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Generation of bulk vorticity and current density in current-vortex sheet models[☆]

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Abstract

Matsuoka et al. [1] present a set of equations governing the evolution of two-dimensional MHD flows using current-vortex sheets. In the resulting model the vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ are zero except on the current-vortex sheets. We show that this is not true in general and that the term $\Delta(\boldsymbol{B} \times \boldsymbol{u})$ in the evolution equation for $\boldsymbol{j}$ does not vanish, leading to the generation of $\boldsymbol{j}$ and $\boldsymbol{\omega}$ in the bulk. This means that the evolution of the system is not governed solely by the dynamics of quantities on the current-vortex sheets. A perturbative solution is derived that shows this explicitly.

Keywords: Non-uniform current-vortex sheet, Surface Alfvén wave

1. Introduction

In this journal Matsuoka et al. [2] investigate the nonlinear motion of vortex sheets with non-uniform current, motivated by magnetohydrodynamic Richtmyer–Meshkov and Kelvin–Helmholtz instabilities. They employ a set of equations governing the evolution of two-dimensional MHD flows using current-vortex sheets originally derived in a previous paper [1] (hereafter MNS2017). An attractive feature of this system is that its evolution is governed solely by the dynamics of quantities on the sheets.

For this to be possible, the vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ must vanish in the bulk. MNS2017 argue that if these quantities initially vanish so that the flow is irrotational and current free in the bulk, they remain zero during the evolution of the system. The purpose of this Short Communication is to show from the governing equations that this is not generally true and obtain an explicit solution with bulk vorticity and current density.

2. Generation of bulk quantities

We use the same equations as MNS2017 for two-dimensional MHD flow in the (x, y) plane. The bulk vorticity $\boldsymbol{\omega} = \omega \boldsymbol{e}_z = \boldsymbol{\nabla} \times \boldsymbol{u}$ and current density $\boldsymbol{j} = j\boldsymbol{e}_z = \boldsymbol{\nabla} \times \boldsymbol{B}$ (both directed out of the plane) are governed by (equations 2.6–2.7 of MNS2017)

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{1}{\rho}(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{j}, \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}t} = (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{j} + \Delta(\boldsymbol{B} \times \boldsymbol{u}), \tag{2}$$

where $\mathrm{d}/\mathrm{d}t$ is the Lagrangian derivative and the magnetic permeability μ has been scaled to unity. The critical issue is that the second term on the right-hand side of (2) is a product of two terms that are not localized on current-vortex sheets (they are given by the Biot–Savart law from terms on the sheets but the resulting expression is not illuminating).

MNS2017 assume that the right-hand sides of (1) and (2) are Lipschitz continuous. Assuming also that $\boldsymbol{j} = \boldsymbol{0}$ and $\Delta(\boldsymbol{B} \times \boldsymbol{u}) = \boldsymbol{0}$ in the bulk at $t = 0$, they argue that the right-hand sides of (1) and (2)

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vanish initially and for all times, and that hence $\boldsymbol{j}(t) = \boldsymbol{\omega}(t) = \mathbf{0}$ in the bulk.

However, this is not necessarily the case, since $\Delta(\boldsymbol{B} \times \boldsymbol{u})$ does not necessarily vanish at later times. Hence the equations obtained by [1] are missing the contributions of bulk current density and vorticity. The presence of bulk quantities means that the evolution of the system is not governed solely by the dynamics of quantities on the current-vortex sheets.

3. Perturbative solution

To make this argument more concrete, we derive a solution to the governing equations that includes bulk current density and vorticity starting from an initial condition consistent with the assumptions of MNS2017. The full system is nonlinear and analytic solutions are intractable, so we present a perturbative solution. Since this solution shows the presence of bulk contributions, these must be present in the full nonlinear solution. The analysis will initially parallel that of § 3 of MNS2017, in which linearized surface Alfvén wave solutions were computed.

As in MNS2017, we consider a situation with an initial magnetic field parallel to the interface between fluids 1 (below) and 2 (above), taken to be at $y = 0$, an initially irrotational velocity field and no initial current density.

We construct a solution with a small velocity perturbation, say $O(\delta)$, and $O(1)$ magnetic field. Then

$$\boldsymbol{B} = (B_0, 0) + \delta \boldsymbol{b}_1 + \delta^2 \boldsymbol{b}_2 + \cdots, \quad \boldsymbol{u} = \delta \boldsymbol{u}_1 + \delta^2 \boldsymbol{u}_2 + \cdots, \tag{3}$$

and other variables use the same suffices. The x- and y-components of $\boldsymbol{u}$ are u and v, and similarly for $\boldsymbol{u}_1$ and so on. The $O(1)$ term equations are satisfied trivially. At $O(\delta)$, (1) and (2) become

$$\frac{\partial \omega_1}{\partial t} = \frac{B_0}{\rho} \frac{\partial j_1}{\partial x}, \qquad \frac{\partial j_1}{\partial t} = B_0 \Delta v_1. \tag{4}$$

Initially the velocity is irrotational, so that $\Delta v_1 = 0$ at $t = 0$. Hence the right-hand sides of (1) and (2) vanish at $t = 0$. This means that ω_1 and j_1 do not change for all times, so that $\omega_1(t) = j_1(t) = 0$ as claimed by MNS2017.

We move to the next order, where

$$\frac{\partial \omega_2}{\partial t} = \frac{B_0}{\rho} \frac{\partial j_2}{\partial x}, \quad \frac{\partial j_2}{\partial t} = B_0 \Delta v_2 + \Delta(b_{1x} v_1 - b_{1y} u_1). \tag{5}$$

We hence need to understand the $O(\delta)$ problem for the magnetic field and velocity. This is a linear problem governed by the equations given in § 3 of MNS2017, which are correct since there is no bulk vorticity or current density at this order, as shown above. Following MNS2017, we consider wave solutions, but taken to have zero initial current density. Then in the lower layer

$$\begin{aligned} \boldsymbol{u}_1 &= (-\sin kx, \cos kx) \cos \varpi t e^{ky}, \\ \boldsymbol{b}_1 &= -\frac{B_0}{v_{a0}} (\cos kx, \sin kx) \sin \varpi t e^{ky}, \end{aligned} \tag{6}$$

where v_{a0} is the Alfvén velocity given by

$$v_{a0}^2 = \frac{2B_0^2}{\rho_1 + \rho_2}. \tag{7}$$

As a result

$$\omega_1 = -2 \sin kx \cos \varpi t, \quad j_1 = -2 \frac{B_0}{v_{a0}} \cos kx \sin \varpi t, \tag{8}$$

respectively, corresponding at $t = 0$ to the initial conditions (6.1) in MNS2017. Hence

$$b_{1x} v_1 - b_{1y} u_1 = \frac{B_0}{2 v_{a0}} \sin 2\varpi t e^{2ky}. \tag{9}$$

This is not a harmonic function, so the term $\Delta(b_{1x} v_1 - b_{1y} u_1)$ does not vanish and appears on the right-hand side of (5) as a forcing term that generates j_2 in the lower layer. Similarly j_2 is generated in the upper layer. Note that it is not possible to cancel (9) by choosing v_2 appropriately since ω_2 and j_2 are coupled.

By combining waves with different wavenumbers, all of which are solutions of the linear problem, i.e. by Fourier synthesis, one can construct initial conditions with arbitrary x-dependence. Each Fourier mode will behave as above.

4. Conclusion

We have shown that the evolution of the system is not governed solely by the dynamics of quantities on the current-vortex sheets. Our perturbative solution identifies explicitly the effect, which occurs at $O(\delta^2)$ and is not present in the linearized Alfén wave analysis.

MNS2017 show current density and vorticity from ideal MHD simulations in their Figures 4(b,c) and 5(b,c). The former has initial conditions resulting from a shock water interaction, the latter is initialized directly by a perturbation of the interface. The resulting solutions show the presence of bulk vorticity and magnetic current density concentrated near the sheet. While the sheet has deformed considerably, so that quantities are no longer well described by a perturbative solution, the form of the vorticity and in particular of the current density is consistent with the present results: largest near the sheet, as in the decay away from the sheet in (9), and with different signs of $\boldsymbol{j}$ on either side of the sheet. The vorticity has spread more because of diffusion in the results of MNS2017, but the bulk current density cannot be solely due to diffusion as it has both signs.

The results given here confirm that the presence of bulk current density and vorticity is not just due to diffusive or numerical effects. The results of MNS2017 indicate that their reduced model can be useful since it matches the full simulations reasonably well, but the model is inconsistent with the governing equations. (This is different from MNS2017's use of a regularization term in the Biot–Savart integral, which is also an approximation, but a purely numerical one.)

References

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