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ABSTRACT

It is shown that use of the TCP theorem leads to the well-known simple structure of Green functions of two variables even if these variables are connected with different fields. For the case of Green functions of three variables, dependence only upon the squares of difference vectors (and not also upon Θ functions) is demonstrated. The triangular inequalities in Minkowski space are treated in the Appendix.

TCP THEOREM AND GREEN FUNCTIONS*

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We want to show that the TCP theorem[1] leads to considerable simplifications of the structure of field theoretical Green functions.

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two different Hermitian spin-0 fields. By inserting intermediate states and invoking invariance with respect to the proper Lorentz group, one shows, in the usual way[2]

$$
\begin{aligned}
< \varphi_1(x_1)\, \varphi_2(x_2) >_0 &= i \int da\, \rho(a)\, \Delta_a^{(+)}(x_1 - x_2) , \\
< \varphi_2(x_2)\, \varphi_1(x_1) >_0 &= i \int da\, \rho^*(a)\, \Delta_a^{(+)}(x_2 - x_1) ,
\end{aligned} \tag{1}
$$

where use has also been made of the Hermiticity of the two fields. The TCP theorem now permits one to conclude that

$$
< \varphi_1(x_1)\, \varphi_2(x_2) >_0 = < \varphi_2(-x_2)\, \varphi_1(-x_1) >_0 = < \varphi_2(x_1)\, \varphi_1(x_2) >_0 . \tag{2}
$$

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1 We understand that Jost as well as Wightman and Schweber were recently able to prove the theorem on the basis of invariance with respect to the proper Lorentz group and microscopic causality alone. Because these two assumptions are usually made in discussions of Green functions, the application of the TCP theorem does not involve an additional postulate.

2 G. Källén, Helv. Phys. Acta 25, 417 (1952); H. Lehmann, Nuovo Cimento 11, 342 (1957). The symbol $< >_0$ stands for the expectation value with respect to the physical vacuum.

Here translational invariance has been employed for the second sign of equality. From this it follows that

$$\rho^*(a) = \rho(a) \quad , \tag{3}$$

so that[3]

$$< T(\varphi_1(x_1)\ \varphi_2(x_2)) >_0 = \frac{1}{2} \int da\ \rho(a)\ \Delta_{aF}(x_1 - x_2) \quad . \tag{4}$$

In the case of Green functions of more than two variables, the TCP theorem does not lead directly to the general structure of such functions. It can be used, however, to prove that these functions depend only upon the squared lengths of difference-four vectors $x_{21} = x_2 - x_1$ etc.[4] A proof of this statement is needed because from relativistic invariance alone one cannot exclude dependence upon $\theta(x_{21})$ etc. for time-like four vectors; vacuum expectation values of nonordered products of field operators do indeed depend upon the θ function (cf. Eq. (1)).

Let us take three Bose fields which might not all be different and analyze the Green function $< T(\varphi_1(x_1)\ \varphi_2(x_2)\ \varphi_3(x_3)) >_0$. To show the general argument, we limit the discussion to the case where all three difference vectors are time-like. From the triangular inequality,[5] it then follows that there must be one of these vectors whose modulus is bigger than the sum of the two others. Let this be x_{13} so that

$$\left| x_{13} \right| \geq \left| x_{32} \right| + \left| x_{21} \right| \quad . \tag{5}$$

---

[3] H. Lehmann kindly informed the author that this relation is known to Professor Pauli and a number of other physicists.

[4] This is implied in Schwinger's recent discussion on the structure of Green functions; cf. Proceedings of Seventh Annual Rochester Conference, 1957 (Interscience, New York, 1957).

[5] Cf. Appendix.

The time order is then such that x_1 is either later or earlier than x_3 but that in any case the time of x_2 is between those of the two other variables. Let $x_1 > x_2 > x_3$ (where $>$ stands for "later than") and let x'_1, x'_2, x'_3 be another set of coordinates with $x'_3 > x'_2 > x'_1$ for which the squared lengths of difference vectors are, however, the same. One can for instance choose $x'_i = -x_i$. Because the TCP theorem gives

$$< \varphi_1(x_1)\ \varphi_2(x_2)\ \varphi_3(x_3) >_0 \ = \ < \varphi_3(-x_3)\ \varphi_2(-x_2)\ \varphi_1(-x_1) >_0 \ , \tag{6}$$

the value of the Green function is in this case indeed the same for primed and unprimed variables. Thus there is no dependence upon the particular over-all time ordering, i.e. upon θ functions. The other possible configurations (i.e. some or all of the vectors space-like) can be treated in a similar manner if microscopic causality is used.

We believe that analogous results will be true for Green functions of more than two variables. For the analysis of such cases, however, a more complicated discussion of the various possible geometrical configurations will be required.

The author wises to acknowledge discussions on the subject with Dr. John Gerald Taylor.

APPENDIX

Triangular inequalities in Minkowski space

Let $\underline{a}$ and $\underline{b}$ be two four-vectors in Minkowski space. We ask for inequalities for the sum vector $\underline{a} + \underline{b}$ and for the difference vector $\underline{a} - \underline{b}$. The inequalities are to be formulated in terms of the dot products $\underline{a}\cdot\underline{a} = \underline{a}^2$, $\underline{b}^2$, $(\underline{a}+\underline{b})^2$, and $(\underline{a}-\underline{b})^2$. Well-known inequalities that hold in Euclidian space cannot directly be applied because of the indefinite metric factor of Minkowski space. We choose this metric factor such that we have $\underline{a}^2 > 0$ for space like $\underline{a}$, and $\underline{a}^2 < 0$ for time like $\underline{a}$.

First let $\underline{a}$ and $\underline{b}$ be two time-like vectors with

$$\Theta(\underline{a}) = \Theta(\underline{b}) = 1 . \tag{A.1}$$

Put

$$|\underline{a}| = \sqrt{-\underline{a}^2} , \quad \text{and} \quad |\underline{b}| = \sqrt{-\underline{b}^2} . \tag{A.2}$$

We keep $\underline{a}$ fixed and vary $\underline{b}$ for constant $|\underline{b}|$. We further choose the coordinate system such that

$$\underline{a} = (0, 0, 0, |\underline{a}|) \quad \text{and} \quad \underline{b} = (\eta, 0, 0, \sqrt{|\underline{b}|^2 + \eta^2}) . \tag{A.3}$$

Because of Eq. (A.1), the root has to be taken with positive sign. From these expressions, one finds that

$$-(\underline{a}+\underline{b})^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2\underline{a}\sqrt{|\underline{b}|^2 + \eta^2} ,$$

and

$$-(\underline{a}-\underline{b})^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2\underline{a}\sqrt{|\underline{b}|^2 + \eta^2} . \tag{A.4}$$

For the sum vector $\underline{a} + \underline{b}$, which always is time-like with

$$\Theta(\underline{a}+\underline{b}) = 1 , \tag{A.5}$$

one so obtains

$$\left| \underline{a} + \underline{b} \right| \geq \left| \underline{a} \right| + \left| \underline{b} \right| . \tag{A.6}$$

The difference vector $\underline{a} - \underline{b}$ is either time-like with

$$\left| \underline{a} - \underline{b} \right| \leq \left| \left| a \right| - \left| b \right| \right| \tag{A.7}$$

and a θ value equal to that of the "longer" of the vectors $\underline{a}$ and $\underline{b}$, or space-like without any limitations as to the modulus.

The case $\theta(\underline{a}) = \theta(\underline{b}) = -1$ leads to exactly the same inequalities with $\theta(\underline{a} + \underline{b}) = -1$ (substitute $\underline{a} \longrightarrow -\underline{a}$, $\underline{b} \longrightarrow -\underline{b}$). Similar considerations for $\theta(\underline{a}) = -\theta(\underline{b}) = 1$ ($\underline{a} \longrightarrow a$, $\underline{b} \longrightarrow -\underline{b}$) give, for the modulus of the time-like difference vector (with $\theta(\underline{a} - \underline{b}) = 1$),

$$\left| \underline{a} - \underline{b} \right| \geq \left| a \right| + \left| b \right| . \tag{A.8}$$

The following inequality is obtained for the sum vector if it is time-like (with θ value equal to that of the longer vector):

$$\left| a + b \right| \leq \left| \left| a \right| - \left| b \right| \right| , \tag{A.9}$$

and no limitation if it is space-like.

Therefore, for any three time-like difference vectors x_{13}, x_{32}, x_{21} there must hold one of the following three mutually exclusive inequalities:

$$\begin{aligned} \left| x_{13} \right| &\geq \left| x_{32} \right| + \left| x_{21} \right| , \\ \left| x_{32} \right| &\geq \left| x_{21} \right| + \left| x_{13} \right| , \\ \text{or} \quad \left| x_{21} \right| &\geq \left| x_{13} \right| + \left| x_{32} \right| . \end{aligned} \tag{A.10}$$

The time coordinate of the point which appears twice on the right-hand side lies between the time coordinates of the two other points.

In a similar way one shows that there are no such inequalities if one or both of the vectors $\underline{a}$ and $\underline{b}$ are space-like. So there are no triangular inequalities between difference vectors if not all three of them are time-like. The usual three-dimensional triangular inequalities are only obtained if $\underline{a}$, $\underline{b}$, and both $\underline{a} + \underline{b}$ and $\underline{a} - \underline{b}$ are space-like.