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2014

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Moduli of Elliptic Curves via Twisted Stable Maps

by

Andrew Norton Niles

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Martin Olsson, Chair
Professor Arthur Ogus
Professor Mary Gaillard

Spring 2014

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Abstract

Moduli of Elliptic Curves via Twisted Stable Maps

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Andrew Norton Niles

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Martin Olsson, Chair

Abramovich, Corti and Vistoli have studied modular compactifications of stacks of curves equipped with abelian level structures arising as substacks of the stack of twisted stable maps into the classifying stack of a finite group, provided the order of the group is invertible on the base scheme. Recently Abramovich, Olsson and Vistoli extended the notion of twisted stable maps to allow arbitrary base schemes, where the target is a tame stack, not necessarily Deligne-Mumford.

In this dissertation, we use this extended notion of twisted stable maps to extend the results of Abramovich, Corti and Vistoli to the case of elliptic curves with level structures over arbitrary base schemes. We prove that we recover the compactified Katz-Mazur regular models, with a natural moduli interpretation in terms of level structures on Picard schemes of twisted curves. Additionally, we study the interactions of the different such moduli stacks contained in a stack of twisted stable maps in characteristics dividing the level.

To Robin

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Acknowledgments

I would like to thank my advisor, Martin Olsson, for his extremely helpful and enthusiastic support throughout my studies at UC Berkeley. In addition, I have learned much about algebraic geometry and number theory from the Berkeley faculty, particularly Matt Baker, Ken Ribet, and Arthur Ogus. Over the years, my understanding of these subjects has been shaped by many stimulating conversations with other graduate students in the department, including Khoa Nguyen, Mike Daub, Peter Mannisto, and Piotr Achinger.

Finally, my graduate career was made possible by the outstanding mentorship I received as an undergraduate from Naomi Jochnowitz, Douglas Ravenel, Scott Chapman, and J. Maurice Rojas.

Chapter 1

Introduction

Modular curves are of fundamental importance in modern number theory and arithmetic geometry. Classically, they are constructed as quotients of the complex upper half-plane by congruence subgroups (notably $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$); these quotients are naturally coarse moduli spaces, whose complex points correspond in a natural way to complex elliptic curves equipped with certain extra structure.

A remarkable result about the classical modular curves is that they have canonical, proper, moduli-theoretic models over $\mathrm{Spec}(\mathbb{Z})$. Specifically, for each positive integer N, there are algebraic stacks $\mathcal{X}_0(N)$, $\mathcal{X}_1(N)$ and $\mathcal{X}(N)$, proper and flat over $\mathrm{Spec}(\mathbb{Z})$, defined as moduli stacks classifying generalized elliptic curves with extra structure, whose coarse moduli spaces $X_0(N)$, $X_1(N)$ and $X(N)$ are proper integral models for the corresponding classical modular curves. This theory is developed in [DR73], [KM85] and [Con07] and will be reviewed in Chapter 2 below.

Originally appearing in the study of string theory, moduli stacks of stable maps from nodal curves to a projective scheme (or even a proper algebraic space) X are natural compactifications of the moduli stacks classifying maps from smooth genus-g curves to X, and are constructed in [Kon95] and [BM96]. If instead one considers stable maps from nodal curves to a stack $\mathcal{X}$, the resulting moduli stacks are often not complete; see Example 3.1.3.

The problem of finding a suitable analogue of the space of stable maps to a stack was addressed by Abramovich and Vistoli in [AV02], where the stack $\mathcal{K}_{g,n}(\mathcal{X})$ of *n-pointed genus-g twisted stable maps* into $\mathcal{X}$ is introduced, for $\mathcal{X}$ a proper tame Deligne-Mumford stack over a base scheme S with a projective coarse moduli space X/S. It is shown in [AV02] that $\mathcal{K}_{g,n}(\mathcal{X})$ is an algebraic stack, proper over the Kontsevich stack $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps into X, giving an appropriate analogue of the usual Kontsevich stack of stable maps when the target is allowed to be a tame stack instead of merely a scheme or algebraic space.

In particular, taking $\mathcal{X} = \mathcal{B}G$ for a finite group G naturally yields a modular compactification of the stack of n-pointed genus-g curves with certain level structure, where the level structure on a curve corresponds to giving a G-torsor over the curve; this is studied extensively in [ACV03]. This of course differs from the approach in [AR12], where the coverings of curves are assumed themselves to be stable curves; here the coverings are generally not

stable curves. In [Pet12] it is shown that in good characteristics, by taking its closure in $\mathcal{K}_{1,1}(\mathcal{B}(\mathbb{Z}/(N))^2)$ we recover the usual compactification of the moduli stack $\mathcal{Y}(N)$ of elliptic curves with full level structure.

However, the algebraic stack $\mathcal{B}G$ is not tame in characteristics dividing $|G|$, so a priori the results of [ACV03] only hold over $\mathbb{Z}[1/|G|]$. The notion of a tame stack is generalized in [AOV08] to algebraic stacks that are not necessarily Deligne-Mumford, and in the subsequent paper [AOV11] the stack $\mathcal{K}_{g,n}(\mathcal{X})$ of twisted stable maps into proper tame algebraic stacks is introduced and shown to be an algebraic stack, proper over $\overline{\mathcal{M}}_{g,n}(X)$. This naturally leads us to replace the finite group G with a group scheme $\mathcal{G}$, agreeing with G over $\mathbb{Z}[1/|G|]$ but such that $\mathcal{B}\mathcal{G}$ is a tame algebraic stack over $\mathrm{Spec}(\mathbb{Z})$; see for example [Abr12]. In particular, the group scheme μ_N is (noncanonically) isomorphic to $\mathbb{Z}/(N)$ over $\mathbb{Z}[1/N, \zeta_N]$, and the classifying stack $\mathcal{B}\mu_N$ (unlike $\mathcal{B}(\mathbb{Z}/(N))$) is a tame stack over $\mathrm{Spec}(\mathbb{Z})$.

The purpose of this dissertation is to record how the results of [ACV03] extend to moduli of elliptic curves with level structure over arbitrary base schemes, using the group scheme μ_N in place of $\mathbb{Z}/(N)$. For example, the moduli stack $\mathcal{Y}_1(N)$ of elliptic curves with $[\Gamma_1(N)]$-structure arises as an open substack in the rigidification $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ (defined below) of $\mathcal{K}_{1,1}(\mathcal{B}\mu_N)$. Explicitly, given an elliptic curve E/S and a $[\Gamma_1(N)]$-structure P on E, via the canonical group scheme isomorphism

$$E \cong \mathrm{Pic}^0_{E/S}$$

over S, we may view P as a "point of exact order N" on $\mathrm{Pic}^0_{E/S}$, determining a group scheme homomorphism

$$\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{E/S}.$$

The stack $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ will be seen to classify certain pairs $(\mathcal{C}, \phi)$ where $\mathcal{C}/S$ is a 1-marked genus-1 twisted stable curve and $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}/S}$ is a group scheme homomorphism, so this construction defines the immersion

$$\mathcal{Y}_1(N) \to \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N).$$

We will define the notion of a $[\Gamma_1(N)]$-structure on a 1-marked genus-1 twisted stable curve, and write $\mathcal{X}_1^{\mathrm{tw}}(N)$ for the stack classifying such data. Our first main result is:

Theorem 1.1.1. *Let S be a scheme and $\mathcal{X}_1^{\mathrm{tw}}(N)$ the stack over S classifying $[\Gamma_1(N)]$-structures on 1-marked genus-1 twisted stable curves with non-stacky marking. Then $\mathcal{X}_1^{\mathrm{tw}}(N)$ is a closed substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$, which contains $\mathcal{Y}_1(N)$ as an open dense substack.*

This will be proved as Theorem 4.2.4 below. The main point is to verify the valuative criterion for properness. We accomplish this by using properness of $\mathcal{X}_1(N)$ and the e_N-pairing on generalized elliptic curves. To complete a family of twisted curves with level structure when the generic fiber is smooth, we first complete it to a family of generalized elliptic curves with level structure. Then we use a quotient construction involving the e_N-pairing to produce a completed family of twisted curves with level structure, modifying our strategy in

characteristics dividing N by exploiting the relationship between cyclotomic torsors and line bundles (Lemma 3.4.2). The same techniques immediately give a corresponding result for $\mathcal{Y}(N)$, which is an open substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$; its closure $\mathcal{X}^{\mathrm{tw}}(N)$ classifies $[\Gamma(N)]$-structures on 1-marked genus-1 twisted stable curves, as we will show in Theorem 5.2.2 below.

Unsurprisingly, it turns out that these closures are isomorphic as algebraic stacks to the stacks $\mathcal{X}_1(N)$ and $\mathcal{X}(N)$ classifying $[\Gamma_1(N)]$-structures and $[\Gamma(N)]$-structures on generalized elliptic curves, as studied in [Con07]:

Theorem 1.1.2. *Over the base $S = \mathrm{Spec}(\mathbb{Z})$, there is a canonical isomorphism of algebraic stacks $\mathcal{X}_1^{\mathrm{tw}}(N) \cong \mathcal{X}_1(N)$ extending the identity map on $\mathcal{Y}_1(N)$, and a canonical isomorphism of algebraic stacks $\mathcal{X}^{\mathrm{tw}}(N) \cong \mathcal{X}(N)$ extending the identity map on $\mathcal{Y}(N)$.*

We will prove this as Theorem 6.1.1 below. We first verify that $\mathcal{X}_1^{\mathrm{tw}}(N)$ and $\mathcal{X}^{\mathrm{tw}}(N)$ are finite over $\overline{\mathcal{M}}_{1,1}$. It then follows from general commutative algebra considerations that these stacks are Cohen-Macaulay over $\mathrm{Spec}(\mathbb{Z})$, at which point we may proceed as in [Con07, §4.1] to identify these stacks with the normalizations of $\overline{\mathcal{M}}_{1,1}$ in $\mathcal{X}_1(N)|_{\mathbb{Z}[1/N]}$ and $\mathcal{X}(N)|_{\mathbb{Z}[1/N]}$. These isomorphisms have natural moduli interpretations, as discussed in Corollaries 6.1.9 and 6.1.10.

The techniques of this paper also yield new moduli interpretations of various moduli stacks of elliptic curves that are not (apparently) contained in a stack of twisted stable maps. These results may be known to experts, but are not recorded in the literature; for completeness we include a careful proof of the modular interpretation of the closure of $\mathcal{Y}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$. We also study how the different moduli stacks of elliptic curves contained in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ and $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$ interact in characteristics dividing N; this easily generalizes an example in [AOV11] but otherwise does not appear in the literature.

In the appendix we recall an example of [CN90] which implies that the techniques of this paper do not generalize nicely to curves of higher genus: using the Katz-Mazur notion of a "full set of sections" we can define a stack over $\mathrm{Spec}(\mathbb{Z})$ classifying genus-g curves with full level N structures on their Jacobians, but this stack is not even flat over $\mathrm{Spec}(\mathbb{Z})$. The corresponding stack over $\mathbb{Z}[1/N]$ is well-behaved, and arises as an open substack of $\overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu_N^{2g})$, but no moduli interpretation for its closure in characteristics dividing N appears to be known.

Chapter 2

Review of moduli of generalized elliptic curves

2.1 Classical level structures on elliptic curves

We begin by reviewing some of the main results of the arithmetic theory of classical modular curves. The theory of moduli spaces of elliptic curves with level structure, in characteristics not dividing the level, was put on a sound footing by Igusa in the papers [Igu59a], [Igu59b], [Igu59c], [Igu68]. We follow the exposition found in [DR73]; for the necessary background on algebraic stacks we refer the reader to [LMB00].

Definition 2.1.1. Fix a positive integer N, let S be a scheme over $\mathrm{Spec}(\mathbb{Z}[1/N])$, and let E/S be an elliptic curve.

A $[\Gamma_1(N)]$-*structure on E* is a section $P \in E(S)$ which has exact order N as an element of the group $E(S)$.

A $[\Gamma_0(N)]$-*structure on E* is a cyclic subgroup $G \subseteq E(S)$ of order N.

A $[\Gamma(N)]$-*structure on E* (or a *full level-N structure on E*) is a pair (P, Q) of sections in $E(S)$ such that the induced group homomorphism

$$(\mathbb{Z}/(N))^2 \to E(S)[N]$$
$$(1, 0) \mapsto P$$
$$(0, 1) \mapsto Q$$

is an isomorphism.

We write $\mathcal{Y}_1(N)_{\mathbb{Z}[1/N]}$ (resp. $\mathcal{Y}_0(N)_{\mathbb{Z}[1/N]}$, $\mathcal{Y}(N)_{\mathbb{Z}[1/N]}$) for the category fibered in groupoids over $\mathrm{Spec}(\mathbb{Z}[1/N])$, associating to a scheme $S \in \mathrm{Sch}/\mathbb{Z}[1/N]$ the groupoid of pairs consisting of an elliptic curve E/S together with a $[\Gamma_1(N)]$-structure (resp. $[\Gamma_0(N)]$-structure, $[\Gamma(N)]$-structure) on E.

Finally, if we fix a primitive N^{th} root of unity ζ, we write $\mathcal{Y}(N)^{\mathrm{can}}_{\mathbb{Z}[1/N]}$ for the category fibered in groupoids over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta])$, associating to S the groupoid of data $(E, (P, Q))$, where E/S is an elliptic curve and (P, Q) is a $[\Gamma(N)]$-structure on E such that $e_N(P, Q) = \zeta$.

Theorem 2.1.2 ([DR73, IV.3.4]). $\mathcal{Y}_1(N)_{\mathbb{Z}[1/N]}$, $\mathcal{Y}_0(N)_{\mathbb{Z}[1/N]}$, $\mathcal{Y}(N)_{\mathbb{Z}[1/N]}$ and $\mathcal{Y}(N)^{\mathrm{can}}_{\mathbb{Z}[1/N]}$ are smooth, separated Deligne-Mumford stacks over $\mathrm{Spec}(\mathbb{Z}[1/N])$.

Analytically, modular curves are constructed as quotients of the upper half-plane $\mathcal{H}$. Specifically, via the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$, we also have natural actions of the congruence subgroups $\Gamma_1(N)$, $\Gamma_0(N)$ and $\Gamma(N)$, and the quotients $\Gamma_1(N)\backslash\mathcal{H}$, $\Gamma_0(N)\backslash\mathcal{H}$ and $\Gamma(N)\backslash\mathcal{H}$ are smooth Riemann surfaces.

Let $Y_1(N)_{\mathbb{Z}[1/N]}$ (resp. $Y_0(N)_{\mathbb{Z}[1/N]}$, $Y(N)^{\mathrm{can}}_{\mathbb{Z}[1/N]}$) denote the coarse moduli space of the corresponding moduli stack. Taking the complex points of these schemes gives us Riemann surfaces $Y_1(N)(\mathbb{C})$, $Y_0(N)(\mathbb{C})$ and $Y(N)^{\mathrm{can}}(\mathbb{C})$.

Theorem 2.1.3 ([DR73, §IV.5]). *There are canonical isomorphisms of Riemann surfaces*

$$Y_1(N)(\mathbb{C}) \cong \Gamma_1(N)\backslash\mathcal{H}$$
$$Y_0(N)(\mathbb{C}) \cong \Gamma_0(N)\backslash\mathcal{H}$$
$$Y(N)^{\mathrm{can}}(\mathbb{C}) \cong \Gamma(N)\backslash\mathcal{H}.$$

However, the stacks $\mathcal{Y}_1(N)_{\mathbb{Z}[1/N]}$, $\mathcal{Y}_0(N)_{\mathbb{Z}[1/N]}$, $\mathcal{Y}(N)_{\mathbb{Z}[1/N]}$ and $\mathcal{Y}(N)^{\mathrm{can}}_{\mathbb{Z}[1/N]}$ (and their coarse moduli spaces, and the associated analytic spaces) are not proper. This is remedied by allowing elliptic curves to degenerate to generalized elliptic curves.

Definition 2.1.4. A *Deligne-Rapoport (DR) semistable curve of genus* 1 over a scheme S is a proper, flat, separated, finitely presented morphism of schemes $f : C \to S$, all of whose geometric fibers are non-empty, connected semistable curves with trivial dualizing sheaves.

By [DR73, §II.1], an equivalent definition for $f : C \to S$ to be a DR semistable curve of genus 1 is that f is a proper flat morphism of finite presentation and relative dimension 1, such that every geometric fiber is either a smooth connected genus-1 curve or a Néron polygon. Recall (loc. cit.) that over a base scheme S, the *standard Néron N-gon C_N/S* (for any $N \geq 1$) is obtained from $\widetilde{C}_N := \mathbb{P}^1_S \times \mathbb{Z}/(N)$ by "gluing" the section 0 in the ith copy of $\mathbb{P}^1_S$ to the section ∞ in the $(i+1)^{\mathrm{th}}$ copy of $\mathbb{P}^1_S$:

Standard Néron N-gon   Standard Néron 1-gon
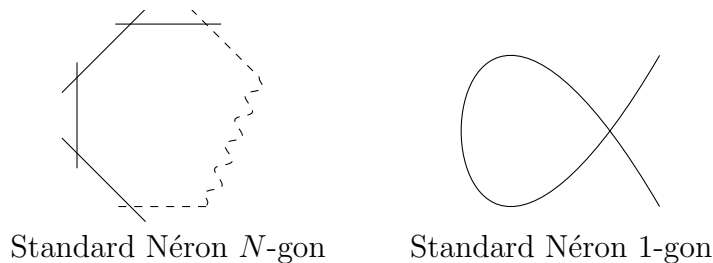
The natural multiplication action of $\mathbb{G}_m$ on $\mathbb{P}^1_S$, together with the action of $\mathbb{Z}/(N)$ on itself via the group law, determines an action of the group scheme $\mathbb{G}_m \times \mathbb{Z}/(N)$ on $\mathbb{P}^1_S \times \mathbb{Z}/(N)$, descending uniquely to an action of $\mathbb{G}_m \times \mathbb{Z}/(N) = C_N^{\mathrm{sm}}$ on C_N ([DR73, II.1.9]).

Definition 2.1.5. A *generalized elliptic curve* over a scheme S is a DR semistable curve E/S of genus 1, equipped with a morphism

$$E^{\mathrm{sm}} \times E \to E$$

and a section

$$0_E \in E^{\mathrm{sm}}(S)$$

such that the restriction

$$E^{\mathrm{sm}} \times E^{\mathrm{sm}} \to E^{\mathrm{sm}}$$

makes E^{sm} a commutative group scheme over S with identity 0_E, and such that on any singular geometric fiber $E_{\bar{s}}$, the translation action by a rational point on $E_{\bar{s}}^{\mathrm{sm}}$ acts by a rotation on the graph $\Gamma(E_{\bar{s}})$ ([DR73, I.3.5]) of the irreducible components of $E_{\bar{s}}$.

By [DR73, II.1.15], over an algebraically closed field a generalized elliptic curve is either a smooth elliptic curve or a Néron N-gon (for some $N \geq 1$) with the action described above. In fact this result says more: for any generalized elliptic curve E/S, there is a locally finite family $(S_N)_{N \geq 1}$ of closed subschemes of S such that $\cup S_N$ is the non-smooth locus in S of $E \to S$, and $E \times_S S_N$ is fppf-locally on S_N isomorphic to the standard Néron N-gon over S_N.

Definition 2.1.6. Fix a positive integer N, let S be a scheme over $\mathrm{Spec}(\mathbb{Z}[1/N])$, and let E/S be a generalized elliptic curve.

A $[\Gamma_1(N)]$-*structure on E* is a section $P \in E^{\mathrm{sm}}(S)$ which has exact order N as an element of the group $E^{\mathrm{sm}}(S)$, and such that the subgroup $\langle P \rangle \subseteq E^{\mathrm{sm}}(S)$ meets every irreducible component of each geometric fiber of $E \to S$.

A $[\Gamma_0(N)]$-*structure on E* is a cyclic subgroup $G \subseteq E^{\mathrm{sm}}(S)$ of order N, which meets every irreducible component of each geometric fiber of $E \to S$.

A $[\Gamma(N)]$-*structure on E* (or a *full level-N structure on E*) is a pair (P, Q) of sections in $E^{\mathrm{sm}}(S)$ such that the induced group homomorphism

$$(\mathbb{Z}/(N))^2 \to E^{\mathrm{sm}}(S)[N]$$
$$(1, 0) \mapsto P$$
$$(0, 1) \mapsto Q$$

is an isomorphism.

We write $\mathcal{X}_1(N)_{\mathbb{Z}[1/N]}$ (resp. $\mathcal{X}_0(N)_{\mathbb{Z}[1/N]}$, $\mathcal{X}(N)_{\mathbb{Z}[1/N]}$) for the category fibered in groupoids over $\mathrm{Spec}(\mathbb{Z}[1/N])$, associating to a scheme $S \in \mathrm{Sch}/\mathbb{Z}[1/N]$ the groupoid of pairs consisting of a generalized elliptic curve E/S together with a $[\Gamma_1(N)]$-structure (resp. $[\Gamma_0(N)]$-structure, $[\Gamma(N)]$-structure) on E.

Finally, if we fix a primitive N^{th} root of unity ζ, we write $\mathcal{Y}(N)_{\mathbb{Z}[1/N]}^{\mathrm{can}}$ for the category fibered in groupoids over $\mathrm{Spec}(\mathbb{Z}[1/N, \zeta])$, associating to S the groupoid of data $(E, (P, Q))$, where E/S is a generalized elliptic curve and (P, Q) is a $[\Gamma(N)]$-structure on E such that $e_N(P, Q) = \zeta$. (Note that the e_N-pairing is well-defined not only for elliptic curves but also for Néron N-gons; see [DR73, 3.21].)

Theorem 2.1.7 ([DR73, IV.3.4]). $\mathcal{X}_1(N)_{\mathbb{Z}[1/N]}$, $\mathcal{X}_0(N)_{\mathbb{Z}[1/N]}$, $\mathcal{X}(N)_{\mathbb{Z}[1/N]}$ *and* $\mathcal{X}(N)_{\mathbb{Z}[1/N]}^{\mathrm{can}}$ *are smooth, proper Deligne-Mumford stacks over* $\mathrm{Spec}(\mathbb{Z}[1/N])$.

Analytically, compactified modular curves are constructed as quotients of the extended upper half-plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Specifically, via the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}^*$, we also have natural actions of the congruence subgroups $\Gamma_1(N)$, $\Gamma_0(N)$ and $\Gamma(N)$, and the quotients $\Gamma_1(N) \backslash \mathcal{H}^*$, $\Gamma_0(N) \backslash \mathcal{H}^*$ and $\Gamma(N) \backslash \mathcal{H}^*$ are smooth, proper Riemann surfaces.

Let $X_1(N)_{\mathbb{Z}[1/N]}$ (resp. $X_0(N)_{\mathbb{Z}[1/N]}$, $X(N)_{\mathbb{Z}[1/N]}^{\mathrm{can}}$) denote the coarse moduli space of the corresponding moduli stack. Taking the complex points of these schemes gives us Riemann surfaces $X_1(N)(\mathbb{C})$, $X_0(N)(\mathbb{C})$ and $X(N)^{\mathrm{can}}(\mathbb{C})$.

Theorem 2.1.8 ([DR73, §IV.5]). *There are canonical isomorphisms of Riemann surfaces*

$$X_1(N)(\mathbb{C}) \cong \Gamma_1(N) \backslash \mathcal{H}^*$$
$$X_0(N)(\mathbb{C}) \cong \Gamma_0(N) \backslash \mathcal{H}^*$$
$$X(N)^{\mathrm{can}}(\mathbb{C}) \cong \Gamma(N) \backslash \mathcal{H}^*.$$

2.2 Drinfeld level structures on generalized elliptic curves

The moduli problems posed in the previous section are well-defined over $\mathrm{Spec}(\mathbb{Z})$, but they lead to poorly behaved algebraic stacks. This is because if E is an elliptic curve over an algebraically closed field k with $\mathrm{char}(k) \neq p$, then $E(k)[p] \cong (\mathbb{Z}/(p))^2$, but if instead $\mathrm{char}(k) = p$ then the group $E(k)[p]$ is isomorphic to either $\mathbb{Z}/(p)$ (if E is ordinary) or is trivial (if E is supersingular). An immediate consequence is that if we were to define a moduli problem of $[\Gamma_1(p)]$-structures over $\mathrm{Spec}(\mathbb{Z})$ using the definitions of level structures in the previous section, the resulting algebraic stacks would not even be flat over $\overline{\mathcal{M}}_{1,1}$; the geometric fibers of the natural morphism to $\overline{\mathcal{M}}_{1,1}$ would have smaller length in characteristic p than in other characteristics.

The idea pursued systematically in [KM85] and [Con07], and originally introduced by Drinfeld in [Dri74], is to replace the naïve notion of level structures in terms of points of order N with more refined level structures defined in terms of Cartier divisors. For now we will focus on the case of $[\Gamma_1(N)]$-structures and $[\Gamma(N)]$-structures; we return to the case of more general level structures in Chapter 7.

Recall that for an S-scheme X, a *relative effective Cartier divisor* in X over S is an effective Cartier divisor in X which is flat over S.

Definition 2.2.1. Let E/S be a generalized elliptic curve. A $[\Gamma_1(N)]$-*structure* on E is a section $P \in E^{\mathrm{sm}}(S)$ such that:

- $N \cdot P = 0_E$;

- the relative effective Cartier divisor

$$D := \sum_{a \in \mathbb{Z}/(N)} [a \cdot P]$$

in E^{sm} is a subgroup scheme; and

- for every geometric point $\overline{p} \to S$, $D_{\overline{p}}$ meets every irreducible component of $E_{\overline{p}}$.

We write $\mathfrak{X}_1(N)$ for the stack over $\mathrm{Spec}(\mathbb{Z})$ associating to a scheme S the groupoid of pairs (E, P), where E/S is a generalized elliptic curve and P is a $[\Gamma_1(N)]$-structure on E. We write $\mathcal{Y}_1(N)$ for the substack classifying such pairs where E/S is a smooth elliptic curve.

Definition 2.2.2. Let E/S be a generalized elliptic curve. A $[\Gamma(N)]$-*structure* on E is an ordered pair (P, Q) of sections in $E^{\mathrm{sm}}[N](S)$ such that:

- the relative effective Cartier divisor

$$D := \sum_{a,b \in \mathbb{Z}/(N)} [a \cdot P + b \cdot Q]$$

in E^{sm} is an N-torsion subgroup scheme, hence $D = E^{\mathrm{sm}}[N]$; and

- for every geometric point $\overline{p} \to S$, $D_{\overline{p}}$ meets every irreducible component of $E_{\overline{p}}$.

We write $\mathfrak{X}(N)$ for the stack over $\mathrm{Spec}(\mathbb{Z})$ associating to a scheme S the groupoid of tuples $(E, (P, Q))$, where E/S is a generalized elliptic curve and (P, Q) is a $[\Gamma(N)]$-structure on E. We write $\mathcal{Y}(N)$ for the substack classifying such tuples where E/S is a smooth elliptic curve.

Definition 2.2.3. Let E/S be a generalized elliptic curve, and let G be a 2-generated finite abelian group, say $G \cong \mathbb{Z}/(n_1) \times \mathbb{Z}/(n_2)$, $n_2 | n_1$. A G-*structure* on E is a homomorphism $\phi : G \to E^{\mathrm{sm}}$ of group schemes over S such that:

- the relative effective Cartier divisor

$$D := \sum_{a \in G} [\phi(a)]$$

in E^{sm} is an n_1-torsion subgroup scheme; and

- for every geometric point $\overline{p} \to S$, $D_{\overline{p}}$ meets every irreducible component of $E_{\overline{p}}$.

Theorem 2.2.4 ([Con07, 3.1.7, 3.2.7, 3.3.1, 4.1.1]). *$\mathfrak{X}_1(N)$ and $\mathfrak{X}(N)$ are regular Deligne-Mumford stacks, proper and flat over $\mathrm{Spec}(\mathbb{Z})$ of pure relative dimension* 1.

In particular, it follows (cf. [Con07, 4.1.5]) that $\mathfrak{X}_1(N)$ (resp. $\mathfrak{X}(N)$) is canonically identified with the normalization of $\overline{\mathcal{M}}_{1,1}$ in the normal Deligne-Mumford stack $\mathfrak{X}_1(N)|_{\mathbb{Z}[1/N]}$ (resp. $\mathfrak{X}(N)|_{\mathbb{Z}[1/N]}$), as in [KM85, §8.6] and [DR73, §VII.2].

2.3   Reductions mod p of the moduli stacks

It will be useful for us to have a description of the "reduction mod p" of the stacks $\mathfrak{X}_1(N)$ and $\mathfrak{X}(N)$ for primes p dividing N. These reductions are described using the "crossings theorem" ([KM85, 13.1.3]) which we now recall.

We work over a fixed field k. Let Y/k be a smooth curve, and let $X \to Y$ be finite and flat. Suppose there is a non-empty finite set of k-rational points of Y (which we will call the *supersingular points* of Y) such that for each supersingular point y_0, X_{y_0} is a single k-rational point, and $\widehat{\mathcal{O}}_{X,x_0} \cong k[\![x,y]\!]/(f)$ for some f (depending on y_0).

Also suppose we have a finite collection of closed immersions $\{Z_i \hookrightarrow X\}_{i=1}^n$, where each Z_i is finite and flat over Y, with Z_i^{red} a smooth curve over k, such that the morphism $\sqcup Z_i \to X$ is an isomorphism over the non-supersingular locus of Y, and such that for each i and each supersingular point $y_0 \in Y$, $(Z_i)_{y_0}$ is a single k-rational point.

Theorem 2.3.1 (Crossings Theorem, [KM85, 13.1.3]). *Under the above assumptions, let $y_0 \in Y$ be a supersingular point and $x_0 = X_{y_0}$. Then*

$$\widehat{\mathcal{O}}_{X,x_0} \cong k[\![x,y]\!]/\prod_{i=1}^m f_i^{e_i},$$

where each f_i is irreducible in $k[\![x,y]\!]$, each f_i and f_j ($i \neq j$) are distinct in $k[\![x,y]\!]$ modulo multiplication by units, and for $z_{i,0} := (Z_i)_{y_0}$ we have

$$\widehat{\mathcal{O}}_{Z_i,z_{i,0}} \cong k[\![x,y]\!]/(f_i^{e_i}).$$

If Y is (geometrically) connected then each Z_i is (geometrically) connected, in which case $\{Z_i\}_{i=1}^n$ is the set of irreducible components of X.

Definition 2.3.2. Under the above assumptions, we say that X is the *disjoint union with crossings at the supersingular points* of the closed subschemes $\{Z_i\}_{i=1}^n$.

If $\mathcal{X} \to \mathcal{Y}$ is a finite (hence representable) flat morphism of algebraic stacks over k, with $\mathcal{Y}$ Deligne-Mumford (hence $\mathcal{X}$ is also Deligne-Mumford), and $\{\mathcal{Z}_i \hookrightarrow \mathcal{X}\}_{i=1}^n$ is a finite collection of closed immersions of algebraic stacks, such that for some étale surjection $Y' \to \mathcal{Y}$ with Y' a scheme, the algebraic spaces Y', $\mathcal{X} \times_{\mathcal{Y}} Y'$, and $\{\mathcal{Z}_i \times_{\mathcal{Y}} Y'\}_{i=1}^n$ are schemes satisfying the assumptions of the Crossings Theorem, we say that $\mathcal{X}$ is the *disjoint union with crossings at the supersingular points* of the closed substacks $\{\mathcal{Z}_i\}_{i=1}^n$.

Remark 2.3.3. We will solely be applying the above theorem in the case $\mathcal{Y} = \overline{\mathcal{M}}_{1,1}$, with $\mathcal{X}$ an algebraic stack known to be quasi-finite and proper over $\overline{\mathcal{M}}_{1,1}$. $\overline{\mathcal{M}}_{1,1}$ is a Deligne-Mumford stack with separated diagonal, so once we know that $\mathcal{X} \to \overline{\mathcal{M}}_{1,1}$ is representable, it follows from [Knu71, II.6.16] that the algebraic spaces $\mathcal{X} \times_{\overline{\mathcal{M}}_{1,1}} Y'$ and $\{\mathcal{Z}_i \times_{\overline{\mathcal{M}}_{1,1}} Y'\}_{i=1}^n$ are schemes.

Definition 2.3.4. Let $S \in \mathrm{Sch}/\mathbb{F}_p$. Let E/S be an elliptic curve, with relative Frobenius $F : E \to E^{(p)}$. Let $G \subset E$ be a finite, locally free S-subgroup scheme of rank p^n, $n \geq 1$. For integers $a, b \geq 0$ with $a + b = n$, we say that G is an (a, b)-*subgroup* if $\ker(F^a) \subset G$, and if in the resulting factorization of $E \to E/G$ as

$$E \xrightarrow{F^a} E^{(p^a)} \xrightarrow{\pi} E/G$$

we have $\ker(\widehat{\pi}) = \ker(F^b_{E/G})$ (where $\widehat{\pi}$ denotes the dual isogeny and

$$F_{E/G} : E/G \to (E/G)^{(p)}$$

is the relative Frobenius). In particular note that

$$E^{(p^a)} \cong (E/G)^{(p^b)}.$$

We say that G is an (a, b)-*cyclic subgroup* if it is an (a, b)-subgroup, and either $a = 0$, $b = 0$, or there exists a closed subscheme $Z \subset S$ defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ with $\mathcal{I}^{p-1} = 0$, such that the isomorphism

$$E^{(p^a)}|_Z \cong (E/G)^{(p^b)}|_Z$$

is induced by an isomorphism

$$E^{(p^{a-1})}|_Z \cong (E/G)^{(p^{b-1})}|_Z.$$

Finally, a $[\Gamma_1(p^n)]$-(a, b)-*cyclic structure* on E is a $[\Gamma_1(p^n)]$-structure $P \in E[p^n](S)$ such that the S-subgroup scheme

$$D := \sum_{m \in \mathbb{Z}/(p^n)} [m \cdot P]$$

in E is an (a, b)-cyclic subgroup of E. We write $\mathcal{Y}_1(p^n)^{(a,b)}_S$ for the substack of $\mathcal{Y}_1(p^n)_S$ associating to a scheme T/S the groupoid of pairs (E, P), where E/T is an elliptic curve and P is a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure on E.

If E is an ordinary elliptic curve over a scheme S of characteristic p, it is considerably easier to describe what is meant by a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure on E. Namely, it is just a $[\Gamma_1(p^n)]$-structure P on E such that the relative effective Cartier divisor

$$D_b := \sum_{m=1}^{p^b} [m \cdot P]$$

is a subgroup scheme of E which is étale over S. Note that for an arbitrary $[\Gamma_1(p^n)]$-structure P on E, D_b will not generally be a subgroup scheme of E if $b < n$, and even if it is a subgroup scheme it might not be étale over S. Being a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure

on E (for E ordinary) means that for any geometric fiber $E_{\bar{s}}$, $P_{\bar{s}}$ has exact order p^b as an element of the group $E_{\bar{s}}[p^n](k(\bar{s})) \cong \mathbb{Z}/(p^n)$.

The following elementary result will be required when we study the interactions in characteristic p of different moduli stacks of elliptic curves contained in a moduli stack of twisted stable maps. We include a proof for lack of a suitable reference.

Lemma 2.3.5. *Let $E/S/\mathbb{F}_p$ be an elliptic curve, and let P be a $[\Gamma_1(p^n)]$-(a,b)-cyclic structure on E. Then P is also a $[\Gamma_1(p^{n+1})]$-structure on E, and is $[\Gamma_1(p^{n+1})]$-$(a+1,b)$-cyclic.*

Proof. Consider the relative effective Cartier divisor

$$G := \sum_{m \in \mathbb{Z}/(p^n)} [m \cdot P]$$

in E. This is an S-subgroup scheme containing $\ker(F^a)$, and in the resulting factorization of the quotient map $E \to E/G$ as

$$E \xrightarrow{F^a} E^{(p^a)} \xrightarrow{\pi} E/G$$

we have $\ker(\widehat{\pi}) = \ker(F^b_{E/G})$.

Consider also the relative effective Cartier divisor

$$G' := \sum_{m \in \mathbb{Z}/(p^{n+1})} [m \cdot P]$$

in E. G is a subgroup of G', and the image of G' in E/G is the relative effective Cartier divisor

$$\sum_{m \in \mathbb{Z}/(p)} [m \cdot 0_{E/G}].$$

This is simply the kernel of

$$F_{E/G} : E/G \to (E/G)^{(p)},$$

so we may identify E/G' with $(E/G)^{(p)}$ and the quotient map $E/G \to E/G'$ with the relative Frobenius

$$F_{E/G} : E/G \to (E/G)^{(p)}.$$

In particular, the quotient map $E \to E/G'$ is a cyclic p^{n+1}-isogeny of elliptic curves with kernel generated by P, so we may already conclude that P is a $[\Gamma_1(p^{n+1})]$-structure on E.

Now, since $G' = p \cdot G$ as Cartier divisors, the fact that $\ker(F^a) \subset G$ implies that $\ker(F^{a+1}) \subset G'$. Factor the quotient map $E \to E/G' = (E/G)^{(p)}$ as

$$E \xrightarrow{F^{a+1}} E^{(p^{a+1})} \xrightarrow{\pi'} (E/G)^{(p)}.$$

We have a diagram

$$
\begin{array}{ccc}
E \xrightarrow{\ F^a\ } E^{(p^a)} \xrightarrow{\ \pi\ } E/G \\
\end{array}
$$

$$
\begin{array}{ccccc}
E & \xrightarrow{\ F^a\ } & E^{(p^a)} & \xrightarrow{\ \pi\ } & E/G \\
 & \searrow^{F^{a+1}} & \downarrow^{F} & & \downarrow^{F_{E/G}} \\
 & & E^{(p^{a+1})} & \xrightarrow{\ \pi'\ } & (E/G)^{(p)}
\end{array}
$$

where the composite $E \to (E/G)^{(p)}$ is the natural quotient map $E \to E/G'$. The outer arrows and the left-hand triangle commute, hence the square commutes as well since $F^a : E \to E^{(p^a)}$ is an epimorphism.

Now consider the following diagram:

$$
\begin{array}{ccccc}
E^{(p^a)} & \xrightarrow{\ \pi\ } & E/G & \xrightarrow{\ \widehat{\pi}\ } & E^{(p^a)} \\
\downarrow^{F} & & \downarrow & & \downarrow^{F} \\
E^{(p^{a+1})} & \xrightarrow{\ \pi'\ } & (E/G)^{(p)} & \xrightarrow{\ \widehat{\pi}'\ } & E^{(p^{a+1})}
\end{array}
$$

The outer rectangle commutes since the horizontal composites are the isogenies $[\deg(\pi)]$ and $[\deg(\pi')]$ (and $\deg(\pi) = \deg(\pi')$), and we have shown that the left-hand square commutes, so the right-hand square commutes as well. Since $\ker(\widehat{\pi}) = \ker(F^b_{E/G})$, we conclude that $\ker(\widehat{\pi}') = \ker(F^b_{(E/G)^{(p)}})$.

Finally, suppose $a, b \geq 1$. Since $E^{(p^a)} \cong (E/G)^{(p^b)}$ we also have

$$
E^{(p^{a+1})} \cong (E/G)^{(p^{b+1})} = ((E/G)^{(p)})^{(p^b)},
$$

and this is already induced by the isomorphism

$$
E^{(p^a)} \cong (E/G)^{(p^b)} = ((E/G)^{(p)})^{(p^{b-1})}.
$$

Therefore P is a $[\Gamma_1(p^{n+1})]$-$(a+1, b)$-cyclic structure on E. $\qquad\square$

Remark 2.3.6. The reason the above argument fails when p is invertible on the base scheme S is that, preserving the notation of the above proof, the image of the relative Cartier divisor G' in E/G is not a subgroup of E/G unless S is an $\mathbb{F}_p$-scheme. Indeed, the image of G' in E/G is the relative effective Cartier divisor $p \cdot [0_{E/G}]$, which is finite flat of rank p over S. If p is invertible on S, any finite flat commutative group scheme of rank p over S is étale (cf. [Sha86, Cor. 4.3]), but $p \cdot [0_{E/G}]$ is obviously not étale.

It is straightforward to extend the definition of a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure, and the result of the above lemma, to the case of generalized elliptic curves:

Definition 2.3.7. Let $E/S/\mathbb{F}_p$ be a generalized elliptic curve; let $S_1 \subset S$ be the open locus where the morphism $E \to S$ is smooth, and let $S_2 \subset S$ be the complement of the closed locus in S where the geometric fibers are supersingular elliptic curves. We say that a $[\Gamma_1(p^n)]$-structure P on E/S is a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure if:

- $P|_{S_1}$ is a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure on the elliptic curve E_{S_1}/S_1, and
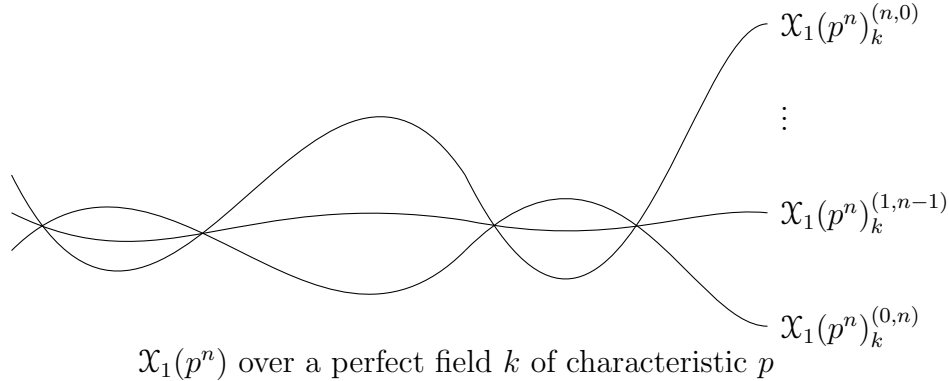
- the relative effective Cartier divisor

$$D_b := \sum_{m=1}^{p^b} [m \cdot P|_{S_2}]$$

  in $E^{\mathrm{sm}}|_{S_2}$ is a subgroup scheme of $E^{\mathrm{sm}}|_{S_2}$ which is étale over S_2.

We write $\mathfrak{X}_1(p^n)_S^{(a,b)}$ for the substack of $\mathfrak{X}_1(p^n)_S$ associating to a scheme T/S the groupoid of pairs (E, P), where E/T is a generalized elliptic curve and P is a $[\Gamma_1(p^n)]$-(a, b)-cyclic structure on E.

Theorem 2.3.8 ([KM85, 13.5.4]). *Let k be a perfect field of characteristic p. $\mathcal{Y}_1(p^n)_k$ (resp. $\mathfrak{X}_1(p^n)_k$) is the disjoint union, with crossings at the supersingular points, of the $n+1$ substacks $\mathcal{Y}_1(p^n)_k^{(a,n-a)}$ (resp. $\mathfrak{X}_1(p^n)_k^{(a,n-a)}$) for $0 \le a \le n$.*

It can be helpful to visualize $\mathfrak{X}_1(p^n)_k$ as follows:

$$\mathfrak{X}_1(p^n)_k^{(n,0)}$$
$$\vdots$$
$$\mathfrak{X}_1(p^n)_k^{(1,n-1)}$$
$$\mathfrak{X}_1(p^n)_k^{(0,n)}$$

$\mathfrak{X}_1(p^n)$ over a perfect field k of characteristic p

Definition 2.3.9. Let $S \in \mathrm{Sch}/\mathbb{F}_p$, and let $H \le (\mathbb{Z}/(p^n))^2$ such that $H \cong \mathbb{Z}/(p^n)$ (hence also $(\mathbb{Z}/(p^n))^2/H \cong \mathbb{Z}/(p^n)$). Let E/S be a generalized elliptic curve. Let (P, Q) be a $[\Gamma(p^n)]$-structure on E, corresponding to a group homomorphism

$$\phi : (\mathbb{Z}/(p^n))^2 \to E[p^n]$$
$$(1, 0) \mapsto P$$
$$(0, 1) \mapsto Q.$$

We say that (P, Q) has *component label* H if:

- $\phi(H) \subseteq \ker(F^n)$, where $F : E \to E^{(p)}$ is the relative Frobenius, and

- the resulting group scheme homomorphism

$$\mathbb{Z}/(p^n) \cong (\mathbb{Z}/(p^n))^2/H \to E^{(p^n)} = E/\ker(F^n)$$

is a $[\Gamma_1(p^n)]$-structure on $E^{(p^n)}$. (Note that this is independent of the choice of iso-morphism $(\mathbb{Z}/(p^n))^2/H \cong \mathbb{Z}/(p^n)$, although the $[\Gamma_1(p^n)]$-structure obtained depends on this choice.)

We define $\mathcal{Y}(p^n)_S^H$ (resp. $\mathcal{X}(p^n)_S^H$) to be the substack of $\mathcal{Y}(p^n)_S$ (resp. $\mathcal{X}(p^n)_S$) associating to a scheme T/S the groupoid of tuples $(E, (P, Q))$, where E/T is an elliptic curve (resp. generalized elliptic curve) and (P, Q) is a $[\Gamma(p^n)]$-structure on E of component label H.

Theorem 2.3.10 ([KM85, 13.7.6]). *Let k be a perfect field of characteristic p. $\mathcal{Y}(p^n)_k$ (resp. $\mathcal{X}(p^n)_k$) is the disjoint union, with crossings at the supersingular points, of the substacks $\mathcal{Y}(p^n)_k^H$ (resp. $\mathcal{X}(p^n)_k^H$) for $H \leq (\mathbb{Z}/(p^n))^2$ with $H \cong \mathbb{Z}/(p^n)$.*

The proof of the above theorem immediately generalizes to a slightly more general setting which we will find useful when studying compactified stacks of $[\Gamma(N)]$-structures below. Let $K \leq (\mathbb{Z}/(p^n))^2$, and write $G_K = (\mathbb{Z}/(p^n))^2/K$. Then there exist integers $m \geq l \geq 0$ with

$$G_K \cong \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l).$$

Suppose that $l \geq 1$. Given a G_K-structure ϕ on an ordinary elliptic curve $E/T/\mathbb{F}_p$ (in the sense of Definition 2.2.3), étale locally on T we can consider the composite

$$G_K = \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l) \xrightarrow{\phi} E[p^n] \cong \mathbb{Z}/(p^n) \times \mu_{p^n} \xrightarrow{\pi_1} \mathbb{Z}/(p^n).$$

Since ϕ is a G_K-structure, the kernel and image of this composite are necessarily cyclic. The same argument used in [KM85, 13.7.6] to prove the above theorem shows that in character-istic p, $\mathcal{Y}_K$ breaks up into a union of substacks indexed in this way by the possible kernels of group homomorphisms $G_K \to \mathbb{Z}/(p^n)$, subject to the condition on the image just described. So we can describe $\mathcal{Y}_K$ as a union of closed substacks indexed by the set

$$L_K := \{H \leq G_K | H \text{ and } G/H \text{ are both cyclic}\}.$$

Since $G_K \cong \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l)$ with $m \geq l \geq 1$, we have that $H \in L_K$ if and only if $H \cong \mathbb{Z}/(p^m)$ or $H \cong \mathbb{Z}/(p^l)$.

The rigorous definition, accounting for elliptic curves that might not be ordinary and for the case where G_K is cyclic, is as follows:

Definition 2.3.11. Let $K \leq (\mathbb{Z}/(N))^2$ with corresponding quotient $G_K = (\mathbb{Z}/(N))^2/K$. We write $\mathcal{Y}_K$ (resp. $\mathcal{X}_K$) for the algebraic stack over $\mathrm{Spec}(\mathbb{Z})$ associating to a scheme S the groupoid of pairs (E, ϕ), where E/S is an elliptic curve (resp. generalized elliptic curve) and ϕ is a G_K-structure on E.

If S is an $\mathbb{F}_p$-scheme, $N = p^n$ and $G_K \cong \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l)$ with $m \geq l \geq 1$, then for any $H \in L_K$, we define $\mathcal{Y}_{K,S}^H \subset \mathcal{Y}_{K,S}$ (resp. $\mathcal{X}_{K,S}^H \subset \mathcal{X}_{K,S}$) to be the substack associating to a scheme T/S the groupoid of pairs (E, ϕ), where E/T is an elliptic curve (resp. generalized elliptic curve) and $\phi : G_K \to E[p^m]$ is a G_K-structure such that:

- $\phi(H) \subset \ker(F^m)$, where $F^m : E \to E^{(p^m)}$ is the m-fold relative Frobenius on E, and

- the resulting group scheme homomorphism

$$G_K/H \to E/\ker(F^m) \cong E^{(p^m)}$$

is a G_K/H-structure on $E^{(p^m)}$ in the sense of [KM85, §1.5].

In this case we say that the G_K-structure ϕ has *component label H*.

If $G_K \cong \mathbb{Z}/(p^m)$ (i.e. $l = 0$), the stack $\mathcal{Y}_K$ (resp. $\mathcal{X}_K$) is isomorphic to $\mathcal{Y}_1(p^m)$ (resp. $\mathcal{X}_1(p^m)$), and for $H \cong \mathbb{Z}/(p^a) \in L_K$ and $S \in \mathrm{Sch}/\mathbb{F}_p$ we define $\mathcal{Y}_{K,S}^H$ (resp. $\mathcal{X}_{K,S}^H$) to be the substack $\mathcal{Y}_1(p^m)_S^{(a,m-a)} \subset \mathcal{Y}_1(p^m)_S$ (resp. $\mathcal{X}_1(p^m)_S^{(a,m-a)} \subset \mathcal{X}_1(p^m)_S$), as in Definition 2.3.4. We still say that $\mathcal{Y}_{K,S}^H$ and $\mathcal{X}_{K,S}^H$ classify G_K-structures of *component label H*.

The result is:

Theorem 2.3.12. *Let k be a perfect field of characteristic p. $\mathcal{Y}_{K,k}$ (resp. $\mathcal{X}_{K,k}$) is the disjoint union, with crossings at the supersingular points, of the closed substacks $\mathcal{Y}_{K,k}^H$ (resp. $\mathcal{X}_{K,k}^H$) for $H \in L_K$.*

We also note that an analogue of Lemma 2.3.5 holds in this case, which we record for future use. It is clear that if

$$K' \leq K \leq (\mathbb{Z}/(p^n))^2,$$

giving a canonical quotient map

$$\pi : G_{K'} \to G_K,$$

and if

$$\phi : G_K \to E^{\mathrm{sm}}$$

is a G_K-structure on a generalized elliptic curve $E/S/\mathbb{F}_p$, then a necessary condition for the composite

$$\phi \circ \pi : G_{K'} \to E^{\mathrm{sm}}$$

to be a $G_{K'}$-structure is that $\ker(\pi)$ is cyclic. In fact, unwinding the definitions we immediately deduce:

Lemma 2.3.13. *Let $E/S/\mathbb{F}_p$ be a generalized elliptic curve with no supersingular fibers. Let*

$$K' \leq K \leq (\mathbb{Z}/(p^n))^2$$

such that the canonical quotient

$$\pi : G_{K'} \to G_K$$

has cyclic kernel. Let $H \in L_K$ and suppose that

$$\phi : G_K \to E$$

is a G_K-structure with component label H. Then $\phi \circ \pi$ is a $G_{K'}$ structure on E if and only if $\pi^{-1}(H) \subseteq G_{K'}$ is cyclic, in which case $\phi \circ \pi$ has component label $\pi^{-1}(H) \in L_{K'}$.

Chapter 3

Generalities/review of twisted stable maps

3.1 Stable maps

Moduli stacks of stable maps to projective schemes are constructed in [Kon95] and [BM96]. We will require the general notion of a stable map to an algebraic space, as discussed in [AV02, 8.4]:

Definition 3.1.1. Let X be a proper, finitely presented algebraic space over a scheme S. A *stable map* from an n-marked nodal curve $(C/S, p_1, ..., p_n)$ to X is an S-morphism $f : C \to X$ such that for every geometric point $\overline{s} \to S$ and every irreducible component $E \subseteq C_{\overline{s}}$ collapsed to a point by f, we have

$$2p_a(E) - 2 + d(E) + n > 0,$$

where $d(E)$ is the number of points of the normalization E^ν lying over $C_{\overline{s}}^{\text{sing}}$. (If X/S is projective, this simply says that the line bundle $\omega_C(\sum p_i)$ is f-ample.)

If X is projective over a field, the *target class* of f is the class of $f_*[C]$ in the Chow group of one-dimensional cycles of X up to numerical equivalence.

We write $\overline{\mathcal{M}}_{g,n}(X)$ for the category fibered in groupoids over S, associating to an S-scheme T the groupoid of stable maps from n-marked, genus-g nodal curves over T to X_T. If X is projective over a field, we write $\overline{\mathcal{M}}_{g,n}(X, \beta)$ for the substack classifying stable maps with target class β.

Theorem 3.1.2 ([AV02, 8.4])**.** *Let X be a proper, finitely presented algebraic space over a scheme S. Then $\overline{\mathcal{M}}_{g,n}(X)$ is an algebraic stack, locally finitely presented over S and with finite inertia. If X is projective over a field k, then $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is an open and closed substack of $\overline{\mathcal{M}}_{g,n}(X)$ and is proper over k.*

In fact, the argument of [AV02, 8.4] shows that for any proper, finitely presented algebraic space X/S, the algebraic stack $\overline{\mathcal{M}}_{g,n}(X)$ is complete, i.e. satisfies the valuative criterion for

properness. (However, it is generally not of finite type over the base S, and therefore not proper.)

A natural next step is to attempt to generalize the notion of a stable map when the target is not merely a scheme or algebraic space, but perhaps an algebraic stack $\mathcal{X}/S$. A naïve definition for a stable map to $\mathcal{X}$ would be an S-morphism $f : (C, \{p_i\}) \to \mathcal{X}$ such that the composite with the coarse space map $\mathcal{X} \to X$ gives a stable map in the above sense. However, the moduli stack of n-marked genus-g stable maps to a stack $\mathcal{X}$ is generally not complete, as we illustrate in the following example.

Example 3.1.3. Let $S = \mathrm{Spec}(\mathbb{C})$, $G = (\mathbb{Z}/(2))^2$ and $\mathcal{X} = \mathcal{B}_S G$. Since the coarse space of $\mathcal{X}$ is S, a morphism from an n-marked genus-g nodal curve C to $\mathcal{X}$ is stable (in the sense of the naïve definition above) if and only if C is a stable curve.

Let $T = \mathrm{Spec}(\mathbb{C}[\![t]\!])$, and let C/T be a family of 1-marked genus-1 stable curves with smooth generic fiber C_η and singular closed fiber C_s (so C_η is an elliptic curve, and C_s is a rational curve with one node).

Fix an isomorphism $C_\eta[2] \cong G$. Then the quotient map

$$C_\eta \to [C_\eta/G] \cong C_\eta$$

makes $P_\eta := C_\eta$ a G-torsor over C_η, corresponding to a stable map $C_\eta \to \mathcal{X}$. This does not extend to a stable map $C \to \mathcal{X}$ (and therefore the stack of naïve 1-marked genus-1 stable maps to $\mathcal{X}$ is not proper).

Indeed, suppose $C \to \mathcal{X}$ is such an extension. This corresponds to a G-torsor $P \to C$, and since $P_\eta = C_\eta$ is connected, P_s is also connected. But a connected G-cover of C_s corresponds to a surjection of abelian groups $\pi_1(C_s) \to G$. Since $\pi_1(C_s) \cong \mathbb{Z}$ and $G = (\mathbb{Z}/(2))^2$, this is impossible.

Therefore, if the goal is to generalize stable maps to the case where the target is a stack, the naïve definition proposed above would not be sufficient. This problem is remedied by introducing twisted stable maps.

3.2 Twisted stable maps

Linearly reductive group schemes and the related notion of a tame Artin stack (introduced in [AOV08]) will play an indispensable role in what follows, so we quickly recall the definitions.

Definition 3.2.1 ([AOV08, 2.2]). Let G be a group scheme over a scheme S. Write $\mathrm{QCoh}^G(S)$ for the category of G-equivariant quasi-coherent sheaves on S; writing $\mathcal{B}G$ for the classifying stack of G over S, this is equivalent to $\mathrm{QCoh}(\mathcal{B}G)$ ([AOV08, §2.1]). We say that G is *linearly reductive* if the functor

$$\mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$$
$$\mathcal{F} \mapsto \mathcal{F}^G$$

is exact, or equivalently if the pushforward

$$\text{QCoh}(\mathcal{B}G) \to \text{QCoh}(S)$$

is exact.

Linearly reductive group schemes are classified in [AOV08, §2.3]. The examples in which we are most interested for this paper are the finite flat commutative linearly reductive group schemes μ_N and μ_N^2 over S.

Definition 3.2.2 ([AOV08, 3.1]). Let $\mathcal{X}$ be a locally finitely presented algebraic stack over a scheme S, with finite inertia. By [KM97], $\mathcal{X}$ has a coarse moduli space $\rho : \mathcal{X} \to X$, with ρ proper. We say that $\mathcal{X}/S$ is *tame* if

$$\rho_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X)$$

is exact.

As observed in [AOV08, §3], for any finite flat group scheme G over a scheme S, the classifying stack $\mathcal{B}G$ over S is tame if and only if G is linearly reductive. So in particular the classifying stacks $\mathcal{B}\mu_N$ and $\mathcal{B}\mu_N^2$ are always tame.

Definition 3.2.3 ([AOV11, §2]). An *n-marked twisted curve* over a scheme S is a proper tame stack $\mathcal{C}$ over S, with connected dimension 1 geometric fibers, and coarse space $f : C \to S$ a nodal curve over S; together with n closed substacks

$$\{\Sigma_i \subset \mathcal{C}\}_{i=1}^n$$

which are fppf gerbes over S mapping to n markings $\{p_i \in C^{\text{sm}}(S)\}$, such that:

- the preimage in $\mathcal{C}$ of the complement $C' \subset C$ of the markings and singular locus of C/S maps isomorphically onto C';

- if $\bar{p} \to C$ is a geometric point mapping to the image in C of a marking $\Sigma_i \subset \mathcal{C}$, then

$$\text{Spec}(\mathcal{O}_{C,\bar{p}}) \times_C \mathcal{C} \simeq [D^{\text{sh}}/\mu_r]$$

  for some $r \geq 1$, where D^{sh} is the strict Henselization at $(\mathfrak{m}_{S,f(\bar{p})}, z)$ of

$$D = \text{Spec}(\mathcal{O}_{S,f(\bar{p})}[z])$$

  and $\zeta \in \mu_r$ acts by $z \mapsto \zeta \cdot z$; and

- if $\bar{p} \to C$ is a geometric point mapping to a node of C, then

$$\text{Spec}(\mathcal{O}_{C,\bar{p}}) \times_C \mathcal{C} \simeq [D^{\text{sh}}/\mu_r]$$

  for some $r \geq 1$, where D^{sh} is the strict Henselization at $(\mathfrak{m}_{S,f(\bar{p})}, x, y)$ of

$$D = \text{Spec}(\mathcal{O}_{S,f(\bar{p})}[x, y]/(xy - t))$$

  for some $t \in \mathfrak{m}_{S,f(\bar{p})}$, and $\zeta \in \mu_r$ acts by $x \mapsto \zeta \cdot x$ and $y \mapsto \zeta^{-1} \cdot y$.

We say a twisted curve $\mathcal{C}/S$ has *genus g* if the geometric fibers of its coarse space C/S have arithmetic genus g, and we say an n-marked genus-g twisted curve $\mathcal{C}/S$ is *stable* if the genus-g curve C/S with the markings $\{p_i\}$ is an n-marked genus-g stable curve over S.

Twisted curves also admit an appealing interpretation in terms of logarithmic geometry; a twisted curve $\mathcal{C}/S$ with coarse space C/S is naturally a stack over S classifying certain logarithmic structures on C. We will not require this in what follows, but we refer the interested reader to [Ols07].

Example 3.2.4. Over any base scheme S, consider a Néron 1-gon C/S as in Chapter 2. We have $C^{\mathrm{sm}} \cong \mathbb{G}_m$, and C admits the structure of a generalized elliptic curve with an action

$$C^{\mathrm{sm}} \times C \to C$$

extending the group scheme structure of $\mathbb{G}_m$. For any positive integer N, the inclusion $\mu_N \subset \mathbb{G}_m$ determines an action of μ_N on C, and the stack quotient
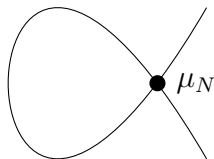
$$\mathcal{C} := [C/\mu_N]$$

is a twisted stable curve over S, with coarse space $f : C' \to S$ again a Néron 1-gon. If $\bar{p} \to C'$ is a geometric point mapping to a node of C', then

$$\mathrm{Spec}(\mathcal{O}_{C',\bar{p}}) \times_{C'} \mathcal{C} \simeq [D^{\mathrm{sh}}/\mu_N],$$

where D^{sh} denotes the strict Henselization of

$$D := \mathrm{Spec}(\mathcal{O}_{S,f(\bar{p})}[x,y]/(xy))$$

at the point $(\mathfrak{m}_{S,f(\bar{p})}, x, y)$ and $\zeta \in \mu_N$ acts by $x \mapsto \zeta \cdot x$ and $y \mapsto \zeta^{-1} \cdot y$. We will refer to this twisted curve as the *standard μ_N-stacky Néron 1-gon over S*.

Standard μ_N-stacky Néron 1-gon

Definition 3.2.5 ([AOV11, §4]). Let $\mathcal{X}$ be a finitely presented algebraic stack, proper over a scheme S and with finite inertia. A *twisted stable map* to $\mathcal{X}$ from an n-marked twisted curve $(\mathcal{C}/S, \{\Sigma_i\})$ over S with coarse space $(C/S, \{p_i\})$ is a morphism $\mathcal{C} \to \mathcal{X}$ of stacks over S such that:

- $\mathcal{C} \to \mathcal{X}$ is a representable morphism, and
- the induced map $C \to X$ is a stable map from $(C, \{p_i\})$ to X.

Over any base scheme S, we write $\mathcal{K}_{g,n}(\mathcal{X})$ for the stack over S associating to a scheme T/S the groupoid of pairs $(\mathcal{C}, \alpha)$, where $\mathcal{C}/T$ is an n-marked twisted curve whose coarse space C/T is a genus-g nodal curve, and $\alpha : \mathcal{C} \to \mathcal{X} \times_S T$ is a twisted stable map.

Proposition 3.2.6 ([AOV11, 4.2]). *For $\mathcal{X}$ as above, $\mathcal{K}_{g,n}(\mathcal{X})$ is a locally finitely presented algebraic stack over S.*

There is a natural morphism of stacks

$$\mathcal{K}_{g,n}(\mathcal{X}) \to \overline{\mathcal{M}}_{g,n}(X)$$

defined by passing to coarse spaces. In particular, note that since a twisted stable map is required to be respresentable, if $\mathcal{X}$ is representable then this map is an equality.

Theorem 3.2.7 ([AOV11, 4.3]). *Let $\mathcal{X}$ be as above, and also assume that $\mathcal{X}$ is tame. Then $\mathcal{K}_{g,n}(\mathcal{X})$ is proper and quasi-finite over $\overline{\mathcal{M}}_{g,n}(X)$.*

A quick word of caution: the twisted stable maps of [AOV11] are a generalization of what are referred to as *balanced* twisted stable maps in [AV02] and [ACV03]; we are only interested in balanced maps for this paper, so hopefully our notation $\mathcal{K}_{g,n}(\mathcal{X})$ will not cause any confusion.

Let $(\mathcal{C}/S, \{\Sigma_i\})$ be an n-marked twisted curve with coarse space C/S. If $\bar{p} \to C$ is a geometric point mapping to the image in C of the gerbe Σ_i, such that

$$\mathrm{Spec}(\mathcal{O}_{C,\bar{p}}) \times_C \mathcal{C} \simeq [D^{\mathrm{sh}}/\mu_r]$$

as in Definition 3.2.3, then the integer r is uniquely determined by $\bar{p}$, and we call r the *local index of $\mathcal{C}$ at $\bar{p}$.* In fact one verifies immediately (cf. [AV02, 8.5.1]) that the local index only depends on i and on the image of $\bar{p}$ in S, and that the function

$$\epsilon_i : S \to \mathbb{Z}_{>0}$$

sending $s = f(\bar{p}) \in S$ to the local index r is locally constant.

Notation 3.2.8. Let $\mathbf{e} = (e_1, ..., e_n) \in \mathbb{Z}_{>0}^n$. We say an n-marked twisted curve $(\mathcal{C}/S, \{\Sigma_i\})$ has *global index* $\mathbf{e}$ if ϵ_i is constant of value e_i for all $i = 1, ..., n$.

If $\mathcal{X}/S$ is a finitely presented algebraic stack, proper over S and with finite inertia, we write $\mathcal{K}_{g,n}^{\mathbf{e}}(\mathcal{X})$ for the substack of $\mathcal{K}_{g,n}(\mathcal{X})$ associating to a scheme T/S the groupoid of pairs $(\mathcal{C}, \alpha)$, where $\mathcal{C}/T$ is an n-marked genus-g twisted curve of global index $\mathbf{e}$, and $\alpha : \mathcal{C} \to \mathcal{X} \times_S T$ is a twisted stable map. Since the functions $\epsilon_i : S \to \mathbb{Z}_{>0}$ are locally constant, we see that $\mathcal{K}_{g,n}^{\mathbf{e}}(\mathcal{X})$ is an open and closed substack of $\mathcal{K}_{g,n}(\mathcal{X})$, and

$$\mathcal{K}_{g,n}(\mathcal{X}) = \coprod_{\mathbf{e} \in \mathbb{Z}_{>0}^n} \mathcal{K}_{g,n}^{\mathbf{e}}(\mathcal{X}).$$

In this paper we will generally only be interested in the case $\mathbf{e} = (1, ..., 1)$, in which case $\mathcal{K}_{g,n}^{\mathbf{e}}(\mathcal{X})$ classifies twisted stable maps to $\mathcal{X}$ from twisted curves whose markings are honest sections. We will write $\mathcal{K}'_{g,n}(\mathcal{X})$ for $\mathcal{K}_{g,n}^{(1,...,1)}(\mathcal{X})$.

Notation 3.2.9. We write $\mathcal{K}_{g,n}^{\circ}(\mathcal{X})$ for the open substack of $\mathcal{K}_{g,n}'(\mathcal{X})$ associating to T/S the groupoid of pairs $(\mathcal{C}, \alpha)$, where $\mathcal{C}/T$ is a smooth n-marked genus-g curve (with no stacky structure) and $\alpha : \mathcal{C} \to \mathcal{X} \times_S T$ is a twisted stable map.

3.3   Twisted covers and Picard schemes of twisted curves

If G/S is a linearly reductive finite flat group scheme, then we have already observed that $\mathcal{B}G$ is tame. So in this case we can consider the algebraic stack $\mathcal{K}_{g,n}(\mathcal{B}G)$, which is proper and quasi-finite over the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ by Theorem 3.2.7, since the coarse space of $\mathcal{B}G$ is S.

Theorem 3.3.1 ([AOV11, 5.1]). *Let G/S be a linearly reductive finite flat group scheme. Then $\mathcal{K}_{g,n}(\mathcal{B}G)$ is flat over S, with local complete intersection fibers.*

Definition 3.3.2. Since a map $\mathcal{C} \to \mathcal{B}G$ is equivalent to a G-torsor $P \to \mathcal{C}$, it is often convenient to view $\mathcal{K}_{g,n}(\mathcal{B}G)$ as classifying such torsors. A G-torsor $P \to \mathcal{C}$ is a *twisted G-cover* of $\mathcal{C}$ if it arises in this way.

Note that since the coarse space of $\mathcal{B}G$ is S, the condition that the map $\mathcal{C} \to \mathcal{B}G$ is stable just says that C/S is stable in the usual sense. So for G as above, $\mathcal{K}_{g,n}(\mathcal{B}G)$ can and will be viewed as the stack classifying twisted G-covers of n-marked genus-g twisted stable curves.

Still writing G for a finite flat linearly reductive group scheme over S, suppose in addition that G is commutative. Then every object of $\mathcal{K}_{g,n}(\mathcal{B}G)$ canonically contains G in the center of its automorphism group scheme (determining a unique subgroup stack H in the center of the inertia stack of $\mathcal{K}_{g,n}(\mathcal{B}G)$ such that for any object $\xi \in \mathcal{K}_{g,n}(\mathcal{B}G)(T)$ the pullback of H to T is G). We can therefore apply the rigidification procedure described in [ACV03, §5.1] and generalized in [AOV08, Appendix A], thereby "removing" G from the automorphism group of each object:

Definition 3.3.3. $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ is the rigidification of $\mathcal{K}_{g,n}(\mathcal{B}G)$ with respect to the copy of G naturally contained in the center of its inertia stack. We refer to the objects of $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ as *rigidified twisted stable maps* into $\mathcal{B}G$, or *rigidified twisted G-covers* of n-marked genus-g twisted stable curves. We write $\overline{\mathcal{K}}_{g,n}^{\circ}(\mathcal{B}G)$ for the open substack corresponding to smooth curves and $\overline{\mathcal{K}}_{g,n}'(\mathcal{B}G)$ for the open and closed substack corresponding to twisted stable curves whose markings are representable.

In [AGV08, §C.2] it is shown that the rigidification of an algebraic stack with respect to a group scheme admits a natural moduli interpretation. In this paper we are only interested in the case where G is diagonalizable, in which case $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ can be given a more concrete moduli interpretation which we will now describe.

Given a twisted curve $\mathcal{C}$ over a scheme S, let $\mathcal{P}ic_{\mathcal{C}/S}$ denote the stack associating to T/S the groupoid of line bundles on $\mathcal{C} \times_S T$.

Proposition 3.3.4 ([AOV11, 2.7]). $\mathcal{P}ic_{\mathcal{C}/S}$ *is a smooth locally finitely presented algebraic stack over S, and for any $\mathcal{L} \in \mathcal{P}ic_{\mathcal{C}/S}(T)$ the group scheme $\mathrm{Aut}_T(\mathcal{L})$ is canonically isomorphic to $\mathbb{G}_{m,T}$.*

Write $\mathrm{Pic}_{\mathcal{C}/S}$ for the rigidification of this stack with respect to $\mathbb{G}_m$; $\mathrm{Pic}_{\mathcal{C}/S}$ is none other than the relative Picard functor of $\mathcal{C}/S$. From the analysis of $\mathrm{Pic}_{\mathcal{C}/S}$ in [AOV11, §2] we have:

Proposition 3.3.5. $\mathrm{Pic}_{\mathcal{C}/S}$ *is a smooth group scheme over S, and if $\pi : \mathcal{C} \to C$ is the coarse space of $\mathcal{C}/S$, then there is a short exact sequence of group schemes over S*

$$0 \to \mathrm{Pic}_{C/S} \xrightarrow{\pi^*} \mathrm{Pic}_{\mathcal{C}/S} \to W \to 0,$$

with W quasi-finite and étale over S.

In fact, since $\pi_* \mathbb{G}_m = \mathbb{G}_m$ we can deduce from the exact sequence of low-degree terms of the fppf Leray spectral sequence

$$E_2^{p,q} = H^p(C, \mathbf{R}^q \pi_* \mathbb{G}_m) \Rightarrow H^{p+q}(\mathcal{C}, \mathbb{G}_m)$$

that W is the sheaf associated to the presheaf $T \mapsto H^0(C_T, \mathbf{R}^1 \pi_* \mathbb{G}_m)$ (where we still write $\pi : \mathcal{C}_T \to C_T$ for the morphism induced by base change from $\pi : \mathcal{C} \to C$).

For any integer N annihilating W, we get a natural morphism

$$\times N : \mathrm{Pic}_{\mathcal{C}/S} \to \mathrm{Pic}_{\mathcal{C}/S}.$$

Definition 3.3.6 ([AOV11, 2.11]). The *generalized Jacobian* of $\mathcal{C}$ is

$$\mathrm{Pic}^0_{\mathcal{C}/S} := \mathrm{Pic}_{\mathcal{C}/S} \times_{\times N, \mathrm{Pic}_{C/S}} \mathrm{Pic}^0_{C/S},$$

where $\mathrm{Pic}^0_{C/S}$ is the fiberwise connected component of the identity in the group scheme $\mathrm{Pic}_{C/S}$.

$\mathrm{Pic}^0_{\mathcal{C}/S}$ is independent of N and is viewed as classifying line bundles of fiberwise degree 0 on $\mathcal{C}/S$.

Remark 3.3.7. Unlike the case of $\mathrm{Pic}^0_{C/S}$ for C/S a (non-stacky) genus-g curve, the geometric fibers of $\mathrm{Pic}^0_{\mathcal{C}/S}$ need not be connected for $\mathcal{C}/S$ a twisted curve. In fact, when $\mathcal{C}/S$ is a 1-marked genus-1 twisted stable curve, $\mathrm{Pic}^0_{\mathcal{C}/S}$ behaves like the smooth part of a generalized elliptic curve over S. For instance, if $\mathcal{C} = [C/\mu_N]$ for C/S a Néron 1-gon as in Example 3.2.4, it is easily verified that $\mathrm{Pic}^0_{\mathcal{C}/S} \cong \mathbb{G}_m \times \mathbb{Z}/(N)$. Standard μ_N-stacky Néron 1-gons will play an analogous role in this paper to that of Néron polygons in [DR73] and [Con07]. In particular, we have:

Lemma 3.3.8. *Let k be an algebraically closed field, and $\mathcal{C}/k$ a 1-marked genus-1 twisted stable curve, with no stacky structure at its marking, such that the coarse space C/k is not smooth. Then $\mathcal{C}$ is a standard μ_N-stacky Néron 1-gon (Example 3.2.4) for some (unique) positive integer N.*

Proof. It follows from [DR73, II.1.15] that the 1-marked genus-1 stable curve C/k is a Néron 1-gon. Write $\pi : \mathcal{C} \to C$ for the map exhibiting C as the coarse space of $\mathcal{C}$. Write $\widetilde{C} = \mathbb{P}^1 \to C$ for the normalization of C, and write $\widetilde{\mathcal{C}}$ for the following fiber product:

$$
\begin{array}{ccc}
\widetilde{\mathcal{C}} & \overset{\tau}{\longrightarrow} & \mathbb{P}^1 \\
{\scriptstyle \nu}\downarrow & & \downarrow \\
\mathcal{C} & \underset{\pi}{\longrightarrow} & C
\end{array}
$$

We have a short exact sequence of fppf sheaves on $\mathbb{P}^1$

$$0 \to \mathcal{O}_{\mathbb{P}^1}^{\times} \to \tau_* \mathcal{O}_{\widetilde{\mathcal{C}}}^{\times} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is the pushforward to $\mathbb{P}^1$ of $\mathcal{O}_Z^{\times}$, for $Z \cong \mathcal{B}\mu_N \sqcup \mathcal{B}\mu_N$ the preimage in $\widetilde{\mathcal{C}}$ of the (stacky) node of $\mathcal{C}$. This induces an exact sequence

$$0 \to \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(\widetilde{\mathcal{C}}) \to \mathrm{Pic}(Z) \to H^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\times}) = 0,$$

inducing an isomorphism $\mathrm{Pic}^0_{\widetilde{\mathcal{C}}/k} \cong \mathrm{Pic}(Z) \cong \mathbb{Z}/(N) \times \mathbb{Z}/(N)$ since $\mathrm{Pic}^0_{\mathbb{P}^1/k} = 0$.

Now consider the short exact sequence of fppf sheaves on $\mathcal{C}$

$$0 \to \mathcal{O}_{\mathcal{C}}^{\times} \to \nu_* \mathcal{O}_{\widetilde{\mathcal{C}}}^{\times} \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is the pushforward to $\mathcal{C}$ of $\mathcal{O}_p^{\times}$ for $p \cong \mathcal{B}\mu_N$ the (stacky) node of $\mathcal{C}$. This induces an exact sequence

$$0 \to k^{\times} \overset{\mathrm{id}}{\to} k^{\times} \overset{0}{\to} k^{\times} \to \mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\widetilde{\mathcal{C}}) \to \mathrm{Pic}(\mathcal{B}\mu_N).$$

We have $\mathrm{Pic}(\mathcal{B}\mu_N) \cong \mathbb{Z}/(N)$ and $\mathrm{Pic}^0_{\mathcal{C}/k} \cong \mathbb{Z}/(N) \times \mathbb{Z}/(N)$, and the map

$$\mathrm{Pic}^0_{\mathcal{C}/k} \cong \mathbb{Z}/(N) \times \mathbb{Z}/(N) \to \mathbb{Z}/(N) \cong \mathrm{Pic}(\mathcal{B}\mu_N)$$

is given by $(a, b) \mapsto a - b$. In particular the kernel is isomorphic to $\mathbb{Z}/(N)$, so we have a short exact sequence

$$0 \to k^{\times} \to \mathrm{Pic}^0_{\mathcal{C}/k} \overset{f}{\to} \mathbb{Z}/(N) \to 0.$$

This sequence splits (noncanonically) since $k^{\times}$ is divisible (as k is algebraically closed), so $\mathrm{Pic}^0_{\mathcal{C}/k} \cong \mathbb{G}_m \times \mathbb{Z}/(N)$.

The fppf exact sequence of sheaves on $\mathcal{C}$

$$0 \to \mu_N \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

then gives us an isomorphism

$$H^1(\mathcal{C}, \mu_N) \cong \ker(\mathrm{Pic}_{\mathcal{C}/k} \overset{\times N}{\longrightarrow} \mathrm{Pic}_{\mathcal{C}/k})$$
$$\cong \mu_N \times \mathbb{Z}/(N).$$

Let $C' \to \mathcal{C}$ be the μ_N-torsor over $\mathcal{C}$ corresponding to the class

$$(1,1) \in \mu_N \times \mathbb{Z}/(N) \cong H^1(\mathcal{C}, \mu_N).$$

Then C' is representable by [AH11, 2.3.10], since $(1,1)$ is the class in $H^1(\mathcal{C}, \mu_N)$ of a uniformizing line bundle on $\mathcal{C}$. Let

$$V = C^{\mathrm{sm}} \cong \mathcal{C} \times_C C^{\mathrm{sm}} \overset{\iota}{\hookrightarrow} \mathcal{C},$$

and consider the resulting μ_N-torsor $C'_V \to V \cong \mathbb{G}_m$. The pullback map

$$\iota^* : H^1(\mathcal{C}, \mu_N) \cong \mu_N \times \mathbb{Z}/(N) \to \mathbb{Z}/(N) \cong H^1(\mathbb{G}_m, \mu_N)$$

is given by $(\zeta, a) \mapsto a$. It therefore follows that $C'_V \cong \mathbb{G}_m$, with the μ_N-action given by the standard multiplication action on $\mathbb{G}_m$. The quotient map

$$\mathbb{G}_m \to [\mathbb{G}_m/\mu_N] \cong \mathbb{G}_m$$

is the group scheme homomorphism $x \mapsto x^N$.

Fix an étale neighborhood W of the node of C of the form $W = \mathrm{Spec}(k[z,w]/(zw))$, such that

$$\mathcal{C} \times_C W \cong [D/\mu_N]$$

for $D = \mathrm{Spec}(k[x,y]/(xy))$, with $\zeta \in \mu_N$ acting by $x \mapsto \zeta x$ and $y \mapsto \zeta^{-1}y$. The composite

$$D \to [D/\mu_N] \to W$$

is given by the ring homomorphism $z \mapsto x^N, w \mapsto y^N$. Since C' is representable and

$$C' \times_C W \to \mathcal{C} \times_C W$$

is a μ_N-torsor, it follows that $C' \times_C W \cong D$ with the above μ_N-action. In particular we see that C' is a nodal curve with one node.

Composing our original $C' \to \mathcal{C}$ with the coarse space map $\mathcal{C} \to C$ gives us a finite morphism of nodal curves $C' \to C$, which restricts to the μ_N-torsor $\mathbb{G}_m \to \mathbb{G}_m \cong C^{\mathrm{sm}}$ and which is totally ramified over the node of C. Riemann-Hurwitz for nodal curves implies that C' has arithmetic genus 1, so C' is a Néron 1-gon with smooth locus $C'^{\mathrm{sm}} = \mathbb{G}_m$. The multiplication action of μ_N on $\mathbb{G}_m$ extends uniquely to an action on C', and by assumption $\mathcal{C} = [C'/\mu_N]$. Thus $\mathcal{C}$ is a standard μ_N-stacky 1-gon. $\square$

3.4 Relationship to moduli of elliptic curves with level structure

Notation 3.4.1. For any finite flat commutative group scheme G over a base scheme S, $\mathcal{H}(G)$ is the stack over S associating to an S-scheme T the groupoid of pairs (E, ϕ), where E/T is an elliptic curve and $\phi : G^* \to E[N]$ is a homomorphism of group schemes over T (for G^* the Cartier dual of G). For the S-scheme $G = \mu_N$ we write $\mathcal{H}_1(N)$ for $\mathcal{H}(\mu_N)$, and for the S-scheme $G = \mu_N^2$ we write $\mathcal{H}(N)$ for $\mathcal{H}(\mu_N^2)$.

As in [AOV08, §2.3], we say a finite commutative group scheme is *diagonalizable* if its Cartier dual is constant, and *locally diagonalizable* if its Cartier dual is étale.

Lemma 3.4.2 ([AOV11, 5.7]). *Let G/S be a finite diagonalizable commutative group scheme, so $\mathcal{B}G$ is tame (since G is linearly reductive) and $X := G^*$ is constant. For any twisted curve $\mathcal{C}/S$ there is an equivalence of categories between the stack $\mathrm{TORS}_{\mathcal{C}/S}(G)$ classifying G-torsors on $\mathcal{C}/S$ and the Picard stack $\mathcal{P}ic_{\mathcal{C}/S}[X]$ of morphisms of Picard stacks $X \to \mathcal{P}ic_{\mathcal{C}/S}$.*

The construction for the above equivalence is as follows:

> Let $\pi : P \to \mathcal{C}$ be a G-torsor over $\mathcal{C}$. G acts on the sheaf $\pi_* \mathcal{O}_P$, yielding a decomposition
> $$\pi_* \mathcal{O}_P = \bigoplus_{\chi \in X} \mathcal{L}_\chi.$$
> Each $\mathcal{L}_\chi$ is invertible since P is a torsor over E, so this determines a morphism of Picard stacks
> $$\phi_P : X \to \mathcal{P}ic_{\mathcal{C}/S}$$
> $$\chi \mapsto [\mathcal{L}_\chi].$$
> And conversely, such a morphism $\phi : X \to \mathcal{P}ic_{\mathcal{C}/S}$ naturally determines an algebra structure on
> $$\bigoplus_{\chi \in X} \phi(\chi),$$
> giving a G-torsor
> $$\underline{\mathrm{Spec}}_{\mathcal{C}} \Big(\bigoplus_{\chi \in X} \phi(\chi) \Big) \to \mathcal{C}$$
> with the G-action determined by the X-grading.

This defines an open immersion from $\mathcal{K}_{g,n}(\mathcal{B}G)$ into the algebraic stack over S associating to an S-scheme T the groupoid of pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/T$ is an n-marked genus-g twisted stable curve and $\phi \in \mathcal{P}ic_{\mathcal{C}/T}[X]$. Rigidifying $\mathcal{K}_{g,n}(\mathcal{B}G)$ and $\mathcal{P}ic_{\mathcal{C}/T}$ with respect to the group schemes G and $\mathbb{G}_m$ respectively, we have an open immersion from $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ into the stack classifying pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/T$ is an n-marked genus-g twisted stable curve and $\phi : X \to \mathrm{Pic}_{\mathcal{C}/T}$ is a homomorphism of group schemes over T.

Writing $f : \mathcal{C} \to S$ for the structural morphism, we have $\mathrm{Pic}_{\mathcal{C}/S} = \mathbf{R}^1 f_* \mathbb{G}_m$. Therefore fppf-locally on T, ϕ corresponds to the choice of an X-torsor $P \to \mathcal{C}$, with P representable if and only if $(\mathcal{C}, \phi)$ comes from an object of $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ (refer for example to [AH11, 2.3.10] to see that a morphism from a twisted curve $\mathcal{C}$ to $\mathcal{B}G$ is representable if and only if the corresponding G-torsor over $\mathcal{C}$ is representable). This gives us:

Corollary 3.4.3. *For a finite flat diagonalizable group scheme G/S, the above construction gives an equivalence between $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$ and the stack over S associating to T/S the groupoid of pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/T$ is an n-marked genus-g twisted stable curve and $\phi : X \to \mathrm{Pic}_{\mathcal{C}/T}$ is a group scheme homomorphism such that, fppf-locally on T, the G-torsor over $\mathcal{C}$ corresponding to ϕ as above is representable.*

In particular, we get an equivalence

$$\overline{\mathcal{K}}_{1,1}^\circ(\mathcal{B}G) \simeq \mathcal{H}(G),$$

since for an elliptic curve E/S every G-torsor over E is representable and we canonically have $E \cong \mathrm{Pic}_{E/S}^0$. This isomorphism sends $Q \in E(S)$ to the class of the line bundle $\mathcal{L}((Q)-(0_E))$, so as a special case we see that if $\phi : \mathbb{Z}/(N) \to E$ is a group scheme homomorphism with $\phi(1) = Q$, the corresponding μ_N-torsor over E is of the form

$$P = \underline{\mathrm{Spec}}_E\Big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q) - (0_E)))\Big).$$

Since $\mathcal{H}_1(N)$ naturally contains a closed substack isomorphic to the stack $\mathcal{Y}_1(N)$ over S classifying $[\Gamma_1(N)]$-structures on elliptic curves, we see that the algebraic stack $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ is a modular compactification of $\mathcal{Y}_1(N)$. Similarly, the algebraic stack $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$ is a modular compactification of the stack $\mathcal{Y}(N)$ classifying (not necessarily symplectic) full level N structures on elliptic curves. The task in both cases is to give a moduli interpretation of the closure of these classical moduli stacks in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}G)$, and we address this in the following sections.

Lemma 3.4.4. *$\overline{\mathcal{K}}_{1,1}^\circ(\mathcal{B}\mu_N)$ is dense in $\overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_N)$, and $\overline{\mathcal{K}}_{1,1}^\circ(\mathcal{B}\mu_N^2)$ is dense in $\overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_N^2)$.*

Proof. Let $\mathcal{C}_0$ be a standard μ_d-stacky 1-gon over an algebraically closed field k, and let

$$\phi_0 : \mathbb{Z}/(N) \to \mathrm{Pic}_{\mathcal{C}_k}^0 \cong \mathbb{G}_m \times \mathbb{Z}/(d)$$

be a group scheme homomorphism such that

$$(\mathcal{C}_0, \phi_0) \in \overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_N)(k)$$

(so $d|N$ and ϕ_0 meets every component of $\mathrm{Pic}_{\mathcal{C}_0/k}^0$). We need to lift $(\mathcal{C}_0/k, \phi_0)$ to a pair
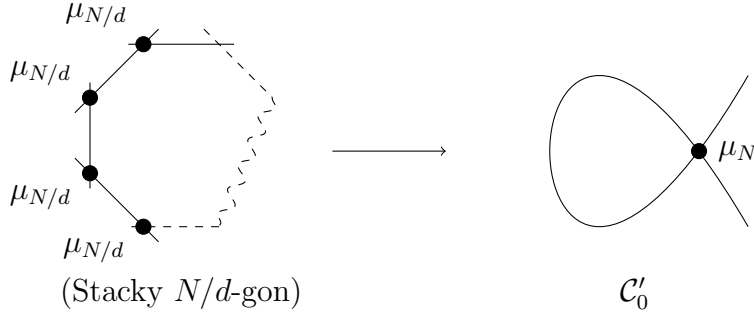
$$(\mathcal{C}, \phi) \in \overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_N)(A)$$

for a local ring A with residue field k, such that the generic fiber of $\mathcal{C}$ is a smooth elliptic curve.

Let $\mathcal{C}_0'/k$ be a standard μ_N-stacky 1-gon, and consider the morphism $\mathcal{C}_0' \to \mathcal{B}\mu_N$ corresponding to the group scheme homomorphism

$$\mathbb{Z}/(N) \to \mathrm{Pic}_{\mathcal{C}_0'/k}^0 \cong \mathbb{G}_m \times \mathbb{Z}/(N)$$

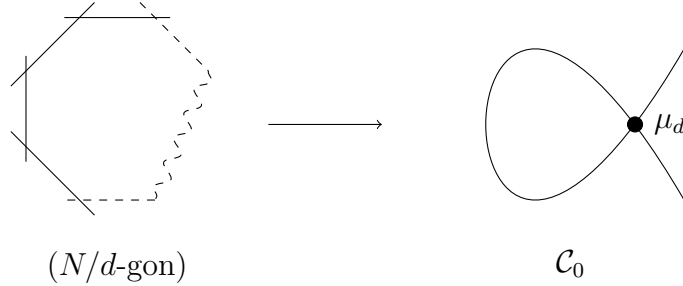sending 1 to $(1, N/d)$. This morphism is not representable; the corresponding μ_N-torsor over $\mathcal{C}_0'$ is as follows:

(Stacky N/d-gon) $\mathcal{C}_0'$

We may factor the morphism $\mathcal{C}_0' \to \mathcal{B}\mu_N$ as

$$\mathcal{C}_0' \to \mathcal{C}_0 \to \mathcal{B}\mu_N,$$

where $\mathcal{C}_0 \to \mathcal{B}\mu_N$ is the relative coarse moduli space ([AOV11, Thm. 3.1]) of $\mathcal{C}_0' \to \mathcal{B}\mu_N$:

(N/d-gon) $\mathcal{C}_0$

Since this $\mathcal{C}_0$ is a standard μ_d-stacky 1-gon, we may identify it with our original twisted curve $\mathcal{C}_0$. This gives us a morphism $\mathcal{C}_0' \to \mathcal{C}_0$, and the resulting pullback map $\mathrm{Pic}_{\mathcal{C}_0/k}^0 \to \mathrm{Pic}_{\mathcal{C}_0'/k}^0$ is the monomorphism

$$\mathbb{G}_m \times \mathbb{Z}/(d) \to \mathbb{G}_m \times \mathbb{Z}/(N)$$

$$(\zeta, a) \mapsto (\zeta, \frac{N}{d} \cdot a).$$

Let $\mathcal{E}/k[\![q^{1/N}]\!]$ be an N-gon Tate curve, so the special fiber of $\mathcal{E}$ is a Néron N-gon, $\mathcal{E} \otimes k((q^{1/N}))$ is a smooth elliptic curve, and we have an isomorphism

$$\mathcal{E}^{\mathrm{sm}}[N] \cong \mu_N \times \mathbb{Z}/(N)$$

of finite flat commutative group schemes over $k[\![q^{1/N}]\!]$. Let $Q = (1, 1) \in \mathcal{E}^{\mathrm{sm}}[N]$. The relative effective Cartier divisor

$$D := \sum_{a \in \mathbb{Z}/(N)} [a \cdot Q]$$

in $\mathcal{E}^{\mathrm{sm}}$ is a subgroup scheme, étale over $k[\![q^{1/N}]\!]$, and the quotient $\overline{\mathcal{E}} := \mathcal{E}/D$ is naturally a generalized elliptic curve whose special fiber is a 1-gon. The stack quotient

$$\mathcal{C} := [\mathcal{E}/\mathcal{E}^{\mathrm{sm}}[N]]$$

is naturally a twisted curve, whose generic fiber is an elliptic curve and whose special fiber is $\mathcal{C}'_0$. Writing $\pi : \overline{\mathcal{E}} \to \mathcal{C}$ for the natural quotient map, we will see in Chapter 6 that for any line bundle $\mathcal{L}$ on $\overline{\mathcal{E}}$ there is a canonical decomposition

$$\pi_* \mathcal{L} \cong \bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}_a,$$

where each $\mathcal{L}_a$ is a line bundle on $\mathcal{C}$.

For a section $R \in \mathcal{E}^{\mathrm{sm}}(k[\![q^{1/N}]\!])$, we write $\overline{R}$ for its image in $\overline{\mathcal{E}}^{\mathrm{sm}}(k[\![q^{1/N}]\!])$. Then we have the degree 0 line bundle

$$\mathcal{L}_R := \mathcal{L}((\overline{R}) - (0_{\overline{\mathcal{E}}}))$$

on $\overline{\mathcal{E}}$, hence a canonical decomposition

$$\pi_* \mathcal{L}_R \cong \bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}_{R,a}.$$

We will see in Chapter 6 that the map $\mathcal{E}^{\mathrm{sm}}[N] \to \mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N]$ sending

$$R = (\zeta, a) \in \mu_N \times \mathbb{Z}/(N) \cong \mathcal{E}^{\mathrm{sm}}[N]$$

to $\mathcal{L}_{R,a}$ is an isomorphism of group schemes over $k[\![q^{1/N}]\!]$.

Returning to our original pair $(\mathcal{C}_0, \phi_0) \in \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)(k)$, write

$$\phi(1) = (\zeta, a) \in \mu_N \times \mathbb{Z}/(d) \cong \mathrm{Pic}^0_{\mathcal{C}_0/k}[N].$$

Via the map $\mathcal{C}'_0 \to \mathcal{C}_0$ constructed above, this corresponds to

$$(\zeta, (N/d) \cdot a) \in \mu_N \times \mathbb{Z}/(N) \cong \mathrm{Pic}^0_{\mathcal{C}'_0/k}[N].$$

$\mathcal{C}'_0$ is the special fiber of the twisted curve $\mathcal{C}$, and we have an isomorphism over $k[\![q^{1/N}]\!]$

$$\mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N] \cong \mathcal{E}^{\mathrm{sm}}[N] \cong \mu_N \times \mathbb{Z}/(N).$$

Now $(\zeta, (N/d) \cdot a)$ lifts to a section of $\mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N]$, corresponding to a group scheme homomorphism $\mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}$, hence to a morphism $\mathcal{C} \to \mathcal{B}\mu_N$. Writing $\overline{\mathcal{C}} \to \mathcal{B}\mu_N$ for the relative coarse moduli space and

$$\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\overline{\mathcal{C}}/k[\![q^{1/N}]\!]}$$

for the corresponding group scheme homomorphism, we see that $\overline{\mathcal{C}}/k[\![q^{1/N}]\!]$ is a twisted curve with special fiber $\mathcal{C}_0$ and generic fiber an elliptic curve. ϕ extends ϕ_0 and

$$(\overline{\mathcal{C}}, \phi) \in \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)(k[\![q^{1/N}]\!])$$

as desired.

A similar argument of course applies to $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\mu_N^2) \subset \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)$. $\square$

We will require a concrete description of the μ_N-torsor corresponding to a particular sort of $[\Gamma_1(N)]$-structure on an elliptic curve:

Lemma 3.4.5. *Let K be a field and E/K an elliptic curve. Let $Q \in E(K)$ be a $[\Gamma_1(N)]$-structure on E such that the relative effective Cartier divisor*

$$D := \sum_{a=0}^{N-1} [a \cdot Q]$$

in E is étale over $\mathrm{Spec}(K)$. Let $P \to E$ be the μ_N-torsor corresponding to Q as in 3.4.3:

$$P = \underline{\mathrm{Spec}}_E \big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q) - (0_E)) \big).$$

Then P can be naturally identified with the quotient E/D, where D is viewed as a subgroup scheme of E, étale of rank N over $\mathrm{Spec}(K)$, with the quotient map $P \to E$ corresponding to the natural map $E/D \to E/E[N] \cong E$.

Proof. Consider the e_N-pairing on $E[N]$, a nondegenerate bilinear pairing of finite flat group schemes over $\mathrm{Spec}(K)$:

$$e_N : E[N] \times E[N] \to \mu_N.$$

Under our assumptions, the composite map

$$E[N]/D = \{Q\} \times E[N]/D \hookrightarrow D \times E[N]/D \overset{e_N}{\to} \mu_N$$

is an isomorphism of group schemes over $\mathrm{Spec}(K)$. Then via this isomorphism, μ_N acts on the quotient E/D through the natural action of the subgroup scheme

$$E[N]/D \subset (E/D)[N],$$

making E/D a μ_N-torsor over $E/E[N] \cong E$ with quotient map the obvious one induced from the factorization of $[N]$ as

$$E \to E/D \to E/E[N] \cong E.$$

By Lemma 3.4.2, we may express the μ_N-torsor $E/D \to E$ as

$$E/D = \underline{\mathrm{Spec}}_E \big(\bigoplus_{a=0}^{N-1} \mathcal{L}_a \big)$$

for some line bundles $\mathcal{L}_a \in \mathrm{Pic}^0_{E/K}[N]$, with the algebra structure determined by isomorphisms

$$\mathcal{L}_a \otimes \mathcal{L}_b \cong \mathcal{L}_{a+b \bmod N}$$

and the μ_N-action corresponding to the grading. We have a natural isomorphism of group schemes over K

$$E \to \mathrm{Pic}^0_{E/K}$$
$$R \in E(K) \mapsto \mathcal{L}((R) - (0_E)),$$

so we conclude that

$$\mathcal{L}_1 \cong \mathcal{L}((Q_0) - (0_E))$$

for some $Q_0 \in E[N]$, and

$$\mathcal{L}_a \cong \mathcal{L}((a \cdot Q_0) - (0_E)).$$

Let $\hat{\pi} : E \to E/D$ be the isogeny dual to

$$\pi : E/D \to E/E[N] \cong E.$$

Identifying $E \cong \mathrm{Pic}^0_{E/K}$ and $E/D \cong \mathrm{Pic}^0_{(E/D)/K}$, $\hat{\pi}$ is simply given by the pullback map

$$\pi^* : \mathrm{Pic}^0_{E/K} \to \mathrm{Pic}^0_{(E/D)/K}.$$

For any line bundle $\mathcal{L}$ on E/K we have

$$\pi^*(\mathcal{L}) = \bigoplus_{a=0}^{N-1} \mathcal{L} \otimes \mathcal{L}((a \cdot Q_0) - (0_E)),$$

viewing the direct sum of line bundles on E as a line bundle on

$$E/D = \underline{\mathrm{Spec}}_E(\oplus \mathcal{L}((a \cdot Q_0) - (0_E))).$$

In particular, for our original $[\Gamma_1(N)]$-structure Q,

$$\pi^*(\mathcal{L}((Q) - (0_E))) = \bigoplus_{a=0}^{N-1} \mathcal{L}((Q + a \cdot Q_0) - (0_E)).$$

But the dual isogeny to $\pi : E/D \to E/E[N] \cong E$ is the natural quotient map $E \to E/D$, and this maps Q to 0_E. Therefore the line bundle $\oplus \mathcal{L}((Q + a \cdot Q_0) - (0_E))$ on E/D is the trivial line bundle on E/D:

$$\bigoplus_{a=0}^{N-1} \mathcal{L}((Q + a \cdot Q_0) - (0_E)) \cong \bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q_0) - (0_E)).$$

and so Q is contained in the subgroup scheme of E generated by Q_0, say $Q = b \cdot Q_0$ for some b. Since $Q_0 \in E[N]$ and N is the minimal positive integer with $N \cdot Q = 0_E$, this implies that $b \in (\mathbb{Z}/(N))^\times$, and in fact by the definition of the e_N-pairing we have $b = 1$, i.e. $Q_0 = Q$. Thus

$$\underline{\mathrm{Spec}}_E\Big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q_0) - (0_E))\Big) = \underline{\mathrm{Spec}}_E\Big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q) - (0_E))\Big),$$

that is, $E/D \cong P$ with the quotient map $P \to E$ of the given μ_N-action becoming identified with the natural quotient map $E/D \to E/E[N] \cong E$. $\qquad\square$

Chapter 4

Moduli of elliptic curves in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$

4.1   Reduction mod p of $\mathcal{H}_1(N)$

We first describe how the different components of $\overline{\mathcal{K}}_{1,1}^\circ(\mathcal{B}\mu_N) \cong \mathcal{H}_1(N)$ interact. These results are direct corollaries of [KM85, §13.5]. Continue working over an arbitrary base scheme S. First consider the case where $N = p^n$ is a prime power. For each $0 \leq m \leq n$ we get a closed immersion

$$\iota^{(p^m)} : \mathcal{Y}_1(p^m) \hookrightarrow \mathcal{H}_1(p^n)$$

sending a pair $(E, \phi) \in \mathcal{Y}_1(p^m)(T)$, where

$$\phi : \mathbb{Z}/(p^m) \to E[p^m]$$

is a $[\Gamma_1(p^m)]$-structure on E/T, to the pair $(E, \widetilde{\phi}) \in \mathcal{H}_1(p^n)(T)$, where

$$\widetilde{\phi} : \mathbb{Z}/(p^n) \to E[p^n]$$

is obtained from ϕ by precomposing with the canonical projection $\mathbb{Z}/(p^n) \to \mathbb{Z}/(p^m)$. These yield a proper surjection of algebraic stacks

$$\mathcal{Y}_1(p^n) \sqcup \mathcal{Y}_1(p^{n-1}) \sqcup ... \sqcup \mathcal{Y}_1(p) \sqcup \mathcal{Y}_1(1) \to \mathcal{H}_1(p^n)$$

which is an isomorphism over $S[1/p]$.

But this is not an isomorphism over $S \otimes \mathbb{F}_p$. Recall that by Theorem 2.3.8, for any perfect field k of characteristic p, $\mathcal{Y}_1(p^m)_k$ is the disjoint union, with crossings at the supersingular points, of components

$$\mathcal{Y}_1(p^m)_k^{(m-b,b)} \ (0 \leq b \leq m),$$

where an object of $\mathcal{Y}_1(p^m)_k^{(m-b,b)}(T)$ is a pair (E, ϕ) where E/T is an elliptic curve and $\phi : \mathbb{Z}/(p^m) \to E[p^m]$ is a $[\Gamma_1(p^m)]$-$(m-b, b)$-cyclic structure on E/T (Definition 2.3.4). Such an object corresponds via $\iota^{(m)}$ to the pair $(E, \widetilde{\phi}) \in \mathcal{H}_1(p^n)(T)$ as described above.

The key observation is Lemma 2.3.5: if $\phi : \mathbb{Z}/(p^m) \to E$ is a $[\Gamma_1(p^m)]$-$(m-b,b)$-cyclic structure on an ordinary elliptic curve $E/T/\mathbb{F}_p$, then

$$\widetilde{\phi} := \phi \circ \pi : \mathbb{Z}/(p^n) \to E$$

is a $[\Gamma_1(p^n)]$-$(n-b,b)$-cyclic structure on E, where $\pi : \mathbb{Z}/(p^n) \to \mathbb{Z}/(p^m)$ is the natural projection. Étale locally on T we have

$$E[p^n] \cong \mu_{p^n} \times \mathbb{Z}/(p^n)$$

and

$$E[p^m] \cong \mu_{p^m} \times \mathbb{Z}/(p^m),$$

in which case the $[\Gamma_1(p^m)]$-$(m-b,b)$-cyclic structure ϕ corresponds to a section of

$$\mu_{p^m}^{\times} \times (\mathbb{Z}/(p^b))^{\times} \quad \text{if } b < m$$
$$\mu_{p^m} \times (\mathbb{Z}/(p^b))^{\times} \quad \text{if } b = m,$$

and the reason that such a section also gives a $[\Gamma_1(p^n)]$-structure is that in characteristic p, unlike in characteristic $\neq p$, if $c < n$ and $\mathbb{Z}/(p^c) \to \mu_{p^c}$ is a $\mathbb{Z}/(p^c)$-generator then the composite

$$\mathbb{Z}/(p^n) \twoheadrightarrow \mathbb{Z}/(p^c) \to \mu_{p^c} \hookrightarrow \mu_{p^n}$$

is a $\mathbb{Z}/(p^n)$-generator. This gives us:

Proposition 4.1.1. *Let k be a perfect field of characteristic p. $\mathcal{H}_1(p^n)_k$ is the disjoint union, with crossings at the supersingular points, of components Z_b for $0 \leq b \leq n$, where*

$$Z_b = \bigcup_{b \leq m \leq n} \mathcal{Y}_1(p^m)_k^{(m-b,b)},$$

identifying each $\mathcal{Y}_1(p^m)_k^{(m-b,b)}$ with a closed substack of $\mathcal{H}_1(p^n)_k$ via $\iota^{(p^m)}$. Each $\mathcal{Y}_1(p^m)_k^{(m-b,b)}$ is "set-theoretically identified with Z_b" in the sense that $(Z_b)_{\mathrm{red}} = \mathcal{Y}_1(p^m)_{k,\mathrm{red}}^{(m-b,b)}$ as substacks of $\mathcal{H}_1(p^n)_{k,\mathrm{red}}$ for all $b \leq m \leq n$.

To illustrate, visualize $\mathcal{H}_1(p^n)_{k,\mathrm{red}}$ as follows:
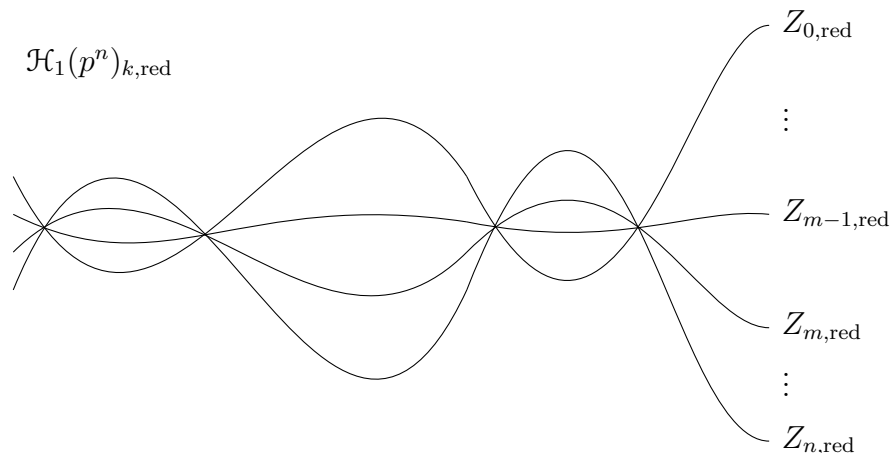
The closed immersion $\iota^{(p^m)}$ sends the following copy of $\mathcal{Y}_1(p^m)_k$ to the obvious closed substack of $\mathcal{H}_1(p^n)_k$, contributing nilpotent structure to the components $Z_0, ..., Z_m$:
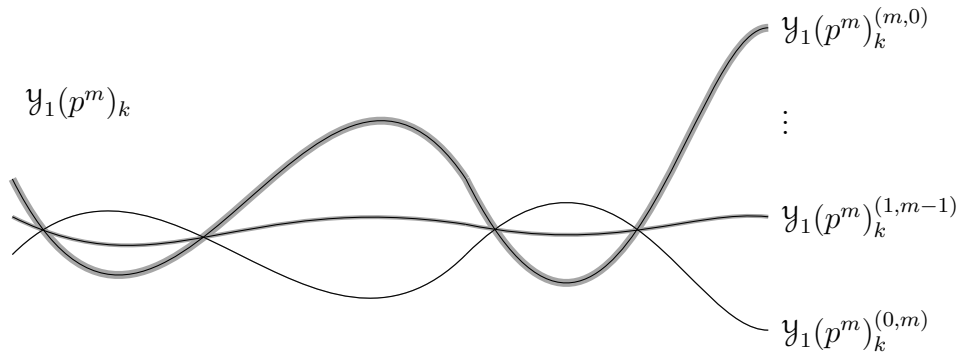
The result is that $\mathcal{H}_1(p^n)_{k,\mathrm{red}} \cup \mathcal{Y}_1(p^m)_k \subseteq \mathcal{H}_1(p^n)_k$ looks something like this:
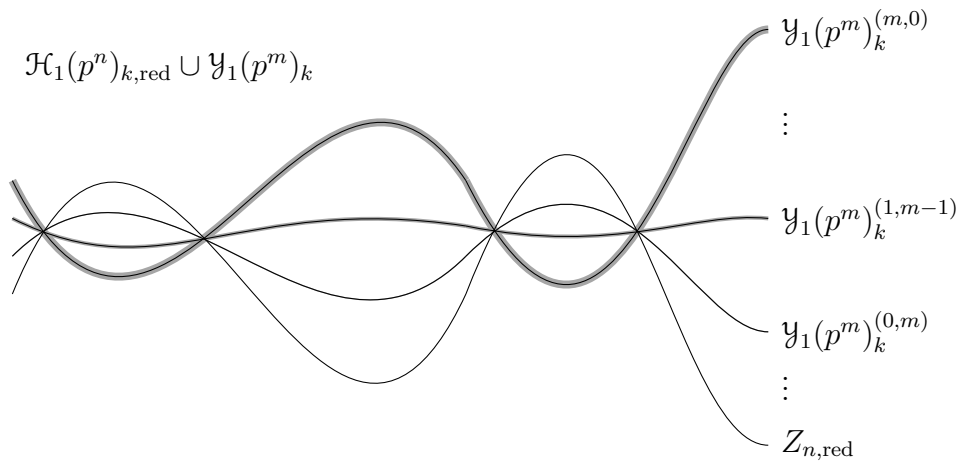
Each $\mathcal{Y}_1(p^m)_k$ (for $0 \leq m \leq n$) contributes additional nilpotent structure, giving us:
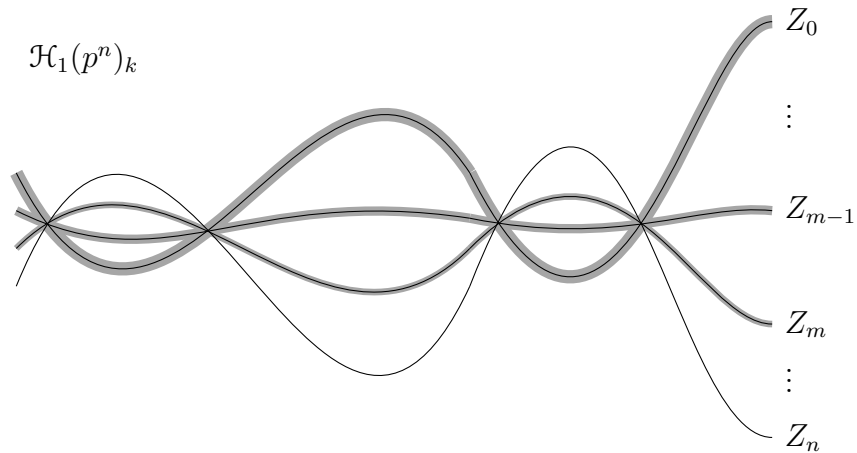
More generally, for arbitrary N we get a closed immersion

$$\iota^{(d)} : \mathcal{Y}_1(d) \hookrightarrow \mathcal{H}_1(N)$$

for each d dividing N, and the resulting map

$$\bigsqcup_{d|N} \mathcal{Y}_1(d) \to \mathcal{H}_1(N)$$

is an isomorphism over $S[1/N]$. It follows immediately from [KM85, §13.5] that if $(r, p) = 1$, then for a perfect field k of characteristic p, $\mathcal{Y}_1(p^m r)_k$ is the disjoint union, with crossings at the supersingular points, of $m + 1$ components $\mathcal{Y}_1(p^m r)_k^{(m-b,b)}$ ($0 \le b \le m$), where

$$\mathcal{Y}_1(p^m r)_k^{(m-b,b)} := \mathcal{Y}_1(r)_k \times_{\overline{\mathcal{M}}_{1,1,k}} \mathcal{Y}_1(p^m)_k^{(m-b,b)}.$$

Now let $N = p^n N'$ where $(N', p) = 1$, and for any $r | N'$ let $\mathcal{H}_1(N)_k^r \subset \mathcal{H}_1(N)_k$ denote the union of the components $\mathcal{Y}_1(p^m r)_k$ for $0 \le m \le n$. In summary:

Corollary 4.1.2. *Let k be a perfect field of characteristic p. For any r dividing N', $\mathcal{H}_1(N)_k^r$ is the disjoint union, with crossings at the supersingular points, of components Z_b^r for $0 \le b \le n$, where*

$$Z_b^r = \bigcup_{b \le m \le n} \mathcal{Y}_1(p^m r)_k^{(m-b,b)},$$

identifying each $\mathcal{Y}_1(p^m r)_k^{(m-b,b)}$ with a closed substack of $\mathcal{H}_1(N)_k$ via $\iota^{(p^m r)}$.

Each $\mathcal{Y}_1(p^m r)_k^{(m-b,b)}$ is "set-theoretically identified with Z_b^r" in the sense that

$$(Z_b)_{\mathrm{red}} = \mathcal{Y}_1(p^m r)_{k,\mathrm{red}}^{(m-b,b)}$$

as substacks of $\mathcal{H}_1(N)_{k,\mathrm{red}}$. $\mathcal{H}_1(N)_k$ is the disjoint union of the closed substacks

$$\{\mathcal{H}_1(N)_k^r\}_{r|N'}.$$

4.2   Closure of $\mathcal{Y}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$

We now want to describe the closure of $\mathcal{Y}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$, as a moduli stack classifying twisted curves with extra structure. This is accomplished in [ACV03, §5.2] over $\mathbb{Z}[1/N]$; let us briefly recall how this is done.

Definition 4.2.1. Let G be a finite group, viewed as a finite étale group scheme over $\mathbb{Z}[1/|G|]$. Fix an isomorphism $G^* \cong G$, after adjoining the necessary roots of unity to $\mathbb{Z}[1/|G|]$. $\mathcal{B}_{g,n}^{\mathrm{tei}}(G)$ is defined as the substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}G)$ over $\mathbb{Z}[1/|G|]$ whose objects are *twisted Teichmüller G-structures* on twisted curves. An object of $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)(T)$ is a pair $(\mathcal{C}, \phi)$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and ϕ is a group scheme homomorphism $G \to \mathrm{Pic}_{\mathcal{C}/T}^0$. By definition, $(\mathcal{C}, \phi) \in \mathcal{B}_{g,n}^{\mathrm{tei}}(G)(T)$ if and only if, whenever $P \to \mathcal{C}$ is a G-torsor corresponding to ϕ (fppf-locally on T), the geometric fibers of P/T are connected.

$\mathcal{B}_{g,n}^{\mathrm{tei}}(G)$ is naturally a closed substack of $\overline{\mathcal{K}}_{g,n}(\mathcal{B}G)$. Working over $\mathbb{Z}[1/N]$, the choice of a primitive N^{th} root of unity ζ_N determines an isomorphism of group schemes $G := \mathbb{Z}/(N) \cong \mu_N$ over $\mathbb{Z}[\zeta_N, 1/N]$. Applying the resulting isomorphism $\overline{\mathcal{K}}_{1,1}(\mathcal{B}G) \cong \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$, we may view $\mathcal{Y}_1(N)$ as a substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}G)$, and the closure of $\mathcal{Y}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}G)$ over $\mathbb{Z}[\zeta_N, 1/N]$ can be shown to be $\mathcal{B}_{1,1}^{\mathrm{tei}}(G)$ (indeed, this follows from 4.2.4 below). Thus:

Corollary 4.2.2. *The closure of* $\mathcal{Y}_1(N)$ *in* $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ *over* $\mathbb{Z}[1/N]$ *is the stack whose objects over a scheme* $T/\mathbb{Z}[1/N]$ *are pairs* $(\mathcal{C}, \phi)$*, where* $\mathcal{C}/T$ *is a 1-marked genus-1 twisted stable curve with non-stacky marking,* $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}_{\mathcal{C}/S}^0$ *is a group scheme homomorphism, and whenever* $P \to \mathcal{C}$ *is a* μ_N*-torsor corresponding to* ϕ *(fppf-locally on* T*) the geometric fibers of* P/T *are connected.*

However, in characteristics dividing N, simply requiring the μ_N-torsors to have connected geometric fibers will not give us a moduli stack isomorphic to the closure of $\mathcal{Y}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$. For example, the group scheme μ_{p^n} is itself connected over any field of characteristic p; so any μ_{p^n}-torsor over an n-marked genus-g twisted stable curve over a field of characteristic p will automatically be connected. So if the above result held over a base scheme $S \in \mathrm{Sch}/\mathbb{F}_p$, it would say the closure of $\mathcal{Y}_1(p^n)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_{p^n})$ is all of $\overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_{p^n})$ (the substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_{p^n})$ where the marking is representable). We will see that this is not true; the closure of $\mathcal{Y}_1(p^n)$ turns out to be finite and flat of constant rank $p^{2n} - p^{2n-2}$ over $\overline{\mathcal{M}}_{1,1}$, while $\overline{\mathcal{K}}_{1,1}'(\mathcal{B}\mu_{p^n})$ turns out to be finite and flat of constant rank p^{2n} over $\overline{\mathcal{M}}_{1,1}$.

A μ_N-torsor over $\mathcal{C}/S$ determines a group scheme homomorphism $\mathbb{Z}/(N) \to \mathrm{Pic}_{\mathcal{C}/S}^0$ (cf. 3.4.2), and over $\mathbb{Z}[1/N]$ a μ_N-torsor with connected geometric fibers corresponds to a group scheme homomorphism which is injective. We are therefore led to consider Drinfeld structures on our twisted curves:

Definition 4.2.3. Let $\mathcal{C}/S$ be a 1-marked genus-1 twisted stable curve with no stacky structure at its marking. A $[\Gamma_1(N)]$-*structure* on $\mathcal{C}$ is a group scheme homomorphism $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}_{\mathcal{C}/S}^0$ such that:

- the relative effective Cartier divisor

$$D := \sum_{a \in \mathbb{Z}/(N)} [\phi(a)]$$

  in $\mathrm{Pic}_{\mathcal{C}/S}^0$ is an S-subgroup scheme, and

- for every geometric point $\overline{p} \to S$, $D_{\overline{p}}$ meets every irreducible component of $(\mathrm{Pic}_{\mathcal{C}/S}^0)_{\overline{p}} = \mathrm{Pic}_{\mathcal{C}_{\overline{p}}/k(\overline{p})}^0$.

Over any scheme S, we define $\mathcal{X}_1^{\mathrm{tw}}(N)$ to be the substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ whose objects over an S-scheme T are pairs $(\mathcal{C}, \phi) \in \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)(T)$ such that ϕ is a $[\Gamma_1(N)]$-structure on $\mathcal{C}$.

Note that if $\mathcal{C}/S$ is a 1-marked genus-1 stable curve (non-twisted), so that every geometric fiber of $\mathcal{C}/S$ is either a smooth elliptic curve or a Néron 1-gon, then this agrees with the definition of a $[\Gamma_1(N)]$-structure on $\mathcal{C}/S$ as given in [Con07, 2.4.1] and in Definition 2.2.1 above, identifying a $[\Gamma_1(N)]$-structure $P \in \mathcal{C}^{\mathrm{sm}}(S)$ with its corresponding group scheme homomorphism

$$\mathbb{Z}/(N) \to \mathcal{C}^{\mathrm{sm}}$$
$$1 \mapsto P,$$

since $\mathrm{Pic}^0_{\mathcal{C}/S} \cong \mathcal{C}^{\mathrm{sm}}$ in this case.

Theorem 4.2.4 (Restatement of 1.1.1). *Let S be a scheme and $\mathfrak{X}_1^{\mathrm{tw}}(N)$ the stack over S classifying $[\Gamma_1(N)]$-structures on 1-marked genus-1 twisted stable curves with non-stacky marking. Then $\mathfrak{X}_1^{\mathrm{tw}}(N)$ is a closed substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$, which contains $\mathcal{Y}_1(N)$ as an open dense substack.*

In particular $\mathfrak{X}_1^{\mathrm{tw}}(N)$ is flat over S with local complete intersection fibers, and is proper and quasi-finite over $\overline{\mathcal{M}}_{1,1}$.

Remark 4.2.5. Although this gives a new modular compactification of $\mathcal{Y}_1(N)$, it should be noted that the proof of the theorem relies in several places on the proof in [Con07] that the moduli stack classifying $[\Gamma_1(N)]$-structures on generalized elliptic curves is a proper algebraic stack over $\overline{\mathcal{M}}_{1,1}$.

Proof of 4.2.4. The main point is to verify the valuative criterion of properness for $\mathfrak{X}_1^{\mathrm{tw}}(N)$, which implies $\mathfrak{X}_1^{\mathrm{tw}}(N)$ is a closed substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$. It follows from Lemma 3.4.4 that $\mathcal{Y}_1(N)$ is dense.

Let R be a discrete valuation ring with $T := \mathrm{Spec}(R) \in \mathrm{Sch}/S$; write $\eta = \mathrm{Spec}(K)$ for the generic point of T and $s = \mathrm{Spec}(k)$ for the closed point. Let

$$(\mathcal{C}_\eta, \phi_\eta) \in \mathfrak{X}_1^{\mathrm{tw}}(N)(\eta),$$

so $\mathcal{C}_\eta/K$ is a 1-marked genus-1 twisted stable curve with non-stacky marking and

$$\phi_\eta : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}_\eta/K}$$

is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_\eta$. Since $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ is proper over S, there is a discrete valuation ring R_1 containing R as a local subring, with corresponding morphism of spectra $T_1 \to T$ over S, such that the pair $(\mathcal{C}_\eta \times_T T_1, (\phi_\eta)_{T_1})$ extends to a pair

$$(\mathcal{C}_{T_1}, \phi_{T_1}) \in \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)(T);$$

and whenever such an extension exists, it is unique. Therefore it suffices to show that for some such R_1 and T_1, there exists a $[\Gamma_1(N)]$-structure ϕ_{T_1} on a 1-marked genus-1 twisted stable curve $\mathcal{C}_{T_1}$ extending the $[\Gamma_1(N)]$-structure $(\phi_\eta)_{T_1}$ on $\mathcal{C}_\eta \times_T T_1$. This is accomplished in Lemmas 4.2.7-4.2.10 below.

Terminology 4.2.6. In the following lemmas and their proofs, "base change on R" will refer to replacing R with a discrete valuation ring R_1 as above.
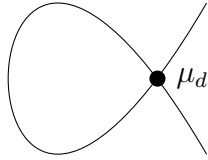
By [DR73, IV.1.6], after a base change on R we may assume that the minimal proper regular model of the coarse space C_η of $\mathcal{C}_\eta$ is a generalized elliptic curve. We maintain this assumption for the rest of the proof.

Lemma 4.2.7. *Suppose $\mathcal{C}_\eta/K$ is a smooth elliptic curve whose minimal proper regular model over R is smooth, and ϕ_η is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_\eta$. Then after base change on R, $(\mathcal{C}_\eta, \phi_\eta)$ extends to a $[\Gamma_1(N)]$-structure ϕ on a smooth elliptic curve $\mathcal{C}/R$.*

Proof. This is immediate from the fact that the stack $\mathfrak{X}_1(N)$ classifying $[\Gamma_1(N)]$-structures on generalized elliptic curves is proper ([Con07, 1.2.1]). $\qquad\square$

Lemma 4.2.8. *Suppose $(\mathcal{C}_\eta, \phi_\eta) \in \mathfrak{X}_1^{\mathrm{tw}}(N)(\eta)$, such that the coarse space C_η of $\mathcal{C}_\eta$ is singular. Then after base change on R, $(\mathcal{C}_\eta, \phi_\eta)$ extends to $(\mathcal{C}, \phi) \in \mathfrak{X}_1^{\mathrm{tw}}(N)(R)$.*

Proof. After a base change on R, we may assume that $\mathcal{C}_\eta/K$ is the standard μ_d-stacky Néron 1-gon as in Example 3.2.4, for some $d \geq 1$; since $\mathcal{C}_\eta$ admits a $[\Gamma_1(N)]$-structure and $\mathrm{Pic}^0_{\mathcal{C}_\eta/K} \cong \mathbb{G}_{m,K} \times \mathbb{Z}/(d)$, we have $d|N$.

Standard μ_d-stacky Néron 1-gon

Let $\mathcal{C}/R$ be the standard μ_d-stacky Néron 1-gon over R, so $\mathcal{C}$ is a genus-1 twisted curve over R extending $\mathcal{C}_\eta$ (and of course the marking of $\mathcal{C}_\eta$ extends to $\mathcal{C}$). After further base change on R we may assume $\mu_N(K) = \mu_N(R)$. Let

$$\phi_\eta(1) = (\zeta, a) \in \mathbb{G}_m(K) \times \mathbb{Z}/(d);$$

since ϕ_η is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_\eta$, we have $\zeta \in \mu_N(K)$ and $a \in (\mathbb{Z}/(d))^\times$. Since $\mu_N(K) = \mu_N(R)$, this extends to a section $(\zeta, a) \in \mathbb{G}_m(R) \times \mathbb{Z}/(d)$, determining a group scheme homomorphism

$$\phi : \mathbb{Z}/(N) \to \mathbb{G}_{m,R} \times \mathbb{Z}/(d)$$

with $\phi(1) = (\zeta, a)$. Since $a \in (\mathbb{Z}/(d))^\times$, ϕ is a $[\Gamma_1(N)]$-structure on $\mathcal{C}$. $\qquad\square$

Lemma 4.2.9. *Suppose R has pure characteristic $p \geq 0$, and $\mathcal{C}_\eta = E_\eta$ is an elliptic curve over K whose minimal proper regular model over R is not smooth; that is, E_η/K is an ordinary elliptic curve with bad reduction. If $\phi_\eta : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}_\eta/K}$ is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_\eta$, then after base change on R, there exists a pair $(\mathcal{C}, \phi) \in \mathfrak{X}_1^{\mathrm{tw}}(N)(R)$ extending $(\mathcal{C}_\eta, \phi_\eta)$.*

Proof. Since for any coprime N and N' we obviously have

$$\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_{NN'}) \cong \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N) \times_{\overline{\mathcal{M}}_{1,1}} \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_{N'})$$

and

$$\mathfrak{X}_1^{\mathrm{tw}}(NN') \cong \mathfrak{X}_1^{\mathrm{tw}}(N) \times_{\overline{\mathcal{M}}_{1,1}} \mathfrak{X}_1^{\mathrm{tw}}(N'),$$

it suffices to consider the cases where **(a)** N is prime to p (including the case $p = 0$) and **(b)** $N = p^n$.

(a) First suppose $(N, p) = 1$ or $p = 0$. After base extension on R we may assume that the finite abelian group $E_\eta(K)[N]$ is isomorphic to $(\mathbb{Z}/(N))^2$. After further base extension on R, the map ϕ_η gives us a μ_N-torsor $P_\eta \to E_\eta$, corresponding to the point

$$Q_\eta = \phi_\eta(1) \in \mathrm{Pic}^0_{E_\eta/K}(K) \cong E_\eta(K)$$

of "exact order N" in the sense of [KM85, §1.4] (and since $(N, p) = 1$, this just says Q_η has exact order N as an element of the group $E_\eta(K)$).

Let $\mathcal{E}/R$ be the minimal proper regular model; after further base extension on R and replacing $\mathcal{E}$ with the new minimal proper regular model of E_η, we may assume $\mathcal{E}$ has the structure of a generalized elliptic curve extending that of E_η ([DR73, IV.1.6]; also cf. [Con07, proof of 3.2.6]). Since $E_\eta(K) = \mathcal{E}^{\mathrm{sm}}(R)$, the finite flat R-group scheme $\mathcal{E}^{\mathrm{sm}}[N]$ has rank N^2, hence is étale over $\mathrm{Spec}(R)$ since $(N, p) = 1$. Therefore the special fiber $\mathcal{E}_s$ is geometrically a Néron mN-gon for some m.

Replacing $\mathcal{E}$ with its contraction away from the subgroup scheme $\mathcal{E}^{\mathrm{sm}}[N] \subset \mathcal{E}^{\mathrm{sm}}$ (cf. [DR73, IV.1]), we get a generalized elliptic curve $\mathcal{E}/R$ extending E_η, whose special fiber is geometrically a Néron N-gon, with $E_\eta(K)[N] \cong \mathcal{E}^{\mathrm{sm}}(R)[N]$.

$\mathcal{E}_\eta$ $\quad$ $\mathcal{E}_{\overline{s}}$ $\quad$ $(N\text{-gon})$
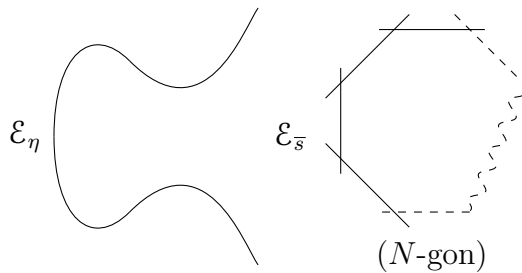
Note that Q_η extends uniquely to a section $Q \in \mathcal{E}^{\mathrm{sm}}(R)[N]$ of exact order N, which Conrad calls a "possibly non-ample $[\Gamma_1(N)]$-structure on $\mathcal{E}$," meaning Q satisfies all the conditions of a $[\Gamma_1(N)]$-structure except that the relative effective Cartier divisor

$$\sum_{a \in \mathbb{Z}/(N)} [a \cdot Q]$$

might not meet every irreducible component of a geometric closed fiber $\mathcal{E}_{\overline{s}}$.

Recall that by Corollary 3.4.3,

$$P_\eta = \underline{\mathrm{Spec}}_{E_\eta}\left(\bigoplus_{a=0}^{N-1}\mathcal{L}((a \cdot Q_\eta) - (0_{E_\eta}))\right)$$

with the μ_N-action on

$$\bigoplus \mathcal{L}((a \cdot Q_\eta) - (0_{E_\eta}))$$

induced by the $\mathbb{Z}/(N)$-grading. Since N is invertible on $\mathrm{Spec}(R)$, the assumptions of 3.4.5 are satisfied, so we may identify P_η with $E_\eta/\langle Q_\eta\rangle$, with the quotient map $P_\eta \to E_\eta$ identified with the quotient map

$$E_\eta/\langle Q_\eta\rangle \to E_\eta/E_\eta[N] \cong E_\eta.$$

Then the μ_N-action on $P_\eta = E_\eta/\langle Q_\eta\rangle$ is determined by the group scheme isomorphism $E_\eta[N]/\langle Q_\eta\rangle \cong \mu_N$ induced by the e_N-pairing and the choice of Q_η:
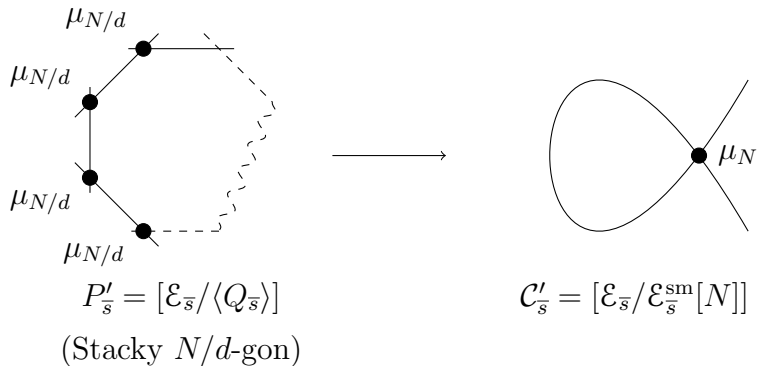
$$E_\eta[N]/\langle Q_\eta\rangle = \{Q_\eta\} \times E_\eta[N]/\langle Q_\eta\rangle \hookrightarrow \langle Q_\eta\rangle \times E_\eta[N]/\langle Q_\eta\rangle \xrightarrow{e_N} \mu_N.$$

By [Con07, Theorem 4.1.1], the e_N-pairing on E_η/K extends (possibly after further base change on R) to a nondegenerate bilinear pairing of finite flat commutative group schemes over R

$$e_N : \mathcal{E}^{\mathrm{sm}}[N] \times \mathcal{E}^{\mathrm{sm}}[N] \to \mu_N.$$

Therefore the isomorphism of group schemes $E_\eta[N]/\langle Q_\eta\rangle \cong \mu_N$ described above may be extended via the same formula to an isomorphism of group schemes $\mathcal{E}^{\mathrm{sm}}[N]/\langle Q\rangle \cong \mu_N$.

This isomorphism of group schemes makes $P' := [\mathcal{E}/\langle Q\rangle]$ a μ_N-torsor over the twisted curve $\mathcal{C}' := [\mathcal{E}/\mathcal{E}^{\mathrm{sm}}[N]]$, extending our original μ_N-torsor over E_η ($\mathcal{C}'$ is indeed a twisted curve: by [AOV11, Prop. 2.3] we may detect this on the geometric fibers, where it is clear, since the geometric closed fiber of $\mathcal{C}'$ is a standard μ_N-stacky Néron 1-gon). But P' is not necessarily representable. Indeed, let d be the minimal positive integer such that $d \cdot Q_{\overline{s}}$ lies in the identity component of the geometric closed fiber $\mathcal{E}_{\overline{s}}$. Then the coarse space $\overline{P}'_{\overline{s}}$ of $P'_{\overline{s}}$ is a Néron N/d-gon, and for any geometric point $\overline{q} \to \overline{P}'_{\overline{s}}$ mapping to a node of $\overline{P}'_{\overline{s}}$, we have $P'_{\overline{s}} \times_{\overline{P}'_{\overline{s}}} \overline{q} \cong (\mathcal{B}\mu_{N/d})_{k(\overline{q})}$.

$$P'_{\overline{s}} = [\mathcal{E}_{\overline{s}}/\langle Q_{\overline{s}}\rangle]$$
(Stacky N/d-gon)

$$\mathcal{C}'_{\overline{s}} = [\mathcal{E}_{\overline{s}}/\mathcal{E}^{\mathrm{sm}}_{\overline{s}}[N]]$$

Let $P \to \mathcal{C}$ be the μ_N-torsor corresponding to the relative coarse moduli space ([AOV11, Thm. 3.1]) of the map $\mathcal{C}' \to \mathcal{B}\mu_N$ coming from the μ_N-torsor $P' \to \mathcal{C}'$. The generic fibers are the same as those of $P' \to \mathcal{C}'$, $P_{\overline{s}}$ is a non-stacky Néron N/d-gon, and $\mathcal{C}_{\overline{s}}$ is a standard μ_d-stacky Néron 1-gon, say with coarse space $\pi : \mathcal{C}_{\overline{s}} \to C_{\overline{s}}$.

$$P_{\overline{s}}$$
$$(N/d\text{-gon})$$

$$\mathcal{C}_{\overline{s}}$$
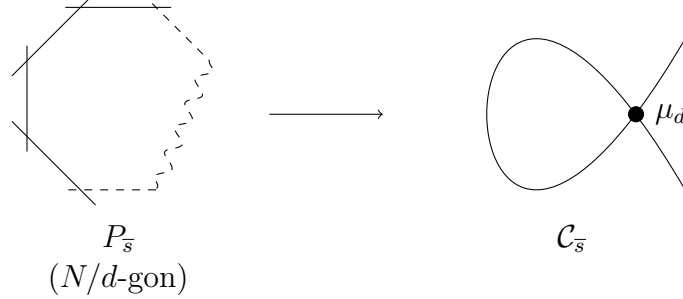
Let $\overline{q} \to C_{\overline{s}}$ be a geometric point mapping to the node of $C_{\overline{s}}$. Then $\mathcal{C}_{\overline{s}} \times_{C_{\overline{s}}} \overline{q} \cong (\mathcal{B}\mu_d)_{k(\overline{q})}$ and $P_{\overline{s}} \times_{C_{\overline{s}}} \overline{q} = \mu_{N/d} \times \overline{q}$ (which as a $k(\overline{q})$-scheme is just N/d disjoint copies of $\overline{q}$ since $(N,p) = 1$), so the resulting μ_N-torsor over $(\mathcal{B}\mu_d)_{k(\overline{q})}$ corresponds to a generator of $H^1(\mathcal{B}\mu_d, \mu_N) \cong \mathbb{Z}/(d)$. Therefore, with respect to the decomposition

$$\operatorname{Pic}^0_{\mathcal{C}_{\overline{s}}/k(\overline{s})} \cong \operatorname{Pic}^0_{C_{\overline{s}}/k(\overline{s})} \times H^0(C_{\overline{s}}, \mathbf{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{Z}/(d),$$

the class of the μ_N-torsor $P_{\overline{s}} \to \mathcal{C}_{\overline{s}}$ projects in the second factor to a generator of $\mathbb{Z}/(d)$, so the map

$$\phi : \mathbb{Z}/(N) \to \operatorname{Pic}^0_{\mathcal{C}/R}$$

induced by P is a $[\Gamma_1(N)]$-structure on the twisted curve $\mathcal{C}/R$ extending the $[\Gamma_1(N)]$-structure ϕ_η on $\mathcal{C}_\eta$.

(b) Now suppose $N = p^n$. After base change on R, the $[\Gamma_1(N)]$-structure ϕ_η gives us a μ_N-torsor $P_\eta \to E_\eta$ corresponding to the point

$$Q_\eta = \phi_\eta(1) \in \operatorname{Pic}^0_{E_\eta/K}(K) \cong E_\eta(K)$$

of "exact order N" (in the sense of [KM85, §1.4], but not necessarily as an element of the group $E_\eta(K)$). Since E_η/K is ordinary, after base change on R we have an isomorphism of group schemes over K

$$E_\eta[N] \cong \mu_N \times \mathbb{Z}/(N),$$

so

$$E_\eta(K)[N] \cong \mu_N(K) \times \mathbb{Z}/(N) = \{1\} \times \mathbb{Z}/(N) = \mathbb{Z}/(N).$$

The μ_N-torsor $P_\eta \to E_\eta$ is

$$P_\eta = \underline{\operatorname{Spec}}_{E_\eta} \left(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q_\eta) - (0_{E_\eta})) \right)$$

with the μ_N-action on

$$\bigoplus \mathcal{L}((Q_\eta) - (0_{E_\eta}))$$

induced by the $\mathbb{Z}/(N)$-grading.

Choose $m \geq 0$ minimal such that the image of $p^m \cdot Q_\eta$ in $\mathbb{Z}/(N)$ is zero. So $p^m \cdot Q_\eta = 0_{E_\eta}$ is viewed as a point of "exact order p^{n-m}" on E_η. Since

$$\mathcal{L}((p^m \cdot Q_\eta) - (0_{E_\eta})) \cong \mathcal{O}_{E_\eta},$$

the corresponding $\mu_{p^{n-m}}$-torsor over E_η is trivial. So if $\mathcal{C}/R$ is a twisted curve extending E_η/K, the $\mu_{p^{n-m}}$-torsor corresponding to $p^m \cdot Q_\eta$ will automatically extend to the trivial $\mu_{p^{n-m}}$-torsor over $\mathcal{C}$.

Now view Q_η as a point of "exact order p^m" on E_η/K. The relative effective Cartier divisor

$$D_\eta := \sum_{a \in \mathbb{Z}/(p^m)} [a \cdot Q_\eta]$$

in E_η is a subgroup scheme which is étale over K. So by 3.4.5, E_η/D_η is the underlying scheme of the μ_{p^m}-torsor corresponding to Q_η. The μ_{p^m}-action on E_η/D_η is given by the group scheme isomorphism

$$E_\eta[p^m]/D_\eta \cong \mu_{p^m}$$

induced by the e_{p^m}-pairing and the choice of Q_η:

$$E_\eta[p^m]/D_\eta = \{Q_\eta\} \times E_\eta[p^m]/D_\eta \hookrightarrow D_\eta \times E_\eta[p^m]/D_\eta \overset{e_{p^m}}{\to} \mu_{p^m}.$$

After a base change on R if necessary, let $\mathcal{E}/R$ be a generalized elliptic curve extending E_η, with a $[\Gamma_1(p^m)]$-structure Q on $\mathcal{E}$ extending Q_η. The special fiber $\mathcal{E}_s/k$ is geometrically a Néron p^m-gon, and $\mathcal{E}^{\mathrm{sm}}[p^m] \cong \mu_{p^m} \times \mathbb{Z}/(p^m)$.

$\mathcal{E}_{\overline{\eta}}$ $\qquad$ $\mathcal{E}_{\overline{s}}$

(p^m-gon)
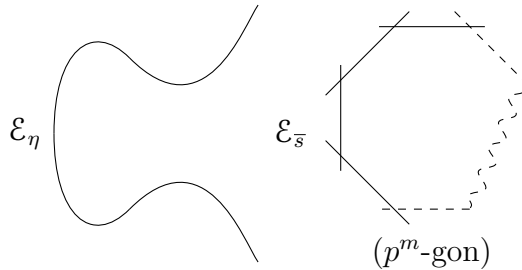
By [Con07, Theorem 4.1.1], the e_{p^m}-pairing on E_η/K extends (possibly after further base change on R) to a nondegenerate bilinear pairing of finite flat commutative group schemes over R

$$e_{p^m} : \mathcal{E}^{\mathrm{sm}}[p^m] \times \mathcal{E}^{\mathrm{sm}}[p^m] \to \mu_{p^m}.$$

Therefore the isomorphism of group schemes $E_\eta[p^m]/D_\eta \cong \mu_{p^m}$ described above may be extended via the same formula to an isomorphism of group schemes

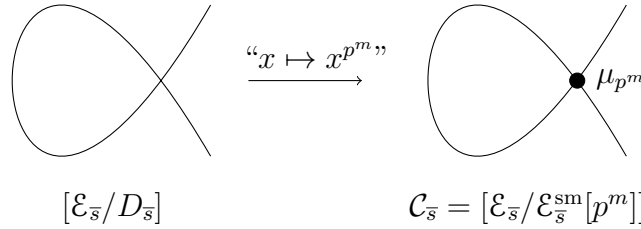$$\mathcal{E}^{\mathrm{sm}}[p^m]/D \cong \mu_{p^m},$$

where D is the relative effective Cartier divisor

$$D = \sum_{a \in \mathbb{Z}/(p^m)} [a \cdot Q]$$

in $\mathcal{E}^{\mathrm{sm}}$. This makes the quotient $[\mathcal{E}/D] = \mathcal{E}/D$ a representable μ_{p^m}-torsor over the twisted curve

$$\mathcal{C} := [\mathcal{E}/\mathcal{E}^{\mathrm{sm}}[p^m]],$$

which extends the given μ_{p^m}-torsor over E_η.

$$[\mathcal{E}_{\overline{s}}/D_{\overline{s}}] \qquad\qquad \mathcal{C}_{\overline{s}} = [\mathcal{E}_{\overline{s}}/\mathcal{E}_{\overline{s}}^{\mathrm{sm}}[p^m]]$$

The geometric special fiber $\mathcal{C}_{\overline{s}}$ is a standard μ_{p^m}-stacky Néron 1-gon, say with coarse space

$$\pi : \mathcal{C}_{\overline{s}} \to C_{\overline{s}}.$$

The quotient $[\mathcal{E}_{\overline{s}}/D_{\overline{s}}] = \mathcal{E}_{\overline{s}}/D_{\overline{s}}$ is a (non-stacky) Néron 1-gon, and the quotient map

$$P_{\overline{s}} := \mathcal{E}_{\overline{s}}/D_{\overline{s}} \to \mathcal{C}_{\overline{s}}$$

extends the map

$$P_{\overline{s}}^{\mathrm{sm}} = \mathbb{G}_m \to \mathcal{C}_{\overline{s}}^{\mathrm{sm}} = \mathbb{G}_m$$
$$x \mapsto x^{p^m}.$$

In particular, if $\overline{q} \to C_{\overline{s}}$ is a geometric point mapping to the node of $C_{\overline{s}}$, then

$$\mathcal{C} \times_C \overline{q} \cong (\mathcal{B}\mu_{p^m})_{k(\overline{q})}$$

and $P_{\overline{s}} \times_C \overline{q} = \overline{q}$, so the resulting μ_{p^m}-torsor over $(\mathcal{B}\mu_{p^m})_{k(\overline{q})}$ corresponds to a generator of

$$H^1(\mathcal{B}\mu_{p^m}, \mu_{p^m}) \cong \mathbb{Z}/(p^m).$$

Therefore, with respect to the decomposition

$$\mathrm{Pic}^0_{\mathcal{C}_{\overline{s}}/k(\overline{s})} \cong \mathrm{Pic}^0_{C_{\overline{s}}/k(\overline{s})} \times H^0(C_{\overline{s}}, \mathbf{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{Z}/(p^m),$$

the class of the μ_{p^m}-torsor $P_{\overline{s}} \to \mathcal{C}_{\overline{s}}$ projects in the second factor to a generator of $\mathbb{Z}/(p^m)$, so the group scheme homomorphism

$$\phi : \mathbb{Z}/(p^m) \to \mathrm{Pic}^0_{\mathcal{C}/R}$$

corresponding to $P := [\mathcal{E}/D] = \mathcal{E}/D$ is a $[\Gamma_1(p^m)]$-structure on the twisted curve $\mathcal{C}/R$.

Finally, write

$$P = \underline{\mathrm{Spec}}_{\mathcal{C}}\Big(\bigoplus_{a=0}^{p^m-1} \mathcal{L}_a \Big),$$

with the grading determined by the μ_{p^m}-action on P. Since P extends the μ_{p^m}-torsor on E_η determined by Q_η, we have $\mathcal{L}_a|_\eta \cong \mathcal{L}((Q_\eta) - (0_{E_\eta}))$. Then

$$P \times \mu_{p^{n-m}} = \underline{\mathrm{Spec}}_{\mathcal{C}}\Big(\bigoplus_{a=0}^{p^n-1} \mathcal{L}_{(a \bmod p^m)} \Big)$$

is a μ_{p^n}-torsor over $\mathcal{C}$ with the μ_{p^n}-action determined by the grading, extending the original μ_{p^n}-torsor over E_η and representable since P is representable. Since the group scheme homomorphism $\phi : \mathbb{Z}/(p^m) \to \mathrm{Pic}^0_{\mathcal{C}/R}$ corresponding to P is a $[\Gamma_1(p^m)]$-structure on $\mathcal{C}$, and the group scheme homomorphism $\phi' : \mathbb{Z}/(p^n) \to \mathrm{Pic}^0_{\mathcal{C}/R}$ corresponding to $P \times \mu_{p^{n-m}}$ is $\phi \circ \pi$ for the canonical projection $\pi : \mathbb{Z}/(p^n) \to \mathbb{Z}/(p^m)$, it follows immediately that ϕ' is a $[\Gamma_1(p^n)]$-structure on $\mathcal{C}$. $\qquad\square$

Lemma 4.2.10. *Suppose R has mixed characteristic $(0, p)$, and $\mathcal{C}_\eta = E_\eta$ is an elliptic curve over K whose minimal proper regular model over R is not smooth; that is, E_η/K is an ordinary elliptic curve with bad reduction. If $\phi_\eta : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}_\eta/K}$ is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_\eta$, then after base change on R, there exists a pair $(\mathcal{C}, \phi) \in \mathcal{X}_1^{\mathrm{tw}}(N)(R)$ extending $(\mathcal{C}_\eta, \phi_\eta)$.*

Proof. As before, we can restrict to the two separate cases where $(N, p) = 1$ and where $N = p^n$.

(a) If $(p, N) = 1$, the same argument as in part (a) of the proof of Lemma 4.2.9 carries through.

(b) Suppose $N = p^n$. As in the proof of Lemma 4.2.9, after base extension on R we can extend E_η to a generalized elliptic curve $\mathcal{E}/R$ whose special fiber is geometrically a Néron N-gon, such that

$$\mathcal{E}^{\mathrm{sm}}(R)[N] \cong E_\eta(K)[N] \cong (\mathbb{Z}/(N))^2$$

(the latter isomorphism being a noncanonical isomorphism of abelian groups). After further base change on R, the $[\Gamma_1(p^n)]$-structure ϕ_η gives us a μ_N-torsor $P_\eta \to E_\eta$, corresponding to

$$Q_\eta = \phi_\eta(1) \in \mathrm{Pic}^0_{E_\eta/K}(K) \cong E_\eta(K)$$

of "exact order N" in the sense of [KM85, §1.4]. Q_η extends to a "possibly non-ample $[\Gamma_1(N)]$-structure" on $\mathcal{E}/R$. Since the special fiber $\mathcal{E}_s$ is geometrically a Néron N-gon, after further base change on R we may assume $\mathcal{E}_s^{\mathrm{sm}}[N] \cong \mu_N \times \mathbb{Z}/(N)$, so

$$\mathcal{E}_s^{\mathrm{sm}}(k(s))[N] \cong \mu_N(k(s)) \times \mathbb{Z}/(N) = \{1\} \times \mathbb{Z}/(N) \cong \mathbb{Z}/(N).$$

Suppose $d \geq 1$ is minimal such that $d \cdot Q_s$ maps to 0 in $\mathbb{Z}/(N)$. Then we can choose $Q^1, Q^2 \in \mathcal{E}^{\mathrm{sm}}(R)[N]$ such that:

- $Q = Q^1 + Q^2$ in $\mathcal{E}^{\mathrm{sm}}(R)$;

- Q^1 has exact order d in the abelian group $\mathcal{E}^{\mathrm{sm}}(R)[N]$, and the relative effective Cartier divisor

$$\sum_{a=0}^{d-1} [a \cdot Q]$$

  in $\mathcal{E}^{\mathrm{sm}}$ is étale over $\mathrm{Spec}(R)$; and

- Q_s^2 maps to 0 in $\mathbb{Z}/(N)$ via the above isomorphism.

In the abelian group $\mathcal{E}^{\mathrm{sm}}(R)[N] \cong (\mathbb{Z}/(N))^2$, Q^2 has exact order e for some $e | N$. Therefore Q_η^1 is a $[\Gamma_1(d)]$-structure on E_η, and Q_η^2 is a $[\Gamma_1(e)]$-structure on E_η. They correspond via Lemma 3.4.2 to the μ_d-torsor

$$P_\eta^1 := \underline{\mathrm{Spec}}_{E_\eta} \Big(\bigoplus_{a=0}^{d-1} \mathcal{L}((a \cdot Q_\eta^1) - (0_{E_\eta})) \Big)$$

and the μ_e-torsor

$$P_\eta^2 := \underline{\mathrm{Spec}}_{E_\eta} \Big(\bigoplus_{a=0}^{e-1} \mathcal{L}((a \cdot Q_\eta^2) - (0_{E_\eta})) \Big)$$

respectively, with the gradings determining the actions of μ_d and μ_e. The μ_N-torsor corresponding to Q_η via Lemma 3.4.2 is

$$P_\eta := \underline{\mathrm{Spec}}_{E_\eta} \Big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q_\eta) - (0_{E_\eta})) \Big).$$

The group law on E_η tells us that

$$\big((Q_\eta^1) - (0_{E_\eta})\big) + \big((Q_\eta^2) - (0_{E_\eta})\big) \sim (Q_\eta^1 + Q_\eta^2) - (0_{E_\eta}) = (Q_\eta) - (0_{E_\eta}),$$

so we conclude that

$$P_\eta = \underline{\mathrm{Spec}}_{E_\eta} \Big(\bigoplus_{a=0}^{N-1} \mathcal{L}((a \cdot Q_\eta^1) - (0_{E_\eta})) \otimes \mathcal{L}((a \cdot Q_\eta^2) - (0_{E_\eta})) \Big),$$

with the μ_N-action induced by the grading.

Consider the μ_d-torsor $P_\eta^1 \to E_\eta$. As in Lemma 4.2.9, factoring the isogeny $[d]$ on E_η as $E_\eta \to E_\eta/\langle Q_\eta^1 \rangle \to E_\eta$, we have $P_\eta^1 = E_\eta/\langle Q_\eta^1 \rangle$ with μ_d acting on P_η^1 through the isomorphism with the group scheme $E_\eta[d]/\langle Q_\eta^1 \rangle$ induced via

$$E_\eta[d]/\langle Q_\eta^1 \rangle \cong \{Q_\eta^1\} \times E_\eta[d]/\langle Q_\eta^1 \rangle \hookrightarrow \langle Q_\eta^1 \rangle \times E_\eta[d]/\langle Q_\eta^1 \rangle \overset{e_d}{\to} \mu_d.$$

Let $\mathcal{E}_1/R$ (possibly after base change on R) be a generalized elliptic curve extending E_η, whose closed fiber is geometrically a Néron d-gon, with $\mathcal{E}_1^{\mathrm{sm}}(R)[d] \cong E_\eta(K)[d]$.

$(d\text{-gon})$
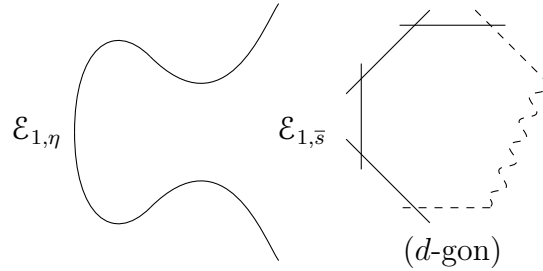
As in Lemma 4.2.9, after further base change on R we may assume that the e_d-pairing on E_η extends to a nondegenerate bilinear pairing of finite flat commutative group schemes over R

$$e_d : \mathcal{E}_1^{\mathrm{sm}}[d] \times \mathcal{E}_1^{\mathrm{sm}}[d] \to \mu_d.$$

Q_η^1 extends to a $[\Gamma_1(d)]$-structure Q^1 on $\mathcal{E}_1/R$, and the relative effective Cartier divisor

$$D := \sum_{a=0}^{d-1} [a \cdot Q^1]$$

in $\mathcal{E}_1^{\mathrm{sm}}$ is étale over R, so via the same formula as above we see that the isomorphism $E_\eta[d]/\langle Q_\eta^1 \rangle \cong \mu_d$ extends to a group scheme isomorphism

$$\mathcal{E}_1^{\mathrm{sm}}[d]/\langle Q^1 \rangle \cong \mu_d.$$

This makes

$$P^1 := [\mathcal{E}_1/\langle Q^1 \rangle] = \mathcal{E}_1/\langle Q_1 \rangle$$

a representable μ_d-torsor over the twisted curve

$$\mathcal{C} := [\mathcal{E}/\mathcal{E}^{\mathrm{sm}}[d]],$$

extending the given μ_d-torsor $P_\eta^1 \to E_\eta$.

The special fiber $\mathcal{C}_s$ is geometrically a standard μ_d-stacky Néron 1-gon, say with coarse space

$$\pi : \mathcal{C}_s \to C_s.$$

The quotient $[\mathcal{E}_{1,s}/D_s] = \mathcal{E}_{1,s}/D_s$ is geometrically a (non-stacky) Néron 1-gon, and the quotient map

$$P_s^1 := \mathcal{E}_{1,s}/D_s \to \mathcal{C}_s$$

extends the map

$$P_s^{1,\mathrm{sm}} = \mathbb{G}_m \to \mathcal{C}_s^{\mathrm{sm}} = \mathbb{G}_m$$
$$x \mapsto x^d.$$

$$P^1_{\overline{s}} = [\mathcal{E}_{1,\overline{s}}/D_{\overline{s}}] \qquad\qquad \mathcal{C}_{\overline{s}} = [\mathcal{E}_{1,\overline{s}}/\mathcal{E}^{\mathrm{sm}}_{1,\overline{s}}[d]]$$

In particular, if $\overline{q} \to C_s$ is a geometric point mapping to the node of C_s, then

$$\mathcal{C} \times_C \overline{q} \cong (\mathcal{B}\mu_d)_{k(\overline{q})}$$

and $P^1_s \times_C \overline{q} = \overline{q}$, so the resulting μ_d-torsor over $(\mathcal{B}\mu_d)_{k(\overline{q})}$ corresponds to a generator of

$$H^1(\mathcal{B}\mu_d, \mu_d) \cong \mathbb{Z}/(d).$$

Therefore, with respect to the decomposition

$$\mathrm{Pic}^0_{\mathcal{C}_s/k(s)} \cong \mathrm{Pic}^0_{C_s/k(s)} \times H^0(C_s, \mathbf{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{Z}/(d),$$

the class of the μ_d-torsor $P^1_s \to \mathcal{C}_s$ projects in the second factor to a generator of $\mathbb{Z}/(d)$. Thus the group scheme homomorphism

$$\mathbb{Z}/(d) \to \mathrm{Pic}^0_{\mathcal{C}/R}$$

defined by the μ_d-torsor

$$P^1 := [\mathcal{E}_1/D] = \mathcal{E}_1/D$$

over $\mathcal{C}$ is a $[\Gamma_1(d)]$-structure on the twisted curve $\mathcal{C}/R$.

Next consider the μ_e-torsor $P^2_\eta \to E_\eta$. Let $\pi : \mathcal{C} \to C$ be the coarse space of the twisted curve $\mathcal{C}/R$ described above. By [DR73, IV.1.6], after further base change on R we may assume that C/R is a generalized elliptic curve, with structure extending that of E_η; note that $C_\eta = E_\eta$ and that C_s is geometrically a Néron 1-gon.
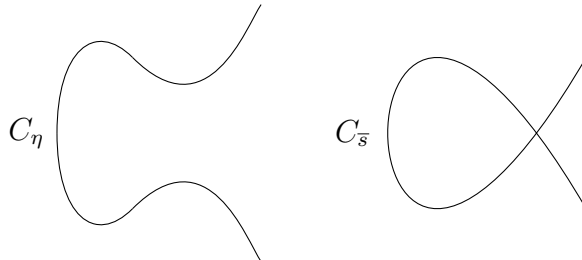
We may take the scheme-theoretic closure of the section $Q^2_\eta \in E_\eta(K)$ to get a unique section $Q^2 \in C^{\mathrm{sm}}(R)$; necessarily $e \cdot Q^2 = 0_C$ since $e \cdot Q^2_\eta = 0_{E_\eta}$. The isomorphisms

$$\mathcal{L}((a \cdot Q^2_\eta) - (0_{E_\eta})) \otimes \mathcal{L}((b \cdot Q^2_\eta) - (0_{E_\eta})) \cong \mathcal{L}(((a+b) \cdot Q^2_\eta) - (0_{E_\eta}))$$

of line bundles on E_η extend uniquely to isomorphisms

$$\mathcal{L}((a \cdot Q^2) - (0_C)) \otimes \mathcal{L}((b \cdot Q^2) - (0_C)) \cong \mathcal{L}(((a+b) \cdot Q^2) - (0_C))$$

of line bundles on C; therefore the μ_e-torsor

$$P_\eta^2 = \underline{\mathrm{Spec}}_{E_\eta}\Big(\bigoplus_{a=0}^{e-1} \mathcal{L}((a \cdot Q_\eta^2) - (0_{E_\eta}))\Big)$$

extends uniquely to a μ_e-torsor

$$P^2 := \underline{\mathrm{Spec}}_C\Big(\bigoplus_{a=0}^{e-1} \mathcal{L}((a \cdot Q^2) - (0_C))\Big)$$

over C, with the μ_e-action induced by the grading. Since $\mathrm{Pic}^0_{C/R}$ has irreducible geometric fibers, this is a $[\Gamma_1(e)]$-structure on the generalized elliptic curve C/R. Pulling this back to $\mathcal{C}$ via the coarse moduli space map $\pi : \mathcal{C} \to C$, we get a μ_e-torsor $\mathcal{P}^2 \to \mathcal{C}$ extending $P_\eta^2 \to E_\eta$, such that the corresponding map $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{C/R}$ lands in the identity component of every geometric fiber.

Finally, we return to the μ_N-torsor $P_\eta \to E_\eta$. Write the μ_d-torsor $P^1 \to \mathcal{C}$ as

$$P^1 = \underline{\mathrm{Spec}}_\mathcal{C}\Big(\bigoplus_{a=0}^{d-1} \mathcal{L}_a\Big),$$

so $\mathcal{L}_a|_\eta = \mathcal{L}((a \cdot Q_\eta^1) - (0_{E_\eta}))$; and write the μ_e-torsor $P^2 \to \mathcal{C}$ as

$$P^2 = \underline{\mathrm{Spec}}_\mathcal{C}\Big(\bigoplus_{a=0}^{e-1} \mathcal{L}'_a\Big),$$

so $\mathcal{L}'_a|_\eta = \mathcal{L}((a \cdot Q_\eta^2) - (0_{E_\eta}))$. Consider the μ_N-torsor

$$P := \underline{\mathrm{Spec}}_\mathcal{C}\Big(\bigoplus_{a=0}^{N-1} \mathcal{L}_{a \bmod d} \otimes \mathcal{L}'_{a \bmod e}\Big)$$

over $\mathcal{C}$, with the μ_N-action induced by the grading. Since

$$\mathcal{L}((a \cdot Q_\eta^1) - (0_{E_\eta})) \otimes \mathcal{L}((a \cdot Q_\eta^2) - (0_{E_\eta})) \cong \mathcal{L}((a \cdot Q_\eta) - (0_{E_\eta})),$$

we conclude that $P \to \mathcal{C}$ extends the original μ_N-torsor $P_\eta \to E_\eta$. Furthermore, with respect to the decomposition

$$\mathrm{Pic}^0_{\mathcal{C}_s/k(s)} \cong \mathrm{Pic}^0_{C_s/k(s)} \times H^0(C_s, \mathbf{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{Z}/(d),$$

the line bundle $\mathcal{L}_1|_s$ projects to a generator of $\mathbb{Z}/(d)$, and the line bundle $\mathcal{L}'_1|_s$ projects to $0 \in \mathbb{Z}/(d)$; therefore the line bundle $(\mathcal{L}_1 \otimes \mathcal{L}'_1)|_s$ projects to a generator of $\mathbb{Z}/(d)$, so the group scheme homomorphism $\mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}/R}$ corresponding to the μ_N-torsor $P \to \mathcal{C}$ is a $[\Gamma_1(N)]$-structure on $\mathcal{C}/R$, extending our original $[\Gamma_1(N)]$-structure $\phi_\eta : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}_\eta/K}$. $\square$

This concludes the proof of our final lemma and thus of Theorem 4.2.4. $\square$

4.3 Reduction mod p of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)$

The analysis of $\mathcal{H}_1(N)$ above immediately generalizes to the compactified case. Recall (3.2.8) that $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N) \subset \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ denotes the closed substack classifying rigidified twisted stable μ_N-covers of twisted curves with non-stacky marking; $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)$ is the closure of $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\mu_N) \simeq \mathcal{H}_1(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N)$ by Lemma 3.4.4.

We have a natural closed immersion

$$\iota^{(d)} : \mathcal{X}^{\mathrm{tw}}_1(d) \hookrightarrow \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)$$

for each d dividing N, precomposing a map $\mathbb{Z}/(d) \to \mathrm{Pic}^0_{\mathcal{C}/S}$ with the canonical projection $\mathbb{Z}/(N) \to \mathbb{Z}/(d)$. The resulting map

$$\bigsqcup_{d|N} \mathcal{X}^{\mathrm{tw}}_1(d) \to \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)$$

is an isomorphism over $S[1/N]$.

Definition 4.3.1. Let p be prime and $S \in \mathrm{Sch}/\mathbb{F}_p$. Let $\mathcal{C}/S$ be a 1-marked genus-1 twisted stable curve with no stacky structure at its marking. Let $n \geq 1$ and $a, b \geq 0$ with $a + b = n$. A $[\Gamma_1(p^n)]$-(a,b)-*cyclic structure* on $\mathcal{C}$ is a $[\Gamma_1(p^n)]$-structure $\phi : \mathbb{Z}/(p^n) \to \mathrm{Pic}^0_{\mathcal{C}/S}$, such that:

- if $S_1 \subset S$ is the maximal Zariski open subset such that $\mathcal{C}_{S_1} \to S_1$ is smooth, $\phi_{S_1} : \mathbb{Z}/(p^n) \to \mathcal{C}_{S_1}$ is a $[\Gamma_1(p^n)]$-(a,b)-cyclic structure in the sense of [KM85]; and

- if $S_2 \subset S$ is the complement of the supersingular locus of $\mathcal{C} \to S$, then the relative effective Cartier divisor

$$D := \sum_{m=1}^{p^b} [\phi(m)]$$

in $\mathrm{Pic}^0_{\mathcal{C}_{S_2}/S_2}$ is a subgroup scheme of $\mathrm{Pic}^0_{\mathcal{C}_{S_2}/S_2}$ which is étale over S_2.

Over the base scheme $S \in \mathrm{Sch}/\mathbb{F}_p$, we define $\mathcal{X}^{\mathrm{tw}}_1(p^n)^{(a,b)} \subset \mathcal{X}^{\mathrm{tw}}_1(p^n)$ to be the closed substack associating to T/S the groupoid of pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/S$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and ϕ is a $[\Gamma_1(p^n)]$-(a,b)-cyclic structure on $\mathcal{C}$.

If $N = p^n N'$ with $(N', p) = 1$, we define $\mathcal{X}^{\mathrm{tw}}_1(N)^{(a,b)} := \mathcal{X}^{\mathrm{tw}}_1(N') \times_{\overline{\mathcal{M}}_{1,1}} \mathcal{X}^{\mathrm{tw}}_1(p^n)^{(a,b)}$.

The same argument as that used to prove Lemma 2.3.5 immediately gives us:

Lemma 4.3.2. *Let $\mathcal{C}/S/\mathbb{F}_p$ be a 1-marked genus-1 twisted stable curve with non-stacky marking, and let $\phi : \mathbb{Z}/(p^n) \to \mathrm{Pic}^0_{\mathcal{C}/S}$ be a $[\Gamma_1(p^n)]$-(a,b)-cyclic structure on $\mathcal{C}$. Then for the canonical projection $\pi : \mathbb{Z}/(p^{n+1}) \twoheadrightarrow \mathbb{Z}/(p^n)$, the composite $\phi \circ \pi : \mathbb{Z}/(p^{n+1}) \to \mathrm{Pic}^0_{\mathcal{C}/S}$ is a $[\Gamma_1(p^{n+1})]$-structure on $\mathcal{C}$, and is $[\Gamma_1(p^{n+1})]$-$(a+1,b)$-cyclic.*

If $N = p^n N'$ with $(N', p) = 1$, for any $r | N'$ write $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)^r \subset \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)$ for the union of the components $\mathfrak{X}^{\mathrm{tw}}_1(p^m r)$ over $0 \leq m \leq n$.

Corollary 4.3.3. *Let k be a perfect field of characteristic p, and let $N = p^n N'$ where $(N', p) = 1$. For any $r | N'$, $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)^r_k$ is the disjoint union, with crossings at the supersingular points, of components $\mathcal{Z}^r_b$ for $0 \leq b \leq n$, where*

$$\mathcal{Z}^r_b = \bigcup_{b \leq m \leq n} \mathfrak{X}^{\mathrm{tw}}_1(p^m r)^{(m-b,b)}_k,$$

identifying each $\mathfrak{X}^{\mathrm{tw}}_1(p^m r)^{(m-b,b)}_k$ with a closed substack of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)_k$ via $\iota^{(p^m r)}$. Each substack $\mathfrak{X}^{\mathrm{tw}}_1(p^m r)^{(m-b,b)}_k$ is "set-theoretically identified with $\mathcal{Z}^r_b$" in the sense that $(\mathcal{Z}^r_b)_{\mathrm{red}} = \mathfrak{X}_1(p^m r)^{(m-b,b)}_{k,\mathrm{red}}$ as substacks of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)_{k,\mathrm{red}}$. $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)_k$ is the disjoint union of the open and closed substacks

$$\{\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N)^r_k\}_{r|N'}.$$

Strictly speaking, to apply the crossings theorem (Theorem 2.3.1) to get the above corollary, we need to know that the morphism $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N) \to \overline{\mathcal{M}}_{1,1}$ is finite. This follows from Lemma 6.1.3 below.

The picture in the case $N = p^n$ is essentially the same as the picture for $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\mu_{p^n})_k$ (as discussed after Proposition 4.1.1), except now each component is proper:
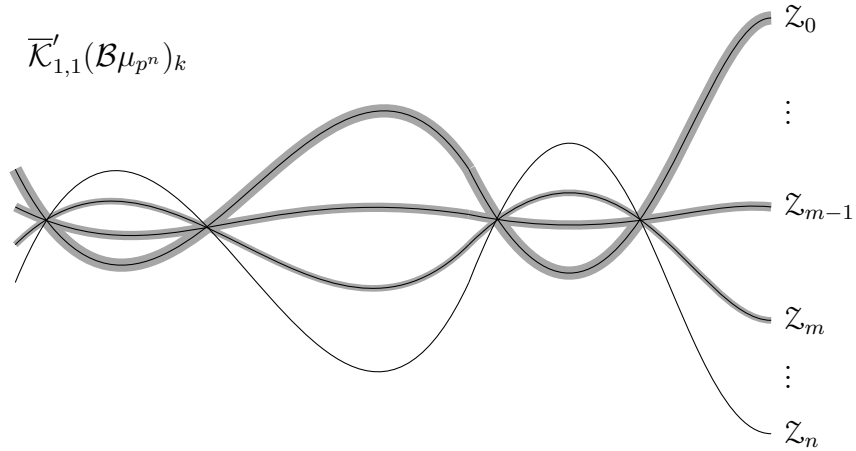
Chapter 5

Moduli of elliptic curves in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$

5.1   Reduction mod p of $\mathcal{H}(N)$

Next we turn our attention to $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$, working as before over an arbitrary base scheme S. Recall that by Corollary 3.4.3, the open substack $\overline{\mathcal{K}}_{1,1}^{\circ}(\mathcal{B}\mu_N^2)$ classifying rigidified twisted μ_N^2-covers of smooth elliptic curves is naturally equivalent to the stack $\mathcal{H}(N)$ associating to a scheme T/S the groupoid of pairs (E, ϕ) where E/T is an elliptic curve and

$$\phi : (\mathbb{Z}/(N))^2 \to E[N]$$

is a homomorphism of group schemes over T.

For any subgroup $K \leq (\mathbb{Z}/(N))^2$ with corresponding quotient $G_K = (\mathbb{Z}/(N))^2/K$ of $(\mathbb{Z}/(N))^2$, recall (Definition 2.3.11) that $\mathcal{Y}_K$ denotes the moduli stack associating to a scheme T/S the groupoid of pairs (E, ψ), where E/T is an elliptic curve and $\psi : G_K \to E$ is a G_K-structure (in the sense of [KM85, §1.5]). So for example, if $G_K \cong \mathbb{Z}/(d)$ for some $d|N$, $\mathcal{Y}_K$ is isomorphic to the stack $\mathcal{Y}_1(d)$, and if $G_K \cong (\mathbb{Z}/(d))^2$ then $\mathcal{Y}_K$ is isomorphic to the stack $\mathcal{Y}(d)$ classifying (not necessarily symplectic) $[\Gamma(d)]$-structures on elliptic curves.

For every such K, we have a closed immersion

$$\iota^K : \mathcal{Y}_K \hookrightarrow \mathcal{H}(N),$$

given by precomposing a G_K-structure $\phi : G_K \to E[N]$ with the canonical projection $(\mathbb{Z}/(N))^2 \to G_K$. Together, these give a proper surjection

$$\bigsqcup_{K \leq (\mathbb{Z}/(N))^2} \mathcal{Y}_K \to \mathcal{H}(N)$$

which is an isomorphism over $S[1/N]$.

But in characteristics dividing N this is not an isomorphism. First we consider the case where $N = p^n$ for some prime p. Any quotient $G_K = (\mathbb{Z}/(p^n))^2/K$ is isomorphic as an abelian group to $\mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l)$ for some $l \leq m \leq n$. The corresponding moduli stack $\mathcal{Y}_K$

classifies G_K-structures on elliptic curves, and we saw in Theorem 2.3.12 that over a perfect field k of characteristic p, $\mathcal{Y}_{K,k}$ is the disjoint union, with crossings at the supersingular points, of substacks $\mathcal{Y}_{K,k}^H$ indexed by the set

$$L_K := \{H \leq G_K | H \text{ and } G_K/H \text{ are both cyclic}\}.$$

The component $\mathcal{Y}_{K,k}^H$ classifies G_K-structures of component label H.

Now consider two subgroups $K' \leq K \leq (\mathbb{Z}/(p^n))^2$, and write $\pi : G_{K'} \to G_K$ for the canonical surjection. If $\phi : G_K \to E[p^n]$ is a G_K-structure on an ordinary elliptic curve $E/T/k$, then $\phi \circ \pi$ may or may not be a $G_{K'}$-structure on E. Indeed, we saw in Lemma 2.3.13 that $\phi \circ \pi$ is a $G_{K'}$ structure if and only if $\pi^{-1}(H) \in L_{K'}$, i.e. if and only if $\pi^{-1}(H) \subseteq G_{K'}$ is cyclic.

Consider the set

$$\{(K, H) | K \leq (\mathbb{Z}/(p^n))^2, H \in L_K\}.$$

Let $\sim$ be the equivalence relation on this set generated by requiring that $(K, H) \sim (K', H')$ if $K' \leq K$ and H, H' are as above, and let $\Lambda = \{(K, H)\}/\sim$.

Proposition 5.1.1. *Let k be a perfect field of characteristic p. $\mathcal{H}(p^n)_k$ is the disjoint union, with crossings at the supersingular points, of components $\mathcal{H}(p^n)_k^\lambda$ for $\lambda \in \Lambda$, where*

$$\mathcal{H}(p^n)_k^\lambda := \bigcup_{[(K,H)]=\lambda} \mathcal{Y}_{K,k}^H,$$

identifying each $\mathcal{Y}_{K,k}^H$ with a closed substack of $\mathcal{H}(p^n)_k^\lambda$ via ι^K. Each $\mathcal{Y}_{K,k}^H$ is "set-theoretically identified with $\mathcal{H}(p^n)_k^\lambda$" in the sense that

$$(\mathcal{H}(p^n)_k^\lambda)_{\mathrm{red}} = \mathcal{Y}_{K,k,\mathrm{red}}^H$$

as substacks of $\mathcal{H}(p^n)_{k,\mathrm{red}}$ for all (K, H) with $[(K, H)] = \lambda \in \Lambda$.

If $N = p^n N'$ with $(p, N') = 1$, for any $A \leq (\mathbb{Z}/(N))^2$ of order prime to p, let

$$\mathcal{H}(N)_k^A \subset \mathcal{H}(N)_k$$

be the union of the substacks $\mathcal{Y}_{K,k}$ for $A \leq K \leq (\mathbb{Z}/(N))^2$ with $(K : A)$ a power of p. Then similarly $\mathcal{H}(N)_k^A$ is the disjoint union, with crossings at the supersingular points, of components $\mathcal{H}(N)_k^{A,\lambda}$ for $\lambda \in \Lambda$, and $\mathcal{H}(N)_k$ is the disjoint union of the open and closed substacks $\mathcal{H}(N)_k^A$ for $A \leq (\mathbb{Z}/(N))^2$ of order prime to p.

As in the case of $\mathcal{H}_1(p^n)$, the reduction mod p of $\mathcal{H}(p^n)$ has an appealing geometric description. To keep our pictures from getting unreasonably large, we restrict our attention to the case $n = 1$. The group $K := (\mathbb{Z}/(p))^2$ has $p + 3$ subgroups, namely the entire group K, $K_0 := 0$, and $p + 1$ subgroups $K_1, ..., K_{p+1}$ isomorphic to $\mathbb{Z}/(p)$, corresponding to

$$\mathcal{Y}_K = \mathcal{Y}(1)$$
$$\mathcal{Y}_{K_0} = \mathcal{Y}(p)$$
$$\mathcal{Y}_{K_i} \cong \mathcal{Y}_1(p) \quad \text{for } i = 1, ..., p + 1.$$

Thus, over $\mathbb{Z}[1/p]$, $\mathcal{H}(p)$ is the disjoint union of $\mathcal{Y}(1)$, $\mathcal{Y}(p)$ and $p+1$ copies of $\mathcal{Y}_1(p)$.

By definition we have

$$
\begin{aligned}
L_K &= \{0\} \\
L_{K_0} &= \{K_1, ..., K_{p+1}\} \\
L_{K_i} &= \{G_{K_i}, 0\} \quad \text{for } i = 1, ..., p+1,
\end{aligned}
$$

where as usual $G_{K_i} = (\mathbb{Z}/(p))^2/K_i$. The set of labels Λ is built by putting an equivalence relation on the set consisting of the following pairs:

$$
\begin{aligned}
&(K, 0) \\
&(K_0, K_i) \quad \text{for } i = 1, ..., p+1 \\
&(K_i, G_{K_i}) \quad \text{for } i = 1, ..., p+1 \\
&(K_i, 0) \quad \text{for } i = 1, ..., p+1.
\end{aligned}
$$

By definition, working over a perfect field k of characteristic p, the pair $(K, 0)$ corresponds to $\mathcal{Y}(1)_k$; the pair (K_0, K_i) corresponds to the component $\mathcal{Y}(p)_k^{K_i}$ of $\mathcal{Y}(p)_k$; the pair (K_i, G_{K_i}) corresponds to the component $\mathcal{Y}_1(p)_k^{(1,0)}$ in $\mathcal{Y}_{K_i,k} \cong \mathcal{Y}_1(p)_k$; and the pair $(K_i, 0)$ corresponds to the component $\mathcal{Y}_1(p)_k^{(0,1)}$ in $\mathcal{Y}_{K_i,k} \cong \mathcal{Y}_1(p)_k$. Unwinding the definition, we see that the equivalence relation defining Λ just says that $(K, 0) \sim (K_i, G_{K_i})$ for $i = 1, ..., p+1$, and that $(K_0, K_i) \sim (K_i, 0)$ for $i = 1, ..., p+1$. Thus the set of labels is $\Lambda = \{\lambda_0, \lambda_1, ..., \lambda_{p+1}\}$, where

$$
\begin{aligned}
\lambda_0 &= [(K, 0)] = [(K_i, G_{K_i})] \quad \text{for } i = 1, ..., p+1 \\
\lambda_i &= [(K_0, K_i)] = [(K_i, 0)] \quad \text{for } i = 1, ..., p+1.
\end{aligned}
$$

$\mathcal{H}(p)_{k,\text{red}}$ can be visualized as follows:
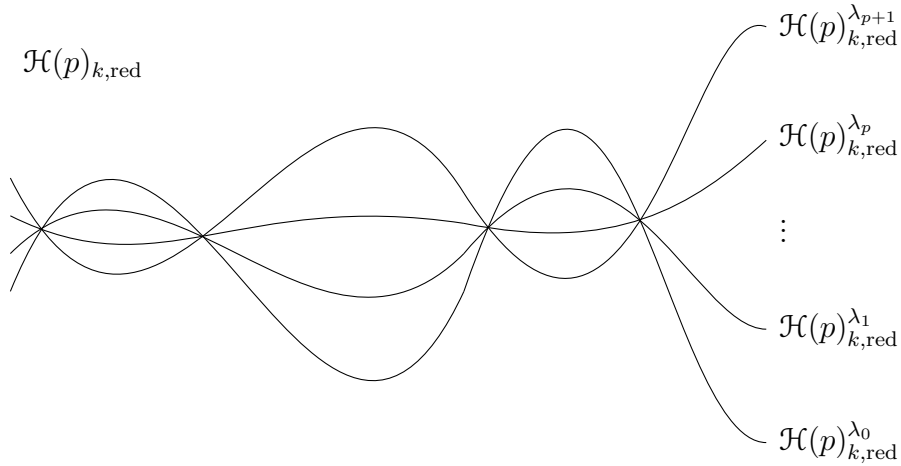
The component $\mathcal{H}(p)_k^{\lambda_0}$ is "set-theoretically identified" with the component $\mathcal{Y}(1)_k$ and with the component $\mathcal{Y}_1(p)_k^{(1,0)}$ in each copy of $\mathcal{Y}_1(p)_k$; each of these contributes additional nilpotent

structure to the component $\mathcal{H}(p)_k^{\lambda_0}$ (each $\mathcal{Y}_1(p)_k^{(1,0)}$ has length $p-1$ over $\mathcal{H}(p)_{k,\text{red}}^{\lambda_0} = \mathcal{Y}(1)_k$, so $\mathcal{H}(p)_k^{\lambda_0}$ has length $(p+1)(p-1)+1 = p^2$ over $\mathcal{H}(p)_{k,\text{red}}^{\lambda_0} = \mathcal{Y}(1)_k$).

$\mathcal{H}(p)_k^{\lambda_0}$
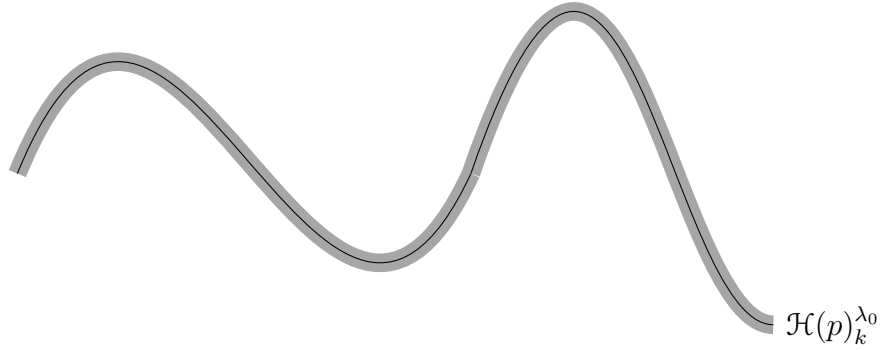
For $i = 1, ..., p+1$, the component $\mathcal{H}(p)_k^{\lambda_i}$ is "set-theoretically identified" with the component $\mathcal{Y}(p)_k^{K_i}$ of $\mathcal{Y}(p)_k$, contributing nilpotent structure to $\m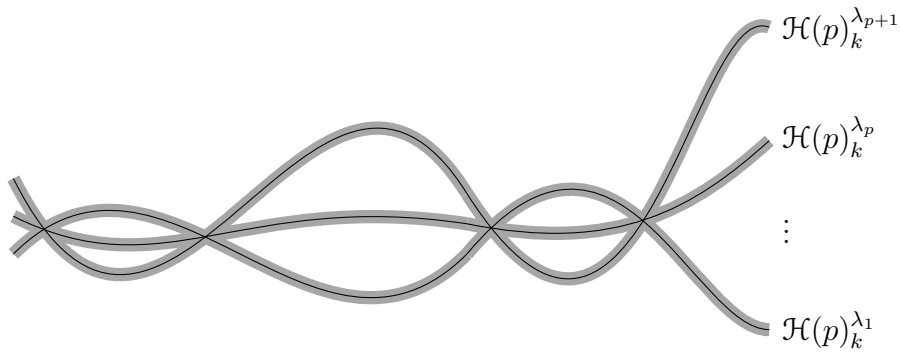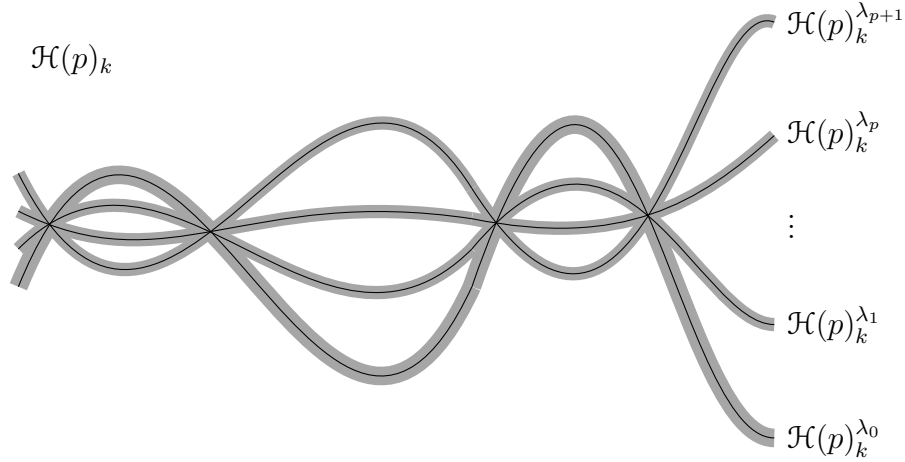athcal{H}(p)_k^{\lambda_i}$ (each $\mathcal{Y}(p)_k^{K_i}$ has length $p-1$ over $\mathcal{H}(p)_{k,\text{red}}^{\lambda_i}$). The component $\mathcal{H}(p)_k^{\lambda_i}$ is also "set-theoretically identified" with the component $\mathcal{Y}_1(p)_k^{(0,1)}$ of $\mathcal{Y}_{K_i,k} \cong \mathcal{Y}_1(p)_k$; each of these is reduced, adding 1 to the length of the component $\mathcal{H}(p)_k^{\lambda_i}$ over $\mathcal{H}(p)_{k,\text{red}}^{\lambda_i}$. Thus $\mathcal{H}(p)_k^{\lambda_i}$ has length p over $\mathcal{H}(p)_{k,\text{red}}^{\lambda_i}$, which is isomorphic to $\mathcal{Y}_1(p)_k^{(0,1)}$ and hence has degree $p^2 - p$ over $\mathcal{Y}(1)_k$ (cf. [KM85, 13.5.6]). The result is that the component $\mathcal{H}(p)_k^{\lambda_i}$ has length p over the underlying reduced stack $\mathcal{Y}_1(p)_k^{(0,1)}$, which has degree $p^2 - p$ over $\mathcal{Y}(1)_k$:

$\mathcal{H}(p)_k^{\lambda_{p+1}}$

$\mathcal{H}(p)_k^{\lambda_p}$

$\vdots$

$\mathcal{H}(p)_k^{\lambda_1}$

This gives us the following picture of $\mathcal{H}(p)_k$:

The figure is labeled with $\mathcal{H}(p)_k$ on the left, and on the right, from top to bottom: $\mathcal{H}(p)_k^{\lambda_{p+1}}$, $\mathcal{H}(p)_k^{\lambda_p}$, $\vdots$, $\mathcal{H}(p)_k^{\lambda_1}$, $\mathcal{H}(p)_k^{\lambda_0}$.

Note that adding up the lengths calculated in the course of the above construction, we recover the fact that the stack $\mathcal{H}(p)$ has length p^4 over $\mathcal{Y}(1) = \mathcal{M}_{1,1}$.

5.2   Closure of $\mathcal{Y}(N)$ in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$

Definition 5.2.1. Let $\mathcal{C}/S$ be a 1-marked genus-1 twisted stable curve over a scheme S, with no stacky structure at its marking. A $[\Gamma(N)]$-*structure* on $\mathcal{C}$ is a group scheme homomorphism $\phi : (\mathbb{Z}/(N))^2 \to \mathrm{Pic}^0_{\mathcal{C}/S}$ such that:

* the relative effective Cartier divisor

$$D := \sum_{a \in (\mathbb{Z}/(N))^2} [\phi(a)]$$

in $\mathrm{Pic}^0_{\mathcal{C}/S}$ is an N-torsion subgroup scheme, hence $D = \mathrm{Pic}^0_{\mathcal{C}/S}[N]$; and

* for every geometric point $\overline{p} \to S$, $D_{\overline{p}}$ meets every irreducible component of $(\mathrm{Pic}^0_{\mathcal{C}/S})_{\overline{p}} = \mathrm{Pic}^0_{\mathcal{C}_{\overline{p}}/k(\overline{p})}$.

We write $\mathcal{X}^{\mathrm{tw}}(N)$ for the substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$ associating to T/S the groupoid of pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/S$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and ϕ is a $[\Gamma(N)]$-structure on $\mathcal{C}$.

Note that if $\mathcal{C}/S$ is a twisted curve admitting a $[\Gamma(N)]$-structure and $\overline{p} \to S$ is a geometric point such that $C_{\overline{p}}$ is singular, then necessarily

$$\mathrm{Pic}^0_{\mathcal{C}_{\overline{p}}/k(\overline{p})} \cong \mathbb{G}_m \times \mathbb{Z}/(N),$$

so by Lemma 3.3.8 $\mathcal{C}_{\overline{p}}$ is a standard μ_N-stacky Néron 1-gon over $k(\overline{p})$, as in 3.2.4.

Applying the methods of our study of $\mathcal{X}_1^{\mathrm{tw}}(N)$ to the stack $\mathcal{X}^{\mathrm{tw}}(N)$, we have:

Theorem 5.2.2. *Let S be a scheme and $\mathcal{X}^{\mathrm{tw}}(N)$ the stack over S classifying $[\Gamma(N)]$-structures on 1-marked genus-1 twisted stable curves with non-stacky marking. Then $\mathcal{X}^{\mathrm{tw}}(N)$ is a closed substack of $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$, which contains $\mathcal{Y}(N)$ as an open dense substack.*

In particular $\mathcal{X}^{\mathrm{tw}}(N)$ is flat over S with local complete intersection fibers, and is proper and quasi-finite over $\overline{\mathcal{M}}_{1,1}$.

Remark 5.2.3. In [Pet12] a direct proof is given that over $\mathbb{Z}[1/N]$, $\mathcal{X}^{\mathrm{tw}}(N)$ agrees with the stack $\mathcal{X}(N)$ classifying $[\Gamma(N)]$-structures on generalized elliptic curves; in this case a μ_N^2-torsor over a 1-marked genus-1 twisted stable curve with non-stacky marking is in fact a generalized elliptic curve, and it is this observation that gives the desired equivalence. This argument does not generalize to characteristics dividing N, because, for example, if $N = p^n$ then the Néron N-gon in characteristic p (which is a generalized elliptic curve admitting various $[\Gamma(N)]$-structures) cannot be realized as a μ_N^2-torsor over a 1-marked genus-1 twisted stable curve.

Proof of 5.2.2. This is proved in exactly the same manner as Theorem 4.2.4. An immediate consequence of Lemma 3.4.5 is that if E/K is an elliptic curve over a field K in which N is invertible, and (Q_1, Q_2) is a $[\Gamma(N)]$-structure on E, then the μ_N^2-torsor

$$P = \underline{\mathrm{Spec}}_E\Big(\bigoplus_{a,b \in \mathbb{Z}/(N)} \mathcal{L}((a \cdot Q_1 + b \cdot Q_2) - (0_E))\Big)$$

over E as in 3.4.3 may be identified with E itself, with the quotient map $P \to E$ corresponding to the isogeny $[N] : E \to E$. We immediately deduce via the methods of Theorem 4.2.4 that over an algebraically closed field k in which N is invertible, the μ_N^2-torsor obtained when we pass to the cusp of $\overline{\mathcal{M}}_{1,1}$ is a Néron N-gon P over a standard μ_N-stacky Néron 1-gon $\mathcal{C}$, with the μ_N^2-action on P induced by some choice of isomorphism $P^{\mathrm{sm}}[N] \cong \mu_N^2$:

$$\mu_N^2 \cong P^{\mathrm{sm}}[N]$$

P $\mathcal{C} = [P/P^{\mathrm{sm}}[N]]$
(N-gon)
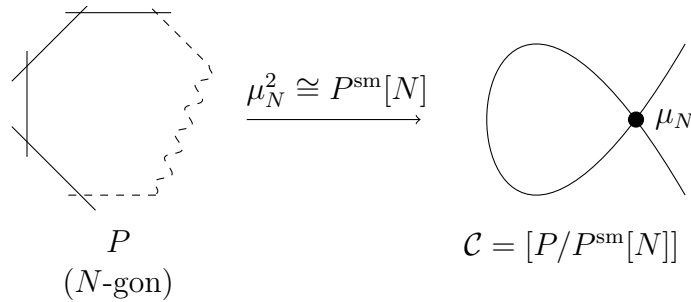
And in the case of $N = p^n$, over an algebraically closed k field of characteristic p, the $\mu_{p^n}^2$-torsor P obtained in passing to the cusp of $\overline{\mathcal{M}}_{1,1}$ may be realized as a trivial μ_{p^n}-torsor over a standard Néron 1-gon C', which in turn is a μ_{p^n}-torsor over a standard μ_{p^n}-stacky Néron 1-gon $\mathcal{C} = [C'/\mu_{p^n}]$ via the choice of an isomorphism $(C')^{\mathrm{sm}}[p^n] \cong \mu_{p^n}$:

$$P = C' \times \mu_{p^n} \qquad\qquad C' \qquad\qquad \mathcal{C}$$
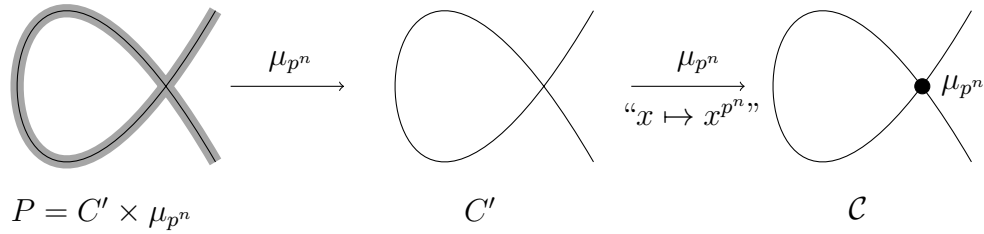
In both of the above cases, it is immediately verified that for each of these μ_N^2-torsors, the corresponding group scheme homomorphism

$$(\mathbb{Z}/(N))^2 \to \operatorname{Pic}^0_{\mathcal{C}/k} \cong \mathbb{G}_m \times \mathbb{Z}/(N)$$

is a $[\Gamma(N)]$-structure on the standard μ_N-stacky Néron 1-gon $\mathcal{C}$ in the sense of Definition 5.2.1, giving the valuative criterion of properness for $\mathfrak{X}^{\mathrm{tw}}(N)$, hence Theorem 5.2.2. $\qquad\square$

5.3 Reduction mod p of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)$

Recall that $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2) \subset \overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$ is the closed substack classifying rigidified twisted stable μ_N^2-covers of twisted curves with non-stacky marking; so $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)$ is the closure of

$$\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\mu_N^2) \simeq \mathcal{H}(N)$$

in $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\mu_N^2)$.

Definition 5.3.1. Let $\mathcal{C}/S$ be a 1-marked genus-1 twisted stable curve with non-stacky marking, and let G be a 2-generated finite abelian group, say $G \cong \mathbb{Z}/(n_1) \times \mathbb{Z}/(n_2)$, $n_1 \geq n_2$. A G-*structure* on $\mathcal{C}$ is a homomorphism $\phi : G \to \operatorname{Pic}^0_{\mathcal{C}/S}$ of group schemes over S such that:

- the relative effective Cartier divisor

$$D := \sum_{a \in G} [\phi(a)]$$

  in $\operatorname{Pic}^0_{\mathcal{C}/S}$ is an n_1-torsion subgroup scheme; and

- for every geometric point $\bar{p} \to S$, $D_{\bar{p}}$ meets every irreducible component of $(\operatorname{Pic}^0_{\mathcal{C}/S})_{\bar{p}} = \operatorname{Pic}^0_{\mathcal{C}_{\bar{p}}/k(\bar{p})}$.

For any subgroup $K \leq (\mathbb{Z}/(N))^2$ with corresponding quotient $G_K = (\mathbb{Z}/(N))^2/K$ of $(\mathbb{Z}/(N))^2$, we write $\mathfrak{X}^{\mathrm{tw}}_K$ for the moduli stack over S associating to a scheme T/S the groupoid of pairs $(\mathcal{C}, \psi)$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and $\psi : G_K \to \operatorname{Pic}^0_{\mathcal{C}/T}$ is a G_K-structure on $\operatorname{Pic}^0_{\mathcal{C}/T}$.

For a twisted curve $\mathcal{C}/S/\mathbb{F}_p$, if $N = p^n$ and $G_K \cong \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l)$ with $m \geq l \geq 1$, we set

$$L_K := \{H \leq G_K | H \text{ and } G_K/H \text{ are both cyclic}\},$$

and for any $H \in L_K$ we say a G_K-structure $\phi : G_K \to \mathrm{Pic}^0_{\mathcal{C}/S}$ has *component label* H if H maps to the kernel of the n-fold relative Frobenius F^n on the group scheme $\mathrm{Pic}^0_{\mathcal{C}/S}$ over S, and the resulting group scheme homomorphism $G_K/H \to \mathrm{Pic}^0_{\mathcal{C}/S}[p^n]/\mathrm{ker}(F^n)$ is a G_K/H-structure in the sense of [KM85, §1.5].

If $G_K \cong \mathbb{Z}/(p^m)$ (i.e. $l = 0$), then $\mathcal{X}_K^{\mathrm{tw}} \cong \mathcal{X}_1^{\mathrm{tw}}(p^m)$, and for $H \cong \mathbb{Z}/(p^a) \in L_K$ we define $\mathcal{X}_K^{\mathrm{tw},H} \subset \mathcal{X}_K^{\mathrm{tw}}$ to be the substack $\mathcal{X}_1^{\mathrm{tw}}(p^m)^{(a,m-a)} \subset \mathcal{X}_1^{\mathrm{tw}}(p^m)$ as in Definition 4.3.1. We still say that $\mathcal{X}_K^{\mathrm{tw},H}$ classifies G_K-structures of *component label* H.

So for example, if $G_K \cong \mathbb{Z}/(d)$ for some $d|N$, $\mathcal{X}_K^{\mathrm{tw}}$ is isomorphic to the stack $\mathcal{X}_1^{\mathrm{tw}}(d)$, and if $G_K \cong (\mathbb{Z}/(d))^2$ then $\mathcal{X}_K^{\mathrm{tw}}$ is isomorphic to the stack $\mathcal{X}^{\mathrm{tw}}(d)$. For every such K, we have a closed immersion

$$\iota^K : \mathcal{X}_K^{\mathrm{tw}} \hookrightarrow \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2),$$

given by precomposing a G_K-structure $\phi : G_K \to \mathrm{Pic}^0_{\mathcal{C}/T}$ with the canonical projection $(\mathbb{Z}/(N))^2 \to G_K$. Together, these give a proper surjection

$$\bigsqcup_{K \leq (\mathbb{Z}/(N))^2} \mathcal{X}_K^{\mathrm{tw}} \to \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)$$

which is an isomorphism over $S[1/N]$.

Let k be a perfect field of characteristic p. The same argument proving Theorem 2.3.13 immediately gives us:

Corollary 5.3.2. *If $K \leq (\mathbb{Z}/(p^n))^2$ such that $G_K := (\mathbb{Z}/(p^n))^2/K \cong \mathbb{Z}/(p^m) \times \mathbb{Z}/(p^l)$ with $l \leq m \leq n$, then $\mathcal{X}_{K,k}^{\mathrm{tw}}$ is the disjoint union, with crossings at the supersingular points, of closed substacks $\mathcal{X}_{K,k}^{\mathrm{tw},H}$ for $H \in L_K$. $\mathcal{X}_{K,k}^{\mathrm{tw},H}$ classifies G_K-structures with component label H.*

As before, we let Λ denote the set $\{(K,H)|K \leq (\mathbb{Z}/(p^n))^2, H \in L_K\}$, modulo the equivalence relation generated by declaring $(K,H) \sim (K', \pi^{-1}(H))$ whenever $K' \leq K$ with corresponding quotient map $\pi : G_{K'} \to G_K$ such that $\pi^{-1}(H) \in L_{K'}$. We conclude:

Corollary 5.3.3. *Let k be a perfect field of characteristic p. $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_k$ is the disjoint union, with crossings at the supersingular points, of components $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_k^\lambda$ for $\lambda \in \Lambda$, where*

$$\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_k^\lambda := \bigcup_{[(K,H)]=\lambda} \mathcal{X}_{K,k}^{\mathrm{tw},H},$$

identifying each $\mathcal{X}_{K,k}^{\mathrm{tw},H}$ with a closed substack of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_k^\lambda$ via ι^K. Each $\mathcal{X}_{K,k}^{\mathrm{tw},H}$ is "set-theoretically identified with $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_k^\lambda$" in the sense that

$$\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n}^2)_{k,\mathrm{red}}^\lambda = \mathcal{X}_{K,k,\mathrm{red}}^{\mathrm{tw},H}$$

as substacks of $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_{p^n})_{k,\mathrm{red}}$ for all (K,H) with $[(K,H)] = \lambda \in \Lambda$.

If $N = p^n N'$ with $(p, N') = 1$, for any $A \leq (\mathbb{Z}/(N))^2$ of order prime to p, let

$$\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k^A \subset \overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k$$

be the union of the substacks $\mathfrak{X}_{K,k}^{\mathrm{tw}}$ for $A \leq K \leq (\mathbb{Z}/(N))^2$ with $(K : A)$ a power of p. Then similarly $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k^A$ is the disjoint union, with crossings at the supersingular points, of components $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k^{A,\lambda}$ for $\lambda \in \Lambda$, and $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k$ is the disjoint union of the open and closed substacks $\overline{\mathcal{K}}'_{1,1}(\mathcal{B}\mu_N^2)_k^A$ for $A \leq (\mathbb{Z}/(N))^2$ of order prime to p.

The picture in the case $N = p$ is essentially the same as the picture for $\overline{\mathcal{K}}^{\circ}_{1,1}(\mathcal{B}\mu_p^2)_k \simeq \mathcal{H}(p)_k$ (as discussed after Proposition 5.1.1), except now each component is proper:
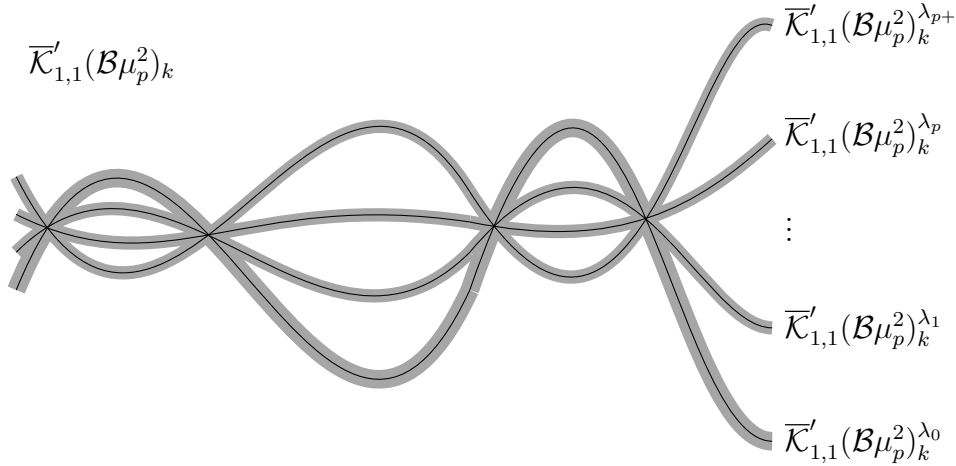
Chapter 6

Comparison with the classical moduli stacks

As promised, we verify that the moduli stacks $\mathfrak{X}_1^{\mathrm{tw}}(N)$ and $\mathfrak{X}^{\mathrm{tw}}(N)$ are isomorphic to the corresponding classical moduli stacks, justifying the claim in [AOV11] that we have recovered the Katz-Mazur regular models:

Theorem 6.1.1 (Restatement of 1.1.2)**.** *Over the base* $S = \mathrm{Spec}(\mathbb{Z})$*, there is a canonical isomorphism of algebraic stacks*

$$\mathfrak{X}_1^{\mathrm{tw}}(N) \cong \mathfrak{X}_1(N)$$

extending the identity map on $\mathcal{Y}_1(N)$*, and a canonical isomorphism of algebraic stacks*

$$\mathfrak{X}^{\mathrm{tw}}(N) \cong \mathfrak{X}(N)$$

extending the identity map on $\mathcal{Y}(N)$*.*

We prove this after some preliminary results; the main point is to demonstrate that $\mathfrak{X}_1^{\mathrm{tw}}(N)$ and $\mathfrak{X}^{\mathrm{tw}}(N)$ are normal.

Proposition 6.1.2. *The morphism* $\pi : \mathfrak{X}_1^{\mathrm{tw}}(N) \to \overline{\mathcal{M}}_{1,1}$ *sending* $(\mathcal{C}, \phi)$ *to the coarse space* C *of* $\mathcal{C}$ *is a representable morphism of stacks. In particular,* $\mathfrak{X}_1^{\mathrm{tw}}(N)$ *is Deligne-Mumford.*
Similarly the natural morphism $\mathfrak{X}^{\mathrm{tw}}(N) \to \overline{\mathcal{M}}_{1,1}$ *is representable, and so* $\mathfrak{X}^{\mathrm{tw}}(N)$ *is Deligne-Mumford.*

Proof. We have already seen that $\mathfrak{X}_1^{\mathrm{tw}}(N)$ is an algebraic stack, so it suffices to show that for any object $(\mathcal{C}, \phi) \in \mathfrak{X}_1^{\mathrm{tw}}(N)(k)$ with k an algebraically closed field, the natural map

$$\mathrm{Aut}(\mathcal{C}, \phi) \to \mathrm{Aut}(C)$$

is a monomorphism of group schemes. Here C/k is the coarse space of $\mathcal{C}$, and automorphisms are required to preserve the marking. It is obvious that $\mathrm{Aut}(\mathcal{C}, \phi) \to \mathrm{Aut}(C)$ is a monomorphism if $\mathcal{C} = C$ is a smooth elliptic curve over k, so by Lemma 3.3.8 we reduce to the case where $\mathcal{C}/k$ is a standard μ_d-stacky Néron 1-gon for some $d|N$.
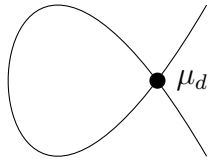
In this case an automorphism of $\mathcal{C}$ is an automorphism of the coarse space C, together with an automorphism of the μ_d-gerbe in $\mathcal{C}$ lying over the node of C. Thus

$$\operatorname{Aut}(\mathcal{C}) \cong \operatorname{Aut}(C) \times \operatorname{Aut}(\mathcal{B}\mu_{d,k}).$$

The only nontrivial automorphism of C preserving the marked point $1 \in C$ is the automorphism $\iota : C \to C$ induced by the inversion automorphism of $\mathbb{G}_m$. We have $\operatorname{Aut}(\mathcal{B}\mu_d) \cong \mu_d$, and the automorphism of $\operatorname{Pic}^0_{\mathcal{C}/k} \cong \mathbb{G}_m \times \mathbb{Z}/(d)$ induced by $(0, \zeta) \in \operatorname{Aut}(\mathcal{C}) \cong \langle \iota \rangle \times \mu_d$ sends (η, a) to $(\zeta^a \eta, a)$. Since

$$\phi : \mathbb{Z}/(N) \to \operatorname{Pic}^0_{\mathcal{C}/k}$$

meets every component of $\operatorname{Pic}^0_{\mathcal{C}/k}$, the only automorphisms of $\mathcal{C}$ than can possibly preserve ϕ are the automorphisms $\langle \iota \rangle \times \{0\} \subset \operatorname{Aut}(\mathcal{C})$ (cf. [Con07, proof of 3.1.8]). Thus

$$\operatorname{Aut}(\mathcal{C}, \phi) \subset \langle \iota \rangle \times \{0\} \cong \operatorname{Aut}(C).$$

The same argument applies to $\mathcal{X}^{\mathrm{tw}}(N)$. $\qquad\qquad\square$

Corollary 6.1.3. $\mathcal{X}_1^{\mathrm{tw}}(N)$ *and* $\mathcal{X}^{\mathrm{tw}}(N)$ *are finite over* $\overline{\mathcal{M}}_{1,1}$.

Proof. By Theorems 4.2.4 and 5.2.2, the natural maps $\mathcal{X}_1^{\mathrm{tw}}(N) \to \overline{\mathcal{M}}_{1,1}$ and $\mathcal{X}^{\mathrm{tw}}(N) \to \overline{\mathcal{M}}_{1,1}$ are proper and quasi-finite; since they are also representable, they are finite. $\qquad\square$

Let $\infty \hookrightarrow \overline{\mathcal{M}}_{1,1}$ denote the closed substack classifying 1-marked genus-1 stable curves whose geometric fibers are singular. Let

$$\mathcal{X}_1^{\mathrm{tw}}(N)^\infty = \mathcal{X}_1^{\mathrm{tw}}(N) \times_{\overline{\mathcal{M}}_{1,1}} \infty$$

and

$$\mathcal{X}^{\mathrm{tw}}(N)^\infty = \mathcal{X}^{\mathrm{tw}}(N) \times_{\overline{\mathcal{M}}_{1,1}} \infty.$$

Exactly analogously to [Con07, 2.1.12], formation of these closed substacks is compatible with arbitrary base change.

Proposition 6.1.4. *The proper flat morphisms* $\mathcal{X}_1^{\mathrm{tw}}(N) \to \operatorname{Spec}(\mathbb{Z})$ *and* $\mathcal{X}^{\mathrm{tw}}(N) \to \operatorname{Spec}(\mathbb{Z})$ *are Cohen-Macaulay (of pure relative dimension 1).*

Proof. Let $\mathcal{X}$ denote $\mathcal{X}_1^{\mathrm{tw}}(N)$ or $\mathcal{X}^{\mathrm{tw}}(N)$. The canonical morphism $\mathcal{X} \to \overline{\mathcal{M}}_{1,1}$ is finite (by Corollary 6.1.3) and flat (by Theorems 4.2.4 and 5.2.2), and the structural morphism $\overline{\mathcal{M}}_{1,1} \to \operatorname{Spec}(\mathbb{Z})$ is Cohen-Macaulay (cf. [Con07, 3.3.1]), so by [Bou07, 2.7.9, Cor. 3], the composite $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ is Cohen-Macaulay. $\qquad\square$

Lemma 6.1.5. $\mathfrak{X}_1^{\mathrm{tw}}(N)^\infty$ *and* $\mathfrak{X}^{\mathrm{tw}}(N)^\infty$ *are relative effective Cartier divisors over* $\mathrm{Spec}(\mathbb{Z})$ *in* $\mathfrak{X}_1^{\mathrm{tw}}(N)$ *and* $\mathfrak{X}^{\mathrm{tw}}(N)$ *respectively.*

Proof. Here we are using the notion of a Cartier divisor on a Deligne-Mumford stack, as in [ACG11, Ch. XIII]. For $\mathfrak{X} = \mathfrak{X}_1^{\mathrm{tw}}(N)$ or $\mathfrak{X} = \mathfrak{X}^{\mathrm{tw}}(N)$, write $\mathfrak{X}^\infty$ for the pullback $\mathfrak{X} \times_{\overline{\mathcal{M}}_{1,1}} \infty$. We know $\infty \subset \overline{\mathcal{M}}_{1,1}$ is a relative effective Cartier divisor over $\mathrm{Spec}(\mathbb{Z})$ (meaning an effective Cartier divisor which is flat over $\mathrm{Spec}(\mathbb{Z})$), and by Theorems 4.2.4 and 5.2.2 the morphism $\mathfrak{X} \to \overline{\mathcal{M}}_{1,1}$ is flat.

Cartier divisors are preserved by flat morphisms (cf. [Ful98, §1.7]), so $\mathfrak{X}^\infty$ is an effective Cartier divisor in $\mathfrak{X}$. Since $\mathfrak{X} \to \overline{\mathcal{M}}_{1,1}$ is flat, so is $\mathfrak{X}^\infty \to \infty$, so $\mathfrak{X}^\infty$ is flat over $\mathrm{Spec}(\mathbb{Z})$, i.e. $\mathfrak{X}^\infty$ is a relative effective Cartier divisor in $\mathfrak{X}$ over $\mathrm{Spec}(\mathbb{Z})$. $\square$

Corollary 6.1.6. $\mathfrak{X}_1^{\mathrm{tw}}(N)$ *and* $\mathfrak{X}^{\mathrm{tw}}(N)$ *are normal.*

Proof. This is proven in an identical manner to [Con07, 4.1.4]. The stacks $\mathfrak{X}_1^{\mathrm{tw}}(N)$ and $\mathfrak{X}^{\mathrm{tw}}(N)$ are Deligne-Mumford, and from [ACV03, 3.0.2] we know $\mathfrak{X}_1^{\mathrm{tw}}(N) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ and $\mathfrak{X}^{\mathrm{tw}}(N) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ are smooth over $\mathrm{Spec}(\mathbb{Z}[1/N])$. In particular, they are regular at any characteristic 0 points. Furthermore, by Proposition 6.1.4 they are Cohen-Macaulay over $\mathrm{Spec}(\mathbb{Z})$ of pure relative dimension 1.

As in [Con07, 4.1.4], we can conclude from Serre's criterion for normality that it suffices to prove that these stacks are regular away from some relative effective Cartier divisor, since such a divisor cannot contain any codimension 1 points of positive residue characteristic. Use the divisors $\mathfrak{X}_1^{\mathrm{tw}}(N)^\infty$ and $\mathfrak{X}^{\mathrm{tw}}(N)^\infty$; their complements are $\mathcal{Y}_1(N)$ and $\mathcal{Y}(N)$, which are regular by [KM85, 5.1.1]. $\square$

Proof of 6.1.1. $\mathfrak{X}_1^{\mathrm{tw}}(N)$ and $\mathfrak{X}^{\mathrm{tw}}(N)$ are finite, flat and normal over $\overline{\mathcal{M}}_{1,1}$, so they are naturally identified with the normalizations (in the sense of [DR73, IV.3.3]) of $\overline{\mathcal{M}}_{1,1}$ in

$$\mathfrak{X}_1^{\mathrm{tw}}(N)|_{\mathcal{M}_{1,1}} = \mathcal{Y}_1(N)$$

and

$$\mathfrak{X}^{\mathrm{tw}}(N)|_{\mathcal{M}_{1,1}} = \mathcal{Y}(N)$$

respectively; cf. [Con07, 4.1.5]. $\square$

We now give a moduli interpretation of the equivalence $\mathfrak{X}_1(N) \simeq \mathfrak{X}_1^{\mathrm{tw}}(N)$. Let S be a scheme, E/S a generalized elliptic curve, and $P \in E^{\mathrm{sm}}(S)[N]$ a $[\Gamma_1(N)]$-structure on E. From this data we want to construct a pair $(\mathcal{C}_P, \phi_P)$, where $\mathcal{C}_P/S$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and

$$\phi_P : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}_P/S}$$

is a $[\Gamma_1(N)]$-structure on $\mathcal{C}_P$.

If E/S is a smooth elliptic curve, there is nothing to show: we simply take $(\mathcal{C}_P, \phi_P) = (E, \phi_P)$ where

$$\phi_P : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{E/S} \cong E$$

sends $1 \mapsto P$. Therefore to construct $(\mathcal{C}_P, \phi_P)$ in general, we may restrict to the open subscheme of S where E/S has no supersingular geometric fibers; once we have constructed $(\mathcal{C}_P, \phi_P)$ in this case, we only need to check that it agrees with our previous construction for ordinary elliptic curves.

For the rest of the construction, assume that E/S has no supersingular geometric fibers.

Note that by [Con07, 4.2.3], fppf-locally on S there exists a generalized elliptic curve E'/S, whose singular geometric fibers are N-gons, together with an open S-immersion

$$\iota : E^{\mathrm{sm}} \hookrightarrow E'^{\mathrm{sm}}$$

of group schemes over S. In particular, by [DR73, II.1.20] the group scheme $E'^{\mathrm{sm}}[N]/S$ is finite and flat of constant rank N^2.

Since E'/S has no supersingular geometric fibers and all its singular geometric fibers are N-gons, it follows that fppf-locally on S, there exists a $[\Gamma(N)]$-structure (Q, R) on E' such that:

- the relative effective Cartier divisor

$$D := \sum_{a \in \mathbb{Z}/(N)} [a \cdot Q]$$

  in E'^{sm} is étale over S, and

- R meets the identity component of every geometric fiber of E'/S.

The choice of Q and the pairing

$$e_N : E'^{\mathrm{sm}}[N] \times E'^{\mathrm{sm}}[N] \to \mu_N$$

induce a canonical isomorphism

$$E'^{\mathrm{sm}}[N]/D \cong E'^{\mathrm{sm}}[N]/D \times \{Q\} \cong \mu_N.$$

Identifying $E'^{\mathrm{sm}}[N]/D$ with its image in the N-torsion of the generalized elliptic curve $C := E'/D$ (a generalized elliptic curve whose singular fibers are 1-gons), the group law of C and the above isomorphism give us an action of μ_N on C, making C a μ_N-torsor over the twisted curve

$$\mathcal{C} := [C/\mu_N] = [E'/E'^{\mathrm{sm}}[N]] = [E/E^{\mathrm{sm}}[N]].$$

Write

$$C = \underline{\mathrm{Spec}}_{\mathcal{C}} \Big(\bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{G}_a \Big) \xrightarrow{\pi} \mathcal{C},$$

where each $\mathcal{G}_a$ is an invertible $\mathcal{O}_{\mathcal{C}}$-module, with the grading and algebra structure corresponding to the structure of C as a μ_N-torsor over $\mathcal{C}$.

The image $\overline{R}$ of R in C is a $[\Gamma_1(N)]$-structure on C, so we get a μ_N-torsor

$$T := \underline{\mathrm{Spec}}_C \Big(\bigoplus_{b \in \mathbb{Z}/(N)} \mathcal{L}((b \cdot \overline{R}) - (0_C)) \Big)$$

over C; the μ_N-action on T corresponds to the $\mathbb{Z}/(N)$-grading and the algebra structure on

$$\bigoplus_{b \in \mathbb{Z}/(N)} \mathcal{L}((b \cdot \overline{R}) - (0_C))$$

comes from the group law on C^{sm} and the canonical isomorphism $C^{\mathrm{sm}} \cong \mathrm{Pic}^0_{C/S}$.

Since C is a μ_N-torsor over $\mathcal{C}$, if $\mathcal{L} \in \mathrm{Pic}(C)$ we have a canonical decomposition

$$\pi_* \mathcal{L} = \bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}_a,$$

where each $\mathcal{L}_a$ is an invertible sheaf on $\mathcal{C}$ and $\zeta \in \mu_N$ acts on $\mathcal{L}_a$ via multiplication by ζ^a. In particular this applies to the invertible sheaf

$$\mathcal{L} = \mathcal{L}((b \cdot \overline{R}) - (0_C)),$$

giving us a canonical decomposition

$$\pi_* \mathcal{L}((b \cdot \overline{R}) - (0_C)) = \bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}_{(a,b)}.$$

We have $\mathcal{L}_{0,0} = \mathcal{G}_0 = \mathcal{O}_{\mathcal{C}}$, and the isomorphisms

$$\mathcal{L}((b_0 \cdot \overline{R}) - (0_C)) \otimes_{\mathcal{O}_C} \mathcal{L}((b_1 \cdot \overline{R}) - (0_C)) \cong \mathcal{L}(((b_0 + b_1) \cdot \overline{R}) - (0_C))$$

(coming from the algebra structure of $\oplus_b \mathcal{L}((b \cdot \overline{R}) - (0_C))$) induce isomorphisms

$$\mathcal{L}_{(a_0,b_0)} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{L}_{(a_1,b_1)} \cong \mathcal{L}_{(a_0+a_1,b_0+b_1)}$$

for all $(a_0, b_0), (a_1, b_1) \in (\mathbb{Z}/(N))^2$, giving us a canonical algebra structure on the direct sum

$$\bigoplus_{(a,b) \in (\mathbb{Z}/(N))^2} \mathcal{L}_{(a,b)}.$$

Identifying our original $[\Gamma_1(N)]$-structure P with its image in $E'^{\mathrm{sm}}(S)[N]$, there exists some $(a_0, b_0) \in (\mathbb{Z}/(N))^2$ with

$$P = a_0 \cdot Q + b_0 \cdot R \in E'^{\mathrm{sm}}[N].$$

This determines a μ_N-torsor

$$\mathcal{T} := \underline{\mathrm{Spec}}_{\mathcal{C}} \Big(\bigoplus_{c \in \mathbb{Z}/(N)} \mathcal{L}_{(ca_0, cb_0)} \Big)$$

over $\mathcal{C}$, corresponding to a morphism $\mathcal{C} \to \mathcal{B}\mu_N$. Here $\oplus \mathcal{L}_{(ca_0, cb_0)}$ is viewed as a sub-$\mathcal{O}_{\mathcal{C}}$-algebra of the algebra $\oplus \mathcal{L}_{(a,b)}$.

Definition 6.1.7. We define $\mathcal{C}_P \to \mathcal{B}\mu_N$ to be the relative coarse moduli space of the above morphism $\mathcal{C} \to \mathcal{B}\mu_N$, and we write $\phi_P : \mathbb{Z}/(N) \to \operatorname{Pic}^0_{\mathcal{C}_P/S}$ for the corresponding group scheme homomorphism.

It is immediate that ϕ_P is a $[\Gamma_1(N)]$-structure on the twisted curve $\mathcal{C}_P$.

Lemma 6.1.8. $(\mathcal{C}_P, \phi_P)$ *is independent of the choice of* (a_0, b_0) *with* $P = a_0 \cdot Q + b_0 \cdot R$, *and of the choice of generalized elliptic curve* E' *and* $[\Gamma(N)]$-*structure* (Q, R) *on* E' *such that* $D = \sum[a \cdot Q]$ *is étale over* S *and* R *meets the fiberwise identity components of* E'/S.

Proof. First of all, if E/S is an ordinary elliptic curve, then $E' = E$. Our construction defines a map
$$E[N] \to \operatorname{Pic}^0_{E/S}[N] \cong E[N],$$
which in fact is simply the identity map (which in particular is independent of the choice of (a_0, b_0) and the $[\Gamma(N)]$-structure (Q, R)). To see this, recall that by Lemma 3.4.5 we may identify the generalized elliptic curve $C = E/\langle Q \rangle$ (viewed as a μ_N-torsor over $E \cong [E/E[N]]$ as discussed above) with the μ_N-torsor
$$\underline{\operatorname{Spec}}_E \left(\bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}((a \cdot Q) - (0_E)) \right)$$
over E. So in the notation of the above construction, $\mathcal{C} = E$ and $\mathcal{G}_a = \mathcal{L}_{(a,0)} = \mathcal{L}((a \cdot Q) - (0_E))$. We have
$$\pi_* \mathcal{L}((\overline{R}) - (0_C)) \cong \bigoplus_{a \in \mathbb{Z}/(N)} \left(\mathcal{G}_a \otimes \mathcal{L}((R) - (0_E)) \right)$$
$$\cong \bigoplus_{a \in \mathbb{Z}/(N)} \mathcal{L}((a \cdot Q + R) - (0_E)).$$

So the map defined in the above construction sends Q to $\mathcal{L}_{(1,0)} = \mathcal{L}((Q) - (0_E))$ and R to $\mathcal{L}_{(0,1)} = \mathcal{L}((R) - (0_E))$. Composing with the usual isomorphism $\operatorname{Pic}^0_{E/S} \cong E$ yields the identity map on $E[N]$.

To complete the proof in the case where E/S is not necessarily smooth, it suffices to consider the case where S is the spectrum of an algebraically closed field k and E/k is a Néron d-gon (for some $d|N$). E'/k is then a Néron N-gon, and our Drinfeld basis (Q, R) was chosen so that $\langle Q \rangle$ meets every irreducible component of E' and $\langle R \rangle$ lies on the identity component. We may therefore choose an isomorphism $E'^{\text{sm}} \cong \mathbb{Z}/(N) \times \mu_N$ such that $Q = (1,1)$ and $R = (0, \zeta)$ for some $\zeta \in \mu_N^\times(k)$.
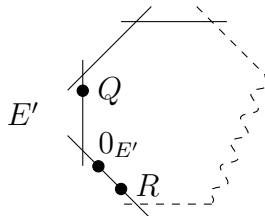
Then if
$$P = a_0 \cdot Q + b_0 \cdot R = a_1 \cdot Q + b_1 \cdot R,$$
it follows that $a_0 = a_1$ and $\zeta^{b_0} = \zeta^{b_1}$, the latter of which implies that $b_0 \cdot \overline{R} = b_1 \cdot \overline{R}$. Thus
$$\mathcal{L}((b_0 \cdot \overline{R}) - (0_C)) = \mathcal{L}((b_1 \cdot \overline{R}) - (0_C)),$$
so $\mathcal{L}_{(a_0,b_0)} = \mathcal{L}_{(a_1,b_1)}$, hence $(\mathcal{C}_P, \phi_P)$ is independent of the choice of (a_0, b_0).

Now we must check that $(\mathcal{C}_P, \phi_P)$ is independent of the choice of $(E', (Q, R))$. E'/k is a Néron N-gon, so the choice of E' is unique up to composition with an automorphism of E' fixing $E^{\mathrm{sm}} \subset E'^{\mathrm{sm}}$. E' is the special fiber of an N-gon Tate curve $\mathcal{E}'/k[\![q^{1/N}]\!]$. Let $\mathcal{C} = [\mathcal{E}'/\mathcal{E}'^{\mathrm{sm}}[N]]$, so $\mathcal{C}_k = [E'/E'^{\mathrm{sm}}[N]] = [E/E^{\mathrm{sm}}[N]]$.
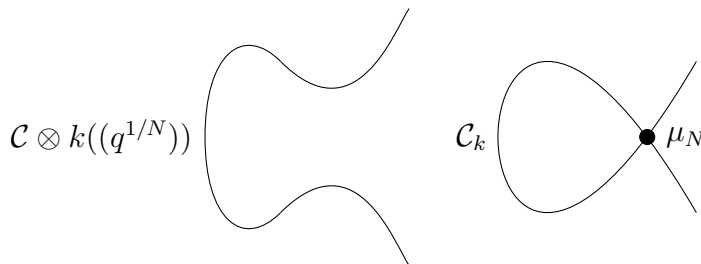
We may choose an isomorphism $\mathcal{E}'^{\mathrm{sm}}[N] \cong \mathbb{Z}/(N) \times \mu_N$ of finite flat group schemes over $k[\![q^{1/N}]\!]$. (Q, R) extends to a $[\Gamma(N)]$-structure $(\mathcal{Q}, \mathcal{R})$ on $\mathcal{E}'$ with $\mathcal{D} = \sum(a \cdot \mathcal{Q})$ étale over $k[\![q^{1/N}]\!]$ (and of course $\mathcal{R}$ meets the identity component of every geometric fiber of $\mathcal{E}'/k[\![q^{1/N}]\!]$). Given such a $[\Gamma(N)]$-structure on $\mathcal{E}'$, our construction defines a group scheme homomorphism
$$\mathcal{E}'^{\mathrm{sm}}[N] \to \mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N].$$
Both of these are finite flat group schemes over $k[\![q^{1/N}]\!]$ which are isomorphic to $\mathbb{Z}/(N) \times \mu_N$, and $\underline{\mathrm{End}}(\mathbb{Z}/(N) \times \mu_N)$ is finite (hence proper) over $k[\![q^{1/N}]\!]$. Since
$$\mathcal{E}'^{\mathrm{sm}}[N] \otimes k((q^{1/N})) \to \mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N] \otimes k((q^{1/N}))$$
is independent of the choice of $[\Gamma(N)]$-structure over $k((q^{1/N}))$ (as $\mathcal{E}' \otimes k((q^{1/N}))$ is an elliptic curve), we conclude that
$$\mathcal{E}'^{\mathrm{sm}}[N] \to \mathrm{Pic}^0_{\mathcal{C}/k[\![q^{1/N}]\!]}[N]$$
is independent of the choice of $(\mathcal{Q}, \mathcal{R})$. Thus in particular $E'^{\mathrm{sm}}[N] \to \mathrm{Pic}^0_{\mathcal{C}_k/k}[N]$ is independent of the choice of (Q, R) and the resulting homomorphism $E^{\mathrm{sm}}[N] \to \mathrm{Pic}^0_{\mathcal{C}_k/k}[N]$ is independent of the choice of E'. $\qquad\square$

Thus by descent, $(\mathcal{C}_P, \phi_P) \in \mathcal{X}_1^{\mathrm{tw}}(N)(S)$ is well-defined globally over our initial base scheme S (even allowing supersingular fibers) and depends only on the pair
$$(E, P) \in \mathcal{X}_1(N)(S).$$
We define our map $\mathcal{X}_1(N) \to \mathcal{X}_1^{\mathrm{tw}}(N)$ by sending (E, P) to $(\mathcal{C}_P, \phi_P)$.

Corollary 6.1.9. *Over any base scheme S, the morphism $\mathfrak{X}_1(N) \to \mathfrak{X}_1^{\mathrm{tw}}(N)$ sending (E, P) to $(\mathcal{C}_P, \phi_P)$ is an isomorphism of algebraic stacks.*

Similarly, given a generalized elliptic curve E/S with a $[\Gamma(N)]$-structure (P_1, P_2), the above procedure produces a $[\Gamma(N)]$-structure $\phi_{(P_1, P_2)}$ on the twisted curve

$$\mathcal{C}_E := [E/E^{\mathrm{sm}}[N]].$$

Corollary 6.1.10. *Over any base scheme S, the morphism*

$$\mathfrak{X}(N) \to \mathfrak{X}^{\mathrm{tw}}(N)$$
$$(E, (P_1, P_2)) \mapsto (\mathcal{C}_E, \phi_{(P_1, P_2)})$$

is an isomorphism of algebraic stacks.

Chapter 7

Other compactified moduli stacks of elliptic curves

It is worth noting that the techniques in the proof of 4.2.4 are easily adapted to prove properness of the natural analogues in our current setting of well-known modular compactifications of other various moduli stacks of elliptic curves with extra structure, even when these moduli stacks do not naturally lie in a moduli stack of twisted stable maps:

Definition 7.1.1. Let $\mathcal{C}/S$ be a 1-marked genus-1 twisted stable curve with non-stacky marking.

(i) A $[\Gamma_0(N)]$-*structure* on $\mathcal{C}$ is a finite locally free S-subgroup scheme G of $\operatorname{Pic}^0_{\mathcal{C}/S}$ of rank N over S which is cyclic (fppf-locally admits a $\mathbb{Z}/(N)$-generator), such that for every geometric point $\overline{p} \to S$, $G_{\overline{p}}$ meets every irreducible component of $\operatorname{Pic}^0_{\mathcal{C}_{\overline{p}}/k(\overline{p})}$. We write $\mathfrak{X}^{\mathrm{tw}}_0(N)$ for the stack over S associating to T/S the groupoid of pairs $(\mathcal{C}, G)$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and G is a $[\Gamma_0(N)]$-structure on $\mathcal{C}$.

(ii) A *balanced* $[\Gamma_1(N)]$-*structure* (cf. [KM85, §3.3]) on $\mathcal{C}$ is an fppf short exact sequence of commutative group schemes over S

$$0 \to K \to \operatorname{Pic}^0_{\mathcal{C}/S}[N] \to K' \to 0, \qquad (\dagger)$$

where K and K' are locally free of rank N over S, together with sections $P \in K(S)$ and $P' \in K'(S)$ which are $\mathbb{Z}/(N)$-generators of K and K' in the sense of [KM85, §1.4]. We write $\mathfrak{X}^{\mathrm{bal,tw}}_1(N)$ for the stack over S associating to T/S the groupoid of pairs $(\mathcal{C}, \dagger)$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and $\dagger$ is a balanced $[\Gamma_1(N)]$-structure on $\mathcal{C}$.

(iii) An $[N\text{-Isog}]$-*structure* (cf. [KM85, §6.5]) on $\mathcal{C}$ is a finite locally free commutative S-subgroup scheme $G \subset \operatorname{Pic}^0_{\mathcal{C}/S}[N]$ of rank N over S, such that for every geometric point $\overline{p} \to S$, $G_{\overline{s}}$ meets every irreducible component of $\operatorname{Pic}^0_{\mathcal{C}_{\overline{p}}/k(\overline{p})}$. We write $\mathfrak{X}^{\mathrm{tw}}(N\text{-Isog})$ for the stack over S associating to T/S the groupoid of pairs $(\mathcal{C}, G)$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and G is an $[N\text{-Isog}]$-structure on $\mathcal{C}$.

(iv) If N and n are positive integers such that $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for all primes p dividing both N and n, a $[\Gamma_1(N; n)]$-*structure* (cf. [Con07, 2.4.3]) on $\mathcal{C}$ is a pair (ϕ, G), where:

- $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{\mathcal{C}/S}$ is a $\mathbb{Z}/(N)$-structure in the sense of [KM85, §1.5];

- $G \subset \mathrm{Pic}^0_{\mathcal{C}/S}$ is a finite locally free S-subgroup scheme which is cyclic of order n;

- the degree Nn relative effective Cartier divisor

$$\sum_{a=0}^{N-1} (\phi(a) + G)$$

  in $\mathrm{Pic}^0_{\mathcal{C}/S}$ meets every irreducible component of each geometric fiber of $\mathrm{Pic}^0_{\mathcal{C}/S}$ over S;

- for all primes p dividing both N and n, for $e = \mathrm{ord}_p(n)$ we have an equality of closed subschemes of $\mathrm{Pic}^0_{\mathcal{C}/S}$

$$\sum_{a=0}^{p^e-1} ((N/p^e) \cdot \phi(a) + G[p^e]) = \mathrm{Pic}^0_{\mathcal{C}/S}[p^e].$$

We write $\mathcal{X}_1^{\mathrm{tw}}(N; n)$ for the stack over S associating to T/S the groupoid of tuples $(\mathcal{C}, (\phi, G))$, where $\mathcal{C}/T$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, and (ϕ, G) is a $[\Gamma_1(N; n)]$-structure on $\mathcal{C}$.

Corollary 7.1.2. *The stacks $\mathcal{X}_0^{\mathrm{tw}}(N)$, $\mathcal{X}_1^{\mathrm{bal,tw}}(N)$, $\mathcal{X}^{\mathrm{tw}}(N\text{-Isog})$, and $\mathcal{X}_1^{\mathrm{tw}}(N; n)$ are algebraic stacks which are flat and locally finitely presented over S, with local complete intersection fibers. They are proper and quasi-finite over $\overline{\mathcal{M}}_{1,1}$, and each is isomorphic to the corresponding moduli stack for generalized elliptic curves.*

As shown explicitly for the stacks $\mathcal{X}_1^{\mathrm{tw}}(N)$ and $\mathcal{X}^{\mathrm{tw}}(N)$ earlier in this paper, one may study the reductions modulo p of these moduli stacks, and one finds that over a perfect field of characteristic p, each stack is a disjoint union with crossings at the supersingular points of various closed substacks, which come naturally equipped with moduli interpretations extending the definitions given in [KM85] (except for $\mathcal{X}_1^{\mathrm{tw}}(N; n)$, which is not studied in [KM85]).

Question 7.1.3. If $\mathcal{C}/S$ is a 1-marked genus-1 twisted stable curve with non-stacky marking, the group scheme $\mathrm{Pic}^0_{\mathcal{C}/S}$ behaves just like the smooth part of a generalized elliptic curve. Over the Zariski open set of S where $\mathcal{C} \to S$ is smooth, it agrees with C (which is in this case a smooth elliptic curve); if $\overline{p} \to S$ is a geometric point such that $C_{\overline{p}}$ is singular, then $\mathrm{Pic}^0_{\mathcal{C}_{\overline{p}}/k(\overline{p})} \cong \mathbb{G}_m \times \mathbb{Z}/(N)$ for some N, and this is just the smooth part of a Néron N-gon over $k(\overline{p})$. Is there a natural way to exhibit $\mathrm{Pic}^0_{\mathcal{C}/S}$ as the smooth part of a generalized elliptic curve, giving an equivalence between the stack of generalized ellipic curves over S and the

stack of 1-marked genus-1 twisted stable curves over S with non-stacky marking? More precisely, is there a natural way of defining compactified Jacobians of twisted curves, such that the degree zero compactified Jacobian of a standard μ_d-stacky Néron 1-gon is a Néron d-gon?

Appendix A

On moduli of curves of higher genus

As in the case of elliptic curves, stacks of twisted stable maps allow for natural compactifications of stacks of genus-g curves equipped with certain extra structure. For example,

$$\overline{\mathcal{K}}^{\circ}_{g,0}(\mathcal{B}\mu_N) := \overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu_N) \times_{\overline{\mathcal{M}}_g} \mathcal{M}_g$$

classifies pairs $(C/S, \phi)$ where C/S is a smooth genus-g curve and $\phi : \mathbb{Z}/(N) \to \mathrm{Pic}^0_{C/S}$, which we view as an N-torsion point in $\mathrm{Pic}^0_{C/S} = \mathrm{Jac}(C/S)$. This stack is naturally contained in the proper algebraic stack $\overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu_N)$ as an open dense substack. However, the situation becomes considerably more complicated when we attempt to use this to obtain proper moduli stacks of curves with level structure, e.g. replacing "N-torsion points" with "points of exact order N."

Over $\mathbb{Z}[1/N]$, we have a stack $\overline{\mathcal{M}}^{(N)}_g$ of twisted curves with level N structure, studied in [ACV03, §6]; this is a smooth proper modular compactification of the stack classifying (not necessarily symplectic) Jacobi level N structures on smooth genus-g curves, as in [DM69, 5.4]. One may be tempted to proceed as follows:

Definition A.1.1. Let $\mathcal{C}/S$ be an unmarked genus-g ($g > 1$) twisted stable curve over a scheme S. A *full level N structure* on $\mathcal{C}$ is a group scheme homomorphism

$$\phi : (\mathbb{Z}/(N))^{2g} \to \mathrm{Pic}^0_{\mathcal{C}/S}$$

such that $\{\phi(a)|a \in (\mathbb{Z}/(N))^2\}$ is a full set of sections for the finite flat group scheme $\mathrm{Pic}^0_{\mathcal{C}/S}[N]$ over S in the sense of [KM85, §1.8].

We write $\overline{\mathcal{M}}^{(N),\mathrm{tw}}_g$ for the substack of $\overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu^{2g}_N)$ associating to T/S the groupoid of pairs $(\mathcal{C}, \phi)$, where $\mathcal{C}/T$ is an unmarked genus-g twisted stable curve and ϕ is a full level N structure on $\mathcal{C}$.

Unfortunately, this is not the "right" definition. By this we mean that we would like the stack $\overline{\mathcal{M}}^{(N),\mathrm{tw}}_g$ to be a closed substack of $\overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu^{2g}_N)$, flat over S; but it follows immediately from the example [CN90, Appendix] that flatness of $\overline{\mathcal{M}}^{(N),\mathrm{tw}}_g$ fails in mixed characteristic,

even over the ordinary locus of $\mathcal{M}_g$. Of course, if N is invertible on S then this definition is the correct one. More precisely, the choice of an isomorphism $(\mathbb{Z}/(N))^{2g} \cong \mu_N^{2g}$ identifies $\overline{\mathcal{M}}_g^{(N),\text{tw}}$ with the stack $\overline{\mathcal{M}}_g^{(N)}$ of [ACV03, §6], which is shown in loc. cit. to be smooth over $\mathbb{Z}[1/N]$ and proper over $\overline{\mathcal{M}}_g$. One would hope to be able to change the above definition to get a closed substack $\overline{\mathcal{M}}_g^{(N),\text{tw}}$ of $\overline{\mathcal{K}}_{g,0}(\mathcal{B}\mu_N^{2g})$, flat over S, agreeing with $\overline{\mathcal{M}}_g^{(N)}$ over $S[1/N]$ and with a natural moduli interpretation in terms of the maps from $(\mathbb{Z}/(N))^{2g}$ to the group schemes $\text{Pic}^0_{\mathcal{C}/S}$.

More generally, we have good properties for the moduli stack $\overline{\mathcal{K}}_{g,0}(\mathcal{B}G)$ whenever G is a finite diagonalizable or locally diagonalizable group scheme over S, so the Cartier dual G^* is commutative and constant or locally constant. Recall that for a finite group G there is a stack $_G\mathcal{M}_g$ over $\mathbb{Z}[1/|G|]$ of Teichmüller structures of level G on smooth curves, (cf. [DM69, 5.6] and [PJ95]). Now if G is a diagonalizable group scheme with $|G|$ invertible on the base scheme S, then after adjoining appropriate roots of unity we may fix an isomorphism $G \cong G^*$. In [ACV03, 5.2.3] it is observed that this determines an isomorphism between $_{G^*}\mathcal{M}_g$ and a substack $\mathcal{B}_g^{\text{tei}}(G)^\circ$ of $\overline{\mathcal{K}}_{g,0}^\circ(\mathcal{B}G)$ whose closure $\mathcal{B}_g^{\text{tei}}(G)$ in $\overline{\mathcal{K}}_{g,0}(\mathcal{B}G)$ is a moduli stack whose geometric objects correspond precisely to G-torsors $P \to \mathcal{C}$ which are connected (where $\mathcal{C}$ is a genus-g twisted stable curve); these are called *twisted Teichmüller G-structures*. One would hope that $\mathcal{B}_g^{\text{tei}}(G)$ can be defined in arbitrary characteristic, with a natural moduli interpretation, but it is not clear to the author how to proceed with this for genus $g > 1$; as discussed above, it does not suffice to simply consider the substack of $\overline{\mathcal{K}}_{g,0}(\mathcal{B}G)$ whose geometric objects correspond to G-torsors which are connected, since μ_{p^n} is connected in characteristic p, and the definition in terms of "full sets of sections" does not give a stack flat over the base scheme in mixed characteristic.

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