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Publication Date

1982-03-01

Peer reviewed

LBL--14228

LBL-14228

DE82 018583

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March 1982

**Presented at the
Symposium on Unsaturated Flow and Transport Modeling
Seattle, Washington, March 23-24, 1982**

**This work was supported by the Nuclear Regulatory Commission
through Battelle Pacific Northwest Laboratories,
Richland, Washington.**

and

The U.S. Department of Energy under Contract DE-AC03-76SF00098.

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ABSTRACT

Partially saturated soil materials undergo consolidation, heave, collapse and failure due to changes in pore fluid pressure. The precise nature of the mechanics of such deformations is only poorly understood at present. Experimental evidence has shown that the volume change behavior of unsaturated soils cannot be adequately explained through changes in effective stress, even when a saturation dependent parameter is incorporated into the definition of effective stress. Two independent stress-state variables, involving combinations of total stress, pore air pressure and pore water pressure, are required to characterize volume changes and saturation changes in the partially saturated state. In general, two coupled conservation equations, one for the water-phase and the other for the air-phase need to be solved in order to predict the deformation behavior of unsaturated soils. If directional displacements and changes in the stress-field are required, then the conservation equations are to be integrated with an additional set of multi-dimensional force balance equations. For lack of a sufficient understanding of elastic constants such as Poisson's Ratio and Lamé's constants as applied to unsaturated soils, little has

been achieved so far in integrating the conservation equations and the force balance equations. For the long-term modeling of consolidation with respect to Uranium mill tailings, it may be acceptable and economical to solve a single conservation equation for water, assuming that the air-phase is continuous and is at atmospheric pressure everywhere in the soil. The greatest challenge to modeling consolidation in the unsaturated zone at the present time is to develop enough experimental data defining the variation of void ratio and saturation with reference to the two chosen stress-state variables.

INTRODUCTION

The tailings from Uranium mills are often disposed off in the form of slurries into lined or unlined ponds. After emplacement, water drains from the slurry material, soon causing the material to desaturate and become partially saturated. Drainage continues for prolonged periods of time in the partially saturated state, until the hydrological regime within the tailings comes to a dynamic equilibrium with the local climatic conditions and the local ground water regime.

As drainage proceeds, the tailings material successively goes through liquid, plastic, semisolid and solid states before attaining a relatively stable configuration. During this transition, the tailings may undergo significant deformation in the form of consolidation, heave or even failure. There is reason to believe that much of this deformation may occur while the material is in a state of partial saturation. Apart from the fact that the deformation of the piles is of immediate concern for safe engineering management, it is of significance that deformation strongly controls permeability variations within the tailings material. Thus the distribution of fluid velocities, which is critical to the modeling of chemical transport, is strongly influenced by the dynamics of the consolidation process. The motivation for realistically simulating the consolidation process in the unsaturated zone stems both from a purely engineering view point of tailings management and the phenomenological view point of simulating chemical transport within the tailings.

The purpose of this paper is to review the current status of knowledge in the area of modeling consolidation in the unsaturated zone. Although the flow of water in the soil zone close to the atmospheric boundary may often be influenced by temperature changes and, in the case of Uranium mill tailings, water movement may also be influenced by chemical potentials, we shall restrict our attention to isothermal fluid flow. We shall be concerned essentially with an unconsolidated porous material with coexisting water and air phases.

THE PROBLEM

We are concerned with a porous medium, varying in grain size from that of fine silt or clay to that of sand, which is emplaced in the pond in the form of a slurry. Beginning with this initial condition, the material undergoes drainage, mainly dictated by gravity and undergoes progressive desaturation. In addition, the material may also lose water to the atmosphere due to evaporation at the surface or may gain water from rainfall. In an active mill tailing pond, new tailings material is built up on older material as the pond fills up. The weight of the newly added material causes the pore water pressure and the pore air pressure in the deeper materials to rise. The pressure rise, in turn, modifies the existing fluid flow dynamics.

The overall problem is one of transient fluid flow in a variably saturated, heterogeneous, deformable porous medium, subjected to periodic incremental boundary loads during the active phase of tailings emplacement. At its upper boundary, the system interacts with the

atmosphere, through any vegetative or other cover material that may have been placed on the tailings. The downward drainage is ultimately dictated by the disposition of the water table of the local ground water system. The drainage-deformation phenomenon is characterized by the presence and movement of two fluid phases, water and air, in the presence of a discrete, particulate, solid phase constituted by the grains of the porous medium.

THE GOVERNING EQUATIONS

The overall problem of isothermal consolidation of a variably saturated soil involves the integration of two fluid flow equations (the fluids being water and air) and a stress strain equation to handle multidimensional deformation. The fluid pressures, which are the independent variables in the flow equations, are controlled, among other factors, by the compressibility of the porous material. The stress-strain equation, on the other hand, involves displacements and stresses in different directions. As has been well established through the work of Biot (1941) and others, the scalar volumetric compressibility governing the fluid flow equations and the vectorial displacements of the stress strain equation can be suitably coupled through the use of fundamental elastic constants such as Lamé's constants. While coupled equations based on Biot's theory have been extensively used to simulate multi-dimensional deformation of saturated soils, they have not so far been extended to variably saturated soils. The reason for this non-extension is that the deformation properties of partially

saturated soils are extremely difficult to measure. At the present time, some successful measurements of volume change properties have been reported in the literature under some simple boundary conditions. However, little is known about the elastic parameters such as Lamé's constants as applied to partially saturated soils or about the dependence of such parameters on the saturation properties. The best that has been achieved in the literature is to solve the flow equations in conjunction with the scalar deformation phenomenon, namely, volume change. The scalar parameter can either correspond to isotropic compression loading, or to confined compression (oedometer) loading. Considering the fact that the movement of water in the partially saturated zone tends to be vertical, especially within the core of the tailings, the one-dimensional oedometer test, which is the simpler of the two, can be considered to be reasonably representative of the field condition. Under the circumstances, we shall not any more be concerned with the coupling of the stress-strain equation with the flow equation.

A brief history

Interest in the deformation of unsaturated soils began establishing itself in the geotechnical engineering field during the late 1950's. Bishop (1959) was one of the earliest to recognize that the deformation theory of the saturated-zone could be extended, with suitable modifications, to the unsaturated zone. As a first step, he incorporated the air-phase pressure into Terzaghi's effective stress relation and introduced a saturation-dependent parameter χ to account for the two-phase

conditions prevalent under variable saturation. However, it was soon learned by many workers through experimental evidence that deformation in the unsaturated zone cannot be accurately related to a single variable such as the effective stress. Indeed, Bishop and Blight (1963) suggested that void ratio should be treated as a function of two variables ($\sigma - p_a$ and $p_a - p_w$) rather than just effective stress. Matyas and Radhakrishna (1968) provided some of the earliest results expressing void ratio and saturation to water as functions of $\sigma - p_a$ and $p_a - p_w$. Fredlund and Morgenstern (1976) carried out a detailed analysis of the stress-state variables for the unsaturated soil, treating it as a four phase system and showed that the combination, $[\sigma - p_a, p_a - p_w]$ was indeed an acceptable set of fundamental stress-state variables for the unsaturated soil. They also provided experimental results to show that the functional relation between void ratio and the two stress-state variables was unique, at least locally. At the present time there is enough evidence available to show that the two-stress-state variable model reasonably accounts for the known behavior of unsaturated soil volume changes, including collapse upon wetting. For a more complete treatment of the literature on this topic, the reader is referred to Fredlund and Morgenstern (1976) and Lloret and Alonso (1980).

Interest in modeling deformation in the unsaturated zone did not commence until the mid-1970's. Narasimhan and Witherspoon (1977) incorporated Bishop's χ parameter into a numerical model for three dimensional fluid flow and one-dimensional consolidation. Fredlund and

Hasan (1979) developed a one-dimensional model for the deformable unsaturated zone in which they solved two mass conservation equations, one for water and the other for air. In this model they also incorporated the generation of pore water pressure and pore air pressure due to undrained response to external loads. Lloret and Alonso (1980) developed a finite element one dimensional model to solve almost the same set of equations as considered by Fredlund and Hasan, except that they investigated the role of air-dissolution in the water and allowed for the non-linear relation between void ratio and saturation on the one hand and the stress state variables on the other.

The Two-Phase Flow Approach

Consider a sufficiently small volume element of the porous medium over which we can realistically define average values of intensive physical quantities such as porosity, saturation, density, compressibility, fluid pressure, fluid potential and so on. Let this volume element be enclosed by the surface Γ . We shall define this element to contain a constant volume of solids, V_s . Under transient flow conditions in the partially saturated state, we need to write two conservation equations, one for the mass of water and the other, for the mass of air accumulating in the element, as has been suggested by Fredlund and Hasan (1979) and Lloret and Alonso (1980):

Water

$$-\int_{\Gamma} \rho_w \mathbf{q}_w \cdot \mathbf{n} \, d\Gamma = \frac{\partial M_w}{\partial t} \quad (1)$$

Air

$$-\int_{\Gamma} \rho_{a,\Gamma} \vec{q}_a \cdot \vec{n} \, d\Gamma + \int_{\Gamma} \rho_{a,\Gamma} (\vec{q}_w \cdot \vec{n} \, d\Gamma) H = \frac{\partial M_a}{\partial t} \quad (2)$$

where $\rho_{w,\Gamma}$ and $\rho_{a,\Gamma}$ denote, respectively, the densities of water and air at the interface $d\Gamma$, q_w and q_a are the Darcy velocities of water and air, M_w and M_a are the masses of water and air present in the volume element and H is Henry's solubility coefficient. The integral on the left hand side of (1) and the first integral on the left hand side of (2) denote the rates of accumulation of water and air due to fluid flow in and out of the volume element. The second integral on the left hand side of (2) denotes the mass of dissolved air convected into the volume element by the flowing water.

The mass of water M_w can be expressed by the relation:

$$M_w = V_s e S_w \rho_w \quad (3)$$

where e is void ratio, and S_w is water saturation. The mass of air in the element consists of two components, free air and dissolved air. As suggested by Lloret and Alonso (1980), one may express the total mass of air in the element by the relation,

$$M_a = V_s e \rho_a [(1-S_w) + HS_w] \quad (4)$$

In equation (4) we recognize that since only two fluid phases are present, $S_a = (1-S_w)$ where S_a is the saturation of air.

In order to make the governing equations (1) and (2) to be of practical utility, we will use, as dependent variables, the easily measured physical quantities, p_w and p_a . We now have to express each of the quantities on the right hand side of (3) and (4) as functions of p_w and p_a . Of these, V_g and H are constants and need no further consideration. As we shall see subsequently, the relation of the fluid densities to fluid pressures is far simpler than that of either of the other two quantities, void ratio and saturation. Hence, to consider the simplest case first, we can write down the dependence of the densities of the fluids on their respective pressures.

Under isothermal conditions, water has a relatively constant compressibility and its equation of state is given by,

$$\rho_w \Big|_{p_w} = \rho_{w,0} \exp c_w (p_w - p_{w,0}) \quad (5)$$

where $\rho_{w,0}$ is the density at pressure $p_{w,0}$ and c_w is the compressibility of water. For air, one may use the Gas law (Fredlund, 1976) and write ,

$$\rho_a = \frac{1}{R_m} \frac{p_a}{T} \quad (6)$$

where R_m is an average value of the Gas Content for moist air and T is absolute temperature.

Assuming V_g is constant in (3) and (4), we now have to consider the dependence of void ratio, e , and water saturation, S_w , on the fluid pressures, p_w and p_a . Historically, the dependence of void

ratio on fluid pressure in fully saturated soils has been a topic of great practical importance in the field of soil mechanics. The now widely used concept of effective stress was proposed nearly sixty years ago by Terzaghi for saturated soils. According to this concept, e is a function of both the total stress and the pore water pressure. For a porous material with incompressible solid grains, the effective stress relation is

$$\sigma' = \sigma - p_w \quad (7)$$

where σ is the total stress and σ' is the effective stress. For oedometer-type loading one may treat σ and σ' as vertical stresses, while for an isotropic loading, σ and σ' may be treated as mean principal stresses. For saturated materials there is enough experimental justification to treat e as a unique function of effective stress, although some path-dependence may be involved (in the form of differences between virgin compression and rebound). The functional dependence between e and σ' for a soil is given in Fig. 1 (from Lambe and Whitman, 1969). If we consider that the total stress over the element does not change with time, the void ratio for that element is purely a function of p_w .

The dependence of water saturation on p_w has been the subject study by the soil physicists nearly a century. Generally assuming that the soil matrix is rigid, and neglecting temperature effects, there is sufficient experimental justification to consider that S_w is a function of the pressure of the water phase. This relation too is characterized by strong path dependence as shown in Fig. 2.

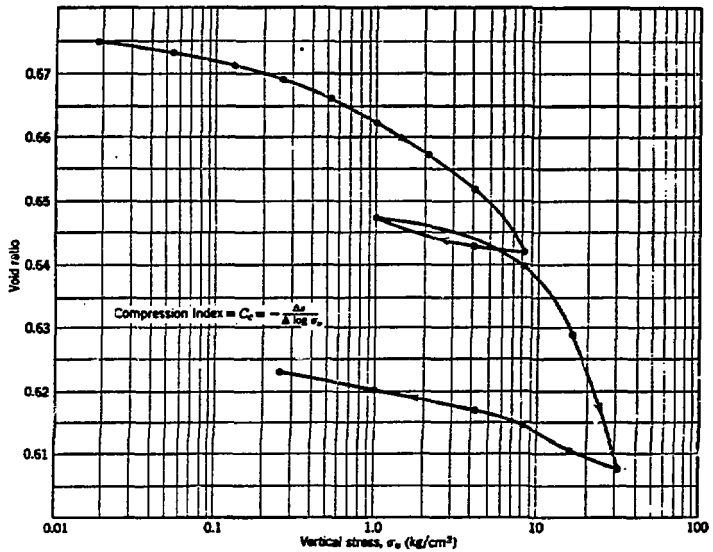


Fig. 1. Experimentally observed functional dependence between void ratio and effective stress (after Lambe and Whitman, 1969).

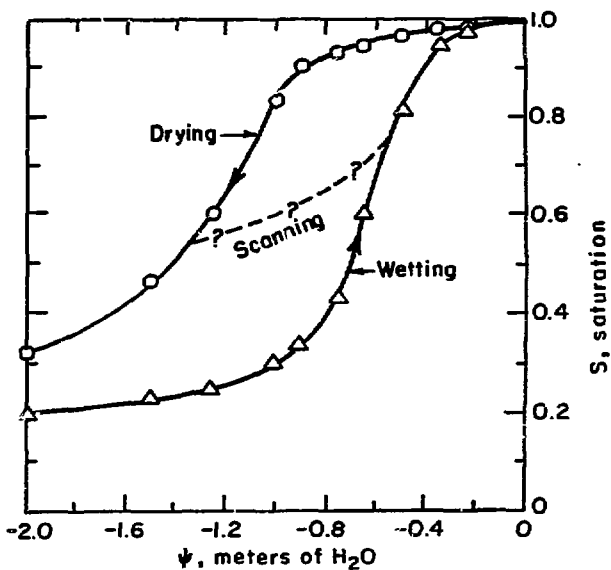


Fig. 2. Experimentally observed functional dependence between water saturation and pressure head showing path-dependence. Scanning curve represents the expected path when the direction of saturation is reversed (after Liakopoulos, 1965).

Interest in the analysis of deformation in partially saturated soils developed commenced in the late 1950's with the work of Bishop (1959), and Aitchison and Bishop (1960). Drawing upon the existing theory for saturated materials, these authors first attempted to extend the concept of effective stress to the zone of partial saturation. They proposed that effective stress is equal to total stress less a weighted sum of the p_w and p_a . Thus,

$$\begin{aligned}\sigma' &= \sigma - [\chi p_w + (1-\chi)p_a] \\ &= \sigma - p_a + \chi(p_a - p_w)\end{aligned}\quad (8)$$

with Bishop's weighting parameter χ being strongly dependent on saturation. Lambe and Whitman (1969) interpret χ as a normalized area of water contact suggesting that χ is positive and varies from zero to 1. A positive χ implies that an increase in effective stress causes a decrease in bulk volume and vice versa. However, subsequent experimental work by many workers (for detailed discussions see, Fredlund, 1979 and Lloret and Alonso, 1980) showed that in certain soils with an unstable structure, a decrease in effective stress due to wetting (an increase in p_w) can cause a decrease in volume due to pore collapse, provided that the total stress is sufficiently high. This indicated that the effective stress, which according to Bishop and Blight (1963) is, "that function of the total stress and pore pressure which controls the mechanical effects of a change in stress, such as volume change," is a function of total stress in addition to the fluid-phase stress differential, $p_a - p_w$. Thus, the effective stress relation (8) is not adequate to describe the volume change behavior of partially saturated

soils. In a partially saturated soil, not only does the water pressure act over progressively decreasing solid surface area with decreasing saturation, but also the skeletal stress field is modified by capillary pressure forces acting radially inward towards the center of curved menisci. Indeed, the term $(p_a - p_w)$ occurring in (8) is the capillary pressure differential between the non-wetting phase (air) and the wetting phase (water) across the meniscus. Following an early suggestion by Bishop and Blight (1963) and the work of Matyas and Radhakrishna (1968), Fredlund and Morgenstern (1976) analyzed constitutive relations for unsaturated soils and showed on the basis of semi-empirical grounds that both e and S_w have to be treated as functions of two independent stress-state variables. After examining three such acceptable pairs, Fredlund (1979) suggests that the combination $(\sigma - p_a)$ and $(p_a - p_w)$, originally used by Matyas and Radhakrishna as the most advantageous combination.

Thus, referring back to (3) and (4), both e and S_w are functions of two variables,

$$e = e(\sigma - p_a, p_a - p_w) \quad (9)$$

$$S_w = S_w(\sigma - p_a, p_a - p_w) \quad (10)$$

We now introduce the term "capillary pressure," p_c , which is the pressure differential that exists across the meniscus between the non-wetting phase, air, and the wetting phase, water, and is always positive. It corresponds to the synonymous terms "matric suction", "matric tension", and "moisture suction" commonly used in the soil physics literature. While p_c has units of pressure, the other three terms

are usually defined in terms of head of water with reference to atmospheric pressure. Thus,

$$p_c = p_a - p_w \quad (11)$$

Figure 3, from Matyas and Radhakrishna (1968), shows the surfaces of variation of porosity and saturation in relation to the two independent state variables, $\sigma - p_a$ and $p_a - p_w$.

In view of the aforesaid considerations, we may now proceed to express that time rates of change of M_w and M_a over the volume element in terms of the fluid pressures.

First, consider the time derivative of M_w from (3),

$$\frac{\partial M_w}{\partial t} = \frac{\partial}{\partial t} (v_s e S_w \rho_w) = v_s \left[\rho_w S_w \frac{\partial e}{\partial t} + \rho_w e \frac{\partial S_w}{\partial t} + e S_w \frac{\partial \rho_w}{\partial t} \right] \quad (12)$$

Now consider $\partial e / \partial t$ in view of (9),

$$\frac{\partial e}{\partial t} = \frac{\partial e}{\partial(\sigma - p_a)} \frac{\partial(\sigma - p_a)}{\partial t} + \frac{\partial e}{\partial p_c} \frac{\partial p_c}{\partial t} \quad (13)$$

Assuming that total stress remains constant over dt,

$$\frac{\partial e}{\partial t} = \left[\frac{\partial e}{\partial p_c} - \frac{\partial e}{\partial(\sigma - p_a)} \right]_{\sigma = \text{const}} \frac{\partial p_a}{\partial t} - \frac{\partial e}{\partial p_c} \frac{\partial p_w}{\partial t} \quad (14)$$

In a similar fashion,

$$\frac{\partial S_w}{\partial t} = \left[\frac{\partial S_w}{\partial p_c} - \frac{\partial S_w}{\partial(\sigma - p_a)} \right]_{\sigma = \text{const}} \frac{\partial p_a}{\partial t} - \frac{\partial S_w}{\partial p_c} \frac{\partial p_w}{\partial t} \quad (15)$$

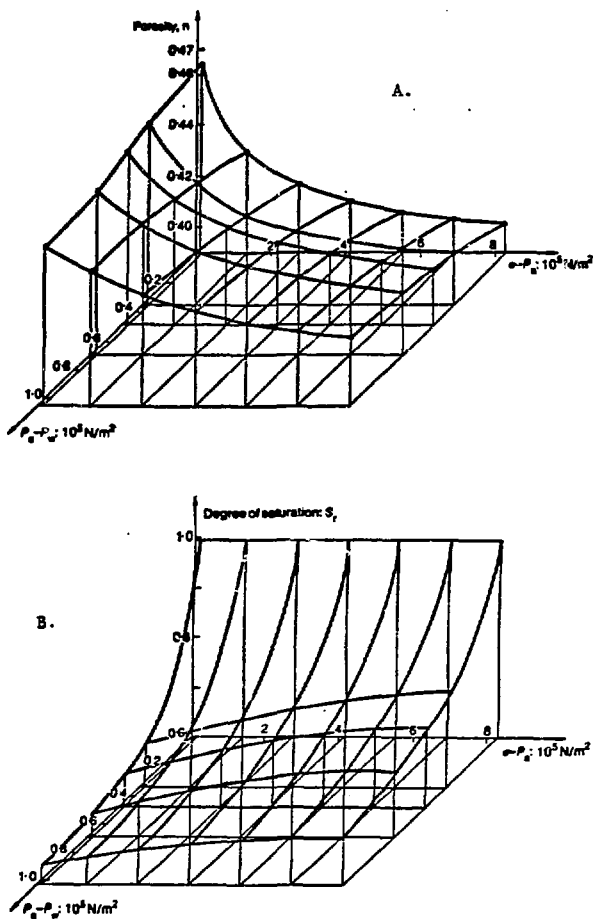


Fig. 3. Variation of porosity (A) and saturation (B) as a function of $\sigma - p_w$ and $p_a - p_w$ (from Matyas and Radhakrishna, 1968).

And, from the equation of state for water, (5),

$$\frac{\partial \rho_w}{\partial t} = \frac{\partial \rho_w}{\partial p_w} \frac{\partial p_w}{\partial t} = \rho_w c_w \frac{\partial p_w}{\partial t} \quad (16)$$

Combining (13), (14), and (15), we now have,

$$\frac{\partial M_w}{\partial t} = V_g \rho_w A_1 \frac{\partial p_a}{\partial t} + A_2 \frac{\partial p_w}{\partial t} \quad (17)$$

where,

$$A_1 = S_w [m_1^e - m_2^e] + e [m_1^w - m_2^w]$$

$$A_2 = S_w m_2^e + e m_2^w + e S_w c_w$$

and, in which,

$$m_1^e = \begin{cases} -\frac{\partial e}{\partial(\sigma-p_a)}, & p_w < \text{atm. press.} \\ 0, & p_w \geq \text{atm. press.} \end{cases} \quad m_2^e = \begin{cases} -\frac{\partial e}{\partial p_c}, & p_w < \text{atm. press.} \\ -\frac{\partial e}{\partial \sigma}, & p_w \geq \text{atm. press.} \end{cases}$$

$$m_1^w = \begin{cases} -\frac{\partial S_w}{\partial(\sigma-p_a)}, & p_w < \text{atm. press.} \\ 0, & p_w \geq \text{atm. press.} \end{cases} \quad m_2^w = \begin{cases} -\frac{\partial S_w}{\partial p_c}, & p_w < \text{atm. press.} \\ 0, & p_w \geq \text{atm. press.} \end{cases}$$

Similarly,

$$\frac{\partial M_a}{\partial t} = v_s \rho_a \left[A_3 \frac{\partial p_a}{\partial t} + A_4 \frac{\partial p_w}{\partial t} \right] \quad (18)$$

where,

$$A_3 = (1 - S_w) \left[m_1^e - m_2^e + \frac{e}{\rho_a R T} \right] - e \left[m_1^w - m_2^w \right]$$

and

$$A_4 = (1 - S_w) m_2^e - e m_2^w$$

The Equation of Motion

An important feature of the unsaturated zone is the strong influence of the phase saturations on fluid transport. We will assume that both air and water move within the soil in accordance with Darcy's Law, subject to the modification of the hydraulic conductivity term to include relative permeability effects. Thus:

$$q_w^z = - \frac{k k_{r,w}}{\mu_w} (\rho_w g \nabla z + \nabla p_w) \quad (19)$$

$$q_a^z = - \frac{k k_{r,a}}{\mu_a} \nabla p_a \quad (20)$$

where k is intrinsic permeability, $k_{r,w}$ and $k_{r,a}$ are relative permeabilities with reference to the water and air phases, μ_w and μ_a are viscosities of water and air and z is elevation above datum.

In (20) the gravitational effects in regard to the air-phase has been omitted.

In a deformable, partially saturated soil k is a function of the geometry of the porous medium, while k_r is a function of the phase saturation. For saturated materials there is enough experimental evidence to suggest that $\log k$ is linearly related to void ratio (Lambe and Whitman, 1969). This leads to the exponential expression,

$$k = k_o \exp \left[\frac{2.303 \Delta e}{C_k} \right] \quad (21)$$

where k_o is the permeability at the reference void ratio e_o and C_k is the slope of the best-fitting straight line on the e versus $\log k$ plot. No reliable data is available to date as to how (21) may be extended to unsaturated soils.

In a similar fashion, there is sufficient experimental evidence to show that the relative permeabilities to the wetting and the non-wetting phases are non-linearly related to saturation. Figure 4 from Corey (1977) is a schematic representation of the dependence of k_r on saturation. In this figure S_r denotes residual saturation of the wetting phase while S_m denotes the critical condition at which the non-wetting phase tends to form disconnected bubbles within the wetting; that is, S_m corresponds to the condition at which p_w equals the air-entry pressure.

Many workers have proposed semi-empirical relationships for the dependence of the relative permeabilities on saturation. Brooks and Corey (1964) proposed an equation of the following form for the wetting phase relative permeability:

$$k_{rw} = S_e^E \quad (22)$$

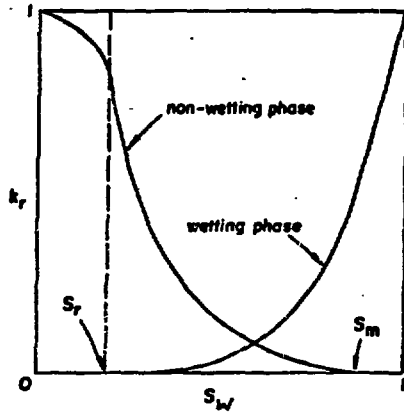


Fig. 4. Sketch of relative permeabilities to the wetting and the non-wetting phases (after Corey, 1977).

where S_e is the effective saturation defined as,

$$S_e = \frac{S_w - S_r}{1 - S_r} \quad (23)$$

The exponent ϵ varies from 3 to 4, as proposed by different workers (see Corey (1964)). According to Corey, this range is typical for soil materials as well as porous rocks. For highly developed structures, ϵ may exceed a value of 4.

It is very well known that the relative permeability curves are strongly subject to hysteresis. Although the phenomenon of hysteresis is a topic of basic interest (e.g., Mualem, 1976), its incorporation into computational models for engineering purposes is far from a reality; the data required and the computational effort needed to implement hysteresis are too large to be economical. Under the circumstances, one simply restricts computational models to unidirectional changes in saturation; that is, one either considers a desaturating medium or a resaturating medium.

At the present time no reliable data is available on the relative permeability characteristics of a porous medium which is, at one and the same time, partially saturated and is undergoing deformation. Nor is much known about the nature of anisotropy to permeability in such media. Although one may incorporate into (19) and (20) the appropriate expressions for k and k_r as given in (21) and (22), such substitutions will have to be justified by future experimental data.

Pore Pressure Generation

The governing equations in (1) and (2) are to be complemented by appropriate source terms. Two types of source terms are of interest. The first relates to the arbitrary addition or withdrawal of fluid from the system; examples include production of water from a well or the infiltration of rainfall. The latter could also be treated as a flux boundary condition. The second kind of source relates to the generation of fluid pressures due to external loading, such as that caused by the gradual increase in the weight of the mill tailing over burden during the active phase of tailing emplacement. The peculiarity of this source term is that the pore pressure is generated instantaneously without the addition or removal of mass from the system. The generation of pore pressure is to be computed assuming that the soil responds in an undrained fashion to the imposed external load and that the change in void volume caused in the volume element by the external load is equal to the sum of the volume changes in the two phases. Obviously, the pore pressures generated in the two phases are functions of the matrix compressibility, the two fluid compressibilities, the porosity of fluid saturations and the magnitude of the total stress change. As pointed out by Fredlund and Hasan (1979) and by Lloret and Alonso (1980), an iterative computational procedure is required to compute the magnitudes of the pore pressures generated in each phase.

The Complete Two-Phase Governing Equation

The mass conservation equations for the water phase and the air phase are presented below in an integral form in equations (24) and (25). These are obviously coupled equations since the pressures of the two phases occur in both the equations. Initial conditions are prescribed for the two phase-pressures as well as for the total stress, as indicated in (26). Boundary conditions may be of the Dirichlet type (27), the Neumann type (28) or may be of a mixed type, such as a seepage face (29),

Water Phase:

$$\rho_w G_w + \int_{\Gamma} \rho_w \frac{k k_{rw}}{u_w} [\rho_w g v_z + \nabla p_w] \cdot \vec{n} d\Gamma = v_s \rho_w A_1 \frac{\partial p_a}{\partial t} + A_2 \frac{\partial p_w}{\partial t} \quad (24)$$

Air Phase:

$$\rho_a G_a + \int_{\Gamma} \rho_a \frac{k k_{ra}}{u_a} \nabla p_a \cdot \vec{n} d\Gamma = v_s \rho_a A_3 \frac{\partial p_a}{\partial t} + A_4 \frac{\partial p_w}{\partial t} \quad (25)$$

where G_w and G_a are source terms representing volumetric fluid generation rates.

Initial Conditions:

$$p_w (t=0) = p_{w0} \quad (26a)$$

$$p_a (t=0) = p_{a0} \quad (26b)$$

$$\sigma (t=0) = \sigma_0 \quad (26c)$$

Boundary Conditions:

(i) Dirichlet condition:

$$p_w = p_w(t) \quad \text{along} \quad \Gamma_1 \quad (27a)$$

$$p_a = p_a(t) \quad \text{along} \quad \Gamma_2 \quad (27b)$$

(ii) Neumann condition:

$$-\vec{q}_w \cdot \vec{n} \, d\Gamma = Q_w(t) \quad \text{across} \quad \Gamma_3 \quad (28a)$$

$$-\vec{q}_a \cdot \vec{n} \, d\Gamma = Q_a(t) \quad \text{across} \quad \Gamma_4 \quad (28b)$$

(iii) Seepage Face:

$$p_a = p_{atm} \quad (29)$$

and water can only flow out }
of the flow region } on Γ_5

The handling of the pore pressure generation term arising due to change in boundary stresses needs special mention. The pore pressure generated due to external stresses are not accompanied by a change in the mass of water within the volume element. It is not possible to incorporate such a pore pressure term into (24) or (25) since the two sides of both the equations have dimensions of mass per unit time while the pore pressure generation term has units of pressure per unit time. As is done sometimes, [e.g., Jacob, 1950; Lambe and Whitman, 1969], one could generate the same pore pressure by injecting an appropriate mass of water into the element. Although this procedure will generate a quantitatively equal amount of pore pressure, the physics of the

phenomena are not the same. Therefore, to maintain dimensional consistency as well as physical realism, we shall use two auxiliary conditions to incorporate the pore pressure generation into the governing equations. These auxiliary conditions relate to the conditions at the end of the time interval dt , which will constitute the new initial conditions for the next time interval. Thus:

Auxiliary Conditions:

$$(i) \quad p_w \Big|_{t+dt} = p_{w0} + \frac{\partial p_w}{\partial t} dt + p_{w,ext} \quad (30a)$$

$$(ii) \quad p_a \Big|_{t+dt} = p_{a0} + \frac{\partial p_a}{\partial t} dt + p_{a,ext} \quad (30b)$$

$$(iii) \quad \sigma \Big|_{t+dt} = \sigma_0 + \frac{\partial \sigma}{\partial t} dt \quad (30c)$$

where $p_{w,ext}$ and $p_{a,ext}$ are fluid pressures generated due to external stress.

Simplification of the Governing Equations

Although the two-phase equation is physically complete, it may be possible, under certain circumstances, to ignore the equation for the air phase, leading to considerable savings in computational effort. In the case of Uranium mill tailings, the time-scale of interest is of the order of several tens of years at least. However, due to the considerable mobility of the air phase, any locally enhanced air-pressure pockets that may exist within the air phase should be expected to dissipate within a matter of a few hours to not more than a few days. Under the circumstances it is practically realistic to reduce the problem to a single conservation equation for water, assuming that

the air-phase is continuous and is at atmospheric pressure. Such an assumption is indeed the basis for the classical unsaturated flow equation, originally suggested by Richards (1931).

The Unsaturated Flow Equation

We may obtain the unsaturated flow equation from (24) by discarding the air-phase pressure term and by eliminating equation 25. This equation as well as its associated initial conditions, boundary conditions and auxiliary conditions are given in equations 31 to 36.

$$\rho_w G_w + \int_{\Gamma} \rho_w \frac{k k_{rw}}{\mu_w} [\rho_w g \nabla z + \nabla p_w] \cdot \vec{n} \, d\Gamma$$

$$= V_s \rho_w \left[S_w \frac{m_e}{m_2} \Big|_{\sigma=p_a} + e \frac{m_w}{m_2} \Big|_{\sigma=p_a} + e S_w c_w \right] \frac{\partial p_w}{\partial t} \quad (31)$$

Initial Conditions:

$$p_w(t=0) = p_{w0} \quad (32a)$$

$$\sigma(t=0) = \sigma_0 \quad (32b)$$

Boundary Conditions:

(i) Dirichlet condition:

$$p_w = p_w(t) \quad \text{along} \quad \Gamma_1 \quad (33)$$

(ii) Neumann condition:

$$-\vec{q}_w \cdot \vec{n} \, d\Gamma = Q_w(t) \quad \text{across} \quad \Gamma_2 \quad (34)$$

(iii) Seepage Face:

$$p_w = \text{atmospheric pressure} \quad (35)$$

and water can only flow out
of the flow region on Γ_3

Auxiliary Conditions:

$$(i) \quad p_w(t+dt) = p_{w0} + \frac{\partial p_w}{\partial t} dt + p_{w,ext} \quad (36a)$$

$$(ii) \quad \sigma(t+dt) = \sigma_0 + \frac{\partial \sigma}{\partial t} dt \quad (36b)$$

Narasimhan and Witherspoon (1977) proposed an integral governing equation for an unsaturated deformable medium. The primary difference between their equation and equation 31 is that in the latter both void ratio and saturation are treated as functions of two variables, $\sigma - p_a$ and $p_a - p_w$. However, Narasimhan and Witherspoon (1977) treated e as a function of Bishop's effective stress using the χ parameter and S_w as a function only of p_c .

NUMERICAL SOLUTION

The governing equations of either the generalized two-phase system or the simpler unsaturated system are best solved numerically. The basic task of numerical solution is to discretize the flow domain into appropriately small volume elements and to apply the conservation equations to these subdomains in a systematic fashion. The subdomains of integration may be defined explicitly (as in the integral finite

difference method or the finite difference method) or implicitly by a procedure of weighted volume integration (as in the finite element method). Spatial gradients of fluid pressures, which are essential to evaluate fluid fluxes according to Darcy's law, can be estimated either uses finite difference approximations or finite element approximations. In order to assure that a stable solution is obtained for the transient problem, it is necessary to use appropriate time-averaged values of fluid pressures to evaluate the fluid fluxes using Darcy's law. These time-averages over a time step may be based on either the central differencing approximation or a backward differencing approximation or a combination of the two. Or, if instability is expected to occur only in certain portions of the flow region, one may use a combination of forward differencing approximation (where stability is not expected to be violated) and central- or backward differencing approximations (where stability is expected to be violated; Narasimhan et al., 1978a, 1978b).

The implicit time-averaged approximations of the fluid pressures, when combined with the spatial gradients (either by finite differences or by finite elements), boundary conditions, initial conditions and the source terms, lead to system simultaneous equations in which the time-derivatives of the fluid pressures over each subdomain constitute the unknowns. The known boundary conditions, initial conditions and the sources together determine the magnitude of the known vector. The coefficients of the unknown, which constitute the elements of the conductance matrix, are functions of the material diffusivity as well as the geometrical properties of the volume element. In as much as

each volume element may only have few elements as its neighbors, the matrix containing the coefficients of the unknowns is usually extremely sparse in nature.

The Matrix Equation for the Two-Phase Problem

The conservation equations (24) and (25) are presented in a discretized form in (37) and (38):

$$\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^w [\rho_{w,\ell m} g z_m + \bar{p}_{w,m}] - (\rho_{w,\ell m} g z_\ell + \bar{p}_{w,\ell}) \left(\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^w \right) + \rho_{w,\ell} G_\ell$$

$$= V_s \rho_{w,\ell} \left[A_1 \frac{\Delta p_{a,\ell}}{\Delta t} + A_2 \frac{\Delta p_{w,\ell}}{\Delta t} \right] \quad (37)$$

$$\ell = 1, 2, 3 \dots L$$

$$m = 1, 2, 3 \dots L$$

$$\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^a \bar{p}_{a,m} - \bar{p}_{a,\ell} \left(\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^a \right) + \rho_{\ell,a} G_a$$

$$= V_s \rho_{a,\ell} \left[A_3 \frac{\Delta p_{a,\ell}}{\Delta t} + A_4 \frac{\Delta p_{w,\ell}}{\Delta t} \right], \quad (38)$$

$$\ell = 1, 2, 3 \dots L$$

$$m = 1, 2, 3 \dots L$$

In (37) and (38),

$$\bar{p}_{w,i} = p_{w,i}^0 + \lambda \Delta p_{w,i}, \quad i=\ell,m \quad (39a)$$

$$\bar{p}_{a,i} = p_{a,i}^0 + \lambda \Delta p_{a,i}, \quad i=l,m \quad (39b)$$

where the superscript 0 implies initial conditions and λ is a weighting factor; $\lambda=0$ for forward differencing, 0.5 for central differencing and 1.0 for backward differencing schemes.

Incorporating boundary conditions, substituting (39a) and (39b) into (37) and (38) and collecting all known quantities leads to,

$$B_{\ell,m}^w \Delta p_{w,m} + B_{\ell,L+\ell}^w \Delta p_{a,\ell} = R_{\ell}^w \quad (40a)$$

$$B_{\ell,\ell}^a \Delta p_{w,\ell} + B_{L+\ell,L+m}^a \Delta p_{a,m} = R_{\ell}^a \quad (40b)$$

The $2L \times 2L$ coefficient matrix B is non-symmetric and sparse, whose structure is schematically depicted in Fig. 5.

The Matrix Equation for the Unsaturated Flow Problem

The discretized form of (31), which expresses the single-phase unsaturated flow equation is as follows:

$$\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^w [\rho_{w,\ell n} g z_m + p_m] - (\rho_{w,\ell n} g z_{\ell} + p_{w,\ell}) \left(\sum_{\substack{m \\ \ell \neq m}} U_{\ell m}^w \right) + \rho_{w,\ell} G_{\ell} \\ = V_s \rho_{w,\ell} \left[S_w m_2^e \Big|_{\sigma-p_a} + e m_2^w \Big|_{\sigma-p_a} \right] + e S_w c_w \Big] \frac{\Delta p_w}{\Delta t} \quad (41)$$

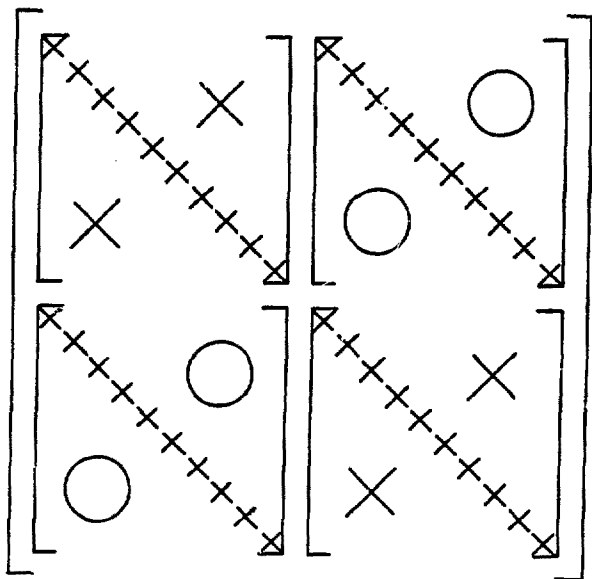


Fig. 5. Schematic description of the sparsity structure of the two-phase problem.

On incorporating boundary conditions and collecting all known quantities to the right hand side, we get,

$$B_{\ell m} \Delta p_{w,m} = R_{\ell} \quad (42)$$

Equation 42 is a sparse, symmetric matrix.

The matrix equations given in (40) and (42) can be solved either by direct solution techniques or by iterative techniques. Because of the strong dependence of permeabilities as well as saturation and void ratio on the fluid pressures, the coefficient matrices in (40) and (42) will in general be very stiff. That is, the terms on the principal diagonal will vary greatly in magnitude. As a result, solving them through iterative techniques may lead to convergence difficulties, to overcome which one may have to sacrifice computational speed. Direct solvers, on the other hand, can handle stiff matrices with greater ease, but are constrained by much larger computer storage requirements than iterative solvers. At present, it appears that for one-dimensional and two-dimensional problems direct solvers are competitive and preferable. For three dimensional problems, iterative methods may still have to be used although computational time may be large. Currently frontal solvers and profile solvers are available in the literature for improved efficiency in solving sparse and poorly structured matrices.

EXISTING ALGORITHMS

Although numerous computer programs have been developed in the literature to solve transient fluid flow in variably saturated systems (Freeze (1971), Neuman (1973)), the attention given to systematically handling deformation in such systems has been minimal. For want of time, it has not been possible in the present study to carry out an extensive literature search for the few models on unsaturated consolidation that are available in the literature. Under the circumstances, details of three models will be given below. The first of these is based on Bishop's parameter, while the other two are based on two stress-state variables.

Narasimhan and Witherspoon (1977) developed a unified model for fluid flow in saturated-unsaturated porous media and incorporated it into a computer program called TRUST (Narasimhan et al., 1978a). The model is based on the following governing equation,

$$\rho_{w,l} G_l + \int_{\Gamma} \rho_{w,l} \frac{k_{r,w}}{\mu_w} \rho_{w,l} g \nabla(z+\psi) \cdot \vec{n} d\Gamma = M_{c,l} \frac{\partial \psi}{\partial t} \quad (43)$$

where ψ is the pressure head and $M_{c,l}$ is the fluid mass capacity of element l defined as,

$$M_{c,l} = V_s \rho_{w,l} \left[S_w e \gamma_w c_w + S \gamma_w \chi' a_v + e \frac{dS_w}{d\psi} \right] \quad (44)$$

In (44) χ' is related to Bishop's χ parameter by,

$$\chi' = \chi + \psi \frac{d\chi}{d\psi} \quad (45)$$

Also, the coefficient of compressibility, a_v is defined by,

$$a_v = - \frac{de}{d\sigma'} \quad (46)$$

where σ' is effective stress when e and σ' are non-linearly related, the dependence of a_v on σ' can be better treated more accurately by using the compression index C_c or the swelling index C_s . Thus,

$$a_v = \frac{C_c}{2.303 \sigma'} \quad , \quad \text{normal consolidation} \quad (47)$$

or

$$a_v = \frac{C_s}{2.303 \sigma'} \quad , \quad \text{swelling or rebound} \quad (48)$$

where C_c and C_s are, respectively, the indices of compression and swelling (Lambe and Whitman, 1969).

The model is based on an integral finite difference method (IPDM; Narasimhan and Witherspoon, 1976) and can handle three dimensional fluid flow in heterogeneous or anisotropic domains under complex geometry and time-variant boundary conditions and sources. The volume element is defined to have constant volume of incompressible solids; its bulk volume and voids volume change in time. Narasimhan and Witherspoon (1978) verified the model by applying it to a variety of saturated-unsaturated flow problems including deformation. Narasimhan (1979) applied the model to a 85-m high desaturating column and carried out parametric studies on the role of the variation of the χ parameter with saturation. A detailed documentation of the program is currently available (Reisenhauer et al., 1982).

A comparison of the TRUST equation (43) with the two-state variable-unsaturated flow equation (31) shows that in the former, the e and S_w are treated as a function of a single independent variable, effective stress defined as,

$$\sigma' = \sigma - \chi \rho_w g \psi \quad (49)$$

However, since TRUST stores the total stress in memory at all times as well as keeps track of moisture suction as a function of time, one can update it to evaluate e , S_w as functions of both $\sigma - p_a$ and p_c . The task of modification involves the replacement of the χ -parameter subroutine by a new subroutine in which the dependence of e and S_w on the two state variables are input either as a table or as a convenient polynomial function.

Fredlund and Hasan (1979) implemented the theory developed by Fredlund and Morgenstern (1976) into a one-dimensional consolidation model. In this model, two continuity equations are solved, one for the water phase and the other for the air phase. The relative permeabilities to the two phases are allowed to vary as functions of fluid pressures, while void ratio and saturation are both non-linearly related to the state variables. Air is treated as an ideal gas and is assumed to obey the gas law. Temperature effects are neglected. In addition, they also developed an auxiliary, iterative algorithm to compute the magnitudes of the water-phase pressure and the air-phase pressure generated due to undrained response of the partially saturated soil to external loads. Using this auxiliary algorithm to provide a step-wise change in initial conditions, they applied their model to a

doubly draining unsaturated column, subjected to a stepwise external load at time $t = 0$. The Fredlund-Hasan model is based on the finite difference technique.

Lloret and Alonso (1980) have also developed an algorithm to model, in one-dimension, the consolidation of an unsaturated soil using the isothermal, two-phase approach. The physics of their model is essentially same as that of Fredlund and Hasan, except that Lloret and Alonso have also incorporated the dissolution of air in water. However, it appears that the consideration of air dissolution may not be worth the added computational effort. The Lloret-Alonso model is based on the finite element method. They applied their model to a partially saturated foundation subjected to a step-wise increment in load and the initial time.

Note that all the models referred to above compute only volumetric changes. They do not attempt to compute the changes in stress field. As such, one could use for the total stress, either the vertical stress (one-dimensional consolidation theory) or the mean principal stress, depending on which of these will closely approximate the field boundary conditions. The determination of linear displacements from volume changes will have to be carried out on the basis of a knowledge of field boundary conditions.

CONCLUDING REMARKS

That soil materials deform in the state of partial saturation is known. It is known, if only qualitatively, that the concepts of deformation related to fully saturated soils could be extended to unsaturated soils. At the present time, theoretical analysis and experimental measurements have been carried out only with respect to volume change behavior of unsaturated soils. Little has been done towards fully analyzing the directional displacements and the stress field in partially saturated soils. Extreme difficulties associated with experimentation on unsaturated soils and their complex behavior due to the presence of several phases, including the effects of capillary stresses, stand in the way of complete stress-strain analysis of unsaturated soils.

Experimental data have shown that the volume-change behavior of unsaturated soils cannot be adequately accounted for through the use of an "effective stress" even though it is defined in terms of a saturation-dependent parameter. Recent work based on a semi-empirical approach shows that volume change of unsaturated soils is more realistically accounted for in terms of two stress-state variables which are combinations of the total stress pore-air pressure and pore-water pressure. Additionally, the undrained response of a partially saturated soil to external loads has to be quantified in terms of an implicit relation between the compressibilities of the porous medium, the liquid-phase and the air-phase. The hydraulic

conductivity of a deforming unsaturated soil is a function of the void ratio as well as the phase saturations.

Mathematical models have been developed and are available to solve the governing equations either in terms of two coupled conservation equations or, under certain simplifying assumptions in terms of a single conservation equation for the water phase. It appears at the moment that our computational abilities exceed our abilities to generate empirical data to be input into the computational model. Perhaps the greatest challenge confronting the simulation of consolidation of unsaturated soils is the measurement of the dependences of void ratio and saturation on the chosen stress-state variables. For long term prediction of consolidation of Uranium mill tailings it may be acceptable and economical to work with the simplified single-phase governing equation rather than the two-phase equation.

ACKNOWLEDGMENTS

I would like to thank Dellwyn Fredlund, Tom Gates, Michael King and Panos Panapopoulos for their review of the manuscript. This work was supported by the Nuclear Regulator Commission through Batelle Pacific Northwest Laboratories, and the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

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