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by

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Abstract

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Professor Federico Ardila and Professor Lior Pachter, Co-chairs

We study generalized permutohedra and some of the geometric properties they exhibit. We decompose matroid polytopes (and several related polytopes) into signed Minkowski sums of simplices and compute their volumes. We define the associahedron and multiplihedron in terms of trees and show them to be generalized permutohedra. We also generalize the multiplihedron to a broader class of generalized permutohedra, and describe their face lattices, vertices, and volumes. A family of interesting polynomials that we call composition polynomials arises from the study of multiplihedra, and we analyze several of their surprising properties. Finally, we look at generalized permutohedra of different root systems and study the Minkowski sums of faces of the crosspolytope.

To Joe and Sue

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Acknowledgments

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—Jeffrey Doker, August 2011

Chapter 1

Introduction

The permutohedron P_n is a polytope whose vertices consist of all permutations of the entries of the vector $(1, 2, \ldots, n)$. Many of the rich geometric properties of P_n are inherited and extended by the larger class of polytopes formed as deformations of P_n, better known as generalized permutohedra. These objects have been studied at length in [23] and [25]. This dissertation extends several notions from these works and presents a survey of the geometry of generalized permutohedra, including matroid polytopes, associahedra, multiplihedra, and generalized permutohedra of other root systems.

Chapter 2 is based on joint work [2] with Federico Ardila and Carolina Benedetti. We review the concept of a matroid M and its associated matroid polytope P_M. We explore the notion of Minkowski sums and differences, and show that any generalized permutohedron can be represented as a signed Minkowski sum of simplices. We describe P_M as a generalized permutohedron and derive a formula for its volume. We then extend these techniques to the independent set polytope I_M and the associated flag matroid polytope.

Chapter 3 focuses on two particular classes of polytopes: the associahedron [19] [11] and the multiplihedron [12]. We show these polytopes to be realizations of posets of trees, and prove that they are generalized permutohedra. We construct the multiplihedron from the associahedron by developing a technique called q-lifting, and we analyze the face structure and volumes of the larger class of q-lifted generalized permutohedra.

Chapter 4 introduces a family of polynomials defined in terms of compositions c of n: the composition polynomial $g_c(q)$ and the associated reduced composition polynomial $f_c(q)$. Composition polynomials compute the volumes the pieces of a subdivision of the q-lifted polytopes introduced in Chapter 3. We produce a recursive definition of reduced composition polynomials and prove their coefficients are strictly positive, as well as several other properties.

Chapter 5 explores the notion of generalized permutohedra defined in terms of the other classical root systems. In particular we describe the parameter spaces of type-B and type-D generalized permutohedra. (The type-C arrangement C_n is equivalent to B_n, and therefore so are their respective sets of generalized permutohedra.) We also fully describe the space of polytopes that can be generated by signed Minkowski sums of faces of the crosspolytope. In particular we show that in even dimension the crosspolytope itself is decomposable into a Minkowski sum of simplices.

Chapter 2

Matroid polytopes and their volumes

2.1 Introduction

The theory of matroids can be approached from many different points of view; a matroid can be defined as a simplicial complex of independent sets, a lattice of flats, a closure relation, etc. A relatively new point of view is the study of matroid polytopes, which in some sense are the natural combinatorial incarnations of matroids in algebraic geometry and optimization. Our paper is a contribution in this direction.

We begin with the observation that matroid polytopes are members of the family of *generalized permutohedra* [23]. With some modifications of Postnikov's beautiful theory, we express the matroid polytope P_M as a signed Minkowski sum of simplices, and use that to give a formula for its volume $\mathrm{Vol}\,(P_M)$. This is done in Theorems 2.2.6 and 2.3.3. Our answers are expressed in terms of the beta invariants of the contractions of M.

Formulas for $\mathrm{Vol}\,(P_M)$ were given in very special cases by Stanley [29] and Lam and Postnikov [18], and a polynomial-time algorithm for finding $\mathrm{Vol}\,(P_M)$ was constructed by de Loera et. al. [9]. One motivation for this computation is the following. The closure of the torus orbit of a point p in the Grassmannian $Gr_{k,n}$ is a toric variety X_p, whose degree is the volume of the matroid polytope P_{M_p} associated to p. Our formula allows us to compute the degree of X_p combinatorially.

One can naturally associate two other polytopes to a matroid M: its independent set polytope and its associated flag matroid polytope. By a further extension of Postnikov's theory, we also write these polytopes as signed Minkowski sums of simplices and give formulas for their volumes. This is the content of Sections 2.4 and 2.5.

Throughout the chapter we assume familiarity with the basic concepts of matroid theory; for further information we refer the reader to [22].

2.2 Matroid polytopes are generalized permutohedra

The *permutohedron* P_n is a polytope in $\mathbb{R}^n$ whose vertices consist of all permutations of the entries of the vector $(1, 2, \ldots, n)$. A *generalized permutohedron* is a deformation of the permutohedron, obtained by moving the vertices of P_n in such a way that all edge directions and orientations are preserved (and some may possibly be shrunken down to a single point) [25]. In Section 3.5 we give a more precise treatment to the concept of a deformation.

Every generalized permutohedron can be written in the following form:

$$P_n(\{z_I\}) = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \text{ for all } I \subseteq [n] \right\}$$

where z_I is a real number for each $I \subseteq [n] := \{1, \ldots, n\}$, and $z_\emptyset = 0$. Different choices of z_I can give the same generalized permutohedron: if one of the inequalities does not define a face of $P_n(\{z_I\})$, then we can increase the value of the corresponding z_I without altering the polytope. When we write $P_n(\{z_I\})$, we will always assume that the z_Is are all chosen minimally; *i.e.*, that all the defining inequalities are tight.

Though every generalized permutohedron has a z_I parameterization, not every list of z_I parameters corresponds to a generalized permutohedron. Morton et. al. proved the forward direction of following criterion for the z_I:

Theorem 2.2.1. *[21, Theorem 17] A set of parameters $\{z_I\}$ defines a generalized permutohedron $P_n(\{z_I\})$ if and only if the z_I satisfy the submodular inequalities*

$$z_I + z_J \leq z_{I \cup J} + z_{I \cap J}$$

for all $I, J \subseteq [n]$.

Postnikov indicated the backward direction of this via personal communication [24], and Aguiar and Ardila wrote down the details in [1].

The *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is defined to be $P + Q = \{p + q \,:\, p \in P, q \in Q\}$. We say that the *Minkowski difference* of P and Q is $P - Q = R$ if $P = Q + R$.[1] The following lemma shows that generalized permutohedra behave nicely with respect to Minkowski sums.

Lemma 2.2.2. *If $P_n(\{z_I\})$ and $P_n(\{z'_I\})$ are generalized permutohedra then their Minkowski sum is a generalized permutohedron and $P_n(\{z_I\}) + P_n(\{z'_I\}) = P_n(\{z_I + z'_I\})$.*

[1] We will only consider Minkowski differences $P - Q$ such that Q is a Minkowski summand of P. More generally, the Minkowski difference of two arbitrary polytopes P and Q in $\mathbb{R}^n$ is defined to be $P - Q = \{r \in \mathbb{R}^n \,|\, r + Q \subseteq P\}$ [23]. It is easy to check that $(Q + R) - Q = R$, so the two definitions agree in the cases that interest us. In this paper, a signed Minkowski sum equality such as $P - Q + R - S = T$ should be interpreted as $P + R = Q + S + T$.

Proof. The polytopes $P_n(\{z_I\})$ and $P_n(\{z'_I\})$ are deformations of P_n, and therefore by [21, Theorem 17] they are each a Minkowski summand of a dilate of P_n. Thus $P_n(\{z_I\})+P_n(\{z'_I\})$ must also be a summand of a dilate of P_n, which implies, again by [21, Theorem 17], that this polytope too is a deformation of P_n and can thus be defined by hyperplane parameters z_I. That the values of these parameters are $z_I + z'_I$ follows from the observation that, if a linear functional w takes maximum values a and b on (faces A and B of) polytopes P and Q respectively, then it takes maximum value $a + b$ on (the face $A + B$ of) their Minkowski sum. □

Let Δ be the standard unit $(n-1)$-simplex

$$\begin{aligned}\Delta &= \{(t_1,\ldots,t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = 1, t_i \geq 0 \text{ for all } 1 \leq i \leq n\} \\ &= \operatorname{conv}\{e_1,\ldots,e_n\},\end{aligned}$$

where $e_i = (0, , \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in its ith coordinate. As J ranges over the subsets of $[n]$, let Δ_J be the face of the simplex Δ defined by

$$\Delta_J = \operatorname{conv}\{e_i : i \in J\} = P_n(\{z(J)_I\})$$

where $z(J)_I = 1$ if $I \supseteq J$ and $z(J)_I = 0$ otherwise. Lemma 2.2.2 gives the following proposition.

Proposition 2.2.3. *[23, Proposition 6.3] For any $y_I \geq 0$, the Minkowski sum $\sum y_I \Delta_I$ of dilations of faces of the standard $(n-1)$-simplex is a generalized permutohedron. We can write*

$$\sum_{A \subseteq E} y_I \Delta_I = P_n(\{z_I\}),$$

where $z_I = \sum_{J \subseteq I} y_J$ for each $I \subseteq [n]$.

We can extend this to encompass signed Minkowski sums as well.

Proposition 2.2.4. *Every generalized permutohedron $P_n(\{z_I\})$ can be written uniquely as a signed Minkowski sum of simplices, as*

$$P_n(\{z_I\}) = \sum_{I \subseteq [n]} y_I \Delta_I$$

where $y_I = \sum_{J \subseteq I} (-1)^{|I|-|J|} z_J$ for each $I \subseteq [n]$.

Proof. First we need to separate the righthand side into its positive and negative parts. By Proposition 2.2.3,

$$\sum_{I \subseteq [n] : y_I < 0} (-y_I)\Delta_I = P_n(\{z_I^-\}) \text{ and } \sum_{I \subseteq [n] : y_I \geq 0} y_I \Delta_I = P_n(\{z_I^+\})$$

where $z_I^- = \sum_{J\subseteq I\,:y_J<0}(-y_J)$ and $z_I^+ = \sum_{J\subseteq I\,:y_J\geq 0} y_J$. Now $z_I + z_I^- = z_I^+$ gives

$$P_n(\{z_I\}) + \sum_{I\subseteq[n]\,:y_I<0}(-y_I)\Delta_I = \sum_{I\subseteq[n]\,:y_I\geq 0} y_I\Delta_I,$$

as desired. Uniqueness is clear. □

Let M be a matroid of rank r on the set E. The *matroid polytope* of M is the polytope P_M in $\mathbb{R}^E$ whose vertices are the indicator vectors of the bases of M. The known description of the polytope P_M by inequalities makes it apparent that it is a generalized permutohedron:

Proposition 2.2.5. *[34] The matroid polytope of a matroid M on E with rank function r is $P_M = P_E(\{r - r(E - I)\}_{I\subseteq E})$.*

Proof. The inequality description for P_M is:

$$P_M = \{\mathbf{x} \in \mathbb{R}^E\ :\ \sum_{i\in E} x_i = r, \sum_{i\in A} x_i \leq r(A) \text{ for all } A \subseteq E\}.$$

It remains to remark that the inequality $\sum_{i\in A} x_i \leq r(A)$ is tight, and may be rewritten as $\sum_{i\in E-A} x_i \geq r - r(A)$, and to invoke the submodularity of the rank function of a matroid. □

The *beta invariant* [8] of M is a non-negative integer given by

$$\beta(M) = (-1)^{r(M)} \sum_{X\subseteq E} (-1)^{|X|} r(X)$$

which stores significant information about M; for example, $\beta(M) = 0$ if and only if M is disconnected and $\beta(M) = 1$ if and only if M is series-parallel. If

$$T_M(x, y) = \sum_{A\subseteq E} (x - 1)^{r(E)-r(A)}(y - 1)^{|A|-r(A)} = \sum_{i,j} b_{ij} x^i y^j$$

is the *Tutte polynomial* [35] of M, then $\beta(M) = b_{10} = b_{01}$ for $|E| \geq 2$.

Our next results are more elegantly stated in terms of the *signed beta invariant* of M, which we define to be

$$\widetilde{\beta}(M) = (-1)^{r(M)+1}\beta(M).$$

Theorem 2.2.6. *Let M be a matroid of rank r on E and let P_M be its matroid polytope. Then*

$$P_M = \sum_{A\subseteq E} \widetilde{\beta}(M/A)\, \Delta_{E-A}. \tag{2.1}$$

Proof. By Propositions 2.2.4 and 2.2.5, $P_M = \sum_{I \subseteq E} y_I \Delta_I$ where

$$\begin{aligned} y_I &= \sum_{J \subseteq I} (-1)^{|I|-|J|}(r - r(E-J)) = -\sum_{J \subseteq I} (-1)^{|I|-|J|} r(E-J) \\ &= -\sum_{E-J \supseteq E-I} (-1)^{|E-J|-|E-I|}(r(E-J) - r(E-I)) \\ &= -\sum_{X \subseteq I} (-1)^{|X|}(r(E-I \cup X) - r(E-I)) \\ &= -\sum_{X \subseteq I} (-1)^{|X|} r_{M/(E-I)}(X) = \widetilde{\beta}(M/(E-I)) \end{aligned}$$

as desired. □

Example 2.2.7. Let M be the matroid on $E = [4]$ with bases $\{12, 13, 14, 23, 24\}$; its matroid polytope is a square pyramid. Theorem 2.2.6 gives $P_M = \Delta_{234} + \Delta_{134} + \Delta_{12} - \Delta_{1234}$, as illustrated in Figure 2.1. The dotted lines in the polytope $\Delta_{234} + \Delta_{134} + \Delta_{12}$ are an aid to visualize the Minkowski difference.

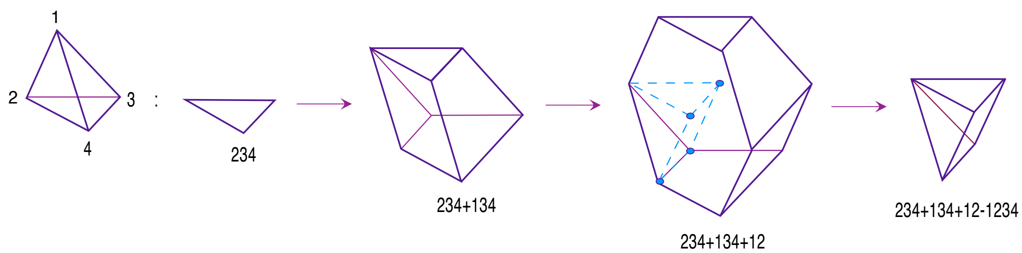


Figure 2.1: A matroid polytope as a signed Minkowski sum of simplices.

One way of visualizing the Minkowski sum of two polytopes P and Q is by grabbing a vertex v of Q and then using it to "slide" Q around in space, making sure that v never leaves P. The region that Q sweeps along the way is $P + Q$. Similarly, the Minkowski difference $P - R$ can be visualized by picking a vertex v of R and then "sliding" R around in space, this time making sure that no point in R ever leaves P. The region that v sweeps along the way is $P - R$. This may be helpful in understanding Figure 2.1.

Some remarks about Theorem 2.2.6 are in order.

- Generally most terms in the sum of Theorem 2.2.6 are zero. The nonzero terms correspond to the *coconnected flats* A, which we define to be the sets A such that M/A is connected. These are indeed flats, since contracting by them must produce a loopless matroid.

- A matroid and its dual have congruent matroid polytopes, and Theorem 2.2.6 gives different formulas for them. For example $P_{U_{1,3}} = \Delta_{123}$ while $P_{U_{2,3}} = \Delta_{12} + \Delta_{23} + \Delta_{13} - \Delta_{123}$.

- The study of the subdivisions of a matroid polytope into smaller matroid polytopes, originally considered by Lafforgue [17], has recently received significant attention [3, 5, 10, 27]. Speyer conjectured [27] that the subdivisions consisting of series-parallel matroids have the largest number of faces in each dimension and proved this [26] for a large and important family of subdivisions: those that arise from a tropical linear space. The important role played by series-parallel matroids is still somewhat mysterious. Theorem 2.2.6 characterizes series-parallel matroids as those whose matroid polytope has no repeated Minkowski summands. It would be interesting to connect this characterization to matroid subdivisions; this may require extending the theory of mixed subdivisions to signed Minkowski sums.

- Theorem 2.2.6 provides a geometric interpretation for the beta invariant of a matroid M in terms of the matroid polytope P_M. In Section 2.5 we see how to extend this to certain families of Coxeter matroids. This is a promising point of view towards the notable open problem [7, Problem 6.16.6] of defining useful enumerative invariants of a Coxeter matroid.

2.3 The volume of a matroid polytope

Our next goal is to present an explicit combinatorial formula for the volume of an arbitrary matroid polytope. Formulas have been given for very special families of matroids by Stanley [29] and Lam and Postnikov [18]. Additionally, a polynomial time algorithm for computing the volume of an arbitrary matroid polytope was recently discovered by de Loera et. al. [9]. Let us say some words about the motivation for this question.

Consider the Grassmannian manifold $Gr_{k,n}$ of k-dimensional subspaces in $\mathbb{C}^n$. Such a subspace can be represented as the rowspace of a $k \times n$ matrix A of rank k, modulo the left action of GL_k which does not change the row space. The $\binom{n}{k}$ maximal minors of this matrix are the *Plücker coordinates* of the subspace, and they give an embedding of $Gr_{k,n}$ as a projective algebraic variety in $\mathbb{CP}^{\binom{n}{k}-1}$.

Each point p in $Gr_{k,n}$ gives rise to a matroid M_p whose bases are the k-subsets of n where the Plücker coordinate of p is not zero. Gelfand, Goresky, MacPherson, and Serganova [14] first considered the stratification of $Gr_{k,n}$ into *matroid strata*, which consist of the points corresponding to a fixed matroid.

The torus $\mathbb{T} = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$ for $t_i \neq 0$; this action extends to an action of $\mathbb{T}$ on $Gr_{k,n}$. For a point $p \in Gr_{k,n}$, the closure of the torus orbit $X_p = \overline{\mathbb{T} \cdot p}$ is a toric variety which only depends on the matroid M_p of p, and the polytope corresponding to X_p under the moment map is the matroid polytope of M_p [14].

Under these circumstances it is known [13] that the volume of the matroid polytope M_p equals the degree of the toric variety X_p as a projective subvariety of $\mathbb{CP}^{\binom{n}{k}-1}$:

$$\mathrm{Vol}\, P_{M_p} = \deg X_p.$$

Therefore, by finding the volume of an arbitrary matroid polytope, one obtains a formula for the degree of the toric varieties arising from arbitrary torus orbits in the Grassmannian.

To prove our formula for the volume of a matroid polytope, we first recall the notion of the *mixed volume* $\mathrm{Vol}\,(P_1, \ldots, P_n)$ of n (possibly repeated) polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^n$. All volumes in this section are normalized with respect to the lattice generated by $e_1 - e_2, \ldots, e_{n-1} - e_n$ where our polytopes live; so the standard simplex Δ has volume $1/(n-1)!$.

Proposition 2.3.1. *[20] Let n be a fixed positive integer. There exists a unique function $\mathrm{Vol}\,(P_1, \ldots, P_n)$ defined on n-tuples of polytopes in $\mathbb{R}^n$, called the* mixed volume *of $P_1, \ldots, P_n$, such that, for any collection of polytopes $Q_1, \ldots, Q_m$ in $\mathbb{R}^n$ and any nonnegative real numbers $y_1, \ldots, y_m$, the volume of the Minkowski sum $y_1Q_1 + \cdots + y_mQ_m$ is the polynomial in $y_1, \ldots, y_m$ given by*

$$\mathrm{Vol}\,(y_1Q_1 + \cdots + y_mQ_m) = \sum_{i_1,\ldots,i_n} \mathrm{Vol}\,(Q_{i_1}, \ldots, Q_{i_n}) y_{i_1} \cdots y_{i_n}$$

where the sum is over all ordered n-tuples $(i_1, \ldots, i_n)$ with $1 \leq i_r \leq m$.

We now show that the formula of Proposition 2.3.1 still holds if some of the y_is are negative as long as the expression $y_1Q_1 + \cdots + y_mQ_m$ still makes sense.

Proposition 2.3.2. *If $P = y_1Q_1 + \cdots + y_mQ_m$ is a* signed *Minkowski sum of polytopes in $\mathbb{R}^n$, then*

$$\mathrm{Vol}\,(y_1Q_1 + \cdots + y_mQ_m) = \sum_{i_1,\ldots,i_n} \mathrm{Vol}\,(Q_{i_1}, \ldots, Q_{i_n}) y_{i_1} \cdots y_{i_n}$$

where the sum is over all ordered n-tuples $(i_1, \ldots, i_n)$ with $1 \leq i_r \leq m$.

Proof. We first show that

$$\mathrm{Vol}\,(A - B) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathrm{Vol}\,(A, \ldots, A, B, \ldots, B) \tag{2.2}$$

when B is a Minkowski summand of A in $\mathbb{R}^n$. Let $A - B = C$. By Proposition 2.3.1, for $t \geq 0$ we have that

$$\mathrm{Vol}\,(C + tB) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(C, \ldots, C, B \ldots, B) t^k =: f(t)$$

and we are interested in computing $\mathrm{Vol}\,(C) = f(0)$. Invoking Proposition 2.3.1 again, for $t \geq 0$ we have that

$$\mathrm{Vol}\,(A + tB) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(A, \dots, A, B, \dots, B) t^k =: g(t). \tag{2.3}$$

But $A + tB = C + (t+1)B$ and therefore $g(t) = f(t+1)$ for all $t \geq 0$. Therefore $g(t) = f(t+1)$ as polynomials, and $\mathrm{Vol}\, C = f(0) = g(-1)$. Plugging into (2.3) gives the desired result.

Having established (2.2), separate the given Minkowski sum for P into its positive and negative parts as $P = Q - R$, where $Q = x_1 Q_1 + \cdots + x_r Q_r$ and $R = y_1 R_1 + \cdots + y_s R_s$ with $x_i, y_i \geq 0$. For positive t we can write $Q + tR = \sum x_i Q_i + \sum t y_j R_j$, which gives two formulas for $\mathrm{Vol}\,(Q + tR)$.

$$\begin{aligned} \mathrm{Vol}\,(Q + tR) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(Q, \dots, Q, R, \dots, R) t^k \\ &= \sum_{\substack{1 \leq i_a \leq r \\ 1 \leq j_b \leq s}} \mathrm{Vol}\,(Q_{i_1}, \dots, Q_{i_{n-k}}, R_{j_1}, \dots, R_{j_k}) x_{i_1} \cdots x_{i_{n-k}} y_{j_1} \cdots y_{j_k} t^k \end{aligned}$$

The last two expressions must be equal as polynomials. A priori, we cannot plug $t = -1$ into this equation; but instead, we can use the formula for $\mathrm{Vol}\,(Q - R)$ from (2.2), and then plug in coefficient by coefficient. That gives the desired result. □

Theorem 2.3.3. *If a connected matroid M has n elements, then the volume of the matroid polytope P_M is*

$$\mathrm{Vol}\, P_M = \frac{1}{(n-1)!} \sum_{(J_1, \dots, J_{n-1})} \widetilde{\beta}(M/J_1) \widetilde{\beta}(M/J_2) \cdots \widetilde{\beta}(M/J_{n-1}),$$

summing over the ordered collections of sets $J_1, \dots, J_{n-1} \subseteq [n]$ such that, for any distinct $i_1, \dots, i_k$, $|J_{i_1} \cap \cdots \cap J_{i_k}| < n - k$.

Proof. Postnikov [23, Corollary 9.4] gave a formula for the volume of a (positive) Minkowski sum of simplices. We would like to apply his formula to the signed Minkowski sum in Theorem 2.2.6, and Proposition 2.3.2 makes this possible. □

There is an alternative characterization of the tuples $(J_1, \dots, J_{n-1})$ considered in the sum above. They are the tuples such that, for each $1 \leq k \leq n$, the collection $([n] - J_1, \dots, [n] - J_{n-1})$ has a system of distinct representatives avoiding k; that is, there exist $a_1 \in [n] - J_1, \dots, a_{n-1} \in [n] - J_{n-1}$ with $a_i \neq a_j$ for $i \neq j$ and $a_i \neq k$ for all i. Postnikov refers to this as the *dragon marriage condition*; see [23] for an explanation of the terminology.

As in Theorem 2.2.6, most of the terms in the sum of Theorem 2.3.3 vanish. The nonzero terms are those such that each J_i is a coconnected flat. Furthermore, since P_M and P_{M^*} are congruent, we are free to apply Theorem 2.3.3 to the one giving a simpler expression.

Example 2.3.4. Suppose we wish to compute the volume of $P_{U_{2,3}}$ using Theorem 2.3.3. The expression $P_{U_{1,3}} = \Delta_{123}$ is simpler than the one for $P_{U_{2,3}}$. So we can obtain $\operatorname{Vol} P_{(U_{1,3})^*} = \operatorname{Vol} P_{U_{1,3}} = \frac{1}{2}\widetilde{\beta}(M)^2 = \frac{1}{2}$.

In Theorem 2.3.3, the hypothesis that M is connected is needed to guarantee that the matroid polytope P_M has dimension $n-1$. More generally, if we have $M = M_1 \oplus \cdots \oplus M_k$ then $P_M = P_{M_1} \times \cdots \times P_{M_k}$ so the $((n-k)$-dimensional) volume of P_M is $\operatorname{Vol} P_M = \operatorname{Vol} P_{M_1} \cdots \operatorname{Vol} P_{M_k}$.

2.4 Independent set polytopes

In this section we show that our analysis of matroid polytopes can be carried out similarly for the *independent set polytope* I_M of a matroid M, which is the convex hull of the indicator vectors of the independent sets of M. The inequality description of I_M is known to be:

$$I_M = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \, : \, x_i \geq 0 \text{ for } i \in [n], \sum_{i \in A} x_i \leq r(A) \text{ for all } A \subseteq E\}. \tag{2.4}$$

This realization of the independent set polytope of a matroid is not a generalized permutohedron. Instead, it is a *Q-polytope.* The class of Q-polytopes are the deformations of the simple polytope Q_n whose vertices are formed by all distinct permutations of entries of the vectors $(1, \ldots, n), (0, 2, \ldots, n), \ldots, (0, \ldots, 0, n)$, and $(0, \ldots, 0)$. After translation, every Q-polytope can be expressed in the form

$$Q_n(\{z_J\}) = \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_i \geq 0 \text{ for all } i \in [n], \sum_{i \in J} t_i \leq z_J \text{ for all } J \subseteq [n]\right\} \tag{2.5}$$

where z_J is a non-negative real number for each $J \subseteq [n]$. Analogously to generalized permutohedra, the parameters z_J which describe a Q-polytope $Q_n(\{z_J\})$ are those which satisfy a submodular inequality, however since our inequalities are reversed in the hyperplane description of Q_n we reverse the inequality on our defining submodular criterion to make the supermodular criterion.

Proposition 2.4.1. *$Q_n(\{z_J\})$ is a Q-polytope if and only if*

$$z_I + z_J \geq z_{I \cup J} + z_{I \cap J}$$

for all $I, J \subseteq [n]$.

Proof. This follows directly from Theorem 2.2.1. We will describe a deformation-preserving bijection from generalized permutohedra to Q-polytopes where submodularity of z_I parameters of $P_n(\{z_I\})$ corresponds to supermodularity of z_J parameters of $Q_n(\{z_J\})$. Given a generalized permutohedron $P_n(\{z_I\}) \subset \mathbb{R}^n$, define $P_{n+1}(\{z'_I\})$ to be the generalized permutohedron in $\mathbb{R}^{n+1}$ defined by $z'_I = 0$ and $z'_{I\cup\{n+1\}} = z_I$ for all $I \subseteq [n]$. Now define $Q_n(\{z''_I\})$ to be the projection of $P_{n+1}(\{z'_I\})$ into $\mathbb{R}^n$, by removal of the last coordinate. This invertible process sends $P_n(\{z_I\})$ to $Q_n(\{z''_I\})$ where $z''_I = z_{[n]} - z_{[n]\setminus I}$. Moreover, it sends the permutohedron P_n to Q_n and deformations of P_n to deformations of Q_n. A polytope P has inequality description $P_n(\{z_I\})$ satisfying the submodular inequalities if and only if P is a deformation of the permutohedron P_n. By the map described above this occurs if and only if the corresponding polytope $Q = Q_n(\{z''_I\})$ is a deformation of Q_n. By the construction of the z''_I, submodularity of the z_I is equivalent to supermodularity of the z''_I. Thus we have a deformation of Q_n if and only if the corresponding z''_I parameters are supermodular. □

We can also express these polytopes as signed Minkowski sums of simplices, though the simplices we use are not the Δ_Js, but those of the form

$$\begin{aligned} D_J &= \mathrm{conv}\{0,\ e_i\ :\ i \in J\} \\ &= Q_n(\{d(J)_I\}) \end{aligned}$$

where $d(J)_I = 0$ if $I \cap J = \emptyset$ and $d(J)_I = 1$ otherwise.

The following lemmas on Q-polytopes are proved in a way analogous to the corresponding lemmas for generalized permutahedra, as was done in Section 2.2.

Lemma 2.4.2. *If $Q_n(\{z_J\})$ and $Q_n(\{z'_J\})$ are Q-polytopes, then so is their Minkowski sum, and $Q_n(\{z_J\}) + Q_n(\{z'_J\}) = Q_n(\{z_J + z'_J\})$*

Proposition 2.4.3. *For any $y_I \geq 0$ we have*

$$\sum_{I\subseteq[n]} y_I D_I = Q_n(\{z_J\})$$

where $z_J = \sum_{I:I\cap J\neq\emptyset} y_I$.

Proposition 2.4.4. *Every Q-polytope $Q_n(\{z_J\})$ can be written uniquely*[2] *as a signed Minkowski sum of D_Is as*

$$Q_n(\{z_J\}) = \sum_{I\subseteq[n]} y_I D_I,$$

where

$$y_J = -\sum_{I\subseteq J} (-1)^{|J|-|I|} z_{[n]-I}.$$

[2] assuming $y_\emptyset = 0$

Proof. We need to invert the relation between the y_Is and the z_Js given by $z_J = \sum_{I:I\cap J\neq\emptyset} y_I$. We rewrite this relation as

$$z_{[n]} - z_J = \sum_{I\subseteq[n]-J} y_I$$

and apply inclusion-exclusion. As in Section 2.2, we first do this in the case $y_I \geq 0$ and then extend it to arbitrary Q-polytopes. □

Theorem 2.4.5. *Let M be a matroid of rank r on E and let I_M be its independent set polytope. Then*

$$I_M = \sum_{A\subseteq E} \widetilde{\beta}(M/A)\, D_{E-A}$$

where $\widetilde{\beta}$ denotes the signed beta invariant.

Proof. This follows from Proposition 2.4.4 and a computation almost identical to the one in the proof of Theorem 2.2.6. □

The great similarity between Theorems 2.2.6 and 2.4.5 is not surprising, since P_M is the facet of I_M which maximizes the linear function $\sum_{i\in E} x_i$, and Δ_I is the facet of D_I in that direction as well. In fact we could have first proved Theorem 2.4.5 and then obtained Theorem 2.2.6 as a corollary.

Theorem 2.4.6. *If a connected matroid M has n elements, then the volume of the independent set polytope I_M is*

$$\operatorname{Vol} I_M = \frac{1}{n!} \sum_{(J_1,\ldots,J_n)} \widetilde{\beta}(M/J_1)\widetilde{\beta}(M/J_2)\cdots\widetilde{\beta}(M/J_n)$$

where the sum is over all $n-$tuples $(J_1,\ldots,J_n)$ of subsets of $[n]$ such that, for any distinct $i_1,\ldots,i_k$, we have $|J_{i_1}\cap\cdots\cap J_{i_k}| \leq n-k$.

Notice that by Hall's marriage theorem, the condition on the J_is is equivalent to requiring that $(E-J_1,\ldots,E-J_n)$ has a system of distinct representatives (SDR); that is, there are $a_1 \in E-J_1,\ldots,a_n \in E-J_n$ with $a_i \neq a_j$ for $i \neq j$.

Proof. By Theorem 2.4.5 and Proposition 2.3.1 it suffices to compute the mixed volume $\operatorname{Vol}(D_{A_1},\ldots,D_{A_n})$ for each n-tuple $(A_1,\ldots,A_n)$ of subsets of $[n]$. Bernstein's theorem [32] tells us that $\operatorname{Vol}(D_{A_1},\ldots,D_{A_n})$ is the number of isolated solutions in $(\mathbb{C}-\{0\})^n$ of the system of equations:

$$\begin{aligned}
\beta_{1,0} + \beta_{1,1}t_1 + \beta_{1,2}t_2 + \cdots + \beta_{1,n}t_n &= 0\\
\beta_{2,0} + \beta_{2,1}t_1 + \beta_{2,2}t_2 + \cdots + \beta_{2,n}t_n &= 0\\
&\vdots\\
\beta_{n,0} + \beta_{n,1}t + \beta_{n,2}t_2 + \cdots + \beta_{n,n}t_n &= 0
\end{aligned}$$

where $\beta_{i,0}$ and $\beta_{i,j}$ are generic complex numbers when $j \in A_i$, and $\beta_{i,j} = 0$ if $j \notin A_i$.

This system of linear equations will have one solution if it is non-singular and no solutions otherwise. Because the $\beta_{i,0}$ are generic, such a solution will be non-zero if it exists. The system is non-singular when the determinant is non-zero, and by genericity that happens when $(A_1, \ldots, A_n)$ has an SDR. We conclude that $\mathrm{Vol}\,(D_{E-J_1}, \ldots, D_{E-J_n})$ is 1 if $(E-J_1, \ldots, E-J_n)$ has an SDR and 0 otherwise, and the result follows. □

Let us illustrate Theorem 2.4.6 with an example.

Example 2.4.7. The independent set polytope of the uniform matroid $U_{2,3}$ is shown in Figure 2.2. We have $I_M = D_{12} + D_{23} + D_{13} - D_{123}$. Theorem 2.4.6 should confirm that its volume is $\frac{5}{6}$; let us carry out that computation.

The coconnected flats of M are $1, 2, 3$ and $\emptyset$ and their complements are $\{23, 13, 12, 123\}$. We need to consider the triples of coconnected flats whose complements contain an SDR. Each one of the 24 triples of the form (a, b, c), where $a, b, c \in [3]$ are not all equal, contributes a summand equal to 1. The 27 permutations of triples of the form $(a, b, \emptyset)$, contribute a -1 each. The 9 permutations of triples of the form $(a, \emptyset, \emptyset)$ contribute a 1 each. The triple $(\emptyset, \emptyset, \emptyset)$ contributes a -1. The volume of I_M is then $\frac{1}{6}(24 - 27 + 9 - 1) = \frac{5}{6}$.

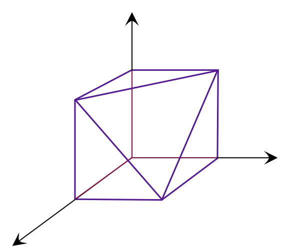


Figure 2.2: The independent set polytope of $U_{2,3}$.

2.5 Truncation flag matroids

We will soon see that any flag matroid polytope can also be written as a signed Minkowski sum of simplices Δ_I. We now focus on the particularly nice family of *truncation flag matroids*, introduced by Borovik, Gelfand, Vince, and White [6], where we obtain an explicit formula for this sum.

The *strong order* on matroids is defined by saying that two matroids M and N on the same ground set E, having respective ranks $r_M < r_N$, are *concordant* if their rank functions satisfy $r_M(Y) - r_M(X) \leq r_N(Y) - r_N(X)$ for all $X \subset Y \subseteq E$. [7].

Flag matroids are an important family of *Coxeter matroids* [7]. There are several equivalent ways to define them; in particular they also have an algebro-geometric interpretation. We proceed constructively. Given pairwise concordant matroids $M_1, \ldots, M_m$ on E of ranks

$k_1 < \cdots < k_m$, consider the collection of flags $(B_1, \ldots, B_m)$, where B_i is a basis of M_i and $B_1 \subset \cdots \subset B_m$. Such a collection of flags is called a *flag matroid*, and $M_1, \ldots, M_m$ are called the *constituents* of $\mathcal{F}$.

For each flag $B = (B_1, \ldots, B_m)$ in $\mathcal{F}$ let $v_B = v_{B_1} + \cdots + v_{B_m}$, where $v_{\{a_1,\ldots,a_i\}} = e_{a_1} + \cdots + e_{a_i}$. The *flag matroid polytope* is $P_{\mathcal{F}} = \text{conv}\{v_B : B \in \mathcal{F}\}$.

Theorem 2.5.1. *[7, Cor 1.13.5] If $\mathcal{F}$ is a flag matroid with constituents $M_1, \ldots, M_k$, then $P_{\mathcal{F}} = P_{M_1} + \cdots + P_{M_k}$.*

As mentioned above, this implies that every flag matroid polytope is a signed Minkowski sum of simplices Δ_I; the situation is particularly nice for truncation flag matroids, which we now define.

Let M be a matroid over the ground set E with rank r. Define M_i to be the rank i *truncation* of M, whose bases are the independent sets of M of rank i. One easily checks that the truncations of a matroid are concordant, and this motivates the following definition of Borovik, Gelfand, Vince, and White.

Definition 2.5.2. [6] The flag $\mathcal{F}(M)$ with constituents $M_1, \ldots, M_r$ is a flag matroid, called the *truncation flag matroid* or *underlying flag matroid* of M.

Our next goal is to present the decomposition of a truncation flag matroid polytope as a signed Minkowski sum of simplices. For that purpose, we define the *gamma invariant* of M to be $\gamma(M) = b_{20} - b_{10}$, where $T_M(x, y) = \sum_{i,j} b_{ij} x^i y^j$ is the Tutte polynomial of M.

Proposition 2.5.3. *The gamma invariant of a matroid is given by*

$$\gamma(M) = \sum_{I \subseteq E} (-1)^{r-|I|} \binom{r - r(I) + 1}{2}.$$

Proof. We would like to isolate the coefficient of x^2 minus the coefficient of x in the Tutte polynomial $T_M(x, y)$. We will hence ignore all terms containing y by evaluating $T_M(x, y)$ at $y = 0$, and then combine the desired x terms through the following operations:

$$\begin{aligned} \gamma(M) &:= \frac{1}{2} \left[\frac{d^2}{dx^2} (1 - x) T_M(x, 0) \right]_{x=0} \\ &= \frac{1}{2} \left[\frac{d^2}{dx^2} \sum_{I \subseteq E} (-1)^{|I| - r(I) + 1} (x - 1)^{r - r(I) + 1} \right]_{x=0} \\ &= \sum_{I \subseteq E} (-1)^{r-|I|} \binom{r - r(I) + 1}{2}, \end{aligned}$$

as we wished to show. □

Unlike the beta invariant, the gamma invariant is not necessarily nonnegative. In fact its sign is not simply a function of $|E|$ and r. For example, $\gamma(U_{k,n}) = -\binom{n-3}{k-1}$, and $\gamma(U_{k,n} \oplus C) = \binom{n-2}{k-1}$ where C denotes a coloop.

As we did with the beta invariant, define the *signed gamma invariant* of M to be $\widetilde{\gamma}(M) = (-1)^{r(M)}\gamma(M)$.

Theorem 2.5.4. *The truncation flag matroid polytope of M can be expressed as:*

$$P_{\mathcal{F}(M)} = \sum_{I \subseteq E} \widetilde{\gamma}(M/I)\Delta_{E-I}.$$

Proof. By Theorems 2.2.6 and 2.5.1, $P_{\mathcal{F}(M)}$ is

$$\sum_{i=1}^{r} P_{M_i} = \sum_{i=1}^{r} \sum_{I \subseteq E} \sum_{J \subseteq I} (-1)^{|I|-|J|}(i - r_i(E-J))\Delta_I,$$

where $r_i(A) = \min\{i, r(A)\}$ is the rank function of M_i. Then

$$\begin{aligned} P_{\mathcal{F}(M)} &= \sum_{I \subseteq E} \left[\sum_{J \subseteq I} (-1)^{|I|-|J|} \sum_{i=r(E-J)+1}^{r} (i - r(E-J)) \right] \Delta_I \\ &= \sum_{I \subseteq E} \left[\sum_{J \subseteq I} (-1)^{|I|-|J|} \binom{r - r(E-J)+1}{2} \right] \Delta_I \\ &= \sum_{I \subseteq E} \left[\sum_{X \subseteq I} (-1)^{|X|} \binom{r_{M/(E-I)} - r_{M/(E-I)}(X)+1}{2} \right] \Delta_I \\ &= \sum_{I \subseteq E} \widetilde{\gamma}(M/(E-I))\Delta_I \end{aligned}$$

as desired. □

Corollary 2.5.5. *If a connected matroid M has n elements, then*

$$\operatorname{Vol} P_{\mathcal{F}(M)} = \frac{1}{(n-1)!} \sum_{(J_1,\ldots,J_{n-1})} \widetilde{\gamma}(M/J_1)\widetilde{\gamma}(M/J_2)\cdots\widetilde{\gamma}(M/J_{n-1}),$$

summing over the ordered collections of sets $J_1, \ldots, J_{n-1} \subseteq [n]$ such that, for any distinct $i_1, \ldots, i_k$, $|J_{i_1} \cap \cdots \cap J_{i_k}| < n-k$.

Proof. This follows from Proposition 2.3.2 and Theorem 2.5.4. □

Example 2.5.6. Let M be the matroid on $[3]$ with bases $\{1,2\}$ and $\{1,3\}$. The flags in $\mathcal{F}(M)$ are: $\{1\} \subseteq \{1,2\}, \{1\} \subseteq \{1,3\}, \{2\} \subseteq \{1,2\}, \{3\} \subseteq \{1,3\}$, so the vertices of $P_{\mathcal{F}(M)}$ are $(2,1,0), (2,0,1), (1,2,0), (1,0,2)$, respectively. Theorem 2.5.4 gives $P_{\mathcal{F}(M)} = \Delta_{123} + \Delta_{23}$. Since $\widetilde{\gamma}(M) = \widetilde{\gamma}(M/1) = 1$, Corollary 2.5.5 gives

$$\operatorname{Vol} P_{\mathcal{F}(M)} = \frac{1}{2!}[\widetilde{\gamma}(M/\emptyset)\widetilde{\gamma}(M/\emptyset) + \widetilde{\gamma}(M/\emptyset)\widetilde{\gamma}(M/1) + \widetilde{\gamma}(M/1)\widetilde{\gamma}(M/\emptyset)] = \frac{3}{2}.$$

Chapter 3

Geometry and generalizations of multiplihedra

3.1 Introduction

We introduce the multiplihedron $\mathcal{J}(n)$ as an $(n-1)$-dimensional polytope that sits in a hyperplane in $\mathbb{R}^n$. We define $\mathcal{J}(n)$ as a specific Minkowski sum of coordinate simplices and show that its face lattice is isomorphic to the poset of painted trees as studied by Forcey [12]. Our definition of the multiplihedron is a simple extension of the Minkowski decomposition of the associahedron defined in [19], and many of the geometric properties of the associahedron are naturally generalized through this construction. The associahedron is a generalized permutohedron, and by our definition as a Minkowski sum of simplices the multiplihedron is as well [23].

The machinery used to lift the associahedron to the multiplihedron is called the *q-lifting operator*, and it can be applied to any generalized permutohedron. We explore this machinery and make precise the geometric properties of q-lifted polytopes, such as their face lattices, inequality descriptions, vertices, and volumes.

3.2 q-lifted polytopes

In this section we define our main object of study, the *q-lifting operator*. The q-lifting operator sends a generalized permutohedron P in $\mathbb{R}^n$ to a higher dimensional generalized permutohedron $P(q)$ in $\mathbb{R}^{n+1}$ through the use of Minkowski sum decompositions. We show that the q-lifting operator sends the permutohedron P_n to a polytope $P_n(q)$ that is combinatorially equivalent to P_{n+1}, and describe the face lattice of $P(q)$ in terms of the face lattice of P.

Before defining $P(q)$ we must define the class of $\tilde{Q}$-polytopes, which are combinatorially equivalent variants of the Q-polytopes introduced in Section 2.4.

Definition 3.2.1. Given a generalized permutohedron $P_n(\{z_I\}) \subset \mathbb{R}^n$, define $\widetilde{Q}_n(\{z_I\}) := P_{n+1}(\{z'_I\})$ to be the generalized permutohedron in $\mathbb{R}^{n+1}$ defined by $z'_J = 0$ and $z'_{J\cup\{n+1\}} = z'_J$ for all $J \subseteq [n]$.

Proposition 3.2.2. *If we write $P_n(\{z_I\})$ in terms of Minkowski sums as $P_n(\{y_I\})$, then the induced $\tilde{Q}$-polytope $\tilde{Q}_n(\{z'_I\})$ may be written*

$$\tilde{Q}_n(\{y'_I\}) = \sum_{I\subseteq[n]} y_I \Delta_{I\cup\{n+1\}}.$$

Proof. This can be verified directly by the linear relation between the z_I and the y_I. □

We will now combine $P_n(\{y_I\})$ and $\tilde{Q}_n(\{y_I\})$, therefore let us assume that $P_n(\{y_I\})$ is embedded in the hyperplane $x_{n+1} = 0$ in $\mathbb{R}^{n+1}$.

Definition 3.2.3. For a generalized permutohedron $P = P_n(\{y_I\})$, along with a constant $q \in [0, 1]$, define the polytope $P(q) \subset \mathbb{R}^{n+1}$ to be the Minkowski sum

$$P(q) := qP_n(\{y_I\}) + (1-q)\tilde{Q}_n(\{y_I\}). \tag{3.1}$$

We say that the generalized permutohedron $P(q)$ is the *q-lifting* of P.

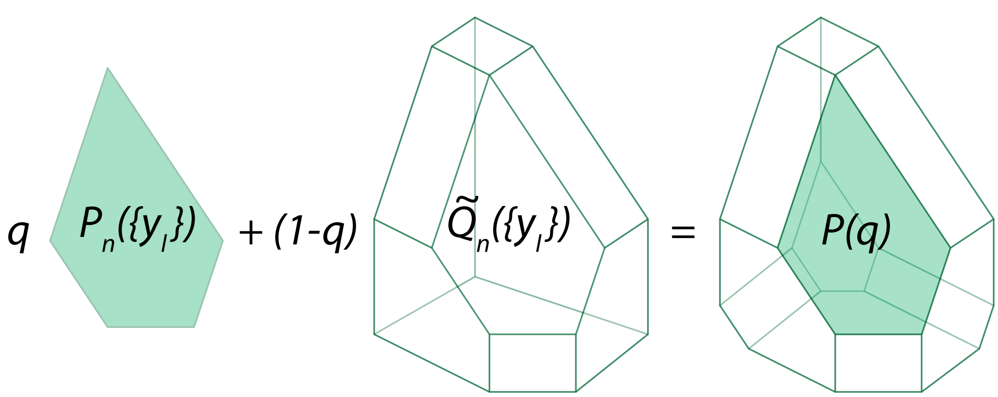


Figure 3.1: The q-lifting of a generalized permutohedron $P_n(\{y_I\})$ shown projected onto the 3-dimensional hyperplane $x_4 = 0$.

The face of $P(q)$ maximized in the $(1, \ldots, 1, 0)$ direction is a copy of P, and the face maximized in the opposite direction is a copy of P scaled by q. The vertices of $P(q)$ will consist of copies of the vertices of P, but with a factor of q applied to certain specific coordinates. We describe them in Section 3.4.

Postnikov [23] showed that any generalized permutohedron $P \subset \mathbb{R}^n$ can be written in terms of hyperplane parameters $\{z_I\}_{I\subseteq[n]}$ as

$$P = P_n(\{z_I\}) = \left\{ x \in \mathbb{R}^n : \sum_{i\in I} x_i \geq z_I \text{ for } I \subset [n], \sum_{i=1}^{n} x_i = z_{[n]} \right\}.$$

Given a Minkowski description of a generalized permutohedron $P = P_n(\{y_I\})$, the associated inequality description is expressed as $P = P_n(\{z_I\})$ where $z_I = \sum_{J\subseteq I} y_J$. Applying this to $P(q)$ we get the following immediate proposition:

Proposition 3.2.4. *Let $P = P_n(\{z_I\})$ be a generalized permutohedron with hyperplane parameters $\{z_I\}_{I\subseteq[n]}$. Then we can express $P(q) = P_{n+1}(\{z'_I\})$ as a generalized permutohedron with hyperplane parameters $\{z'_I\}_{I\subseteq[n+1]}$ given by $z'_J = qz_J$ and $z'_{J\cup\{n+1\}} = z_J$ for $J \subseteq [n]$ and $q \in [0,1]$.*

We now look into the face structure of q-lifted polytopes. One property enjoyed by generalized permutohedra is that their face lattices are always coarsenings of the face lattice of the permutohedron P_n. Moreover, all facets of a generalized permutohedron are parallel to facets of P_n. We discuss these properties in more detail in Section 3.5.

Definition 3.2.5. Consider the linear functional $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$. We partition the a_i into blocks $A_1, \ldots, A_k$ such that $a_i = a_j$ if and only if a_i and a_j both belong to the same block A_s, and $a_i < a_j$ if and only if $a_i \in A_s$ and $a_j \in A_t$ for some $s < t$. If we let $\pi = A_1|\cdots|A_k$ be an ordered partition of $[n]$ then we label $f(x)$ as $f_\pi(x)$ and we say that the functional f_π is of *type* π. We denote the face of a polytope P that maximizes $f_\pi(x)$ by P_π.

We wish to investigate the poset structure of the faces of $P(q)$. First recall that the face lattice $\mathcal{L}(P_n)$ of the permutohedron P_n is isomorphic to the poset $(\mathcal{P}^n, \prec)$, where $\mathcal{P}^n$ is the set of all ordered partitions of the set $[n]$, and $\pi \prec \pi'$ if and only if π' coarsens π [23]. First we show that the q-lifted permutohedron $P_n(q)$ is combinatorially equivalent to P_{n+1}.

Proposition 3.2.6. *The q-lifting of the permutohedron P_n is combinatorially equivalent to the permutohedron P_{n+1}.*

Proof. By definition $P_n(q)$ is a generalized permutohedron in $\mathbb{R}^{n+1}$, and hence its face lattice is a coarsening of the poset of ordered partitions on a set of size $n+1$. We will show that this coarsening is trivial, i.e. that every strict containment of faces in P_{n+1} corresponds to a strict containment of faces in $P_n(q)$. The permutohedron P_n is a zonotope. In particular it can be represented as the Minkowski sum of all coordinate 1-simplices Δ_{ij} for $1 \leq i < j \leq n$. Using our established notation, we write $P_n = P_n(\{y_I\})$ where $y_I = 1$ if I has size 2, and 0 otherwise. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n+1]$, and let f_π be a linear functional in $\mathbb{R}^{n+1}$ of type π. Coarsen π by joining blocks B_i and B_{i+1}. Call this new ordered partition σ. For every pair $b_1, b_2 \in [n+1]$ the Minkowski decomposition of $P_n(q)$ contains a simplex with $\Delta_{b_1 b_2}$ as a face. Take $b_1 \in B_i$ and $b_2 \in B_{i+1}$. Then the Minkowski decomposition of the face $P_n(q)_\sigma$ will include a contribution from $\Delta_{b_1 b_2}$, whereas the decomposition of $P_n(q)_\pi$ did not. Thus $P_n(q)_\pi$ is properly contained in $P_n(q)_\sigma$. We conclude that the face lattice is isomorphic to the poset of ordered partitions. □

Now we extend our focus to face lattices of general q-liftings. Let us assume that the generalized permutohedra P we are analyzing have nonempty intersection with the interior of the positive orthant of $\mathbb{R}^n$. If not, then simply project out the unused coordinate(s) until this condition is satisfied. This will make our proofs easier later.

Definition 3.2.7. Let P be a generalized permutohedron in $\mathbb{R}^n$, and let π and μ be ordered partitions of $[n]$. Then we say that $\pi \sim \mu$ if $P_\pi = P_\mu$. We can write the face lattice of P as

$$\mathcal{L}(P) \cong (\mathcal{P}^n, \prec) \,/ \sim .$$

The order $\prec$ is generated by cover relations on equivalence classes: the equivalence class $[\mu]$ covers $[\pi]$ if and only if P_μ is one dimension greater than P_π and some element of $[\mu]$ coarsens some element of $[\pi]$. Equivalently, this last statement states that there exist $\pi \in [\pi]$ and $\mu \in [\mu]$ such that $(P_n)_\pi \subset (P_n)_\mu$.

Definition 3.2.8. Let $\pi = A_1|\cdots|A_{k_1}$ and $\mu = B_1|\cdots|B_{k_2}$ be ordered partitions of $[n]$. Augment π and μ to construct π' and μ' by adding the element $\{n+1\}$ to the (possibly new, in which case we relabel the blocks) blocks A_{j_1} and B_{j_2}, respectively. Let $P(q)$ be the q-lifting of P. Then we say that $\pi' \sim' \mu'$ if the following conditions hold:

1. $\pi \sim \mu$, and
2. $A_{j_1} = B_{j_2}$ and $\bigcup_{i>j_1} A_i = \bigcup_{i>j_2} B_i$.

Proposition 3.2.9. *Using the notation established above, the face lattice of $P(q)$ is given by*

$$\mathcal{L}(P(q)) \cong \left(\mathcal{P}^{n+1}, \prec\right) / \sim' .$$

Proof. Write $P = P_n(\{y_I\})$. The assumption that P intersects the positive orthant implies that for every $i \in [n]$ there is some $I \subseteq [n]$ that contains i such that $y_I \neq 0$. Now decompose $P(q)_{\pi'}$ into

$$\begin{aligned} P(q)_{\pi'} &= qP_n(\{y_I\})_{\pi'} + (1-q)\tilde{Q}_n(\{y_I\})_{\pi'} \\ &= q \sum_{I \subseteq [n]} y_I (\Delta_I)_\pi + (1-q) \sum_{I \subseteq [n]} y_I (\Delta_{I \cup \{n+1\}})_{\pi'}. \end{aligned}$$

In the last expression we wrote $(\Delta_I)_\pi$ instead of $(\Delta_I)_{\pi'}$ because all of the Δ_I belong to the hyperplane $x_{n+1} = 0$. The decomposition for $P(q)_{\sigma'}$ is analogous. Now if condition 1 is not satisfied, then in the above expression the sums $\sum_{I\subseteq[n]} y_I(\Delta_I)_\pi$ and $\sum_{I\subseteq[n]} y_I(\Delta_I)_\sigma$ will be unequal. If 2 is not satisfied, then similarly $(\Delta_{I\cup\{n+1\}})_{\pi'}$ and $(\Delta_{I\cup\{n+1\}})_{\sigma'}$ must differ for some I with $y_I \neq 0$ by our positive orthant assumption. The reader can verify that both of these implications are reversible. □

There are other applications of the q-lifting operator. For example, if P_M is a matroid polytope, then $P_M(0)$ is combinatorially equivalent to the independent set polytope I_M as seen in Section 2.4. The above propositions provide an expression for the Minkowski decomposition of I_M in terms of the Minkowski decomposition P_M.

The associahedron can be q-lifted to form the multiplihedron. The next section is devoted to this.

3.3 The associahedron and the multiplihedron

In this section we define the *associahedron* $\mathcal{K}(n)$ both geometrically and in terms of planar trees, as in [19]. We then define the *multiplihedron* $\mathcal{J}(n)$ in terms of painted trees as studied by Forcey [12], and we show that this polytope is the q-lifting of the associahedron.

Definition 3.3.1. The *associahedron* $\mathcal{K}(n) \subset \mathbb{R}^{n-1}$ is an $(n-2)$-dimensional polytope whose face lattice is isomorphic to the poset of rooted planar trees with n leaves, ordered by reverse refinement, or coarsening. It is not trivial that such a polytope exists. A polytopal realization was first discovered by Haiman in 1984 [15].

We say the tree T' *coarsens* the tree T if T' can be formed by shortening branch lengths of T between some adjacent internal nodes until those nodes coincide. The vertices of $\mathcal{K}(n)$ correspond to the binary rooted planar trees, and the top face of $\mathcal{K}(n)$ corresponds to the rooted tree with only 1 internal node. A planar rooted tree with only one internal node will be called a *sapling*. We call the space between two adjacent interior edges growing upward from the interior node where they meet a *crook* of the tree. Crooks are in bijection with the n spaces between leaves, as a raindrop that falls between two adjacent leaves will roll down to a unique crook. Figure 3.2 (right) shows a tree with its labeled crooks.

It is known [23] that the associahedron is a generalized permutohedron with an elegant Minkowski decomposition. We will use the following equivalence relation from [33] to help derive this decomposition.

Definition 3.3.2. [33] Let $B_1|\cdots|B_k$ be an ordered partition of the set $[n]$. We say that blocks B_{m-1} and B_m are *independent* if there exists $x \in \bigcup_{i>m} B_i$ such that $\max B_{m-1} < x < \min B_m$ or $\max B_m < x < \min B_{m-1}$. Then define $\sim$ to be the equivalence relation generated by

$$(B_1|\cdots|B_k) \sim (B_1|\cdots|B_{m-1}\cup B_m|\cdots|B_k)$$

whenever B_{m-1} and B_m are independent. (Here parentheses are added purely for visual clarity.)

We will now show that $(\mathcal{P}^n, \prec)/\sim$, the poset of ordered partitions of $[n]$ modulo the above equivalence relation, is isomorphic to the poset of rooted planar trees on $n+1$ leaves, as well as to the face lattice of a particular polytope.

Proposition 3.3.3. *[23] The associahedron $\mathcal{K}(n+1)$ can be represented as the generalized permutohedron $P_n(\{y_I\})$ where $y_I = 1$ if I is an interval of consecutive integers, and $y_I = 0$ otherwise.*

For example $\mathcal{K}(4) = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_{12} + \Delta_{23} + \Delta_{123}$. Here the only omitted simplex is Δ_{13} because 1 and 3 are not consecutive. The bijection between faces of $\mathcal{K}(n)$ and rooted planar trees on n leaves can be visualized through the following example.

Example 3.3.4. Consider the face of $\mathcal{K}(4)$ that maximizes a linear functional f_π of type $\pi = 12|3$. We write the decomposition of this face as

$$(\mathcal{K}(4))_{12|3} = \begin{pmatrix} \Delta_1 & + & \Delta_2 & + & \Delta_3 \\ + & \Delta_{12} & + & \Delta_{23} & \\ & + & \Delta_{123} & & \end{pmatrix}_{12|3} = \begin{pmatrix} \Delta_1 & + & \Delta_2 & + & \Delta_3 \\ + & \Delta_{12} & + & \Delta_3 & \\ & + & \Delta_3 & & \end{pmatrix}.$$

Now in this last array, for each integer $i = 1, \ldots, n-1$ find the vertically lowest summand Δ_I such that $i \in I$, and draw a binary sapling in its place. If there's already a sapling in that spot, then just add an additional leaf to its one internal node. After all of these saplings are drawn, graft them, root-to-leaf, based on their position to form one large tree.

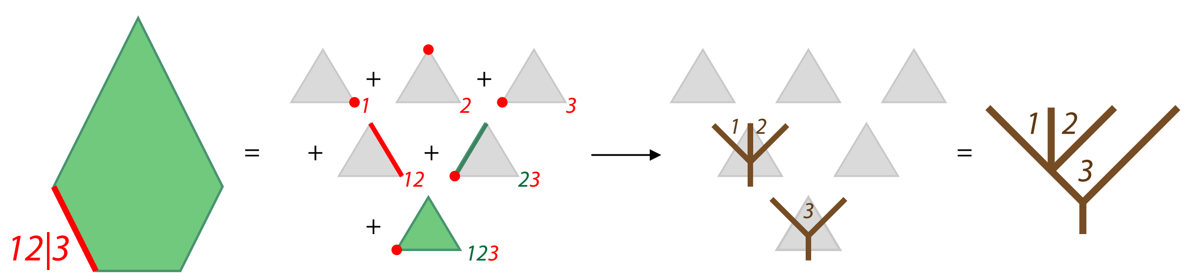


Figure 3.2: An illustration of the correspondence between faces of the associahedron and rooted planar trees. The Minkowski decomposition of the face $\mathcal{K}(4)_{12|3}$ is pictured in red. (The summands of $\mathcal{K}(4)$ omitted in the decomposition of that face are pictured in green.) The brown tree at the right is assembled by grafting the two pictured saplings. To construct the ordered set partition 12|3 from the brown tree use the labeled crooks at each vertex to construct the blocks of 12|3.

Though Proposition 3.3.3 was proven by Postnikov, we will present another proof here that will help us to later prove Theorem 3.3.6.

Proof of Proposition 3.3.3. We will create a bijection between the equivalence classes of $(\mathcal{P}^n, \prec)/\sim$ and the faces of $P_n(\{y_I\})$, where $y_I = 1$ whenever I is an interval of consecutive integers and $y_I = 0$ otherwise. We will also create a bijection between those same equivalence classes and the set of planar rooted trees on $n+1$ leaves. Then for each bijection we will show that cover relations are preserved and conclude that all three posets are isomorphic. In particular, if $[\pi']$ covers $[\pi]$ in $(\mathcal{P}^n, \prec)/\sim$, then some representative $\pi' \in [\pi']$ is formed by merging two adjacent dependent blocks of some $\pi \in [\pi]$. On the other hand merging two adjacent *independent* blocks of π will yield another member of $[\pi]$. For each bijection we will show that merging two dependent blocks yields a cover relation on faces (resp. trees), and merging two independent blocks yields the same face (resp. tree).

Define a map from $\mathcal{P}^n$ to $\mathcal{L}(P_n(\{y_I\}))$ by $\pi \mapsto P_n(\{y_I\})_\pi$, where $\pi = B_1|\cdots|B_k$ is an ordered partition of $[n]$. Let $\pi' = B_1|\cdots|B_{m-1} \cup B_m|\cdots|B_k$ be the ordered partition formed by combining adjacent blocks B_{m-1} and B_m. If these blocks are independent, then any simplex Δ_I, where I is an interval that contains elements from both B_{m-1} and B_m, will maximize functionals f_π and $f_{\pi'}$ of type π and π', respectively, on a the same face $\Delta_J \subset \Delta_I$, where all elements of J belong to blocks of π (and π') situated to the right of B_m. Therefore $P_n(\{y_I\})_\pi = P_n(\{y_I\})_{\pi'}$. On the other hand, if B_{m-1} and B_m are not independent then such a simplex Δ_I will maximize f_π and $f_{\pi'}$ on faces Δ_J and $\Delta_{J'}$, respectively, such that $\Delta_J \subsetneq \Delta_{J'}$. Therefore $P_n(\{y_I\})_\pi \subsetneq P_n(\{y_I\})_{\pi'}$, and we have our isomorphism of posets.

Now we show that $(\mathcal{P}^n, \prec)/\sim$ is isomorphic to the poset of planar rooted trees on $n+1$ leaves. Proceed by induction on n. In the case of $n = 1$ the polytope $P_1\{y_I\}$ is a point, and there is only one ordered partition of $\{1\}$. Now assume the two posets are isomorphic up to dimension $n-1$, and let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$ as before. We will construct a tree T from π as follows. Let S be a sapling with $|B_k|+1$ leaves. Label the crooks of S with the integers of B_k from left to right in increasing order. Partition the block B_k into maximal intervals $B_k = B_k^1 \cup \cdots \cup B_k^j$ where each B_k^i is an interval of consecutive integers and $i_1 < i_2$ iff $b_1 < b_2$ for every $b_1 \in B_k^{i_1}$ and $b_2 \in B_k^{i_2}$. Let A^i be the set of integers that fall between the integers in the sets B_k^i and B_k^{i+1}. Each A^i induces an ordered set partition $\pi|_{A^i}$, which by inductive hypothesis maps to some rooted planar tree T^i on $|A^i|+1$ leaves. For each A^i, graft the root of the corresponding tree T^i onto the unique leaf of S that separates crooks labeled with integers from B_k^i and B_k^{i+1}. Let π' be defined as before, and let it correspond to the tree T'. Suppose B_{m-1} and B_m are independent. We may assume these blocks are separated by some integer $x \in B_k$. Then the elements in B_{m-1} belong to A^i sets which are disjoint from the A^i sets that contain the elements of B_m. Hence the blocks B_{m-1} and B_m contribute to disjoint subtrees, and thus $T = T'$. Suppose instead that B_{m-1} and B_m are not independent. We may assume that $m = k$. In this case, to get T' from T all of the secondary nodes of T created by B_{m-1} will be identified with the base node B_m, thus coarsening T.

In the other direction, start with a rooted planar tree T on $n+1$ leaves and k internal nodes. Label the n crooks of T from left to right with the integers from 1 to n in increasing order. This labeling is unique. The labels of the crooks adjacent to each internal node will form a block of an ordered partition $\pi = B_1|\cdots|B_k$, with the bottom node corresponding to the rightmost block B_k. As we move up the tree, each new node we encounter will form a block to be appended onto the left of the existing list of blocks of π. If two nodes are incomparable then their corresponding blocks may be appended in any order, or combined. These different configurations correspond to partitions that are equivalent according to $\sim$. The face $P_n(\{y_I\})_\pi$ corresponds to T.

We have shown that $\mathcal{L}(P_n(\{y_I\}))$ and the poset of rooted planar trees are both isomorphic to $(\mathcal{P}^n, \prec)/\sim$, thus $P_n(\{y_I\})$ is a realization of the associahedron. □

Definition 3.3.5. Define the polytope $\mathcal{J}(n)$ as the q-lifting of $\mathcal{K}(n)$,

$$\mathcal{J}(n) = \mathcal{K}(n)(q).$$

Just as the faces of $\mathcal{K}(n)$ correspond to rooted planar trees, we will show that the faces of $\mathcal{J}(n)$ correspond to *painted* rooted planar trees, as defined in [12].

A painted tree is formed by taking a rooted planar tree and applying a connected region of paint, starting at the root, and branching upward subject to the constraint that the local region around each internal node must either

(1) be totally unpainted,

(2) be totally painted, or

(3) be painted below the node, but unpainted on all branches above the node.

We say nodes are painted of *type* (1), (2), or (3).

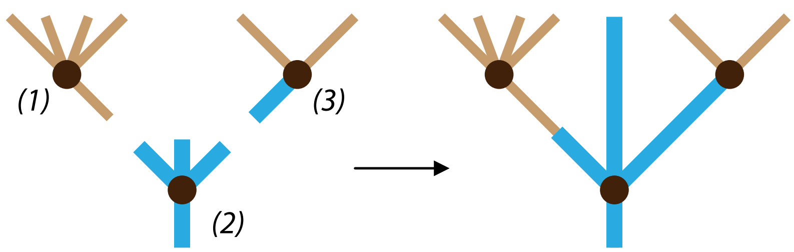


Figure 3.3: A painted tree with nodes of type (1), (2), and (3) constructed by grafting together three saplings.

Another way to think of painted trees is by constructing them out of painted saplings. Start with an unpainted sapling. Then either leave the sapling entirely unpainted, paint the entire sapling, or paint just the root below the node. These painting options correspond respectively to node types (1), (2), and (3) above. Any painted tree on multiple internal nodes can be assembled from these painted saplings by attaching roots to leaves, provided that no painted roots are attached to unpainted leaves (see Figure 3.3).

The space of painted rooted planar trees on n leaves has a poset structure that extends that of the poset of unpainted trees. If T and T' are painted trees that are topologically equivalent, then we say that the painting of T' coarsens the painting of T if T can be formed by taking some nodes in T' of type (3) and converting them to types (1) or (2) through local painting or unpainting of branches immediately adjacent to those nodes. Suppose the painted trees T and T' are not topologically equivalent, but that T' coarsens T topologically. Let the map ψ represent this coarsening, i.e. $\psi(T) = T'$. Extend ψ to act on painted trees by following the rule that if ψ identifies two nodes of differing painted type, then the resulting node should be painted to be type (3). Now for any painted rooted planar trees on n leaves T and T', we say that T' coarsens T if

1. T' coarsens T topologically, and
2. the painting of $\psi(T')$ coarsens the painting of T.

Forcey [12] proved that this is the face lattice of a polytope, the "multiplihedron." We will show that this multiplihedron can be realized as the q-lifting of the associahedron.

Theorem 3.3.6. *The face lattice of the polytope $\mathcal{J}(n)$ is isomorphic to the poset of painted trees. Moreover, $\mathcal{J}(n)$ is the $(n-1)$-dimensional multiplihedron.*

The correspondence between faces of the multiplihedron $\mathcal{J}(n)$ and painted trees is a simple extension of the correspondence between faces of the associahedron and unpainted trees. We illustrate it in the following example.

Example 3.3.7. Let us now consider the face of $\mathcal{J}(4)$ that maximizes a linear functional f_π of type $\pi = 124|3$. This process is illustrated in Figure 3.4. Write this face as

$$\begin{aligned}
\mathcal{J}(4)_{124|3} =& qP_n(\{y_I\})_{12|3} + (1-q)\tilde{Q}_n(\{y_I\})_{124|3} \\
=& q\begin{pmatrix} \Delta_1 & + & \Delta_2 & + & \Delta_3 \\ + & \Delta_{12} & + & \Delta_{23} & \\ & + & \Delta_{123} & & \end{pmatrix}_{124|3} \\
&+(1-q)\begin{pmatrix} \Delta_{14} & + & \Delta_{24} & + & \Delta_{34} \\ + & \Delta_{124} & + & \Delta_{234} & \\ & + & \Delta_{1234} & & \end{pmatrix}_{124|3} \\
=& q\begin{pmatrix} \Delta_1 & + & \Delta_2 & + & \Delta_3 \\ + & \Delta_{12} & + & \Delta_3 & \\ & + & \Delta_3 & & \end{pmatrix} + (1-q)\begin{pmatrix} \Delta_{14} & + & \Delta_{24} & + & \Delta_3 \\ + & \Delta_{124} & + & \Delta_3 & \\ & + & \Delta_3 & & \end{pmatrix}.
\end{aligned}$$

Now we will construct a tree using a slight modification of the associahedron technique from Example 3.3.4. First draw into the array corresponding to $P_n(\{y_I\})_{124|3}$ the saplings based on the associahedron method. Now for each sapling we've drawn, look at the corresponding entry $(\Delta_{I\cup\{n+1\}})_{124|3}$ in the array for $\tilde{Q}_n(\{y_I\})_{124|3}$. If the subscript on that simplex contains only the singleton $n+1$, then leave that sapling unpainted. If the subscript does not contain $n+1$, then paint that sapling entirely. Finally if the subscript contains $n+1$ along with some other integers, then paint only the root of that sapling. Then connect up the saplings to form the painted tree corresponding to the face $\mathcal{J}(4)_{124|3}$.

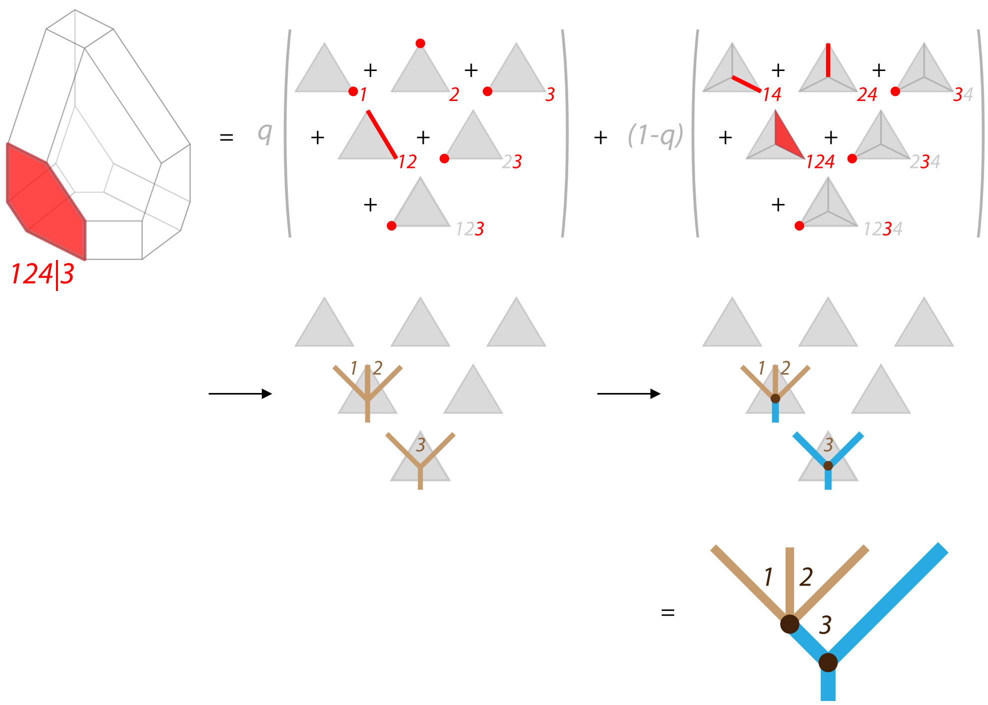


Figure 3.4: An illustration of the construction of a painted tree from a Minkowski decomposition of a face of the multiplihedron. The decomposition of the face $\mathcal{J}(4)_{124|3}$ is pictured in red. (The omitted indices of the decomposition of the full multiplihedron are written in gray.) The first grouped term of the decomposition is the same as that of Example 3.3.4, and determines the topology of the tree. The second grouped term decomposes the associated $\tilde{Q}$-polytope, and determines the painting of the tree. Labels on crooks are provided to reverse the process.

We are now ready to prove Theorem 3.3.6.

Proof of Theorem 3.3.6. Consider the equivalence $\sim'$ from Definition 3.2.8 as it applies to $\mathcal{J}(n) = \mathcal{K}(n)(q)$. We will construct a bijection between the equivalence classes of $(\mathcal{P}^{n+1}, \prec)/\sim'$ and the set of painted rooted planar trees on $n+1$ leaves. Like in the proof of Proposition 3.3.3 we will show this bijection preserves order by examining cover relations, taking care to distinguish between actions that produce actual covers and those which produce other elements in an equivalence class.

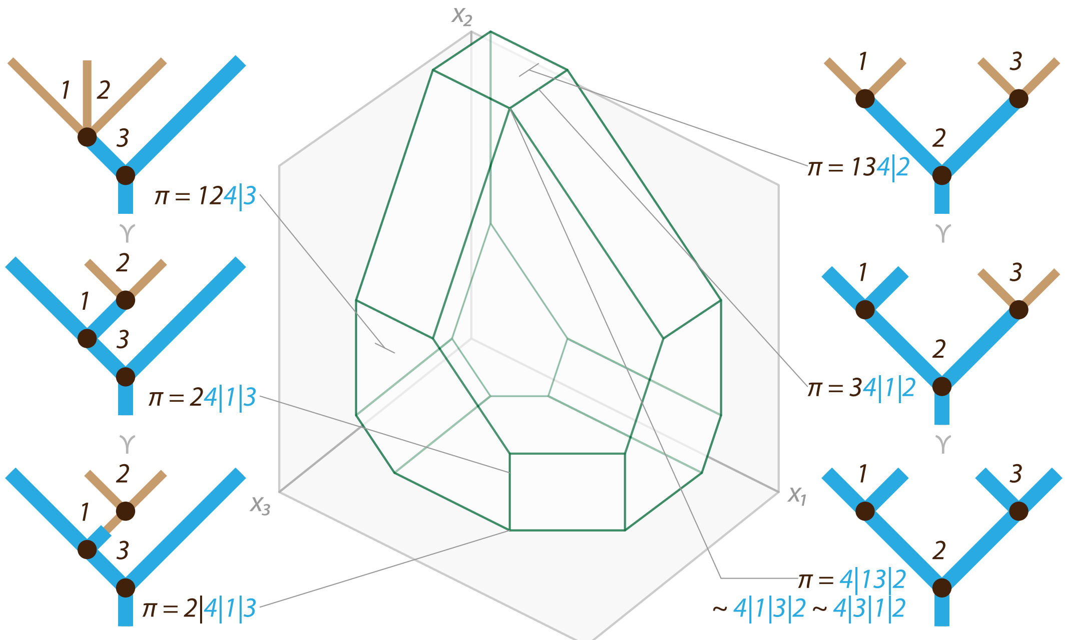


Figure 3.5: Two chains in the poset of painted trees, with labeled crooks, and the corresponding faces of the multiplihedron, projected onto the hyperplane $x_4 = 0$ for visualization. Blocks of π corresponding to painted nodes are shown in blue, blocks corresponding to unpainted nodes are in brown, and blocks corresponding to nodes of type (3) are written in both colors.

Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$, and let π' be its augmentation by the element $\{n+1\}$ via insertion into the (possibly new, in which case we relabel the blocks) block B_j. Using the construction from Proposition 3.3.3 build the unpainted tree $T(\pi)$ that corresponds to the partition π . Each block B_i of π corresponds to some internal node(s) of $T(\pi)$. If $i > j$ then then paint the nodes of B_i according to type (2). For the nodes of B_j apply paint of type (3). Leave all remaining nodes unpainted, type (1). Consider the equivalence $\sim'$ from Definition 3.2.8 as it applies to $\mathcal{J}(n) = \mathcal{K}(n)(q)$. Suppose $\pi' \sim' \mu'$. The first condition of $\sim'$ states that the unpainted trees $T(\pi)$ and $T(\mu)$ are topologically congruent. The second condition ensures that there are no differences in the paintings of nodes of $T(\pi')$ and $T(\mu')$ corresponding to differing blocks of π' and μ'. Hence $T(\pi') = T(\mu')$.

To reverse this map, start with a painted tree T. Ignoring the paint of T build the corresponding ordered partition $\pi(T)$. By the definition of painted trees all nodes of T painted of type (3) are incomparable. Hence according to the equivalence $\sim$ on unpainted trees we may combine all blocks of $\pi(T)$ corresponding to such nodes into a single block. Call this block B_j, and augment it with the element $\{n+1\}$ to form $\pi'(T)$. The definition of painted trees ensures that all nodes painted of type (1) will correspond to blocks to the left of B_j and all nodes painted of type (2) will correspond to blocks to the right of B_j. If two incomparable nodes of T are either both painted of type (1) or both of type (2), then their corresponding blocks in $\pi'(T)$ may be reordered or combined, as in Proposition 3.3.3.

Now we show that this correspondence preserves order. Suppose μ' coarsens π' by combining blocks B_{m-1} and B_m of π', and let B_j be the block containing $\{n+1\}$. We've already addressed the situation where $\pi' \sim' \mu'$, so assume that this isn't the case. Let $T(\pi')$ and $T(\mu')$ be the corresponding painted trees. First, if $j \neq m, m-1$ then the combining of blocks affects only the topology of the tree and not its painting. By Proposition 3.3.3 the unpainted tree $T(\mu)$ coarsens $T(\pi)$, and therefore $T(\mu')$ coarsens $T(\pi')$. Suppose now that $j = m$. If B_{m-1} and $B_m \setminus \{n+1\}$ are independent blocks in π, then the corresponding nodes in $T(\pi')$ are incomparable. Combining blocks to make μ' corresponds to repainting nodes of B_m in $T(\pi')$ to be type (3) instead of their existing type (1), which coarsens the painting of the tree $T(\pi')$. If B_{m-1} and $B_m \setminus \{n+1\}$ are not independent, then combining blocks corresponds to collapsing the internal edge of $T(\pi')$ that connects the nodes of B_m with the nodes of B_{m-1}, and then repainting the identified node to be type (3). This is also a coarsening action on $T(\pi')$. An analogous argument covers the case where $j = m - 1$. □

3.4 Face q-liftings and volumes

We will now modify the q-lifting operator $P(q)$ and define the *face q-lifting operator* $P^\pi(q)$, which acts on a specific face P_π of a generalized permutohedron instead of on P as a whole. This operator is useful in that it subdivides the polytope $P(q)$ into pieces whose volumes are easy to compute, i.e.

$$P(q) = \bigcup_{\pi \in \mathcal{P}^n} P^\pi(q),$$

and

$$\operatorname{Vol}_n(P(q)) = \sum_{\pi \in \mathcal{P}^n} \operatorname{Vol}_n(P^\pi(q)),$$

where $\operatorname{Vol}_n(P^\pi(q))$ is a degree-n polynomial in q. The family of polynomials described in this volume formula are defined in terms of compositions of n, and we explore them in greater depth in Section 4.2.

For the sake of visualizations and the cleanliness of formulas, for this section let us treat $P(q)$ as a full-dimensional polytope in $\mathbb{R}^n$ via projection onto the hyperplane $x_{n+1} = 0$, rather than as a codimension-1 polytope in $\mathbb{R}^{n+1}$. Thus if $P = P_n(\{z_I\})$ then it follows from Proposition 3.2.4 that $P(q)$ will have hyperplane description

$$P(q) = \left\{ x \in \mathbb{R}^n : qz_I \leq \sum_{i \in I} x_i \leq z_{[n]} - z_{[n]\setminus I} \text{ for all } I \subseteq [n] \right\}.$$

Definition 3.4.1. Define the face q-lifting of P as follows. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$ and let P_π be the face of P that maximizes a linear functional of type π. Now for $i = 0, \ldots, k$ construct a modified copy of P_π by applying a factor of q to the coordinates of the vertices of P_π whose indices belong to the first i blocks of π, $B_1 \cup \cdots \cup B_i$. The convex hull of all of these modified copies of P_π is the *face q-lifting* of P_π, and we denote it as $P^\pi(q)$.

Example 3.4.2. Consider the associahedron $\mathcal{K}(4)$. The face q-lifting

$$\mathcal{K}(4)^{1|3|2}(q) = \mathrm{conv}\{(1,4,1),(q,4,1),(q,4,q),(q,4q,q)\}.$$

This and other face q-liftings are pictured in Figure 3.6.

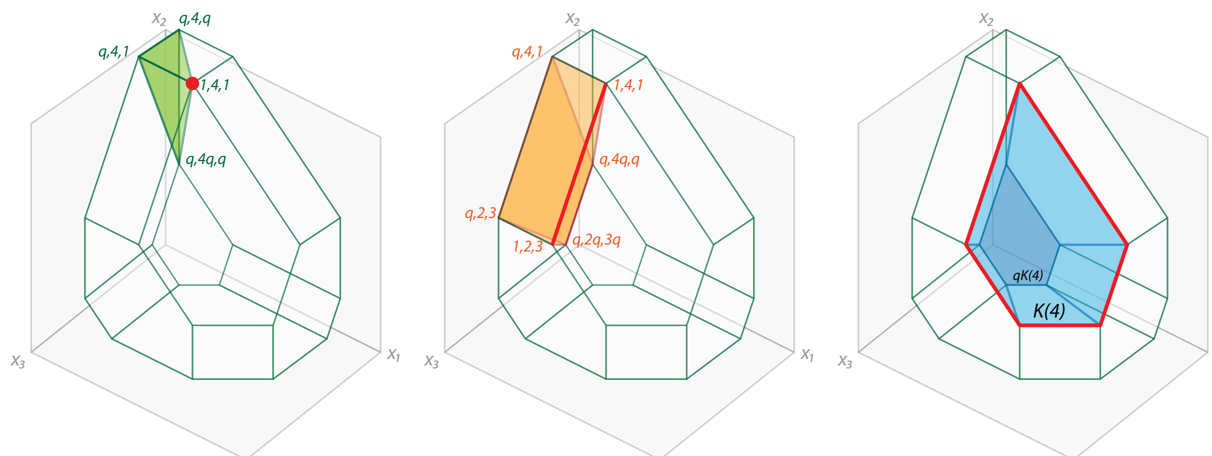


Figure 3.6: Three face q-liftings of the associahedron $\mathcal{K}(4)$: $\mathcal{K}(4)^{1|3|2}(q)$, $\mathcal{K}(4)^{1|23}(q)$, and $\mathcal{K}(4)^{123}(q)$. The red regions represent the faces $\mathcal{K}(4)_\pi$.

For the notation $P^\pi(q)$ we use the superscript on π instead of the subscript to avoid ambiguity. Take for example $P = \mathcal{K}(4)$. Then $P_{123} = P$, and thus $P_{123}(q)$ is interpreted as the regular q-lifting $P(q)$. However $P^{123}(q)$, pictured in Figure 3.6, is properly contained in $P(q)$.

Next we will prove that the face q-liftings of P subdivide the polytope $P(q)$ into simpler pieces. In particular, we will calculate the volume of a face q-lifting $P^\pi(q)$ directly using integrals, and thus obtain a formula for the volume of $P(q)$.

Definition 3.4.3. For a subset $I \subseteq [n]$ define $x_I := \sum_{i \in I} x_i$. For a generalized permutohedron $P = P_n(\{z_I\})$ and an ordered partition $\pi = B_1|\cdots|B_k$ define

$$z^{B_i} := z_{B_1 \cup \cdots \cup B_i} - z_{B_1 \cup \cdots \cup B_{i-1}}.$$

If B_i can be written as a disjoint union $B_i' \cup B_i''$ then define

$$z^{B_i'} := z_{B_1 \cup \cdots \cup B_{i-1} \cup B_i'} - z_{B_1 \cup \cdots \cup B_{i-1}} \text{ and}$$

$$z^{B_i''} := z_{B_1 \cup \cdots \cup B_i} - z_{B_1 \cup \cdots \cup B_{i-1} \cup B_i'}.$$

Proposition 3.4.4. *For a generalized permutohedron $P = P_n(\{z_I\})$ and an ordered partition $\pi = B_1|\cdots|B_k$ the face q-lifting $P^\pi(q)$ has the following hyperplane description:*

$$\frac{x_{B_i'}}{z^{B_i'}} \geq \frac{x_{B_i''}}{z^{B_i''}}$$

for all B_i expressible as a disjoint union $B_i' \cup B_i''$, and

$$q \leq \frac{x_{B_1}}{z^{B_1}} \leq \cdots \leq \frac{x_{B_k}}{z^{B_k}} \leq 1.$$

Let these first types of inequalities be called *facial inequalities*, and let the second types be called *simplicial inequalities*.

Proof. The face P_π consists of the points x in P that satisfy $x_{B_i} = z^{B_i}$ for $i = 1, \ldots, k$. The face q-lifting $P^\pi(q)$ is combinatorially the product of a k-dimensional simplex Δ and the face P_π. First we define the facets of $P^\pi(q)$ of the form $\Delta' \times P_\pi$, where Δ' is a facet of Δ. The inequality $q \leq \frac{x_{B_1}}{z^{B_1}}$ defines the facet of $\Delta \times P_\pi$ that contains all copies of P_π that have been scaled by q in the x_{B_1} coordinates. The inequality $\frac{x_{B_k}}{z^{B_k}} \leq 1$ defines the facet of $\Delta \times P_\pi$ that contains all copies of P_π that have not been scaled by q in the x_{B_k} coordinates. Finally, the inequality $\frac{x_{B_i}}{z^{B_i}} \leq \frac{x_{B_{i+1}}}{z^{B_{i+1}}}$ defines the facet that contains all copies of P_π for which the B_i and B_{i+1} coordinates have either both been scaled by q or have both not been scaled by q.

Next we define the facets of $P^\pi(q)$ of the form $\Delta \times P_\mu$ where P_μ is a facet of P_π. We may assume that π covers μ in the poset of ordered partitions, and particularly that μ is formed by splitting a block B_i of π into blocks B_i' and B_i'' so that $\mu = B_1|\cdots|B_i'|B_i''|\cdots|B_k$. Then all points x in P_μ will satisfy $x_{B_j} = z^{B_j}$ for $j = 1, \ldots, k$ as above, as well as the additional equations $x_{B_i'} = z^{B_i'}$ and $x_{B_i''} = z^{B_i''}$. Moreover, all other points on P_π will satisfy the inequality $\frac{x_{B_i'}}{z^{B_i'}} \geq \frac{x_{B_i''}}{z^{B_i''}}$, and one can check that this inequality defines the hyperplane that contains the facet $\Delta \times P_\mu$. □

Proposition 3.4.5. *The set of face q-liftings $\{P^\pi(q) : \pi$ an ordered partition of $[n]\}$ forms a subdivision of the q-lifted polytope $P(q)$.*

Proof. Let $\pi = B_1|\cdots|B_k$ be an ordered partition and let $A_i = B_1 \cup \cdots \cup B_i$. Let us assume that P has been translated to sit in the interior of the positive orthant of $\mathbb{R}^n$. This means that every $x \in P$ will have all strictly positive coordinates, and moreover that $z_I < z_J$ for $I \subsetneq J$. We will now reinterpret the inequality description parameters of $P^\pi(q)$ in terms of slopes. For a point $x \in \mathbb{R}^n$ let $v_I = (z_I, x_I) \in \mathbb{R}^2$, where $x_I = \sum_{i \in I} x_i$ as above. For $x \in P^\pi(q)$ the term $\frac{x_{B_i}}{z^{B_i}} = \frac{x_{A_i} - x_{A_{i-1}}}{z_{A_i} - z_{A_{i-1}}}$ is the slope of the segment joining $v_{A_{i-1}}$ and v_{A_i}. Thus the simplicial inequalities in Proposition 3.4.4 can be interpreted as stating that, starting at the origin $v_{A_0} = v_\emptyset$, the points $v_{A_0}, v_{A_1}, v_{A_2}, \ldots, v_{A_k}$ form a broken line of ascending slopes. Similarly, the facial inequalities state that all points v_C with $A_{i-1} \subset C \subset A_i$ lie above the segment connecting $v_{A_{i-1}}$ and v_{A_i}.

Now given a point $x \in P(q)$ construct a partition π as follows. Draw the 2^n points v_I, take the convex hull to create a polygon Q, and look at the "lower hull" of Q. By lower hull we mean all faces of Q that maximize a linear functional whose second component is nonpositive. This will form a broken line of ascending slopes connecting vertices $v_{A_0}, v_{A_1}, \ldots, v_{A_k}$. Because the x_i are strictly positive we know v_{A_0} will be the origin, and because of the increasing condition on the z_I we know $A_k = [n]$. Now we claim that $A_{i-1} \subset A_i$ for all i. Suppose by way of contradiction that, ordered from left to right, v_A and v_B are consecutive vertices in the lower hull of Q, but that $A \not\subset B$. By the increasing condition on the z_I we have $z_{A\cap B} < z_A < z_B < z_{A\cup B}$. Moreover, because v_A and v_B are vertices of the lower hull of Q we know that the slope of the line segment connecting $v_{A\cap B}$ and v_A is strictly less than the slope of the segment between v_A and v_B, which is in turn strictly less than the slope of the segment between v_B and $v_{A\cup B}$. Thus

$$\frac{x_A - x_{A\cap B}}{z_A - z_{A\cap B}} < \frac{x_{A\cup B} - x_B}{z_{A\cup B} - z_B}.$$

Notice that the numerators on both sides of this inequality are equal, hence we may rearrange terms to get

$$z_A + z_B > z_{A\cup B} + z_{A\cap B},$$

which violates the submodularity condition on the z_I. This is a contradiction. Now we may let $\pi = B_1|\cdots|B_k$ where $B_i = A_i \backslash A_{i-1}$. By construction x satisfies the simplicial inequalities of $P^\pi(q)$, and by the increasing property of the z_I, x satisfies the facial inequalities as well. $\square$

Note that each ordered partition π corresponds to a unique face q-lifting $P^\pi(q)$. Even if $P_\pi = P_\mu$, the face q-liftings $P^\pi(q)$ and $P^\mu(q)$ will be distinct for $\pi \neq \mu$.

Theorem 3.4.6. *Let P be a generalized permutohedron in $\mathbb{R}^n$. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$. Then the volume of the face q-lifting $P^\pi(q)$ is a polynomial in q given by*

$$\mathrm{Vol}_n(P^\pi(q)) = z^\pi \mathrm{Vol}_{n-k}(P_\pi) \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{|B_1|-1} \cdots t_k^{|B_k|-1} dt_1 \cdots dt_k,$$

where $z^\pi = z^{B_1} \cdots z^{B_k}$.

Proof. The face q-lifting $P^\pi(q)$ is combinatorially the product of P_π and a $(k+1)$-dimensional simplex Δ. From Proposition 3.4.4 the inequality description of Δ is given by the simplicial inequalities

$$q \leq \frac{x_{B_1}}{z^{B_1}} \leq \cdots \leq \frac{x_{B_k}}{z^{B_k}} \leq 1.$$

Furthermore, there exists a projection map f onto Δ where the fiber $f^{-1}(x)$ is congruent to $S(x)P_\pi$, where $S(x)$ is some scaling factor. For any point $x \in \Delta$, the value of $\frac{x_{B_i}}{z^{B_i}}$ varies linearly between q and 1, thus the scaling factor $S(x_{B_i})$ restricted to those coordinates is simply the linear term $\frac{x_{B_i}}{z^{B_i}}$. Because this linear variation occurs in $|B_i|$ coordinates, this scales the relative volume $\text{Vol}_{n-k}(P_\pi)$ by a monomial factor of degree $|B_i|-1$: $\text{Vol}_{n-k}(S(x_{B_i})P_\pi) = \left(\frac{x_{B_i}}{z^{B_i}}\right)^{|B_i|-1}\text{Vol}_{n-k}(P_\pi)$. Since all coordinate blocks x_{B_i} are orthogonal, we may combine the restricted scaling factors independently and write $S(x) = \frac{x_{B_1}}{z^{B_1}}\cdots\frac{x_{B_k}}{z^{B_k}}$. Moreover, the relative volume of the fibre $f^{-1}(x)$ is equal to

$$\text{Vol}_{n-k}(S(x)P_\pi) = \left(\frac{x_{B_1}}{z^{B_1}}\right)^{|B_1|-1}\cdots\left(\frac{x_{B_k}}{z^{B_k}}\right)^{|B_k|-1}\text{Vol}_{n-k}(P_\pi).$$

Thus the volume of $P^\pi(q)$ can be computed by

$$\text{Vol}_n(P^\pi(q)) = \int_\Delta \text{Vol}_{n-k}(S(x)P_\pi)dx.$$

The bounds of integration are defined by the simplicial inequalities, and $\text{Vol}_{n-k}(S(x)P_\pi)$ is given above. Then using the substitution $t_i := \frac{x_{B_i}}{z^{B_i}}$ we attain the integral formula as desired. □

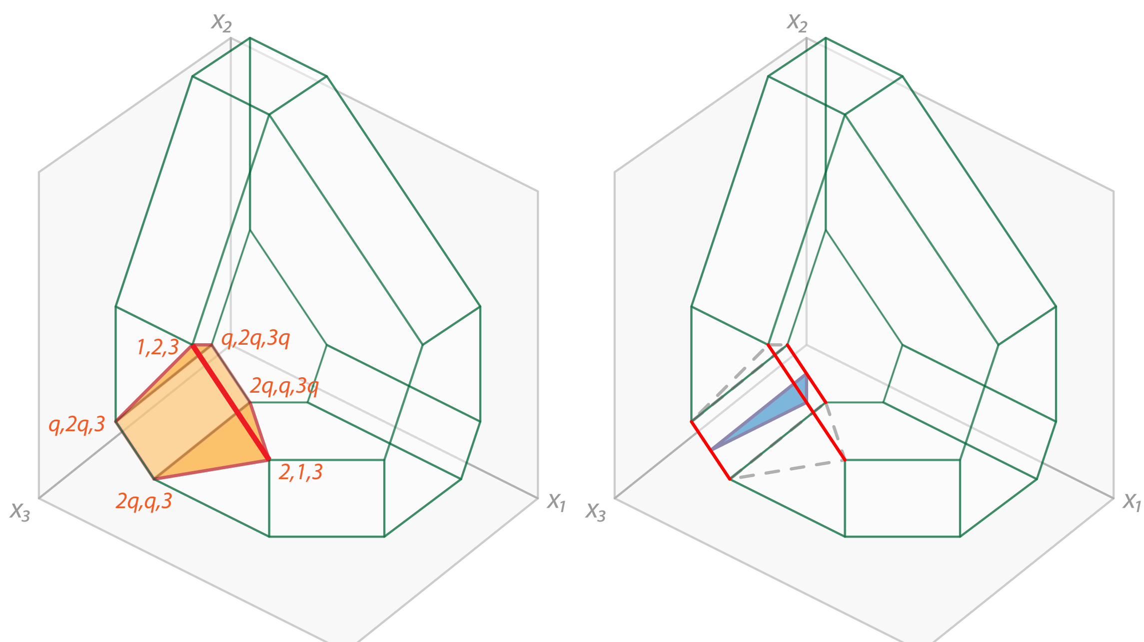


Figure 3.7: The face q-lifting $U_q^{12|3}(\mathcal{K}(4))$ (left) along with its simplicial cross section (right) shown in blue.

Observe that the above integral itself evaluates to a polynomial in q and depends only on the sizes of the blocks of π. The sequence of these block sizes can be thought of as a composition $c(\pi)$ of the integer n. Let us call this polynomial $g_{c(\pi)}(q)$. Then because the

$P^\pi(q)$ form a subdivision of $P(q)$, summing over all ordered partitions π in $\mathcal{P}^n$ we can express the volume of the q-lifted polytope $P(q)$ in terms of these polynomials:

Corollary 3.4.7. *The volume of the q-lifted polytope $P(q)$ is given by*

$$\mathrm{Vol}_n(P(q)) = \sum_{\pi \in \mathcal{P}^n} z^\pi \mathrm{Vol}_{n-k}(P_\pi) g_{c(\pi)}(q).$$

We investigate the properties of the polynomials $g_{c(\pi)}(q)$ in Chapter 4. We can also decompose a q-lifted face $P^\pi(q)$ into a Minkowski sum.

Proposition 3.4.8. *The face q-lift $P^\pi(q)$ can be decomposed into the Minkowski sum*

$$P^\pi(q) = qP_\pi + (1-q)P^\pi(0).$$

Proof. The normal fans of both $P_\pi = P^\pi(1)$ and $P^\pi(0)$ coarsen the normal fan of $P^\pi(q)$, and hence so does the normal fan of the right hand side. Thus the hyperplane directions used to define $P^\pi(q)$ can also be used to adequately define the other two polytopes. Moreover, the respective hyperplane parameters on the right hand side will be additive over Minkowski addition. Substituting in $q = 1$ and $q = 0$ into the inequalities given in Proposition 3.4.4 gives the parameters for these respective polytopes, and summing them with respective coefficients q and $1-q$ yields the hyperplane parameters for $P^\pi(q)$. □

This decomposition preserves the flavor of our definition of the regular q-lifting $P(q)$, and may provide geometric insight for future work with these polytopes.

3.5 Vertex q-liftings and deformation maps

Here we define q-lifted polytopes in terms of their vertices and we show that this definition agrees with Definition 3.2.3 through the analysis of vertex deformation maps. This approach is used in the definition of the multiplihedron in [12].

Generalized permutohedra can be defined and presented in many ways. In Section 3.2 we defined the q-lifting $P(q)$ in terms of Minkowski sums. An alternate approach is to define $P(q)$ in terms of *vertex deformation maps*, as described in [25].

Definition 3.5.1. [25] Let $Q \subset \mathbb{R}^n$ be a simple polytope with vertex set $V(Q)$ and edge set $E(Q) \subset \binom{V(Q)}{2}$. We say that a polytope P is a *deformation* of Q if there exists a *vertex deformation map* $b : V(Q) \to V(P)$ that preserves edge directions and orientations. By this we mean that

$$b(v_1) - b(v_2) \in \mathbb{R}_{\geq 0}(v_1 - v_2), \text{ for every edge } v_1v_2 \in E(Q).$$

We say P is *deformation equivalent* to Q if there is a deformation map between Q and P that preserves combinatorial type.

An important application of deformation maps is that generalized permutohedra are deformations of permutohedra.

Theorem 3.5.2. *[23] A polytope $P \subset \mathbb{R}^n$ is a generalized permutohedron if and only if it is a deformation of the permutohedron P_n .*

To approach q-liftings in terms of deformation maps, let us define the *vertex q-lifting operator*. This operator takes a vertex v of a polytope in $\mathbb{R}^n$, applies a factor of q to certain coordinates of v, and embeds the result in a hyperplane in $\mathbb{R}^{n+1}$. This is similar to the face q-lifting operator, but it is designed specifically to work well with vertex deformation maps.

Definition 3.5.3. Let v be a vertex of a generalized permutohedron $P \subset \mathbb{R}^n$, where P sits on the hyperplane $x_1 + \cdots + x_n = z$ for some constant z. Then for a permutation $\sigma \in S_n$, an integer $i \in \{0, \ldots, n\}$, and a constant $q \in [0, 1]$, define the *vertex q-lifting* of v, written $v^{\sigma,i}(q)$, to be the point obtained by applying a factor of q to the coordinates of v indexed by the first i entries in σ, and then embedding that result in the hyperplane $x_1 + \cdots + x_{n+1} = z$ in $\mathbb{R}^{n+1}$. This embedding is performed via the affine linear map that leaves the first n coordinates fixed and then adjusts x_{n+1} appropriately. See Figure 3.8 for a depiction of this.

When combined with a vertex deformation map b, the vertex q-lifting operator defines the vertices of the q-lifted polytope $P(q)$.

Theorem 3.5.4. *Let $P \subset \mathbb{R}^{n+1}$ be a generalized permutohedron defined by the vertex deformation map $b : V(P_n) \to V(P)$, and let $P(q)$ be the q-lifting of P, as defined in Definition 3.2.3. Then*

$$P(q) = conv\left\{b(\sigma)^{\sigma,i} : \sigma \in S_n, i \in \{0, \ldots, n\}\right\}.$$

One way to prove Theorem 3.5.4 is to simply show that the vertices of $P(q)$ are the same as the vertices produced by the vertex q-liftings of P. We will instead take a more circuitous route so as to demonstrate the utility of vertex deformation maps. First we show that applying the vertex q-lifting operator to the permutohedron P_n yields a polytope that is deformation equivalent to P_{n+1}. Then we show that applying the vertex q-lifting operator to any generalized permutohedron $P \subset \mathbb{R}^n$ produces a deformation of P_{n+1}, i.e. a generalized permutohedron in $\mathbb{R}^{n+1}$. Finally with this fact established we can then express our polytope in terms of supporting hyperplanes using Postnikov's $P(\{z_I\})$ notation of generalized permutohedra, which we will show to match the hyperplane description of $P(q)$.

Proposition 3.5.5. *Let A be the polytope defined by q-lifting the vertices of P_n:*

$$A := conv\left\{\sigma^{\sigma,i} : \sigma \in S_n, i \in \{0, \ldots, n\}\right\}. \tag{3.2}$$

Then A is deformation equivalent to the permutohedron P_{n+1} when $q \in (0, 1)$.

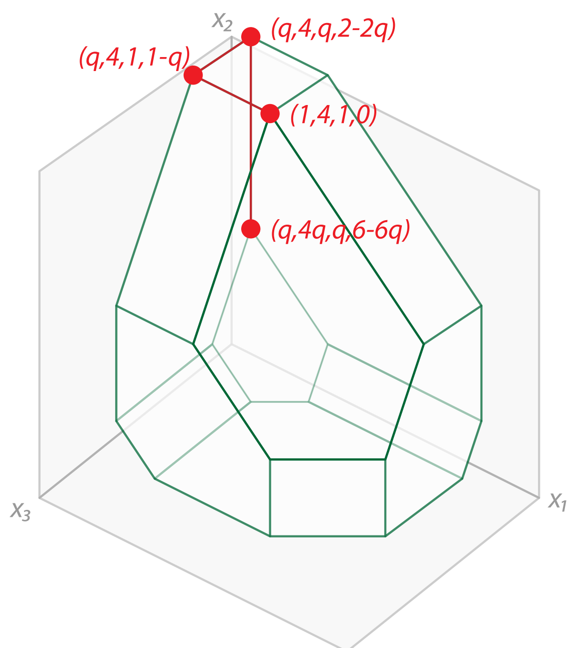


Figure 3.8: Pictured in red, the vertex q-liftings $v^{132,i}(q)$ for $v = (1,4,1)$ and $i = 0,1,2,3$, projected onto the first 3 coordinates for visualization. Here $(1,4,1)$ is the vertex of the associahedron $\mathcal{K}(4)$, which is the image of the vertex $(1,3,2)$ of the permutohedron P_3 under the deformation map that defines $\mathcal{K}(4)$.

Proof. We will define an explicit vertex map from the vertices of P_{n+1} to the vertices of A, and then show that the map is bijective and preserves edge directions and orientations. Consider the projection map of permutations $p : S_{n+1} \to S_n$ defined by

$$p(\sigma)_i = \begin{cases} \sigma_i & : \sigma_i < \sigma_{n+1} \\ \sigma_i - 1 & : \sigma_i > \sigma_{n+1} \end{cases},$$

where $p(\sigma) = (p(\sigma)_1, \ldots, p(\sigma)_n)$. Now define the vertex map $a : V(P_{n+1}) \to V(A)$ by

$$a(\sigma) := p(\sigma)^{p(\sigma),\sigma_{n+1}-1}.$$

Now we show that a is a deformation equivalence map. For convenience, the coordinates of $a(\sigma)$ can be written explicitly as

$$a(\sigma)_i = \begin{cases} q\sigma_i & : \sigma_i < \sigma_{n+1} \\ \sigma_i - 1 & : \sigma_i > \sigma_{n+1} \end{cases} \text{ for } i \in [n]$$

$$a(\sigma)_{n+1} = (1-q)\binom{\sigma_{n+1}}{2}.$$

First we observe that a is a bijection between the vertex sets of the two polytopes. Indeed, the map $\sigma \mapsto (p(\sigma), \sigma_{n+1} - 1)$ is a bijection between $V(P_{n+1}) = S_{n+1}$ and $S_n \times \{0, \ldots, n\}$, and furthermore a is injective onto its image $V(A)$.

Now we show that a preserves edge directions. Suppose $\sigma, \tau \in S_n$ are vertices of P_{n+1} that form an edge. Then σ and τ differ by an adjacent transposition $(s, s+1)$, and $\sigma - \tau = e_h - e_k$ where $\sigma_k = \tau_h = s$ and $\sigma_h = \tau_k = s + 1$.

Suppose $k < h < n + 1$. Then $\sigma_{n+1} = \tau_{n+1}$ and $p(\sigma)$ and $p(\tau)$ differ by the transposition $(s, s+1)$. If $s > \sigma_{n+1}$ then by the definition of a we have $a(\sigma) - a(\tau) = e_h - e_k$. Similarly, if $s+1 < \sigma_{n+1}$ then $a(\sigma) - a(\tau) = q(e_h - e_k)$. Because we have taken $q > 0$, the edge direction and orientation are preserved.

Suppose instead that $k < h = n + 1$. Then $p(\sigma) = p(\tau)$, $\tau_{n+1} = s$, and $\sigma_{n+1} = s + 1$. We then have

$$\begin{aligned} a(\sigma) - a(\tau) &= e_k(qs - s) + e_{n+1}\left((1-q)\binom{s+1}{2} - (1-q)\binom{s}{2}\right) \\ &= s(1-q)(e_{n+1} - e_k). \end{aligned}$$

Because we have taken $q < 1$, we see that a also preserves edge direction and orientation in this case, thus completing the proof. □

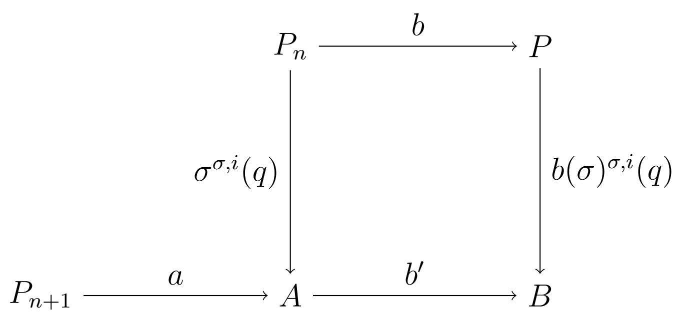
Now we can show that the vertex q-lifting operator sends generalized permutohedra in $\mathbb{R}^n$ to generalized permutohedra in $\mathbb{R}^{n+1}$.

Proposition 3.5.6. *Let P be a generalized permutohedron that sits in the positive orthant $\{x_i \geq 0 : i = 1, \ldots n\}$ of $\mathbb{R}^n$. Let $b : V(P_n) \to V(P)$ be the vertex deformation map that defines P as a deformation of P_n. Let $B \subset \mathbb{R}^{n+1}$ be the polytope formed by applying the vertex q-lifting operator to P, given by*

$$B := conv\left\{b(\sigma)^{\sigma,i} : \sigma \in S_n, i \in \{0, \ldots, n\}\right\}.$$

Then B is a deformation of P_{n+1}, i.e. *B is itself a generalized permutohedron.*

Proof. Let A be the vertex q-lifting of P_n, as defined in (3.2). We will lift the vertex deformation map b to an induced map $b' : V(A) \to V(B)$ and show that b' preserves edge directions and orientations. Proposition 3.5.5 furnished us with a vertex deformation map a from P_{n+1} to A. We will show that the composition $b' \circ a : V(P_{n+1}) \to B$ is itself a vertex deformation map and hence B is a generalized permutohedron.



Define the vertex map $b' : V(A) \to V(B)$ by

$$b'(\sigma^{\sigma,i})(q) := b(\sigma)^{\sigma,i}(q).$$

Surjectivity of b' is inherited from surjectivity of b. Now we show that b' preserves edge directions and orientations. Let $v = \sigma^{\sigma,i}$ and $w = \tau^{\tau,j}$ be vertices of A. By the proof of Proposition 3.5.6 v and w share an edge in A exactly when either $\sigma = \tau$ and $|i-j| = 1$, or when σ and τ differ by a transposition $(s, s+1)$ and $i = j \neq s+1$.

First suppose that vw is an edge of A with $\sigma = \tau$ and $j = i+1$. In this case, as in the proof of Proposition 3.5.6, we have that $v - w$ is a nonnegative multiple of $e_{n+1} - e_k$. Suppose that $\sigma_k = i$. In coordinate k the vectors $b'(\sigma^{\sigma,i})$ and $b'(\sigma^{\sigma,i-1})$ will differ by $(q-1)b(\sigma)_k e_k$. Recall that the vertex q-lifting operator embeds points into the hyperplane $x_1 + \cdots + x_{n+1} = z$. From this we see that in the last coordinate $b'(v)_{n+1} - b'(w)_{n+1} = b'(w)_k - b'(v)_k = (1-q)b(\sigma)_k$. So $b'(v) - b'(w) = (1-q)b(\sigma)_k(e_{n+1} - e_k)$. By our assumption that P sits in the positive orthant of $\mathbb{R}^n$, we have $b(\sigma)_k \geq 0$ and hence edge direction is preserved.

In the second case, suppose vw is an edge of A with $\sigma = \tau \cdot (s, s+1)$ and $i = j \neq s+1$. We may write $\sigma - \tau = e_h - e_k$ for some h, k, and because b is a deformation we have $b(\sigma) - b(\tau) = c(e_h - e_k)$ for some constant $c \geq 0$. Now suppose $i > s+1 = \sigma_h = \tau_k$. Then the vertex q-lifting operator will apply a factor of q to the h^{th} and k^{th} coordinates of both $b(\sigma)$ and $b(\tau)$. Thence $b'(v)_h - b'(w)_h = q(b(\sigma)_h - b(\tau)_h) = qc$, and similarly $b'(v)_k - b'(w)_k = (b(\sigma)_k - b(\tau)_k) = -qc$. Applying the affine embedding into $\mathbb{R}^{n+1}$ we see that $b'(v)_{n+1} - b'(w)_{n+1} = b'(w)_h - b'(v)_h + b'(w)_k - b'(v)_k = 0$. It is clear that $b'(v)$ and $b'(w)$ agree on all other coordinates. Therefore $b'(v) - b'(w) = qc(e_h - e_k)$. If instead we took $i < s+1$, then a completely analogous argument produces $b'(v) - b'(w) = c(e_h - e_k)$. From the proof of Proposition 3.5.6 we know $v - w$ is a nonnegative multiple of $e_h - e_k$, so edge directions are preserved.

We have shown that $b' : V(A) \to V(B)$ is a vertex deformation map, and it follows from the definition that vertex deformation maps are closed under composition. Hence $b' \circ a : V(P_{n+1}) \to V(B)$ is a vertex deformation map, and B is a generalized permutohedron. $\square$

Now that we have shown that the vertex q-lifting operator sends P to the generalized permutohedron $B \in \mathbb{R}^{n+1}$ we can write down the hyperplane description of B and show that it agrees with the hyperplane description of the regular q-lifting $P(q)$. This proves that the vertex q-lifting operator, when applied to all vertices of a generalized permutohedron P, is actually the same as the regular q-lifting of P.

Proof of Theorem 3.5.4. Recall from Section 3.2 that we can write any generalized permutohedron $P \in \mathbb{R}^n$ as $P(\{z_I\})$ using Postnikov's inequality description

$$P(\{z_I\}) = \left\{ x \in \mathbb{R}^n : \sum_{i \in I} x_i \geq z_I \text{ for } I \subset [n], \sum_{i=1}^{n} x_i = z_{[n]} \right\}.$$

Let $P = P(\{z_I\})$ have hyperplane parameters $\{z_I\}_{I \subseteq [n]}$. Recall from Proposition 3.2.4 that the q-lifting $P(q)$ has hyperplane parameters $\{z'_I\}_{I \subseteq [n+1]}$ given by $z'_J = qz_J$ and $z'_{J \cup \{n+1\}} = z_J$ for $J \subseteq [n]$ and $q \in [0, 1]$. We will show that this agrees with the hyperplane parameters of B.

First notice that, by the definition of the vertex q-lifting operator, B will contain as a face a copy of P as well as a copy of qP (P scaled by the constant q). That $z'_J = qz_J$ for $J \subseteq [n]$ follows from the fact that any functional $f(x) = \sum x_{i_j}$ that does not involve x_{n+1} will be minimized somewhere on the face of B equal to qP. To see that $z'_{J \cup \{n+1\}} = z_J$ it suffices to observe that any functional $f(x) = \sum x_{i_j}$ that involves x_{n+1} is weakly increasing over application of factors of q to coordinates of vertices of P, as in the construction of B. Indeed, let $f_I(x) = x_{i_1} + \cdots + x_{i_k} + x_{n+1}$ for $i_j \in I \subseteq [n]$, and let v be a vertex of B. Let w be the vertex obtained by applying a factor of q to some coordinate v_k of v for $k \leq n$. If $k \in I$ we have $f_I(w) - f_I(v) = (q - 1)v_k + (1 - q)v_k = 0$, and if $k \notin I$ then $f_I(w) - f_I(v) = (1 - q)v_k \geq 0$. Thus any such functional is minimized somewhere on the face of B equal to P and we can take $z'_{I \cup \{n+1\}} = z_I$. □

Chapter 4

Composition polynomials

4.1 Introduction

Here we define the *composition polynomial* $g_c(q)$, and the related *reduced composition polynomial* $f_c(q) = (1-q)^{-k} g_c(q)$, both of which depend only on a composition $c = (c_1, \ldots, c_k)$ of n. In Section 3.4 we defined $g_{c(\pi)}(q)$. This turned out to be a constant multiple of the volume of the face q-lifting $P^\pi(q)$. In Theorem 4.2.3 we detail several other properties of (reduced) composition polynomials, some of which are illustrated in the following examples:

- $f_{(1,1,1,1)}(q) = \frac{1}{24}$.
- $f_{(2,2,2,2)}(q) = \frac{1}{384}(1+q)^4$.
- $f_{(1,2,2)}(q) = \frac{1}{120}(8 + 9q + 3q^2)$.
- $f_{(2,2,1)}(q) = \frac{1}{120}(3 + 9q + 8q^2)$.
- $f_{(5,3)}(q) = \frac{1}{120}(5 + 10q + 15q^2 + 12q^3 + 9q^4 + 6q^5 + 3q^6)$.
- $f_{(a,b)}(q) = \frac{1}{ab(a+b)}(b+2bq+\cdots+(a-1)bq^{a-2}+abq^{a-1}+a(b-1)q^a+\cdots+2aq^{a+b-3}+aq^{a+b-2})$ for a and b relatively prime.

4.2 Composition polynomials

Let us begin by reviewing the notion of a composition of an integer.

Definition 4.2.1. A *composition* c is a finite ordered tuple of positive integers, denoted $c = (c_1, \ldots, c_k)$. We call the c_i the *parts* of c, and the sum $c_1 + \cdots + c_k$ the *size* of c. If $c = (c_1, \ldots, c_k)$ has size n, we say that c is a composition of n into k parts. The *reverse* of the composition c is defined as $\bar{c} = (c_k, \ldots, c_1)$. Define the *truncated compositions* $c^L :=$

$(c_2, \ldots, c_k)$ and $c^R := (c_1, \ldots, c_{k-1})$. For $m \in \{1, \ldots, k-1\}$ we define the *merged composition* c^m as the composition formed by combining the parts c_m and c_{m+1} into a single part:

$$c^m := (c_1, \ldots, c_{m-1}, c_m + c_{m+1}, c_{m+2}, \ldots, c_k).$$

For an ordered set partition $\pi = B_1|\cdots|B_k$, we define the *induced composition* $c(\pi) := (|B_1|, \ldots, |B_k|)$.

Definition 4.2.2. For a composition $c = (c_1, \ldots, c_k)$ we define its associated *composition monomial* to be $\mathbf{t}^{\mathbf{c-1}} := t_1^{c_1-1} \cdots t_k^{c_k-1}$, where $t = (t_1, \ldots, t_k)$. This monomial has degree $n - k$ and belongs to $\mathbb{Q}[t_1, \ldots, t_k]$. Then define the *composition polynomial* $g_c(q)$ by

$$g_c(q) := \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c-1}} dt_1 \cdots dt_k.$$

This polynomial has degree n and belongs to $\mathbb{Q}[q]$.

The main goal for this section is to prove the following theorem about composition polynomials.

Theorem 4.2.3. *Let $c = (c_1, \ldots, c_k)$ be a composition of n. Then the following are true:*

1. *$g_c(q)$ factors into $g_c(q) = (1-q)^k f_c(q)$, where $deg(f_c(q)) = n - k$ and $f_c(1) \neq 0$.*
2. *the coefficients of $f_c(q)$ are strictly positive,*
3. *$f_c(1) = 1/k!$,*
4. *$f_{\bar{c}}(q) = q^{n-k} f_c(1/q)$, and*
5. *$g_{\alpha c}(q) = \frac{1}{\alpha^k} g_c(q^\alpha)$ and $f_{\alpha c}(q) = \frac{1}{\alpha^k}(1 + q + \cdots + q^{\alpha-1})^k f_c(q^\alpha)$ for any positive integer α.*

The polynomials $g_c(q)$ and $f_c(q)$ also arise as solutions to particular polynomial interpolation problems. We present this in Theorem 4.3.1.

Definition 4.2.4. We will refer to the polynomial $f_c(q)$ as the *reduced composition polynomial* for the composition c.

Our proof of Theorem 4.2.3 relies on a recursive construction of the polynomial $g_c(q)$. The integral definition of $g_c(q)$ hints at this recursion.

$$\begin{aligned}
g_c(q) &= \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{c_1-1} \cdots t_k^{c_k-1} dt_1 \cdots dt_k \\
&= \frac{1}{c_1} \int_q^1 \int_q^{t_k} \cdots \int_q^{t_3} t_2^{c_2-1} \cdots t_k^{c_k-1}(t_2^{c_1} - q^{c_1}) dt_2 \cdots dt_k \\
&= \frac{1}{c_1} g_{(c_1+c_2, c_3, \ldots, c_k)}(q) - \frac{q^{c_1}}{c_1} g_{(c_2, c_3, \ldots, c_k)}(q) \\
&= \frac{1}{c_1} g_{c^1}(q) - \frac{q^{c_1}}{c_1} g_{c^L}(q).
\end{aligned}$$

This observation does not give us enough to prove Theorem 4.2.3, however it is useful in deriving an explicit formula for $g_c(q)$, which in turn will help to produce the stronger recursive formula that does prove Theorem 4.2.3.

First let us introduce notation related to partial sums of the c_i. Let $c = (c_1, \ldots, c_k)$ be a composition of size n. Now define the sequence of partial sums $0 = \beta_0 < \cdots < \beta_k = n$ by $\beta_i = c_1 + \cdots + c_i$ for $i = 1, \ldots, k$.

Let (β) denote the Vandermonde matrix

$$(\beta) = \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^k \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^k \end{pmatrix}.$$

We will index the rows and columns of this matrix from 0 to k. It is well known that

$$\det(\beta) := \prod_{0 \le i < j \le k} (\beta_j - \beta_i)$$

Let $[\beta_i]$ be a factor of $\det(\beta)$ defined by

$$[\beta_i] := (-1)^i \prod_{j \ne i} (\beta_j - \beta_i).$$

Lastly define $[\hat{\beta}_i] := \det(\beta)/[\beta_i]$. Notice that $[\hat{\beta}_i]$ is the unsigned minor of (β) obtained by removing row i and column k. Moreover, $[\hat{\beta}_i]$ is itself a Vandermonde determinant of degree $k-1$.

Proposition 4.2.5. *The composition polynomial $g_c(q)$ has closed form*

$$g_c(q) = \sum_{i=0}^{k} (-1)^i \frac{q^{\beta_i}}{[\beta_i]}.$$

Proof. Let $0 = \beta_0 < \cdots < \beta_k$ be the sequence of partial sums of the parts of the composition $c = (c_1, \ldots, c_k)$. Define $[\beta_i]$ as above, and define $[\beta_i^k]$ analogously for the truncated composition $c^R = (c_1, \ldots, c_{k-1})$. Proceed by induction on k. If $k = 1$ then

$$\int_q^1 t_1^{c_1 - 1} dt_1 = \frac{1}{c_1} - \frac{q^{c_1}}{c_1} = \frac{q^{\beta_0}}{[\beta_0]} - \frac{q^{\beta_1}}{[\beta_1]}.$$

Now assume that the formula holds up to $k-1$. Then

$$g_{c^R}(q) = \int_q^1 \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c^R - 1}} dt_1 \cdots dt_{k-1} = \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i}}{[\beta_i^k]}.$$

Changing the upper bound of the outer integral produces

$$\int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c^R-1}} dt_1 \cdots dt_{k-1} = \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i} t_k^{\beta_{k-1}-\beta_i}}{[\beta_i^k]}.$$

This follows from the observation that this integral must evaluate to a homogeneous polynomial in t_k and q of total degree $c_1 + \cdots + c_{k-1} = \beta_{k-1}$. Now the original integral we wish to compute becomes

$$\begin{aligned} g_c(q) &= \int_q^1 t_k^{c_k-1} \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i} t_k^{\beta_{k-1}-\beta_i}}{[\beta_i^k]} dt_k \\ &= \int_q^1 \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i} t_k^{\beta_k-\beta_i-1}}{[\beta_i^k]} dt_k \\ &= \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i}}{[\beta_i]} - q^{\beta_k} \sum_{i=0}^{k-1} \frac{(-1)^i}{[\beta_i]}. \end{aligned}$$

Here we changed the denominators by using the fact that $(\beta_k - \beta_i)[\beta_i^k] = [\beta_i]$. Observe that the sum $(\beta) \sum_{i=0}^{k} (-1)^i/[\beta_i] = \sum_{i=0}^{k} (-1)^i [\hat{\beta}_i]$ computes, up to sign, the determinant of the matrix formed by replacing the last column in the Vandermonde matrix (β) with a column of 1s. This determinant is clearly zero, hence $\sum_{i=0}^{k-1} (-1)^i/[\beta_i] = (-1)^{k+1}/[\beta_k]$. This gives us the desired result. □

We can prove a similar formula corresponding to each of the merged compositions c^m and the truncated compositions c^L and c^R.

Corollary 4.2.6. *Given a composition $c = (c_1, \ldots, c_k)$, the closed formulas for the composition polynomials of the associated merged and truncated compositions are given by*

$$\begin{aligned} g_{c^m}(q) &= \sum_{i=0}^{k} (-1)^i \frac{q^{\beta_i}(\beta_m - \beta_i)}{[\beta_i]}, \\ g_{c^R}(q) &= \sum_{i=0}^{k} (-1)^i \frac{q^{\beta_i}(n - \beta_i)}{[\beta_i]}, \textit{ and} \\ q^{c_1} g_{c^L}(q) &= -\sum_{i=0}^{k} (-1)^i \frac{q^{\beta_i}\beta_i}{[\beta_i]}. \end{aligned}$$

Proof. For the merged composition c^m, the partial sums β_i^m are given by $\beta_i^m = \beta_i$ for $i < m$, and $\beta_i^m = \beta_{i+1}$ for $i \geq m$. From this observe that $[\beta_i^m] = [\beta_i]/(\beta_m - \beta_i)$ for $i < m$ and $[\beta_i^m] = [\beta_{i+1}]/(\beta_{i+1} - \beta_m)$ for $i \geq m$. Notice that the coefficient of q^{β_m} is zero, as it should be.

For the truncated composition c^R the partial sums β_i^R follow this same pattern. Finally, for the truncation c^L we have $\beta_i^L = \beta_{i+1} - \beta_1$ for $i \geq 1$, and $\beta_0^L = 0$. From this we observe that $[\beta_i^L] = [\beta_{i+1}]/\beta_{i+1}$ for all i. Substituting each of these partial sum lists into Proposition 4.2.5 yields the desired formulas. □

Now we can write down a recursive formula for $g_{c^m}(q)$ that will be the key to proving Theorem 4.2.3.

Corollary 4.2.7. *Let $c = (c_1, \ldots, c_k)$ be a composition of n into k parts. Let c^m be the merged composition $(c_1, \ldots, c_m + c_{m+1}, \ldots, c_k)$, and let $c^L = (c_1, \ldots, c_k)$ and $c^R = (c_0, \ldots, c_{k-1})$ be the truncated compositions. Then the composition polynomial $g_{c^m}(q)$ follows the recursion*

$$g_{c^m}(q) = \frac{\beta_m}{n} g_{c^R}(q) + \left(1 - \frac{\beta_m}{n}\right) q^{c_1} g_{c^L}(q). \tag{4.1}$$

Proof. This follows immediately from the formulas of Corollary 4.2.6. □

This result is significant because every composition c except for the trivial composition $(1, \ldots, 1)$ can be thought of as a merged composition. Also notice that the sizes of c^L and c^R are each strictly less than the size of c^m, though the number of parts remains constant. This means we have actually produced a recursive expression for an arbitrary nontrivial composition polynomial in terms of composition polynomials of strictly smaller degree. With this tool in hand we can prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Proceed by induction on the size of c for a fixed k. Let $c = (1, \ldots, 1)$ be the trivial composition of k into k parts. From Proposition 4.2.5 observe that

$$g_{(1,\ldots,1)}(q) = \sum_{i=0}^{k} (-1)^i \frac{q^i}{i!(k-i)!} = \frac{1}{k!}(1-q)^k.$$

Hence $f_{(1,\ldots,1)}(q) = 1/k!$ and all of the properties of the theorem are trivially satisfied. Now suppose c has size $n > k$. Then some part of c is nontrivial, and we can write c as some merged composition c'^m, and by the recursive formula in (4.1) we can express $g_c(q)$ as

$$g_c(q) = g_{c'^m}(q) = \frac{\beta'_m}{n} g_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1} g_{c'^L}(q).$$

Notice that c'^R and c'^L are compositions both of strictly smaller size than c. Therefore by induction we may write

$$\begin{aligned} g_c(q) &= \frac{\beta'_m}{n}(1-q)^k f_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1}(1-q)^k f_{c'^L}(q) \\ &= (1-q)^k \left(\frac{\beta'_m}{n} f_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1} f_{c'^L}(q)\right) \\ &:= (1-q)^k f_c(q). \end{aligned}$$

Notice that this function we've defined as $f_c(q)$ is indeed of degree $n-k$. Also both $\frac{\beta_m}{n}$ and $\left(1-\frac{\beta_m}{n}\right)$ are positive and they sum to 1. This, combined with the fact that $f_{(1,\ldots,1)}(q) = 1/k!$ gives us the properties that for arbitrary c with k parts the coefficients of $f_c(q)$ will be strictly positive and sum to $1/k!$.

To show the reversal property, first observe that it holds trivially for the trivial composition. Then assume by induction that the property holds for all compositions of size less than n. Let $c = (c_1, \ldots, c_k)$ be a composition of size n and let $c^m = (c_1, \ldots, c_m + c_{m+1}, \ldots, c_k)$ be a merging of c. The reverse $\overline{c^m} = (c_k, \ldots, c_{m+1} + c_m, \ldots, c_1)$ can also be written as

$$\overline{c^m} = \bar{c}^{k-m} = (\bar{c}_1, \ldots, \bar{c}_{k-m} + \bar{c}_{k-m+1}, \ldots, \bar{c}_k)$$

with partial sums $\bar{\beta}_i = \bar{c}_1 + \cdots + \bar{c}_i$. Similarly $\overline{c^L} = \bar{c}^R$ and $\overline{c^R} = \bar{c}^L$. Using the recursive formula in (4.2) we then write

$$\begin{aligned} f_{\overline{c^m}}(q) = f_{\bar{c}^{k-m}}(q) &= \frac{\bar{\beta}_{k-m}}{n} f_{\bar{c}^R}(q) + \left(1 - \frac{\bar{\beta}_{k-m}}{n}\right) q^{\bar{c}_1} f_{\bar{c}^L}(q) \\ &= \frac{n-\beta_m}{n} f_{\overline{c^L}}(q) + \frac{\beta_m}{n} q^{c_k} f_{\overline{c^R}}(q) \\ &= \left(1 - \frac{\beta_m}{n}\right) q^{n-c_1-k+1} f_{c^L}(1/q) + \frac{\beta_m}{n} q^{n-k+1} f_{c^R}(q) \\ &= q^{n-k+1} f_{c^m}(1/q), \end{aligned}$$

as desired. In the penultimate step we used the inductive hypothesis to write $f_{\overline{c^L}}(q)$ and $f_{\overline{c^R}}(q)$ in terms of $f_{c^L}(q)$ and $f_{c^L}(q)$, respectively.

To prove the scaling formula, let α be a positive integer. Observe that $[\alpha\beta_i] = \alpha^k[\beta_i]$ for $i = 0, \ldots, k$. Applying the closed formula for $g_c(q)$ gives us the desired formula $g_{\alpha c}(q) = \frac{1}{\alpha^k} g_c(q^\alpha) = \frac{1}{\alpha^k}(1-q^\alpha)^k f_c(q^\alpha)$, and the formula for $f_{\alpha c}(q)$ follows by removing the factor $(1-q)^k$ from $(1-q^\alpha)^k$. □

Let us look further at the recurrence relation for $f_c(q)$:

$$f_{c^m}(q) = \frac{\beta_m}{n} f_{c^R}(q) + \left(1 - \frac{\beta_m}{n}\right) q^{c_1} f_{c^L}(q). \tag{4.2}$$

This relation provides a combinatorial interpretation of $f_c(q)$ as the sum of monomials over a choice of paths in a tree. This can be best seen by translating compositions into the language of sets of positive integers. Observe that the set of compositions $c = (c_1, \ldots, c_k)$ with k parts is in bijection with the set of k-element sets $\{\beta_1, \ldots, \beta_k\}$ of positive integers, as seen by taking partial sums β_i of the parts of c.

Let S be a set of k positive integers, with greatest element s_k. Select an integer $a \in \{1, \ldots, s_k\} \setminus S$, and define the set $\gamma_L(S)$ by

$$\gamma_L(S) = S \cup \{a\} \setminus \{s_k\}.$$

Now take s_0 to be the smallest element of $S \cup \{a\}$, and define $\gamma_R(S)$ to be

$$\gamma_R(S) = \{s - s_0 : s \in S \cup \{a\} \setminus \{s_0\}\}.$$

The functions γ_L and γ_R both take sets of k distinct positive integers and "compress" them by reducing gaps between nonconsecutive elements. We will call γ_L the *left compression function* and γ_R the *right compression function*. Repeated application of γ_L and γ_R will always eventually yield the set $\{1, \ldots, k\}$. Moreover, these functions induce a binary rooted tree $T(S)$, defined recursively by letting S be the root, letting $\gamma_L(S)$ and $\gamma_R(S)$ be the left and right leaves of S, and populating the rest of the tree by recursively applying γ_L and γ_R to all existing leaves until each remaining leaf of $T(S)$ is the set $\{1, \ldots, k\}$. We call this tree a *compression tree* of S. Example 4.2.9 shows a compression tree for $S = \{1, 3, 5\}$.

If N is a node of $T(S)$ with smallest element n_0 and biggest element n_k, then define the monomial $N(q)$ by

$$N(q) = \begin{cases} \frac{n_0}{n_k} & \text{if } N \text{ is a left child, and} \\ \left(1 - \frac{n_0}{n_k}\right) q^{n_0} & \text{if } N \text{ is a right child.} \end{cases}$$

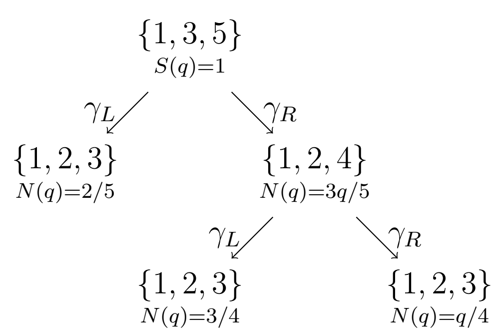
Take by convention $S(q) := 1$. Define the *weight* of a leaf L to be the product of the monomials $N(q)$ for all nodes N between L and the root S, inclusive. Denote this by $\mathrm{wt}(L)$.

Corollary 4.2.8. *Let $c = (c_1, \ldots, c_k)$ be a composition, and let $S = \{\beta_1, \ldots, \beta_k\}$ be the set of partial sums $\beta_i = c_1 + \cdots + c_i$ of the parts of c. Let $T(S)$ be a compression tree of S. Then the reduced composition polynomial $f_c(q)$ can be expressed as*

$$f_c(q) = \frac{1}{k!} \sum_{L \text{ leaf of } T(S)} wt(L).$$

Proof. This is an exact rephrasing of the recursive equation (4.2) using the $T(S)$ notation established above, and the reader is invited to verify the correspondence. □

Example 4.2.9. Let $S = \{1, 3, 5\}$ be the set corresponding to the composition $c = (1, 2, 2)$. Then a compression tree $T(S)$ is shown below:



A monomial $N(q)$ is written below each node of the tree. The weight of each leaf is calculated by taking the product of the chain of the $N(q)$ from leaf to root, and the sum of these weights yields $f_c(q) = \frac{1}{120}(8 + 9q + 3q^2)$.

Further examples suggest that the sequence of coefficients of $f_c(q)$ may be log-concave, meaning $a_i^2 \geq a_{i-1}a_{i+1}$ for all coefficients a_i of $f_c(q) = \sum_{i=0}^{n-k} a_i q^i$. We have computed $f_c(q)$ for all compositions of at most 7 parts and sizes of parts at most 6, and in all cases the sequence of coefficients is log-concave. We state this as a conjecture.

Conjecture 4.2.10. *The sequence of coefficients of $f_c(q)$ is log-concave.*

Since $g_c(q)$ measures the volume of a Minkowski sum of two polytopes (Proposition 3.4.8), it looks like log-concavity might follow from the Aleksandrov-Fenchel inequalities [30] [31], however this relationship does not work trivially and the question remains open.

4.3 Vandermonde matrices and polynomial interpolation

Here we interpret the composition polynomial $g_c(q)$ as the determinant of a slightly altered Vandermonde matrix, and through this perspective obtain alternate proofs of some results of the previous section, as well as two explicit formulas for the coefficients of $f_c(q)$. This Vandermonde interpretation of $g_c(q)$ also can be thought of as a polynomial interpolation problem, which provides an alternate meaning to some of the properties of $g_c(q)$ and $f_c(q)$.

Recall from Section 4.2 that (β) denotes the Vandermonde matrix

$$(\beta) = \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^k \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^k \end{pmatrix}$$

with rows and columns indexed from 0 to k, and that $[\hat{\beta}_i] = \det(\beta)/[\beta_i]$ is the unsigned minor of (β) obtained by removing row i and column k. Moreover, $[\hat{\beta}_i]$ is itself a Vandermonde determinant of degree $k-1$.

Applying this to the closed formula for $g_c(q)$ from Proposition 4.2.5 we obtain $\det(\beta)g_c(q) = \sum_{i=0}^{k}(-1)^i q^{\beta_i}[\hat{\beta}_i]$, and can write

$$[\beta]g_c(q) = (-1)^k \det \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k-1} & q^{\beta_0} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k-1} & q^{\beta_k} \end{pmatrix}. \tag{4.3}$$

We will use this expression to provide an alternate proof that 1 is a root of multiplicity k in $g_c(q)$.

Composition polynomials can also be viewed in the language of polynomial interpolation.

Theorem 4.3.1. *Given a composition $c = (c_1, \ldots, c_k)$ with partial sums $\beta_i = c_1 + \cdots + c_i$, consider the degree-$k$ polynomial $h(t) = x_0 + \cdots + t^k x_k$ that passes through the $k+1$ points (β_i, q^{β_i}) for $i = 0, \ldots, k$. Here the coefficients x_i are functions of q. Then, up to sign, the the lead coefficient x_k of $h(t)$ is the composition polynomial $g_c(q)$.*

Proof. In equation (4.3) we observed that $g_c(q)$ can be expressed as the determinant of a matrix:

$$g_c(q) = (-1)^k \det \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k-1} & q^{\beta_0} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k-1} & q^{\beta_k} \end{pmatrix} / [\beta].$$

This can be viewed as an execution of Cramer's rule, where we are solving for x_k in the linear system

$$\begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} q^{\beta_0} \\ \vdots \\ q^{\beta_k} \end{pmatrix}.$$

This system is equivalent to the polynomial interpolation problem of finding the coefficients of the degree-k polynomial $h(t) = x_0 + \cdots + t^k x_k$ that passes through the $k+1$ points (β_i, q^{β_i}) for $i = 0, \ldots, k$. The polynomial $g_c(q)$ is simply the lead coefficient x_k of $h(t)$, up to sign. □

That 1 is a root of $g_c(q)$ with multiplicity k can be loosely related to the fact that all of the coefficients x_i of $h(t)$ will be zero if $q = 1$ in the interpolation problem. Indeed, in that situation the points $(\beta_i, 1)$ all fall on the horizontal line $t = 1$. The coefficients of $f_c(q)$ can be interpreted using interpolation as well.

Proposition 4.3.2. *Let $f_c(q) = \sum_{i=0}^{n-k} a_i q^i$. Then $(-1)^k a_i$ is the lead coefficient of the degree-k polynomial $p_i(t)$ that passes through the points $(\beta_j, 0)$ for $\beta_j > i$ and $\left(\beta_j, \binom{k-1+i-\beta_j}{k-1}\right)$ for $\beta_j \leq i$. The fact that the a_i are positive implies that $(-1)^k p_i(t)$ has positive lead coefficient.*

Proof. This follows from an analogous construction of the proof of Theorem 4.3.1. □

Let us note a useful property of Vandermonde determinants.

Lemma 4.3.3. *Let $(\beta)^p$ be the matrix formed from the Vandermonde matrix (β) by replacing the entries β_i^k of the last column of (β) with a polynomial $p(\beta_i)$ of degree $d \leq k$. If $d = k$ and the lead coefficient of this polynomial is c, then $det((\beta)^p) = c \cdot det(\beta)$. If $d < k$ then $det((\beta)^p) = 0$.*

Proof. For $d = k$ we simply observe that $(\beta)^p$ can be obtained from (β) via elementary column operations. The only such operation that affects the determinant is multiplying the last column of (β) by c. If $d < k$ then then the last column of $(\beta)^p$ is a linear combination of the previous columns, and thus the matrix is singular. □

Proposition 4.3.4 (Alternate approach to part (1) of Theorem 4.2.3)**.** *If c is a composition of n into k parts, then 1 is a root of multiplicity exactly k of the composition polynomial $g_c(q)$.*

Proof. Consider the matrix expression for $g_c(q)$ in (4.3). Since all entries involving q reside in the same column, the operation of differentiation by q factors through the determinant. Thus we can write the i^{th} derivative of $[\beta]g_c(q)$ at $q = 1$ as

$$[\beta]g_c^{(i)}(1) = (-1)^k \det \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k-1} & \beta_0(\beta_0 - 1)\cdots(\beta_0 - i + 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k-1} & \beta_k(\beta_k - 1)\cdots(\beta_k - i + 1) \end{pmatrix}.$$

Note that 1 is a root of $g_c(q)$ of multiplicity at least k if and only if $g_c^{(i)}(1) = 0$ for $i = 0, \ldots, k-1$. Here we see that for $i = 0, \ldots, k-1$ the final column of the above matrix is a polynomial in β_i of degree strictly less than k, so by Lemma 4.3.3 the determinant will be zero. For $i = k$ this final column is a degree-k polynomial in β_i, so here by the lemma the determinant is nonzero. In particular $g_c^{(k)}(1) = (-1)^k$. We conclude that 1 is a root of multiplicity exactly k in $g_c(q)$. □

We will now discuss some properties of the coefficients of $f_c(q)$. First let us derive an explicit formula for these coefficients.

Proposition 4.3.5. *Let $f_c(q) = \sum_{i=0}^{n-k} a_i q^i$ be a reduced composition polynomial corresponding to the composition $c = (c_1, \ldots, c_k)$. Then the coefficients of $f_c(q)$ are given by*

$$a_i = \sum_{j:\beta_j \le i} (-1)^j \binom{k-1+i-\beta_j}{k-1} / [\beta_j].$$

Proof. We can write $f_c(q) = g_c(q)/(1-q)^k$ and then expand the power series:

$$\begin{aligned} f(q) &= g(q)(1 + q + q^2 + \cdots)^k \\ &= \left(\sum_{j=0}^{k} (-1)^j \frac{q^{\beta_j}}{[\beta_j]} \right) \left(\sum_{i=0}^{\infty} q^i \binom{k-1+i}{k-1} \right) \\ &= \sum_{i=0}^{n-k} q^i \sum_{j:\beta_j \le i} (-1)^j \binom{k-1+i-\beta_j}{k-1} / [\beta_j]. \end{aligned}$$

□

Corollary 4.3.6. *An alternate expression for a_i is given by*

$$a_i = \sum_{j:\beta_j > i} (-1)^j \binom{\beta_j - i - 1}{k-1} / [\beta_j].$$

Proof. Extend the expression for a_i in the previous proposition to sum over all of the β_j, and multiply by the Vandermonde determinant (β):

$$(\beta)\sum_{j=0}^{k}(-1)^j\binom{k-1+i-\beta_j}{k-1}/[\beta_j] = \sum_{j=0}^{k}(-1)^j\binom{k-1+i-\beta_j}{k-1}[\hat{\beta}_j]$$

This is an expression for the determinant of the matrix obtained from (β) by replacing the entries β_j^k in the last column with the degree-$(k-1)$ polynomial $\binom{k-1+i-\beta_j}{k-1}$. By Lemma 4.3.3 the determinant is zero. After rewriting the binomial coefficient using the identity $\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$ so that the top argument will always be positive, we obtain the desired alternate expression for a_i. □

Now we present an alternate proof that $f_c(1) = 1/k!$.

Proposition 4.3.7 (Alternate approach to part (3) of Theorem 4.2.3)**.** *The sum of the coefficients a_i of $f_c(q)$ is $1/k!$.*

Proof. It will be easier to work with the polynomial $[\beta]f_c(q)$. Using the explicit formula for a_i we obtain

$$\begin{aligned}[\beta]f_c(1) = [\beta]\sum_{i=0}^{n-k}a_i &= \sum_{i=0}^{n-k}\sum_{j:\beta_j\le i}(-1)^j[\hat{\beta}_j]\binom{k-1+i-\beta_j}{k-1} \\ &= \sum_{j=0}^{k}(-1)^i[\hat{\beta}_j]\sum_{i=k-1}^{n-\beta_j-1}\binom{i}{k-1} \\ &= (-1)^k\sum_{j=0}^{k}(-1)^{j+k}[\hat{\beta}_j]\binom{n-\beta_j}{k}.\end{aligned}$$

Notice that $\binom{n-\beta_j}{k}$ is a polynomial in β_j of degree k with lead coefficient $(-1)^k/k!$. Using Lemma 4.3.3 we observe that this last expression is equal to the signed determinant $(-1)^k(\beta)^{\binom{n-x}{k}} = [\beta]/k!$. Therefore $f_c(1) = 1/k!$ as desired. □

Chapter 5

Generalized permutohedra for classical reflection groups

5.1 Introduction

Up to this point we've been only discussing generalized permutohedra corresponding to type-A hyperplane arrangements A_n. However we can similarly define generalized permutohedra for the other classical reflection groups of types B, C, and D. We will discuss the constructions of the respective hyperplane arrangements and analyze their respective orbit polytopes [7].

We lend particular focus to the crosspolytope and fully describe the space of signed Minkowski sums of its faces. Through this we prove a general indecomposability theorem about pyramids, produce a class of geometric identities on Minkowski sums of simplices, and show that the crosspolytope is decomposable into a signed Minkowski sum of its faces if and only if its dimension is even.

5.2 Basic constructions

Definition 5.2.1. For a reflection arrangement Σ define the *orbit polytope* $\Sigma(v)$ at a point v to be the convex hull of all points generated by reflections of v across the hyperplanes of Σ. Since Σ is a reflection arrangement there will be finitely many such points. Let $\widetilde{\Sigma}$ be the hyperplane arrangement formed by modding out by any lineality space of Σ, if one exists. If we take v to be on a ray of $\widetilde{\Sigma}$, then we say $\Sigma(v)$ is an *extremal orbit polytope* of Σ.

The type-A arrangement $A_n \subset \mathbb{R}^n$ is the set of hyperplanes $x_i = x_j$ for $i \neq j$. The orbit polytope $A(1, \ldots, n)$ is the traditional permutohedron P_n, and for any other choice of v, $A_n(v)$ will be a generalized permutohedron. If v does not belong to any of the hyperplanes in A_n then $A_n(v)$ will be combinatorially equivalent to P_n. A_n has 1-dimensional lineality space generated by $(1, \ldots, 1)$.

In characteristic 0 the type-B and type-C hyperplane arrangements are equal. We will thus consider them together, and call the arrangement BC_n.

Definition 5.2.2. The type-BC hyperplane arrangement BC_n in $\mathbb{R}^n$ is the set of all hyperplanes of the form $x_i = 0$, $x_i = x_j$, or $x_i = -x_j$ (for $i \neq j$). The orbit polytope $BC_n(1, \ldots, n)$ is the convex hull of all signed permutation vectors, and we call this the *type-BC permutohedron*.

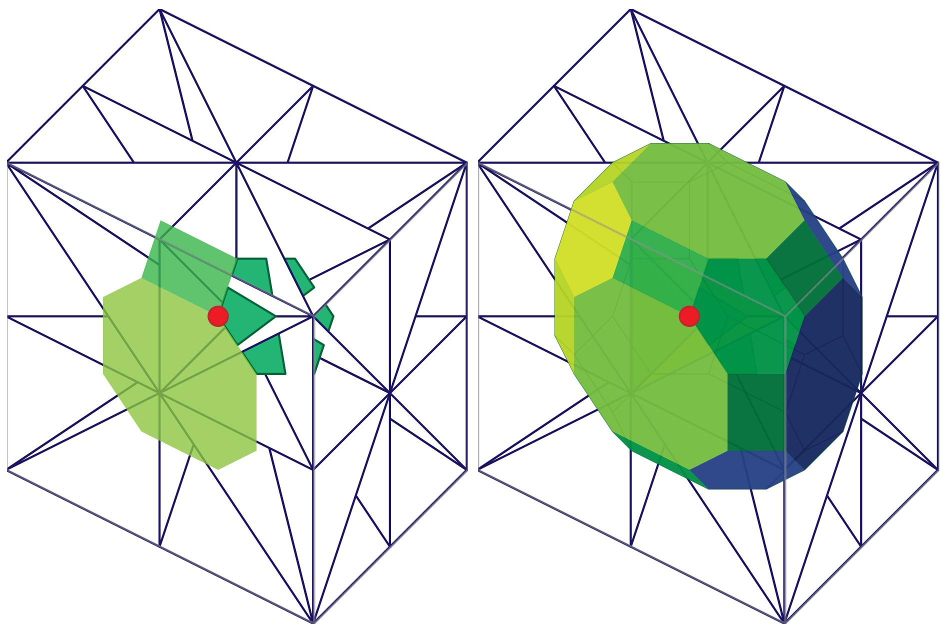


Figure 5.1: The type-BC permutohedron is formed by reflecting the point (1,. . . ,n) (shown in red) about all the hyperplanes in the arrangement. This is equivalent to taking the convex hull of all signed permutation vectors, or 2^n copies of the type-A permutohedron P_n.

Definition 5.2.3. The type-D hyperplane arrangement in $\mathbb{R}^n$ is the set of all hyperplanes of the form $x_i = x_j$ or $x_i = -x_j$ (for $i \neq j$). The orbit polytope $D_n(1, \ldots, n)$ is the convex hull of all signed permutation vectors with an even number of negative entries, and we call this the *type-D permutohedron*.

It should be noted that the orbit polytopes are generalized permutohedra. Generalized permutohedra of type A are defined to be all polytopes formed by deformations of the permutohedron P_n. The concept of a deformation has several equivalent definitions (see the Appendix in [25]). We may think of a deformation as a parallel shifting of facet hyperplanes, with care taken that no hyperplanes move "beyond" any vertices, as described in Section 2.2. We extend this definition to types BC and D.

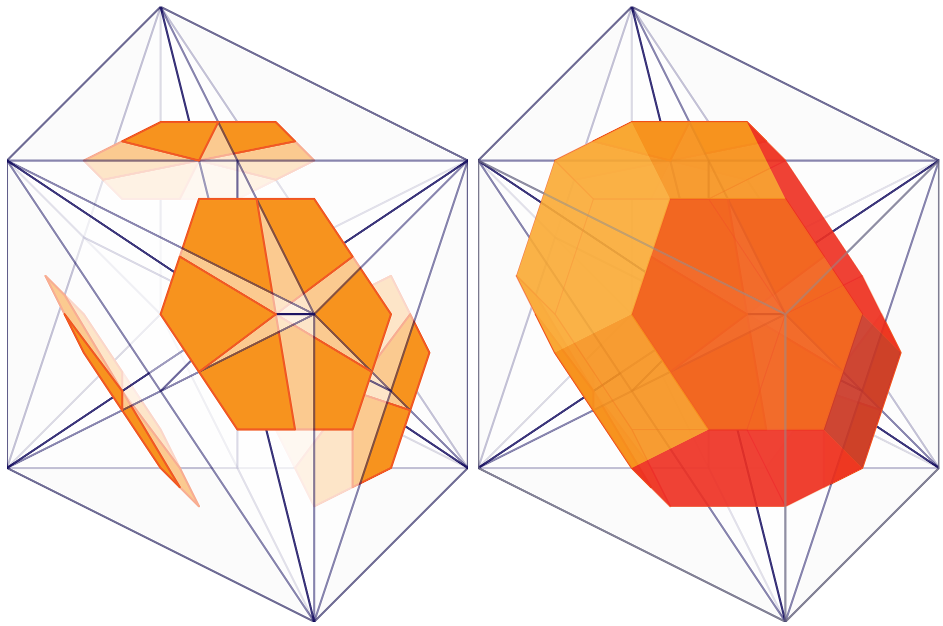


Figure 5.2: The type-D permutohedron is formed by taking the convex hull of all signed permutation vectors with an even number of negative entries, or 2^{n-1} copies of the type-A permutohedron P_n.

Definition 5.2.4. We say a polytope P is a *type-A (resp. type-BC, type-D) generalized permutohedron* if it is a deformation of the type-A (resp. type-BC, type-D) permutohedron. The set of generalized permutohedra in $\mathbb{R}^n$ is denoted $\mathcal{A}^n$ (resp. $\mathcal{BC}^n$, $\mathcal{D}^n$).

Because polytope deformation is a linear operation, the sets $\mathcal{A}^n$, $\mathcal{BC}^n$, and $\mathcal{D}^n$ will be identified with linear cones whose dimensions are equal to the number of free deformation parameters of each respective permutohedron, *i.e.* the number of facets of each permutohedron or equivalently the number of rays (modulo lineality) of each respective hyperplane arrangement. It is easy to check that

$$\dim(\mathcal{A}^n) = 2^n - 2, \text{ and}$$

$$\dim(\mathcal{BC}^n) = 3^n - 1.$$

The type-D case is less obvious:

$$\dim(\mathcal{D}^n) = 3^n - n2^{n-1} - 1.$$

The rays of D_n are generated by the $\{-1, 0, 1\}$-vectors apart from the origin for whom the number of nonzero coordinates (support) is not equal to $n-1$. We show this in the following lemma.

Lemma 5.2.5. *The number of rays of D_n is*

$$3^n - n2^{n-1} - 1.$$

Proof. Let R^D be the subset of vectors in $\{-1,0,1\}^n$ with support of size not equal to 0 or $n-1$. Let us represent each ray of the type-D hyperplane arrangement by its intersection with the boundary of the cube $[-1,1]^d$. This defines a subset of $\{-1,0,1\}^n$. Let $v \in \{-1,0,1\}^n$ be a vector whose zero coordinates are indexed by $I \subset [n]$, i.e, $v_i = 0$ if and only if $i \in I$. We do not want to consider the origin as a ray, so assume $I \neq [n]$.

First suppose $|I| > 1$. Then v lies on the intersection of the $|I|$ hyperplanes defined by

$$x_{i_1} = x_{i_2}, x_{i_2} = x_{i_3}, \ldots, x_{i_{|I|-1}} = x_{i_{|I|}}, x_{i_{|I|}} = -x_{i_1}.$$

Each of these $|I|$ hyperplanes belongs to the type-D arrangement. The normal vectors to these hyperplanes are linearly independent, and thus this intersection is $(n-|I|)$-dimensional. If $|I| = n-1$ then we have already constructed the line in the arrangement containing v, so let's assume $|I| < n-1$. Now $v_j = \pm 1$ for all $j \in I^c$, so v also lies on the intersection of the hyperplanes given by

$$v_{j_1}x_{j_1} = v_{j_2}x_{j_2}, \ldots, v_{j_{|I^c|-1}}x_{j_{|I^c|-1}} = v_{j_{|I^c|}}x_{j_{|I^c|}},$$

where $\{j_1, ..., j_{|I^c|}\} = I^c$. These $|I^c| - 1$ hyperplanes intersect properly and all belong to the type-D arrangement as well. Together this and the above set of hyperplanes form a set of $n-1$ hyperplanes with proper intersection, *i.e.* they intersect in a line, which by construction contains v as desired.

Now conversely suppose that $|I| = 1$. Without loss of generality, let us assume $v_n = 0$ and $v_i \neq 0$ for $i < n$. Then, as above, we know that v lies on the $(n-2)$-dimensional intersection of hyperplanes expressed by

$$v_1x_1 = v_2x_2 = \cdots = v_{n-1}x_{n-1}.$$

In order to construct a line we need to intersect this space with one more hyperplane from the type-D arrangement. If this hyperplane relates x_i and x_j with $i, j < n$, then the intersection will either be improper or it will force all the coordinates of the intersection to be zero (and thence it will no longer contain v). If, however, this hyperplane relates x_n and x_i with $i \neq n$, then this will force x_n to be nonzero which contradicts the assumption that $v_n = 0$, so the ray we desire does not exist.

We conclude that the nonzero vector v lies on a ray of the type-D arrangement if and only if $|I| \neq 1$, i.e. if $v \in R^D$. There are $n \cdot 2^{n-1}$ nonzero integer vectors in $[-1,1]^n$ with exactly one zero coordinate. Excluding the origin from our total count yields $3^n - n \cdot 2^{n-1} - 1$ total rays, as desired. □

Let us now recall the definitions of some common polytopes that appear as extremal orbit polytopes of the above arrangements.

Definition 5.2.6. The *demihypercube* (resp. *odd demihypercube*) is the convex hull of all $\{-1, 1\}$-vectors with an even (resp. odd) number of negative coordinates. The *crosspolytope* $\Diamond_n \in \mathbb{R}^n$ is the convex hull of all standard basis vectors e_i and their negatives $-e_i$. The *k-th rectification* of a polytope P is the convex hull of the barycenters of the dimension-k faces of P.

We now describe the extremal orbit polytopes of of each classical reflection group Σ. There is one such polytope for each ray of the fundamental chamber of Σ. Since that chamber is simplicial there are $\mathrm{rk}(\Sigma)$ extremal orbit polytopes.

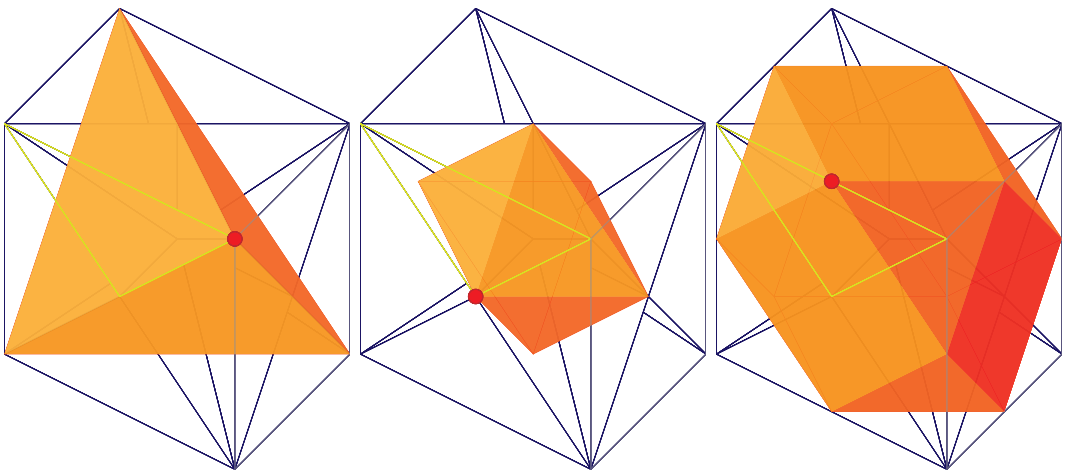


Figure 5.3: Orbit polytopes of the type-D arrangement: the extremal orbit polytopes of the demihypercube (left) and the crosspolytope (center), as well a dilation of the first rectification of the crosspolytope (right), which is not extremal in D_n.

Proposition 5.2.7. *The $n-1$ extremal orbit polytopes of A_n are the $n-1$ hypersimplices $\Delta_{k,n}$, for $k = 1, \ldots, n-1$.*

Proof. The fundamental chamber $x_1 \leq \cdots \leq x_n$ has $n-1$ rays generated by

$$(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1, 0).$$

The orbits of these points generate the hypersimplices $\Delta_{k,n}$, where k corresponds to the number of 1s in the generating ray. □

Proposition 5.2.8. *The n extremal orbit polytopes of BC_n are the hypercube, the crosspolytope, and the first $n-2$ rectifications of the crosspolytope.*

Proof. The fundamental chamber $0 \leq x_1 \leq \cdots \leq x_n$ has n rays generated by

$$(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1).$$

If v is such a point that has k nonzero coordinates, then the orbit of v will be the set of all points in $\{-1,0,1\}^n$ with k nonzero coordinates. If $k = 1$ then this describes the crosspolytope, if $2 \leq k \leq n-1$ then this is a dilation of the k-th rectification of the crosspolytope, and if $k = n$ then we have the hypercube. □

Proposition 5.2.9. *The n extremal orbit polytopes of D_n are the even and odd demihypercubes, the crosspolytope, and the first $n-3$ rectifications of the crosspolytope.*

Proof. The fundamental chamber $-x_2 \leq x_1 \leq \cdots \leq x_n$ has n rays generated by

$$(1,0,\ldots,0),(1,1,0,\ldots,0),\ldots,(1,\ldots,1,0,0),(1,\ldots,1),(-1,1,\ldots,1).$$

If v is such a point with $k \leq n-2$ nonzero coordinates, then the orbit of v will be the set of all points in R^D with k nonzero coordinates. The convex hull of this orbit is by definition the crosspolytope for $k = 1$ and a dilation of the k-th rectification of the crosspolytope for $2 \leq k \leq n-2$. If all coordinates of v are nonzero, then the orbit of v will preserve parity of sign of the coordinates of v, as reflection across any hyperplane in the D_n always either switches two coordinates or does so while negating both. If the parity is even, this defines the even demihypercube. If the parity is odd, we have the odd demihypercube. □

5.3 Geometry of the crosspolytope

In A_n the faces of the simplest extremal orbit polytope—the simplex Δ_n—generate the space $\mathcal{A}^n$ through signed Minkowski sums, and when taken together all of the extremal orbit polytopes of A_n can generate $\mathcal{A}^n$ via non-negative Minkowski sums. One could hope the same to be true in types BC and D, but it is not so. Computer testing has shown that above dimension 3 there is no extremal polytope of BC_n (respectively D_n) whose faces generate $\mathcal{BC}^n$ (respectively $\mathcal{D}^n$) via signed Minkowski sums. Moreover even when all faces of all extremal polytopes of the arrangement and its subarrangements are included, the space is not generated. This investigation does however lead to an interesting subclass of type BC and D generalized permutohedra, which we now study.

The crosspolytope $\Diamond_n$ is an extremal orbit polytope of both BC_n and D_n, and as such it is both a type-D and a type-BC generalized permutohedron. We show that through signed Minkowski sums the faces of the crosspolytope generate a subspace of $\mathcal{BC}^n$ and $\mathcal{D}^n$ of dimension $\frac{1}{2}(3^n - (-1)^n)$. In a sense, this can be thought of as meaning that about half of all type-BC generalized permutohedra can be expressed as a signed Minkowski sum of faces of the crosspolytope. In proving this result we also uncover some properties of Minkowski arithmetic of simplices and of faces of pyramids.

First we must establish the fact that Minkowski summation of generalized permutohedra characterizes a vector space. Recall that any type-A generalized permutohedron P can be parameterized by parameters $\{z_I\}_{I\subseteq[n]}$ which describe how the facet hyperplanes of P_n are deformed to create P:

$$P = P_n(\{z_I\}) = \left\{ x \in \mathbb{R}^n : \sum_{i\in I} x_i \geq z_I \text{ for } I \subset [n], \sum_{i=1}^{n} x_i = z_{[n]} \right\}.$$

Definition 5.3.1. If we define R^A to be the set of vectors in $\{0,1\}^n$ minus the origin, then we may rewrite this parameterization as

$$P = A_n\{z_v\} := \left\{ x \in \mathbb{R}^n : x \cdot v \geq z_v \text{ for } v \in R^A, x \cdot (1,\dots,1) = z_{(1,\dots,1)} \right\}.$$

We can write similar parameterizations for generalized permutohedra of types BC and D as well. Recall that R^{BC} is the set of vectors in $\{-1,0,1\}^n$ minus the origin, and R^D is the set of vectors in $\{-1,0,1\}^n$ whose support size is not equal to 0 or $n-1$. If P' is a deformation of the type-BC permutohedron $BC_n(1,\dots,n)$ then P' can be parameterized by parameters $\{z_v\}_{v\in R^{BC}}$, and we write

$$P' = BC_n\{z_v\} := \left\{ x \in \mathbb{R}^n : x \cdot v \leq z_v \text{ for } v \in R^{BC} \right\}.$$

Finally if P'' is a deformation of the type-D permutohedron $D_n(1,\dots,n)$ then we can write P'' as

$$P'' = D_n\{z_v\} := \left\{ x \in \mathbb{R}^n : x \cdot v \leq z_v \text{ for } v \in R^{D} \right\}.$$

As in Chapter 1, we assume that the parameters z_v are chosen minimally.

Recall from Chapter 1 that if $A_n\{z_v\}$ and $A_n\{z'_v\}$ are both deformations of P_n, then their hyperplane deformation parameters z_v are additive across Minkowski sums, meaning

$$A_n\{z_v\} + A_n\{z'_v\} = A_n\{z_v + z'_v\}.$$

This means that the set of deformations of P_n forms a cone via Minkowski addition. A Minkowski sum equation comprised of polytopes that are deformations of P_n is equivalent to a linear dependence relation amongst those summands' corresponding z_v parameter vectors. Conversely, if a set of z_v parameter vectors, all of whom correspond to deformations of P_n, are linearly independent, then there is no nontrivial Minkowski sum equation relating those polytopes. This follows from Theorem 15.3 in [25], and holds analogously for types BC and D as well.

The faces of a generalized permutohedron are themselves generalized permutohedra. Therefore based on the preceding discussion, in order to determine the dimension of the subcone of $\mathcal{BC}^n$ (or $\mathcal{D}^n$) generated by the faces of $\Diamond_n$ it suffices to determine the dimension of the subspace of $\mathbb{R}^{3^n-1}$ (or $\mathbb{R}^{3^n-n2^{n-1}-1}$) generated by the z_v parameter vectors of these faces. We will calculate this dimension to be $\frac{1}{2}(3^n-(-1)^n)$ by producing a basis for the span of faces of $\Diamond_n$ comprised of $\frac{1}{2}(3^n-(-1)^n)$ specific faces of $\Diamond_n$.

For the rest of this section we will work in $\mathcal{BC}^n$ because R^{BC} is an easier indexing set to work with than R^D, though either treatment will produce the same results. The 3^n faces of $\Diamond_n$ are in bijection with the vectors w in $\{-1,0,1\}^n$, with w corresponding to the face $(\Diamond_n)_w$ that maximizes the functional of taking the inner product (x, w). We will now describe the z_v parameter vectors of each such face. First we introduce some notation.

Definition 5.3.2. For $v \in \{-1,0,1\}^n$ let $v^s \subset [-n] \cup [n]$ be the *signed support* of v, as defined by

$$v^s = \{iv_i : v_i \neq 0\}.$$

For example, if $v = (1, 0, -1, -1)$ then $v^s = \{1, -3, -4\}$.

Proposition 5.3.3. *If $BC_n\{z_v\}$ is the face $(\Diamond_n)_w$ of $\Diamond_n$, then*

$$z_v = \begin{cases} 1 & \text{if } w^s \cap v^s \neq \emptyset, \\ -1 & \text{if } w^s \subseteq (-v)^s, \\ 0 & \text{otherwise} \end{cases}$$

for every $v \in R^{BC}$.

Proof. First note that if $BC_n\{z_v\}$ is a face of the crosspolytope $\Diamond_n$, then $z_v \in \{0, 1, -1\}$ for every $v \in R^{BC}$. This follows from the fact that v is a $\{-1,0,1\}$-vector, z_v is the maximum value of the functional defined by the inner product (x, v) applied to all vertices of a face of $\Diamond_n$, and each such vertex is a signed standard basis vector $\pm e_i$.

Next observe that the vector sum of the vertices of the face $(\Diamond_n)_w$ equals the vector w. The functional (x, v) will have maximum value 1 if and only if some vertex u of $(\Diamond_n)_w$ satisfies $u^s \cap v^s \neq \emptyset$. The functional (x, v) taken over $(\Diamond_n)_w$ will therefore have maximum value 1 if and only if $w^s \cap v^s \neq \emptyset$. The functional (x, v) will have maximum value -1 if and only if for every vertex u of $(\Diamond_n)_w$ we have $u^s \subseteq (-v)^s$. Thus taken over the entire face $(\Diamond_n)_w$ this condition translates to $w^s \subseteq (-v)^s$. □

Apart from the crosspolytope itself, the set of faces of $\Diamond_n$ come in antipodal pairs, i.e. $-(\Diamond_n)_w = (\Diamond_n)_{-w}$.

Definition 5.3.4. Let $H(\Diamond_n)$ be the set of proper faces of the n-crosspolytope indexed by vectors w whose last nonzero coordinate is positive. Note that the set $H(\Diamond_n)$ contains exactly one member of each antipodal face pair, and $H(\Diamond_n)$ has cardinality $\frac{1}{2}(3^n - 1)$.

Note that we must be careful with our notation here. The negation $-(\Diamond_n)_w$ of a face $(\Diamond_n)_w$ is not the same as its additive inverse in the vector space of Minkowski addition. In particular, negating $(\Diamond_n)_w$ does not negate its corresponding z_v parameter vector. Also geometrically one can see that the Minkowski sum of a polytope P with its negation $-P$ will be the zero polytope (the point at the origin) if and only if P is a point. See Proposition 2.2.4 on Minkowski differences for more information.

Proposition 5.3.5. *Any proper face of $\Diamond_n$ that does not belong to $H(\Diamond_n)$ can be written as a signed Minkowski sum of faces in $H(\Diamond_n)$.*

Proof. One easily checks that if the proper face $(\Diamond_n)_w \notin H(\Diamond_n)$ is not a facet, then it lies on the hyperplane $x_i = 0$ for some i. Moreover, it is a proper face of the crosspolytope $\Diamond_{n-1}$ formed by projecting $\Diamond_n$ onto the hyperplane $x_i = 0$. Hence by induction $(\Diamond_n)_w$ is in the Minkowski span of faces in $H(\Diamond_n)$ that live on the hyperplane $x_i = 0$. Now suppose $(\Diamond_n)_w \notin H(\Diamond_n)$ is a facet. We need now only show that $(\Diamond_n)_w$ is in the Minkowski span of the facets in $H(\Diamond_n)$ along with all other faces of $\Diamond_n$ that lie on the hyperplane $x_n = 0$. This describes the set of all faces of $\Diamond_n$ that lie in the half space $x_n \geq 0$. By symmetry then we may assume that $w = (-1, \ldots, -1)$. We present an explicit formula for the Minkowski decomposition and check that it holds in each coordinate. The formula is as follows:

$$(\Diamond_n)_{(-1,\ldots,-1)} = \sum_{w_i \geq 0 \text{ for all } i} (-1)^{\dim((\Diamond_n)_w)+1} (\Diamond_n)_w.$$

Let z^w be the parameter vector corresponding to the face $(\Diamond_n)_w$. By Proposition 5.3.3 the entries of the parameter vector $z^{(-1,\ldots,-1)}$ for the facet $(\Diamond_n)_{(-1,\ldots,-1)}$ is given by

$$z_v^{(-1,\ldots,-1)} = \begin{cases} 1 & \text{if } v \text{ contains a } -1 \text{ in some coordinate,} \\ -1 & \text{if } v = (1, \ldots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, suppose v contains a -1 in some coordinate. We may assume that $v = e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{k+m}$. Suppose w is such that $w_i \geq 0$ for all i. Then z_v^w will be 1 whenever w contains a 1 somewhere in its first k coordinates. Counting all such w, signed according to dimension, we get

$$\sum_{i=1}^{k} \binom{k}{i} \sum_{j=0}^{n-k} (-1)^{i+j} \binom{n-k}{j} = 0.$$

Note that we are using the fact that when $(\Diamond_n)_w$ is a proper face, $\dim((\Diamond_n)_w) + 1$ equals the support of w. We also note that z_v^w will be -1 whenever all nonzero entries of w are in coordinates $x_{k+1}, \ldots, x_{k+m}$. Counting this with signed dimension yields

$$-\sum_{i=1}^{m} (-1)^i \binom{m}{i} = 1.$$

Hence the z_v parameter of the right hand side is 1 as expected.

Now suppose the coordinates of v are nonnegative. If $v = (1, \ldots, 1)$ then $z_v^w = 1$ for all w such that $w_i \geq 0$ for all i. Summing with signed dimension yields

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} = -1.$$

If on the other hand v has some 0 entry, then our signed sum becomes

$$\sum_{i=0}^{|v^s|}(-1)^i\binom{|v^s|}{i}\sum_{j=0}^{n-|v^s|}(-1)^j\binom{n-|v^s|}{j}=0$$

because $n-|v^s|\geq 1$. This corresponds to our formula for $z_v^{(-1,\ldots,-1)}$, thus completing the proof. □

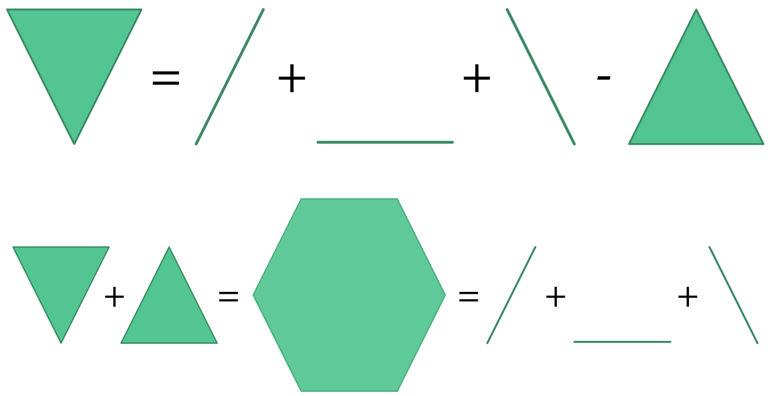


Figure 5.4: Depiction of Proposition 5.3.5 as as signed Minkowski sum for $n=3$ (top), and a rearrangement of terms (bottom) to illustrate a familiar geometric identity in terms of positive Minkowski sums (zero-faces omitted).

Every proper face of $\Diamond_n$ is a simplex, so Proposition 5.3.5 provides an infinite class of geometric identities that express an n-simplex as a signed Minkowski sum of an oppositely-oriented n-simplex and its faces. Figure 5.4 depicts this for $n=3$.

We have shown that those faces outside of $H(\Diamond_n)$ are linearly dependent in the cone of Minkowski addition on the faces of $H(\Diamond_n)$. We also need to show that $H(\Diamond_n)$ is itself a linearly independent set. The crosspolytope $\Diamond_n$ may be thought of as a union of two pyramids that share a common base. The faces of $H(\Diamond_n)$ are contained in one such pyramid. The following theorem on pyramids will provide us with this linear independence of $H(\Diamond_n)$.

Definition 5.3.6. Let P be a pyramid with apex p. A *lateral face* of P is a face that contains p. A *basal face* of P is a face that does not contain p.

Theorem 5.3.7. *No lateral face of a pyramid P, including P itself, can be decomposed into a signed Minkowski sum of other faces of P.*

Proof. If f is a linear functional let P_f be the face of P that maximizes f.

Claim 1: It suffices to only look at decompositions of lateral faces into Minkowski sums of other lateral faces. Indeed, suppose some set of linear functionals $\{f_i\}$ maximizes lateral faces of P and some other set of functionals $\{g_j\}$ maximizes basal faces of P. Say a lateral face P_f is decomposable as $P_f = \sum y_i P_{f_i} + \sum y_j P_{g_j}$. Let f_p be the linear functional that is normal to the base of P and that maxmizes on the apex p of P. Applying f_p to both sides of the decomposition induces the decomposition among faces given by $p = \sum y_i p + \sum y_j P_{g_j}$. Thus the signed Minkowski sum $\sum y_j P_{g_j}$ is trivial, and we may ignore it.

Claim 2: The pyramid P is indecomposable into a signed Minkowski sum of its lateral faces. Proceed by induction. The claim is true in dimension 2 because a triangle is not a zonotope, and any signed Minkowski sum of two 1-faces is a zonotope. Now assume the claim is true for dimension $n-1$, and let P be a pyramid of dimension n. Let f be a linear functional that is maximized on a lateral facet P_f of P. Suppose by way of contradiction that P is decomposable into the Minkowski sum $P = \sum y_i P_{f_i}$, where the f_i are some collection of linear functionals that maximize on lateral faces of P and the y_i are constants. Apply f to both sides to get $P_f = \sum y_i (P_{f_i})_f$. The maximum value of f over the pyramid P is attained on P_f, which contains the pyramid's apex. Therefore if we apply f to any subset of P and attain this same maximum value, then that subset must also be contained in P_f. Since each face P_{f_i} also contains the apex we know the functional f attains its global maximum on each P_{f_i}. Thus $(P_{f_i})_f$ must be a lateral subface of P_f for each f_i. Hence we've presented the $(n-1)$-dimensional pyramid P_f as a signed Minkowski sum of its lateral faces. Contradiction.

Claim 3: The lateral face P_f of the pyramid P is indecomposable into a signed Minkowski sum of other lateral faces. Suppose not. Then we can write $P_f = \sum y_i P_{f_i}$. Apply f to both sides and we get $P_f = \sum y_i (P_{f_i})_f$, which violates Claim 2. □

Corollary 5.3.8. *The faces of $H(\Diamond_n)$ are linearly independent in the vector space of Minkowski addition.*

Proof. Notice $H(\Diamond_n)$ belongs to a pyramid P with apex e_n. The faces $(\Diamond_n)_w$ of $H(\Diamond_n)$ for whom $w_n = 1$ are the lateral faces of P, and by Theorem 5.3.7 they are all indecomposable. The base of P is a copy of the crosspolytope $\Diamond_{n-1}$ sitting on the hyperplane $x_n = 0$, and all members of $H(\Diamond_n)$ belonging to this base are themselves contained in a pyramid with apex e_{n-1}. Recursively we conclude that all remaining faces of $H(\Diamond_n)$ are indecomposable. □

Our final step in describing the signed Minkowski span of faces of $\Diamond_n$ is to address the decomposability of the crosspolytope itself.

Theorem 5.3.9. *The n-dimensional crosspolytope $\Diamond_n$ is decomposable into a Minkowski sum of its faces if and only if n is even. For even n the unique decomposition is given by*

$$\Diamond_n = \sum_{w_n=1} (-1)^{dim((\Diamond_n)_w)+1} (\Diamond_n)_w.$$

Proof. Each face $(\Diamond_n)_w$ of the crosspolytope can be represented by its z-vector of hyperplane parameters z^w. Since all faces of $\Diamond_n$ are BC_n-generalized permutohedra Minkowski decomposability is equivalent to linear dependency of z-vectors. We will exhibit a linear equation that vanishes on all faces of $\Diamond_n$ except for $\Diamond_n$ itself when n is odd. Thus the polytope $\Diamond_n$ is only in the Minkowski span of its faces when n is even. We will then verify the decomposition formula for even n.

Each z-vector belongs to $\mathbb{R}^{3^n-1}$, with coordinates indexed by the vectors v in R^{BC}. Let z_v^w denote the coordinate of z^w indexed by v. Consider then the linear function

$$l(z^w) := \sum_{v \in R^{BC}} (-1)^{|v^s|} z_v^w.$$

Let $w \in R^{BC}$ represent a proper face of $\Diamond_n$. Without loss of generality let us assume that $w = e_1 + \cdots + e_k$ for some nonzero $k \leq n$. Now we evaluate $l(z^w)$. The coordinate z_v will be 1 whenever v and w have nonzero entries in common. Let us compute the contribution of those v to the right hand side. Build up such a vector v as follows. Choose i of the first k coordinates to be nonzero, but subtract all such configurations that contain no 1s. Then choose j of the remaining $n-k$ coordinates to be nonzero. Counting with parity of the support of v, the signed sum of these z_v^w is

$$\begin{aligned} &\left(\sum_{i=0}^{k}(-1)^i 2^i \binom{k}{i} - \sum_{i=0}^{k}(-1)^i \binom{k}{i}\right) \sum_{j=0}^{n-k}(-1)^j 2^j \binom{n-k}{j} \\ =&\left((-1)^k - 0\right)(-1)^{n-k} \\ =&(-1)^n. \end{aligned}$$

The coordinate z_v will be -1 whenever $w^s \subseteq (-v)^s$; consider their contribution to the right hand side. Let w be as before. Build up such a vector v as follows. Let the first k coordinates of v be -1. Then select j of the remaining coordinates to be 0, and let the rest be any combination of 1s and -1s. Counting these unique vectors with value $z_v = -1$, signed according to parity of support, we get

$$\begin{aligned} &-\sum_{j=0}^{n-k}(-1)^{n-j}\binom{n-k}{j}2^{n-k-j} \\ =&(-1)^{n+1}. \end{aligned}$$

Thus for all proper faces $(\Diamond_n)_w$ we have $l(z^w) = 0$. The z_v-vector z^0 for $\Diamond_n$ is the vector of all 1s. So $l(z^0) = \sum_{j=1}^{n}(-2)^j\binom{n}{j} = (-1)^n - 1$. This will be zero if and only if n is even. Hence if n is odd $\Diamond_n$ is indecomposable.

Now we prove the decomposition formula for n even. We must show that for any choice of $v \in R^{BC}$ with n even we have $\sum_{w_n=1}(-1)^{|w^s|}(z_v)_w = 1$. We can break this into three cases, based upon the last coordinate v_n of v.

First suppose $v_n = 1$. Since we are only working with vectors w such that $w_n = 1$, all such vectors will satisfy $w^s \cap v^s \neq \emptyset$, and so $z_v^w = 1$ for all w in question. Summing with signed parity of support we get $\sum_{w_n=1}(-1)^{|w^s|} = 1$.

Next suppose $v_n = 0$. Note that no w with $w_n = 1$ will satisfy $w^s \subseteq (-v)^s$. To count vectors w that satisfy $w^s \cap v^s \neq \emptyset$, we employ the same technique as above and our signed sum adds up to $(-1)^n$, which equals 1 by assumption that n is even.

Finally suppose $v_n = -1$. The vectors w that satisfy $w^s \subseteq (-v)^s$ correspond to subsets of $(-v)^s$. Counting these with signed parity of support will always sum to zero. To count vectors w that satisfy $w^s \cap v^s \neq \emptyset$ we again use the inclusion-exclusion technique and arrive at a signed sum of $(-1)^n = 1$, as in the previous case. □

Corollary 5.3.10. *The space of Minkowski sums of faces of the n-crosspolytope has dimension $\frac{1}{2}(3^n - (-1)^n)$. A basis for this parameter space is $H(\lozenge_n)$ for n even, and $H(\lozenge_n) \cup \{\lozenge_n\}$ for n odd.*

Proof. The preceding results indicate this directly. □

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