. The spectrum of A, denoted by $\sigma(A)$ is the set of all complex numbers λ that are not in $\rho(A)$.*

The following theorem is well known and can be found (for the present case of self-adjoint operators discussed in Theorem 65) in [Sc, Theorem 2.2.1]. We include its proof, for completeness.

Theorem 65 *If A is a self-adjoint unbounded linear operator in a complex Hilbert space $\mathcal{H}$, then a real number λ is in $\sigma(A)$ if and only if there is a sequence $\{u_n\}_{n=1}^{\infty}$ of elements in $D(A)$ such that*

$$||u_n|| = 1 \quad \text{and } ||(\lambda - A)u_n|| \to 0 \text{ as } n \to \infty. \tag{5.1}$$

Proof. Suppose (5.1) holds. If λ were in in $\rho(A)$ (the resolvent set of A), we would have $u_n = R(\lambda)(\lambda - A)u_n \to 0$ by the second part of (5.1). This contradicts the first part according to which $||u_n|| = 1$ for all $n \geq 1$.

Conversely, suppose (5.1) does not hold. Then there is a constant C such that

$$||u|| \leq C||(\lambda - A)u||, \quad u \in D(A). \tag{5.2}$$

For if (5.2) did not hold, there would be a sequence $\{v_n\} \subset D(A)$ of nonzero vectors such that

$$||(\lambda - A)v_n||^{-1}||v_n|| \to \infty \quad \text{as } n \to \infty.$$

Set $u_n = \frac{v_n}{||v_n||}$. Then, u_n satisfies (5.1). Thus (5.2) holds. This inequality tells us two things. The first is that for each $f \in R(\lambda - A)$, there is only one $u \in D(A)$ such that $(\lambda - A)u = f$. For if there were two, their difference v would satisfy $(\lambda - A)v = 0$. But v would have to vanish, by (5.2). The second bit of information we can obtain from (5.2) is that $R(\lambda - A)$ is closed. For if $\{f_n\}$ is a sequence of elements in $R(\lambda - A)$ and $f_n \to f$

in $\mathcal{H}$, let u_n be the unique solution of $(\lambda - A)u_n = f_n$. By (5.2), $\{u_n\}_{n=1}^{\infty}$ form a Cauchy sequence. Thus it converges to some element $u \in \mathcal{H}$. It follows that, if $v \in D(A)$ we have

$$< u, (\lambda - A)v > = \lim < u_n, (\lambda - A)v > = \lim < f_n, v > = < f, v > \text{ as } n \to \infty$$

It follows that, $u \in D(A)$ and $(\lambda - A)u = f$. Once we know this, we can show that $R(\lambda - A)$ is the whole space $\mathcal{H}$. This in turn implies that $\lambda \in \rho(A)$. For we can define $R(\lambda)f$ to be the unique solution of $(\lambda - A)u = f, u \in D(A)$. It clearly satisfies (5.1) and it is bounded by the right-hand side of (5.2). Thus it remains only to show that $R(\lambda - A) = \mathcal{H}$. Let v be the unique element of $D(A)$ such that $(\lambda - A)v = w$. Set $Fw = (v, F)$. This defines a linear functional on $R(\lambda - A)$, which is a Hilbert space (since it is a closed subspace of a Hilbert space). F is also bounded. For we have by (5.2)

$$||Fw|| \leq ||v||||f|| \leq C||f||||w||.$$

By the Riesz representation theorem, there is a $u \in R(\lambda - A)$ such that $Fw = (w, u)$ for all $w \in R(\lambda - A)$. This gives

$$< u, (\lambda - A)v > = < f, v >, \quad v \in D(A).$$

Since A is self-adjoint, this implies that $u \in D(A)$ and $(\lambda - A)u = f$. Hence $f \in R(\lambda - A)$, and the proof is complete. ■

Next, we show that the counterpart of the result obtained in Theorem 65 holds when A is an unbounded *normal* operator. This result is probably known, but we could not find an explicit statement in the litterature. We note that in light of the functional calculus of the spectral theorem (for unbounded normal operators), Theorem 66 is really a corollary (of the proof) of Theorem 65. (To follow the proof of Theorem

66, it is useful to recall that if A is an unbounded self-adjoint operator, then $\sigma(A) \subset R$, which explains in particular why λ was assumed to be real in the statement of Theorem 65.)

Theorem 66 *Let A be an unbounded* normal *linear operator. Then a complex number λ is in $\sigma(A)$ iff there is a sequence $\{u_n\}$ of elements in D(A) such that*

$$||u_n|| = 1 \quad \textit{and} \; ||(\lambda - A)u_n|| \to 0 \; \textit{as} \; n \to \infty. \tag{5.3}$$

Proof. Suppose (5.3) holds. If λ were in in $\rho(A)$ (the resolvent set of A), we would have $u_n = R(\lambda)(\lambda - A)u_n \to 0$ by the second part of (5.3). This contradicts the first part.

Now, assume that $\lambda = \lambda_1 + i\lambda_2 \in \sigma(A)$ (λ_1, λ_2 are real numbers). We want to show that there is a sequence $\{u_n\}_{n=1}^{\infty}$ of elements in D(A) such that $||u_n|| = 1$ and $||(\lambda - A)u_n|| \to \infty$ as $n \to \infty$.

Since A is normal, then $A = A_1 + iA_2$ (A_1, A_2 are commuting unbounded self-adjoint operators). In fact, in the sense of the functional calculus for unbounded operators, we have $A_1 = Re(A)$ and $A_2 = Im(A)$, from which it follows that A_1 and A_2 commute and $D(A) = D(A_1) \cap D(A_2)$. Furthermore, the spectral mapping theorem for unbounded normal linear operators implies that $\lambda_1 \in \sigma(A_1)$ and $\lambda_2 \in \sigma(A_2)$. Moreover, according to (the proof of) Theorem 65 and because A_1 and A_2 commute (since they core functions of the same normal operator A), there exists a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ in $D(A) = D(A_1) \cap D(A_2)$ such that

$$(\lambda_1 - A_1)u_n \to 0 \, \textit{and} \, (\lambda_2 - A_2)u_n \to 0 \quad \text{as } n \to \infty.$$

Since $\lambda - A = (\lambda_1 - A_1) + i(\lambda_2 - A_2)$, it follows that, as $n \to \infty$, $(\lambda - A)u_n \to 0$ as desired, which completes the proof. ∎

5.2 The spectrum of the differentiation operator ∂_c.

Let $c \geq 0$ be arbitrary and let $A = \partial_c$ be the differentiation operator, as defined in Section 4.2. We will next precisely determine its spectrum, which will turn out to be purely continuous.

Theorem 67 *The spectrum of the differentiation operator $A = \partial_c$ is the closed vertical line of the complex plane passing through $c \geq 0$, it is equal to the essential spectrum of A :*

$$\sigma(A) = \sigma_e(A) = \{\, \lambda \in C |\, Re(\lambda) = c\,\}. \tag{5.4}$$

Finally, the point spectrum of A is empty (i.e., A does not have any eigenvalues).

Proof. We recall that A is an unbounded normal linear operator whose adjoint is given by $A^* := 2c - A$, where $c \geq 0$ and $D(A^*) = D(A)$. Thus, $\frac{A+A^*}{2} = c$ or equivalently, $Re(A) = \frac{A+A^*}{2} = c$. Since $Im(A) = \frac{A-A^*}{2i}$, then $Im(A) = \frac{A-(2c-A)}{2i} = \frac{A-c}{i}$. As a consequence, $A = Re(A) + iIm(A) = c + i\left(\frac{A-c}{i}\right)$. If $\lambda \in \sigma(A)$, then (using the last equation and the spectral mapping theorem for unbounded normal linear operators), we have $Re(\lambda) = c$. (Indeed, by the spectral mapping theorem, we have $Re(\sigma(A)) = \sigma(Re(A))$ $= \sigma(cI) = \{c\}$, where I is the identity operator. Note that the spectrum of I consists of a single eigenvalue 1.) This shows that $\sigma(A) \subset \{c + i\theta;\, \theta \in R\}$.

Next, we will show that $\{c + i\theta;\, \theta \in R\} \subset \sigma(A)$. Given an unbounded normal linear operator A, one can decompose its spectrum as a disjoint union of the following spectra

$$\sigma(A) = \sigma_p(A) \cup \sigma_e(A),$$

where $\sigma_p(A)$ is the point spectrum of A and $\sigma_e(A)$ is its essential spectrum. The residual spectrum is empty in this case since A is a linear normal unbounded operator. The point spectrum is empty as well. Indeed, if $\sigma_p(A) \neq \emptyset$, then there is an eigenvalue $\lambda \in \sigma_p(A)$ corresponding to some nonzero eigenvector $\psi(t) = Ce^{\lambda t}$, for some nonzero constant C.. In order for ψ to be in $D(A)$ it must satisfy the boundary conditions, in particular the one at the origin. Such a boundary condition is satisfied only if the constant $C = 0$, which is contradiction. Therefore, $\sigma_p(A) =$ and hence $\sigma(A) = \sigma_e(A)$.

Let $\lambda \in \{c+i\theta,\ c > 0$ and $\theta \in R\}$. We want to show that $\lambda \in \sigma(A)$. (Our goal is to construct a sequence $\{U_n\}_n \in D(A)$ such that $||U_n|| = 1$ and $||(\lambda - A)U_n|| \to 0$ as $n \to \infty$.)

Note that a solution of the equation $A\psi = \lambda\psi$, where $\lambda \in C$, is of the form $\psi(t) = e^{\lambda t}$. (Any other solution is just given as the previous one multiplied by some constant). Note that $|\psi(t)| = e^{ct}$ implies that $\psi \notin L^2((0, +\infty), \mu_c)$. Clearly, ψ does not vanish on the left side of zero (i.e. does not satisfy the boundary condition at the origin) and is also not in $C_{abs}(R)$. Next, we set $\phi(t) = \widetilde{c}e^{-t}$, where $\widetilde{c}$ is a real constant chosen so that $\int_0^{+\infty} \phi(t)e^{-2ct}dt = 1$, and let

$$\widetilde{\phi}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \widetilde{c}e^{-1}.t, & \text{if } 0 < t < 1; \\ \phi(t), & \text{if } t \geq 1. \end{cases}$$

Then $\widetilde{\phi}(t) \in C(R)$. Note also that $\widetilde{\phi(t)} \in C_{abs}(R)$ which will simplify the proof later on that $\psi_n \in C_{abs}(R)$. Now, let $\{\psi_n\}_n$ be the sequence of functions defined as

$\psi_n(t) := n^{\frac{1}{2}}\widetilde{\phi}(\frac{t}{n})\frac{\psi(t)}{|\psi(t)|}$. Then $\psi_n \in L^2((0,+\infty),\mu_c(dt))$. Indeed, we have successively:

$$\begin{aligned}
||\psi_n||^2 &= \int_0^{+\infty} |\psi_n(t)|^2 e^{-2ct} dt \\
&= \int_0^{+\infty} n|\widetilde{\phi}(\frac{t}{n})|^2 e^{-2ct} dt \\
&= n^2 \int_0^{+\infty} |\widetilde{\phi}(u)|^2 e^{-2cnu} du \quad \text{(by making the change of variable } t = nu) \\
&= n^2 \left[\int_0^1 |\widetilde{\phi}(u)|^2 e^{-2cnu} du + \int_1^{+\infty} |\widetilde{\phi}(u)|^2 e^{-2cnu} du\right] \\
&= n^2 \left[\int_0^1 \widetilde{c}^2 e^{-2} u^2 e^{-2cnu} du + \int_1^{+\infty} |\phi(u)|^2 e^{-2cnu} du\right] \\
&= \widetilde{c}^2 n^2 \left[e^{-2} \int_0^1 u^2 e^{-2cnu} du + \int_1^{+\infty} e^{-2u} e^{-2cnu} du\right] \\
&= \widetilde{c}^2 n^2 \left[\frac{-e^{-2-2cn}}{2cn}\left(1 + \frac{1}{cn} + \frac{1}{c^2n^2}\right) + \frac{e^{-2(1+cn)}}{2(1+cn)}\right] < \infty.
\end{aligned}$$

Note that we used in the last line a double integration by parts to get

$$\int_0^1 u^2 e^{-2cnu} du = \frac{-e^{-2cn}}{2cn}\left[1 + \frac{1}{cn} + \frac{1}{c^2n^2}\right].$$

Therefore, $\psi_n \in L^2((0,+\infty),\mu_c(dt))$. Note also that $\psi_n' \in L^2((0,+\infty),\mu_c(dt))$.

Indeed, we have

$$\begin{aligned}
||\psi_n'||^2 &= \int_0^{+\infty} |\psi_n'(t)|^2 e^{-2ct} dt \\
&= \int_0^{+\infty} \left| n^{\frac{1}{2}}\widetilde{\phi}'(\frac{t}{n}) e^{iIm(\lambda)t} + n^{\frac{1}{2}}\widetilde{\phi}(\frac{t}{n}) i\theta e^{iIm(\lambda)t}\right|^2 e^{-2ct} dt \\
&\leq \int_0^{+\infty} \left[n|\widetilde{\phi}'(\frac{t}{n})|^2 + n|\widetilde{\phi}(\frac{t}{n})|^2|\theta|^2 + 2n|\widetilde{\phi}'(\frac{t}{n})||\widetilde{\phi}(\frac{t}{n})||\theta|\right] e^{-2ct} dt \\
&\leq \int_0^{+\infty} n|\widetilde{\phi}'(\frac{t}{n})|^2 e^{-2ct} dt + n|\theta|^2 \int_0^{+\infty} |\widetilde{\phi}(\frac{t}{n})|^2 e^{-2ct} dt \\
&\quad + 2n|\theta| \int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})||\widetilde{\phi}(\frac{t}{n})| e^{-2ct} dt < \infty,
\end{aligned}$$

since each of the integrals $J_1 := \int_0^{+\infty} n|\widetilde{\phi}'(\frac{t}{n})|^2 e^{-2ct} dt$, $J_2 := n|\theta|^2$

$.\int_0^{+\infty} |\widetilde{\phi}(\frac{t}{n})|^2 e^{-2ct} dt$ and $J_3 := 2n|\theta| \int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})||\widetilde{\phi}(\frac{t}{n})| e^{-2ct} dt$ is finite. (See Lemma 89 in

Appendix B for the rather lengthy proof of this fact.)

The sequence $\{\psi_n\}_n$ is also in $C_{abs}(R)$. Note that $\psi_n \in C(R)$. To show that ψ_n is absolutely continuous on R, it suffices to prove that it has a derivative ψ_n' a.e., this derivative is Lebesgue integrable and $\psi_n(x) = \psi_n(a) + \int_a^x \psi_n'(t)dt$ (See [Fo, Thm. 3.35].) Note that the derivative of ψ_n exists a.e. and is defined as follows:

$$\psi_n'(t) = \begin{cases} 0, & \text{for } t \leq 0; \\ n^{-\frac{1}{2}}\widetilde{c}e^{-1} + in^{-\frac{1}{2}}\widetilde{c}.t.\theta.e^{i\theta t}, & \text{for } 0 < t < 1; \\ n^{-\frac{1}{2}}\widetilde{c}e^{-\frac{t}{n}}e^{i\theta t} + in^{\frac{1}{2}}\widetilde{c}.e^{-\frac{t}{n}}\theta.e^{i\theta t}, & \text{for } t \geq 1. \end{cases}$$

Next, we will show that ψ_n' is Lebesgue integrable on R. Indeed, we have

$$\begin{aligned} \int_R |\psi_n'(t)|dt &= \int_0^1 |\psi'(t)|dt + \int_1^{+\infty} |\psi'(t)|dt \\ &= \int_0^1 |n^{-\frac{1}{2}}\widetilde{c}e^{-1} + n^{-\frac{1}{2}}\widetilde{c}e^{\frac{-t}{n}} i\theta e^{i\theta t}|dt \\ &\leq \int_0^1 (n^{-\frac{1}{2}}|\widetilde{c}|e^{-1} + n^{-\frac{1}{2}}|\widetilde{c}||\theta|t)dt + \int_1^{+\infty} (n^{-\frac{1}{2}}|\widetilde{c}|e^{\frac{-t}{n}} \\ &\quad + n^{\frac{1}{2}}|\widetilde{c}|e^{\frac{-t}{n}}|\theta|)dt \\ &\leq n^{-\frac{1}{2}}|\widetilde{c}| + n^{-\frac{1}{2}}|\widetilde{c}||\theta| + n^{-\frac{1}{2}}|\widetilde{c}|\int_1^{+\infty} e^{\frac{-t}{n}}dt + n^{\frac{1}{2}}|\widetilde{c}||\theta| \\ &\quad . \int_1^{+\infty} e^{\frac{-t}{n}}dt < \infty. \end{aligned}$$

We also have $\psi_n(x) = \psi_n(a) + \int_a^x \psi_n'(t)dt$ for any $a \in R$ and $x \in R$. Therefore, $\{\psi_n\}_n$ is an absolutely continuous sequence of functions on R. Hence, we have shown that $\{\psi_n\}_n \in D(A)$. Now, let $\widetilde{\psi_n(t)} = \frac{\psi_n(t)}{||\psi_n||_c}$. Then, $||\widetilde{\psi_n}|| = 1$, for any $n \geq 1$. We also have

$$A\widetilde{\psi_n(t)} = \frac{d\widetilde{\psi_n(t)}}{dt} = \frac{n^{-\frac{3}{2}}}{||\psi_n||_c}.\widetilde{\phi}'(\frac{t}{n}).\frac{\psi(t)}{|\psi(t)|} + \left[(c - \frac{1}{n}) + iIm(\lambda)\right].\widetilde{\psi_n}(t),$$

As a consequence, we have

$$
\begin{aligned}
||A\widetilde{\psi}_n(t) - \lambda\widetilde{\psi}_n(t)||_c^2 &= \frac{n^{-\frac{3}{2}}}{||\psi_n||_c} \int_0^{+\infty} |\widetilde{\phi'}(\frac{t}{n})|^2 e^{-2ct} dt \\
&= \frac{n^{-\frac{3}{2}}}{||\psi_n||_c} \int_0^{+\infty} |\widetilde{\phi'(u)}|^2 e^{-2nu} n du \quad \text{we let } t = nu \\
&= \frac{n^{-\frac{1}{2}}}{||\psi_n||_c} \left(\int_0^1 |\widetilde{\phi'}(u)|^2 e^{-2cnu} du + \int_1^{+\infty} |\widetilde{\phi'}|^2 e^{-2cnu} du \right) \\
&= \frac{n^{-\frac{1}{2}}}{||\psi_n||_c} \left(\int_0^1 \widetilde{c}^2 e^{-2} e^{-2cnu} du + \int_1^{+\infty} \widetilde{c}^2 e^{-2u} e^{-2cnu} du \right) \\
&= \frac{n^{-\frac{1}{2}}}{||\psi_n||_c} \left(e^{-2}\widetilde{c}^2 \int_0^1 e^{-2cnu} du + \widetilde{c}^2 \int_1^{+\infty} e^{-2u(1+cn)} du \right) \\
&= \frac{n^{-\frac{1}{2}}}{||\psi_n||_c} \left(e^{-2}\widetilde{c}^2 \left[\frac{e^{-2cnu}}{-2cn} \right]_0^1 + \widetilde{c}^2 \left[\frac{e^{-2u(1+cn)}}{-2(1+cn)} \right]_1^{+\infty} \right) \\
&= \frac{n^{-\frac{1}{2}}}{||\psi_n||_c} \left(e^{-2}\widetilde{c}^2 \left[\frac{e^{-2cn}}{-2cn} + \frac{1}{2cn} \right] + \widetilde{c}^2 \left[\frac{e^{-2(1+cn)}}{-2(1+cn)} \right] \right) \\
&\to 0 \text{ as } n \to \infty \text{ (see the value of } ||\psi_n||_c.)
\end{aligned}
$$

Therefore, by Theorem 66, $\lambda = c + i\theta \in \sigma(A)$, for $c > 0$ and any $\theta \in R$. Thus, $\{c + i\theta,\ c > 0,\ \theta \in R\} \subset \sigma(A)$. Finally, we showed that $\sigma(A) = \{\lambda \in C : Re(\lambda) = c\}$. ■

5.3 The strongly continuous semigroup of operators $\{e^{-t\partial_c}\}_{t\geq 0}$

Next, we will show that $\{e^{-t\partial_c}\}_{t\geq 0}$ is a strongly continuous semigroup of operators on H_c and in fact that it is a translation semigroup, namely, $(e^{-t\partial})(f)(u) = f(u-t)$ for any $u \in (0, +\infty)$ and $t \geq 0$. This will justify the representation ,we obtained earlier, of the spectral operator as a composite map of the Riemann zeta function and the first order differential operator ∂_c.

Definition 68 *Let $A : \mathcal{H} \to \mathcal{H}$ be an unbounded linear operator and define $U(t) = e^{tA}$. Then, the operator-valued function $U(t)$ is called a* strongly continuous one-parameter group *if*

1. $U(0) = Id$*, the identity map on* $\mathcal{H}$*.*

2. $U(t+s) = U(t)U(s)$ *for all* $s, t \in R$*.*

3. If $\phi \in \mathcal{H}$ *and* $t \to t_0$*, then* $U(t)\phi \to U(t_0)\phi$*.*

4. For $\psi \in D(A)$*,* $\frac{U(t)\psi - \psi}{t} \to A\psi$ *as* $t \to 0$*.*

Lemma 69 $\{e^{-t\partial}\}_{t\geq 0}$ *is a strongly continuous semigroup of operators and* $||e^{-t\partial}|| = e^{-tc}$ *for any* $t \geq 0$ *and any* $c \geq 0$*. The adjoint semigroup* $\{(e^{-t\partial})^*\}_{t\geq 0}$ *is given by* $\{e^{-t\overline{\partial^*}}\}_{t\geq 0} = \{e^{-t(2c-\partial)}\}_{t\geq 0}$*.*

Proof. Let $t \geq 0$ be fixed and let $G_t(s) := e^{-ts}$. We recall that ∂ is an unbounded normal linear operator whose spectrum is $\sigma(\partial) \subset \{s \in C | Re(s) = c\}, c \geq 0$. Note that $|e^{-ts}| = e^{-tRe(s)} \leq e^{-tc} < \infty$ for any $t \geq 0$ and any $c \geq 0$. Since for each $t \geq 0$ the map G_t is complex-valued. Then, using the holomorphic functional calculus and the spectral theorem applied to the normal linear operator ∂ defined on $D(\partial) \subset H_c$, the map $G_t(\partial) = e^{-t\partial}$ is a well-defined bounded linear operator on H_c. Note that $||G_t||_{\mathcal{L}(H_c)} = ||G_t||_\infty = \sup_{s\in\sigma(\partial)} |e^{-ts}| \leq e^{-tc} < \infty$. As a consequence, the adjoint semigroup $\{(e^{-t\partial})^*\}_{t\in R}$ is given by $(e^{-t\partial})^* = (G_t(\partial))^* = \overline{G_t}(\partial^*) = (e^{-t\overline{s}})(\partial^*) = e^{-t\partial^*} = e^{t(2c-\partial)}$. Finally, we show that $\{e^{-t\partial}\}_{t\in R}$ is strongly continuous. First, note that for any $t_1, t_2 \in R$ and any $s \in C$ we have $e^{-t_1 s}e^{-t_2 s} = e^{-(t_1+t_2)}$; hence, $G_{t_1} \circ G_{t_2} = G_{t_1+t_2}$; i.e, for $t_1, t_2 \geq 0$, $e^{-(t_1+t_2)\partial}$ $= e^{-t_1\partial}e^{-t_2\partial}$. Second, it follows that from the functional calculus for normal operators that for any $f \in H_c$, $G_t(\partial)f = e^{-t\partial}f \to f$ as $t \to 0^+$. Therefore, $\{e^{-t\partial}\}_{t\geq 0}$ is a strongly continuous semigroup of operators. (See also the discussion surrounding Theorem 13.38 in [Ru2].) ■

Lemma 70 *The strongly continuous semigroup of operators* $\{e^{-t\partial}\}_{t\geq 0}$ *is a translation semigroup. That is, for every* $t \geq 0$, $(e^{-t\partial})(f)(u) = f(u-t)$, *for all* $f \in H_c$ *and* $u \in R$. *(For a fixed* $t \geq 0$ *this equality holds for elements on* H_c *and hence; for a.e.* $u \in (0, +\infty)$*.)*

Proof. Assume that $f \in D(\partial)$. Let $g(t) = e^{-t\partial}f(t)$; then $g(t)$ is differentiable at $t = 0^+$ and $\frac{dg(t)}{dt}|_{t=0^+} = g'(0^+) = \partial f = f' \in H_c$ a.e. That is, $\lim\limits_{h\to 0^+} \frac{g(h)-g(0)}{h} = f' \in H_c$ and $g(0^+) = f$. Hence, g is the unique solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \\ u(0) = u(0,t) = f. \end{cases}$$

But for any $t \geq 0$, $(T(t)f)(u) := f(u-t)$ is a strongly continuous semigroup on H_c and its generator is also ∂; indeed, $\lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h} = f' = \partial f$. Therefore, $T(t) = e^{-t\partial}$ for any $t \geq 0$. ■

Remark 71 *Note that* $\partial_c = Re(\partial_c) + iIm(\partial_c)$, *where* $Re(\partial_c) = c$ *and* $Im(\partial_c) = \frac{\partial_c - c}{i}$. *Clearly,* $Im(\partial_c)$ *is self-adjoint, being the imaginary part of an unbounded normal linear operator. As a consequence, it generates a unitary semigroup* $\{e^{itIm(\partial_c)}\}_{t\in R}$. *In fact, we have* $e^{-t\partial_c} = e^{-t(Re(\partial_c)+iIm(\partial_c))} = e^{-t(c+i(\frac{\partial_c-c}{i}))} = e^{-ct}e^{-it(\frac{\partial_c-c}{i})}$. *Thus,* $||e^{-t\partial_c}|| = e^{-ct}||e^{-it(\frac{\partial_c-c}{i})}|| = e^{-ct}$. *It follows that* $\{e^{-t\partial_c}\}_{t\in R}$ *is a strongly continuous semigroup of bounded linear operators on* H_c, *with* $||e^{-t\partial_c}|| = e^{-tc}$ *for all* $t \in R$ *(hence,* $\{e^{tc}e^{-t\partial_c}\}_{t\in R}$ *is a one-parameter group of isometries on* H_c*); see e.g., [Kat]. Its adjoint group is given by* $\{e^{-t(c-iIm(\partial_c)}\}_{t\in R}$. *Furthermore,* $(e^{-t\partial_c}f)(u) = f(u-t)$ *for all* $t \in R$ *and* $f \in H_c$.

Finally, note that $\{e^{-tc}\}_{t\geq 0}$ *is a contraction semigroup for every* $c \geq 0$*, while* $\{e^{-it\partial_c}\}_{t\in R}$ *is a unitary group if and only if* $c = 0$*.*

Remark 72 *In light of Lemma 70 and Remark 71, we may call* $\partial = \partial_c$ *the* infinitesimal shift*. Indeed, it is the infinitesimal generator of the one-parameter group of translations of* R*.*

Chapter 6

Riemann Zeroes and Invertibility of the Spectral Operator

6.1 Precise definition of the spectral operator: $a = \zeta(\partial)$

In Section 3.2, we have given a semi-heuristic description of the spectral operator, as it was provided in [La-vF3]. In short, it is the operator that sends the geometric counting function of a generalized fractal string onto its spectral counting function. Now that we have precisely defined the differentiation operator (or 'infinitesimal shift') $\partial = \partial_c$ in Section3, and given a detailed description of its spectrum (in Section 5.2) of its associated semigroup (in Section 5.3), we are able to give a precise definition of the spectral operator a. We will do so briefly now, and then use the results of Sections 2, 3and 4 to determine its spectrum (in Section 6.2) and partly justified (in Section 6.1) the definition of a adapted here, as well as connect it with the one given in [La-vF3,§6.3.2] and Section 3.2.

We will often write ∂ instead of ∂_c or A, and a (instead of a_c), where $c \geq 0$

is fixed.

Definition 73 *Let $c \geq 0$ be fixed. Then* the spectral operator *a is defined (via the functional calculus for unbounded normal operators, see, e.g., [Ru2]) as follows:*[1]

$$a = \zeta(\partial) \tag{6.1}$$

Remark 74 *Note that since, by Theorem 63, $\partial = \partial_c$ is an unbounded normal operator on H_c (with domain $D(\partial) = D(A)$, as given in Section 4.2) and ζ is meromorphic in all of C (and hence, is measurable), it follows from Definition 73 and the measurable functional calculus for unbounded normal operators that $a = \zeta(\partial)$ is an unbounded normal operator on H_c, with domain given as follows:*

$$D(a) = \{f \in H_c : \zeta(\partial)f \in H_c\}. \tag{6.2}$$

As we will see in Section 6.3, for $c > 1$, the definition of a coincides with the one given in [La-vF3,§6.3.2] and section 6.3; of Theorem 75. Namely, $a = \sum_{k=1}^{\infty} k^{-\partial}$, for all $c > 1$. Moreover, in general (i.e., for any $c > 0$), a coincides with $\sum_{k=1}^{\infty} k^{-\partial}$ on a suitable subspace of H_c, see Theorem 77. Hence, a is a normal extension of the operator $\sum_{k=1}^{\infty} k^{-\partial}$.

In some sense, much as $\zeta = \zeta(s)$ is the analytic (i.e., meromorphic) extension of $\sum_{k=1}^{\infty} k^{-s}$ to all of C (and in particular, to $Re(s) > 0$) the spectral operator $a = \zeta(\partial)$ as given in Definition 73 can be thought of as an analytic extension of $\sum_{k=1}^{\infty} k^{-\partial}$ to a nice operator (i.e. to an unbounded normal operator) on H_c. Note that by analogy with

[1] The operator ∂ is unbounded since (by Theorem 67 above) its spectrum $\sigma(\partial)$ is unbounded (because it is a line). Similarly, $a = \zeta(\partial)$ is an unbounded operator because by Theorem 75 its spectrum is unbounded (since $|\zeta(c + it)| \to \infty$ as $|t| \to \infty$, see [Tit]).

the fact that the analytic continuation of a complex-valued meromorphic function is unique, we conjecture that $a = \zeta(\partial)$ is the unique (operator-valued) 'analytic extension' of $\sum_{k=1}^{\infty} k^{-\partial}$ as a normal unbounded operator on H_c; see Conjecture 79 at the end of Section 6.3.

6.2 The spectrum of a

The following result follows from the spectral mapping theorem for unbounded normal operators (see, e.g., [Ru2]).

Theorem 75 *For any $c \geq 0$, we have*

$$\sigma(a) = \zeta(\sigma(\partial)) = \zeta(\{s \in C | \, Re(s) = c, \, c > 0\}) = \{\zeta(s) : Re(s) = c\}, \tag{6.3}$$

where σ is the spectrum of a, and σ_e is the essential spectrum of ∂.

Proof. The proof is an immediate consequence of the measurable functional calculus for unbounded normal linear operators. Indeed, ζ admits a meromorphic continuation to the whole complex plane and is measurable. Since ∂ is an unbounded normal linear operator, then by the measurable functional calculus we have $\zeta(\sigma(\partial)) = \sigma(\zeta(\partial))$. Therefore, Theorem 5.4 implies that $\sigma(a) = \zeta(\sigma_e(\partial)) = \zeta(\{\lambda \in C | \, Re(\lambda) = c, \, c > 0\})$. ■

6.3 Justification of the definition of a

In this section, we establish two results (Theorems 77 and 76) which partially justify the definition of the spectral operator given in Section 6.1, namely, $a = \zeta(\partial)$. The results of Section 6.3 will not be used in the rest of the paper but will be useful as a motivation for [HerLa2].

Theorem 76 *Assume that* $c > 1$. *Then, for any* $f \in D(a)$, *we have*

$$a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k). \tag{6.4}$$

In other words, for $c > 1$,

$$a = \zeta(\partial) = \sum_{k=1}^{\infty} k^{-\partial}. \tag{6.5}$$

Proof. According to Theorem 67 and since $c > 1$, we have that $\sigma(\partial) = \{s \in C : Re(s) > 1\}$. Hence, $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, for $s \in \sigma(\partial)$. Therefore, in view of Theorem 75 , for $f \in D(a)$,

$$\begin{aligned} a(f)(t) &= \zeta(\partial)(f)(t) \\ &= \left(\sum_{k=1}^{\infty} k^{-\partial}\right)(f)(t) \\ &= \sum_{k=1}^{\infty} f(t - \log k) \end{aligned}$$

Where in the last equality, we have used the fact that for any $f \in H_c$,

$$\left(k^{-\partial}\right)(f)(t) = e^{-(\log k)\partial}(f)(t) = f(t - \log k), \tag{6.6}$$

in light of Lemma 70 and Remark 71.

■

In order to state the next result, we introduce some notation. Given $f \in H_c$, we let $g = \left(\sum_{k=1}^{\infty} k^{-\partial}\right)(f)$. It is easy to check that g is a well-defined measurable function on R such that $supp(g) \subset [0, +\infty)$. Furthermore, in light of Lemma 70 and Remark

71, we have (as in the proof of Theorem 76 above)

$$g(t) = \sum_{k=1}^{\infty} f(t - \log k).$$

We now define the following linear subspace of H_c:

$$\mathcal{Q} = \{f \in H_c : g \in H_c\}. \tag{6.7}$$

Theorem 77 *Assume that $c > 0$. Then, for any $f \in D(a) \cap \mathcal{Q}$, we have*

$$a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k) = \zeta(\partial)(f)(t) = \left(\sum_{n=1}^{\infty} n^{-\partial}\right)(f)(t) = g(t). \tag{6.8}$$

Proof. We recall that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $Re(s) > 1$ and that ζ has a meromorphic extension to all of C, with a single simple pole (with residue 1) located at the point s=1. In particular, the meromorphic continuation of ζ to the region $Re(s) > 0$ is given by:

$$\zeta(s) = \frac{1}{s-1} + \int_1^{\infty} \left([t]^{-s} - t^{-s}\right) ds \quad \text{for } Re(s) > 0. \tag{6.9}$$

This identity is easy to check for $Re(s) > 1$ and the integral on the right-hand side is analytic for $Re(s) > 0$. Hence, by analytic continuation, this identity continues to hold for $Re(s) > 0$. Note that this same identity was used by M. L. Lapidus and C. Pomerance in their proof of the modified Weyl–Berry conjecture for fractal strings (i.e., in one dimension); see [LaPo1,LaPo2].

Since $c > 0$, it follows from Theorem 76 that $\sigma(\partial) \subset \{s \in C : Re(s) > 0\}$. Hence, the above form of the analytic continuation of ζ can be used to express $a(f) = \zeta(\partial)(f)$. Furthermore, note that since ∂ does not have any eigenvalues, $\partial - 1$ is injective.

Next, let $f \in D(a) \cap \mathcal{Q}$. Then we have successively:

$$\begin{aligned}
\zeta(\partial)(f)(t) &= \left[\frac{1}{\partial - 1} + \int_1^{+\infty} ([t]^{-\partial} - t^{-\partial}) dt\right](f)(t) \\
&= \left[\frac{1}{\partial - 1} + \sum_{n=1}^{\infty} \left(\int_n^{n+1} [t]^{-\partial} dt - \int_1^{\infty} t^{-\partial} dt\right)\right](f)(t) \\
&= \left[\sum_{n=1}^{\infty} n^{-\partial} + \frac{1}{\partial - 1} - \int_1^{\infty} t^{-\partial} dt\right](f)(t).
\end{aligned}$$

We wish to show that $\left[\int_1^{\infty} t^{-\partial} dt - \frac{1}{\partial - 1}\right](f)(t) = 0,$ or that $\int_1^{\infty} t^{-\partial} dt (f)(t) = \frac{1}{\partial - 1}(f)(t).$

Note that the last equality is equivalent to

$$\left[(\partial - 1)\int_1^{\infty} t^{-\partial} dt\right](f)(t) = \left[\int_1^{\infty} t^{-\partial}(\partial - 1) dt\right](f)(t) = f(t) \quad \text{for } f \in D(A).$$

Hence, it suffices to show that for any $f \in D(\partial)$, we have

$$\int_1^{\infty} \big(f'(u - \log t) - f(u - \log t)\big)\, dt = f(u).$$

Note that each of these integrals separately makes sense because $f(v)$ vanishes for all $v \leq 0$.

Let $\phi(t) := \int_1^{\infty} (f'(u - \log t) - f(u - \log t))\, dt$. Making the change of variable $v = u - \log t$, we obtain

$$\phi(u) = -\int_u^{-\infty} \big(f'(v) - f(v)\big)\, e^{u-v} dv = e^u \int_{-\infty}^{u} \big(f'(v) - f(v)\big)\, e^{-v} dv.$$

Hence, showing that $\phi(u) = f(u)$ is equivalent to showing that

$$\int_{-\infty}^{u} f'(v) e^{-v} dv = \int_{-\infty}^{u} f(v) e^{-v} dv + e^{-u} f(u).$$

Let $\psi(u) := \int_{-\infty}^{u} f'(v)e^{-v}dv.$ Note that this integral makes sense since $f(v)$ vanishes for all $v \leq 0.$ Using an integration by parts, we obtain successively:

$$\begin{aligned}\psi(u) &= [e^{-v}f(v)]_{-\infty}^{u} + \int_{-\infty}^{u} f(v)e^{-v}dv \\ &= e^{-u}f(u) - \lim_{v\mapsto+\infty} e^{-v}f(v) + \int_{-\infty}^{u} f(v)e^{-v}dv \\ &= e^{-u}f(u) + \int_{-\infty}^{u} f(v)e^{-v}dv.\end{aligned}$$

Note that in the passage from the second to the third equality, we have used the fact (proved in Theorem A.1 of Appendix A) that any $f \in D(\partial)$,

$\lim_{v\to+\infty} e^{-v}f(v) = 0.$

Therefore, for any $f \in D(a)\cap\mathcal{Q}$, we have $\zeta(\partial)(f) = a(f) = \sum_{k=1}^{\infty} f(t-\log k).$

■

Corollary 78 *$a = \zeta(\partial)$ is an* extension *of the operator $\sum_{k=1}^{\infty} k^{-\partial}$ as an* unbounded normal operator *on H_c. Moreover, any extension that is normal must coincide with a.*

Proof. This follows at once from Theorem 77 since a is a normal unbounded operator, it cannot have any further strict normal extensions. ■

Conjecture 79 *$a = \zeta(\partial)$ is the* unique normal extension *of $\sum_{k=1}^{\infty} k^{-\partial}$ on the Hilbert space H_c.*

Then, if Conjecture 79 is proved to be correct, it will follow that $a = \zeta(\partial)$ can indeed be viewed as *the unique operator-valued analytic continuation* of $\sum_{k=1}^{\infty} k^{-\partial}$ that makes it a 'nice' operator, namely an unbounded *normal* operator on H_c. (See the discussion following following Definition in Section 5.1.)

We note that in light of Theorem 77 and Corollary 78, proving Conjecture amounts to showing that the subspace $D(a) \cap \mathcal{Q}$ is an operator *core* for a (See .[Kat]).

6.4 Invertibility of the spectral operator

Recall that a (possibly unbounded) linear operator on the Hilbert space H_c is said to be *invertible* if it is injective (i.e., has a trivial kernel $\{0\}$) and is surjective (i.e., if its range is all of H_c), and in addition if the inverse operator is bounded on all of H_c.

Furthermore, it is well known (and follows immediately from the definitions) that T is invertible if and only if zero is in its resolvent set $\rho(T)$ (i.e., if and only if $0 \notin \sigma(T)$.)

Remark 80 *We recall that the Riemann zeta function has trivial zeroes which are simple and located at the even negative integers,* -2, -4, -6, ...*; and that it also has nontrivial zeroes (also called* the critical zeroes*) located in the vertical strip of the complex plane* $0 < Re(s) < 1$[2]*. The Riemann hypothesis states that the critical zeroes of* ζ *are all located on the critical line* $Re(s) = \frac{1}{2}$.

Next, we provide a necessary and sufficient condition ensuring the invertibility of the spectral operator $a = \zeta(\partial)$. We will show that such a condition is related to whether the Riemann zeta function ζ have any zeroes on a suitable vertical line.

[2]It follows from Hadamard's theorem and the classic functional equation satisfied by ζ that ζ does not have any zeroes along the vertical lines $Re(s) = 0$ and $Re(s) = 1$.

Theorem 81 *Assume that* $c > 0$. *Then, the spectral operator* $a = \zeta(\partial)$ *is invertible if and only if the Riemann zeta function* $\zeta = \zeta(s)$ *does not vanish on the vertical line* $\{s \in C : Re(s) = c\}$. *In short,* $a = a_c$ *is invertible if and only if* $\zeta(s) \neq 0$ *for* $Re(s) = c$.

Proof. Since, in light of Theorem 77, the spectrum of a is equal to the range of ζ on the vertical line $Re(s) = c$ and since by definition of the spectrum of an operator, a is invertible if and only if $0 \notin \sigma(a)$, it follows that $a = a_c$ is invertible if and only if $\zeta(s) \neq 0$ for $Re(s) = c$. ∎

Corollary 82 The spectral operator $a = a_c$ *is* invertible *for all* $c \geq 1$. *Moreover, it is* not invertible *for* $c = \frac{1}{2}$.

Proof. In light of Theorem 81, a is invertible for $c > 1$ (since $\zeta(s)$ does not vanish for $Re(s) > 1$ because it is given by the Euler product $\prod_{p \in \mathcal{P}}(1 - p^{-s})^{-1}$ and for $c = 1$ (since by the aformentioned Hadamard's theorem, $\zeta(s)$ does not have any zeroes on the vertical line $\{s \in C : Re(s) = 1\}$).

Finally, according to Hardy's theorem (see [Tit], for example), $\zeta(s)$ has infinitely many zeroes on the critical line $\{s \in C : Re(s) = \frac{1}{2}\}$.[3] Hence, it follows from Theorem 81 that a is not invertible for $c = \frac{1}{2}$.

∎

6.5 A spectral reformulation of the Riemann hypothesis

Our next result, Theorem 83, provides a necessary and sufficient condition for the invertibility of the spectral operator. It shows that such a condition is intimetaly

[3] Of course, in the present case, it suffices to know the existence of a single zero on the line $\{s \in C : Re(s) = \frac{1}{2}\}$ to conclude that $a_{c=\frac{1}{2}}$ is not invertible.

related to the critical zeroes of the Riemann zeta function, hence allowing us to obtain (within this functional analytic framework) a spectral reformulation of the Riemann hypothesis.

Theorem 83 *The spectral operator* $a = a_c$ *is invertible for all* $c \in (0,1) - \frac{1}{2}$ *if and only if* $0 \notin \sigma(a)$ *if and only if* $c \in (0,1) - \frac{1}{2}$ *if and only if the Riemann hypothesis is true.*

Proof. This follows at once from Theorem 81 and the second part of Corollary 82. ■

Corollary 84 *The Riemann hypothesis is true if and only if the spectral operator* $a = a_c$ *is invertible for all* $c \in (0, \frac{1}{2})$*.*

Proof. This follows from the functional equation for ζ, according to which $\zeta(s) = 0$ if and only if $\zeta(1-s) = 0$ for $0 < Re(s) < 1$. ■

Chapter 7

Concluding comments

We close this research project by indicating several future research directions suggested by this work.

In our next paper [HerLa2], we will obtain an operator-valued Euler product for $a = \zeta(\partial)$, which is convergent in a suitable sense for all values of c in $(0, +\infty)$. In fact we will show that

$$a(f)(t) = \zeta(\partial)(f)(t) = \prod_{p \in \mathcal{P}} \left(\frac{1}{1 - p^{-\partial}} \right) (f)(t).$$

Since the reciprocal of ζ is expressed in terms of a Dirichlet series involving the Möbius function (see [Edw, Ivi, Tit]), it is then natural to expect that the inverse of $a = a_c$ (when it exists as a bounded or unbounded operator) can be expressed in terms of a suitable *extension* of an operator version of that Möbius Dirichlet series

In light of Corollary 82, a is invertible for every value of $c \geq 1$. We will see in [HerLa2] how to express its inverse (which is a bounded operator defined on all of H_c)

in terms of the above Möbius-type Dirichlet series. (The case when $c = 1$ will also be explored.)

$$a^{-1}(f)(t) := \frac{1}{\zeta(\partial)}(f)(t) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\partial}}(f)(t).$$

Moreover, in the light of Corollary 82, the truth of the Riemann hypothesis guarantees the existence of a bounded inverse for a, and we will then be also able to study the inverse in much the same way as suggested above. In general, unconditionally, we would also like to explore the existence of the inverse of a as a possibly unbounded operator and explore its properties in terms of a suitable operator-valued (normal) extension of the Möbius Dirichlet series.

A number of additional problems can be investigated along these lines, now that we have begun to establish a dictionary between aspects of analytic number theory and operator theory along with the spectral theorem. In particular, it is expected that many identities for various number-theoretic *L-functions* and Dirichlet series can be given an operator-theoretic counterpart

$$\widetilde{a}(f)(t) = \mathcal{L}(\chi, \partial_c)(f)(t) = \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\partial}} \right) (f)(t),$$

where χ is a Dirichlet character and hence we expect to obtain an extended representation for the operator-valued Euler product to the class of *L-functions*

$$\widetilde{a}(f)(t) = \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\partial}} \right) (f)(t) = \prod_{p \in \mathcal{P}} \left(\frac{1}{1 - \chi(p)p^{-\partial}} \right) (f)(t).$$

Appendix A

Appendix

The goal of this appendix is to prove the following theorem, which was used in a crucial way in Section 4.3 (to prove the normality of A) and elsewhere in the paper.

Theorem 85 (Natural boundary conditions at $+\infty$).
Let $A = \partial_c$ be the differential operator, as defined in Section 2.4. Then, for every $f \in D(A)$, the limit of $f(t)e^{-ct}$, as $t \to +\infty$, exists an is equal to zero.

The proof of this theorem is a consequence of the following two lemmas.

Lemma 86 *Let $f \in L^1_{loc}[0, +\infty)$ be such that $f \in C_{abs}(0, +\infty)$. Let $g(t) := f(t)e^{-ct}$, where $c \geq 0$. Then, $g \in L^1_{loc}(0, +\infty)$ and $g \in C_{abs}[0, +\infty)$.*

Proof. The fact that $g \in L^1_{loc}(0, +\infty)$ is obvious since $|g(t)| \leq |f(t)|$ for a.e. $t > 0$ and $f \in L^1_{loc}[0, +\infty)$. (Recall that $L^1_{loc}[0, +\infty)$ is the space of all measurable functions on $[0, +\infty)$ whose restriction to any bounded subinterval is Lebesgue integrable.)

Next, we claim that for any bounded interval $I \subset [0, +\infty)$, $g \in C_{abs}(I)$; i.e.,$g \in$

$C_{abs}(0, +\infty)$. To see this, it suffices to show that $g' \in L^1(I)$. But this is clear since (for a.e. $t > 0$),

$$g'(t) = e^{-ct} f'(t) - cg(t),$$

which is the sum of two integrable Lebesgue functions on I. (Indeed, note that $\phi(t) := e^{-ct} f'(t)$ satisfies $|\phi(t)| \leq |f'(t)|$ for a.e. $t \geq 0$ and that $f' \in L^1(I)$ since $f \in C_{abs}(I)$.) This concludes the proof of the lemma. ■

Lemma 87 *Let $f \in C_{abs}[0, +\infty)$ such that $f(0) = 0$ and suppose that f and f' are in $L^2[0, +\infty)$. Then, $f(t) \to 0$ as $t \to +\infty$.*

Proof. We begin by showing that $\lim |f(t)|$ exists as $t \to +\infty$. Let f be a function satisfying the above hypotheses. Then $t \mapsto f(t)f'(t)$ is absolutely integrable on $[0, +\infty)$. Let $t \in I \subset [0, +\infty)$. Then using the integration by parts formula for absolutely continuous functions (see, e.g. [Fo, Thm.3.36]), we obtain the following identity,

$$\begin{aligned}
\int_0^t f(s)\overline{f'(s)}ds &= \left[|f(s)|^2\right]_0^t - \int_0^t \overline{f(s)}f'(s)ds \\
&= |f(t)|^2 - |f(0)|^2 - \int_0^t \overline{f(s)}f'(s)ds \\
&= |f(t)|^2 - \int_0^t \overline{f(s)}f'(s)ds \quad \text{(since f(0)=0.)}
\end{aligned}$$

Consequently, $\lim |f(t)|$ exists as $t \to +\infty$. Indeed, since both $f\overline{f'}$ and $\overline{f'}f$ belong to $L^1(0, +\infty)$, their integrals over $[0, t]$ converge to $\int_0^{+\infty} f(t)\overline{f'(t)}dt$ and $\int_0^{+\infty} \overline{f'(t)}f(t)dt$, respectively, as $t \to +\infty$. Hence, $\lim_{t \to +\infty} |f(t)|^2 =$
$2Re(\int_0^{+\infty} f(s)\overline{f'(s)}ds)$.

Next, we will show that this limit is equal to zero. We proceed by contradiction. Suppose that $\lim |f(t)| > 0$ as $t \to +\infty$. Then there exist an $M > 0$ and $T > 0$ such that $|f(t)|^2 > M$ for all $t \geq T$. As a consequence, we have (since $|f|^2 \in L^1(0, +\infty)$),

$$\lim_{t\to+\infty}\int_0^t |f(t)|^2 dt > \lim_{t\to+\infty}\int_T^t |f(t)|^2 dt \geq \lim_{t\to+\infty} tM = +\infty,$$

contradicting the fact that $f \in L^2(0,+\infty)$. ■

Remark 88 (Proof of Theorem A.1.)

Let $f \in D(A)$. It follows that $f \in H_c$ and hence, $f = 0$ in $(-\infty, 0]$ and the restriction of f to $[0,+\infty)$ is absolutely continuous. Furthermore, $f' \in H_c$. Then, we have $g(t) := f(t)e^{-ct} \in L^2[0,+\infty)$ and $g'(t) = e^{-ct}f'(t) - cg(t) \in L^2[0,+\infty)$. [The fact that $g \in L^2[0,+\infty)$ follows since $\int_0^{+\infty} |f(t)|^2 e^{-2ct} dt < +\infty$ because $f \in H_c$. Moreover, $g' \in L^2[0,+\infty)$ since it is the sum of two (Lebesgue) square–integrable functions, $e^{-ct}f'(t) \in L^2[0,+\infty)$ (since $f' \in H_c$ implies that $\int_0^{+\infty} |f'(t)|^2 e^{-2ct} dt < +\infty$) and $g \in L^2[0,+\infty)$, as we noted just above).]

Hence, by applying Lemma A.3 to the function g,we deduce that $\lim |g(t)| = 0$ as $t \to +\infty$, or equivalently, that $\lim |f(t)|e^{-ct} = 0$ as $t \to +\infty$. (Note that Lemma A.3 can be applied to g since by Lemma A.2, $f \in C_{abs}[0,+\infty)$ implies that $g \in C_{abs}[0,+\infty)$; indeed, $f \in L^1_{loc}[0,+\infty)$ since $f \in L^2[0,+\infty)$ and $L^2[0,+\infty) \subset L^1_{loc}[0,+\infty)$.) This completes the proof of Theorem A.1.

Appendix B

Appendix

The following lemma justifies our claim (see the proof of Theorem 5.4) that $||\psi_n'||_c^2$ is finite and hence, that $\psi_n' \in L^2((0,+\infty),\mu_c(dt))$.

Lemma 89 *Let*

$$\widetilde{\phi}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \widetilde{c}e^{-1}t, & \text{if } 0 < t < 1; \\ \phi(t), & \text{if } t \geq 1, \end{cases}$$

where $\phi(t) = \widetilde{c}e^{-t}$ *and* $\widetilde{c}$ *is a real number. Then, for any* $\theta \in R$ *and any* $n \geq 1$, *each of the integrals:* $J_1 := \int_0^{+\infty} n|\widetilde{\phi}'(\frac{t}{n})|^2 e^{-2ct}dt$, $J_2 := n|\theta|^2 \int_0^{+\infty} |\widetilde{\phi}(\frac{t}{n})|^2$ $.e^{-2ct}dt$ *and* $J_3 := 2n|\theta| \int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})||\widetilde{\phi}(\frac{t}{n})|e^{-2ct}dt$ *is finite.*

Proof.

$$
\begin{aligned}
J_1 &= n\int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})|^2 e^{-2ct}dt \\
&= n^2\int_0^{+\infty} |\widetilde{\phi}'(u)|^2 e^{-2cnu}du \quad \text{(by making the change of variable } t = nu\text{)} \\
&= n^2\left[\int_0^{1} |\widetilde{\phi}(u)|^2 e^{-2cnu}du + \int_1^{+\infty} |\widetilde{\phi}(u)|^2 e^{-2cnu}du\right] \\
&= n^2\left[\int_0^{1} \widetilde{c}^2 e^{-2} e^{-2cnu}du + \int_1^{+\infty} |\phi'(u)|^2 e^{-2cnu}du\right] \\
&= \widetilde{c}^2 n^2\left[e^{-2}\int_0^{1} e^{-2cnu}du + \int_1^{+\infty} e^{-2u} e^{-2cnu}du\right] \\
&= \widetilde{c}^2 n^2\left[e^{-2}\left[\frac{e^{-2cnu}}{-2cn}\right]_0^1 + \left[\frac{e^{-2u(1+cn)}}{-2(1+cn)}\right]_1^{+\infty}\right] \\
&= \widetilde{c}^2 n^2\left[e^{-2}\left(\frac{-e^{-2cn}}{2cn} + \frac{1}{2cn}\right) + \frac{e^{-2(1+cn)}}{2(1+cn)}\right] < \infty.
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= n|\theta|\int_0^{+\infty} |\widetilde{\phi}(\frac{t}{n})|^2 e^{-2ct}dt \\
&= n^2|\theta|\int_0^{+\infty} |\widetilde{\phi}(u)|^2 e^{-2cnu}du \quad \text{(by making the change of variable } t = nu\text{)} \\
&= n^2|\theta|\left[\int_0^{1} |\widetilde{\phi}(u)|^2 e^{-2cnu}du + \int_1^{+\infty} |\widetilde{\phi}(u)|^2 e^{-2cnu}du\right] \\
&= n^2|\theta|\left[\int_0^{1} \widetilde{c}^2 e^{-2} u^2 e^{-2cnu}du + \int_1^{+\infty} |\phi(u)|^2 e^{-2cnu}du\right] \\
&= \widetilde{c}^2 n^2|\theta|\left[e^{-2}\int_0^{1} u^2 e^{-2cnu}du + \int_1^{+\infty} e^{-2u} e^{-2cnu}du\right] \\
&= \widetilde{c}^2 n^2|\theta|\left[\frac{-e^{-2-2cn}}{2cn}\left(1 + \frac{1}{cn} + \frac{1}{c^2n^2}\right) + \frac{e^{-2(1+cn)}}{2(1+cn)}\right] < \infty.
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= 2n|\theta| \int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})|.|\widetilde{\phi}(\frac{t}{n})|e^{-2ct}dt \\
&= 2n|\theta| \left[\int_0^{+\infty} |\widetilde{\phi}'(\frac{t}{n})|.|\widetilde{\phi}(\frac{t}{n})|e^{-2ct}dt + \int_1^{+\infty} |\widetilde{\phi}'(\frac{t}{n})|.|\widetilde{\phi}(\frac{t}{n})|e^{-2ct}dt\right] \\
&= 2n|\theta| \left[\int_0^1 \widetilde{c}^2 e^{-2}\frac{t}{n}e^{-2ct}dt + \int_1^{+\infty} \phi'(\frac{t}{n})\phi(\frac{t}{n})e^{-2ct}dt\right] \\
&= 2n|\theta| \left[e^{-2}\widetilde{c}^2 \int_0^1 \frac{t}{n}e^{-2ct}dt + (-\frac{\widetilde{c}^2}{n}) \int_1^{+\infty} e^{-\frac{2t}{n}}e^{-2ct}dt\right] \\
&= 2n|\theta| \left[\frac{e^{-2}\widetilde{c}^2}{n} \int_0^1 te^{-2ct}dt - \frac{\widetilde{c}^2}{n} \int_1^{+\infty} e^{-2t(\frac{1}{n}+c)}dt\right] \\
&= 2n|\theta| \left[\frac{e^{-2}\widetilde{c}^2}{n}\left(\frac{-e^{-2c}}{2c} + \frac{1}{2c}\left[\frac{1}{2c} - \frac{e^{-2c}}{2c}\right]\right) - \frac{\widetilde{c}^2}{n}.\frac{e^{2(\frac{1}{n}+c)}}{2(\frac{1}{n}+c)}\right] < \infty.
\end{aligned}
$$

■

Appendix C

Appendix

Lemma 90 *Let $\psi \in L^2((0,+\infty),\mu_c)$. Then, $t \mapsto \int_0^t |\psi(s)|ds \in L^2((0,+\infty),\mu_c)$.*

Proof. Since $\psi \in L^2((0,+\infty),\mu_c)$, then ψ is locally integrable. Indeed, for any $t > 0$, we have

$$\begin{aligned}
\int_0^t |\psi(s)|ds &= \int_0^t \left(|\psi(s)|e^{-cs}\right).e^{cs}ds \\
&\leq \left(\int_0^t |\psi(s)|^2 e^{-2cs}ds\right)^{\frac{1}{2}}.\left(\int_0^t e^{cs}ds\right)^{\frac{1}{2}} \\
&\leq ||\psi||_c^2.C_t < \infty \quad (C_t \text{ is a constant dependent on } t \text{ and } c).
\end{aligned}$$

As a consequence, $t \mapsto \int_0^t |\psi(s)|ds$ is absolutely continuous. We also have

$$\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right)^2 e^{-2ct}dt$$
$$= \left[\left(\int_0^t |\psi(s)|ds\right)^2 . \left(\frac{-e^{-2ct}}{2c}\right)\right]_0^{+\infty} + \frac{1}{c}\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right).|\psi(t)|e^{-2ct}dt$$
$$= \frac{1}{c}\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right).|\psi(t)|e^{-2ct}dt$$
$$= \frac{1}{c}\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right)e^{-ct}.|\psi(t)|e^{-ct}dt$$
$$\leq \frac{1}{c}\left(\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right)^2 e^{-2ct}dt\right)^{\frac{1}{2}} . \left(\int_0^{+\infty} |\psi(t)|^2 e^{-2ct}dt\right)^{\frac{1}{2}}.$$

Clearly, this implies that $\left(\int_0^{+\infty}\left(\int_0^t |\psi(s)|ds\right)^2 e^{-2ct}dt\right)^{\frac{1}{2}} < \frac{M}{c} < \infty$, where $M = \left(\int_0^{+\infty} |\psi(s)|^2 e^{-2ct}dt\right)^{\frac{1}{2}} < \infty$. Hence, the map $t \mapsto \int_0^t |\psi(s)|ds$ is in $L^2((0,+\infty),\mu_c)$.

■

Lemma 91 *The function defined for $t \geq 0$ by $\phi_0(t) = 0$ and*

$$\phi_0(t) = \int_0^t \left(-2c\int_0^\tau \psi(s)ds + \psi(\tau) + \int_0^\tau \psi^*(s)ds + \gamma\right)e^{-2c\tau}d\tau \tag{C.1}$$

is in $D(\partial)$.

Proof. To prove the lemma, we must show that the function ϕ_0 is absolutely continuous on $(0,+\infty)$, that it belongs to $L^2((0,+\infty),\mu_c)$ and that its derivative is in $L^2((0,+\infty),\mu_c)$.

First, we proceed showing that ϕ_0 is absolutely continuous. For that, we need to show that the function $\tau \mapsto -2c\int_0^\tau \psi(s)ds + \psi(\tau) + \int_0^\tau \psi^*(s)ds + \gamma$ is μ_c integrable over $(0,+\infty)]$ i.e.,

$$\int_0^{+\infty}\left|-2c\int_0^\tau \psi(s)ds + \psi(\tau) + \int_0^\tau \psi^*(s)ds + \gamma\right|e^{-2c\tau}d\tau < \infty. \tag{C.2}$$

Note that,

$\int_0^{+\infty}\left|-2c\int_0^\tau \psi(s)ds + \psi(\tau) + \int_0^\tau \psi^*(s)ds + \gamma\right|e^{-2c\tau}d\tau$

$\leq \int_0^{+\infty}(2c\int_0^\tau |\psi(s)|ds + |\psi(\tau)| + \int_0^\tau |\psi^*(s)|ds + |\gamma|e^{-2c\tau}d\tau$

$\leq\ 2c\int_0^{+\infty}\left(\int_0^\tau |\psi(s)|ds\right)e^{-2c\tau}d\tau + \int_0^{+\infty}|\psi(\tau)|e^{-2c\tau}d\tau + \int_0^{+\infty}\left(\int_0^\tau |\psi^*(s)|ds\right)e^{-2c\tau}d\tau +$

$|\gamma|\int_0^{+\infty}e^{-2c\tau}d\tau$ We show that each one of these integrals is finite.

Indeed, $\int_0^{+\infty}(\int_0^\tau |\psi(s)|ds)e^{-2c\tau}d\tau \leq (\int_0^{+\infty}\left(\int_0^\tau |\psi(s)|ds\right)^2 e^{-2c\tau}d\tau)^{\frac{1}{2}}.(\int_0^{+\infty}e^{-2c\tau}d\tau)^{\frac{1}{2}} <$ $+\infty$ (Using Holder's inequality and Lemma 90.)

Note that $\int_0^{+\infty}|\psi(\tau)|e^{-2c\tau}d\tau \leq (\int_0^{+\infty}|\psi(\tau)|^2e^{-2c\tau}d\tau)^{\frac{1}{2}}(\int_0^{+\infty}e^{-2c\tau}d\tau)^{\frac{1}{2}} < +\infty$ (using Holder's inequality and the fact that $\psi \in L^2((0,+\infty),\mu_c)$.

Note also that $\int_0^{+\infty}(\int_0^\tau |\psi^*(s)|ds)e^{-2c\tau}d\tau \leq (\int_0^{+\infty}(\int_0^\tau |\psi^*(s)|ds)^2e^{-2c\tau}d\tau)^{\frac{1}{2}}(\int_0^{+\infty}e^{-2c\tau}d\tau)^{\frac{1}{2}}$ $< +\infty$ and that $\int_0^{+\infty}e^{-2c\tau}d\tau < +\infty$.

Henceforth, the function $\tau \mapsto -2c\int_0^\tau \psi(s)ds + \psi(\tau) + \int_0^\tau \psi^*(s)ds + \gamma$ is μ_c integrable and as a consequence, the function ϕ_0 is absolutely continuous on $(0,+\infty)$.

Next, we will show that $\phi_0 \in L^2((0,+\infty),\mu_c)$. Note that

$$\begin{aligned}
&\int_0^{+\infty}|\phi_0(t)|^2e^{-2ct}dt \\
&\leq \int_0^{+\infty}[\int_0^t\left(2c\left(\int_0^\tau |\psi(s)|ds\right) + |\psi(\tau)| + \left(\int_0^\tau |\psi^*(s)|ds\right) + |\gamma|e^{-2c\tau}d\tau\right)^2 \\
&.\left(\int_0^t e^{-2c\tau}d\tau\right)].e^{-2ct}dt \\
&\leq \int_0^{+\infty}\int_0^t[4c^2\left(\int_0^\tau |\psi(s)|ds\right)^2 + |\psi(\tau)|^2 + \left(\int_0^\tau |\psi^*(s)|ds\right)^2 + |\gamma|^2 \\
&+ 4c\left(\int_0^\tau |\psi(s)|ds\right).|\psi(\tau)| + 4c\left(\int_0^\tau |\psi(s)|ds\right).\left(\int_0^\tau |\psi^*(s)ds\right) + 4c|\gamma| \\
&.\left(\int_0^\tau |\psi(s)|ds\right) + 2|\psi(\tau)|\left(\int_0^\tau |\psi^*(s)|ds\right)].e^{-2c\tau}d\tau.\left(\int_0^t e^{-2c\tau}d\tau.e^{-2ct}dt\right).
\end{aligned} \tag{C.3}$$

The previous term is less or equal to the sum of the eight integrals listed below, we will show that each one of them is finite (clearly, this will imply the desired result) :

$$I_1 = \int_0^{+\infty} \left[\int_0^t 4c^2 (\int_0^\tau |\psi(s)|ds)^2 e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_2 = \int_0^{+\infty} \left[\int_0^t |\psi(\tau)|^2 e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_3 = \int_0^{+\infty} \left[\int_0^t (\int_0^\tau |\psi^*(s)|ds)^2 e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_4 = |\gamma|^2 \int_0^{+\infty} \left[\int_0^{+\infty} e^{-2c\tau} d\tau \int_0^{+\infty} e^{-ec\tau} d\tau \right] e^{-2ct} dt,$$

$$I_5 = 4c \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds |\psi(\tau)| \right) e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_6 = 4c \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds \int_0^\tau |\psi^*(s)|ds \right) e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_7 = 4c|\gamma| \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds \right) e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt,$$

$$I_8 = 2 \int_0^{+\infty} \left[\int_0^t |\psi(\tau)| \left(\int_0^\tau |\psi^*(s)|ds \right) e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt.$$

Note that

$$\begin{aligned}
I_1 &= \int_0^{+\infty} \left[\int_0^t 4c^2 (\int_0^\tau |\psi(s)|ds)^2 e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt \\
&= \int_0^{+\infty} \int_0^{+\infty} 4c^2 (\int_0^\tau |\psi(s)|ds)^2 e^{-2c\tau} d\tau \left[\frac{-1e^{-2c\tau}}{2c} \right]_0^t e^{-2ct} dt \\
&= \int_0^{+\infty} \int_0^t 4c^2 \left(\int_0^\tau |\psi(s)|ds \right)^2 e^{-2c\tau} \left[\frac{1-e^{-2ct}}{2c} \right] e^{-2ct} dt \\
&\le \int_0^{+\infty} \int_0^t \left(\int_0^\tau |\psi(s)|ds \right)^2 e^{-2c\tau} d\tau e^{-2ct} dt \\
&\le 2c \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds \right)^2 e^{-2c\tau} d\tau \left(\frac{-e^{-2ct}}{2c} \right) \right]_0^{+\infty} \\
&+ \int_0^{+\infty} \left(\int_0^{+\infty} |\psi(s)|ds \right)^2 e^{-2ct} e^{-2ct} dt \quad \text{(using an integration by parts)} \\
&\le \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right)^2 e^{-2ct} dt < \infty \quad \text{(see Lemma 90).}
\end{aligned}$$

We also have

$$
\begin{aligned}
I_2 &= \int_0^{+\infty} \left[\int_0^t |\psi(\tau)|^2 e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} |\psi(\tau)|^2 e^{-2c\tau} d\tau \left(\frac{1 - e^{-2ct}}{2c} \right) \right] e^{-2ct} dt \\
&\leq \int_0^{+\infty} \left(\int_0^t |\psi(\tau)|^2 e^{-2c\tau} d\tau \right) e^{-2ct} dt \\
&\leq \left[\int_0^t |\psi(\tau)|^2 e^{-2c\tau} d\tau \left(\frac{-e^{-2ct}}{2c} \right) \right]_0^{+\infty} + \int_0^{+\infty} |\psi(t)|^2 e^{-2ct} . e^{-2ct} dt \\
&\leq \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right)^2 e^{-2ct} dt < +\infty \quad \text{(see Lemma 90).}
\end{aligned}
$$

Note that using a similar argument as the one above one can show that $I_3 < \infty$. Next, we show that $I_4 < \infty$. We have

$$
\begin{aligned}
I_4 &= |\gamma|^2 \int_0^{+\infty} \left[\int_0^t e^{-2c\tau} d\tau \int_0^t e^{-2c\tau} d\tau \right] e^{-2ct} dt \\
&= |\gamma|^2 \int_0^{+\infty} \left[\frac{-e^{-2c\tau}}{2c} \right]_0^t . \left[\frac{-e^{-2c\tau}}{2c} \right]_0^t e^{-2ct} dt \\
&= |\gamma|^2 \int_0^{+\infty} \left[\frac{1 - e^{-2ct}}{2c} \right] . \left[\frac{1 - e^{-2ct}}{2c} \right] . e^{-2ct} dt \\
&\leq \frac{|\gamma|^2}{4c^2} \int_0^{+\infty} e^{-2ct} dt < \infty \quad \text{(since } 1 - e^{-2ct} < 1 \text{ for } t > 0 \text{ and } c > 0\text{).}
\end{aligned}
$$

Note that $I_5 < \infty$, since

$$
\begin{aligned}
I_5 &= 4c \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds.|\psi(\tau)| \right) e^{-2c\tau} d\tau. \int_0^t e^{-2c\tau} d\tau \right] .e^{-2ct} dt \\
&= 4c \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds.|\psi(\tau)| e^{-2c\tau} d\tau \right) \right] . \left(\frac{1 - e^{-2c}}{2c} \right) .e^{-2ct} dt \\
&\leq 2 \int_0^{+\infty} \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds.|\psi(\tau)| e^{-2c\tau} d\tau \right) \right] .e^{-2ct} dt \\
&\leq 2 \left[\int_0^t \left(\int_0^\tau |\psi(s)|ds|\psi(\tau)| \right) e^{-2c\tau} d\tau. (\frac{-e^{-2ct}}{2c}) \right]_0^{+\infty} \\
&+ \frac{1}{c} \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds|\psi(t)| \right) .e^{-2ct} (\frac{e^{-2ct}}{2c}) dt \quad \text{(we integrated by parts)} \\
&\leq \frac{1}{2c^2} \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds.|\psi(t)| \right) e^{-4ct} dt \\
&\leq \frac{1}{2c^2} \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds.e^{-ct} \right) . \left(|\psi(t)| e^{-3ct} \right) dt \\
&\leq \frac{1}{2c^2} \left(\int_0^{+\infty} \left(\int_0^{+\infty} |\psi(s)|ds \right)^2 .e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} |\psi(t)|^2 e^{-6ct} dt \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2c^2} \left(\int_0^{+\infty} \left(\int_0^{+\infty} |\psi(s)|ds \right)^2 .e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} |\psi(t)|^2 e^{-2ct} dt \right)^{\frac{1}{2}}
\end{aligned}
$$

$< \infty$ (using Holder's inequality and Lemma 90).

Using a similar argument as the one above, one can also show that $I_8 < \infty$. Next, we proceed to show that $I_6 < \infty$.

$$I_6 = 4c \int_0^{+\infty} \int_0^t \left[\left(\int_0^\tau |\psi(s)| ds \right) \left(\int_0^\tau |\psi^*(s)| ds \right) \right] e^{-2c\tau} d\tau \left(\int_0^t e^{-2c\tau} d\tau \right)$$

$$.e^{-2ct} dt$$

$$= 4c \int_0^{+\infty} \int_0^t \left[\left(\int_0^\tau |\psi(s)| ds \right) \left(\int_0^\tau |\psi^*(s)| ds \right) \right] e^{-2c\tau} d\tau \left(\frac{1 - e^{-2ct}}{2} \right)$$

$$.e^{-2ct} dt$$

$$\leq 2c \int_0^{+\infty} \int_0^t \left[\left(\int_0^\tau |\psi(s)| ds \right) \left(\int_0^\tau |\psi^*(s)| ds \right) \right] e^{-2c\tau} d\tau . e^{-2ct} dt$$

$$\leq 2c \left[\int_0^t \int_0^\tau |\psi(s)| ds \int_0^\tau |\psi^*(s)| ds e^{-2c\tau} d\tau \left(\frac{-e^{-2ct}}{2c} \right) \right]_0^{+\infty}$$

$$+ \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) \left(\int_0^t |\psi^*(s)| ds \right) . e^{-2ct} \left(\frac{e^{-2ct}}{2c} \right) dt$$

$$\leq \frac{1}{2c} \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) \left(\int_0^t |\psi^*(s)| ds \right) e^{-4ct} dt$$

$$\leq \frac{1}{2c} \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) \left(\int_0^t |\psi^*(s)| ds \right) e^{-2ct} dt$$

$$\leq \frac{1}{2c} \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) \left(\int_0^t |\psi^*(s)| ds \right) e^{-2ct} dt$$

$$\leq \frac{1}{2c} \left(\int_0^{+\infty} \left(\int_0^t |\psi(s)| ds e^{-ct} \right)^2 dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} \left(\int_0^t |\psi^*(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}}$$

$< \infty$ (using Holder's inequality and Lemma 90).

Finally, we will show that $I_7 < \infty$.

$$
\begin{aligned}
I_7 &= 4c|\gamma| \int_0^{+\infty} \int_0^t \left(\int_0^\tau |\psi(s)| ds \right) e^{-2c\tau} d\tau . \int_0^t e^{-2c\tau} d\tau . e^{-2ct} dt \\
&= 4c|\gamma| \int_0^{+\infty} \int_0^t \left(\int_0^\tau |\psi(s)| ds \right) e^{-2c\tau} d\tau . \left(\frac{1 - e^{-2ct}}{2} \right) e^{-2ct} dt \\
&\leq 2c|\gamma| \int_0^{+\infty} \int_0^t \left(\int_0^\tau |\psi(s)| ds \right) e^{-2c\tau} d\tau . e^{-2ct} dt \\
&\leq 2c|\gamma| \left[\int_0^t \left(\int_0^\tau |\psi(s)| ds \right) e^{-2c\tau} d\tau . \left(\frac{-e^{-2ct}}{2c} \right) \right]_0^\infty \\
&+ \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-2ct} . e^{-2ct} dt \\
&\leq |\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-4ct} dt \\
&\leq |\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-2ct} dt \\
&\leq |\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds e^{-ct} \right) e^{-ct} dt \\
&\leq |\gamma| \left(\int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} e^{-2ct} dt \right)^{\frac{1}{2}} \\
&< \infty \quad \text{(using Holder's inequality and Lemma 90).}
\end{aligned}
$$

Therefore, $\phi_0 \in L^2((0, +\infty), \mu_c)$.

Next, we show that $\phi_0' = \frac{d\phi_0(t)}{dt} \in L^2((0, +\infty), \mu_c)$. Note that $\phi_0'(t) = -2c \int_0^t \psi(s) ds + \psi(t) + \int_0^t \psi^*(s) ds + \gamma$. Hence, we have

$$\int_0^{+\infty} |\phi_0'(t)|^2 e^{-2ct} dt$$

$$\leq \int_0^{+\infty} \left[4c^2 | \int_0^t \psi(s)ds| + |\psi(t)| + | \int_0^t \psi^*(s)ds| + |\gamma| \right] e^{-2ct} dt$$

$$\leq 4c^2 \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right)^2 e^{-2ct} dt + \int_0^{+\infty} |\psi(t)|^2 e^{-2ct} dt$$

$$+ \int_0^{+\infty} \left(\int_0^t |\psi^*(s)|ds \right)^2 e^{-2ct} dt + |\gamma|^2 \int_0^{+\infty} e^{-2ct} dt$$

$$+ 4c \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right) .|\psi(s)| e^{-2ct} dt + 4c \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right)$$

$$. \left(\int_0^t |\psi^*(s)|ds \right) e^{-2ct} dt$$

$$+ 4c|\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right) e^{-2ct} dt + 2 \int_0^{+\infty} |\psi(s)|. \left(\int_0^t |\psi^*(s)|ds \right) e^{-2ct} dt.$$

To prove the desired result, we will show that each one of the integrals, listed below is finite:

$$J_1 = 4c^2 \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right)^2 e^{-2ct} dt,$$

$$J_2 = \int_0^{+\infty} |\psi(t)|^2 e^{-2ct} dt,$$

$$J_3 = \int_0^{+\infty} \left(\int_0^t |\psi^*(s)|ds \right)^2 e^{-2ct} dt,$$

$$J_4 = |\gamma|^2 \int_0^{+\infty} e^{-2ct} dt,$$

$$J_5 = \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right) .|\psi(s)| e^{-2ct} dt,$$

$$J_6 = 4c \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right) . \left(\int_0^t |\psi^*(s)|ds \right) e^{-2ct} dt,$$

$$J_7 = 4c|\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)|ds \right) e^{-2ct} dt,$$

$$J_8 = 2 \int_0^{+\infty} |\psi(s)|. \left(\int_0^t |\psi^*(s)|ds \right) e^{-2ct} dt.$$

Note that J_1 and J_3 are both finite, by Lemma 90. Since $\psi \in L^2((0,+\infty),\mu_c)$, then $J_2 < \infty$. Also, J_4 is clearly finite. Next, we show that $J_5 < \infty$. Indeed,

$$\begin{aligned}
J_5 &= \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) .|\psi(t)| e^{-2ct} dt \\
&= \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-ct}.|\psi(t)| e^{-ct} dt \\
&\leq \left(\int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} |\psi(s)|^2 e^{-2ct} dt \right)^{\frac{1}{2}} \\
&< \infty \quad \text{(using Holder's inequality and Lemma 90).}
\end{aligned}$$

An argument similar to the above one can be used to show that $J_8 < \infty$. Next, we show that J_6 is finite

$$\begin{aligned}
J_6 &= 4c \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) . \left(\int_0^t |\psi(s)| ds \right) e^{-2ct} dt \\
&= 4c \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-ct}. \left(\int_0^t |\psi^*(s) ds \right) e^{-ct} dt \\
&\leq 4c \left(\int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} \left(\int_0^t |\psi^*(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}} \\
&< \infty \quad \text{(using Holder's inequality and Lemma 90).}
\end{aligned}$$

Finally, we show that $J_7 < \infty$,

$$\begin{aligned}
J_7 &= 4c|\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-2ct} dt \\
&= 4c|\gamma| \int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right) e^{-ct}.e^{-ct} dt \\
&\leq \left(\int_0^{+\infty} \left(\int_0^t |\psi(s)| ds \right)^2 e^{-2ct} dt \right)^{\frac{1}{2}} . \left(\int_0^{+\infty} e^{-2ct} dt \right)^{\frac{1}{2}} \\
&< \infty \quad \text{(using Holder's inequality and Lemma 90).}
\end{aligned}$$

Therefore, we have shown that $\phi_0 \in D(\partial)$.

■

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