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Title

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Permalink

Journal

ISSN

Authors

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Publication Date

DOI

Supplemental Material

Copyright Information

Peer reviewed

INTERSECTING PRINCIPAL BRUHAT IDEALS AND GRADES OF SIMPLE MODULES

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Abstract. We prove that the grades of simple modules indexed by boolean permutations, over the incidence algebra of the symmetric group with respect to the Bruhat order, are given by Lusztig's **a**-function. Our arguments are combinatorial, and include a description of the intersection of two principal order ideals when at least one permutation is boolean. An important object in our work is a reduced word written as minimally many runs of consecutive integers, and one step of our argument shows that this minimal quantity is equal to the length of the second row in the permutation's shape under the Robinson–Schensted correspondence. We also prove that a simple module over the above-mentioned incidence algebra is perfect if and only if its index is the longest element of a parabolic subgroup.

Keywords. Incidence algebra, grade, boolean permutation, a-function, principal ideal, symmetric group

Mathematics Subject Classifications. 20F55, 06A07, 05E15

1. Introduction and description of results

Homological invariants are very helpful tools for understanding both structure and properties of algebraic objects. The most common such invariants used in representation theory of finite dimensional algebras are projective and injective dimensions that describe the lengths of the minimal projective resolution and injective coresolution of a module, respectively. A slightly less common such invariant is the *grade* of a module; that is, the minimal degree of a non-vanishing extension to a projective module. The latter invariant is important in the theory of Auslander regular algebras, see [12].

Incidence algebras of finite posets are important examples of finite dimensional algebras. The main result of [12] asserts that the incidence algebra of a finite lattice is Auslander regular if and only if the lattice is distributive.

Finite Weyl groups play an important role in modern representation theory. They come equipped with a natural partial order called the *Bruhat order*. Unfortunately, with the exception of a handful of degenerate cases, the Bruhat order on a Weyl group is not a lattice. In March 2021, Rene Marczinzik gave a talk at Uppsala Algebra Seminar in which he addressed the problem of Auslander regularity of incidence algebras of Weyl groups with respect to the Bruhat order (to the best of our knowledge, the problem is still open for symmetric groups in the general case). In connection to this, he presented results of computer calculations of grades of simple modules over the incidence algebras of the symmetric group in small ranks. From these lists, one could observe that the grades of simple modules are often (but not always) given by Lusztig's a-function from [18]. In the case of the symemtric group (i.e. in type A), Lusztig's a-function is uniquely determined by the properties that it is constant on all two-sided Kazhdan–Lusztig cells and coincides with the usual length function on the longest elements in all parabolic subgroups. In several contexts, see [14, 20, 21], this function describes homological invariants of algebraic objects naturally indexed by the elements of the symmetric group (or, more generally, of a finite Weyl group).

The main result of the present paper is the following theorem.

Theorem 1.1. *The grades of simple modules indexed by boolean permutations, over the incidence algebra of the symmetric group with respect to the Bruhat order, are given by Lusztig's a-function.*

We note that the original definition of the a-function reflects some subtle numerical properties of the multiplication of the elements in the Kazhdan–Lusztig bases of the Hecke algebra of a Coxeter group. We do not see any immediate connection between the Kazhdan–Lusztig bases of the Hecke algebra and the incidence algebra of the symmetric group with respect to the Bruhat order. Therefore appearance of the a-function in Theorem 1.1 is rather mysterious.

Our proof of this result is combinatorial. Projective resolutions of simple modules over the incidence algebras of Weyl groups can be constructed using the BGG complex from [1] (that is, the singular homology complex). We use the Serre functor to relate the grade of the simple module L_v, where v is a permutation, to the homology of the complex obtained by restricting the BGG complex to the intersection $B(v) \cap B(w)$ of two principal ideals in the symmetric group, where w is an arbitrary permutation. For boolean v, we describe $B(v) \cap B(w)$ in Proposition 3.7 with a more precise version in Corollary 3.8 under the additional assumption that w is also boolean.

Using this explicit description, we proceed with combinatorial analysis of the restricted BGG complex. In fact, we show that this restricted BGG complex is either exact or has exactly one non-zero homology which, moreover, is one-dimensional. For a fixed v, the extreme degree in which such non-zero homology can appear is given by a combinatorial invariant of v that we call *the number of runs in v*, introduced in Subsection 5.2. The connection between the homology and the number of runs is established in Theorem 5.14.

It is an easy combinatorial exercise to show that the number of runs for a boolean permutation

coincides with Lusztig's **a**-function, and this is presented in Section 6. This is, essentially, what one needs to prove Theorem 1.1.

The paper is organized as follows. In Section 2, we collected some basics on the Bruhat order and boolean permutations. In Section 3, we study combinatorics of intersections $B(v) \cap B(w)$, for boolean v, and show how they can be determined either from reduced words or from the permutations' one-line notations. In Section 4, we describe in detail the algebraic motivation and setup of the problem we study. Section 5 contains combinatorial analysis of the homology of the BGG complex restricted to $B(v) \cap B(w)$. The crucial results of that section give a method for deleting letters from a reduced word without losing the boolean elements in an order ideal, and description of a permutation $w := w(v)$ for which the intersection complex has the desired homology. Finally, in Section 6 we combine all of the pieces necessary for the proof of Theorem 1.1. In the last section of the paper we briefly address what little is known for non-boolean permutations. In particular, we show that Lusztig's **a**-function gives the grade of the simple module indexed by the longest elements of a parabolic subgroup. From this we deduce that a simple module is perfect (in the sense that its grade coincides with its projective dimension) if and only if the index of this module is the longest element of a parabolic subgroup.

2. Bruhat order and boolean elements

The symmetric group $\mathfrak{S}_n$ of permutations of $[1, n] := \{1, 2, \ldots, n\}$ is a Coxeter group with the natural distinguished set of Coxeter generators given by the simple reflections $\{\sigma_i := (i, i+1) : i \in [1, n-1]\}$. A *reduced decomposition* of a permutation w is a product $w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ such that ℓ is minimal (in which case it is called the *length* of w). To save notation, we can equivalently consider *reduced words* of a permutation by looking only at the subscripts in a reduced decomposition. In this paper, we will let $R(w)$ denote the set of reduced words of a permutation w. Because both permutations and reduced words can be represented by strings of integers, we will write $[s]$ to indicate that a string s represents a reduced word. We think of permutations as maps, and we compose maps from right to left.

Example 2.1. For the permutation with one-line notation $4132 \in \mathfrak{S}_4$, we have

$$R(4132) = \{[3213], [3231], [2321]\}.$$

Reduced words represent products of simple reflections, so we can use them interchangeably with the permutations they represent. For example, we can write

$$4132 = [3213] = [3231] = [2321].$$

It is well-known that any two reduced words for a given permutation are related by a sequence of commutation and braid moves [19, 33], as we can see in the previous example.

The Bruhat order gives a poset structure to the symmetric group, and it can be defined in terms of reduced words.

Theorem 2.2 ([2, Theorem 2.2.2])**.** *Let u, w be permutations, and $[s] \in R(w)$. Then $u \leqslant w$ in the Bruhat order if and only if a subword of $[s]$ is a reduced word for u.*

Various structural aspects of this poset have been studied, related to its principal order ideals and intervals (see, for example, [2, 3, 7, 9, 10, 27, 28, 31]). Despite this interest and literature, there has been very little attention paid to the intersection of principal order ideals in this poset.

Definition 2.3. For a permutation $w \in \mathfrak{S}_n$, write $B(w)$ for the principal order ideal of w in the Bruhat order.

As studied by Ragnarsson and the second author [23, 24], and Hultman and Vorwerk in the case of involutions [11], the so-called *boolean* elements of the symmetric group have particularly interesting properties.

Definition 2.4. A permutation v is *boolean* if its principal order ideal $B(v)$ is isomorphic to a boolean algebra.

Although boolean elements can be defined analogously for any Coxeter group, we are focused on permutations in this work. As shown in [27], boolean permutations can be characterized in several ways.

Theorem 2.5. *The following statements are equivalent:*

- *the permutation v is boolean,*

- *the permutation v avoids the patterns 321 and 3412, and*

- *reduced words for the permutation v contain no repeated letters.*

In this work we will consider intersections of principal order ideals $B(v) \cap B(w)$, when v is boolean. First, we will describe the elements of this intersection, and then we will look at its topology.

3. Intersection ideals

3.1. Orientation and intersecting ideals

Throughout this section, let $v \in \mathfrak{S}_n$ be a boolean permutation.

Definition 3.1. The *support* of a permutation w is the set of distinct letters appearing in its reduced words. This will be denoted $\mathsf{supp}(w)$.

Thus a permutation is boolean if and only if its length is equal to the size of its support. For an arbitrary permutation w, we can make the following observation about how $\mathsf{supp}(w)$ might impact the poset $B(w)$.

Let x be a word and I a subset of letters appearing in x. We will denote by x_I the subword of x consisting of all letters from I.

Lemma 3.2. *Suppose that w is a permutation with $\mathsf{supp}(w) = X \sqcup Y$, such that either*

- *x and y commute for all $x \in X$ and $y \in Y$, in which case let $[s] \in R(w)$ be any reduced word; or*

- *there is exactly one pair of noncommuting letters* $(x_0, y_0) \in X \times Y$, *and there exists* $[s] \in R(w)$ *in which all appearances of* x_0 *are to the left of all appearances of* y_0.

Then both $[s_X]$ *and* $[s_Y]$ *are reduced words, and* $B(w)$ *can be written as a direct product:*

$$B(w) \cong B([s_X]) \times B([s_Y]).$$

Proof. The $[s] \in B(w)$ described in the statement of the lemma can be transformed via commutations into $[s_X s_Y] \in B(w)$. The result follows. $\square$

Theorem 2.5 means that Lemma 3.2 can be applied to any boolean permutation.

The goal of this section is to describe $B(v) \cap B(w)$ for arbitrary permutations w. Because v is boolean, the ideal $B(v)$ is determined by:

- the support of v,

- the pairs of noncommuting (i.e., consecutive) letters appearing in the support, and

- the order in which noncommuting letters appear in elements of $R(v)$.

Note that this last item is well-defined because all letters are distinct. Therefore, if k appears to the left of $k+1$ in one element of $R(v)$, then in fact k appears to the left of $k+1$ in all elements of $R(v)$.

When looking at the principal order ideal of a boolean permutation v, we might want to consider the permutations covering an element u in that ideal. By Theorems 2.2 and 2.5 we can think of such an element as being obtained from some $[s] \in R(u)$ by inserting a letter k in such a way as to be consistent with elements of $R(v)$. The lack of repeated letters among those elements means that there is no ambiguity about how to insert k. When such an element exists, we will write it as

$$u \curlywedge \sigma_k.$$

The support of a permutation can be read off from any of its reduced words. It can also be detected from the one-line notation of the permutation, as described in the following lemma.

Lemma 3.3 (cf. [29, Lemma 2.8]). *For any* $w \in \mathfrak{S}_n$, *the following statements are equivalent:*

- $k \in \mathsf{supp}(w)$,

- $\{w(1), \ldots, w(k)\} \neq \{1, \ldots, k\}$, *and*

- $\{w(k+1), \ldots, w(n)\} \neq \{k+1, \ldots, n\}$.

Several facts about $B(v) \cap B(w)$ follow directly from Theorem 2.2.

Lemma 3.4. *Consider* $v, w \in \mathfrak{S}_n$, *where* v *is boolean.*

(a) *The length 1 elements of* $B(v) \cap B(w)$ *are* $\mathsf{supp}(v) \cap \mathsf{supp}(w)$.

(b) *Each element of* $B(v) \cap B(w)$ *is itself a boolean permutation.*

(c) The intersection $B(v) \cap B(w)$ is an order ideal.

Part (c) of Lemma 3.4 means that we will understand $B(v) \cap B(w)$ once we can describe its maximal elements. To do this, we must understand which subsets of $\mathsf{supp}(v) \cap \mathsf{supp}(w)$ will describe an element of $B(v) \cap B(w)$. The only concern arises from consecutive letters in $\mathsf{supp}(v) \cap \mathsf{supp}(w)$. If there are such letters, then they appear in a particular order in all elements of $R(v)$. If they can appear in the same order in an element of $R(w)$, then they can appear together in an element of $B(v) \cap B(w)$. If they never appear in that same order, in any elements of $R(w)$, then these two letters cannot appear together in any element of $B(v) \cap B(w)$.

Definition 3.5. Consider a permutation w with $k, k + 1 \in \mathsf{supp}(w)$. If all appearances of k are to the left of all appearances of $k + 1$ in reduced words for w, then k and $k + 1$ have *increasing orientation* in w. If all appearances of k are to the right of all appearances of $k + 1$, then they have *decreasing orientation*. Otherwise, their orientation is *interlaced*. Two orientations *match* unless one is increasing and the other is decreasing.

3.2. Maximal selfish subsets

For a positive integer k, let us consider the set of integers $[1, k]$. Denote by $\mathscr{Q}_k$ the set of all subsets $X \subset [1, k]$ that are maximal with respect to inclusions and that have the following property, which we call *selfishness*:

- if $i \in X$, then $i \pm 1 \notin X$.

Here is the list of $\mathscr{Q}_k$, for $k = 1, 2, 3, 4, 5$:

$$\begin{aligned}
\mathscr{Q}_1 &= \big\{\{1\}\big\}, \\
\mathscr{Q}_2 &= \big\{\{1\}, \{2\}\big\}, \\
\mathscr{Q}_3 &= \big\{\{1, 3\}, \{2\}\big\}, \\
\mathscr{Q}_4 &= \big\{\{1, 3\}, \{2, 4\}, \{1, 4\}\big\}, \text{ and} \\
\mathscr{Q}_5 &= \big\{\{1, 3, 5\}, \{2, 5\}, \{2, 4\}, \{1, 4\}\big\}.
\end{aligned}$$

We denote by $\mathscr{Q}_k'$ the set of all $X \in \mathscr{Q}_k$ such that $k \in X$ and we set $\mathscr{Q}_k'' := \mathscr{Q}_k \setminus \mathscr{Q}_k'$.

Proposition 3.6.

(a) *For $k > 2$, the map $X \mapsto X \setminus \{k\}$ is a bijection from $\mathscr{Q}_k'$ to $\mathscr{Q}_{k-2}$.*

(b) *For $k > 3$, the map $X \mapsto X \setminus \{k - 1\}$ is a bijection from $\mathscr{Q}_k''$ to $\mathscr{Q}_{k-3}$.*

(c) *For $k > 3$, we have $|\mathscr{Q}_k| = |\mathscr{Q}_{k-2}| + |\mathscr{Q}_{k-3}|$.*

Proof. If $k \in X$, then $k - 1 \notin X$ due to selfishness. Therefore $Y := X \setminus \{k\}$ is a selfish subset of $[1, k - 2]$. If Y were not maximal, we would be able to add to it some element $i \in [1, k - 2]$ preserving its selfishness. Since $k - 1 \notin X$, the subset $X \cup \{i\}$ would also be selfish, contradicting the maximality of X. Therefore $Y \in \mathscr{Q}_{k-2}$.

Conversely, given $Y \in \mathscr{Q}_{k-2}$, the set $X := Y \cup \{k\}$ is, clearly, selfish. By the maximality of Y, we cannot add to X any $i \in [1, k-2]$ without violating selfishness. Clearly, we cannot add $k-1$ either since $k \in X$. Therefore $X \in \mathscr{Q}'_k$. This proves Claim (a).

Note that $X \in \mathscr{Q}''_k$ implies $k-1 \in X$, for otherwise $X \cup \{k\}$ would be selfish, contradicting the maximality of X. Therefore $\mathscr{Q}''_k = \mathscr{Q}'_{k-1}$ and Claim (b) follows from Claim (a) applied to $\mathscr{Q}'_{k-1}$. Claim (c) follows from Claims (a) and (b). $\qquad\square$

From Proposition 3.6(c), it follows that the sequence $\{|\mathscr{Q}_k| \,:\, k \geqslant 1\}$ is the (suitably offsetted) Padovan sequence [22, A000931].

The above concept has a natural generalization to an arbitrary finite subset $\mathcal{U}$ of $\mathbb{Z}_{>0}$ (a *universe*). We can consider selfish subsets of such a $\mathcal{U}$, maximal with respect to inclusions. We denote the set of all such subsets by $\mathscr{Q}(\mathcal{U})$. The set $\mathcal{U}$ has a unique decomposition

$$\mathcal{U} = \coprod_i \mathcal{U}_i$$

of minimal length into a disjoint union of interval subsets (i.e., each $\mathcal{U}_i$ is of the form $[p, p+q]$). The minimality of the length means that $\mathcal{U}_i \cup \mathcal{U}_j$ is not an interval subset for any $i \neq j$.

It is clear that maximal selfish subsets of $\mathcal{U}$ are just the unions of maximal selfish subsets of the individual $\mathcal{U}_i$'s, and the maximal selfish subsets in each $\mathcal{U}_i$ are completely described by Proposition 3.6.

3.3. Intersections with principal ideals of boolean elements

Our next step is to describe $B(v) \cap B(w)$, for an arbitrary permutation w. As initially presented, this will require understanding properties about the reduced words for v and w. However, following the theorem, we will show how it can also be determined from the one-line notations for v and w.

Let $v, w \in \mathfrak{S}_n$ be such that v is boolean and fix some $[s] \in R(v)$. Denote by $\mathcal{W}(v, w)$ the set of all minimal subwords of $[s]$ of the form $\big[i(i+1) \cdots (i+j)\big]$ or $\big[(i+j) \cdots (i+1)i\big]$ with the following properties:

- if $\big[i(i+1) \cdots (i+j)\big]$ is a subword of $[s]$, then $\big[i(i+1) \cdots (i+j)\big] \not\leqslant w$ while both $\big[i(i+1) \cdots (i+j-1)\big] \leqslant w$ and $\big[(i+1)(i+2) \cdots (i+j)\big] \leqslant w$;

- if $\big[(i+j) \cdots (i+1)i\big]$ is a subword of $[s]$, then $\big[(i+j) \cdots (i+1)i\big] \not\leqslant w$ while both $\big[(i+j-1) \cdots (i+1)i\big] \leqslant w$ and $\big[(i+j) \cdots (i+2)(i+1)\big] \leqslant w$.

Further, denote by $\mathcal{W}(v, w)^\uparrow$ the set of all maximal subwords of $[s]$ which do not have any of the elements in $\mathcal{W}(v, w)$ as subwords. In other words, $\mathcal{W}(v, w)^\uparrow$ is the set of maximal elements in the complement to the filter generated by $\mathcal{W}(v, w)$ inside $B(v)$. Note that $\mathcal{W}(v, w)$ contains all simple reflections from $\mathsf{supp}(v) \setminus \mathsf{supp}(w)$.

For example, if $v = [321]$ and $w = [2132]$, then we have $\mathcal{W}(v, w) = \{[321]\}$ and, consequently, $\mathcal{W}(v, w)^\uparrow = \{[21], [31], [32]\}$. Similarly, if $v = [32145]$ and $w = [4521324]$, then $\mathcal{W}(v, w) = \{[321], [345]\}$ and $\mathcal{W}(v, w)^\uparrow = \{[2145], [314], [315], [324], [325]\}$.

Proposition 3.7. *For $v, w \in \mathfrak{S}_n$ with v boolean and $[s] \in R(v)$, the set $\mathcal{W}(v, w)^{\uparrow}$ is exactly the set of all maximal elements in $B(v) \cap B(w)$.*

Proof. Each minimal element in $\mathfrak{S}_n \setminus B(w)$ is, by definition, join irreducible. It is well-known, see [17], that join irreducible elements in $\mathfrak{S}_n$ are exactly the elements with unique left descent and unique right descent (i.e. the so-called bigrassmannian elements, see also [15, 16] for more details). From the classification of all bigrassmannian elements in $\mathfrak{S}_n$ (see, for example, [15, Figure 7]), it follows that the boolean bigrassmannian elements are exactly the elements of the form $\big[i(i+1)\cdots(i+j)\big]$ or $\big[(i+j)\cdots(i+1)i\big]$.

   This implies that the minimal elements in the complement to $B(v) \cap B(w)$ in $B(v)$ are all of the form $\big[i(i+1)\cdots(i+j)\big]$ or $\big[(i+j)\cdots(i+1)i\big]$, for some i and j. Now the claim of the proposition follows directly from the definitions of the sets $\mathcal{W}(v, w)$ and $\mathcal{W}(v, w)^{\uparrow}$.   □

   In the special case of the intersection of two boolean principal ideals, we can make the statement of Theorem 3.7 more precise. Fix permutations $v, w \in \mathfrak{S}_n$ such that v is boolean. Choose any element $[s] \in R(v)$. Denote by $\mathcal{V}(v, w)$ the set of all letters k for which there is $x \in \{k \pm 1\}$ such that the orientation of the pair $\{k, x\}$ in v and w does not match.

Corollary 3.8. *For $v, w \in \mathfrak{S}_n$ with both v and w boolean and $[s] \in R(v)$, the maximal elements in the order ideal $B(v) \cap B(w)$ are exactly the subwords of $[s]$ whose support is of the form*

$$\big((\mathsf{supp}(v) \cap \mathsf{supp}(w)) \setminus \mathcal{V}(v, w)\big) \cup X, \quad \textit{for some} \quad X \in \mathscr{Q}(\mathcal{V}(v, w)). \tag{3.1}$$

Proof. For boolean w, the two conditions

$$\big[i(i+1)\cdots(i+j-1)\big] \leqslant w \text{ and } \big[(i+1)(i+2)\cdots(i+j)\big] \leqslant w, \text{ with } j > 1,$$

imply $\big[i(i+1)\cdots(i+j)\big] \leqslant w$. Similarly, the two conditions

$$\big[(i+j-1)\cdots(i+1)i\big] \leqslant w \text{ and } \big[(i+j)\cdots(i+2)(i+1)\big] \leqslant w, \text{ with } j > 1,$$

imply $\big[(i+j)\cdots(i+1)i\big] \leqslant w$. This means that the set $\mathcal{W}(v, w)$ consists of elements of the form $[i(i+1)]$ or $[(i+1)i]$. It follows directly from the definitions that Formula (3.1) describes exactly the elements of $\mathcal{W}(v, w)^{\uparrow}$. Now the claim follows from Proposition 3.7.   □

   In fact, the previous argument shows that we can use this construction whenever there are no minimal subwords having $j > 1$, using the notation at the beginning of this section.

Corollary 3.9. *For $v, w \in \mathfrak{S}_n$ with v boolean and $[s] \in R(v)$, if all elements of $\mathcal{W}(v, w)$ have $j = 1$, then the maximal elements in the order ideal $B(v) \cap B(w)$ are exactly the subwords of $[s]$ whose support is of the form*

$$\big((\mathsf{supp}(v) \cap \mathsf{supp}(w)) \setminus \mathcal{V}(v, w)\big) \cup X, \quad \textit{for some} \quad X \in \mathscr{Q}(\mathcal{V}(v, w)). \tag{3.2}$$

3.4. The same property via one-line notation

The relative order(s) of k and $k+1$ can be read off by looking at reduced words of a permutation, but it would be nice if they could also be determined directly from its one-line notation. We use the next results to show how that can be done.

Proposition 3.10. *Consider a permutation w, with $\{k, k+1\} \subseteq \mathsf{supp}(w)$. Then, exactly one of the following possibilities holds.*

(i) k *and* $k+1$ *are interlaced in* w, *meaning that* w *has a reduced word with one of the following forms:*

$$\big[\cdots k \cdots (k+1) \cdots k \cdots\big] \qquad or \qquad \big[\cdots (k+1) \cdots k \cdots (k+1) \cdots\big]$$

(ii) k *and* $k+1$ *have either increasing or decreasing orientation in* w, *meaning that* w *has a reduced word with one of the following forms:*

$$\left[\Big(\ letters\ \leqslant k\ \Big)\Big(\ letters\ \geqslant k+1\ \Big)\right] \qquad or \qquad \left[\Big(\ letters\ \geqslant k+1\ \Big)\Big(\ letters\ \leqslant k\ \Big)\right]$$

Proof. Reduced words for w contain both k and $k+1$. Suppose that some $[s] \in R(w)$ interlaces k and $k+1$. Then no sequence of commutation and braid moves can produce a word in which all appearances of k are to one side of all appearances of $k+1$, so w has no reduced words of the forms shown in (ii). Alternatively, suppose that no elements of $R(w)$ interlace k and $k+1$. Thus, in each element of $R(w)$, all appearances of k are to one side of all appearances of $k+1$. Suppose, without loss of generality, that $[s] \in R(w)$ has the form $s = \alpha\beta$, where α contains k but not $k+1$, and β contains $k+1$ but not k. Then, as in [30], we can use commutations to push all $x > k+1$ in α to the right and all $x < k$ in β to the left, in order to find an element of $R(w)$ having one of the forms depicted in (ii). $\qquad\qquad\square$

As discussed previously, these increasing or decreasing orientations restrict which elements can appear in an intersection of principal order ideals. We now describe how to identify a permutation having this property.

Theorem 3.11. *Consider a permutation $w \in \mathfrak{S}_n$, with $\{k, k+1\} \subseteq \mathsf{supp}(w)$.*

(a) *A reduced word for* w *has the form*

$$\left[\Big(\ letters\ \leqslant k\ \Big)\Big(\ letters\ \geqslant k+1\ \Big)\right]$$

(that is, k and $k+1$ have increasing orientation in w) if and only if there exists $x < k+1$ such that $\{w(1), \ldots, w(k)\} = [1, k+1] \setminus \{x\}$ and $w^{-1}(x) > k+1$.

(b) *A reduced word for* w *has the form*

$$\left[\Big(\ letters\ \geqslant k+1\ \Big)\Big(\ letters\ \leqslant k\ \Big)\right]$$

(that is, k and $k+1$ have decreasing orientation in w) if and only if there exists $y > k+1$ such that $\{w(k+2), \ldots, w(n)\} = [k+1, n] \setminus \{y\}$ and $w^{-1}(y) < k+1$.

(c) The letters k and $k + 1$ are interlaced in w if and only if there exists $i \in [1, k]$ and $j \in [k + 2, n]$ such that $w(i) > k + 1$ and $w(j) < k + 1$.

Proof. Consider part (a) of the theorem.

Suppose that w has such a reduced word. This reduced word indicates that $w = tu$, where $\mathsf{supp}(t) \subseteq [1, k]$ and $\mathsf{supp}(u) \subseteq [k + 1, n - 1]$. Moreover, because $\{k, k + 1\} \subseteq \mathsf{supp}(w)$, we must have $k \in \mathsf{supp}(t)$ and $k + 1 \in \mathsf{supp}(u)$. The permutation u fixes all $i < k + 1$. On the other hand, by Lemma 3.3, $u(k + 1) > k + 1$ and $u^{-1}(k + 1) = y > k + 1$. Similarly, the permutation t fixes all $i > k + 1$, with $t(k + 1) < k + 1$ and $t^{-1}(k + 1) = x < k + 1$. Thus in the product $w = tu$, we have

$$w(y) = tu(y) = t(k + 1) < k + 1.$$

Moreover, for all $z \neq y$, either u fixes z or t fixes $u(z)$. In particular, $w(x) = tu(x) = t(x) = k + 1$, completing the proof of this direction.

Now suppose that there exists $x < k + 1$ such that $\{w(1), \ldots, w(k)\} = [1, k + 1] \setminus \{x\}$ and $i := w^{-1}(x) > k + 1$. Consider the permutation

$$u := (\sigma_k \sigma_{k-1} \cdots \sigma_x) w (\sigma_{i-1} \sigma_{i-2} \cdots \sigma_{k+1}). \tag{3.3}$$

The factor on the right in Equation (3.3) slides x leftward in the one-line notation for w, swapping it with $w(j) > k + 1 > x$ at each step, until x is sitting in position $k + 1$ of the permutation. Recall the hypotheses on w. The factor on the left in Equation (3.3) swaps the value x with the value $x + 1$ in the one-line notation, then $x + 1$ with $x + 2$, and so on, always moving the larger value into position $k + 1$ from somewhere to the left of that position. Therefore $\ell(u) = \ell(w) - (k - x + 1) - (i - k - 1) = \ell(w) + x - i$. After all of these transpositions, the resulting permutation u fixes $k + 1$, and $u(i) \in [1, k]$ for all $i \in [1, k]$. Therefore, by Lemma 3.3, we have $k, k + 1 \notin \mathsf{supp}(u)$, and thus there exists some $[\alpha\beta] \in R(u)$ in which α contains only letters less than k and β contains only letters that are greater than $k + 1$. Hence

$$\big[x(x + 1) \cdots (k - 1) k \alpha\beta(k + 1)(k + 2) \cdots (i - 2)(i - 1)\big] \in R(w),$$

and this is the desired reduced word.

Part (b) follows from part (a) using conjugation by w_0 (which acts on S_n by reversing the one-line notation).

Now consider part (c) of the theorem.

Suppose, first, that there is no such i. Because $\{k, k + 1\} \subseteq \mathsf{supp}(v)$, Lemma 3.3 says that $\{w(1), \ldots, w(k)\} \neq [1, k]$ and $\{w(1), \ldots, w(k + 1)\} \neq [1, k + 1]$. Thus, if there is no such i, then we must have $w(h) = k + 1$ for some $h < k + 1$, $w(k + 1) > k + 1$, and $w(q) \in [1, k]$ for all $q \in [1, k] \setminus \{h\}$. But then w has the form described in part (a) of the current theorem, and so, by Proposition 3.10, the letters k and $k + 1$ are not interlaced in w. Similarly, if there is no such j, then the result will follow from part (b) of the current theorem.

Now suppose that there are such i and j. Then the permutation w has neither form from parts (a) or (b) of the current theorem, so, by Proposition 3.10, the letters k and $k + 1$ are interlaced in w. $\qquad\square$

Lemma 3.3 gave a method for detecting the support of a permutation from its one-line notation. Theorem 3.11 gives a method for determining the orientation of any $\{k, k+1\} \subseteq \mathsf{supp}(w)$ from the one-line notation of w, as well. In some ways, this is an analogy to Theorem 2.5, which equates reduced word properties with pattern-avoiding (one-line notation) properties.

We can use this orientation detection to construct the maximal elements of $B(v) \cap B(w)$ when v and w are both boolean, following Corollary 3.8. Moreover, by Corollary 3.9, we can also use it with no conditions on w, when all subwords discussed at the beginning of Section 3.3 have $j = 1$.

Example 3.12. Consider the boolean permutation $v = 312647895 \in \mathfrak{S}_9$ and the non-boolean permutation $w = 325184769 \in \mathfrak{S}_9$, both written in one-line notation. Using Lemma 3.3, we can compute

$$\mathsf{supp}(v) = \{1, 2, 4, 5, 6, 7, 8\} \quad \text{and} \quad \mathsf{supp}(w) = \{1, 2, 3, 4, 5, 6, 7\}.$$

The intersection of these sets is $\{1, 2, 4, 5, 6, 7\}$, so we check four orientations using Theorem 3.11.

Consecutive generators	Orientation in v	Orientation in w
$\{\sigma_1, \sigma_2\}$	decreasing	interlaced
$\{\sigma_4, \sigma_5\}$	decreasing	increasing
$\{\sigma_5, \sigma_6\}$	increasing	decreasing
$\{\sigma_6, \sigma_7\}$	increasing	interlaced

The middle column of the table tells us that the only subword we need to worry about is [567], but the rightmost column shows that, in fact, the subwords discussed at the beginning of Section 3.3 all have $j = 1$. Thus we can use Corollary 3.9, with $\mathcal{V}(v, w) = \{4, 5, 6\}$. Therefore, the maximal elements of $B(v) \cap B(w)$ are defined from any $[s] \in R(v)$ by deleting 8 (which is not in $\mathsf{supp}(w)$), and then by deleting the complement of a maximal selfish subset of $\{4, 5, 6\}$ (i.e., either deleting 5 or deleting both 4 and 6). So if we take $[s] = [5214678]$, then the maximal elements of the intersection are defined by

$$[521\cancel{4}67\cancel{8}] = 312465879 \quad \text{and} \quad [\cancel{5}21467\cancel{8}] = 312547869.$$

In other words,

$$B(312647895) \cap B(325184769) = B([5217]) \cup B([21467]).$$

The most extreme case of non-matching orientation is that of the boolean element v and $w = v^{-1}$, here is an example.

Example 3.13. Consider $v = 312647895 = [5214678] \in \mathfrak{S}_9$, as in Example 3.12. Then, following Subsection 3.2 and Corollary 3.8, we have $\mathsf{supp}(v) = [1, 2] \cup [4, 8]$. The eight maximal elements of $B(v) \cap B(v^{-1})$ have reduced decompositions defined by the product $\{1, 2\} \times \{468, 47, 57, 58\}$, as discussed in Section 3.2. That is,

$$B(v) \cap B(v^{-1}) = B([1468]) \cup B([147]) \cup B([157]) \cup B([158])$$
$$\cup B([2468]) \cup B([247]) \cup B([257]) \cup B([258]).$$

4. Motivation: Incidence algebras and grades of simple modules

4.1. Incidence algebras and their modules

Let us fix an algebraically closed field $\mathbb{k}$. As usual, we denote by $*$ the classical $\mathbb{k}$-duality $\mathrm{Hom}_{\mathbb{k}}(\_, \mathbb{k})$.

Let $(\mathbf{P}, \prec)$ be a finite poset. Consider the incidence algebra $\mathcal{I}(\mathbf{P})$ over $\mathbb{k}$. The algebra $\mathcal{I}(\mathbf{P})$ can be described by its Gabriel quiver Γ that has

- the elements of $\mathbf{P}$ as vertices;

- the arrows $p \to q$, for each pair $(p, q) \in \mathbf{P}^2$ such that p covers q;

and the relations that, for any $(p, q) \in \mathbf{P}^2$, all paths from p to q coincide.

As usual, the simple $\mathcal{I}(\mathbf{P})$-modules are in bijection with the elements in $\mathbf{P}$. Given $p \in \mathbf{P}$, the corresponding simple module L_p is one-dimensional at p and zero-dimensional at all other vertices. Furthermore, all arrows from the Gabriel quiver act on L_p as the zero linear maps.

The indecomposable projective cover P_p of L_p is supported on the ideal $\mathbf{P}_{\preceq p}$, is one-dimensional at each point of this ideal and zero-dimensional at all other points, and all arrows between the elements of $\mathbf{P}_{\preceq p}$ act as the identity linear transformations.

Dually, the indecomposable injective envelope I_p of L_p is supported on the coideal (that is, a filter) $\mathbf{P}_{\succeq p}$, is one-dimensional at each point of this coideal and zero-dimensional at all other points, and all arrows between the elements of $\mathbf{P}_{\succeq p}$ operate as the identity linear transformations. See Subsection 4.5 for an example.

We denote by $\mathcal{I}(\mathbf{P})$-mod the category of finite dimensional (left) $\mathcal{I}(\mathbf{P})$-modules, which we identify with the category of modules over the above quiver satisfying the above relations.

4.2. Grades of simple modules

Since the Gabriel quiver of $\mathcal{I}(\mathbf{P})$ is acyclic, the algebra $\mathcal{I}(\mathbf{P})$ has finite global dimension. In particular, for each $0 \neq M \in \mathcal{I}(\mathbf{P})$-mod, the following invariant, called *grade*, is well-defined and finite:

$$\mathbf{grade}(M) := \min\{i \,:\, \mathrm{Ext}^i_{\mathcal{I}(\mathbf{P})}(M, \mathcal{I}(\mathbf{P})) \neq 0\} \leqslant \mathrm{proj.dim}(M).$$

Of special interest for us will be the grades of the simple modules L_p, where $p \in \mathbf{P}$.

4.3. Grades via the Serre functor

Consider the bounded derived category $\mathcal{D}^b(\mathcal{I}(\mathbf{P}))$ of $\mathcal{I}(\mathbf{P})$-mod. Since $\mathcal{I}(\mathbf{P})$ has finite global dimension, the category $\mathcal{D}^b(\mathcal{I}(\mathbf{P}))$ has a Serre functor $\mathbb{S}$ given by the left derived functor of tensoring with the dual bimodule $\mathcal{I}(\mathbf{P})^*$. The functor $\mathbb{S}$ is a self-equivalence of $\mathcal{D}^b(\mathcal{I}(\mathbf{P}))$, see [5,8]. We have $\mathbb{S}P_p \cong I_p$, for each $p \in \mathbf{P}$.

Lemma 4.1. *For $0 \neq M \in \mathcal{I}(\mathbf{P})$, the grade of M coincides with the minimal i such that the $-i$-th homology of the complex $\mathbb{S}M$ is non-zero.*

Proof. By definition, the grade of M is the minimal value of i such that

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{I}(\mathbf{P}))}(M, \mathcal{I}(\mathbf{P})[i]) \neq 0.$$

Applying the equivalence $\mathbb{S}$, we obtain

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{I}(\mathbf{P}))}(\mathbb{S}M, \mathbb{S}\mathcal{I}(\mathbf{P})[i]) \neq 0.$$

It remains to note that $\mathbb{S}\mathcal{I}(\mathbf{P})$ is an injective cogenerator of $\mathcal{I}(\mathbf{P})$-mod and hence taking homomorphisms into it detects the homology. $\qquad\square$

Lemma 4.1 suggests that, to determine the grade of L_p, one needs to take a minimal projective resolution of L_p, apply $\mathbb{S}$ to it and then understand the homology of the obtained complex.

4.4. Incidence algebras for Bruhat posets of finite Weyl groups

Assume now that $\mathrm{char}(\Bbbk) = 0$.

Let W be a *finite Weyl group* and S a fixed set of simple reflections in W. Then W is a poset with respect to the *Bruhat order* $\leqslant$. This poset has the minimum element e and the maximum element w_0, the longest element of W. We denote by $\ell : W \to \mathbb{Z}_{\geqslant 0}$ the associated *length function*. For simplicity, we denote by A the $\Bbbk$-algebra $\mathcal{I}((W, \leqslant))$. The algebra A is the main protagonist in our motivation.

We would like to determine $\mathbf{grade}(L_w)$, for each $w \in W$. Taking into account the observations in the previous subsection, let us start with a description of projective resolutions of the modules L_w, where $w \in W$.

For $i \geqslant 0$, denote by V_i the formal vector space with basis $\{v_w : \ell(w) = i\}$. By [1], for each i, there exists a linear map $d_i : V_i \to V_{i+1}$ such that

- the v_x-v_y-coefficients of d_i is non-zero if and only if $y \leqslant x$;

- all such non-zero coefficients are ± 1;

- $d_{i+1} \circ d_i = 0$, for all i.

The associated complex

$$0 \to V_0 \to V_1 \to \cdots \to V_{\ell(w_0)} \to 0$$

is exact and is called a *BGG complex*. It has the property that its restriction to the part supported at a principal (co)ideal is exact (unless it is the ideal of the minimum element, respectively

the coideal of the maximum element). This complex can also be interpreted as the singular homology complex for W.

For $i \geqslant 0$, denote by $Q(w, i)$ the direct sum of all P_x, where $x \leqslant w$ and $\ell(w) - \ell(x) = i$.

Proposition 4.2. *There is a projective resolution of L_w of the form*

$$\cdots \to Q(w, 2) \to Q(w, 1) \to Q(w, 0) \to 0, \tag{4.1}$$

where, for a summand P_x in $Q(w, i)$ and P_y in $Q(w, i - 1)$ such that $x \leqslant y$, the map from P_x to P_y is given by the corresponding coefficient in the BGG complex.

Proof. This follows directly from the properties of the BGG complex listed above. $\qquad \square$

For $i \geqslant 0$, denote by $F(w, i)$ the direct sum of all I_x, where $x \leqslant w$ and $\ell(w) - \ell(x) = i$. Applying $\mathbb{S}$ to (4.1) results in a complex

$$\cdots \to F(w, 2) \to F(w, 1) \to F(w, 0) \to 0, \tag{4.2}$$

with the property that, for a summand I_x in $Q(w, i)$ and I_y in $Q(w, i - 1)$ such that $x \leqslant y$, the map from I_x to I_y is given by the corresponding coefficient in the BGG complex.

Now we want to understand the homology of (4.2), or, more precisely, the rightmost degree in which non-zero homology appears. This determines the grade of L_w.

For $u \in W$, let us restrict (4.2) to the vertex u. Recall that the injective module I_x, for $x \in W$, is supported at $W_{\geqslant x}$. Therefore, each $I_x \in Q(w, i)$ such that $x \leqslant u$ contributes one dimension for the vertex u at position $-i$ of the complex (4.2). That is, the restriction of (4.2) to u is the complex

$$\cdots \to F(w, 2)_u \to F(w, 1)_u \to F(w, 0)_u \to 0, \tag{4.3}$$

where $F(w, 2)_u$ is the sum of one-dimensional spaces indexed by x such that $x \leqslant w$, $x \leqslant u$ and $\ell(w) - \ell(x) = i$, and the differential is the restriction of the differential from the BGG complex. In other words, this is exactly the restriction of the BGG complex to $W_{\leqslant w} \cap W_{\leqslant u}$, and our goal is to minimize, over all possible u, the absolute value of the rightmost degree in which this complex has non-zero homology.

Our setup is such that the vertex w is placed at the homological position 0. In particular, the vertex e is placed at position $-\ell(w)$. Taking $u = e$, we get a complex concentrated in position $-\ell(w)$, which means that $\ell(w)$ is an upper bound for our answer.

If $e \neq w \leqslant u$, then $W_{\leqslant w} \cap W_{\leqslant u} = W_{\leqslant w}$ which implies that (4.3) is exact. Similarly, the case $e \neq u \leqslant w$ gives an exact complex (4.3). Therefore the interesting case to consider is $e \neq u$, $e \neq w$, and u and w are not comparable with respect to the Bruhat order.

As was pointed out to us by Axel Hultman, if there is a simple reflection s such that $sw < w$ and $su < u$ or such that $ws < w$ and $us < u$, then the description of the differential in the BGG complex implies, by induction, that (4.3) is exact (we will explain this in more detail in Subsection 4.7 below). Therefore the really interesting case is when u and w neither have any common elements in their left descent sets nor any common elements in their right descent sets.

4.5. A_2 **example**

Consider W of Weyl type A_2 with $S = \{s, t\}$. In this case we have $W = \{e, s, t, st, ts, w_0 = sts = tst\}$. The Gabriel quiver of the corresponding incidence algebra and the coefficients in the associated BGG complex look as follows:
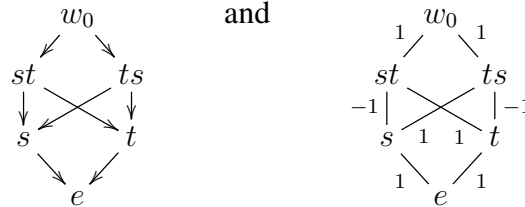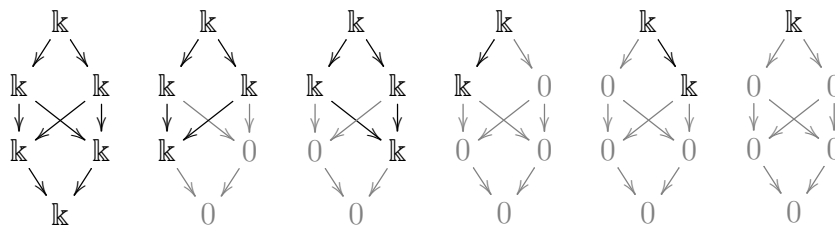

Here is the list of simple modules over the incidence algebra:
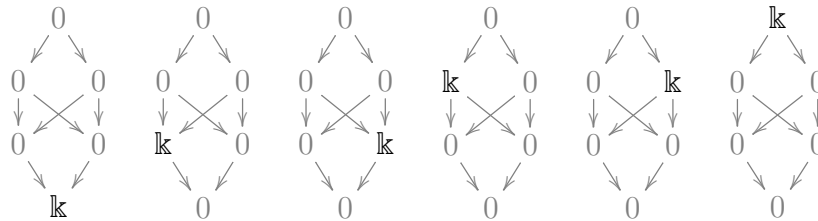
And here is the list of indecomposable projective modules over the incidence algebra (all black arrows represent the identity linear transformations):
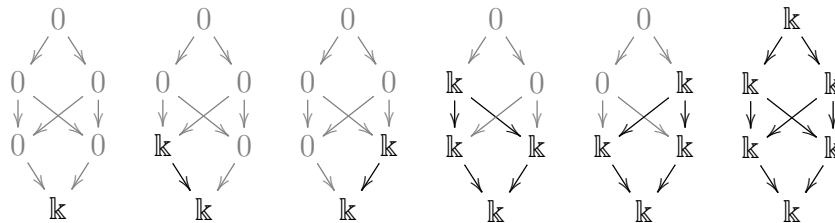
Finally, here is the list of indecomposable injective modules over the incidence algebra:

For $w = e$, we have $L_e = P_e$. Applying $\mathbb{S}$, we get the complex $0 \to I_e \to 0$ with I_e at the homological position 0. This implies that $\mathbf{grade}(L_e) = 0$.

For $w = s$, the projective resolution of L_s is $0 \to P_e \to P_s \to 0$. Applying $\mathbb{S}$, we get the complex $0 \to I_e \to I_s \to 0$ with I_s at the homological position 0. Since the map $I_e \to I_s$ is surjective, we have that $\mathbf{grade}(L_s) = 1$. Similarly, $\mathbf{grade}(L_t) = 1$.

For $w = st$, the projective resolution of L_s is $0 \to P_e \to P_s \oplus P_t \to P_{st} \to 0$. Applying $\mathbb{S}$, we get the complex

$$0 \to \quad \mathbb{k} \quad \to \quad \mathbb{k} \quad \oplus \quad \mathbb{k} \quad \to \quad \mathbb{k} \quad \to 0$$

with I_{st} at the homological position 0. The restriction of this complex to the vertex ts (shown in red) gives the complex

$$0 \to \mathbb{k} \to \mathbb{k} \oplus \mathbb{k} \to 0 \to 0,$$

supported at $W_{\leqslant st} \cap W_{\leqslant ts} = \{e, s, t\}$. It has non-zero homology at position -1. The only other restriction to a vertex resulting in a non-zero homology is that to e (shown in blue) which gives the complex

$$0 \to \mathbb{k} \to 0 \oplus 0 \to 0 \to 0,$$

supported at $W_{\leqslant st} \cap W_{\leqslant e} = \{e\}$. It has non-zero homology at position -2. By taking the minimum of 1 and 2, we obtain $\mathbf{grade}(L_{st}) = 1$. Similarly, $\mathbf{grade}(L_{ts}) = 1$.

For $w = w_0$, the projective resolution of L_{w_0} is $0 \to P_e \to P_s \oplus P_t \to P_{st} \oplus P_{ts} \to P_{w_0} \to 0$. Applying $\mathbb{S}$, we get the complex

$$0 \to \quad \mathbb{k} \quad \to \quad \mathbb{k} \quad \oplus \quad \mathbb{k} \quad \to \quad \mathbb{k} \quad \oplus \quad \mathbb{k} \quad \to \quad \mathbb{k} \quad \to 0$$

with I_{w_0} at the homological position 0. The restriction of this complex to the vertex e (shown in blue) gives the complex

$$0 \to \mathbb{k} \to 0 \oplus 0 \to 0 \oplus 0 \to 0 \to 0,$$

supported at $W_{\leqslant w_0} \cap W_{\leqslant e} = \{e\}$. It has non-zero homology at position -3 and hence

$$\mathbf{grade}(L_{st}) = 3.$$

To sum up, here are the values of the grade function in type A_2:

w	e	s	t	st	ts	w_0
$\mathbf{grade}(L_w)$	0	1	1	1	1	3

One could observe that these values coincide with the values of Lusztig's $\mathbf{a}$-function from [18] in this case. We will come back to this observation in more detail at the end of the paper.

4.6. Connection to Auslander regularity

Our interest in the grades of simple modules stems from the theory of Auslander regular algebras, see [12]. It is shown in [12] that the incidence algebra of a lattice is Auslander regular if and only if the lattice is distributive. The poset $(W, \leqslant)$ is not a lattice, in general, and, outside type A there are reasons why, generically, the incidence algebra of $(W, \leqslant)$ is not Auslander regular. In type A this question is still open (as was mentioned by Rene Marczinzik at a seminar talk in Uppsala in March 2021) and the research presented in this paper is originally motivated by that problem. Grades of simple modules are essential homological invariants for this theory and they behave especially nicely for some Auslander regular algebras, see [12] for details.

4.7. Matchings

Let Q be a convex subset of W in the sense that $x, y \in Q$ with $x \leqslant y$ implies $[x, y] \subseteq Q$. Then we can restrict the BGG complex to its part supported at Q; i.e., to the linear span of all vectors indexed by the elements in Q. Let us denote this complex by $V_\bullet^Q$.

Lemma 4.3. *Assume that the poset $(Q, \leqslant)$ admits a filtration*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_l = Q \tag{4.4}$$

by coideals such that each $Q_i \setminus Q_{i-1} = \{x_i, y_i\}$, where $x_i < y_i$ and $\ell(x_i) = \ell(y_i) - 1$. Then $V_\bullet^Q$ is exact.

Proof. The filtration (4.4) gives rise to a filtration of $V_\bullet^Q$ by subcomplexes with subquotients of the form

$$0 \to \mathbb{C}\langle y_i \rangle \to \mathbb{C}\langle x_i \rangle \to 0. \tag{4.5}$$

From the definition of the BGG complex we see that the map $\mathbb{C}\langle y_i \rangle \to \mathbb{C}\langle x_i \rangle$ is non-zero and therefore (4.5) is homotopic to zero. The claim follows. $\qquad\square$

We will call the decomposition of Q into the subsets $\{x_i, y_i\}$ given by Lemma 4.3 a *perfect matching*. The already mentioned observation by Axel Hultman was that existence of a simple reflection s such that $sv < v$ and $sw < w$ obviously implies that $B(v) \cap B(w)$ has a perfect matching by the pairs $\{x, sx\}$. In particular, $V_\bullet^{B(v) \cap B(w)}$ is exact in this case. Similarly in the case of right descents; i.e., for $vs < v$ and $ws < w$.

Lemma 4.4. *Assume that the poset $(Q, \leqslant)$ admits a filtration*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_l = Q$$

by coideals such that each $Q_i \setminus Q_{i-1} = \{x_i, y_i\}$, where $x_i < y_i$ and $\ell(x_i) = \ell(y_i) - 1$, with the exception of one i for which we get a singleton z. Then $V_\bullet^Q$ has exactly one non-zero homology, namely in the homological position $\ell(z)$ and this homology is one-dimensional.

Proof. Similarly to the proof of Lemma 4.3, all matched pairs will give subquotient complexes that are homotopic to zero, so the one-dimensional homology will be concentrated at the unique unmatched singleton. $\qquad\square$

We will call a decomposition of Q given by Lemma 4.4 an *almost perfect matching*.

5. Matchings in intersections

5.1. Preliminaries

Throughout this section, let $v \in \mathfrak{S}_n$ be a boolean permutation. As discussed in Section 4, we want to find a permutation $w \in \mathfrak{S}_n$ such that there is an almost perfect matching of the elements of $B(v) \cap B(w)$, in which the singleton element is of highest possible rank. Such a w will be called an *optimal partner* for v, and that highest possible rank is the *optimal rank* of v, denoted $\mathsf{ork}(v)$. Note that there always exists an intersection $B(v) \cap B(w)$ that can be almost perfectly matched, because we could let w be the identity permutation. So $\mathsf{ork}(v)$ always exists.

We will also show in Proposition 5.12 that, for any w, the poset $B(v) \cap B(w)$ has either a perfect matching or an almost perfect matching. We begin by formalizing a fact mentioned in the previous section.

Lemma 5.1. *An optimal partner for $v \neq e$ can never be greater than or equal to v in the Bruhat order.*

Proof. A principal order ideal containing more than one element always has zero homology. If $v \in B(w)$, then $B(v) \cap B(w) = B(v)$, which would mean that w is not an optimal partner for v. $\qquad\square$

As an immediate corollary, we record a property of the optimal rank function.

Corollary 5.2. *For any v that is not the identity permutation, $\mathsf{ork}(v) < \ell(v)$.*

Recall Lemma 3.2, which gives a way to decompose order ideals based on their commuting subsets of support. Suppose, for the moment, that $\mathsf{supp}(v) = X \sqcup Y$ as in that lemma. For $[s] \in R(w)$, define $[s_X]$ and $[s_Y]$ accordingly, and we then observe that $B(v) \cap B(w)$ would be isomorphic to $(B([s_X]) \cap B(w)) \times (B([s_Y]) \cap B(w))$. Therefore, if there is a perfect matching of $B([s_X]) \cap B(w)$, then we can construct a perfect matching of $B(v) \cap B(w)$, as a product of the matching filtration of $B([s_X]) \cap B(w)$ with any filtration of $B([s_Y]) \cap B(w)$ by coideals with singleton subquotients. Therefore, we can often assume that $\mathsf{supp}(v) = [1, n-1]$.

5.2. Runs

We begin with an influential special case.

Definition 5.3. A sequence of consecutive integers that is either increasing or decreasing is a *run*.

Lemma 5.4. *Suppose that the run $\big[a(a+1)\cdots(a+b)\big]$ (resp., $\big[(a+b)\cdots(a+1)a\big]$) is a reduced word for v. Then an optimal partner for v is*

$$\big[(a+1)(a+2)\cdots(a+b)a(a+1)\cdots(a+b-1)\big]$$

(resp., $\big[(a+b-1)\cdots(a+1)a(a+b)\cdots(a+2)(a+1)\big]$) if $b > 0$, and an optimal partner is the identity if $b = 0$.

Proof. It is sufficient to consider $a = 1$.

The case $b = 0$ follows from Lemma 5.1.

Now suppose that $b > 0$, and, without loss of generality, that $v = \left[12 \cdots (1+b)\right]$. Consider the permutation $w := \left[23 \cdots (1+b)12 \cdots b\right]$. Using Theorem 2.2, we see that

$$B(v) \cap B(w) = B(v) \setminus \{v\}.$$

The set $B(v)$ is a union of left cosets for the parabolic subgroup generated by σ_1. We can consider the restriction of the Bruhat order to the set of minimal coset representatives. Any filtration of the latter poset by coideals with singleton subquotients extends in the obvious way to a filtration of $B(v)$ by coideals with subquotients being exactly the cosets. This gives a perfect matching for $B(v)$. Removing v, which must belong to the first coset in the filtration, we obtain an almost perfect matching for $B(v) \setminus \{v\}$. The unmatched element $\sigma_1 v$ has length $\ell(v) - 1$, and hence this must be the optimal rank, by Corollary 5.2. Thus w must be an optimal partner for v. $\square$

We demonstrate Lemma 5.4 with an example.

Example 5.5. Consider $v = 51234 = [4321]$. Then $w = 45123 = [321432]$ is an optimal partner for v. The poset $B(v) \cap B(w)$ is shown in Figure 5.1, with thick lines indicating the almost perfect matching from Lemma 5.4.
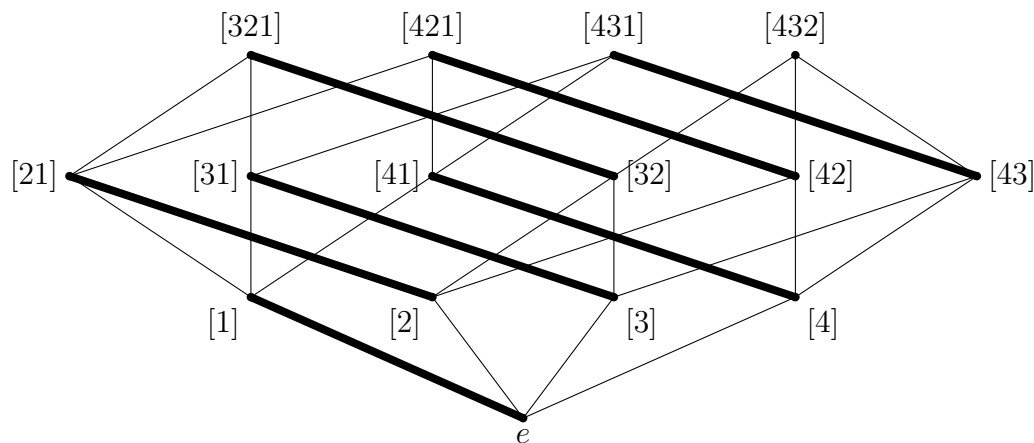
Figure 5.1: A matching of the elements of $B([432]) \cap B([321432])$, in which only $[432]$ is unmatched.

The cases $b = 0$ and $b > 0$ in the previous lemma share an important property.

Corollary 5.6. *If v is boolean and has a reduced word that is a run, then $\mathrm{ork}(v) = \ell(v) - 1$.*

Lemma 5.4 seems to consider a very particular situation. However, because we are assuming that v is boolean, we can actually view any $[s] \in R(v)$ as a product of disjoint runs.

Example 5.7. $R(24153) = \{[1324], [3124], [1342], [3142], [3412]\}$. The first element of that set can be viewed as the concatenation of three runs: $1 \cdot 32 \cdot 4$, whereas the last element can be viewed as the concatenation of two runs: $34 \cdot 12$. Of course, we could also write $1 \cdot 3 \cdot 2 \cdot 4$ and so on, but this inefficiency is not helpful, as described below.

5.3. Optimal ranks

Corollary 5.6 suggests that the optimal rank of v might be related to the fewest number of runs needed to form a reduced word for v, and indeed that is the main result of this section.

Definition 5.8. Fix a boolean permutation v. Let $\mathsf{run}(v)$ be the fewest number of runs needed in any concatenation forming a reduced word for $R(v)$. A reduced word $[s] \in R(w)$ that can be written as the concatenation of $\mathsf{run}(v)$ runs is an *optimal run word* for v.

Recalling Example 5.7, we have that $\mathsf{run}(24153) = 2$, and $[3412]$ is an optimal run word for 24153.

Example 5.9. If v is boolean and, additionally, a product of pairwise commuting simple reflections, then $\mathsf{run}(v) = \ell(v)$ and any reduced word of v is an optimal run word. In this case it is also easy to prove that $\mathsf{ork}(v) = 0$. Indeed, for any w, we obviously have $B(v) \cap B(w) = B(u)$, where u is the product of simple reflections in $\mathrm{supp}(v) \cap \mathrm{supp}(w)$. Hence, if $u \neq e$, the set $B(u)$ has a perfect matching. If $u = e$, then $B(u)$ has an almost perfect matching with singleton of length 0.

The main result of this section will determine the optimal rank and an optimal partner for any boolean permutation. Before doing so, we will give an upper bound on the optimal rank, using a handy lemma.

Lemma 5.10. *Fix a reduced word $[s] = [s_1 \cdots s_l] \in R(w)$ and $i \in [1, l]$. Set $\hat{s} := s_1 \cdots \hat{s}_i \cdots s_l$. There exists a unique maximal $w' \in B(w)$ having a reduced word that is a subword of $\hat{s}$. In other words, the permutations whose reduced words are subwords of $\hat{s}$ form a principal order ideal. Moreover, the boolean elements of $B(w)$ with a reduced word that can be written as a subword of $\hat{s}$ are exactly the boolean elements of $B(w')$.*

Proof. It is easy to check that the lemma holds for permutations of small lengths. In particular, if $\ell(w) = 1$ then $w' = e$. Suppose now that the result is true for all permutations u with $\ell(u) < \ell(w)$.

If $i = l$, then $\hat{s}$ is necessarily reduced. Therefore we have $w' = [\hat{s}]$, and the property for boolean elements follows immediately.

Now suppose that $i < l$. Define the string $t := s_1 \cdots s_{l-1}$ and set $u := [t]$, noting that $\ell(u) = l - 1 < \ell(w)$. Set $\hat{t} := s_1 \cdots \hat{s}_i \cdots s_{l-1}$, and let $u' \in B(u)$ be the permutation produced by the inductive hypothesis. Define a permutation $\overline{w}$ as follows:

$$\overline{w} := \begin{cases} u'\sigma_{s_l} & \text{if } \ell(u'\sigma_{s_l}) > \ell(u'), \text{ and} \\ u' & \text{if } \ell(u'\sigma_{s_l}) < \ell(u'). \end{cases} \tag{5.1}$$

By definition, this $\overline{w}$ has a reduced word that is a subword of $\hat{s}$. It remains to establish that $\overline{w}$ is maximal with this property and that $B(\overline{w}) \subset B(w)$ contains the necessary boolean elements.

Let x be maximal among permutations having reduced words that are subwords of $\hat{s}$. Fix $[q] \in R(x)$ such that q is a subword of $\hat{s}$ and, if possible, the rightmost letter of q is not s_l.

If q does not end with s_l, then q is a subword of $\hat{t}$. Maximality of u' means that $x = u'$. Because this x is maximal, we must have that $\ell(u'\sigma_{s_l}) < \ell(u')$, and so $x = \overline{w}$.

On the other hand, suppose that it is impossible to write q in this way, and so $q = q's_l$ for a subword q' of $\hat{t}$. This means that $[q'] \leqslant u'$ in the Bruhat order, and σ_{s_l} is not a right descent of u'. But then $\overline{w} = u'\sigma_{s_l}$, and so we must have $[q'] = u'$ by maximality. Therefore $x = [q's_l] = \overline{w}$.

Therefore, $w' := \overline{w}$ is the desired permutation.

We now use the inductive hypothesis and Equation (5.1) to prove the second half of the lemma. Let v be a boolean permutation with a reduced word that is a subword of $\hat{s}$. If v actually has a reduced word that is a subword of $\hat{t}$, then $v \in B(u') \subseteq B(w')$ by the inductive hypothesis. If v has no such reduced word, then it must be the case that $w' = u'\sigma_{s_l}$. Fix $[r] \in R(v)$ such that r is a subword of $\hat{s}$. Then r is a concatenation of $r' \subseteq \hat{t}$ and s_l. By the inductive hypothesis, $[r'] \in B(u')$ and hence $v = [r' \cdot s_l] \in B(u'\sigma_{s_l}) = B(w')$. □

We demonstrate Lemma 5.10 with an example.

Example 5.11. Let $w = 4321$ and $[s] = [321232] \in R(w)$, with $i = 3$. Thus $\hat{s} = 32232$. The proof of Lemma 5.10 builds the permutation w' inductively from permutations $u'_1, \ldots, u'_5 = w'$ as follows.

j	1	2	3	4	5
u'_j	[3]	[32]	[32]	[323]	[323]

The ideal $B(w)$ has thirteen boolean elements, five of which can be formed from subwords of $\hat{s}$:

$$\{e, [2], [3], [23], [32]\}. \tag{5.2}$$

The boolean elements of $B([232])$ are exactly the five permutations listed in (5.2).

The ability to delete letters from a reduced word without losing particular elements from its principal order ideal is important for the inductive step in the following proposition.

Proposition 5.12. *For any boolean permutation v, we have $\mathsf{ork}(v) \leqslant \ell(v) - \mathsf{run}(v)$.*

Proof. We prove this result by induction on $\ell(v)$, and it is easy to verify that the proposition holds for $\ell(v) \leqslant 2$. In fact, we also know that the proposition holds whenever $\mathsf{run}(v) = 1$, by Corollary 5.6. Suppose, inductively, that for any boolean u with $\ell(u) < \ell(v)$, any intersection $B(u) \cap B(y)$ either has a perfect matching or has an almost perfect matching with a single unmatched element of rank at most $\ell(u) - \mathsf{run}(u)$.

Fix $[s] \in R(v)$ and an arbitrary permutation w. We want to show that $B(v) \cap B(w)$ either has a perfect matching or has an almost perfect matching with a single unmatched element of rank at most $\ell(v) - \mathsf{run}(v)$. Set $m := \max(\mathsf{supp}(v) \cap \mathsf{supp}(w))$. For $z \in B(v) \cap B(w)$ with $m \notin \mathsf{supp}(z)$, we match $z \longleftrightarrow z \curlywedge \sigma_m$ whenever $z \curlywedge \sigma_m$ exists. Similarly to the proof of Lemma 5.4, this matching is inherited from the filtration with respect to the Bruhat order on the set of shortest coset representatives for the parabolic subgroup generated by σ_m. If $z \curlywedge \sigma_m$ always exists, then this matches all elements of $B(v) \cap B(w)$ and we are done.

Now suppose that this matching does not account for all elements of $B(v) \cap B(w)$, and let X be the set of as-yet-unmatched elements. None of these elements have m in their supports.

Moreover, because it is impossible to introduce m anywhere in their reduced words remaining inside $B(v) \cap B(w)$, they must all have (at least) $m - 1$. Let $m' < m$ be maximal such that the subword $[s]_{[m',m]}$ does not appear in any element of $R(w)$. (Note that $[m', m] \subseteq \mathsf{supp}(v) \cap \mathsf{supp}(w)$, by maximality of m'.) Then

$$q := \left[s_{[m', m-1]} \right] \in X,$$

and any $x \in X$ must include $\left[s_{[m', m]} \right]$ in its reduced words, so x is greater than or equal to q in the Bruhat order. Thus X is a filter; that is, X is the principal coideal of $B(v) \cap B(w)$ generated by q. This is very good news as it now allows us to construct the matching we are looking for, inductively. Note that

$$\ell(q) = m - 1 - m' + 1 = m - m'.$$

Let $v' := \left[s_{[1, m'-1]} \right]$ be the permutation obtained by deleting the letters $[m', m]$ from reduced words for v. Without loss of generality, suppose that $m' - 1$ does not appear to the right of m' in elements of $R(v)$. The permutation v was boolean, so v' is boolean and

$$\ell(v') = \ell(v) - (m - m' + 1).$$

Take a reduced word for w, look for all substrings that match reduced words for q, and mark the rightmost copy of m' used in any of these. Working iteratively, delete each $m' - 1$ that appears to the right of the marked m', using Lemma 5.10 to produce a reduced word after each deletion. When there are no more copies of $m' - 1$ appearing to the right of the marked m', write w' for the permutation described by the resulting reduced word.

We defined v' and w' for the purpose of our inductive argument: the filter X is isomorphic to $B(v') \cap B(w')$, with $[x_1 \cdots x_l] \in B(v') \cap B(w')$ corresponding to $q \curlywedge \sigma_{x_1} \curlywedge \cdots \curlywedge \sigma_{x_l}$. By the inductive hypothesis, $\mathsf{ork}(v') \leqslant \ell(v') - \mathsf{run}(v')$, so the poset $B(v') \cap B(w')$ has either a perfect matching or an almost perfect matching with an unmatched element of rank at most $\ell(v') - \mathsf{run}(v')$.

By definition of m' and q, the sequences $\left[s_{[m'+1, m]} \right]$ and $\left[s_{[m', m-1]} \right]$ both appear in reduced words for w, but $\left[s_{[m', m]} \right]$ does not. The only way for this to happen is for $[m', m]$ to be a run in $[s]$ (and for reduced words for w to contain a subsequence as in Lemma 5.4). Thus

$$\mathsf{run}(v') \geqslant \mathsf{run}(v) - 1.$$

By the inductive hypothesis, the intersection $B(v') \cap B(w')$ either has a perfect matching or it has an almost perfect matching with an unmatched element of rank at most $\ell(v') - \mathsf{run}(v')$. Transfer this matching onto $X \subset B(v) \cap B(w)$. This, together with the initial matching $z \longleftrightarrow z \curlywedge \sigma_m$, produces either a perfect matching of $B(v) \cap B(w)$ or an almost perfect matching whose single unmatched element has rank at most

$$
\begin{aligned}
\ell(q) + \mathsf{ork}(v') &\leqslant \ell(q) + \ell(v') - \mathsf{run}(v') \\
&\leqslant (m - m') + \ell(v) - (m - m' + 1) - (\mathsf{run}(v) - 1) \\
&= \ell(v) - \mathsf{run}(v),
\end{aligned}
$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We demonstrate how this bound can work, along with the inductive argument.

Example 5.13. We proceed with two examples, using the notation of Proposition 5.12.

(a) Let $v = 2341 = [123]$ and $w = 4123 = [321]$, so $m = 3$ and $m' = 2$. We match

$$\varnothing \longleftrightarrow [3]$$
$$[1] \longleftrightarrow [13]$$

and $X = \{[2]\}$. Then $q := \left[123_{[2,2]}\right] = [2]$ and $v' := \left[123_{[1,1]}\right] = [1]$, and $\mathsf{ork}(v') = 0$. Lemma 5.10 gives $w' = [32]$, and hence $B(v') \cap B(w') = \varnothing \cong X$. This example is depicted in Figure 5.2.
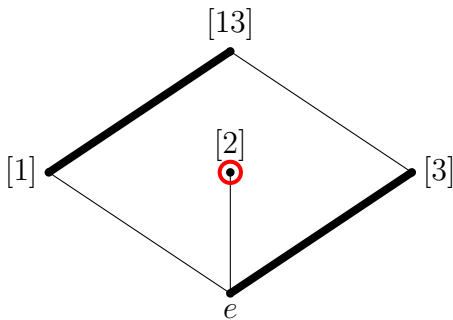

Figure 5.2: The intersection $B([123]) \cap B([321])$, and the matching described by Proposition 5.12. The sole element of the filter X, described in the proof of that proposition, is circled in red. The unmatched element in this almost perfect matching has rank $1 < \ell([123]) - \mathsf{run}([123])$.

(b) Let $v = 314562 = [23451]$ and $w = 235614 = [412534]$, so $m = 5$ and $m' = 3$. Then we match

$$\varnothing \longleftrightarrow [5]$$
$$[1] \longleftrightarrow [15]$$
$$[2] \longleftrightarrow [25]$$
$$[3] \longleftrightarrow [35]$$
$$[4] \longleftrightarrow [45]$$
$$[13] \longleftrightarrow [135]$$
$$[14] \longleftrightarrow [145]$$
$$[23] \longleftrightarrow [235]$$
$$[24] \longleftrightarrow [245]$$

and $X = \{[34], [134], [234]\}$. Then $q := \left[23451_{[3,4]}\right] = [34]$ and $v' := \left[23451_{[1,2]}\right] = [21]$, and $\mathsf{ork}(v') = 1$. Lemma 5.10 gives $w' = w$, and hence

$$B(v') \cap B(w') = \{\varnothing, [1], [2]\} \cong X.$$

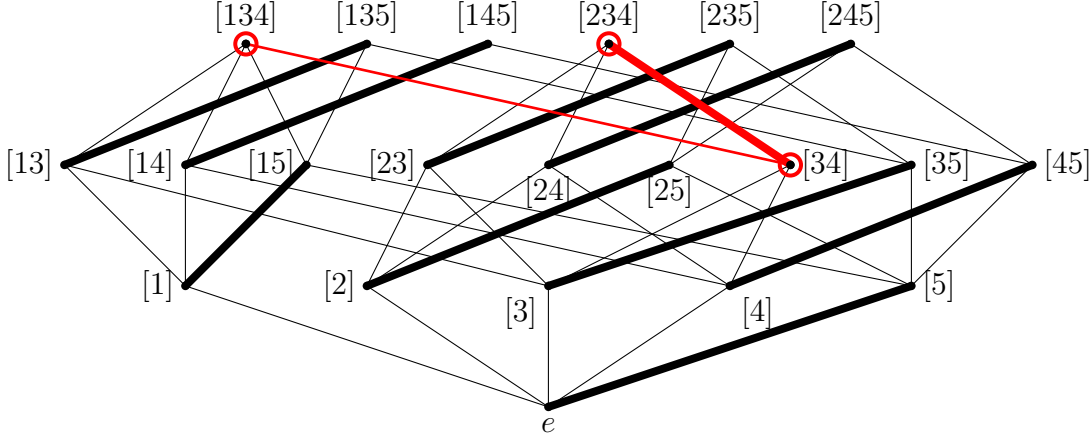This example is depicted in Figure 5.3.

Figure 5.3: The intersection $B([23451]) \cap B([412534])$, and the matching described by Proposition 5.12. The filter X, described in the proof of that proposition, and its matching are marked in red. The unmatched element in this almost perfect matching has rank $3 = \ell([23451]) - \mathsf{run}([23451])$.

5.4. Optimal rank and optimal partner

We are now ready to describe the optimal rank and an optimal partner for any boolean permutation.

Theorem 5.14. *Let v be a boolean permutation, and let $[s]$ be an optimal run word for v. Write s as a concatenation*

$$r_1 \cdots r_{\mathsf{run}(v)},$$

where the r_i are runs. For each i, let $[t_i]$ be the optimal partner for $[r_i]$ as determined by Lemma 5.4. Then $w := \left[t_1 \cdots t_{\mathsf{run}(v)} \right]$ is an optimal partner for v, and

$$\mathsf{ork}(v) = \ell(v) - \mathsf{run}(v).$$

Proof. With v and w as described, we have that $B(v) \cap B(w) = B(v) \setminus Q$, where Q is the set of elements involving at least one full run r_i.

For each i, set $[a_i, a_i + b_i] := \mathsf{supp}([r_i])$. We can now describe an almost perfect matching of $B(v) \cap B(w)$. Consider an element $z \in B(v) \cap B(w)$. We define the matching by examining how much of $[a_i, a_i + b_i]$ is contained in $\mathsf{supp}(z)$, starting with $i = 1$ and increasing i as needed. If we reach an i for which $b_i = 0$, we immediately increase i because $t_i = \varnothing$ and $[a_i] \notin B(v) \cap B(w)$. So in the following outline, assume that each $b_i > 0$.

- Consider $z \in B(v) \cap B(w)$. If $a_1 \notin \mathsf{supp}(z)$ and $[a_1 + 1, a_1 + b_1] \not\subseteq \mathsf{supp}(z)$, then match $z \longleftrightarrow z \curlywedge \sigma_{a_1}$ similarly to the proof of Lemma 5.4.

- The elements in $B(v) \cap B(w)$ that are not yet matched are exactly those that contain all of $[a_1 + 1, a_1 + b_1]$ in their supports. These form a coideal, so we can proceed inductively. Now let z be such an element. If $a_2 \notin \mathsf{supp}(z)$ and $[a_2 + 1, a_2 + b_2] \not\subseteq \mathsf{supp}(z)$, then match $z \longleftrightarrow z \curlywedge \sigma_{a_2}$ similarly to the proof of Lemma 5.4.

- The elements in $B(v) \cap B(w)$ that are not yet matched are exactly those that contain all of $[a_1 + 1, a_1 + b_1] \cup [a_2 + 1, a_2 + b_2]$ in their supports. These form a coideal, so we can proceed inductively. Now let z be such an element and repeat the process with $i = 3$.

- And so on.

At the end of this process, after $i = \mathsf{run}(v)$, we have an almost perfect matching of $B(v) \cap B(w)$, and the only unmatched element is $u := \left[\hat{r}_1 \cdots \hat{r}_{\mathsf{run}(v)}\right]$, where $\hat{r}_i$ is the run i with the element a_i removed. Note that $\ell(u) = \ell(v) - \mathsf{run}(v)$. By Proposition 5.12, this completes the proof. □

It is illuminating to see Theorem 5.14 demonstrated in an example. The example is, perhaps, too big for drawing the full poset, but we can describe the key pieces.

Example 5.15. Let $v = 5123678(12)49(10)(11) = \left[(11)43(10)5216798\right] \in \mathfrak{S}_{12}$. Our first step is to find an optimal run word for v. There are several options for this, including the words $\left[(11)(10)945678321\right]$ and $\left[(11)(10)943215678\right]$. In particular, $\ell(v) = 11$ and $\mathsf{run}(v) = 3$, so the theorem predicts $\mathsf{ork}(v) = 8$. Using $\left[(11)(10)945678321\right]$, the theorem produces the optimal partner $w = \left[(10)9(11)(10)567845672132\right]$, and the single unmatched element in the almost perfect matching of $B(v) \cap B(w)$ described in the proof of Theorem 5.14 is $\left[(11)(10)567832\right]$, which does indeed have length 8.

6. The main result

6.1. The minimal number of runs via the Robinson–Schensted correspondence

The *Robinson–Schensted correspondence* provides a bijection

$$\mathbf{RS} : \mathfrak{S}_n \to \coprod_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$$

between $\mathfrak{S}_n$ and the set of pairs of standard Young tableaux of the same shape (this shape is supposed to be a partition of n). We fix such a bijection given by Schensted's insertion algorithm, see [25, 26]. For $w \in W$, we have $\mathbf{RS}(w) = (P, Q)$, where P is the insertion tableau and Q is the recording tableau, see [25] for details. We denote by $\lambda(w)$ the shape of P, by $\mathsf{Row}_i(w)$ the contents of its ith row, and $\lambda_i(w) := |\mathsf{Row}_i(w)|$.

Recall from Theorem 2.5 that boolean permutations avoid the pattern 321. The main result of [26] therefore gives restrictions on their shapes.

Corollary 6.1.

(a) *For any permutation w, the number $\lambda_1(w)$ is the length of a longest increasing subsequence in w.*

(b) *If w is boolean, then $\lambda(w)$ has at most two rows.*

We start with the following observation.

Lemma 6.2. *Let $w, x \in \mathfrak{S}_n$ be permutations such that $x = [r]$ for a run r. Then we have $|\lambda_1(wx) - \lambda_1(w)| \leqslant 1$.*

Proof. Let $w = a_1 \cdots a_n$ in one-line notation, and suppose that $x = \big[p(p+1) \cdots (p+q)\big]$. Then the one-line notation of wx is as follows, using red to mark the part that changes when multiplying $w \mapsto wx$:

$$a_1 \cdots a_{p-1} a_{p+1} a_{p+2} \cdots a_{p+q+1} a_p a_{p+q+2} \cdots a_n. \tag{6.1}$$

This means that we only change the relative order of one element, a_p, compared to the elements in $\{a_{p+1}, \ldots, a_{p+q+1}\}$, leaving all other relative orders intact. Therefore the length of the longest increasing subsequence can change by at most 1. Similar arguments apply to the run $\big[p(p-1) \cdots (p-q)\big]$. $\square$

The previous lemma lets us relate the size $\lambda_1(v)$ to the number of runs $\mathsf{run}(v)$.

Corollary 6.3. *For any boolean $v \in \mathfrak{S}_n$, we have $n - \lambda_1(v) \leqslant \mathsf{run}(v)$. In particular, we have $\lambda_2(v) \leqslant \mathsf{run}(v)$.*

We are now ready to equate $\lambda_2(v)$ to a statistic we have already encountered, when v is boolean.

Theorem 6.4. *For any boolean permutation v, we have $\lambda_2(v) = \mathsf{run}(v)$.*

Proof. It is enough to prove the claim under the assumption that $\mathsf{supp}(v)$ contains all simple reflections, for otherwise we can write v as a product of two shorter commuting boolean elements and use induction.

We induct on the length of v. If v contains just one simple reflection, the claim is obvious. If $\ell(v) > 1$ then, up to taking the inverse of v, we may assume that 2 appears to the left of 1 in any reduced word for v. Let $k \geqslant 2$ be maximal with the property that $i+1$ appears to the left of i in any reduced word for v, for all $i < k$. Then $v = \big[k(k-1) \cdots 21\big] v'$, where $\mathsf{supp}(v') = [k+1, n-1]$ and, clearly, $\mathsf{run}(v) \leqslant \mathsf{run}(v') + 1$.

The permutation v' fixes all $i \leqslant k$, and hence $12 \cdots k$ belongs to any longest increasing subsequence in the one-line notation of v'. Multiplying v' by $\big[k(k-1) \ldots 21\big]$ moves 1 rightward past $2, 3, \ldots, k, v'(k+1)$ in the one-line notation. Since 1 is the smallest element, this operation can only keep or reduce the length of an increasing subsequence, compared to what had appeared in v'. In fact, because $k \geqslant 2$, this produces the inversion $2 > 1$ in v and hence necessarily makes all longest increasing subsequences in v strictly shorter than what had been in v'. Specifically, removing the 1 in a longest increasing subsequence for v' produces an increasing subsequence for v which is shorter by exactly one term. It follows that $\mathsf{row}_2(v) = \mathsf{row}_2(v') + 1$. Combining this with Corollary 6.3 and $\mathsf{run}(v) \leqslant \mathsf{run}(v') + 1$, by induction we have $\mathsf{run}(v) = \mathsf{run}(v') + 1$. This implies $\mathsf{row}_2(v) = \mathsf{run}(v)$, proving the claim. $\square$

6.2. Lusztig's a-function for the symmetric group

In [18], Lusztig introduced the function $\mathbf{a} : W \to \mathbb{Z}_{\geqslant 0}$, where W is a Coxeter group, with the following properties:

- $\mathbf{a}$ is constant on two-sided Kazhdan–Lusztig cells in W;

- $\mathbf{a}(w) \leqslant \ell(w)$, for all $w \in W$;

- $\mathbf{a}(w) = \ell(w)$ if w is the longest element of some parabolic subgroup of W.

In the special case of a symmetric groups, it is well-known, see [13], that two permutations v and w belong to the same two-sided Kazhdan–Lusztig cell if and only if $\mathsf{sh}(v) = \mathsf{sh}(w)$.

Given a partition $\lambda \vdash n$, consider the transposed partition $\mu := \lambda^t = (\mu_1, \mu_2, \ldots, \mu_m)$. Consider the parabolic subgroup W_μ of $\mathfrak{S}_n$ given by $\mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_m}$. Then it is easy to check that the Robinson–Schensted correspondent of the longest element in W_μ has shape λ. In particular, each two-sided Kazhdan–Lusztig cell contains the longest element of some parabolic subgroup. Therefore the properties of $\mathbf{a}$ listed above determine the function $\mathbf{a}$ for $\mathfrak{S}_n$ uniquely.

Lemma 6.5. *Let $w \in \mathfrak{S}_n$ be such that $\mathsf{sh}(w) = \lambda$. Then, for $\mu = \lambda^t$, we have*

$$\mathbf{a}(w) = \sum_{i=1}^{m} \frac{\mu_i(\mu_i - 1)}{2}.$$

Proof. This follows directly from the above and the fact that the length of the longest element in $\mathfrak{S}_k$ equals $\frac{k(k-1)}{2}$, for any k. $\qquad\square$

Corollary 6.6. *If $v \in \mathfrak{S}_n$ is boolean and $\mathsf{sh}(v) = \lambda$, then $\mathbf{a}(v) = \lambda_2(v)$.*

Proof. If v is boolean, λ has at most two rows. Therefore $\mu = \lambda^t = (2^{\lambda_2}, 1^{n-2\lambda_2})$. Now the claim follows directly from Lemma 6.5. $\qquad\square$

6.3. Grades of simple modules for boolean elements via Lusztig's a-function

We can now prove our main result.

Theorem 6.7. *Let $v \in \mathfrak{S}_n$ be boolean. Then $\mathbf{grade}(L_v) = \mathbf{a}(v)$.*

Proof. After the discussion in Section 4, the claim follows by combining Theorems 5.14 and 6.4 with Corollary 6.6. $\qquad\square$

We note that Lusztig's $\mathbf{a}$-function describes various homological invariants in BGG category $\mathcal{O}$, see [14, 20, 21].

7. Grades of simple modules for non-boolean elements

7.1. Longest elements in parabolic subgroups

We conclude by remarking upon how this work does and does not extend to non-boolean elements. Rene Marczinzik has computed the grades of all simple modules for $\mathfrak{S}_4$ (over $\mathbb{C}$) using a computer. In that case it turns out that $\mathbf{grade}(L_w) \neq \mathbf{a}(w)$, for the (non-boolean) permutations $w = [2132]$ and $w = [12321]$. This means that Theorem 6.7 does not generalize to all elements of W. It does, however, hold true in another special case, in some sense, the "opposite extreme" of the boolean elements.

Theorem 7.1. *Let $v \in \mathfrak{S}_n$ be the longest element of some parabolic subgroup. Then*

$$\mathbf{grade}(L_v) = \mathbf{a}(v) = \ell(v).$$

Proof. If $v = e$, then the claim is obvious. Therefore we assume $v \neq e$. Let $w \in \mathfrak{S}_n$. Consider some reduced word $[s] \in R(w)$. Let $[x]$ be the shortest prefix of $[s]$ with the property that no simple reflection in the suffix of $[s]$, defined as the complement to $[x]$, belongs to the support of v. From the subword property (see Theorem 2.2), it follows that $B(v) \cap B([x]) = B(v) \cap B(w)$. We have to consider two cases.

Case 1: $[x] = e$. In this case $B(v) \cap B([x]) = \{e\}$ and hence the corresponding complex (4.3) is concentrated in one degree, namely, in degree $-\ell(v)$.

Case 2: $[x] \neq e$. In this case, due to the minimality of $[x]$, the rightmost letter of x belongs to the support of v. Therefore the right descent set of $[x]$ contains a simple reflection that belongs to the support of v. Since v is the longest element in some parabolic subgroup, its support coincides with both its left descent set and its right descent set. In particular, $[x]$ and v have a common simple reflection in the right descent set. As explained in Subsection 4.7, this implies exactness of (4.3). Consequently, this case does not effect the computation of $\mathbf{grade}(L_v)$.

It follows that $\mathbf{grade}(L_v) = \ell(v)$ and the claim of the theorem now follows from the property $\mathbf{a}(v) = \ell(v)$, for v the longest element of a parabolic subgroup. $\qquad\square$

Remark 7.2. One could observe that the permutations $w = [2132]$ and $w = [12321]$ are exactly the two elements of $\mathfrak{S}_4$ for which the corresponding Kazhdan–Lusztig polynomial $P_{e,w}$ is non-trivial. By a result of Deodhar, see [6], this condition is equivalent to nonsingularity of the Schubert variety for w.

Unfortunately, at the present stage we do not know whether it is reasonable to extrapolate this observation to a guess for higher ranks. In order to investigate this kind of guess, we need a better understanding of the combinatorial structure of intersections of principal Bruhat ideals. Maybe the recent preprints [4, 32], which appeared after the preprint version of the present paper, will be helpful.

7.2. Classification of perfect simple module

Recall that a module is called *perfect* if its grade coincides with its projective dimension. Theorem 7.1 leads to the following classification of perfect simple modules.

Theorem 7.3. *For $w \in \mathfrak{S}_n$, the module L_w is perfect if and only if w is the longest element in some parabolic sugroup of $\mathfrak{S}_n$.*

Proof. By Proposition 4.2, the projective dimension of L_w is given by $\ell(w)$. This means that the "if" part of the claim follows directly from Theorem 7.1.

To prove the "only if" part, suppose that w is not the longest element in any parabolic subgroup. Let G be the minimal parabolic subgroup containing w. Let $u \in G$ be minimal having the property that $u \notin B(w)$. Because w is not the maximum element of G, such u exists. Since G is the minimal parabolic subgroup containing w, all simple reflections generating G belong to the support of w, in particular, they are all less than w in the Bruhat order and hence cannot coincide with u. This implies that $\ell(u) > 1$.

Consider $B(w) \cap B(u)$. By construction, we have $B(w) \cap B(u) = B(u) \setminus \{u\}$. We know that there is a perfect matching, call it F, of the elements of $B(u)$. Restricting F to $B(u) \setminus \{u\}$ gives an almost perfect matching whose only unmatched element is u's partner under F. This unmatched element of $B(w) \cap B(u)$ has length $\ell(u) - 1$. Therefore the grade of w is at most $\ell(w) - (\ell(u) - 1) < \ell(w)$ which means that L_w is not perfect. $\qquad\square$

We note that both Theorem 7.1 and 7.3 are true, with the same proofs, for arbitrary finite Weyl groups.

Acknowledgments

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