Title
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Journal
International Mathematics Research Notices, 2018(3)

ISSN
1687-3017

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Publication Date
2018

DOI
10.1093/imrn/rnw269

Peer reviewed

X. Li *et al.* (2016) "Four-Dimensional Gradient Shrinking Solitons with Positive Isotropic Curvature,"
International Mathematics Research Notices, Vol. 2018, No. 3, pp. 949–959
Advance Access Publication December 24, 2016
doi: 10.1093/imrn/rnw269

Four-Dimensional Gradient Shrinking Solitons with Positive Isotropic Curvature

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We show that a four-dimensional complete gradient shrinking Ricci soliton with positive isotropic curvature is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. This gives a clean classification result removing the earlier additional assumptions in [14] by Wallach and the second author. The proof also gives a classification result on gradient shrinking Ricci solitons with non-negative isotropic curvature.

1 Introduction

A gradient shrinking Ricci soliton is a triple (M, g, f), a complete Riemannian manifold with a smooth potential function f whose Hessian satisfying

$$\mathrm{Ric} + \nabla\nabla f = \frac{1}{2} g. \tag{1.1}$$

Gradient shrinking solitons arise naturally in the study of the singularity analysis of the Ricci flow [4, 15]. It also attracts the study since (1.1) generalizes the notion of Einstein metrics. The main purpose of this article concerns a classification of such four-manifolds with positive isotropic curvature. The positive isotropic curvature condition was first introduced by Micalleff and Moore [7] in applying the index computation of harmonic spheres to the study of the topology of manifolds. The Ricci flow on four-manifolds with

Received March 10, 2016; Revised October 3, 2016; Accepted October 21, 2016

positive isotropic curvature was studied by Hamilton [5]. This condition was proven to be invariant under Ricci flow in dimension four by Hamilton and in high dimensions by Brendle and Schoen [1] and Nguyen [12]. It is hence then interesting to understand the solitons under the positive isotropic curvature condition. In [14], Wallach and the second author classified the four-dimensional gradient shrinking solitons with positive isotropic curvature under some extra assumptions, including the nonnegative curvature operator, a pinching condition and a curvature growth condition. Since then, there have been much progresses in understanding the general shrinking solitons [3, 6, 8, 10, 11] and particularly the four-dimensional ones [9]. In particular, a classification result was obtained in [10] for solitons with nonnegative curvature operator for all dimensions. The purpose of this article is to prove the following classification result on shrinking solitons with positive isotropic curvature by removing all the additional assumptions in [14].

Theorem 1.1. Any four-dimensional complete gradient shrinking Ricci soliton with positive isotropic curvature is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. □

It remains an interesting question whether or not the same result holds in high dimensions. We plan to return to this in the future.

2 Preliminaries

In this section, we collect some results on gradient shrinking Ricci solitons that will be used in this article.

After normalizing the potential function f via translating, we have the following identities [4]:

Lemma 2.1.

$$S + \Delta f = \frac{n}{2},$$
$$S + |\nabla f|^2 = f,$$

where S denotes the scalar curvature of M. □

Regarding the growth of the potential function f and the volume of geodesic balls, Cao and Zhou [3] showed that

Lemma 2.2. Let (M^n, g) be a complete gradient shrinking Ricci soliton of dimension n and $p \in M$. Then there are positive constants c_1, c_2 and C such that

$$\frac{1}{4}\left(d(x,p)-c_1\right)_+^2 \le f(x) \le \frac{1}{4}\left(d(x,p)+c_2\right)^2,$$
$$\mathrm{Vol}(B_p(r)) \le Cr^n. \qquad \square$$

Munteanu and Sesum [8] proved the following integral bound for the Ricci curvature.

Lemma 2.3. Let (M, g) be a complete gradient shrinking Ricci soliton. Then for any $\lambda > 0$, we have

$$\int_M |\,\mathrm{Ric}\,|^2 e^{-\lambda f} < \infty. \qquad \square$$

The above mentioned results are valid in all dimensions. In the following, we recall some special properties in four dimensions. It is well known that, in dimension four, the curvature operator R can be written as

$$R = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

according to the natural splitting $\wedge^2(\mathbb{R}^4) = \wedge_+ \oplus \wedge_-$, where $\wedge_+$ and $\wedge_-$ are the self-dual and anti-self-dual parts respectively. It is easy to see that A and C are symmetric. Denote $A_1 \le A_2 \le A_3$ and $C_1 \le C_2 \le C_3$ the eigenvalues of A and C, respectively. Also let $0 \le B_1 \le B_2 \le B_3$ be the singular eigenvalues of B. A direct consequence of the first Bianchi identity is that $\mathrm{tr}(A) = \mathrm{tr}(C) = \frac{S}{4}$, where S is the scalar curvature. Moreover, M has positive isotropic curvature amounts to $A_1 + A_2 > 0$ and $C_1 + C_2 > 0$. In [14], Wallach and the second author computed that $\overset{\circ}{\mathrm{Ric}}$ can be expressed in terms of components of B, where $\overset{\circ}{\mathrm{Ric}}$ is the traceless part of the Ricci tensor. In particular, $4\|B\|^2 = |\overset{\circ}{\mathrm{Ric}}|^2$ and $\sum_1^4 \lambda_i^3 = 24 \det B$, where λ_i's are the eigenvalues of $\overset{\circ}{\mathrm{Ric}}$. It was also observed in [14, Proposition 3.1] that these components of the curvature operator satisfy certain differential inequalities, which play a significant role in the classification results.

Proposition 2.1. If $(M, g(t))$ is a solution to the Ricci flow, then we have the following differential inequalities

$$\left(\frac{\partial}{\partial t} - \Delta\right)(A_1 + A_2) \ge A_1^2 + A_2^2 + 2(A_1 + A_2)A_3 + B_1^2 + B_2^2,$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)(C_1 + C_2) \geq C_1^2 + C_2^2 + 2(C_1 + C_2)C_3 + B_1^2 + B_2^2,$$
$$\left(\frac{\partial}{\partial t} - \Delta\right) B_3 \leq A_3 B_3 + C_3 B_3 + 2B_1 B_2$$

in the distributional sense. □

We now summarize the ideas used to prove the classification result in [14] and explain our strategy of removing all the assumptions except positive isotropic curvature. First of all, it was shown in [13, Proposition 4.2] that if $S \neq 0$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|R_{ijkl}|^2}{S^2}\right) = \frac{4P}{S^3} - \frac{2}{S^4}|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 + \left\langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2 \right\rangle,$$

where P is defined by

$$P = 4S\langle R^2 + R^\sharp, R\rangle - |\operatorname{Ric}|^2 |R_{ijkl}|^2.$$

Here # is the operator defined using the Lie algebra structure of $\wedge^2 T_p M$ (see e.g., [4]). Under a certain curvature growth assumption, the classification result follows from the proof of the main theorem in [13] if one can show that $P \leq 0$. In dimension 4, P can be expressed in terms of A, B, and C. Let $\mathring{A}$ and $\mathring{C}$ be the traceless parts of A and C, respectively. By choosing suitable basis of $\wedge_+$ and $\wedge_-$, we may diagonalize $\mathring{A}$ and $\mathring{C}$ such that

$$A = \begin{pmatrix} \frac{S}{12} + a_1 & 0 & 0 \\ 0 & \frac{S}{12} + a_2 & 0 \\ 0 & 0 & \frac{S}{12} + a_3 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{S}{12} + c_1 & 0 & 0 \\ 0 & \frac{S}{12} + c_2 & 0 \\ 0 & 0 & \frac{S}{12} + c_3 \end{pmatrix}.$$

Then P can be written as (see [14])

$$P = -S^2\left(\frac{1}{6}\sum_1^4 \lambda_i^2 + \sum_1^3 a_i^2 + \sum_1^3 c_i^2\right)$$
$$+ 4S\left(\sum_1^3 (a_i^3 + c_i^3) + 6a_1a_2a_3 + 6c_1c_2c_3 - \frac{1}{2}\sum_1^4 \lambda_i^3\right)$$

$$+ 12S\left(a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + c_1\tilde{b}_1^2 + c_2\tilde{b}_2^2 + c_3\tilde{b}_3^2\right)$$
$$- 2\left(\sum_1^4 \lambda_i^2\right)^2 - 4\left(\sum_1^4 \lambda_i^2\right)\left(\sum_1^3 (a_i^2 + c_i^2)\right),$$

where $\sum_1^3 a_i = \sum_1^3 c_i = \sum_1^4 \lambda_i = 0$, $b_i^2 = \sum_{j=1}^3 B_{ij}^2$ and $\tilde{b}_i^2 = \sum_{j=1}^3 B_{ji}^2$.

A key observation in [14] is that for several basic examples, one always has $BB^t = b^2\,\mathrm{id}$ for some constant b and $P = 0$. By considering the evolution equation of the quantity $\frac{B_3^4}{(A_1+A_2)^2(C_1+C_2)^2}$ and applying integration by parts, Wallach and the second author proved that $BB^t = b^2\,\mathrm{id}$ is indeed valid for all four-dimensional shrinking Ricci solitons satisfying two very weak curvature assumptions:

$$\frac{B_3^2}{(A_1 + A_2)(C_1 + C_2)}(x) \le \exp(a(r(x) + 1)), \tag{2.1}$$

$$|R_{ijkl}|(x) \le \exp(a(r(x) + 1)) \tag{2.2}$$

for some $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. These two assumptions are only needed to ensure that all the integrals involved in the integration by parts argument are finite. To remove these two curvature assumptions, we show that $BB^t = b^2\,\mathrm{id}$ in fact holds for all four-dimensional shrinking Ricci solitons with positive isotropic curvature. Our strategy is to consider the evolution equation of $\frac{B_3}{\sqrt{(A_1+A_2)(C_1+C_2)}}$ and sharpen the integration by parts argument in [14]. In the proof, in order to guarantee that all the integrals involved are finite, we use the integral bound of the Ricci curvature in Lemma 2.3.

Now when $BB^t = b^2\,\mathrm{id}$, P has a much simpler expression:

$$P = -S^2\left(\sum_1^3 a_i^2 + \sum_1^3 c_i^2\right) + 12S\left(\sum_1^3 (a_i^3 + c_i^3)\right)$$
$$- 2b^2\,(S + 12b)^2 - 48b^2\left(\sum_1^3 a_i^2 + \sum_1^3 c_i^2\right)$$
$$\le -S\left(S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3\right) - S\left(S\sum_1^3 c_i^2 - 12\sum_1^3 c_i^3\right),$$

where we have used the following

$$3b^2 = \sum_1^3 b_i^2 = \sum_1^3 \tilde{b}_i^2 = \frac{1}{4}\sum_1^4 \lambda_i^2,$$

$$\sum_1^3 a_i^3 + 6a_1a_2a_3 = 3\sum_1^3 a_i^3,$$

$$\sum_1^4 \lambda_i^3 = 24\det B = 24b^3.$$

Then by further assuming that M has non-negative curvature operator, namely, A and C are positive semidefinite, $P \le 0$ was proved in [14] by solving two optimization problems with constraints. More precisely, they showed that under the constraints $\sum_1^3 a_i = 0$ and $\frac{S}{12} + a_i \ge 0$, the maximum of $\sum_1^3 a_i^2$ is $\frac{S^2}{24}$, while under further normalization $\sum_1^3 a_i^2 = 1$ it holds that

$$\frac{\sum_1^3 a_i^3}{\sum_1^3 a_i^2} \le \frac{1}{\sqrt{6}}\left(\sum_1^3 a_i^2\right)^{\frac{1}{2}}.$$

Combining these results, one easily obtains

$$S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \ge 0.$$

The terms with c_i' s can be handled similarly and thus they arrived at that $P \le 0$.

By solving another optimization problem under weaker constraints, we achieve in this paper that $P \le 0$ without assuming M has non-negative curvature operator. In the last step we also sharpen the integration by parts argument in [13] instead of appealing the result in [13] as in [14] since we no longer assume a curvature growth condition (2.2). These three improvements allow us to prove the main result without any additional assumptions.

3 Proof of Theorem 1.1

For brevity, we introduce the same notations as in [14]: $\psi_1 = A_1 + A_2$, $\psi_2 = C_1 + C_2$, $\varphi = B_3$ and

$$\begin{aligned} -E = &-\frac{4B_1(B_3 - B_2)}{B_3} - \frac{(A_1 - B_1)^2 + (A_2 - B_2)^2 + 2A_2(B_2 - B_1)}{A_1 + A_2} \\ &- \frac{(C_1 - B_1)^2 + (C_2 - B_2)^2 + 2C_2(B_2 - B_1)}{C_1 + C_2}. \end{aligned}$$

It is clear that $-E \le 0$ with equality holds only if $A_1 = C_1 = B_1 = B_2 = A_2 = C_2 = B_3$. In particular we have $BB^t = b^2\,\mathrm{id}$ for some b. Also notice that M has positive isotropic curvature amounts to $\psi_1 > 0$ and $\psi_2 > 0$.

Proposition 3.1. The following differential inequality holds in the sense of distribution:

$$\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} \le -\frac{1}{2}\frac{\varphi}{\sqrt{\psi_1\psi_2}}E-\frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}} + \left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle. \tag{3.1}$$

□

Proof. Straightforward calculations yield

$$\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} = \frac{\left(\frac{\partial}{\partial t}-\Delta\right)\varphi}{\sqrt{\psi_1\psi_2}}-\frac{1}{2}\frac{\varphi\psi_1\left(\frac{\partial}{\partial t}-\Delta\right)\psi_2+\varphi\psi_2\left(\frac{\partial}{\partial t}-\Delta\right)\psi_1}{(\psi_1\psi_2)^{\frac{3}{2}}} - \frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}}+\left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle.$$

Substituting the differential inequalities in Proposition 2.1 into the above equation gives, after some cancelations, that

$$\left(\frac{\partial}{\partial t}-\Delta\right)\frac{\varphi}{\sqrt{\psi_1\psi_2}} \le -\frac{1}{2}\frac{\varphi}{\sqrt{\psi_1\psi_2}}E-\frac{1}{4}\frac{\varphi|\psi_1\nabla\psi_2-\psi_2\nabla\psi_1|^2}{(\psi_1\psi_2)^{\frac{5}{2}}} + \left\langle\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right),\nabla\log(\psi_1\psi_2)\right\rangle.$$

This proves the proposition. ■

Proposition 3.2. Let (M,g,f) be a four-dimensional gradient shrinking Ricci soliton with positive isotropic curvature. Then $BB^t = b^2\,\mathrm{id}$. □

Proof. On a gradient Ricci soliton, it holds that

$$\frac{\partial}{\partial t}\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right)=\left\langle\nabla f,\nabla\left(\frac{\varphi}{\sqrt{\psi_1\psi_2}}\right)\right\rangle.$$

Fix a point x_0 in M. For any $r > 0$, one can choose a smooth cut-off function η with support in $\{x \in M : d(x,x_0) \le r\}$ and $|\nabla\eta| \le C/r$. Then multiplying both sides of (3.1) by

$\frac{\varphi}{\sqrt{\psi_1\psi_2}} e^{-f+\log(\psi_1\psi_2)}\eta^2$ and integrating over M,

$$\int_M \left(\left\langle \nabla f, \nabla \left(\frac{\varphi}{\sqrt{\psi_1\psi_2}} \right) \right\rangle - \Delta \left(\frac{\varphi}{\sqrt{\psi_1\psi_2}} \right) \right) \frac{\varphi}{\sqrt{\psi_1\psi_2}} \mathrm{e}^{-f+\log(\psi_1\psi_2)}\eta^2$$
$$\leq -\frac{1}{2}\int_M \varphi^2 E \mathrm{e}^{-f}\eta^2 + \int_M \left\langle \nabla \left(\frac{\varphi}{\sqrt{\psi_1\psi_2}} \right), \nabla \log(\psi_1\psi_2) \right\rangle \frac{\varphi}{\sqrt{\psi_1\psi_2}} \mathrm{e}^{-f+\log(\psi_1\psi_2)}\eta^2.$$

After integration by parts and some cancelations, we arrive at

$$0 \leq -\int_M \left| \nabla \left(\frac{\varphi}{\sqrt{\psi_1\psi_2}} \right) \right|^2 \mathrm{e}^{-f+\log(\psi_1\psi_2)}\eta^2 - \frac{1}{2}\int_M \varphi^2 E \mathrm{e}^{-f}\eta^2$$
$$-2\int_M \left\langle \nabla \left(\frac{\varphi}{\sqrt{\psi_1\psi_2}} \right), \nabla\eta \right\rangle \frac{\varphi}{\sqrt{\psi_1\psi_2}} \mathrm{e}^{-f+\log(\psi_1\psi_2)}\eta$$
$$\leq \int_M \varphi^2 |\nabla\eta|^2 \mathrm{e}^{-f} - \frac{1}{2}\int_M \varphi^2 E \mathrm{e}^{-f}\eta^2$$
$$\leq \frac{C^2}{r^2}\int_M \varphi^2 \mathrm{e}^{-f} - \frac{1}{2}\int_M \varphi^2 E \mathrm{e}^{-f}\eta^2.$$

Since $4\varphi^2 = 4B_3^2 \leq 4\|B\|^2 = |\mathring{\mathrm{Ric}}|^2 = |\,\mathrm{Ric}\,|^2 - \frac{S^2}{4}$, we obtain, in view of Lemmas 2.1, 2.2, and 2.3,

$$\int_M \varphi^2 e^{-f} \leq \frac{1}{4}\int_M \left(|\,\mathrm{Ric}\,|^2 - \frac{S^2}{4} \right) \mathrm{e}^{-f} < \infty.$$

Letting $r \to \infty$ implies that $\int_M \varphi^2 E e^{-f} = 0$. Therefore, we must have either $B_3 = 0$ or $E = 0$. It then follows that $BB^t = b^2\,\mathrm{id}$. ■

Proposition 3.3. Let (M, g, f) be a four-dimensional gradient shrinking Ricci soliton with positive isotropic curvature. Then $P \leq 0$. □

Proof. Recall from Section 2, since $BB^t = b^2\,\mathrm{id}$, we have

$$P \leq -S\left(S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \right) - S\left(S\sum_1^3 c_i^2 - 12\sum_1^3 c_i^3 \right).$$

In order to prove $P \leq 0$, it suffices to show that

$$S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \geq 0.$$

With suitable choices of the orthonormal basis for $\wedge_+$, we can assume that $A_i = \frac{S}{12} + a_i$. We have the constraints $\sum_1^3 a_i = 0$ and $A_i + A_j = \frac{S}{6} + a_i + a_j > 0$ for $i \neq j$ because M

has positive isotropic curvature. By the change of variables $x_i = 1 - \frac{6}{S}a_i$, the constraints become $\sum_1^3 x_i = 3$ and $x_i > 0$ for $1 \le i \le 3$, and the objective function becomes

$$F(x_1, x_2, x_3) := S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 = \frac{S^3}{36}\left(\sum_1^3 (1-x_i)^2 - 2(1-x_i)^3\right)$$
$$= \frac{S^3}{36}\left(2\sum_1^3 x_i^3 - 5\sum_1^3 x_i^2 + 9\right).$$

Using Lagrange multipliers, we find two critical points $Z = (1,1,1)$ and $W = \left(\frac{1}{3}, \frac{4}{3}, \frac{4}{3}\right)$ with $F(Z) = 0$ and $F(W) = \frac{S^3}{162}$. On the boundary, we have $x_i = 0$ for some i. Since F is symmetric, we can assume without loss of generality that $x_1 = 0$. Then we have, using $x_3 = 3 - x_2$, that

$$F(x_1, x_2, x_3) = \frac{S^3}{36}\left(2(x_2^3 + (3-x_2)^3) - 5(x_2^2 + (3-x_2)^2) + 9\right) = \frac{S^3}{18}(2x_2 - 3)^2 \ge 0.$$

Therefore, under the constraints $\sum_1^3 a_i = 0$ and $\frac{S}{6} + a_i + a_j \ge 0$ for $i \ne j$,

$$F(x_1, x_2, x_3) = S\sum_1^3 a_i^2 - 12\sum_1^3 a_i^3 \ge 0.$$

The terms involving c_i's can be handled similarly. Hence $P \le 0$. ■

Proof of Theorem 1.1. Recall, it was shown in [13, Proposition 4.2] that if $S \ne 0$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|R_{ijkl}|^2}{S^2}\right) = \frac{4P}{S^3} - \frac{2}{S^4}|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 + \left\langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2\right\rangle. \tag{3.2}$$

Trying to use integration by parts here would require a stronger integral bound of the Ricci curvature than we actually have in Lemma 2.3. To overcome this difficulty, we adopt a similar idea that was used to prove $BB^t = b^2\,\mathrm{id}$. We consider, instead, $u = \frac{|R_{ijkl}|}{S}$ and $T = \frac{R_{ijkl}}{S}$. A direct calculation shows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)u = \frac{2P}{uS^3} + \langle \nabla u, \nabla \log S^2\rangle + \frac{|\nabla u|^2 - |\nabla T|^2}{u}$$
$$\le \frac{2P}{uS^3} + \langle \nabla u, \nabla \log S^2\rangle,$$

where we have used Kato's inequality in the last line. Now let η be a smooth cut-off function with support in $\{x \in M : d(x, x_0) \le r\}$ and $|\nabla\eta| \le C/r$. Multiplying the above inequality by $ue^{-f+\log S^2}\eta^2$ and integrating over M, we obtain,

$$\begin{aligned}
&\int_M \langle \nabla f, \nabla u\rangle u e^{-f+\log S^2}\eta^2 - \int_M \Delta u \; u e^{-f+\log S^2}\eta^2 \\
&\quad \le \int_M \frac{2P}{S^3} e^{-f+\log S^2}\eta^2 + \int_M \langle \nabla u, \nabla \log S^2\rangle u e^{-f+\log S^2}\eta^2.
\end{aligned}$$

Integration by parts then yields

$$\begin{aligned}
&\frac{1}{2}\int_M |\nabla u|^2 e^{-f+\log S^2}\eta^2 - 2\int_M \frac{P}{S} e^{-f} \\
&\quad \le -\frac{1}{2}\int_M |\nabla u|^2 e^{-f+\log S^2}\eta^2 - 2\int_M \langle \nabla u, \nabla \eta\rangle u\eta e^{-f+\log S^2} \\
&\quad \le 8\int_M |\nabla \eta|^2 u^2 e^{-f+\log S^2} \\
&\quad \le \frac{C}{r^2}\int_M |R_{ijkl}|^2 e^{-f}.
\end{aligned}$$

The integral $\int_M |R_{ijkl}|^2 e^{-f}$ is finite in view of Lemma 2.3, since if M has positive isotropic curvature, then the components of curvature operator A, B and C can be estimated by

$$\begin{aligned}
&-\frac{S}{4} \le A_1 \le A_2 \le A_3 \le \frac{S}{4}, \\
&-\frac{S}{4} \le C_1 \le C_2 \le C_3 \le \frac{S}{4}, \\
&4\|B\| \le |\overset{\circ}{\mathrm{Ric}}|.
\end{aligned}$$

Therefore, we know that u is a positive constant and $P = 0$ by letting $r \to \infty$. Then it follows from (3.2) that $|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}|^2 = 0$. Theorem 1.1 then follows from the proof of the main theorem in [13]. ■

The strong maximum principle of [2] together with the classification of positive case implies the following corollary for the solitons with nonnegative isotropic curvature.

Corollary 3.1. If (M, g, f) is a complete gradient shrinking soliton with nonnegative isotropic curvature then its universal cover must be one of the following spaces $\mathbb{R}^4$, $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$ and $\mathbb{S}^3 \times \mathbb{R}$. □

This provides a more general classification result than that of [11], where the shrinking solitons are assumed to have bounded non-negative curvature operator.

Funding

This work was partially supported by an Inamori fellowship and National Science Foundation [DMS-1401500 to X.L.]; National Science Foundation [DMS-1401500 to L.N.]; National Natural Science Foundation of China [11601359 to K.W.]; and Natural Science Foundation of Jiangsu Province [BK20160301to K.W.].

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