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### Publication Date

1952-03-04

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La Jolla, California

OPTIMAL RECTIFIER SYSTEMS FOR THE DETECTION OF STEADY SIGNALS

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Sponsored by  
Bureau of Ships  
Contract NObsr-43356  
(Marine Physics Research)

Reference 52-11  
4 March 1952

Approved for distribution:

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OPTIMAL RECTIFIER SYSTEMS FOR THE DETECTION OF STEADY SIGNALS

by

Carl Eckart

ABSTRACT

The concept of detection as a corollary to measurement is applied to wide-band signals obscured by ~~wide~~<sup>wide</sup>-band noise. The detection system consists of a linear amplifier and filter, followed by a quadratic rectifier and its associated low-pass filter.

It is assumed that both signal and background are both wide-band noise, and that both are normal or Gaussian. This means that the autocorrelation of the rectified noise is determined by that of the unrectified, according to a simple formula. Moreover, it implies that both signal and background are steady -- perhaps steadier than is actually the case.

Formulae are derived for the admittance  $A(\omega)$  of the optimal filter, which is shown to be determined by the power spectra of the signal and background. If these are  $S_s(\omega)$  and  $S_b(\omega)$ , respectively, then

$$|A|^2 = k S_s / S_b^2,$$

$k$  being an arbitrary constant. The threshold of detection for this system is given by

$$(V/B)^4 = (8\pi/T) \int_{-\infty}^{\infty} [S_s / S_b]^2 d\omega,$$

$T$  being the averaging time of the low-pass filter, and  $V$  and  $B$  the rms signal and background voltages. Every other filter will result in a higher threshold of detection. These formulae are applied to a detailed analysis of five special cases.

It must be emphasized that these conclusions are strongly dependent on the assumptions made about the signal and noise. The extent to which the conclusions must be modified when applied to actual conditions is not yet known.

In conclusion, the function of the filter in a detection system is discussed.

INTRODUCTION

In a previous paper (Ref. 1), the detection of simple a.c. signals in a background of noise was discussed. In this paper, the same concepts will be applied to the detection of wide-band signals. In Ref. 1, the detection problem was considered as a corollary to the measurement problem: the threshold of detection is that value of the signal:noise ratio for which the error of measurement is equal to the quantity measured. The quantity to be measured, in the case of a wide-band signal, is its mean-square value, or power.

Since the noise background will also supply power to the system, some method must be devised for distinguishing between the two sources of power. This is most readily accomplished by turning the signal on and off at intervals and determining the power increment. Since the actual signal generator is inaccessible in all passive detection problems, the "on" and "off" conditions must be produced at the receiver. If the latter is directional, the desired effect can most easily be obtained by sweeping its beam across the target. The noise background at the bearing of the target is then to be determined by interpolation; a bearing recorder or similar device may make this easier. However, this interpolation process will not be considered in detail; the problem will be simplified by treating the on and off conditions as if they were brought about by a switch under control of the operator.

Both the signal and background will be considered to be steady normal noises. Actually, fading and fluctuation will occur, and these can be serious complications. They may be so serious that, under some conditions, the present theory becomes highly unrealistic. As was shown in Ref. 1, sensitivity is

paid for in elapsed time. The following theory supposes that the levels of both signal and noise remain constant during successive "on" and "off" periods. Fluctuation therefore imposes an upper limit on the elapsed time, and consequently a lower limit on the sensitivity.

If the signal is turned on and off by training the receiver, this implies a lower limit on the training rate. Since high training rates can be achieved by some form of sector scan the practical difficulties do not seem serious at this time. The following analysis therefore ignores all such problems.

1. Schematic of the Detection System.

For initial simplicity, the signal will be considered as an electromotive force, accessible at the terminals of a hydrophone. Moreover, the hydrophone's internal impedance will be included in the input impedance of the amplifier-filter system, so that the hydrophone may be treated as an electrical signal generator of zero impedance. In addition to generating the signal, it will be supposed to generate an interfering background.

It will be supposed that the spectrum of the signal is known, and that detection is based on the measurement of its power, or mean-square value. Consequently, the signal emf will be written  $Vv(t)$ . If  $\langle v^2 \rangle = 1$ ,  $V$  will be the rms value of the emf; however, the precise definition of  $V$  is of no importance. The power spectrum of  $v$  will be\*  $S_v(\omega)$ ,  $\omega$  being the frequency in radians per second. The interfering background will be written simply as  $b(t)$ , with spectrum  $S_b(\omega)$ .

The schematic diagram of the detection system is then as shown in Fig. 1.

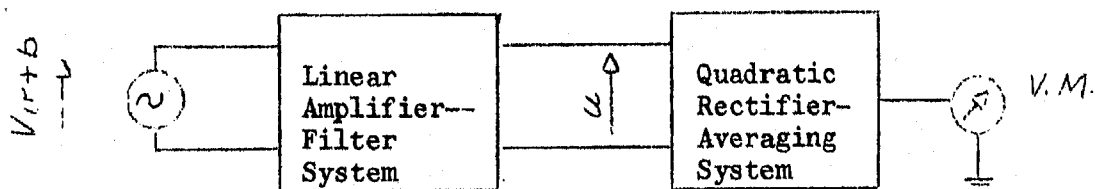


Figure 1.

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\*The general notation and conventions of this paper are those of Ref. 1. For the complete definition  $S_v(\omega)$ , cf Sec. 1.3 of Ref. 1.



The voltmeter is calibrated to read the mean square of  $u$  :

$$\chi = \langle u^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_u d\omega. \quad (1)$$

2. The Threshold of Detection.

It is assumed that the voltages  $v$  and  $b$  are normal, i.e., that the probability distribution of all orders up to and including the fourth are Gaussian. This will then also be true of  $u$ , the input to the rectifier, if  $S_u(\omega)$  is its spectrum, and  $T$  the averaging time of the rectifier-averaging system, the rms estimate of the reading error in  $x$  will be\*  $\Delta x$ , where

$$(\Delta x)^2 = (1/\pi T) \int_{-\infty}^{\infty} S_u^2 d\omega. \quad (2)$$

It will be supposed that the functions  $v$  and  $b$  are incoherent, and that the transfer admittance of the amplifier filter system is  $A(\omega)$ . Then

$$S_u = (V^2 S_v + S_b) |A|^2, \quad (3)$$

so that

$$(\Delta x)^2 = (1/\pi T) \int_{-\infty}^{\infty} (V^2 S_v + S_b)^2 |A|^4 d\omega. \quad (4)$$

According to what has been said in the Introduction, the signal can be turned off ( $V = 0$ ) at the will of the operator. Then both quantities  $x$  and  $y$ , where

$$y = (1/2\pi) \int_{-\infty}^{\infty} S_b |A|^2 d\omega, \quad (5)$$

---

\*Cf, Sec. 3.2 of Ref. 1.

can be measured, and the reading error in  $y$  will be given by

$$(\Delta y)^2 = (1/\pi T) \int_{-\infty}^{\infty} S_b^2 |A|^4 d\omega. \quad (4.1)$$

The error in

$$x - y = (V^2/2\pi) \int_{-\infty}^{\infty} S_v |A|^2 d\omega \quad (6)$$

will be the square root of  $(\Delta x)^2 + (\Delta y)^2$ , and the error  $\Delta V^2$  will be given by

$$(\Delta V^2)^2 = (4\pi/T) \int_{-\infty}^{\infty} [(V^2 S_v + S_b)^2 + S_b^2] |A|^4 d\omega / \left\{ \int_{-\infty}^{\infty} S_v |A|^2 d\omega \right\}^2. \quad (7)$$

The threshold of detection,  $V_m$ , is obtained from Eq. (7) by setting  $V^2 = \Delta V^2 = V_m^2$ , and solving for  $V_m$ . If  $T$  can be chosen large enough,  $V_m$  will be sufficiently small so that it can be neglected on the right side of Eq. (7), and

$$V_m^4 = (8\pi/T) \int_{-\infty}^{\infty} S_b^2 |A|^4 d\omega / \left\{ \int_{-\infty}^{\infty} S_v |A|^2 d\omega \right\}^2 \quad (8)$$

will be an adequate approximation. This requires that

$$T \gg 2\pi \left\{ \int_{-\infty}^{\infty} S_u S_b |A|^4 d\omega / \int_{-\infty}^{\infty} S_u |A|^2 d\omega \right\}^2 / \int_{-\infty}^{\infty} S_b^2 |A|^4 d\omega, \quad (9)$$

$$T \gg 4\pi \int_{-\infty}^{\infty} S_u^2 |A|^4 d\omega / \left\{ \int_{-\infty}^{\infty} S_u |A|^2 d\omega \right\}^2.$$

As has been remarked above, the phenomena of fading and fluctuation will change the level of the signal in the course of time. Unless these changes are negligible during the interval  $T$ , the following considerations will not be applicable. It will appear that values of  $T$  measured in fractions of a second are often adequate, so that there is a domain in which the following calculations are realistic.

These equations are all very cumbersome; they can be much simplified by introducing the two positive functions

$$\begin{aligned} g &= S_u / S_b \\ f &= |A|^2 S_b^2 / S_u. \end{aligned} \quad (10)$$

Then

$$S_u = (V^2 g^2 + g) f, \quad (3.1)$$

$$2\pi y = \int_{-\infty}^{\infty} g f d\omega, \quad (5.1)$$

$$2\pi(x-y) = V^2 \int_{-\infty}^{\infty} g^2 f d\omega, \quad (6.1)$$

$$V_m^4 T / 8\pi = \int_{-\infty}^{\infty} g^2 f^2 d\omega / \left\{ \int_{-\infty}^{\infty} g^2 f d\omega \right\}^2. \quad (8.1)$$

The Eq. (8.1) is a valid approximation only when

$$T / 2\pi \gg \left\{ \int_{-\infty}^{\infty} g^3 f^2 d\omega / \int_{-\infty}^{\infty} g^2 f d\omega \right\}^2 / \int_{-\infty}^{\infty} g^2 f^2 d\omega, \quad (9.1)$$

$$T / 4\pi \gg \int_{-\infty}^{\infty} g^4 f^2 d\omega / \left\{ \int_{-\infty}^{\infty} g^2 f d\omega \right\}^2.$$

3. Determination of the Optimal Amplifier-Filter.

The Eqs. (8) or (8.1) exhibit the dependence of the threshold of detection,  $V_m$ , on the transfer admittance  $|A|^2$  of the amplifier-filter system. The mathematical problem of determining that value of  $|A|^2$  which makes  $V_m$  a minimum is formally simple, but the infinite limits of integration in Eq. (8.1) introduce certain pitfalls. To eliminate them, one may arbitrarily assume that  $|A|^2$  (and therefore  $f$  also) is zero when  $|\omega| > \Omega$ .

Then Eq. (8.1) becomes

$$(V_m^4 T / 8\pi) = \int_{-\Omega}^{\Omega} g^2 f^2 d\omega / \left\{ \int_{-\Omega}^{\Omega} g^2 f d\omega \right\}^2. \quad (8.2)$$

The problem is, given  $g(\omega)$ , to find that  $f(\omega)$  which minimizes the right side of this equation. This problem can be solved by noting that if  $k$  is any real number independent of  $\omega$ ,

$$\int_{-\Omega}^{\Omega} g^2 (f - k)^2 d\omega \geq 0, \quad (10)$$

and that the equality is obtained only when  $f(\omega) = k$  for all  $|\omega| \leq \Omega$ . By giving  $k$  the value

$$k = \int_{-\Omega}^{\Omega} g^2 f d\omega / \int_{-\Omega}^{\Omega} g^2 d\omega, \quad (11)$$

the inequality becomes

$$\int_{-\Omega}^{\Omega} g^2 f^2 d\omega - \left\{ \int_{-\Omega}^{\Omega} g^2 f d\omega \right\}^2 / \int_{-\Omega}^{\Omega} g^2 d\omega \geq 0, \quad (10.1)$$

and the equality is obtained only when  $f(\omega)$  is constant. Combining Eq. (8.2) and (10.1), it is seen that

$$V_m^4 \geq V_o^4 = 8\pi / \left\{ T \int_{-\Omega}^{\Omega} g^2 d\omega \right\}, \quad (12)$$

and that this lower bound is attained when the amplifier filter has the optimal admittance

$$\begin{aligned} |A|^2 &= k S_u(\omega) / [S_b(\omega)]^2, \quad |\omega| < \Omega \\ &= 0, \quad |\omega| > \Omega. \end{aligned} \quad (13)$$

The other quantities of interest are, under these optimal conditions,

$$S_u = k (V^2 g^2 + g) = V^2 S_u' + S_b', \quad (3.2)$$

$$2\pi y = k \int_{-\Omega}^{\Omega} g d\omega = \int_{-\Omega}^{\Omega} S_b' d\omega, \quad (5.2)$$

$$2\pi (x-y) = k V^2 \int_{-\Omega}^{\Omega} g^2 d\omega = V^2 \int_{-\Omega}^{\Omega} S_u' d\omega, \quad (6.2)$$

and Eq. (12) is valid when

$$\begin{aligned} \tau/2\pi \gg \left\{ \int_{-\Omega}^{\Omega} g^3 d\omega \right\}^2 / \left\{ \int_{-\Omega}^{\Omega} g^2 d\omega \right\}^3, \\ \tau/4\pi \gg (1/2) \int_{-\Omega}^{\Omega} g^4 d\omega / \left\{ \int_{-\Omega}^{\Omega} g^2 d\omega \right\}^2. \end{aligned} \tag{9.2}$$

The mathematical difficulties mentioned above arise when  $\Omega \rightarrow \infty$ ; it is then possible that some of the integrals in these equations become infinite. In such cases, the difficulty can usually be resolved by considering the physics of the situation. This will become clearer after several examples have been examined in detail.



4. Example I. Signal and Background have the Same Spectrum.

One of the arguments formerly urged against passive sonar, was that the signals involuntarily emitted by the targets were so nearly similar to the background that almost their only difference is "that one is wanted, the other not." The implication was that this rendered detection very difficult. It is interesting to examine the merits of this argument in terms of the present analysis; let

$$\begin{aligned} S_v &= S(\omega), \\ S_b &= B^2 S(\omega). \end{aligned} \tag{16}$$

Then

$$g = 1/B^2 \tag{16.1}$$

and, if  $k = B^4$ , the optimal filter has the characteristic

$$\begin{aligned} |A_o|^2 &= 1/S(\omega), & |\omega| < \Omega; \\ &= 0, & |\omega| > \Omega. \end{aligned} \tag{16.2}$$

This supplies the rectifier with the signal and noise voltages whose spectra are

$$\begin{aligned} V^2 S'_v &= V^2, \\ S'_b &= B^2 \end{aligned} \quad |\omega| < \Omega. \tag{16.3}$$

Hence the filter equalizes the original spectra until they are "white," with the arbitrary cut-off at the frequency  $\Omega$ .

The Eq. (12) yields the optimal threshold

$$V_o^4 = 4\pi B^4 / \Omega T, \quad (16.4)$$

which is valid when  $\Omega T \gg 8\pi$ . The power supplied to the rectifier during the "on" and "off" periods of the measurement is determined by Eqs. (5.2) and (6.2):

$$\begin{aligned} y &= B^2 \Omega / \pi, \\ x - y &= V^2 \Omega / \pi. \end{aligned} \quad (16.5)$$

Consequently, the detection threshold can be made as small as desired, under these circumstances; but, if the threshold is lowered by increasing the arbitrary cut-off frequency  $\Omega$ , the power requirements increase. In fact, the threshold voltage is inversely proportional to the 1/4th power of the total energy supplied to the rectifier during the on and off periods. Thus, there seems to be no theoretical foundation for the belief that similarity of signal and background spectra imposes a limitation on the detection process.

5. Example II. The Effect of Transmission Loss.

Under the simplified conditions specified by Eq. (16), it appears that the optimal detection threshold depends only on the bandwidth of the amplifier-filter, and is limited only by this and the power that can be supplied by the amplifier, and rectified. To see how departures from these over-simple conditions modify the conclusions, one may suppose that the signal spectrum differs from that of the background, but only because of selective attenuation by the medium.

Let the transmission (in fractions of the initial intensity) be  $G(\omega, \kappa)$ ,  $\kappa$  being the length of the transmission path. Then one may set

$$\begin{aligned} V^2 S_v(\omega) &= V^2 S(\omega) G(\omega, \kappa), \\ S_b(\omega) &= B^2 S(\omega), \end{aligned} \quad (17)$$

and obtains

$$g = G(\omega, \kappa) / B^2. \quad (17.1)$$

With  $k = k' B^4$ , the optimal amplifier-filter is

$$|A_b|^2 = k' G(\omega, \kappa) / S(\omega), \quad (17.2)$$

and supplies the rectifier with voltages whose spectra are

$$\begin{aligned} V^2 S_v' &= k' V^2 [G(\omega, \kappa)]^2, \\ S_b' &= k' B^2 G(\omega, \kappa). \end{aligned} \quad (17.3)$$

This optimal system therefore depends on the range to the target, and equalizes the original background spectrum until its power at any frequency is proportional to the transmission of the medium. If the integrated transmission is finite, one may let  $\Omega \rightarrow \infty$ , and obtains the optimal threshold

$$V_0^4 = (8\pi B^4 / T) / \int_{-\infty}^{\infty} G^2 d\omega. \quad (17.4)$$

To make these results more concrete, let

$$G(\omega, r) = e^{-\omega^2 \alpha r} / r^2, \quad (18)$$

which is characteristic of an infinite medium whose viscosity is the only cause of absorption. In the frequency range below 60 kc, this formula would also apply to an infinite homogeneous mass of sea water; with  $r$  in kiloyards,  $\omega / 2\pi$  in kilocycles per second, the numerical value is  $\alpha = 6 \times 10^{-5}$ . For values of  $\omega$  much greater than

$$\omega_r = 1 / (\alpha r)^{1/2} \quad (19)$$

the transmission becomes very poor. It will be seen that  $\omega_r$  takes the place of the arbitrary cut-off frequency of Section 4, and thus imposes a natural limit on the pass-band of the optimal filter.

In fact, evaluation of the integral in Eq. (17.4) yields the optimal threshold

$$V_0^4 = (2\pi)^{1/2} 8B^4 r^4 / \omega_r T, \quad (20.1)$$

while the optimal filter (with  $k' = r^2$ ) becomes

$$|A_0|^2 = e^{-\alpha \omega^2 r} / S(\omega) \quad (20.2)$$

and

$$V^2 S_w' = V^2 e^{-2\alpha \omega^2 r} / r^2$$
$$S_b' = B^2 e^{-\alpha \omega^2 r} \quad (20.3)$$

Consequently the power required by the rectifier is given by

$$y = B^2 \omega_n / 2^{1/2},$$
$$x - y = V^2 \omega_n / 2 r^2. \quad (20.4)$$

These equations are valid when

$$\omega_n T \gg 7.5. \quad (20.5)$$

(If  $\omega_n / 2\pi$  is in kilocycles per second,  $T$  must, of course, be in milliseconds.)

The values of  $\omega_n / 2\pi$ , as a function of range, are tabulated below

$r$ (kyd)	$\omega_n / 2\pi$ (kcps)
1	20.
10	6.7
100	2.0
1000	0.67

Since the optimal filter does not pass frequencies much above  $\omega_n$ , the significance of this table is clear: at the longer ranges, only the lower frequencies can contribute to detection. This conclusion has been derived only under the simplified conditions implied by Eqs. (17) and (18), but probably remains valid under actual conditions.

6. Example III. The Effect of Circuit Noise.

The noise generated by the filter-amplifier imposes another limit on the detection threshold. The circuit noise can be represented as an additional emf at the input. In the frequency range under consideration, the spectrum of this noise is "white," so that one may examine its effect by setting

$$\begin{aligned} S_v &= S(\omega), \\ S_b &= B^2 S(\omega) + N^2, \end{aligned} \quad (21)$$

the term  $B^2 S$  representing the water noise,  $N^2$  the circuit noise. Then one obtains:

$$V_o^4 = (8\pi B^4 / T) \int_{-\infty}^{\infty} [S / (S + N^2 / B^2)]^2 d\omega, \quad (21.1)$$

$$|A_o|^2 = S(\omega) / [S(\omega) + N^2 / B^2]^2, \quad (21.2)$$

$$y = (B^2 / 2\pi) \int_{-\infty}^{\infty} [S / (S + N^2 / B^2)] d\omega, \quad (21.3)$$

$$x - y = (V^2 / 2\pi) \int_{-\infty}^{\infty} [S / (S + N^2 / B^2)]^2 d\omega.$$

The inputs to the rectifier are

$$\begin{aligned} V^2 S_v' &= V^2 [S / (S + N^2 / B^2)]^2, \\ S_b' &= B^2 [S / (S + N^2 / B^2)]. \end{aligned} \quad (21.4)$$

For definiteness, the original definition,  $V^2 = \langle v^2 \rangle$  will be used, so that

$$\int_{-\infty}^{\infty} S(\omega) d\omega = 2\pi.$$

Two cases are to be distinguished:

$$A : S(\omega) < N^2/B^2 \text{ for all } \omega ;$$

$$B : S(\omega) > N^2/B^2 \text{ for some } \omega .$$

For simplicity, only the extreme of Case A will be examined,  $N^2/B^2$  being so great that  $S$  may be neglected in the denominators of the above equations. This optimal filter is then seen to be a selective attenuator that shapes the white noise spectrum until it is the same as that of the input signal. The signal is correspondingly attenuated, so that those frequencies that carry much power are emphasized relatively to those that carry little power. The smallest attainable threshold becomes

$$V_o^4 = (8\pi N^4/T) / \int_{-\infty}^{\infty} S^2 d\omega. \quad (22)$$

To have an example that is not too unrealistic, let

$$S(\omega) = 2\omega_o / (\omega^2 + \omega_o^2); \quad (23)$$

then

$$V_o^4 = 4N^4\omega_o/T. \quad (24)$$



In Case B, it is to be supposed that  $S(\omega) \rightarrow 0$  as  $|\omega| \rightarrow 0$ ; suppose further that the equation

$$S(\omega) = N^2/B^2$$

has only one positive root,  $\omega = \omega_N$ , which is very large. Then, for frequencies much less than  $\omega_N$ , the term  $N^2/B^2$  can be neglected in Eq. (21.4), and it is seen that the signal and water noise spectra are equalized until they are "white" in this range, (Cf, Section 4). For frequencies much above  $\omega_N$ , they are attenuated as in Case A.

For the spectrum of Eq. (23),

$$\omega_N^2 = (2 \omega_0 B^2/N^2) - \omega_0^2, \quad (25.1)$$

so that  $\omega_N$  increases as the circuit noise decreases. The optimal threshold is, in this case

$$V_0^4 = (4 N^4 \omega_0 / T) [1 + 2 B^2 / N^2 \omega_0]^{3/2}. \quad (25.2)$$

For very small values of  $N/B$ , this may be written

$$V_0^4 = 16 B^4 / \omega_N T, \quad (25.3)$$

so that a low circuit noise has the same effect on the optimal threshold as an arbitrary cut-off frequency  $\Omega = \pi \omega_N / 4$ .

7. Graphical Determination of the Optimal Amplifier-Filter.

The characteristic of the optimal amplifier-filter can readily be determined graphically. It is customary to plot the spectra on log-log paper ( $S$  in decibels,  $\omega$  in octaves or decades). The admittance of the optimal filter can be written

$$\log |A_o|^2 = \log S_u - 2 \log S_b + \text{const.}, \quad (26)$$

and the inputs to the rectifier are given by

$$\begin{aligned} \log S_u' &= 2(\log S_u - \log S_b) + \text{const.}, \\ \log S_b' &= (\log S_u - \log S_b) + \text{const.} \end{aligned} \quad (27)$$

The mathematical operations involved in Eqs. (26) and (27) can readily be carried out with the aid of a compass and the log-log plots of  $S_u$  and  $S_b$ .

This graphical method has been used for the construction of Figs. 2, 4, and 5. It is much simpler to discuss more elaborate problems graphically than analytically.

8. Example IV. Narrow-Band Signal.

Figure 2a represents a narrow-band signal, centered at  $\omega_0$ , and a wide-band background noise. This is shown to have a negative slope for  $\omega < \omega_N$ , much as does the ambient noise in the sea. The horizontal portion of the graph for  $\omega > \omega_N$  represents the effect of circuit noise. For definiteness, it is assumed that the side-bands of the signal have the same slope as the ambient noise.

Figure 2b shows the characteristic of the corresponding optimal amplifier-filter system. The graph suggests that the circuit could be constructed by connecting two filters in parallel: the one, a wide-band filter centered at  $\omega_N$  and having the smooth characteristic indicated by the dashed line, and the other a narrow-band filter centered at the frequency  $\omega_0$ . The combined effect of the two filters is shown by Fig. 2c. The wide-band filter equalizes the portion of the signal (and noise) between  $\omega_0$  and  $\omega_N$ , and suppresses the noise at both higher and lower frequencies. The narrow-band filter emphasizes both signal and noise in the region near  $\omega_0$ , where the signal:noise ratio is very favorable.

The question of the relative importance of the two filters naturally arises, for either alone would enable the signal to be detected. The problem is to determine their relative effectiveness in lowering the threshold. This cannot be done by inspection of Fig. 2c, since the threshold is inversely proportional to

$$\int_{-\infty}^{\infty} S'_v d\omega. \quad (28)$$

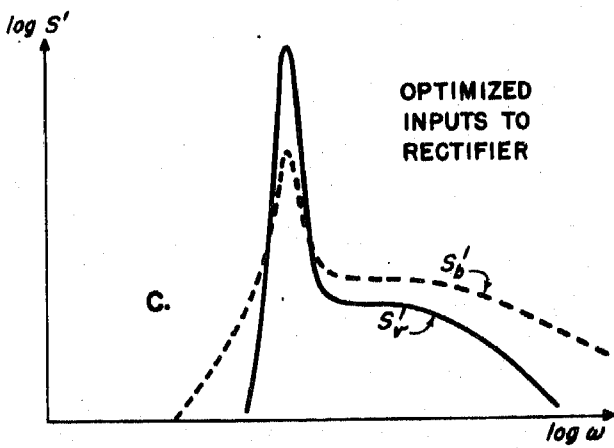
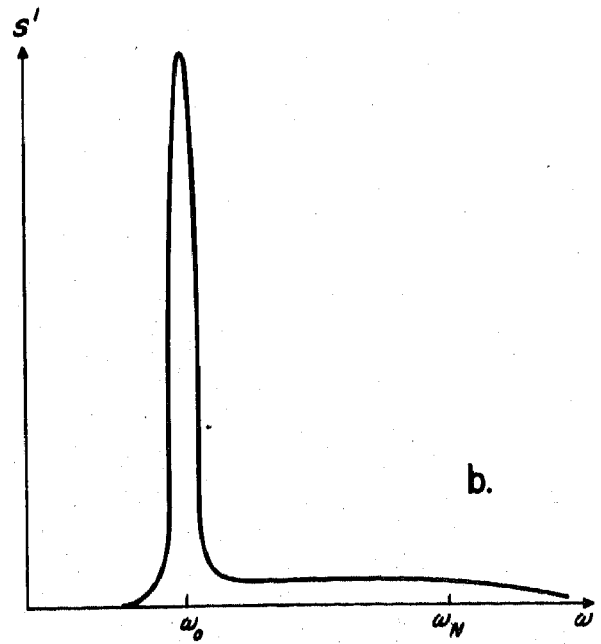
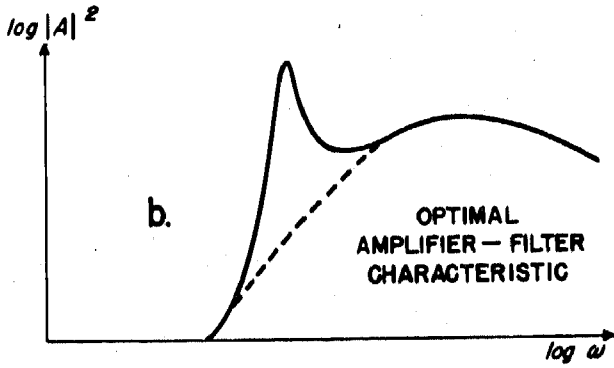
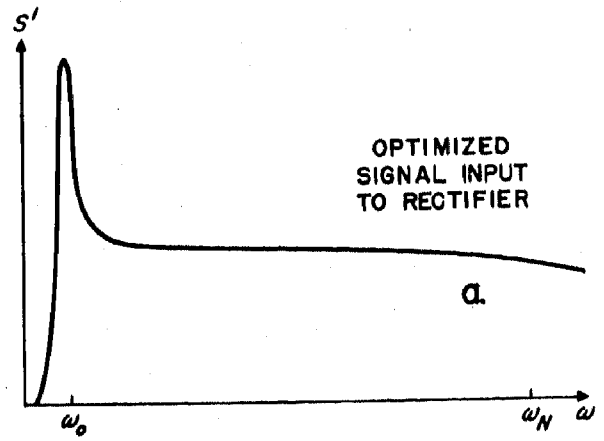
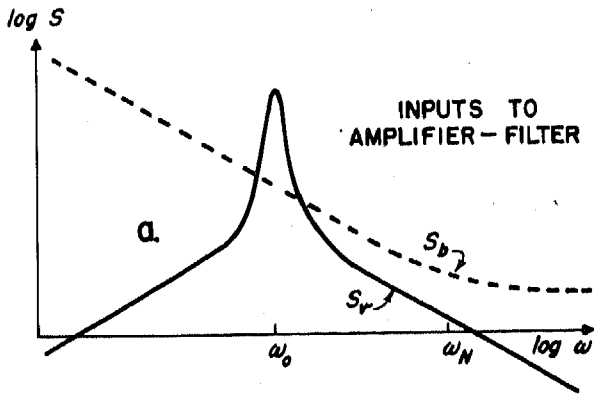


Figure 2.

Figure 3.

Consequently, the graph of Fig. 2c must be replotted on arithmetic scales. This replotting in turn depends on the units of  $\log S$  and  $\log \omega$  on Fig. 2c. Since these are not specified on the figure, many arithmetic graphs are possible.

Figure 3a shows one such graph. The units have been deliberately chosen to produce a large contribution from the equalized upper side-band, which is the output of the wide-band filter. In such cases, the narrow-band filter could be omitted without appreciably raising the detection threshold above its minimal value. Figure 3b represents the result of another choice of scale; in such cases, the peak and the upper side-band have approximately equal areas, and the omission of either filter would rise the threshold several decibels. While it is difficult to represent them graphically, there are other cases in which the power in the equalized side-band is negligible, and the wide-band filter serves no useful purpose.

This example will serve to indicate the manner in which any empirically specified pair of spectra may be analyzed quantitatively. The remaining examples will not be discussed in as much detail.

9. Example V. Wide-Band Signal with Superposed Peak.

In many cases, the emission of a narrow-band signal is accompanied by a wider band at a level that is not so low as to be negligible. This is illustrated in Fig. 4a. The optimal amplifier-filter in this case again consists of the wide- and narrow-band components, the former equalizing the wide-band components below the noise cut-off  $\omega_N$ , while the latter exploits the peak in the signal:noise ratio. Again, the relative importance of the two components requires quantitative investigation.

If the background also contains a peak, the optimal filter consists of three components, one of which rejects the noise peak. This is illustrated in Fig. 5. It should be noted that the rejection of the noise peak is weighted more heavily than is the admission of the signal peak. If the two peaks are equal, on the logarithmic scale, the optimal filter converts the noise peak into a minimum equal in magnitude to the peak which it produces under the signal. This is clearly illustrated in Fig. 5c.

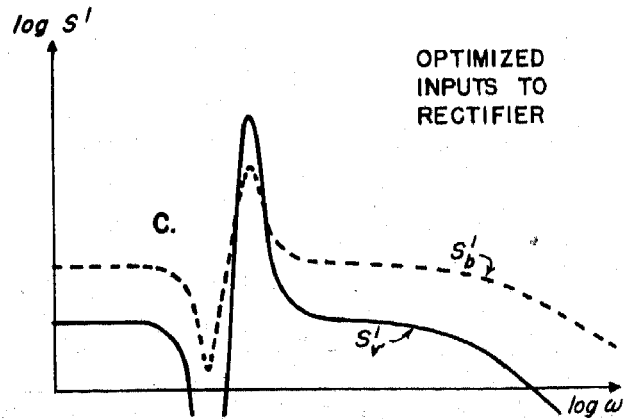
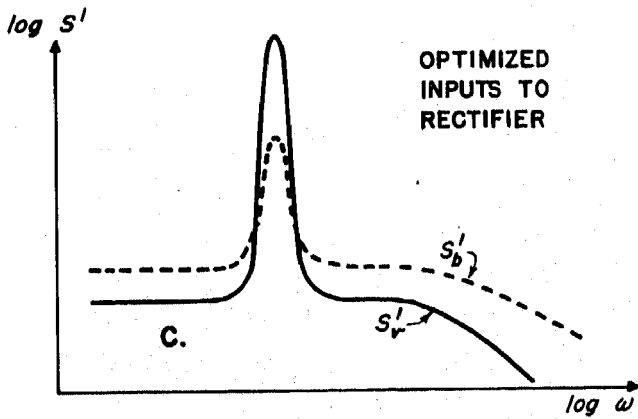
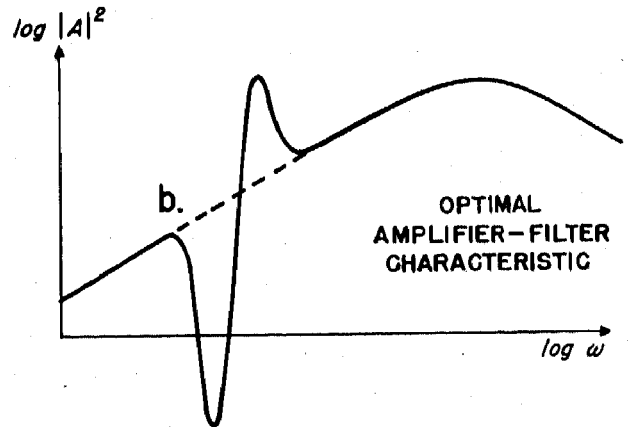
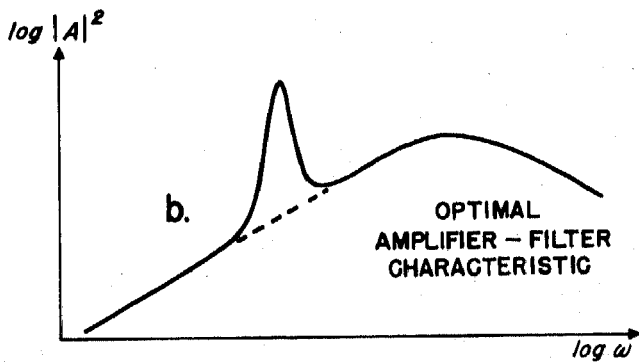
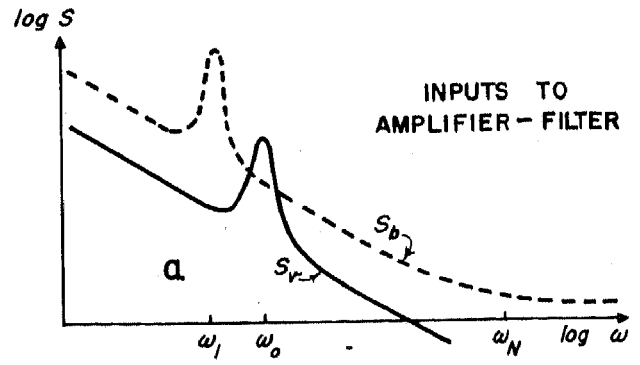
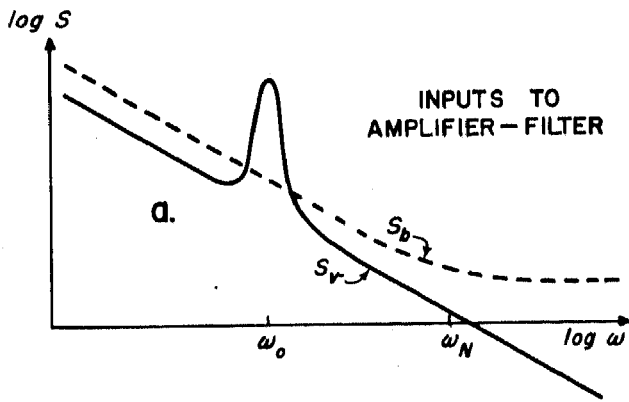


Figure 4.

Figure 5.

10. General Considerations Concerning the Function of the Filter in a Detection System.

The theory of detection used in the present discussion appears to differ considerably from current ideas used in the design of detection systems. This is most clearly indicated by considering the function of the filter-amplifier in the system. It is generally agreed, of course, that its purpose is to lower the detection threshold, but divergences of opinion occur in regard to the manner of its functioning. Three different concepts are distinguishable, and will be discussed separately. The differences between these concepts are most apparent when the signal and background have identical spectra. As has been remarked, this extreme case is often approached in practice.

Concept I. The filter is to improve the fidelity with which the signal is presented to a listener's ear. The optimal filter for the elimination of distortion by noise has been discussed in Refs. 2, 3, 4, 5 and 6. In terms of the present notation, the distortion of the input,  $Vu + b$ , considered as a replica of  $u$ , is

$$\Delta = \langle b^2 \rangle / (V^2 + \langle b^2 \rangle) \quad (29)$$

provided  $\langle u^2 \rangle = 1$ . If the output of the filter is  $u$ , its distortion will be least when the admittance is

$$A(\omega) = k V^2 S_u(\omega) / [V^2 S_u(\omega) + S_b(\omega)] \quad (30)$$



and will then be

$$\delta = \int_{-\infty}^{\infty} \frac{S_v S_b}{V^2 S_v + S_b} d\omega \bigg/ \int_{-\infty}^{\infty} S_v d\omega \quad (31)$$

If these equations are applied to a signal whose spectrum is identical with that of the background:

$$\begin{aligned} S_v &= S(\omega), \\ S_b &= B^2 S(\omega), \end{aligned} \quad (16)$$

one easily obtains

$$\Delta = \delta = B^2 / (V^2 + B^2)$$

and

$$A(\omega) = \text{const.}$$

Consequently, under these conditions, no filter can reduce the distortion of the signal below its value at the terminals of the hydrophone, and the best "filter" is a simple resistance.

According to this conception of the filter's function, identity of signal and noise spectra results in complete frustration: nothing whatever can be done to lower the threshold (defined by  $\delta=1$ ) below the value  $V_m^2 = B^2$ .

Concept II. The filter is to improve the over-all signal:noise ratio.

At the hydrophone terminals, the signal:noise ratio is

$$\sigma = V^2 \int_{-\infty}^{\infty} S_v(\omega) d\omega / \int_{-\infty}^{\infty} S_b(\omega) d\omega, \quad (32)$$

while at the terminals of the filter, it is

$$\sigma = V^2 \int_{-\infty}^{\infty} S_v |A|^2 d\omega / \int_{-\infty}^{\infty} S_b |A|^2 d\omega. \quad (33)$$

To discuss the problem of making  $\sigma$  as large as possible, let

$$g = S_b |A|^2,$$

$$f = S_v / S_b,$$

so that

$$\sigma = V^2 \int_{-\infty}^{\infty} g f d\omega / \int_{-\infty}^{\infty} g d\omega. \quad (33.1)$$

It is easily seen that in general this problem has no solution, even though  $\sigma$  always has a smallest upper bound, which cannot be exceeded for any function  $g$ . For, Eq. (10.1) may be rearranged into

$$\int_{-\infty}^{\infty} g f^2 d\omega / \int_{-\infty}^{\infty} g f d\omega \geq \int_{-\infty}^{\infty} g f d\omega / \int_{-\infty}^{\infty} g d\omega. \quad (10.2)$$

and the equality is obtained only when  $f(\omega) = \text{const.}$  If  $f \neq \text{const.}$ , suppose that  $g = g_0$  makes  $\sigma$  a maximum; then  $g = g_0 f$  will, by Eq. (10.2), result in a larger value of  $\sigma$ ; this is contrary to the assumption, which must therefore be false. One can, however, construct the infinite sequence of functions

$$g_0 = S_b, \quad g_1 = g_0 f, \quad g_2 = g_0 f^2, \quad \dots$$

and calculate the corresponding  $\sigma$ 's:  $\sigma_0 = \Delta$ ,  $\sigma_1$ ,  $\sigma_2$ , .... These are, by Eq. (10.2), an increasing sequence of positive numbers, and, unless they become infinite, must have an upper limit. This is, in turn the smallest upper bound for  $\sigma$ , and can be shown to be the maximum value assumed by  $f$  for any  $\omega$ .

It follows, therefore, that unless  $S_v(\omega)/S_b(\omega)$  is a constant, one can always find a function,  $|A(\omega)|^2$ , such that  $\sigma > \Delta$ . But, when  $S_b(\omega) = B^2 S_v(\omega)$ , one easily sees that, for all functions  $g$ ,

$$\sigma = \Delta = V^2/B^2$$

Consequently, when signal and noise have identical spectra, nothing whatever can be done to improve the signal:noise ratio, and again, the threshold defined by  $\sigma = 1$  is  $V_m^2 = B^2$ .

Concept III. The filter is to reduce the error in an "on-off" measurement of the signal intensity. This is the basic concept of the preceding sections, and has been shown to yield positive conclusions.

ACKNOWLEDGMENT

The writer is indebted to numerous colleagues for much stimulating and helpful discussion of these matters. I wish particularly to thank Professor Arnold T. Nordsieck and Mr. Tom A. Magness for their contributions, and Dr. Leonard N. Liebermann for critically reading the manuscript.

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