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Classification and Characterization of Exotic Quantum Systems: From Band Theory to Black Holes

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Classification and Characterization of Exotic Quantum Systems: From Band Theory to Black Holes

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June 2018

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To Tim Battista

Acknowledgements

Science is, fundamentally, a collaborative endeavor. This is true not only of the direct exchange of scientific ideas but also of the creation of a community of people who produce, encourage, and ultimately enjoy science. I find myself extremely lucky to be surrounded by many talented and excellent individuals in this community; regrettably, space constrains me to only thanking a small number individually.

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Curriculum Vitæ

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Zhen Bi, Alex Rasmussen, Cenke Xu, "Line defects in Three dimensional Symmetry Protected Topological Phases" Phys. Rev. B 89, 184424 (2014)

Yi-Zhuang You, Zhen Bi, Alex Rasmussen, Kevin Slagle, Cenke Xu, "Wave Function and Strange Correlator of Short Range Entangled states" Phys. Rev. Lett. 112, 247202 (2014)

Yi-Zhuang You, Zhen Bi, Alex Rasmussen, Meng Cheng, Cenke Xu, "Bridging Fermionic and Bosonic Short Range Entangled States" New J. Phys. 17, 075010 (2015)

Alex Rasmussen, Yi-Zhuag You, Cenke Xu, "Stable Gapless Bose Liquid Phases without any Symmetry" arxiv:1601.08235

Zhen Bi, Alex Rasmussen, Yoni Ben Tov, Cenke Xu, "Stable Interacting (2+1)d Conformal Field Theories at the Boundary of a class of (3+1)d Symmetry Protected Topological Phases" arXiv:1605.05336

Alex Rasmussen and Adam Jermyn, "Gaplesss topological order, Gravity, and Black holes" Phys. Rev. B 97, 165141 (2018)

Chao-Ming Jian, Alex Rasmussen, Yi-Zhuang You, Cenke Xu, "Emergent Symmetry and Tricritical Points near the deconfined Quantum Critical Point" arXiv:1708.03050

Chao-Ming Jian, Alex Thomson, Alex Rasmussen, Zhen Bi, Cenke Xu, "Deconfined Quantum Critical Point on the Triangular Lattice " arXiv:1710.04668

Abstract

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Exotic quantum systems – those with macroscopic quantum behavior with no classical analogue – have been a mainstay of condensed matter theory since the discovery of the quantum Hall effect. Since then, several families of related systems have been uncovered. Some, such as topological insulators, were predicted first and found experimentally later. Others are more elusive, like strongly correlated bosonic symmetry protected topological phases in high dimension, of which concrete evidence is still lacking. In this dissertation, we study several examples of exotic quantum systems. For SPT phases, we present a physically motivated classification scheme for interacting bosons and a bulk signature independent of boundary. We then construct a new, beyond Landau-Ginzburg second-order phase transition between two ordered phases of the Heisenberg magnet on the triangular lattice. Finally, we investigate an infinite family of spin liquid states, and conjecture on their connection to black holes.

Contents

Chapter 1

Introduction

1.1 What is an Exotic Quantum System?

A very brief history of quantum mechanics could be composed of examples where a physical system is thought of as "essentially" quantum mechanical, only to admit a semiclassical picture that captures the relevant physics. Starting from Planck's blackbody radiation and continuing through specific heat of crystals, magnetism, and superfluidity, the macroscopic measurements in these systems are reproduced by considering semiclassical expansions or saddle-point approximations.

However, this trend has been broken in more modern times. Starting with the discovery of the quantum hall effect (QHE) and continuing on to spin liquids, topological order, symmetry protected topological (SPT) phases, and non-Landau-Ginzburg-Wilson-Fisher (LGWF) phase transitions, it has become clear that there exist systems with macroscopic quantum mechanical behavior that does not admit a semiclassical understanding.

Unfortunately (from a definitional standpoint), the mechanisms obstructing a semiclassical expansion in each of the examples above are largely distinct. For the purposes of this dissertation, we define an "exotic quantum system" as a system with macroscopic quantum behavior and no classical analogue. This definition is vague by construction, given the varied and distinct phenomena which we want to include under its umbrella[1]. And, due to this vagueness, it is inevitable that we will have to manually include or exclude specific systems due to heuristic properties.

An important property of these exotic quantum systems is universality. When discussing phase transitions, this refers to the idea that many critical points are described by the same conformal field theory (CFT). The notion of universality relevant to exotic quantum systems, instead, is that many of the macroscopic properties are independent of particular microscopic realizations. In SPT phases, for example, this is due to the topological term in the Lagrangian being insensitive to the lattice. This type of universality, ultimately, is why this area of research is relevant to real systems.

In this dissertation, we explore several varieties of exotic quantum systems. We focus on two particular aspects: classification and characterization. The former concerns our attempts to identify, label, and count the distinct sub-types of a particular system (such as SPT phases with a particular symmetry). The latter involves understanding the physical properties of the system, such as ground state degeneracy, boundary terms, or local indistinguishability.

1.2 SPT Phases

The history of SPT phases has a rather unique property when compared to other major discoveries: the theory came first[2] – twice! The first prediction involved $2d$ graphene and then quantum wells, while the second involved $3d$ crystals. This already hints at an important part of SPT physics, the dimension of spacetime.

The theoretical prediction of the $2d$ quantum spin Hall (QSH) effect was put forth by

[1]Compare to how a spin liquid is not a banana.

[2]The integer QHE may be roughly considered a "gravitational SPT," but only in the loosest sense.

Kane and Mele, following some work by Haldane[1, 2]. In this state, there are two counter-propagating spin-up and spin-down channels, similar to the ordinary quantum Hall effect. However, this state requires conservation of S_z, which can be broken by magnetic fields or impurities. Even in the presence of S_z-breaking impurities, it was shown[3] that there is nevertheless a topological invariant that distinguishes this insulating state from the ordinary ground state.

This new phase (distinct from QSH due to lack of S_z symmetry) can be considered the first SPT phase[4]. In this phase, the "protecting" symmetry is time reversal[3]. The key physics that distinguishes this phase from the trivial insulator is the existence of gapless boundary edge modes that are stable in the presence of time reversal symmetry. Remarkably, this effect been observed in two-dimensional quantum wells[6].

In three dimensions, the story is similar. A topological invariant can be defined for $3d$ band structures[7, 8] in the same fashion using time reversal invariance. The boundary physics of a $3d$ topological insulator also includes a special gapless boundary, which is a single Dirac cone. This was observed experimentally in heavy spin-orbit coupled materials[9].

Generalizing these ideas requires analyzing several aspects of SPT phases. First, we need a definition of "phase" that is agnostic to symmetries and order parameters. Second, we need to determine what bulk physics are necessary to ensure that the boundary is nontrivial. Finally, we need to know what distinguishes an SPT boundary from a trivial boundary. Once all of these are determined, we can attempt to classify all SPT phases with a particular symmetry.

There are two important parts of defining an SPT phase: symmetry and bulk gap. Two Hamiltonians are said to be in the same phase if we can continuously deform one into the other without closing the bulk gap. This can involve tuning interaction strengths,

[3] In the ten-fold way, this is class AII[5]

hopping coefficients, and so on. However, the path connecting two different Hamiltonians can generically require breaking a symmetry present at either endpoint. For SPT phases, we define two phases to be distinct if they *cannot* be connected smoothly without closing the bulk gap or breaking a symmetry. It is in this sense that a symmetry is "protecting" – for as long as the symmetry and bulk gap are preserved, the system cannot leave the phase.

We note here that SPT physics (like many examples in condensed matter) is entirely a question of quantum ground states. Much of this physics is only accessible at extremely low temperature, so that the thermal occupation of excited states does not ruin the delicate quantum correlations. The bulk gap is essential for this, since the low-energy dynamics will be exponentially suppressed. It also allows for a well-defined notion of "bulk" and "boundary," which would be impossible in the presence of gapless excitations.

As has been seen above, SPT physics necessitates a bulk gap. We want to compare SPT phases to trivial insulating phases, so we also impose several other conditions. In fact, the SPT bulk is in many ways indistinguishable from a trivial bulk. We require that both SPT and trivial bulk have not only bulk gap but also be short range entangled and have a unique ground state on a torus. We also require that the relevant symmetry[4] act locally, linearly, and on-site[10]. This prevents systems with topological order, spontaneous symmetry breaking, or gauge structure from being SPT phases, which is reasonable since the trivial phase is just an ordinary band gap insulator.

Though the bulk is uninteresting in many ways, the boundary of an SPT bulk differs wildly from the boundary of the trivial bulk. The trivial boundary is much like the bulk: gapped, nondegenerate, and short-range entangled. SPT boundaries necessarily violate these conditions. Provided that the dimension is large enough, SPT boundaries can

[4]I.e., the one that "protects" the boundary. There may be many symmetries in the system, not all relevant to the SPT

be gapless, spontaneously break discrete or continuous symmetries, or be topologically ordered. The relevant symmetries may also act projectively, or not on-site, or even nonlocally as well. Most importantly, the $d-1$-dimensional boundary of a d-dimensional SPT bulk *cannot* be realized as a consistent lattice model by itself in $d-1$ dimensions.

These conditions on the boundary are enough to justify the "exotic" label, but they conceal a much richer structure than at first glance. Specifically, an SPT boundary always supports a 't Hooft anomaly for the symmetry[11]. Like all anomalies, it means that the classical Lagrangian has a symmetry but the quantum path integral does not. In this case, the anomaly is exposed by attempting to gauge the symmetry. When a regulator is imposed, the gauging procedure is obstructed, which precludes a lattice model. To avoid this obstruction, the symmetry action can be realized projectively or not on-site, but this will necessarily be different from the symmetry action in the bulk. Correspondingly, the bulk of an SPT has so-called "anomaly inflow," which allows the anomalous boundary to be regularized in the presence of the bulk.

The two famous systems described above and found experimentally fit into this picture exactly. For the QSH-like phase with just time reversal, there is a protected gapless boundary state. For the topological insulator, the single Dirac cone makes sense as a field theory, but any real lattice model always generates two Dirac cones in the IR. In this case, the anomaly is due to the parity symmetry.

SPT phases are notable for their variety, both in type and boundary physics. A large, concerted effort in the community has yielded classifications for how many SPT's of a given symmetry G exist, for both fermions and bosons. In Ch 2, analyze this problem by considering symmetry action on nonlinear sigma models (NLSMs). While it is known that the NLSM classification is incomplete, more opaque mathematical constructions have claimed to be more complete. Whether or not there are still SPT phases beyond the known lists, and how they are all realized physically, remain open questions.

1.3 Non-LGWF Phase Transitions

Possibly one of the most important paradigms in condensed matter physics is the Landau-Ginzburg-Wilson-Fisher theory of second order phase transitions[12]. By bringing symmetry to the forefront and using very general arguments about scale invariance, LGWF theory is able to predict a large number of experimentally or numerically verifiable observations. Most importantly, it works for both classical and quantum systems, where the latter is implemented in one higher dimension.

The grand success of LGWF theory can be attributed to two key facts. First, many physical systems can have the same symmetry. In LGWF theory, the critical point field theory is built out of a symmetry-breaking order parameter and several phenomenological constants. In conjunction with the renormalization group (RG), we can vastly restrict the form of the effective field theory by only considering symmetry-respecting operators.

Second, the renormalization group allows us to construct the effective field theory for a given phase without needing to know the precise lattice theory. If we can find a small set of relevant operators, this allows us to construct the phase diagram proximate to a given critical point and approximate the scaling dimensions of operators. In conjunction with symmetry, this allows us to make very precise predictions about experimentally accessible measurements.

In LGWF theory, phases and critical points are distinguished by an order parameter, which is the expectation value of an operator charged under a symmetry. Ordered and disordered phases are distinguished by whether or not this expectation value takes non-zero values, which correspond to symmetry broken and symmetric phases, respectively. Importantly, the field theory at the critical point is constructed using a field with the same symmetry properties as the order parameter.

Universality is perhaps the most appealing aspect of LGWF theory. Since the critical behavior is dependent solely on symmetries of the order parameter, the dimension, and the relevant operators in the Lagrangian, many systems with very different microscopic theories may have their phase transition described by the same conformal field theory. This enables us to make very general statements about the properties of symmetry broken phases (along with Goldstone's theorem).

Since the critical field theory depends on a single order parameter, it is difficult and non-universal to describe phase transitions between two ordered phases that break different symmetries. The critical point of the LGW phase transition happens when the gap to excitations closes and the order parameter vanishes, so with more than one symmetry either the gaps have to close together (fine-tuning) or separately (two critical points and an intermediate phase).

However, it is nevertheless possible to derive a generic, second-order transition between two ordered phases by moving beyond the LGWF paradigm. Such non-LGWF transitions have been established for the square lattice Heisenberg antiferromagnet[13] by including a dynamical gauge field in the critical action and analyzing the scaling dimensions of monopoles. This approach is beyond LGWF because it is an unfine-tuned critical point between two phases that break very different symmetries, and its behavior is described by fractionalized degrees of freedom instead of an order parameter.

Much like how the topological term in an SPT obstructs a classical interpretation (due to the anomaly), the non-LGW transitions with the aforementioned structure are also exotic quantum systems. In fact, this connection is nearly exact, because the same topological terms that give rise to the boundary anomaly on an SPT are present in these types of non-LGWF transitions[14]. However, the systems in question exist as well-defined quantum field theories, and the "anomaly" is not realized in the same way.

This is a relatively new field, but there is ample numerical and physical evidence[13] to suggest that the non-LGWF behavior is more general than planar antiferromagnets.

Moreover, there may be other ways to modify LGW theory than have been studied, so likely this field is much larger than expected. In Ch. 4 we examine a previously unknown but physically relevant phase transition between two ordered states of the triangular lattice antiferromagnet, which indeed cannot fit into LGWF theory.

1.4 Spin Liquids and Gauge Theories

The role of quantum fluctuations in destroying ordered states at zero temperature is generically a very hard problem. In the case of antiferromagnetism, for example, frustration and lattice effects can lead to new lower-energy states than the ordered Néel state. Originally studied in the context of high-Tc superconductivity, a “spin liquid” state has no magnetic order, usually one electron per unit cell, and fractionalized excitations.

One particular type of spin liquid state is the short-range resonating valence bond (RVB) state[15]. This state is formed by pairing nearest-neighbor spin-$\frac{1}{2}$ into spin singlets across lattice links, and then considering superpositions of assignments of bonds on the lattice. Provided that the bonds are fluctuating (and do not settle into a particular crystalline order), this state breaks neither the spin rotation symmetry nor the point-group symmetries.

But, is the RVB state actually the ground state of a frustrated quantum magnet? This is a very common question for candidate spin liquid states, as the interactions generally require numerics to calculate energies. For the triangular[16] and kagome[17], there is good evidence to suggest that this $\mathbb{Z}_2$ short-range RVB state can indeed exist. A similar state was proposed for the frustrated antiferromagnet on the square lattice[18].

The most important example of $3+1d$ spin liquid to this dissertation is the pyrochlore $U(1)$ spin liquid. Like the RVB states described earlier, the Heisenberg antiferromagnet on a pyrochlore lattice can be described using a short-range RVB-based dimer model.

This model has a stable phase[19] that is, remarkably, gapless. The gapless mode[5] does not come about from an a spontaneously broken symmetry that acts on sites. In fact, no symmetry is strictly necessary to stabilize the phase – only the local dimer constraint.

Many spin liquid phases share one very important property: their low energy dynamics are captured by gauge theories[15]. Indeed, even the names are chosen to reflect the gauge group. In the Hamiltonian formulation, a gauge theory is the natural description of a theory with a local constraint (i.e. conserved quantity) that generates a local action of a symmetry. In spin liquids arising from short-range RVB or dimer model, this local constraint is that each spin is paired to exactly one other spin; equivalently, each site touches exactly one dimer.

While quantum gauge theories, especially the $U(1)$ Maxwell theory, may be seem semi-classical at first glance, they should also be classified as "exotic matter." The most poignant example is $2+1d$ QED, where monopole events present in the path integral proliferate and obliterate the gauge structure[20, 21]. Since these events are not controlled by terms in the classical Lagrangian, the full quantum theory has to be analyzed to ensure that the gauge theory exists, and that it is the correct description for a stable phase of matter.

In $3+1d$, the gauge theory picture is even weirder. The pyrochlore $U(1)$ spin liquid is so named because the low-energy dynamics are best described in terms of a Maxwell-like gauge field. The gapless, collective S_z mode is the photon. Unlike the $2+1d$ case, this phase is describe by a stable RG fixed point. While there are magnetic monopoles, self-duality of the theory prevents them from proliferating. This stands in stark contrast to almost all other known gapless phases, which are either Goldstone modes or critical points between two phases.

Experimental detection of spin liquids is considerably difficult. There has been some

[5]This mode is closely related to the Rokhsar-Kivelson resonon.

progress in using large organic molecules to create triangular lattices, but the physical signatures of the spin liquid probe the physics indirectly. Thus, we cannot simply look for a bulk excitation gap, but instead look for things like ground state entropy. For the pyrochlore $U(1)$ phase, things are even more complicated – the gapless photon is difficult to separate from the phonon modes inherent in the crystal.

In this dissertation, we present work in Ch. 5 that generalizes the pyrochlore phase to an infinite family of related phases. Notable for being stable, gapless phases in $3+1d$ with lattice models, these phases have recently become the subject of intense interest due to the excitations possessing a fracton structure. They also generalize the ideas of topological order, despite the presence of a gapless gauge boson.

The connection between spin liquids and gauge theories points towards even more structure. In Ch 6, we discuss how the asymptotic symmetries of Minkowski spacetime are intimately connected to certain flux integrals in gauge theories emerging from spin liquids, which in turn are connected to particular classes of dimer coverings of the lattice. These flux integrals have come under recent investigation due to their connection to so-called "higher form symmetries," where spontaneous breaking of a 1-form symmetry is the same as the gauge theory deconfining[22]. As such, spin liquid phases may correspond to symmetry broken phases, albeit of a much more complicated symmetry.

1.5 Topological Order

The last type of intrinsically quantum system relevant to this dissertation is topological order. These systems have been of particular interest in recent years following the realization[23, 24] that they can be used to perform quantum computations. As we have noted above, if the boundary of an SPT is fully symmetric and gapped, then it is topologically ordered; the stability of the SPT phase ensures that the computation state is protected from disorder.

Topological order gets its name from the system having a ground state degeneracy on a manifold with nonzero genus[25, 26, 27, 21]. Often we consider the d-dimensional torus, T^d. Alternatively, the system can be described by a topological quantum field theory (TQFT) that is independent of the metric. We require that there be an energy gap to the first excited state, which ensures that the degeneracy is exponentially small in the limit of infinite system size.

Another definition is that the system has several locally indistinguishable ground states. By this, we mean that the ground states cannot be connected via local operators or measurements. However, they can be connected by *global* operators, which are by definition sensitive to the topological properties of the manifold.

In $2+1d$, topologically ordered systems can support special type of excitations that are neither fermions nor bosons. These "anyons" can have any statistics but are not truly local particles. Nevertheless, it is thought that they can be manipulated using local operations to braid and fuse[6], and by doing so perform quantum computation.

The most commonly invoked example of topological order is the deconfined $\mathbb{Z}_2$ gauge theory. For a particular choice of couplings, this is also known as the Kitaev toric code. The degenerate ground states on a torus are reached by winding electric charges around the large loops (or, equivalently, threading magnetic flux). The anyons are the electric charge and flux, which in $2+1d$ are both point-like objects on the lattice.

For discrete gauge groups, there are analogues of the toric code that are also topologically ordered. But what about continuous groups? Flux-winding arguments are common[28, 29] for analyzing integer and fractional quantum hall states with real electrons. They also appear in the pyrochlore $U(1)$ spin liquid and its related family, as we examine in Ch. 5.

[6]The braiding and fusion data is contained within a modular tensor category.

Topological order, as defined above, requires a gap[7]. Thus, the gapless photon in the pyrochlore $U(1)$ spin liquid should seemingly disqualify it from being called topologically ordered. However, with an appropriate analysis of low-lying photon excitations and flux-winding operators, we see in Ch. 6 that there is nevertheless a notion of "gapless" topological order that incorporates the all-important local indistinguishability.

[7]When considering real crystals, one may ignore the phonons[30]

1.6 Permissions and Attributions

- The content of chapter 2 is the result of a collaboration with Zhen Bi, Kevin Slagle, and Cenke Xu, and has previously appeared in Physical Review B 91 134404 (2015) and arXiv:1309.0515. It is reproduced here with the permission of APS (https://journals.aps.org/copyrightFAQ.html#thesis).

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- The content of chapter 4 is the result of a collaboration with Chao-Ming Jian, Alex Thomson, Zhen Bi, and Cenke Xu, and has previously appeared in arXiv:1710.04668 and is in review at several journals.

- The content of chapter 5 is the result of a collaboration with Yi-Zhuang You and Cenke Xu, and has previously appeared in arXiv:1601.08235.

- The content of chapter 6 is the result of a collaboration with Adam Jermyn and has previously appeared in Physical Review B 97 165141 (2018) and arXiv:1703.04772. It is reproduced here with the permission of APS.

Chapter 2

SPT's and Sigma Models

2.1 Why NLSM?

Symmetry protected topological (SPT) phases are one of the most important broad classes of exotic matter, and have recently been an area of intense research. Starting with the theoretical predication and subsequent experimental verification of such phases, an enormous effort has gone into trying to classify and understand these new, disordered phases of matter.

The mathematical structure of these phases is particularly rich. Early attempts to classify bosonic SPT phases used the group cohomology of the symmetry group G, $H^{d+1}(G, U(1))$, by studying how group elements act on local sites and then requiring a consistency condition[31]. Later work extended this by considering the cobordism group of BG, the classifying space of G[32].

However, these formal models are somewhat difficult to understand physically, and so we would like a more familiar construction. For bosons[1], this is easily achieved using

[1]Bosonic SPT phases are necessarily interacting, otherwise the bosons would simply condense in to a BEC.

a nonlinear sigma model (NLSM) description. In addition to describing order-disorder transitions in magnets, a NLSM field theory can incorporate a topological term. While these topological terms do not modify the bulk equations of motion, they have dramatic effects on the boundary physics, giving rise to exactly the right behavior to be an SPT.

2.2 Nonlinear Sigma Model Classification of Bosonic SPT Phases

Symmetry protected topological (SPT) phase is a new type of quantum disordered phase. It is intrinsically different from a trivial direct product state, when and only when the system has certain symmetry G. In terms of its phenomena, a SPT phase on a $d-$dimensional lattice should satisfy at least the following three criteria:

(i). On a $d-$dimensional lattice without boundary, this phase is fully gapped, and nondegenerate;

(ii). On a $d-$dimensional lattice with a $(d-1)-$dimensional boundary, if the Hamiltonian of the entire system (including both bulk and boundary Hamiltonian) preserves certain symmetry G, this phase is either gapless, or gapped but degenerate.

(iii). The boundary state of this $d-$dimensional system cannot be realized as a $(d-1)$-dimensional lattice system with the same symmetry G.

Both the $2d$ quantum spin Hall insulator [1, 3, 33] and $3d$ Topological insulator [8, 7, 34] are perfect examples of SPT phases protected by time-reversal symmetry and charge U(1) symmetry. In this paper we will focus on bosonic SPT phases. Unlike fermion systems, bosonic SPT phases are always strongly interacting phases of boson systems.

Notice that the second criterion (ii) implies the following two possibilities: On a lattice with a boundary, the system is either gapless, or gapped but degenerate. For example, without interaction, the boundaries of $2d$ QSH insulator and $3d$ TBI are both gapless; but with interaction, the edge states of $2d$ QSH insulator, and $3d$ TBI can both be gapped out through spontaneous time-reversal symmetry breaking at the boundary, and this spontaneous time-reversal symmetry breaking can occur through a boundary transition, without destroying the bulk state [35, 36, 37]. When $d \geq 3$, the degeneracy of the boundary can correspond to either spontaneous breaking of G, or correspond to certain topological degeneracy at the boundary. Which case occurs in the system will depend on the detailed Hamiltonian at the boundary of the system. For example, with strong interaction, the boundary of a 3d TBI can be driven into a nontrivial topological phase [38, 39, 40, 41].

The concept of SPT phase was pioneered by Wen and his colleagues. A mathematical paradigm was developed in Ref. [31, 42] that systematically classified SPT phases based on the group cohomology of their symmetry G. But this approach was unable to reveal all the physical properties of the SPT phases. In the last few years, SPT phase has rapidly developed into a very active and exciting field [31, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 40, 54, 55, 56], and besides the general mathematical classification, other approaches of understanding SPT phases were also taken. In 2d, it was demonstrated that the SPT phases can be thoroughly classified by the Chern-Simons field theory [47], although it is unclear how to generalize this approach to 3d. Nonlinear Sigma model field theories were also used to describe some SPT phases in 3d and 2d [50, 48, 49], but a complete classification based on this field theory is still demanded.

The goal of this paper is to systematically classify and describe bosonic SPT phases with various continuous and discrete symmetries in *all dimensions*, using semiclassical nonlinear Sigma model (NLSM) field theories. At least in one dimensional systems, semiclassical NLSMs have been proved successful in describing SPT phases. The O(3)

NLSM plus a topological $\Theta-$term describes a spin-1 Heisenberg chain when $\Theta = 2\pi$:

$$\mathcal{S}_{1d} = \int dx d\tau \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c, \tag{2.1}$$

and it is well-known that the spin-1 antiferromagnetic Heisenberg model is a SPT phase with 2-fold degeneracy at each boundary [57, 58, 59, 60, 61, 62].

In this paper we will discuss SPT phases with symmetry Z_2^T, Z_2, $Z_2 \times Z_2$, $Z_2 \times Z_2^T$, $U(1)$, $U(1) \times Z_2$, $U(1) \rtimes Z_2$, $U(1) \times Z_2^T$, $U(1) \rtimes Z_2^T$, Z_m, $Z_m \times Z_2$, $Z_m \rtimes Z_2$, $Z_m \times Z_2^T$, $Z_m \rtimes Z_2^T$, $SO(3)$, $SO(3) \times Z_2^T$, $Z_2 \times Z_2 \times Z_2$. Here we use the standard notation: Z_2^T stands for time-reversal symmetry, $G \times Z_2^T$ and $G \rtimes Z_2^T$ stand for direct and semidirect product between unitary group G and time-reversal symmetry. A semidirect product between two groups means that these two group actions do not commute with each other. More details will be explained when we discuss the classification of these states. We will demonstrate that a $d-$dimensional SPT phase with any symmetry mentioned above can always be described by an O$(d+2)$ NLSM in $(d+1)-$dimensional space-time, namely all the 1d SPT phases discussed in this paper can be described by Eq. 2.1, all the 2d and 3d SPT phases can be described by the following two field theories:

$$\begin{aligned} \mathcal{S}_{2d} &= \int d^2x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \frac{i2\pi k}{\Omega_3}\epsilon_{abcd} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \mathcal{S}_{3d} &= \int d^3x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \frac{i2\pi}{\Omega_4}\epsilon_{abcde} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d \partial_z n^e, \end{aligned} \tag{2.3}$$

The $O(d+2)$ vector is a Landau order parameter with a unit length constraint: $(\vec{n})^2 = 1$. Ω_d is the surface area of a $d-$dimensional unit sphere. The $2d$ action Eq. 2.2 has a level$-k$ in front of its $\Theta-$term, whose reason will be explained later. Different SPT phases in the same dimension are distinguished by the transformation of the $O(d+2)$ vector under the symmetry. The classification of SPT phases on a $d-$dimensional lattice is given by all the *independent* symmetry transformations of $\vec{n}$ that keep the entire Lagrangian (including the $\Theta-$term) invariant. This classification rule will be further clarified in the next section.

An $O(d+2)$ NLSM can support maximally $O(d+2)$ symmetry and other discrete symmetries such as time-reversal. We choose the 17 symmetries listed above, because they can all be embedded into the maximal symmetry of the field theory, and they are the most physically relevant symmetries. Of course, if we want to study an SPT phase with a large Lie group such as SU(N), the above field theories need to be generalized to NLSM defined with a symmetric space of that Lie group. But for all these physically relevant symmetries, our NLSM is already sufficient.

In principle, a NLSM describes a system with a long correlation length. Thus a NLSM plus a $\Theta-$term most precisely describes a SPT phase tuned *close to* a critical point (but still in the SPT phase). When a SPT phase is tuned close to a critical point, the NLSM not only describes its topological properties (*e.g.* edge states *etc.*), but also describes its dynamics, for example excitation spectrum above the energy gap (much smaller than the ultraviolet cut-off). When the system is tuned deep inside the SPT phase, namely the correlation length is comparable with the lattice constant, this NLSM can no longer describe its dynamics accurately, but since the topological properties of this SPT phase is unchanged while tuning, these topological properties (like edge states) can still be described by the NLSM. The NLSM is an effective method of describing the universal topological properties, as long as we ignore the extra nonuniversal information about dynamics, such as the exact dispersion of excitations, which depends on the details of the lattice Hamiltonian and hence is not universal.

Besides the classification, our NLSMs in all dimensions can tell us explicit physical information about this SPT phase. For example, the boundary states of 1d SPT phases can be obtained by explicitly solving the field theory reduced to the 0d boundary. The boundary of a 3d SPT phase could be a 2d topological phase, and the NLSMs can tell us the quantum number of the anyons of the boundary topological phases. The boundary topological phases of 3d SPT phases with $U(1)$ and time-reversal symmetry were discussed in Ref. [48]. We will analyze the boundary topological phases for some other 3d SPT phases in the current paper.

Our formalism not only can study each individual SPT phase, it also reveals the relation between different SPT phases. For example, using our formalism we are able to show that there is a very intriguing relation between SPT phases with $U(1) \times (\rtimes) G$ symmetry and SPT phases with $Z_m \times (\rtimes) G$ symmetry, where G is another discrete group such as Z_2, Z_2^T. Our formalism demonstrates that after breaking U(1) to Z_m, whether the SPT phase survives or not depends on the parity of integer m. We also demonstrate that when m is an even number, we can construct some extra SPT phases with $Z_m \times (\rtimes) G$ symmetry that *cannot* be deduced from SPT phases with $U(1) \times (\rtimes) G$ symmetry by breaking U(1) down to Z_m. Our field theory also gives many of these SPT states a natural "decorated defect" construction, which will be discussed in more detail in the next section.

NLSMs with a $\Theta-$term can also give us the illustrative universal bulk ground state wave function of the SPT phases. This was discussed in Ref. [50]. These wave functions contain important information for both the boundary and the bulk defects introduced by coupling the NLSM to an external gauge field [50, 63]. It was also demonstrated that the NLSMs are useful in classifying and describing symmetry enriched topological (SET) phases [64], but a complete classification of SET phases based on NLSMs will be studied in the future.

In the current paper we will only discuss SPT states within cohomology. It is now understood that the group cohomology classification is incomplete, and in each dimension there are a few examples beyond cohomology classification [65, 32, 66]. These beyond-cohomology states all involve gravitational anomalies [67] or mixed gauge-gravitational anomalies [66]. Generalization of our field theory to the cases beyond group cohomology can be found in another paper [68].

2.3 Strategy and Clarification

2.3.1 Edge states of NLSMs with $\Theta-$term

In $d-$dimensional theories Eq. 2.1,2.2 and 2.3 (d denotes the spatial dimension), when $\Theta = 2\pi$, their boundaries are described by $(d-1)+1-$dimensional O($d+2$) NLSMs with a Wess-Zumino-Witten (WZW) term at level-1. When $d = 1$, the boundary of Eq. 2.1 with $\Theta = 2\pi$ is a 0+1d O(3) NLSM with a Wess-Zumino-Witten term at level $k = 1$ [62]:

$$\mathcal{S}_b = \int d\tau \frac{1}{g}(\partial_\tau \vec{n})^2 + \int d\tau du \frac{i2\pi}{8\pi} \epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c. \tag{2.4}$$

The WZW term involves an extension of $\vec{n}(\tau)$ to $\vec{n}(\tau, u)$:

$$\vec{n}(\tau, 0) = (0, 0, 1), \quad \vec{n}(\tau, 1) = \vec{n}(\tau). \tag{2.5}$$

The boundary action $\mathcal{S}_b$ describes a point particle moving on a sphere S^2, with a 2π magnetic flux through the sphere. The ground state of this single particle quantum mechanics problem is two fold degenerate. The two fold degenerate ground states have the following wave functions on the unit sphere:

$$U = (\cos(\theta/2)e^{i\phi/2}, \;\; \sin(\theta/2)e^{-i\phi/2})^t,$$

$$\vec{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))\,. \qquad (2.6)$$

The boundary doublet U transforms projectively under symmetry of the SPT phase, and its transformation can be derived explicitly from the transformation of $\vec{n}$. For example if $\vec{n}$ transforms as $\vec{n} \to -\vec{n}$ under time-reversal, then this implies that under time-reversal $\phi \to \phi$, $\theta \to \pi + \theta$, and $U \to i\sigma^y U$.

When $d = 2$, the boundary is a 1+1-dimensional O(4) NLSM with a WZW term at level $k = 1$, and it is well-known that this theory is a gapless conformal field theory if the system has a full O(4) symmetry [69, 70]. The 1d boundary could be gapped but still degenerate if the symmetry of $\vec{n}$ is discrete (the degeneracy corresponds to spontaneous discrete symmetry breaking); when $d = 3$, the boundary is a 2+1d O(5) NLSM with a WZW at level $k = 1$, which can be reduced to a 2+1d O(4) NLSM with $\Theta = \pi$ after the fifth component of $\vec{n}$ is integrated out [48]. This $2 + 1d$ boundary theory should either be gapless or degenerate, and one particularly interesting possibility is that it can become a topological order, which will be discussed in more detail in section IIF. Starting with this topological order, we can prove that this $2 + 1d$ boundary system cannot be gapped without degeneracy.

All components of $\vec{n}$ in Eq. 2.1,2.2 and 2.3 must have a nontrivial transformation under the symmetry group G, namely it is not allowed to turn on a linear “Zeeman" term that polarizes any component of $\vec{n}$. Otherwise the edge states can be trivially gapped, and the bulk $\Theta-$term plays no role.

2.3.2 Phase diagram of NLSMs with a $\Theta-$term

In our classification, the NLSM including its $\Theta-$term is invariant under the symmetry of the SPT phase, for arbitrary value of Θ. For special values of Θ, such as $\Theta = k\pi$ with integer k, some extra discrete symmetry may emerge, but these symmetries are *unimportant* to the SPT phase. However, these extra symmetries guarantee that $\Theta = k\pi$ is a fixed point under renormalization group (RG) flow. In 1+1d NLSMs, the RG flow of Θ was calculated explicitly in Ref. [71, 72] and it was shown that $\Theta = 2\pi k$ are stable fixed points, while $\Theta = (2k+1)\pi$ are instable fixed points, which correspond to phase transitions; in higher dimensions, similar explicit calculations are possible, but for our purposes, we just need to argue that $\Theta = 2\pi k$ are stable fixed points under RG flow. The bulk spectrum of the NLSM with $\Theta = 2\pi k$ is identical to the case with $\Theta = 0$: in the quantum disordered phase the bulk of the system is fully gapped without degeneracy. Now if Θ is tuned away from $2\pi k$: $\Theta = 2\pi k \pm \epsilon$, this perturbation cannot close the bulk gap, and since the essential symmetry of the SPT phase is unchanged, the SPT phase including its edge states should be stable against this perturbation. Thus a SPT phase corresponds to a finite phase $\Theta \in (2\pi k - \delta_1, 2\pi k + \delta_2)$ in the phase diagram.

There is a major difference between $\Theta-$term in NLSM and the $\Theta-$term in the response action of the external gauge field. In our description, a SPT phase corresponds to the entire phase whose stable fixed point is at $\Theta = 2\pi$ (or $2\pi k$ with integer k). Tuning slightly away from these stable fixed points will not break any essential symmetry that protects the SPT state, and hence it does not change the main physics. The theory will always flow back to these stable fixed points under RG (this RG flow was computed explicitly in $1 + 1d$ in Ref. [71, 72], and a similar RG flow was proposed for higher dimensional cases [73]). The $\Theta-$term of the external gauge field after integrating out the matter fields is protected by the symmetry of the SPT phase to be certain discrete value.

For example $\Theta = \pi$ for the ordinary 3d topological insulator [74, 75] is protected by time-reversal symmetry. Tuning Θ away from π will necessarily break the time-reversal symmetry.

2.3.3 $\mathbb{Z}_k$ or $\mathbb{Z}$ classification?

In the classification table in Ref. [31, 42], one can see that in even dimensions, there are many SPT states with $\mathbb{Z}$ classifications, but in odd dimensions, $\mathbb{Z}$ classification never appears. This fact was a consequence of mathematical calculations in Ref. [31, 42], but in this section we will give a very simple explanation based on our field theories.

The manifold of O$(d+2)$ NLSM is S^{d+1}, which has a $\Theta-$term in $(d+1)-$dimensional space-time due to homotopy group $\pi_{d+1}[S^{d+1}] = \mathbb{Z}$. However, this does *not* mean that the $\Theta-$term will always give us $\mathbb{Z}$ classification, because more often than not we can show that $\Theta = 0$ and $\Theta = 2\pi k$ with certain nonzero integer k can be connected to each other without any bulk transition.

For example, let us couple two Haldane phases to each other:

$$\begin{aligned}\mathcal{L} &= \frac{1}{g}(\partial_\mu \vec{n}^{(1)})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n_a^{(1)}\partial_\mu n_b^{(1)}\partial_\nu n_c^{(1)} \\ &+ 1 \rightarrow 2 + A(\vec{n}^{(1)} \cdot \vec{n}^{(2)}). \end{aligned} \tag{2.7}$$

When $A < 0$, effectively $\vec{n}^{(1)} = \vec{n}^{(2)} = \vec{n}$, then the system is effectively described by one O(3) NLSM with $\Theta = 4\pi$; while when $A > 0$, effectively $\vec{n}^{(1)} = -\vec{n}^{(2)} = \vec{n}$, the effective NLSM for the system has $\Theta = 0$. When parameter A is tuned from negative to positive, the bulk gap does not close. The reason is that, since $\Theta = 2\pi$ in both Haldane phases, the $\Theta-$term does not affect the bulk spectrum at all. To analyze the bulk spectrum (and bulk phase transition) while tuning A, we can just ignore the $\Theta-$term. Without the

$\Theta-$term, both theories are just trivial gapped phases, and an inter-chain coupling can not qualitatively change the bulk spectrum unless it is strong enough to overcome the bulk gap in each chain. We have explicitly checked this phase diagram using a Monte Carlo simulation of two coupled O(3) NLSMs, and the result is exactly the same as what we would expect from the argument above. Thus the theory with $\Theta = 4\pi$ and $\Theta = 0$ are equivalent.

By contrast, if we couple two chains with $\Theta = \pi$ each, then the cases $A > 0$ and < 0 correspond to effective $\Theta = 0$ and 2π respectively, and these two limits are separated by a bulk phase transition point $A = 0$, when the system becomes two decoupled chains with $\Theta = \pi$ each. And it is well-known that a $1 + 1d$ O(3) NLSM with $\Theta = \pi$ is the effective field theory that describes a spin-1/2 chain [57, 58], and according to the Lieb-Shultz-Matthis theorem, this theory must be either gapless or degenerate [76]. This conclusion is consistent with the RG calculation in Ref. [71, 72], and a general nonperturbative argument in Ref. [73].

In fact when $\Theta = 4\pi$ the boundary state of Eq. 2.1 is a spin-1 triplet, and by tuning A, at the boundary there is a level crossing between triplet and singlet, while there is no bulk transition. This analysis implies that with SO(3) symmetry, $1d$ spin systems have two different classes: there is a trivial class with $\Theta = 4\pi k$, and a nontrivial Haldane class with $\Theta = (4k + 2)\pi$.

If we cannot connect $\Theta = 4\pi$ to $\Theta = 0$ without closing the bulk gap, then the classification would be bigger than $\mathbb{Z}_2$. For example, let us consider the 2d SPT phase with U(1) symmetry which was first studied in Ref. [44]. This phase is described by Eq. 2.2. $B \sim n_1 + in_2$ and $B' \sim n_3 + in_4$ ($n_1 \cdots n_4$ are the four components of O(4) vector $\vec{n}$ in Eq. 2.2) are two complex boson (rotor) fields that transform identically under the global U(1) symmetry. Now suppose we couple two copies of this systems together through symmetry allowed interactions:

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + A_1 B_1 B_2^\dagger + A_2 B_1 B_2'^\dagger$$
$$+ A_3 B_1' B_2^\dagger + A_4 B_1' B_2'^\dagger + H.c. \quad (2.8)$$

No matter how we tune the parameters A_i, the resulting effective NLSM *always* has $\Theta = 4\pi$ instead of $\Theta = 0$ (this is simply because $(-1)^2 = (-1)^4 = +1$). This implies that we cannot smoothly connect $\Theta = 4\pi$ to 0 without any bulk transition. Thus the classification of 2d SPT phases with U(1) symmetry is $\mathbb{Z}$ instead of $\mathbb{Z}_2$. This is why in $2d$ (and all even dimensions), many SPT states have $\mathbb{Z}$ classification, while in odd dimensions there is no $\mathbb{Z}$ classification at all, namely all the nontrivial SPT phases in odd dimensions correspond to $\Theta = 2\pi$. Thus in Eq. 2.2 we added a level$-k$ in the $\Theta-$term.

2.3.4 NLSM and “decorated defect" construction of SPT states

Ref. [53] has given us a physical construction of some of the SPT states in terms of the “decorated domain wall" picture. For example, one of the $3d$ $Z_2^A \times Z_2^B$ SPT state can be constructed as follows: we first break the Z_2^B symmetry, then restore the Z_2^B symmetry by proliferating the domain wall of Z_2^B, and each Z_2^B domain wall is decorated with a $2d$ SPT state with Z_2^A symmetry. This state is described by Eq. 2.3 with transformation

$$Z_2^B \ : \ n_{1,2} \rightarrow -n_{1,2}, \quad n_a \rightarrow n_a (a = 3, 4, 5);$$
$$Z_2^A \ : \ n_1, \rightarrow n_1, \quad n_a \rightarrow -n_a (a = 2, \cdots 5). \quad (2.9)$$

Here n_i is the ith component of vector $\vec{n}$. To visualize the "decorated domain" wall picture, we can literally make a domain wall of n_1, and consider the following configuration of vector $\vec{n}$: $\vec{n} = (\cos\theta, \sin\theta N_2, \sin\theta N_3, \sin\theta N_4, \sin\theta N_5)$, where $\vec{N}$ is a O(4) vector with unit length, and θ is a function of coordinate z only:

$$\theta(z = +\infty) = \pi, \quad \theta(z = -\infty) = 0. \tag{2.10}$$

Plug this parametrization of $\vec{n}$ into Eq. 2.3, and integrate along z direction, the $\Theta-$term in Eq. 2.3 precisely reduces to the $\Theta-$term in Eq. 2.2 with $k = 1$, and the O(4) vector $\vec{n} = \vec{N}$. This is precisely the $2d$ SPT with Z_2 symmetry. This implies that the Z_2^B domain wall is decorated with a $2d$ SPT state with Z_2^A symmetry.

Many SPT states can be constructed with this decorated domain wall picture. Some $3d$ SPT states can also be understood as "decorated vortex", which was first discussed in [48]. This state has $U(1) \times Z_2^T$ symmetry, and the vector $\vec{n}$ transforms as

$$\begin{aligned} U(1) &: (n_1 + in_2) \to (n_1 + in_2)e^{i\theta}, \quad n_{3,4,5} \to n_{3,4,5}, \\ Z_2^T &: \vec{n} \to -\vec{n}. \end{aligned} \tag{2.11}$$

If we make a vortex of the U(1) order parameter (n_1, n_2), Eq. 2.3 reduces to Eq. 2.1 with O(3) order parameter (n_3, n_4, n_5). Thus this SPT can be viewed as decorating the U(1) vortex with a $1d$ Haldane phase, and then proliferating the vortices.

2.3.5 Independent NLSMs

Let us take the example of 1d SPT phases with $Z_2 \times Z_2^T$ symmetry. As we claimed, all 1d SPT phases in this paper are described by the same NLSM Eq. 2.1. With $Z_2 \times Z_2^T$

symmetry, there seems to be three different ways of assigning transformations to $\vec{n}$ that make the entire Lagrangian invariant:

$$
\begin{aligned}
(1) &\ : \ Z_2 : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n}. \\
(2) &\ : \ Z_2 : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3 \\
& \quad\ Z_2^T : \vec{n} \to -\vec{n} \\
(3) &\ : \ Z_2 : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3 \\
& \quad\ Z_2^T : n_3 \to -n_3, \quad n_{1,2} \to n_{1,2}.
\end{aligned}
\tag{2.12}
$$

However the NLSMs defined with these three different transformations are not totally independent from each other. Let us parameterize the O(3) vectors $\vec{n}^{(i)}$ with transformations (1), (2) and (3) as:

$$
\begin{aligned}
\vec{n}^{(i)}(\vec{r}) = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)}) = \\
\left(\sin(\theta_{\vec{r}}^{(i)})\cos(\phi_{\vec{r}}^{(i)}), \sin(\theta_{\vec{r}}^{(i)})\sin(\phi_{\vec{r}}^{(i)}), \cos(\theta_{\vec{r}}^{(i)})\right),
\end{aligned}
\tag{2.13}
$$

$\phi_{\vec{r}}^{(i)}$ and $\theta_{\vec{r}}^{(i)}$ are functions of space-time. Under Z_2 and Z_2^T symmetry, $\theta^{(i)}$ and $\phi^{(i)}$ transform as

$$
Z_2 \ : \ \theta^{(i)} \to \theta^{(i)},
$$

$$\phi^{(1)} \to \phi^{(1)}, \quad \phi^{(i)} \to \phi^{(i)} + \pi, \ (i = 2, 3);$$

$$Z_2^T \ : \ \theta^{(i)} \to \pi - \theta^{(i)},$$

$$\phi^{(i)} \to \phi^{(i)} + \pi, \ (i = 1, 2), \quad \phi^{(3)} \to \phi^{(3)}. \tag{2.14}$$

First of all, since $\theta^{(i)}$ have the same transformation for all i, we can turn on strong coupling between the three NLSMs to make $\theta^{(1)} = \theta^{(2)} = \theta^{(3)} = \theta$. Now we can construct $\vec{n}^{(3)}$ using the parametrization of $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$:

$$\begin{aligned} n_1^{(3)} &= \sin(\theta)\cos(\phi^{(1)} + \phi^{(2)}), \\ n_2^{(3)} &= \sin(\theta)\sin(\phi^{(1)} + \phi^{(2)}), \\ n_3^{(3)} &= \cos(\theta). \end{aligned} \tag{2.15}$$

It is straightforward to prove that $\vec{n}^{(3)}$ defined this way transforms identically with the case (3) in Eq. 2.12, also the topological number of $\vec{n}^{(3)}$ in 1+1d space-time is the sum of topological numbers of $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$. More explicitly, an instanton of $\vec{n}^{(a)}$ is a domain wall of $n_3^{(a)}$ decorated with a vortex of $\phi^{(a)}$. As we explained above, with appropriate coupling between these vectors, we can make $\theta^{(1)} = \theta^{(2)} = \theta^{(3)} = \theta$, and $\phi^{(3)} = \phi^{(1)} + \phi^{(2)}$. Thus a domain wall of $n_3^{(3)}$ is also a domain wall of $n_3^{(1)}$ and $n_3^{(2)}$, while the vortex number of $\phi^{(3)}$ is the sum of vortex number of $\phi^{(1)}$ and $\phi^{(2)}$. Thus the $\Theta-$term of $\vec{n}^{(3)}$ reduces to the sum of $\Theta-$terms of $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$. In this example we have shown that NLSMs (1) and (2) in Eq. 2.12 can "merge" into NLSM (3). Thus the three NLSMs defined with transformations (1), (2) and (3) are not independent from each other. [2] The consequence of this analysis is that if all three theories exist in one system, although each theory is a nontrivial SPT phase individually, we can turn on some symmetry allowed couplings between these NLSMs and cancel the bulk topological terms completely, and drive the entire coupled system to a trivial state.

Also, for either NLSM (1) or (2) in Eq. 2.12, we can show that $\Theta^{(i)} = 0$ and 4π can be connected to each other without a bulk transition (using the same method as the previous subsection). Then eventually the 1d SPT phase with $Z_2 \times Z_2^T$ symmetry is parametrized by two independent $\Theta-$terms, the fixed point values of $\Theta^{(1)}$ and $\Theta^{(2)}$ can be either 0 or 2π, thus this SPT phase has a $(\mathbb{Z}_2)^2$ classification, which is consistent with the classification using group cohomology. NLSMs with transformations (1), (2) are two "root phases" of 1d SPT phases with $Z_2 \times Z_2^T$ symmetry. All the other SPT phases can be constructed with these two root phases.

For most SPT phases, we can construct the NLSMs using the smallest representation (fundamental representation) of the symmetry groups G, because usually (but not always!) NLSMs constructed using higher representations can reduce to constructions with the fundamental representation with a different Θ. For example, the 1d SPT phase with $U(1) \rtimes Z_2$ symmetry can be described by Eq. 2.1 with the following transformation

$$U(1) \ : \ (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \quad n_3 \to n_3,$$

$$Z_2 \ : \ n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}, \tag{2.16}$$

namely $B \sim (n_1 + in_2)$ is a charge-1 boson under the U(1) rotation, and the edge state

[2] The "merging" argument is usually easy to implement for systems with simple symmetries, but we should admit that for higher dimensions and complicated symmetries, the "merging" argument can become rather involved.

of this SPT phase carries charge-1/2 of boson B. We can also construct an O(3) NLSM using charge-2 boson $B' \sim (n'_1 + in'_2) \sim (n_1 + in_2)^2$ that transforms as $B' \to B'e^{2i\alpha}$, then mathematically we can demonstrate that the NLSM with $\Theta = 2\pi$ for order parameter $\vec{n}' = (n'_1, n'_2, n_3)$ reduces to a NLSM of $\vec{n}$ with $\Theta = 4\pi$, hence it is a trivial phase.

More explicitly, let us take unit vector $\vec{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, and vector $\vec{n}' = (\sin(\theta)\cos(2\phi), \sin(\theta)\sin(2\phi), \cos(\theta))$, then we can show that when $\vec{n}$ has topological number 1 in 1+1d space-time, $\vec{n}'$ would have topological number 2. This means that if there is a $\Theta-$term for $\vec{n}'$ with $\Theta = 2\pi$, it is equivalent to a $\Theta-$term for $\vec{n}$ with $\Theta = 4\pi$.

Physically, the edge state of NLSM of $\vec{n}'$ with $\Theta = 2\pi$ carries a half-charge of B', which is still a charge-1 object, so it can be screened by another charge-1 boson B. Hence in this case NLSM constructed using charge-2 boson B' would be trivial.

However, later we will also show that when the symmetry group involves Z_m with even integer $m > 2$, then using higher representations of Z_m we can construct SPT phases that *cannot* be obtained from the fundamental representation of Z_m.

2.3.6 Boundary topological order of 3d SPT phases

The $(d-1)-$dimensional boundary of a $d-$dimensional SPT phase must be either degenerate or gapless. When $d = 3$, its 2d boundary can spontaneously break the symmetry, or have a topological order [48]. We can use the bulk field theory Eq. 2.3 to derive the quantum numbers of the anyons at the boundary.

Let us take the 3d SPT phase with $Z_2 \times Z_2^T$ symmetry as an example. One of the SPT phases has the following transformations:

$$Z_2 \;:\; n_a \to -n_a (a = 1, \cdots 4), \quad n_5 \to n_5;$$

$$Z_2^T \;\; : \;\; \vec{n} \to -\vec{n}. \tag{2.17}$$

The 2+1d boundary of the system is described by a 2+1d O(5) NLSM with a Wess-Zumino-Witten (WZW) term at level $k = 1$:

$$\begin{aligned} S &= \int d^2x d\tau \; \frac{1}{g}(\partial_\mu \vec{n})^2 \\ &+ \int_0^1 du \; \frac{i2\pi}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_\tau n^e, \end{aligned} \tag{2.18}$$

where $\vec{n}(x, \tau, u)$ satisfies $\vec{n}(x, \tau, 0) = (0, 0, 0, 0, 1)$ and $\vec{n}(x, \tau, 1) = \vec{n}(x, \tau)$. If the time-reversal symmetry is preserved, namely $\langle n_5 \rangle = 0$, we can integrate out n_5, and Eq. 2.18 reduces to a 2+1d O(4) NLSM with $\Theta = \pi$:

$$S = \int d^2x d\tau \; \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\pi}{\Omega_3} \epsilon_{abcd} n^a \partial_\tau n^b \partial_x n^c \partial_y n^d. \tag{2.19}$$

In Eq. 2.19 $\Theta = \pi$ is protected by time-reversal symmetry.

In the following we will argue that the topological terms in Eq. 2.18 and Eq. 2.19 guarantee that the $2d$ boundary cannot be gapped without degeneracy. One particularly interesting possibility of the boundary is a phase with 2d Z_2 topological order [48]. A 2d Z_2 topological phase has e and m excitations that have mutual semion statistics [23]. The semion statistics can be directly read off from Eq. 2.19: if we define complex boson fields $z_1 = n_1 + in_2$ and $z_2 = n_3 + in_4$, then the $\Theta-$term in Eq. 2.19 implies that a vortex of (n_3, n_4) carries half charge of z_1, while a vortex of (n_1, n_2) carries half charge of z_2, thus vortices of z_1 and z_2 are bosons with mutual semion statistics. This statistics survives after z_1 and z_2 are disordered by condensing the *double vortex* (vortex with vorticity 4π) of either z_1 or z_2 at the boundary, then the disordered phase must inherit the statistics and become a Z_2 topological phase [48]. The vortices of (n_1, n_2) and (n_3, n_4) become the e and m excitations respectively. Normally a vortex defect is discussed in systems with a U(1) global symmetry. We do not assume such U(1) global symmetry in our case, this symmetry reduction is unimportant in the Z_2 topological phase.

At the vortex core of (n_3, n_4), namely the m excitation, Eq. 2.18 reduces to a $0+1d$ O(3) NLSM with a WZW term at level 1 [77]:

$$\mathcal{S}_m = \int d\tau \frac{1}{g}(\partial_\tau \vec{N})^2 + \int_0^1 du \frac{i2\pi}{8\pi} \epsilon_{abc}\epsilon_{\mu\nu} N^a \partial_\mu N^b \partial_\nu N^c, \tag{2.20}$$

where $\vec{N} \sim (n_1, n_2, n_5)$. This 0+1d field theory describes a single particle moving on a 2d sphere with a magnetic monopole at the origin. It is well known that if there is a SO(3) symmetry for $\vec{N}$, then the ground state of this 0d problem has two fold degeneracy, with two orthogonal solutions

$$u_m = \cos(\theta/2) e^{i\phi/2}, \quad v_m = \sin(\theta/2) e^{-i\phi/2},$$

$$\vec{N} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \tag{2.21}$$

Likewise, the vortex of (n_1, n_2) (e excitation) also carries a doublet (u_e, v_e). Under the Z_2 transformation, $\phi \to \phi + \pi$, thus $u_{e,m}$ and $v_{e,m}$ carry charge $\pm 1/2$ of the Z_2 symmetry, namely under the Z_2 transformation:

$$Z_2 : U_{e,m} \to i\sigma^z U_{e,m}, \tag{2.22}$$

where $U_{e,m} = (u_{e,m}, v_{e,m})^t$.

Under time-reversal transformation $\mathcal{T}$, $\vec{N} \to -\vec{N}$, $\theta \to \theta + \pi$. Thus the e and m

doublets transform as

$$Z_2^T : U_{e,m} \to i\sigma^y U_{e,m}, \tag{2.23}$$

thus the e and m anyons at the boundary carry projective representation of Z_2^T which satisfies $\mathcal{T}^2 = -1$.

Based on this Z_2 topological order, we can derive the phase diagram around the Z_2 topological order, and show that this boundary cannot be gapped without degeneracy. For example, starting with a $2d$ Z_2 topological order, one can condense either e or m excitation and kill the topological degeneracy. However, because $U_{e,m}$ transform nontrivially under the symmetry group, condensate of either e or m will always spontaneously break certain symmetry and lead to degeneracy. For example, the condensate of e excitation has nonzero expectation value of $(n_3, n_4, n_5) \sim U_e^\dagger \vec{\sigma} U_e$, which necessarily spontaneously breaks the Z_2 or Z_2^T symmetry.

We also note that one bulk BSPT state can have different boundary states, which depends on the details of the boundary Hamiltonian. Recently a different boundary topological order of BSPT state was derived in Ref. [78], but the bulk state is the same as ours.

2.3.7 Rule of classification

With all these preparations, we are ready to lay out the rules of our classification:

1. In $d-$dimensional space, all the SPT phases discussed in this paper are described by a $(d+1)-$dimensional O$(d+2)$ NLSM with a $\Theta-$term. The O$(d+2)$ vector field $\vec{n}$ is an order parameter, namely it must carry a nontrivial representation of the given symmetry. In other words, no component of the vector field transforms completely trivially under the symmetry, because otherwise it is allowed to turn on a strong linear “Zeeman" term to the trivial component, and then the system will become a trivial direct product state.

2. The classification is given by all the possible *independent* symmetry transformations on vector order parameter $\vec{n}$ that keep the $\Theta-$term invariant, for *arbitrary* value of Θ. Independent transformations mean that any NLSM defined with one transformation cannot be obtained by "merging" two (or more) other NLSMs defined with other transformations. SPT phases constructed using independent NLSMs are called "root phases". All the other SPT phases can be constructed with these root phases.

3. With a given symmetry, and given transformation of $\vec{n}$, if $\Theta = 2\pi k$ and $\Theta = 0$ can be connected without a bulk transition, this transformation will contribute classification $\mathbb{Z}_k$; otherwise the transformation will contribute classification $\mathbb{Z}$.

Using the rule and strategy discussed in this section, we can obtain the classification of all SPT phases in all dimensions. In this paper we will systematically study SPT phases in one, two and three spatial dimensions with symmetries Z_2^T, Z_2, $Z_2 \times Z_2$, $Z_2 \times Z_2^T$, $U(1)$, $U(1) \times Z_2$, $U(1) \rtimes Z_2$, $U(1) \times Z_2^T$, $U(1) \rtimes Z_2^T$, Z_m, $Z_m \times Z_2$, $Z_m \rtimes Z_2$, $Z_m \times Z_2^T$, $Z_m \rtimes Z_2^T$, $SO(3)$, $SO(3) \times Z_2^T$, $Z_2 \times Z_2 \times Z_2$. The final classification of the SPT phases we study in this paper is consistent to the classification based on group cohomology [31, 42].

2.4 1d SPT phase with $Z_2 \times Z_2 \times Z_2^T$ symmetry

Before we discuss our full classification, let us carefully discuss 1d SPT phases with $Z_2 \times Z_2 \times Z_2^T$ symmetry as an example. These SPT phases were discussed very thoroughly in Ref. [79]. There are in total 16 different phases (including the trivial phase). The goal of this section is to show that all these phases can be described by the *same* equation Eq. 2.1 with certain transformation of $\vec{n}$, and the projective representation of the boundary states given in Ref. [79] can be derived explicitly using Eq. 2.6.

For the consistency of notation in this paper, R_z and R_x in Ref. [79] will be label-led Z_2^A and Z_2^B here. Let us consider one example, namely Eq. 2.1 with the following transformation:

$$Z_2^A \quad : \quad n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3;$$

$$Z_2^B \quad : \quad n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1;$$

$$Z_2^T \quad : \quad n_2 \to -n_2, \quad n_{1,3} \to n_{1,3}. \tag{2.24}$$

Now let us parametrize $\vec{n}$ as

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\,, \tag{2.25}$$

then θ and ϕ transform as

$$Z_2^A \quad : \quad \theta \to \theta, \quad \phi \to \phi + \pi,$$

$$Z_2^B \quad : \quad \theta \to \pi - \theta, \quad \phi \to -\phi,$$

$$Z_2^T \quad : \quad \theta \to \theta, \quad \phi \to -\phi. \tag{2.26}$$

These transformations lead to the following projective transformation of edge state Eq. 2.6:

$$Z_2^A \quad : \quad U \to i\sigma^z U,$$

$$Z_2^B \quad : \quad U \to \sigma^x U,$$

$$Z_2^T \quad : \quad U \to U. \tag{2.27}$$

Thus this NLSM corresponds to phase E_5 in Ref. [79].

The 16 phases in Ref. [79] correspond to the following transformations of O(3) vector $\vec{n}$:

$$E_0 \quad : \quad \text{Trivial phase, } \Theta = 0;$$

$$E_0' \quad : \quad Z_2^A, Z_2^B : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n};$$

$$E_1 \quad : \quad Z_2^A : \vec{n} \to \vec{n},$$

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^T : \vec{n} \to -\vec{n},$$

$$E_1' \quad : \quad Z_2^A : \vec{n} \to \vec{n},$$

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3;$$

$$E_3 \quad : \quad Z_2^B : \vec{n} \to \vec{n},$$

$$Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^T : \vec{n} \to -\vec{n},$$

$$E_3' \quad : \quad Z_2^B : \vec{n} \to \vec{n},$$

$$Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3;$$

$$E_5 \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3;$$

$$Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1;$$

$$Z_2^T : n_2 \to -n_2, \quad n_{1,3} \to n_{1,3};$$

$$E_5' \quad : \quad E_5 \oplus E_0';$$

$$E_7 \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3;$$

$$E_7' \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3,$$

$$
\begin{aligned}
& & Z_2^T : \vec{n} \to -\vec{n}; \\
E_9 & : & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& & Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& & Z_2^T : n_3 \to -n_3, \quad n_{1,2} \to n_{1,2}; \\
E_9' & : & E_9 \oplus E_0', \\
E_{11} & : & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& & Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& & Z_2^T : n_1 \to -n_1, \quad n_{2,3} \to n_{2,3}; \\
E_{11}' & : & E_{11} \oplus E_0'; \\
E_{13} & : & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& & Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_1 \to n_1; \\
& & Z_2^T : \vec{n} \to -\vec{n}; \\
E_{13}' & : & E_{13} \oplus E_0'.
\end{aligned}
\tag{2.28}
$$

All the phases except for the trivial phase E_0 have $\Theta = 2\pi$ in Eq. 2.1. Here $E_5 \oplus E_0'$ means it is a spin ladder with symmetry allowed weak interchain couplings, and the two chains are E_5 phase and E_0' phase respectively. For all the 16 phases above, we can compute the projective representations of the boundary states using Eq. 2.6, and they all precisely match with the results in Ref. [79].

2.5 Full classification of SPT phases

2.5.1 SPT phases with Z_2 symmetry

In 1d and 3d, there is no Z_2 symmetry transformation that we can assign vector $\vec{n}$ that makes the actions Eq. 2.1 and Eq. 2.3 invariant, thus there is no SPT phase in 1d and 3d with Z_2 symmetry. However, in 2d there is obviously one and only one way to assign the Z_2 symmetry:

$$Z_2 : (n_1, n_2, n_3, n_4) \rightarrow -(n_1, n_2, n_3, n_4). \tag{2.29}$$

Then when $\Theta = 2\pi$ this 2+1d O(4) NLSM describes the Z_2 SPT phase studied in Ref. [43]. Using the method in section IIC, one can show that with the transformation Eq. 2.29, the 2+1d O(4) NLSM Eq. 2.2 with $\Theta = 4\pi$ is equivalent to $\Theta = 0$, thus the classification in 2d is $\mathbb{Z}_2$.

In Ref. [50], the authors also used this NLSM to derive the ground state wave function of the SPT phase:

$$|\Psi\rangle = \sum (-1)^{dw} |C\rangle, \tag{2.30}$$

where $|C\rangle$ standards for an arbitrary Ising field configuration, while dw is the number of

Ising domain walls of this configuration. This wave function was also derived in Ref. [43] with an exactly soluble model for this SPT phase.

The classification of SPT phases with Z_2 symmetry is:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}_2, \quad 3d : \mathbb{Z}_1. \tag{2.31}$$

Here $\mathbb{Z}_1$ means there is only one trivial state, and $\mathbb{Z}_2$ means there is one trivial state and one nontrivial SPT state.

2.5.2 SPT phases with Z_2^T symmetry

In 2d, there is no way to assign Z_2^T symmetry to the O(4) NLSM order parameter in Eq. 2.2 to make the $\Theta-$term invariant, thus there is no bosonic SPT phase in 2d with Z_2^T symmetry. In 1d and 3d, there is only one way to assign the Z_2^T symmetry to vector $\vec{n}$:

$$Z_2^T : \vec{n} \to -\vec{n}, \tag{2.32}$$

and $\Theta = 0$ and $\Theta = 4\pi$ are equivalent. Thus in both 1d and 3d, the classification is $\mathbb{Z}_2$. Notice that time-reversal is an antiunitary transformation, thus $i \to -i$ under Z_2^T; also since our NLSMs are defined in Euclidean space-time, the Euclidean time $\tau = it$ is invariant under Z_2^T.

Using the method in section II.F, one can demonstrate that the boundary of the $3d$ SPT state with Z_2^T symmetry is a $2d$ Z_2 topological order, whose both e and m excitations are Kramers doublet, i.e. the so called $eTmT$ state.

The classification of SPT phases with Z_2^T symmetry is:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}_1, \quad 3d : \mathbb{Z}_2. \tag{2.33}$$

Now it is understood that in $3d$ there is bosonic SPT state with Z_2^T symmetry that is beyond the group cohomology classification [48], and there is a explicit lattice construction for such state [80]. This state is also beyond our current NLSM description. However, a generalized field theory which involves both the NLSM and Chern-Simons theory can describe at least a large class of BSPT states beyond group cohomology. This will be discussed in a different paper [68].

2.5.3 SPT phases with $U(1)$ symmetry

In 1d and 3d, there is no way to assign U(1) symmetry to vector $\vec{n}$ that keeps the entire Lagrangian invariant. But in 2d, bosonic SPT phase with U(1) symmetry was discussed in Ref. [44], and its field theory is given by Eq. 2.2. And since in this case we cannot connect $\Theta = 2\pi k$ and $\Theta = 0$ without a bulk transition, the classification is $\mathbb{Z}$.

The classification of SPT phases with $U(1)$ symmetry is:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}, \quad 3d : \mathbb{Z}_1. \tag{2.34}$$

2.5.4 SPT phases with $U(1) \rtimes Z_2$ symmetry

$U(1) \rtimes Z_2$ is a subgroup of SO(3). In 1d, there is only one way of assigning the symmetry to vector $\vec{n}$ that keeps the entire Lagrangian invariant:

$$U(1) \ : \ (n_1 + in_2) \rightarrow e^{i\theta}(n_1 + in_2), \quad n_3 \rightarrow n_3,$$

$$Z_2 \quad : \quad n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}. \tag{2.35}$$

Here Z_2 is a particle-hole transformation of rotor/boson field $b \sim n_1 + in_2$. n_3 can be viewed as the boson density, which changes sign under particle-hole transformation. One can check that the U(1) and Z_2 symmetry defined above do not commute with each other. The boundary state of this 1d SPT phase is given in Eq. 2.6. Under U(1) and Z_2 transformation, the boundary doublet U transforms as

$$U(1) : U \to e^{i\theta\sigma^z/2}U, \quad Z_2 : U \to \sigma^x U. \tag{2.36}$$

In 3d, there is also only one way of assigning the symmetry to the O(5) vector:

$$U(1) \quad : \quad (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \quad n_b \to n_b, \; b = 3, 4, 5;$$

$$Z_2 \quad : \quad n_1 \to n_1, \quad n_b, \to -n_b, \quad b = 2, \cdots 5. \tag{2.37}$$

In both 1d and 3d, $\Theta = 4\pi$ is equivalent to $\Theta = 0$, thus in both 1d and 3d the classification is $\mathbb{Z}_2$.

In 2d, there are two independent ways of assigning $U(1) \rtimes Z_2$ transformations to the O(4) vector $\vec{n}$:

$$(1) \quad : \quad U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2),$$

$$(n_3 + in_4) \to e^{i\theta}(n_3 + in_4);$$

$$Z_2 : n_1, n_3 \to n_1, n_3, \quad n_2, n_4 \to -n_2, -n_4;$$

$$(2) \quad : \quad U(1) : \vec{n} \to \vec{n}, \quad Z_2 : \vec{n} \to -\vec{n}. \tag{2.38}$$

The transformation (1) contributes $\mathbb{Z}$ classification, while transformation (2) contributes $\mathbb{Z}_2$ classification, *i.e.* in 2d the classification is $\mathbb{Z} \times \mathbb{Z}_2$. *The final classification of SPT phases with* $U(1) \rtimes Z_2$ *symmetry is*:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z} \times \mathbb{Z}_2, \quad 3d : \mathbb{Z}_2. \tag{2.39}$$

2.5.5 SPT phases with $U(1) \times Z_2$ symmetry

In both 1d and 3d, there is no way of assigning $U(1) \times Z_2$ transformations to vector $\vec{n}$ that keeps the Θ term invariant. But in 2d, we can construct three root phases:

$$\begin{aligned}
(1) \quad &: \quad U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \\
& \quad (n_3 + in_4) \to e^{i\theta}(n_3 + in_4); \\
& \quad Z_2 : \vec{n} \to \vec{n}; \\
(2) \quad &: \quad U(1) : \vec{n} \to \vec{n}, \quad Z_2 : \vec{n} \to -\vec{n}; \\
(3) \quad &: \quad U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \\
& \quad n_{3,4} \to n_{3,4}; \\
& \quad Z_2 : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}.
\end{aligned} \tag{2.40}$$

The first transformation contributes classification $\mathbb{Z}$, while transformations (2) and (3) both contribute classification $\mathbb{Z}_2$, *thus the final classification of SPT phases with $U(1)\times Z_2$ symmetry is*:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z} \times (\mathbb{Z}_2)^2, \quad 3d : \mathbb{Z}_1. \tag{2.41}$$

2.5.6 SPT phases with $U(1) \rtimes Z_2^T$ symmetry

A boson operator b with $U(1) \rtimes Z_2^T$ symmetry transforms as $b \to b$ under Z_2^T. In 1d, the only $U(1) \rtimes Z_2^T$ symmetry transformation that keeps Eq. 2.1 invariant is the same transformation as Z_2^T SPT phase, namely vector $\vec{n}$ does not transform under $U(1)$, but changes sign under Z_2^T.

In 2d, the only transformation that keeps Eq. 2.2 invariant is

$$\begin{aligned} U(1) &: (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \ n_{3,4} \to n_{3,4}; \\ Z_2^T &: n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4), \end{aligned} \tag{2.42}$$

and this NLSM gives classification $\mathbb{Z}_2$.

The NLSMs for $U(1) \rtimes Z_2^T$ SPT phases in 3d have been discussed in Ref. [48], and in 3d the classification is $(\mathbb{Z}_2)^2$. *Thus the final classification of SPT phases with $U(1) \rtimes Z_2^T$ symmetry is*:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}_2, \quad 3d : (\mathbb{Z}_2)^2. \tag{2.43}$$

2.5.7 SPT phases with $U(1) \times Z_2^T$ symmetry

In 1d, there are two independent transformations that keep Eq. 2.1 invariant:

$$
\begin{aligned}
(1) \quad &: \quad U(1) : (n_1 + in_2) \to e^{i\theta}(n_1 + in_2), \quad n_3 \to n_3; \\
&\quad Z_2^T : n_{1,2} \to n_{1,2}, \quad n_3 \to -n_3, \\
(2) \quad &: \quad U(1) : \vec{n} \to \vec{n}, \\
&\quad Z_2^T : \vec{n} \to -\vec{n}.
\end{aligned} \tag{2.44}
$$

In 2d there is no $U(1) \times Z_2^T$ transformation that keeps Eq. 2.2 invariant. In 3d the NLSMs for $U(1) \times Z_2^T$ SPT phases were discussed in Ref. [48]. *The final classification of SPT phases with $U(1) \times Z_2^T$ symmetry is*:

$$
1d : (\mathbb{Z}_2)^2, \quad 2d : \mathbb{Z}_1, \quad 3d : (\mathbb{Z}_2)^3. \tag{2.45}
$$

2.5.8 SPT phases with $Z_2 \times Z_2$ symmetry

In 1d, there is only one $Z_2 \times Z_2$ transformation that keeps Eq. 2.1 invariant:

$$
\begin{aligned}
Z_2^A \quad &: \quad n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3, \\
Z_2^B \quad &: \quad n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}.
\end{aligned} \tag{2.46}
$$

The boundary state U defined in Eq. 2.6 transforms as

$$Z_2^A : U \to i\sigma^z U, \quad Z_2^B : U \to \sigma^x U. \tag{2.47}$$

Thus Z_2^A and Z_2^B no longer commute with each other at the boundary.

In 2d, there are three independent $Z_2 \times Z_2$ transformations (three different root phases):

$$\begin{aligned}
(1) &: \quad Z_2^A : \vec{n} \to -\vec{n}, \quad Z_2^B : \vec{n} \to \vec{n}; \\
(2) &: \quad Z_2^A : \vec{n} \to \vec{n}, \quad Z_2^B : \vec{n} \to -\vec{n}; \\
(3) &: \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\
& \quad\;\; Z_2^B : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}.
\end{aligned} \tag{2.48}$$

In 3d, there are also two independent $Z_2 \times Z_2$ transformations that keep Eq. 2.3 invariant (two root phases):

$$\begin{aligned}
(1) &: \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5); \\
& \quad\;\; Z_2^B : n_1, \to n_1, \quad n_a \to -n_a (a = 2, \cdots 5); \\
(2) &: \quad Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5); \\
& \quad\;\; Z_2^A : n_1, \to n_1, \quad n_a \to -n_a (a = 2, \cdots 5).
\end{aligned} \tag{2.49}$$

As we discussed in section II.F, the boundary of these 3d SPT phases can have 2d Z_2 topological order. A 2d Z_2 topological phase has e and m anyon excitations, and these anyons correspond to vortices of certain components of order parameter $\vec{n}$. If the e and m anyons correspond to vortices of (n_3, n_4) and (n_1, n_2) respectively, then according to Eq. 2.20, the e excitation corresponds to a $0+1d$ O(3) WZW model for vector (n_1, n_2, n_5), and the m excitation corresponds to a $0+1d$ WZW model for vector (n_3, n_4, n_5). The boundary anyons of phase (1) transform as:

$$(1) \;\; : \;\; Z_2^A : U_e \to i\sigma^z U_e, \quad U_m \to U_m;$$

$$Z_2^B : U_e \to \sigma^x U_e, \quad U_m \to i\sigma^y U_m^*. \tag{2.50}$$

Notice that under Z_2^B, a vortex of (n_1, n_2) becomes an antivortex, thus the transformation of U_m under Z_2^B involves a complex conjugation. The transformation of boundary anyons of phase (2) is the same as Eq. 2.50 after interchanging Z_2^A and Z_2^B.

The final classification of SPT phases with $Z_2 \times Z_2$ symmetry is:

$$1d : \mathbb{Z}_2, \quad 2d : (\mathbb{Z}_2)^3, \quad 3d : (\mathbb{Z}_2)^2. \tag{2.51}$$

2.5.9 SPT phases with $Z_2 \times Z_2^T$ symmetry

In 1d and 3d, the SPT phases with $Z_2 \times Z_2^T$ symmetry are simply SPT phases with $U(1) \times Z_2^T$ symmetry after reducing U(1) to its subgroup Z_2. The classification is the same as the $U(1) \times Z_2^T$ SPT phases discussed in the previous subsection. In 2d, there are two different root phases that correspond to the following transformations:

$$(1) \;\; : \;\; Z_2 : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4},$$

$$Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4);$$

$$(2) \quad : \quad Z_2 : \vec{n} \to -\vec{n},$$

$$Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4). \tag{2.52}$$

The final classification of SPT phases with $Z_2 \times Z_2^T$ symmetry is:

$$1d : (\mathbb{Z}_2)^2, \quad 2d : (\mathbb{Z}_2)^2, \quad 3d : (\mathbb{Z}_2)^3. \tag{2.53}$$

2.5.10 SPT phases with Z_m symmetry

In 1d and 3d, there are no nontrivial Z_m transformations that can keep Eq. 2.1 and Eq. 2.3 invariant. In 2d, we can construct the following root phase:

$$Z_m \quad : \quad (n_1 + in_2) \to e^{i2\pi k/m}(n_1 + in_2);$$

$$(n_3 + in_4) \to e^{i2\pi k/m}(n_3 + in_4),$$

$$k = 0, \cdots m - 1 \tag{2.54}$$

Using the method in section II, we can demonstrate that with these transformations, Eq. 2.2 with $\Theta = 2\pi m$ and $\Theta = 0$ are equivalent to each other, thus the classification is $\mathbb{Z}_m$ in 2d.

The final classification of SPT phases with Z_m symmetry is:

$$1d : \mathbb{Z}_1, \quad 2d : \mathbb{Z}_m, \quad 3d : \mathbb{Z}_1. \tag{2.55}$$

2.5.11 SPT phases with $Z_m \rtimes Z_2$ symmetry

In 1d, there is one SPT phase with $U(1) \rtimes Z_2$ symmetry. Naively one would expect that when U(1) is broken down to Z_m, this SPT phase survives and becomes a SPT phase with $Z_m \rtimes Z_2$ symmetry. However, this statement is only true for even m, and when m is odd the $U(1) \rtimes Z_2$ SPT phase becomes trivial once U(1) is broken down to Z_m.

The 1d $U(1) \rtimes Z_2$ SPT phase is described by a 1d O(3) NLSM of vector $\vec{n}$ with $\Theta = 2\pi$, and $B \sim (n_1 + in_2)$ is a charge-1 boson under the U(1) rotation. Because the classification of 1d $U(1) \rtimes Z_2$ SPT phase is $\mathbb{Z}_2$, $\Theta = 2\pi$ is equivalent to $\Theta = 2\pi m$ for odd m. As we discussed in section IID, this NLSM with $\Theta = 2\pi m$ is equivalent to another NLSM defined with $\vec{n}'$ and $\Theta = 2\pi$, where $B' \sim (n'_1 + in'_2) \sim (n_1 + in_2)^m$ is a charge-m boson. Under Z_2 transformation, $n'_1 \to n'_1$, $n'_2 \to -n'_2$.

Now let us break U(1) down to its subgroup Z_m. B' transforms trivially under Z_m, thus we are allowed to turn on a Zeeman term $\mathrm{Re}[B'] \sim n'_1$ which fully polarizes n'_1 and kills the SPT phase. Thus the original $U(1) \rtimes Z_2$ SPT phase is instable under U(1) to Z_m breaking with odd m.

The discussion above is very abstract, let us understand this result physically, and we will take $m = 3$ as an example. With a full SO(3) symmetry and $\Theta = 2\pi$ in the bulk, the ground state of the boundary is a spin-1/2 doublet in Eq. 2.6. The excited states of the boundary include a spin-3/2 quartet. When $\Theta = 6\pi$ in the bulk, the boundary ground state is a spin-3/2 quartet. The spin-3/2 and spin-1/2 states can have a boundary transition (level crossing at the boundary) without closing the bulk gap, thus $\Theta = 2\pi$

and 6π are equivalent in the bulk. Now let us take $\Theta = 6\pi$ in the bulk, and break the SO(3) down to $Z_3 \rtimes Z_2$. Then we are allowed to turn on a perturbation $\cos(3\phi)$ at the boundary (which precisely corresponds to the Zeeman coupling $\mathrm{Re}[B'] \sim n'_1$ discussed in the previous paragraph), which will mix and split the two states $S^z = \pm 3/2$ at the boundary, and the boundary ground state can become nondegenerate. Thus when m is odd, the $U(1) \rtimes Z_2$ SPT phase does not survive the symmetry breaking from U(1) to Z_m.

The same situation occurs in 2d and 3d. There is a 3d SPT phase with $U(1) \rtimes Z_2$ symmetry, but once we break the U(1) down to Z_m, this SPT phase does not survive when m is odd. When m is even, besides the phase deduced from $U(1) \rtimes Z_2$ SPT phase, one can construct another root phase:

$$
\begin{aligned}
Z_2 \ &: \ n_{1,2} \to -n_{1,2}, \quad n_a \to n_a \ (a = 3, 4, 5); \\
Z_m \ &: \ n_1, \to n_1, \quad n_a \to (-1)^k n_a \ (a = 2, \cdots 5), \\
& k = 0, \cdots m - 1.
\end{aligned} \tag{2.56}
$$

Here $n_a (a = 2, \cdots 5)$ still carries a nontrivial representation of Z_m for even integer m. n_a with $a = 3, 4, 5$ can be viewed as the real parts of charge-$m/2$ bosons, while n_2 is the imaginary part of such charge-$m/2$ boson. This construction does not apply for odd m.

In 2d, for arbitrary $m > 1$, the $U(1) \rtimes Z_2$ SPT phases survive under $U(1)$ to Z_m symmetry breaking. With even m, another root phase can be constructed

$$
\begin{aligned}
Z_m \ &: \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\
Z_2 \ &: \ n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4},
\end{aligned}
$$

$$k = 0, \cdots m - 1. \tag{2.57}$$

Here n_1 and n_2 are both the real parts of the charge-$m/2$ bosons.

The final classification of SPT phases with $Z_m \rtimes Z_2$ symmetry is:

$$1d : \mathbb{Z}_{(2,m)}, \quad 2d : \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 3d : (\mathbb{Z}_{(2,m)})^2. \tag{2.58}$$

2.5.12 SPT phases with $Z_m \times Z_2$ symmetry

The case $m = 2$ has already been discussed. When $m > 2$, one would naively expect these SPT phases can be interpreted as $U(1) \times Z_2$ SPT phases after breaking U(1) to its Z_m subgroup, but again this is not entirely correct. In 1d there is no SPT phase with $U(1) \times Z_2$ symmetry, simply because we cannot find a nontrivial transformation of $\vec{n}$ under $U(1) \times Z_2$ that keeps Eq. 2.1 invariant. But when m is an even number, we can construct one SPT phase with $Z_m \times Z_2$ symmetry using Eq. 2.1:

$$Z_m \;\; : \;\; n_{1,2} \to (-1)^k n_{1,2}, \quad n_3 \to n_3,$$

$$Z_2 \;\; : \;\; n_1 \to n_1, \quad n_{2,3} \to -n_{2,3},$$

$$k = 0, \cdots m - 1. \tag{2.59}$$

The Z_m and Z_2 transformations on $\vec{n}$ commute with each other.

Again this construction applies to even integer m only. The boundary states of this

1d SPT phase have the following transformations:

$$Z_m \quad : \quad U \to (i\sigma^z)^k U, \quad Z_2 : U \to \sigma^x U;$$

$$k = 0, \cdots m-1. \tag{2.60}$$

Thus the boundary states carry projective representations of $Z_m \times Z_2$, and the transformations of Z_m and Z_2 do not commute.

Similar situations occur in 3d. In 3d, we can construct two root phases for even m, even though there is no SPT phase with $U(1) \times Z_2$ symmetry in 3d :

$$(1) \quad : \quad Z_m : n_{1,2} \to (-1)^k n_{1,2}, \quad n_a \to n_a (a = 3,4,5);$$

$$Z_2 : n_1, \to n_1, \quad n_a \to -n_a \ (a = 2, \cdots 5);$$

$$(2) \quad : \quad Z_2 : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a (a = 3,4,5);$$

$$Z_m : n_1, \to n_1, \quad n_a \to (-1)^k n_a (a = 2, \cdots 5);$$

$$k = 0, \cdots m-1. \tag{2.61}$$

The boundary of these 3d SPT phases can have 2d Z_2 topological order. If the e and m anyons correspond to vortices of (n_3, n_4) and (n_1, n_2) respectively, then the boundary anyons of phase (1) transform as:

$$(1) \quad : \quad Z_m : U_e \to (i\sigma^z)^k U_e, \quad U_m \to U_m;$$

$$Z_2 : U_e \to \sigma^x U_e, \quad U_m \to i\sigma^y U_m^*. \tag{2.62}$$

The transformation of boundary anyons of phase (2) can be derived in the same way.

In 2d all the $Z_m \times Z_2$ SPT phases can be deduced from $U(1) \times Z_2$ SPT phases, by breaking U(1) down to its Z_m subgroup. Thus cases (1), (2) and (3) in Eq. 2.40 seem to reduce to SPT phases with $Z_m \times Z_2$ symmetry after breaking U(1) down to Z_m. However, case (3) in Eq. 2.40 becomes the trivial phase when m is odd. In case (3) of $U(1) \times Z_2$ SPT phase (Eq. 2.40), the NLSM is constructed with a charge-1 boson $B \sim (n_1 + in_2)$, and because case (3) contributes classification $\mathbb{Z}_2$, $\Theta = 2\pi m$ is equivalent to $\Theta = 2\pi$ for odd m. Also, the NLSM with $\Theta = 2\pi m$ is equivalent to the NLSM with $\Theta = 2\pi$ constructed using a charge-m boson $B' \sim (n_1' + in_2') \sim (n_1 + in_2)^m$. Now let us break the U(1) symmetry down to Z_m. Because B' is invariant under Z_m and Z_2, we can turn on a linear Zeeman term that polarizes $\mathrm{Re}[B'] \sim n_1'$, and destroy the boundary states. Thus the NLSM constructed with the charge-m boson B' is trivial once we break U(1) down to Z_m. This implies that when m is odd, case (3) in Eq. 2.40 becomes a trivial phase once U(1) is broken down to Z_m.

The final classification of SPT phases with $Z_m \times Z_2$ symmetry is:

$$1d : \mathbb{Z}_{(2,m)}, \quad 2d : \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 3d : (\mathbb{Z}_{(2,m)})^2. \tag{2.63}$$

2.5.13 SPT phases with $Z_m \rtimes Z_2^T$ symmetry

Again, the situation depends on the parity of m. If m is odd, then in 1d and 3d the only SPT phase is the SPT phase with Z_2^T only. In 2d and 3d the $U(1) \rtimes Z_2^T$ SPT phases (except for the one with Z_2^T symmetry only) do not survive when U(1) is broken down to Z_m with odd m. The reason is similar to what we discussed in the previous two subsections.

When m is even, then in 1d besides the Haldane phase with Z_2^T symmetry, we can construct another SPT phase:

$$
\begin{aligned}
Z_m &: \ n_{1,2} \to (-1)^k n_{1,2}, \quad n_3 \to n_3, \\
& k = 0, \cdots m-1; \\
Z_2^T &: \ \vec{n} \to -\vec{n}.
\end{aligned} \tag{2.64}
$$

Here n_1 and n_2 are both imaginary parts of charge-$m/2$ bosons. The boundary state is a Kramers doublet and transforms as

$$
\begin{aligned}
Z_m &: \ U \to (i\sigma^z)^k U, \quad Z_2^T : U \to i\sigma^y U; \\
& k = 0, \cdots m-1.
\end{aligned} \tag{2.65}
$$

In 2d, we can construct two different root phases:

$$
\begin{aligned}
(1) \qquad & Z_m : (n_1 + in_2) \to (n_1 + in_2) e^{i2\pi k/m}, \\
& n_3, n_4 \to n_3, n_4; \\
& Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4); \\
(2) \qquad & Z_m : \vec{n} \to (-1)^k \vec{n};
\end{aligned}
$$

$$Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2, 3, 4);$$

$$k = 0, \cdots m - 1. \tag{2.66}$$

Phase (1) is the same phase as the 2d $U(1) \rtimes Z_2^T$ SPT phase, after breaking U(1) to Z_m; phase (2) is a new phase, where n_1 is the real part of a charge-$m/2$ boson, while $n_{2,3,4}$ are the imaginary parts of such charge-$m/2$ bosons.

Using similar methods, we can construct three root phases in 3d for even m. Two of the phases can be deduced from the 3d $U(1) \rtimes Z_2^T$ SPT phases. The third root phase has the following transformation:

$$Z_m \quad : \quad n_{1,2} \to (-1)^k n_{1,2}, \quad n_a \to n_a (a = 3, 4, 5);$$

$$Z_2^T \quad : \quad \vec{n} \to -\vec{n};$$

$$k = 0, \cdots m - 1. \tag{2.67}$$

Both n_1 and n_2 are imaginary parts of charge-$m/2$ bosons.

Just like the 3d SPT phase with $U(1) \rtimes Z_2^T$ symmetry, the 2d boundary of the 3d $Z_m \rtimes Z_2^T$ SPT phase described by Eq. 2.67 can have a Z_2 topological order with electric and magnetic anyons. The electric and magnetic anyons are both Kramers doublet, and only one of them has a nontrivial transformation under Z_m: $Z_m : U \to (i\sigma^z)^k U$, $(k = 0, \cdots m - 1)$.

The final classification of SPT phases with $Z_m \rtimes Z_2^T$ symmetry is:

$$1d : \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 2d : (\mathbb{Z}_{(2,m)})^2, \quad 3d : \mathbb{Z}_2 \times (\mathbb{Z}_{(2,m)})^2. \tag{2.68}$$

2.5.14 SPT phases with $Z_m \times Z_2^T$ symmetry

In 1d and 3d, the SPT phases with $Z_m \times Z_2^T$ symmetry can all be deduced from $U(1) \times Z_2^T$ symmetry by breaking U(1) down to Z_m. Again, when m is odd, some of the SPT phases become trivial, for the same reason as what we discussed before.

In 2d there is no SPT phase with $U(1) \times Z_2^T$ symmetry, but when m is even we can construct two root phases, which *cannot* be deduced from $U(1) \times Z_2^T$ SPT phases:

$$\begin{aligned} (1) \quad : \quad & Z_m : \vec{n} \to (-1)^k \vec{n}; \\ & Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2,3,4); \\ (2) \quad : \quad & Z_m : n_{1,2} \to (-1)^k n_{1,2}, \quad n_{3,4} \to n_{3,4}; \\ & Z_2^T : n_1 \to n_1, \quad n_a \to -n_a (a = 2,3,4); \\ & k = 0, \cdots m-1. \end{aligned} \tag{2.69}$$

The final classification of SPT phases with $Z_m \times Z_2^T$ symmetry is:

$$1d : \mathbb{Z}_2 \times \mathbb{Z}_{(2,m)}, \quad 2d : (\mathbb{Z}_{(2,m)})^2, \quad 3d : \mathbb{Z}_2 \times (\mathbb{Z}_{(2,m)})^2. \tag{2.70}$$

2.5.15 SPT phases with $SO(3)$ symmetry

In 1d, the SO(3) symmetry leads to the Haldane phase, which is described by Eq. 2.1 with $\Theta = 2\pi$. In 3d, there is no way to assign SO(3) symmetry to the five-component vector $\vec{n}$ which makes the $\Theta-$term invariant, thus there is no 3d SPT phase with SO(3) symmetry.

In 2d, Ref. [46] has given a nice way of describing SPT phase with SO(3) symmetry, which is a principal chiral model defined with group elements $SO(3)$. We will argue that the SO(3) principal chiral model in Ref. [46] can be formally rewritten as the O(4) NLSM Eq. 2.2, because we can represent every group element G_{ab} (3×3 orthogonal matrix) as a SU(2) matrix $\mathcal{Z}$:

$$G_{ab} = \frac{1}{2}\mathrm{tr}[\mathcal{Z}^\dagger \sigma^a \mathcal{Z} \sigma^b], \tag{2.71}$$

and the SU(2) matrix $\mathcal{Z}$ is equivalent to an O(4) vector $\vec{n}$ with unit length: $\mathcal{Z} = n^4 I_{2\times 2} + i\vec{n}\cdot\vec{\sigma}$. We propose that the minimal SO(3) SPT phase discussed in Ref. [46] can be effectively described by Eq. 2.2 with $\Theta = 8\pi$:

$$\begin{aligned} \mathcal{S}_{2d} &= \int d^2x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i8\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho} n^a \partial_\mu n^b \partial_\nu n^c \partial_\rho n^d \\ &= \int d^2x d\tau \ \frac{1}{g}\mathrm{tr}[\partial_\mu \mathcal{Z}^\dagger \partial_\mu \mathcal{Z}] + \frac{i8\pi}{24\pi^2}\mathrm{tr}[(\mathcal{Z}^\dagger d\mathcal{Z})^3]. \end{aligned} \tag{2.72}$$

Physically, Eq. 2.72 with $\Theta = 8\pi$ gives SU(2) Hall conductivity $\sigma_{SU(2)} = 8$, or equivalently SO(3) Hall conductivity $\sigma_{SO(3)} = 2$, which is the same as the principal chiral model in Ref. [46]. Mathematically, when the field $\mathcal{Z}$ has a instanton number $\int d^3x \ \mathrm{tr}[(\mathcal{Z}^\dagger d\mathcal{Z})^3]/(24\pi^2) = +1$ in the 2+1d space-time, the SO(3) matrix field G_{ab} defined in Eq. 2.71 will have instanton number $\int d^3x \ \mathrm{tr}[(G^{-1}dG)^3]/(24\pi^2) = +4$. This factor of 4 is precisely why $\Theta = 8\pi$ in Eq. 2.72.

In order to represent G_{ab} as $\mathcal{Z}$, we need to introduce a Z_2 gauge field that couples to $\mathcal{Z}$, because $\mathcal{Z}$ is a "fractional" representation of G_{ab}, and G_{ab} is invariant under gauge transformation $\mathcal{Z} \to -\mathcal{Z}$. In the language of lattice gauge theory, our statement in the previous paragraph implies that one of the possible confined phases of this Z_2 gauge field is trivial in the bulk without any extra symmetry breaking or topological degeneracy, namely the vison (a dynamical Z_2 $\pi-$flux coupled to $\mathcal{Z}$) in the bulk can condensed without breaking any symmetry. This is indeed possible, because if we weakly break the SU(2) symmetry down to U(1), Eq. 2.72 describes a bosonic integer quantum Hall state with Hall conductivity 8. A $\pi-$flux in this system carries charge 4, which can be fully screened by four bosons, while maintaining its bosonic statistics. Thus a vison can safely condense in the bulk, confine the field $\mathcal{Z}$, and drive the system into a SO(3) SPT phase.

The final classification of SPT phases with $SO(3)$ symmetry is:

$$1d : \mathbb{Z}_2, \quad 2d : \mathbb{Z}, \quad 3d : \mathbb{Z}_1. \tag{2.73}$$

2.5.16 SPT phases with $SO(3) \times Z_2^T$ symmetry

In 1d, there are two different SPT root phases with $SO(3) \times Z_2^T$ symmetry, which correspond to the following transformations of O(3) vector $\vec{n}$:

$$(1) \ : \ SO(3) : n_a \to G_{ab} n_b, \quad Z_2^T : \vec{n} \to -\vec{n};$$

$$(2) \ : \ SO(3) : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n}. \tag{2.74}$$

In 2d, the SPT phases with $SO(3) \times Z_2^T$ symmetry were discussed in Ref. [49], and it is described by Eq. 2.2 with transformation

$$SO(3) \quad : \quad n_a \to G_{ab} n_b (a, b = 1, 2, 3), \quad n_4 \to n_4;$$

$$Z_2^T \quad : \quad n_a \to n_a (a = 1, 2, 3), \quad n_4 \to -n_4. \tag{2.75}$$

In 3d, there are three root phases for $SO(3) \times Z_2^T$ SPT phases, two of which have the following field theory:

$$(1) \quad : \quad SO(3) : \vec{n} \to \vec{n}, \quad Z_2^T : \vec{n} \to -\vec{n};$$

$$(2) \quad : \quad SO(3) : n_a \to G_{ab} n_b (a, b = 1, 2, 3), \quad n_{4,5} \to n_{4,5}$$

$$Z_2^T : \vec{n} \to -\vec{n}; \tag{2.76}$$

phase (1) is simply the SPT phase with Z_2^T symmetry only. After we break the SO(3) symmetry down to its inplane O(2) subgroup, phase (2) will reduce to a SPT phase with $U(1) \times Z_2^T$ symmetry discussed in Ref. [48], which is a phase whose bulk vortex line is a 1d Haldane phase with Z_2^T symmetry.

Besides the two phases discussed above, there should be another root phase (3) that will reduce to the $U(1) \times Z_2^T$ SPT phase whose boundary is a bosonic quantum Hall state with Hall conductivity ± 1, when time-reversal symmetry is broken at the boundary [48]. In the next two paragraphs we will argue *without* proof that this third root phase can be described by Eq. 2.3 with the following definition and transformation of O(5) vector order parameter $\vec{n}$:

$$(3) \quad : \quad \mathcal{Z} = n^4 I_{2\times 2} + \sum_{a=1}^{3} i n_a \sigma^a,$$

$$Z_2^T : \mathcal{Z} \to i\sigma^y \mathcal{Z}, \quad n_5 \to -n_5;$$

$$\Theta = 8\pi \text{ in bulk.} \tag{2.77}$$

Here $\mathcal{Z}$ is still the "fractional" representation of SO(3) matrix G_{ab} introduced in Eq. 2.71. If we break the Z_2^T symmetry at the boundary of phase (3), the boundary becomes a 2d SO(3) SPT phase with SO(3) Hall conductivity ± 1 (when SO(3) is broken to U(1), the boundary becomes a bosonic integer quantum Hall state with Hall conductivity ± 1), thus it cannot be realized in a pure 2d bosonic system without degeneracy.

In principle $\mathcal{Z}$ is still coupled to a Z_2 gauge field. We propose that the confined phase of this Z_2 gauge field is the desired $SO(3) \times Z_2^T$ SPT phase. In the confined phase of a 3d Z_2 gauge field, the vison loops of the Z_2 gauge field proliferate. Since the Z_2 gauge field is coupled to the fractional field $\mathcal{Z}$, a vison loop of this Z_2 gauge field is bound with a vortex loop of SO(3) matrix field G_{ab} [81], which is defined based on homotopy group $\pi_1[SO(3)] = \mathbb{Z}_2$, thus the confined phase of the Z_2 gauge field is a phase where the SO(3) vortex loops proliferate. If we reduce the SO(3) symmetry down to its inplane U(1) symmetry, the vison loop reduces to the vortex loop of the U(1) phase. When a bulk vortex (vison) loop ends at the boundary, it becomes a 2d vortex (vison). This 2d vortex is a fermion, because according to the previous paragraph, once the Z_2^T is broken at the boundary, the boundary becomes a boson quantum Hall state with Hall conductivity ± 1. This is consistent with the results for $U(1) \times Z_2^T$ SPT phase discussed in Ref. [48, 50, 40]. Thus the SPT phase described by Eq. 2.77 is a phase where SO(3) vortex loops proliferate, and the SO(3) vortices at the boundary are fermions.

The final classification of SPT phases with $SO(3) \times Z_2^T$ symmetry is:

$$1d : (\mathbb{Z}_2)^2, \quad 2d : \mathbb{Z}_2, \quad 3d : (\mathbb{Z}_2)^3. \tag{2.78}$$

2.5.17 SPT phases with $Z_2 \times Z_2 \times Z_2$ symmetry

In 1d, we can construct three different root phases:

$$\begin{aligned}
(1) \quad : \quad & Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& Z_2^B : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\
& Z_2^C : \vec{n} \to \vec{n}; \\
(2) \quad : \quad & Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& Z_2^C : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\
& Z_2^A : \vec{n} \to \vec{n}; \\
(3) \quad : \quad & Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_3 \to n_3; \\
& Z_2^A : n_1 \to n_1, \quad n_{2,3} \to -n_{2,3}; \\
& Z_2^B : \vec{n} \to \vec{n}.
\end{aligned} \tag{2.79}$$

In 2d there are seven different root phases:

$$(1) \quad : \quad Z_2^A : \vec{n} \to -\vec{n}, \quad Z_2^B, Z_2^C : \vec{n} \to \vec{n};$$

$$(2) \quad : \quad Z_2^B : \vec{n} \to -\vec{n}, \quad Z_2^C, Z_2^A : \vec{n} \to \vec{n};$$

$$(3) \quad : \quad Z_2^C : \vec{n} \to -\vec{n}, \quad Z_2^A, Z_2^B : \vec{n} \to \vec{n};$$

$$(4) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4};$$

$$Z_2^B : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4};$$

$$Z_2^C : \vec{n} \to \vec{n};$$

$$(5) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4};$$

$$Z_2^C : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4};$$

$$Z_2^B : \vec{n} \to \vec{n};$$

$$(6) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4};$$

$$Z_2^B : n_{1,3} \to -n_{1,3}, \quad n_{2,4} \to n_{2,4};$$

$$Z_2^C : n_{1,4} \to -n_{1,4}, \quad n_{2,3} \to n_{2,3};$$

$$(7) \quad : \quad Z_2^A : n_{2,3} \to -n_{2,3}, \quad n_{1,4} \to n_{1,4}$$

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_{3,4} \to n_{3,4},$$

$$Z_2^C : n_{1,2} \to n_{1,2}, \quad n_{3,4} \to -n_{3,4}. \tag{2.80}$$

In 3d there are six different root phases:

$$(1) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3, 4, 5);$$

$$Z_2^B : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^C : \vec{n} \to \vec{n};$$

$$(2) \quad : \quad Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3, 4, 5);$$

$$Z_2^A : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^C : \vec{n} \to \vec{n};$$

$$(3) \quad : \quad Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3, 4, 5);$$

$$Z_2^C : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^A : \vec{n} \to \vec{n};$$

$$(4) \quad : \quad Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$

$$Z_2^B : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^A : \vec{n} \to \vec{n};$$

$$(5) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$

$$Z_2^C : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^B : \vec{n} \to \vec{n};$$

$$(6) \quad : \quad Z_2^C : n_{1,2} \to -n_{1,2}, \quad n_a \to n_a, (a = 3,4,5);$$

$$Z_2^A : n_1 \to n_1, \quad n_a \to -n_a, (a = 2, \cdots 5);$$

$$Z_2^B : \vec{n} \to \vec{n};$$

$$(7) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4,5} \to n_{3,4,5};$$

$$Z_2^B : n_{2,3} \to -n_{2,3}, \quad n_{1,4,5} \to n_{1,4,5};$$

$$Z_2^C : n_{4,5} \to -n_{4,5}, \quad n_{1,2,3} \to n_{1,2,3};$$

$$(8) \quad : \quad Z_2^A : n_{1,2} \to -n_{1,2}, \quad n_{3,4,5} \to n_{3,4,5};$$

$$Z_2^C : n_{2,3} \to -n_{2,3}, \quad n_{1,4,5} \to n_{1,4,5};$$

$$Z_2^B : n_{4,5} \to -n_{4,5}, \quad n_{1,2,3} \to n_{1,2,3}. \tag{2.81}$$

All the other SPT phases can be constructed with these root phases above. Here we will show one construction explicitly. For example, one may think the following state should also exist in $3d$:

$$Z_2^B : n_{1,2} \to -n_{1,2}, \quad n_{3,4,5} \to n_{3,4,5},$$

$$Z_2^C : n_{2,3} \to -n_{2,3}, \quad n_{1,4,5} \to n_{1,4,5},$$

$$Z_2^A : n_{4,5} \to -n_{4,5}, \quad n_{1,2,3} \to n_{1,2,3}. \tag{2.82}$$

But this state can be obtained by "merging" state (7) and (8) in Eq. 2.81. First of all, since $n^{(7)}_{1,3,5}$ transform exactly equivalently to $n^{(8)}_{1,5,3}$ under all symmetries, we can turn on coupling between $\vec{n}^{(7)}$ and $\vec{n}^{(8)}$ to make $n^{(7)}_{1,3,5} = n^{(8)}_{1,5,3}$. Now without loss of generality these two vectors can be written as

$$\vec{n}^{(7)} = (\cos\theta N_1, \ \sin\theta\cos\alpha^{(7)}, \ \cos\theta N_2,$$

$$\sin\theta\sin\alpha^{(7)}, \ \cos\theta N_3);$$

$$\vec{n}^{(8)} = (\cos\theta N_1, \ \sin\theta\cos\alpha^{(8)}, \ \cos\theta N_3,$$

$$\sin\theta\sin\alpha^{(8)},\ \cos\theta N_2); \qquad (2.83)$$

where $\vec{N}$ is a unit three-component vector. All the symmetries transformations act on $\vec{N}$ and $\alpha^{(7)}$, $\alpha^{(8)}$, while θ is invariant under all symmetries.

Now let us define a new vector $\vec{n}^{(9)}$ using the parametrization of $\vec{n}^{(7)}$ and $\vec{n}^{(8)}$:

$$\vec{n}^{(9)} = (\cos\theta N_2,\ \sin\theta\cos(\alpha^{(7)}+\alpha^{(8)}),\ \cos\theta N_3,$$

$$\sin\theta\sin(\alpha^{(7)}+\alpha^{(8)}),\ \cos\theta N_1); \qquad (2.84)$$

Obviously, the O(5) instanton number of $\vec{n}^{(9)}$ is exactly the sum of instantons of $\vec{n}^{(7)}$ and $\vec{n}^{(8)}$. More importantly, $\vec{n}^{(9)}$ transforms under all the symmetries as Eq. 2.82, and since it can be "merged" from phase (7) and (8), it should not be viewed as an independent root phase.

The final classification of SPT phases with $Z_2 \times Z_2 \times Z_2$ symmetry is:

$$1d : (\mathbb{Z}_2)^3, \quad 2d : (\mathbb{Z}_2)^7, \quad 3d : (\mathbb{Z}_2)^8. \qquad (2.85)$$

2.6 Summary and Comments

In this work we systematically classified and described bosonic SPT phases with a large set of physically relevant symmetries for all physical dimensions. We have demonstrated that all the SPT phases discussed in this paper can be described by three universal NLSMs Eq. 2.1, 2.2 and 2.3, and the classification of these SPT phases based on NLSMs is completely identical to the group cohomology classification [31, 42]. However, we have not built the general connection between these two classifications, and it is likely that SPT phases with some other symmetry groups (for example symmetry much larger than $O(d+2)$) can no longer be described by these three NLSMs any more. In Ref. [50, 49], SPT phases that involve a large symmetry group PSU(N)= $SU(N)/Z_N$ were discussed, and in these systems it was necessary to introduce NLSMs with a larger target manifold. But it is likely that all the SPT phases with arbitrary symmetry groups (continuous or discontinuous) can be described by a NLSM with certain continuous target manifold.

As we already mentioned, now it is clear that there is a series of BSPT states beyond the group cohomology classification, and a generalized field theory description for such states will be given in Ref. [68]. Our NLSM can also be very conveniently generalized to the cases that involve lattice symmetry such as inversion, as was discussed in Ref. [82], as long as we require our order parameter $\vec{n}$ transform nontrivially under lattice symmetry. We leave a thorough study of SPT states involving lattice symmetry to future studies.

Recently it was pointed out that after the $3d$ SPT state is coupled to gauge field, the gauge defects, which in $3d$ can be loop excitations, can have a novel loop braiding statistics [83]. In a recent work we showed that this loop statistics can also be computed using our NLSM field theory discussed in this work [84].

Chapter 3

The Strange Correlator

3.1 Detecting SPT Phases

SPT phases are often described in terms of their anomalous boundary, which is to say that the boundary field theory cannot be realized as a lattice model in the same number of dimensions with a local, on-site action of the symmetry group. There are many such boundaries for a given SPT phase, and they may be gapless, spontaneously break symmetries, or be topologically ordered. The boundary field theory has an anomaly, where the symmetry of the classical Lagrangian is not a symmetry of the quantum path integral. This anomaly is necessary to cancel the bulk anomaly inflow, rendering the full system (bulk and boundary) well-defined in the presence of a regulator.

In addition to this, the bulk of the SPT is "trivial," which is to say short-range entangled, gapped, and unique on a torus. Thus, since the interesting physics is entirely at the surface, many discussions of SPT phases are entirely focused on characterizing surface behavior. For example, comparison of boundary topological order[51] can distinguish different SPT phases with the same symmetries.

However, the boundary physics is entirely derived from the bulk topological terms (see Ch 2), so one may naturally wonder if there is a computation that can distinguish trivial bulk from SPT bulk without reference to the boundary. By considering certain correlation functions calculated in the bulk of both SPT and trivial phases separated in imaginary time, we can indeed distinguish the two phases.

3.2 Bulk detection via Strange Correlator

A short range entangled (SRE) state is a ground state of a quantum many-body system that does not have ground state degeneracy or bulk topological entanglement entropy. But a SRE state (for example the integer quantum Hall state) can still have protected stable gapless edge states. Thus it appears that the bulk of all the SRE states are identically trivial, and a nontrivial SRE state is usually characterized by its edge states. In this paper we propose a diagnosis to determine whether a given many-body wave function defined on a lattice without boundary is a nontrivial SRE state or a trivial one. This diagnosis is based on the following quantity called "strange correlator" [1]:

$$C(r, r') = \frac{\langle \Omega | \phi(r) \phi(r') | \Psi \rangle}{\langle \Omega | \Psi \rangle}. \tag{3.1}$$

Here $|\Psi\rangle$ is the wave function that needs diagnosis, $|\Omega\rangle$ is a direct product trivial disordered state defined on the same Hilbert space. Our conclusion is that if $|\Psi\rangle$ is a nontrivial SRE state in one or two spatial dimensions, then for some local operator $\phi(r)$, $C(r, r')$ will either saturate to a *constant* or decay as a *power-law* in the limit $|r - r'| \to +\infty$, even though both $|\Omega\rangle$ and $|\Psi\rangle$ are disordered states with short-range correlation.

[1] In the thermodynamic limit, both numerator and denominator of the strange correlator approach zero, while their ratio remains a finite constant. All the calculations in this paper were based on finite system size first, the thermodynamic limit is taken *after* taking the ratio.

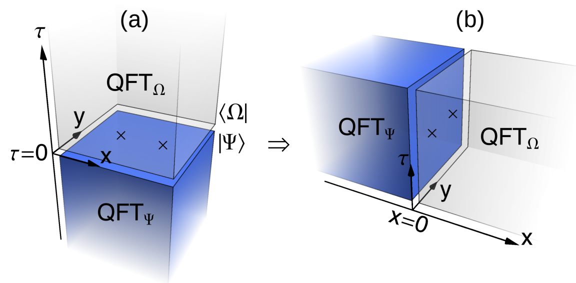

Figure 3.1: (color online). (a) $|\Psi\rangle$ and $\langle\Omega|$ are given by infinite time evolution of their quantum field theories (QFT) from below and above respectively. The strange correlator can be viewed as the correlator at the $\tau = 0$ interface. (b) Under the Lorentz rotation, the two QFT's are separated by the $x = 0$ interface, and the strange correlator can be interpreted as the correlation function on this spatial interface.

Another possible way of diagnosing a SRE wave function is through its entanglement spectrum [85]. If a SRE state has nontrivial edge states, an analogue of its edge states should also exist in its entanglement spectrum [86]. However, many SRE states are protected by certain symmetry, some SRE states are protected by lattice symmetries (for example the spin-2 AKLT state on the square lattice requires translation symmetry). If the cut we make to compute the entanglement spectrum breaks such lattice symmetry, then the entanglement spectrum would be trivial, even if the original state is a nontrivial SRE state. By contrast, the strange correlator in Eq. (3.1) is defined on a lattice without edge, thus it already preserves all the symmetries of the system, including all the lattice symmetries. Thus the strange correlator can reliably diagnose SRE states protected by lattice symmetries as well.

The strange correlator can be roughly understood as follows: $|\Psi\rangle$ can be viewed as a generic initial state evolved with a constant nontrivial SRE Hamiltonian from $\tau = -\infty$ to $\tau = 0$; $\langle\Omega|$ is a state evolved from $\tau = +\infty$ to $\tau = 0$ with a trivial Hamiltonian, thus the strange correlator can be viewed as a “correlation function" at a temporal domain wall of the QFT's at $\tau = 0$, see Fig. 3.1(a). If there is an approximate Lorentz invariant description of the system, a space-time rotation can transform Eq. (3.1) to a space-time correlation at a spatial interface between nontrivial and trivial SRE systems, see Fig. 3.1(b). And for one and two spatial dimensions, a spatial interface between trivial and nontrivial SRE states should have either long range or power-law correlation between certain local operators, which after Lorentz rotation will lead to the conclusion of this paper. A similar observation of Lorentz rotation was used to derive the bulk wave function of topological superconductors [87].

For bosonic SRE states that are protected by certain symmetry (so called symmetry protected topological (SPT) states [31, 42]), the argument above can be demonstrated more explicitly. In Ref. [88], it was demonstrated that a large class of $1d$ and $2d$ bosonic SPT states can be described by the following two nonlinear Sigma model (NLSM) field theories:

$$\mathcal{S}_{1d} = \int dx d\tau \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c, \tag{3.2}$$

$$\mathcal{S}_{2d} = \int d^2x d\tau \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i2\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho} n^a \partial_\mu n^b \partial_\nu n^c \partial_\rho n^d. \tag{3.3}$$

Here $\vec{n}(x)$ is an O(3) or O(4) vector order parameter with unit length constraint: $(\vec{n})^2 = 1$. Different SPT phases are distinguished from each other based on different implementations of the symmetry group on the vector order parameter $\vec{n}$. In both $1d$ and $2d$, ground state wave functions of SPT phases can be derived based on Eq. (3.2) and Eq. (3.3) (see Ref. [50]):

$$|\Psi\rangle_d \sim \int \ D\vec{n}(x) \ \exp^{-\int_{S^d} d^dx \ \frac{1}{G}(\nabla\vec{n})^2 + \mathrm{WZW}_d[\vec{n}]} \, |\vec{n}(x)\rangle, \tag{3.4}$$

where S^d is the compactified real space manifold, and $\mathrm{WZW}_d[\vec{n}]$ is a real space Wess-

Zumino-Witten term:

$$\begin{aligned}\mathrm{WZW}_1[\vec{n}] &= \int_0^1 du \ \frac{i2\pi}{8\pi}\epsilon_{\mu\nu}\epsilon_{ab}n^a\partial_\mu n^b\partial_\nu n^c, \ \ \mu,\nu = x,u \\ \mathrm{WZW}_2[\vec{n}] &= \int_0^1 du \ \frac{i2\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}n^a\partial_\mu n^b\partial_\nu n^c\partial_\rho n^d, \\ \mu,\nu,\rho &= x,y,u. \end{aligned} \tag{3.5}$$

In contrast, the trivial state wave function is a superposition of all configurations of $|\vec{n}(x)\rangle$ without a WZW term. Based on the wave functions in Eq. (3.4), the strange correlator of order parameter $\vec{n}(x)$ reads

$$C(r,r') = \frac{\int D\vec{n}(x) \ n^a(r)n^b(r')e^{-\int_{S^d} d^dx \ \frac{1}{G}(\nabla\vec{n})^2+\mathrm{WZW}_d[\vec{n}]}}{\int D\vec{n}(x) \ e^{-\int_{S^d} d^dx \ \frac{1}{G}(\nabla\vec{n})^2+\mathrm{WZW}_d[\vec{n}]}}. \tag{3.6}$$

Mathematically, this strange correlator can be viewed as an ordinary space-time correlation function of a $(d-1)+1$ dimensional field theory with a WZW term, as long as we view one of the spatial coordinate as the time direction. When $d=1$, this strange correlator is effectively a spin-spin correlation of one isolated free spin-1/2, and the correlation is always long range. When $d=2$, this strange correlator is effectively a space-time correlation function of a $(1+1)d$ O(4) NLSM with a WZW term, and when this model has a full SO(4) symmetry, this theory is a $\mathrm{SU}(2)_1$ conformal field theory with power-law correlation [89, 69]; when the symmetry of the system is a subgroup of SO(4), as long as the residual symmetry prohibits any linear Zeeman coupling to order parameter $\vec{n}$, this $(1+1)d$ system either remains gapless, or spontaneously breaks the symmetry and develop long range order. Thus the strange correlator is either long range or decays with a power-law.

The two arguments above both rely on a certain continuum limit description of the SRE state. However, for a fully gapped system, when the gap is comparable with the ultraviolet energy scale of the system, a continuum limit description may not be appropriate. In the rest of the paper, we will compute the strange correlator for several examples of SRE states *far from* the continuum limit, *i.e.* the gap of the SRE states is comparable with UV cut-off. We will see that in some examples, the strange correlator is indeed different from the physical edge of the SRE state, but our qualitative conclusion is still valid.

The first example we study is the AKLT state [59, 90] of the Haldane phase of spin-1 chain. In the S^z basis, the AKLT wave function is a "dilute" Néel state, namely it is an equal weight superposition of all the S^z configurations with an alternate distribution of $|+\rangle = |S^z = +1\rangle$ and $|-\rangle = |S^z = -1\rangle$, sandwiched with arbitrary numbers of $|0\rangle = |S^z = 0\rangle$ [91]:

$$|\Psi\rangle = \sum \frac{1}{N} |+0\cdots 0 - 0 \cdots 0 + \cdots\rangle \tag{3.7}$$

We choose the trivial state to be $|\Omega\rangle = |000\cdots\rangle$. Straightforward calculation leads to the following answer of the strange correlator:

$$C(r, r') = \frac{\langle\Omega|S^+_r\ S^-_{r'}|\Psi\rangle}{\langle\Omega|\Psi\rangle} = 2, \tag{3.8}$$

which is the expected long range correlation.

The second example we study is the two dimensional quantum spin Hall (QSH) insulator with a Rashba spin orbit coupling. We will use the same notation as Ref. [3].

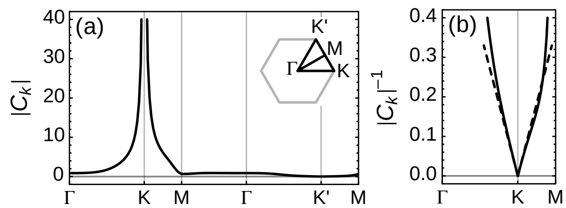

Figure 3.2: (a) The amplitude of strange correlator in the momentum space. The inset shows the Brillouin zone and the high symmetry points. (b) $|C_k|^{-1}$ exhibits nice linearity around the K point, establishing the $1/|k|$ divergence in $|C_k|$.

The QSH insulator Hamiltonian reads

$$H = t\sum_{\langle i,j\rangle} c_i^\dagger c_j + i\lambda_{SO}\sum_{\langle\langle i,j\rangle\rangle}\nu_{ij}c_i^\dagger s^z c_j$$

$$+\lambda_R\sum_{\langle i,j\rangle} c_i^\dagger(\mathbf{s}\times\hat{\mathbf{d}}_{ij})_z c_j + \lambda_v\sum_i \xi_i c_i^\dagger c_i. \tag{3.9}$$

λ_{SO} is the spin-orbit coupling that leads to the QSH topological band structure, λ_R is the Rashba coupling that breaks the conservation of s^z, and λ_v is a staggered potential that leads to charge density wave. The electron operator c_i carries spin and sublattice indices, thus the strange correlator $C(r, r')$ is a 4×4 matrix. For QSH state $|\Psi\rangle$, we choose $\lambda_{SO} = t$, $\lambda_R = 0.5t$, $\lambda_v = 0$; trivial state $|\Omega\rangle$ is chosen to be a strong CDW state with $\lambda_{SO} = t$, $\lambda_R = 0.5t$, $\lambda_v = 10t$. These two states are far from the continuum limit, namely the gap is comparable with the UV cut-off.

Fig. 3.2(a) shows the amplitude of strange correlator $|C_k| = |\langle\Omega|c^\dagger_{A,\uparrow,k}c_{B,\uparrow,k}|\Psi\rangle/\langle\Omega|\Psi\rangle|$ plotted in the momentum space. There is one clear singularity at the corner of the Brillouin zone, which diverges as $\sim 1/|k|$. This implies that in the real space the strange correlator decays as $|C(r, r')| \sim 1/|\vec{r} - \vec{r}'|$, which is consistent with the result obtained from Lorentz transformation, despite the large bulk gap.

The third example we will study is the spin-2 AKLT state on the square lattice,[90, 92] which is a SPT state protected by the on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the lattice translation symmetry,[93] whose wave function has a tensor product state (TPS) representation[94, 95]

$$|\Psi\rangle = \sum_{\{m_i\}} \mathrm{tTr}(\otimes_i T^{m_i})|\{m_i\}\rangle. \tag{3.10}$$

Here $m_i = 0, \pm 1, \pm 2$ labels the S^z quantum number of the spin-2 object on site i, and $|\{m_i\}\rangle$ is the state for the configuration $\{m_i\}$ over the lattice. tTr traces out the internal legs in the tensor network shown in Fig. 3.3(a), in which the vertex tensor is given by

$$T^m_{s_1 s_2 s_3 s_4} = \begin{cases} 4 s_1 s_2 & : -s_1 - s_2 + s_3 + s_4 = m, \\ 0 & : \text{otherwise}, \end{cases} \tag{3.11}$$

with $s_j = \pm 1/2$ labeling the spin-1/2 internal degrees of freedom. While the trivial state $|\Omega\rangle = |\{\forall i : m_i = 0\}\rangle$ is chosen to be the direct product state of $S^z = 0$ on every site. We look into the strange correlator

$$C(r, r') = \frac{\langle\Omega|S^+_r S^-_{r'}|\Psi\rangle}{\langle\Omega|\Psi\rangle} = \frac{\mathrm{tTr}(T^0 \cdots T^1(r) T^{-1}(r') \cdots)}{\mathrm{tTr}(T^0 \cdots)}, \tag{3.12}$$

which can be expressed as a ratio between two tensor networks: the denominator is a uniform network of the tensor T^0 on each site, and the numerator is the same network except for impurity tensors $T^{\pm 1}$ on site r and r' respectively.

The evaluation of the tensor trace in Eq. (3.12) over the $2d$ lattice can be reformulated as an (1+1) dimensional quantum mechanics problem in terms of the transfer matrix for each row, which can then be studied by the density matrix renormalization group (DMRG) method.[96, 97] The calculation is performed on an $128 \times \infty$ lattice with periodic boundary condition along both directions. We found that the strange correlator decays with oscillation (as in Fig. 3.3(b)), and its amplitude follows a power-law behavior

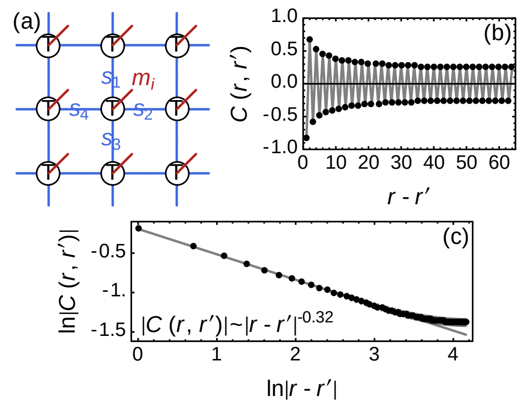

Figure 3.3: (color online). (a) Tensor network representation of the $2d$ AKLT state. The red (blue) legs represent the physical (internal) degrees of freedom. (b) Strange correlator of the $2d$ AKLT state measured along the horizontal direction. (c) The amplitude follows a power-law behavior in the log-log plot. The final deviation is due to the finite-size effect.

$|C(r, r')| \sim |r - r'|^{-\eta}$ with the exponent $\eta \simeq 0.32$, see Fig. 3.3(c), which is consistent with our previous field theory argument.

The last example we will study is the two dimensional bosonic SPT phase with Z_2 symmetry which was first studied in Ref. [43]. The ground state wave function of this SPT phase is

$$|\Psi\rangle = \sum_{\{\sigma_i\}} (-1)^{N_d} \exp\Big(-\frac{\beta}{2} \sum_{\langle i,j\rangle} \sigma_i \sigma_j \Big) |\{\sigma_i\}\rangle, \tag{3.13}$$

which is a superposition of all the configurations of the Ising degree of freedom $|\{\sigma_i\}\rangle$ with a factor (-1) associated with each closed Ising domain wall (with N_d being the number of domain wall loops). The trivial state $|\Omega\rangle$ is simply an Ising paramagnet, whose wave function is similar to Eq. (3.13) but without the domain wall sign structure $(-1)^{N_d}$. Compared with Ref. [43], we have added a factor $\exp(-\beta/2 \sum_{\langle i,j\rangle} \sigma_i \sigma_j)$ to each Ising configuration to adjust the spin correlation length.

The strange correlator of the Z_2 bosonic SPT phase can be viewed as a correlation

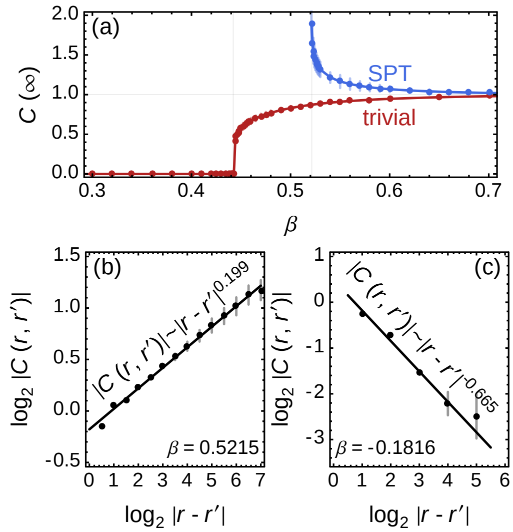

Figure 3.4: (color online). (a) The strange correlator of the SPT state (in blue) at infinite distance $|r - r'| \to \infty$, in comparison with that of the trivial state (in red). The SPT strange correlator follows the power-law behavior (b) at the critical point and (c) in the dense loop phase.

function of a “classical statistical mechanics model":

$$C(r, r') = \frac{\sum_{\{\sigma_i\}} \sigma_r \sigma_{r'} (-1)^{N_d} e^{-\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j}}{\sum_{\{\sigma_i\}} (-1)^{N_d} e^{-\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j}}. \tag{3.14}$$

Our goal is to show that this is either a long range or power-law correlation for arbitrary β. In other words, Eq. 3.14 is less likely to disorder than the ordinary 2d Ising model. This result can be naively understood as follows: the ordinary 2d Ising model is disordered at high temperature (small β) due to the proliferation of Ising domain walls. But in the current model, due to the (-1) sign associated with each domain wall, the proliferation of domain walls is suppressed, thus eventually the current Ising model Eq. (3.14) is not completely disordered even for small β.

This Ising model is dual to a loop model with the following partition function:

$$Z = \sum_{\mathcal{C}} K^L n^{N_d}, \tag{3.15}$$

where loops are the domain walls of the original Ising model, $K = \exp(-2\beta)$ is the loop tension, $n = -1$ is the loop fugacity, L is the total length of loops, and N_d is the total number of closed loops. If the loops do not cross, then according to Ref. [98], by tuning K there is a phase transition between a small loop phase (which corresponds to the Ising ordered phase) for small K, and a dense loop phase for large K. The critical point and the dense loop phase are both critical with power-law correlations, and they correspond to two different conformal field theories with central charges $c = -3/5$ and $c = -7$ respectively. If the loops are allowed to cross, the dense loop phase is driven to a different conformal field theory with $c = -2$, which is described by free symplectic fermions.[99]

The Ising order parameter σ_i corresponds to the “twist" operator of the loop model, because σ_i changes its sign when it crosses a loop. The twist operator is well-studied at the critical point of loop models, and in our case with $n = -1$, at the critical point between small and dense loop phases the scaling dimension of the twist operator is $-1/10$ [100], which is confirmed by our numerical calculation.

The tensor renormalization group (TRG) method[101, 102] has been applied to loop models in Ref. [103]. Here we use the same approach to study the twist operator correlations for the loop model in Eq. (3.15). For simplicity we forbid the loops to cross, so the model never develops antiferromagnetic order even for negative β. For positive large β, the strange correlator is long-ranged, see Fig. 3.4(a). As β decreases, the correlator grows and diverges at the critical point $\beta_c \simeq 0.521$ with a power-law $C(r, r') \sim |r - r'|^{0.199}$ as shown in Fig. 3.4(b), which confirms the theoretical prediction of scaling dimension $-1/10$

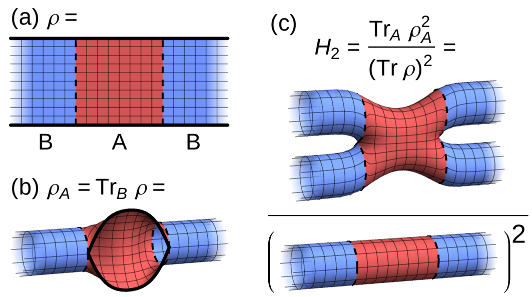

Figure 3.5: (color online). (a) Under Lorentz transformation, the density matrix of the SRE edge states is mapped to the overlap between bulk ground state wave functions on a manifold with open boundaries in one direction. The edge manifold may be partitioned into the regions A (red) and B (blue). (b) The reduced density matrix in the region A of the edge states corresponds to joining the boundaries of B together. (c) $\mathrm{Tr}\rho_A^2$ is given by doubling ρ_A and sealing the boundaries of regions A with each other, resulting in the pants (double torus) topology. $\mathrm{Tr}\rho$ is simply obtained by rolling up (a). Their ratio gives the Rényi entropy H_2.

of twist operator [100]. Theoretically the entire dense loop phase (when $\beta < \beta_c$) should be controlled by one stable conformal field theory fixed point. Our numerical results suggest that this fixed point is likely around $\beta \sim -0.1816$, the power-law behavior of $C(r, r')$ at this point (Fig. 3.4) is qualitatively consistent with the conclusion of this paper. [2]

We have checked that the ordinary free electron $3d$ topological insulator also gives us a very clear power-law decay of strange correlator. However, in general a strongly interacting SRE state in three dimensional space can be more complicated, because its two dimensional edge can be (1) a gapless $(2+1)d$ conformal field theory, (2) long range order that spontaneously breaks symmetry, (3) two dimensional topological phase [48]. Based on our Lorentz transformation argument, it is possible that $\langle\Omega|\Psi\rangle$ is mapped to the

[2]For β far away from this fixed point, the finite system size and error bar, as well as the incommensurate oscillation of the strange correlator make it more difficult to extract a conclusive scaling dimension of σ_i. But we expect $C(r, r')$ to crossover back to the same scaling behavior as the stable fixed point $\beta \sim -0.1816$ in the infrared limit for arbitrary $\beta < \beta_c$. Our result may have also been strongly affected by our choice of microscopic rules for loops close to each other. More recent studies by Scaffidi and Ringel [104] on the Levin-Gu model on a triangular lattice have successfully extracted a scaling dimension consistent with the Coulomb gas prediction of the dense loop phase [100].

partition function of a topological phase, then in this case the strange correlator $C(r, r')$ may also be short ranged. Thus for 3d SRE states, besides the strange correlator, we also need another method that diagnoses the situation when $\langle\Omega|\Psi\rangle$ corresponds to a topological phase partition function.

The method we propose is illustrated in Fig. 3.5, where the horizontal direction represents the XY plane, while the vertical direction is the z axis of the three dimensional space. We can first calculate the overlap between the given 3d wave function $|\Psi\rangle$ and the trivial wave function on a 3d "pants"-like manifold in Fig. 3.5c ($\langle\Omega|\Psi\rangle_{\mathrm{pants}}$), which after Lorentz transformation becomes $\mathrm{Tr}\rho_A^2$ at the edge, where ρ_A is the reduced density matrix of subsystem A at the boundary. The following quantity after Lorentz transformation becomes the Rényi entanglement entropy of the edge topological phase:

$$S = -\log\left(\frac{\langle\Omega|\Psi\rangle_{\mathrm{pants}}}{(\langle\Omega|\Psi\rangle_{\mathrm{cylinder}})^2}\right). \tag{3.16}$$

This quantity should scale as $S = \alpha L - \gamma$, where γ is the analogue of the topological entanglement entropy of the edge topological phase [105, 106]. Thus a $3d$ wave function $|\Psi\rangle$ is still a nontrivial SRE state as long as γ defined above is nonzero, even if this wave function has a short range strange correlator. We will leave the detailed study of this proposal to future work.

In summary, we have proposed a general method to diagnose $1d$ and $2d$ SRE states based on their bulk ground state wave functions. We expect our method to be useful for future numerical studies of SRE states. In Ref. [107, 108, 109, 110], it was proposed that interacting fermionic topological insulators and topological superconductors can be characterized by the full fermion Green's function; Ref. [111] proposed a method to diagnose bosonic SPT states characterized by group cohomology. The method proposed in our current paper is applicable to both fermionic and bosonic SRE states.

Chapter 4

Exotic Critical Points on SPT Boundaries

4.1 Anomalies and DQCP

The theory of second order phase transitions pioneered by Landau, Ginzburg, Wilson, and Fisher has had remarkable success describing order-to-disorder transitions in both quantum and thermal systems. By using the order parameter field, the symmetries, and the renormalization group, we can calculate many experimentally-verifiable quantities for a wide variety of systems.

However, the theory is not suitable for building a generic second order phase transition between two ordered phases. Doing so within the realm of LGWF theory requires either fine tuning the coupling constants so both order parameters vanish at the same point, or using an intermediate phase with two critical points. On the square lattice Heisenberg antiferromagnet, numerics imply[13] that a generic second order critical point should exist between the Néel and VBS phases.

The original construction[13] was called "deconfined criticality" due to the fact that the critical point between Néel and VBS phases included a new, dynamical $U(1)$ gauge field. The two ordered phases are reached by condensing either monopoles or pairs of spinons.

A much more general approach was found later[14] that combined the two order parameters into a larger "superspin." Then, the action of this new field includes the ordinary kinetic term plus a topological WZW term. All of the beyond-LGWF physics is built into the WZW term, which drastically modifies the critical point properties. Importantly, it also modifies the symmetry properties of the defects in the ordered phases.

On the triangular lattice, the picture is analogous. There is both magnetic order and a VBS phase, but the VBS order parameter often encountered in experiments and numerics is more complicated than for the square lattice. To find the correct the critical point action, we use intuition from SPT field theories to build a NLSM into a large Grasmannian manifold. This new manifold can support a WZW term, and this term correctly modifies the defect symmetry properties. Finally, we find a UV fermion action for this manifold, and analyze the relevant terms to see that it describes a direct, second-order transition.

4.2 DQCP on Triangular Lattice

The deconfined quantum critical point (dQCP) [13, 112] was proposed as the first explicit example of a direct unfine-tuned quantum critical point [1] beyond the standard Landau's paradigm, because the dQCP is sandwiched between two very different ordered phases with completely unrelated broken symmetries [13]. More precisely, the symmetry that is spontaneously broken on one side of the transition is completely independent from the symmetry that is broken on the other side. This scenario was forbidden in

[1]Here unfine-tuned means that there is only one relevant operator allowed by the symmetry at the dQCP, which is the tuning parameter.

the standard Landau's paradigm, but was proposed to be possible in quantum spin systems [13, 112]. A lot of numerical work has been devoted to investigating the dQCP with a full spin rotation symmetry [113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124], as well as spin models with only in-plane spin symmetry [125, 126, 127, 128]. Recently developed duality between strongly interacting QCPs in $(2+1)d$ have further improved our understanding of the dQCP [129, 130, 131, 132, 133, 134], and the predictions made by duality has received numerical supports [135, 136].

Let us first summarize the key ingredients of the original dQCP on the square lattice [13, 112]:

(1) This is a quantum phase transition sandwiched between the standard antiferromagnetic Néel state and the valence bond solid (VBS) state. The Néel state has ground state manifold (GSM) equivalent to a two dimensional sphere (S^2), i.e. all the configurations of the Néel vector form a manifold S^2. Although the VBS only has four fold degeneracy on the square lattice, there is a strong evidence that the four fold rotation symmetry of the square lattice is enlarged to a U(1) rotation symmetry right at the dQCP, and the VBS state has an approximate GSM S^1 (one dimensional ring), which is *not* a submanifold of the GSM of the Néel state on the other side of the dQCP. Thus we can view the dQCP on the square lattice as a "S^2-to-S^1" transition.

In another proposed realization of the dQCP [77], the Néel order and the VBS order are replaced by the quantum spin Hall order parameter and the $s-$wave superconductor, thus in this realization the dQCP is literally a transition between S^2 and S^1.

(2) The vortex of the VBS order parameter carries a bosonic spinor (spin-1/2) of the spin symmetry, and the Skyrmion of the Néel order carries lattice momentum. This physics can be described by the NCCP^1 model [13, 112]: $\mathcal{L} = \sum_\alpha |(\partial_\mu - i a_\mu) z_\alpha|^2 + r|z_\alpha|^2 + \cdots$, where the Neél order parameter is $\vec{N} = z^\dagger \vec{\sigma} z$, the flux of a_μ is the Skyrmion density of $\vec{N}$, and the flux condensate (which is dual to the photon phase of a_μ [137, 138, 139]

based on the standard photon-superfluid duality) is the VBS order. Thus there is an "intertwinement" between the Néel and VBS order: the defect of one order parameter is decorated with the quantum number of the symmetry that defines the other order, thus the condensation of the defect leads to the other order. This unusual quantum phase transition is considered "deconfined" because the field theory above is not formulated in terms of the standard Landau order parameter, but in terms of "fractionalized" degrees of freedom such as the spinon field z_α.

(3) If we treat the Néel and the VBS orders on equal footing, we can introduce a five component unit vector $\vec{n} \sim (N_x, N_y, N_z, V_x, V_y)$, and the "intertwinement" between the two order parameters is precisely captured by a topological Wess-Zumino-Witten (WZW) term of the nonlinear sigma model defined in the target space S^4 (four dimensional sphere) where $\vec{n}$ lives [14, 77].

All the previous works on dQCP have focused on the example proposed in Ref. [13, 112], which is a theory specially designed for the square lattice. In this work we propose a possible dQCP on the triangular lattice (and the Kagome lattice) for spin-1/2 systems with a full SU(2) spin rotation symmetry. Soon we will see that due to the fundamentally different structure of the magnetic order and VBS order from the square lattice, the dQCP on frustrated lattices demands a completely different formalism, with a very different universality class, and an unexpected emergent symmetry.

Let us first summarize the standard phases for spin-1/2 systems with a full spin rotation symmetry on the triangular lattice. On the triangular lattice, the standard antiferromagnetic order is no longer a collinear Néel order, it is the $\sqrt{3} \times \sqrt{3}$ noncollinear spin order (or the so-called 120° order) with ground state manifold (GSM) SO(3), which is fundamentally different from the GSM S^2 of the collinear magnetic order.

The VBS order most often discussed and observed in numerical simulations is the so-called $\sqrt{12} \times \sqrt{12}$ VBS pattern [16, 140, 141]. This VBS order is the most natural pattern that can be obtained from the condensate of the vison (or the m excitation) of a Z_2 spin liquid on the triangular lattice. The dynamics of visons on the triangular lattice is equivalent to a fully frustrated Ising model on the dual honeycomb lattice [142], and it has been shown that with nearest neighbor hopping on the dual honeycomb lattice, there are four symmetry protected degenerate minima of the vison band structure in the Brillouin zone, and that the GSM of the VBS order can be most naturally embedded into manifold SO(3) (just like the VBS order on the square lattice can be embedded in S^1) [142]. Thus the $\sqrt{3}\times\sqrt{3}$ noncollinear spin order and the $\sqrt{12}\times\sqrt{12}$ VBS order have a "self-dual" structure, *i.e.* the magnetic order and the VBS order are dual to each other. Conversely on the square lattice, the self-duality between the Néel and VBS order only happens in the easy-plane limit [143].

The self-duality structure on the triangular lattice was noticed in Ref. [81] and captured by a mutual Chern-Simons (CS) theory:

$$\mathcal{L} = |(\partial - ia)z|^2 + r_z|z|^2 + |(\partial - ib)v|^2 + r_v|v|^2 + \frac{i}{\pi}a \wedge db + \cdots \tag{4.1}$$

z_α and v_β carry a spinor representation of $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$ groups respectively, and when they are both gapped ($r_z, r_v > 0$), they are the e and m excitations of a symmetric Z_2 spin liquid on the triangular lattice, with a mutual semion statistics enforced by the mutual Chern-Simons (CS) term [81]. Physically z_α is the Schwinger boson of the standard construction of spin liquids on the triangular lattice [144, 145, 146], while v_β is the low energy effective modes of the vison.

Eq. 4.1 already unifies much of the physics for spin-1/2 systems on the triangular lattice [81]. For example, when both z_α and v_β are gapped, the system is in the Z_2 spin liquid mentioned above. The $\sqrt{3}\times\sqrt{3}$ noncollinear spin order, and the VBS order can be obtained from the self-dual Z_2 spin liquid by condensing z_α and v_β respectively, and both transitions have an emergent O(4) symmetry [147, 142].

The problem of finding a dQCP on the triangular lattice between the noncollinear magnetic order and the VBS order, is equivalent to finding *a direct unfine-tuned transition between two different orders each with GSM SO(3)*, or in our notation an “SO(3)-to-SO(3) transition".

4.3 topological term of effective field theory

As we discussed in the introduction, the physical picture of the dQCP is the “intertwinement" between the two ordered phases, namely the defect of one order is decorated with the quantum number of the other order, hence once we “melt" one ordered phase by proliferating its defects, the system will automatically be driven into the other order. On the square lattice, if we treat the Néel and the VBS orders on equal footing, we can introduce a five component unit vector $\vec{n} \sim (N_x, N_y, N_z, V_x, V_y)$, then the “intertwinement" between the two order parameters is precisely captured by a topological Wess-Zumino-Witten (WZW) term of the nonlinear sigma model defined in the target space S^4 (four dimensional sphere) where $\vec{n}$ lives [14, 77]:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du \ \frac{2\pi i}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d \partial_u n^e, \tag{4.2}$$

where Ω_4 is the volume of S^4. $\vec{n}(x, \tau, u)$ is any smooth extension of $\vec{n}(x, \tau)$ such that $\vec{n}(x, \tau, 0) = (1, 0, 0, 0, 0)$ and $\vec{n}(x, \tau, 1) = \vec{n}(x, \tau)$.

In Eq. 4.1, v_β is the vison of the spin liquid, and it carries a $\pi-$flux of a_μ due to the mutual CS term in Eq. 4.1. The $\pi-$flux of a_μ is bound with the Z_2 vortex of the $\mathrm{SO}(3)_e$ GSM of the $\sqrt{3} \times \sqrt{3}$ spin order. Due to the homotopy group $\pi_1[\mathrm{SO}(3)] = Z_2$, any ordered phase with GSM SO(3) has Z_2 vortex excitations, namely two vortices can

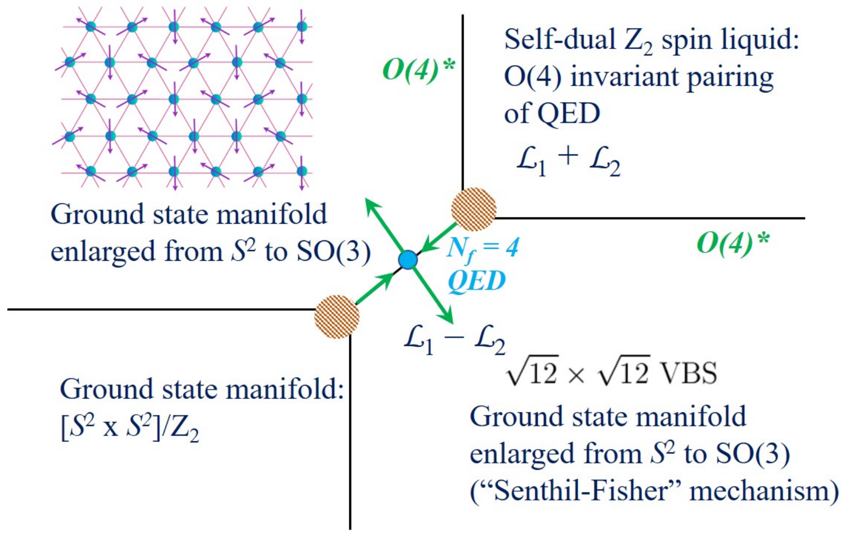

Figure 4.1: The global phase diagram of spin-1/2 systems on the triangular lattice. The intertwinement between the order parameters is captured by the WZW term Eq. 4.3. Our RG analysis concludes that there is a direct unfine-tuned SO(3)-to-SO(3) transition, which is a direct unfine-tuned transition between the noncollinear magnetic order and the VBS order. The detailed structure of the shaded areas demands further studies

annihilate each other. Similarly z_α is also the Z_2 vortex of the $\mathrm{SO}(3)_m$ GSM of the VBS order, analogous to the vortex of the VBS order on the square lattice. This mutual "decoration" of topological defects means that there is also an "intertwinement" between the noncollinear $\sqrt{3}\times\sqrt{3}$ magnetic order and the $\sqrt{12}\times\sqrt{12}$ VBS orders on the triangular lattice.

To capture the "intertwinement" of the two phases both with GSM SO(3), *i.e.* to capture the mutual decoration of topological defects, we need to design a topological term for these order parameters, just like the O(5) WZW term for the dQCP on the square lattice [14]. The topological term we design is as follows:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du \ \frac{2\pi i}{256\pi^2} \epsilon_{\mu\nu\rho\lambda} \mathrm{tr}[\mathcal{P}\partial_\mu\mathcal{P}\partial_\nu\mathcal{P}\partial_\rho\mathcal{P}\partial_\lambda\mathcal{P}]. \tag{4.3}$$

Here $\mathcal{P}$ is a 4×4 Hermitian matrix field:

$$\mathcal{P} = \sum_{a,b=1}^{3} N_e^a N_m^b \sigma^{ab} + \sum_{a=1}^{3} M_e^a \sigma^{a0} + \sum_{b=1}^{3} M_m^b \sigma^{0b}, \tag{4.4}$$

where $\sigma^{ab} = \sigma^a \otimes \sigma^b$, and $\sigma^0 = \mathbf{1}_{2\times 2}$. All vectors $\vec{N}_e$, $\vec{N}_m$, $\vec{M}_e$ and $\vec{M}_m$ transform as vectors under $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$ depending on their subscripts. And we need to also impose some extra constraints:

$$\mathcal{P}^2 = \mathbf{1}_{4\times 4}, \quad \vec{N}_e \cdot \vec{M}_e = \vec{N}_m \cdot \vec{M}_m = 0. \tag{4.5}$$

Then $\vec{N}_e$ and $\vec{M}_e$ together will form a SO(3) “tetrad", which is equivalent to the SO(3) manifold. $\vec{N}_m$ and $\vec{M}_m$ form another SO(3) manifold. With the constraints in Eq. 4.5, the matrix field $\mathcal{P}$ is embedded in the manifold

$$\mathcal{M} = \frac{U(4)}{U(2) \times U(2)}. \tag{4.6}$$

The maximal symmetry of the WZW term Eq. 4.3 is $\mathrm{PSU}(4) = \mathrm{SU}(4)/Z_4$ (which contains both $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$ as subgroups), as the WZW term is invariant under a SU(4) transformation: $\mathcal{P} \to U^\dagger \mathcal{P} U$ with $U \in \mathrm{SU}(4)$, while the Z_4 center of SU(4) does not change any configuration of $\mathcal{P}$. The WZW term Eq. 4.3 is well-defined based on its homotopy group $\pi_4[\mathcal{M}] = \mathbb{Z}$, just like $\pi_4[S^4] = \mathbb{Z}$. Obviously the SU(4) symmetry contains both $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$ as subgroups.

The topological WZW term in Eq. 4.3 is precisely the boundary theory of a $3d$ symmetry protected topological (SPT) state with a PSU(4) symmetry [148]. We will discuss this further later.

Let us test that this topological term captures the correct intertwinement. *i.e.* it must capture the physics that the Z_2 vortex of $\mathrm{SO}(3)_e$ carries spinor of $\mathrm{SO}(3)_m$, and vice versa. This effect is most conveniently visualized after breaking $\mathrm{SO}(3)_m$ down to $\mathrm{SO}(2)_m$, and the Z_2 vortex of the $\mathrm{SO}(3)_m$ manifold becomes the ordinary vortex of an SO(2) order parameter. This symmetry breaking allows us to take $\vec{N}_m = (0,0,1)$, *i.e.* $N_m^1 = N_m^2 = 0$, $N_m^3 = 1$. Because $\vec{N}_m \cdot \vec{M}_m = 0$ (Eq. 4.5), $\vec{M}_m = (M_m^1, M_m^2, 0)$. Then one allowed configuration of $\mathcal{P}$ is

$$\mathcal{P} = \sum_{a=1}^{3} N_e^a \sigma^{a3} + \sum_{b=1}^{2} M_m^b \sigma^{0b} = \vec{n} \cdot \vec{\Gamma}, \tag{4.7}$$

where $\vec{n}$ is a five component vector and $|\vec{n}| = 1$ due to the constraint $\mathcal{P}^2 = \mathbf{1}_{4\times 4}$. $\vec{\Gamma}$ are five anticommuting Gamma matrices. Now the WZW term Eq. 4.3 reduces *precisely* to the standard O(5) WZW at level-1 in $(2+1)d$, and it becomes manifest that the vortex of (M_m^1, M_m^2) (the descendant of the Z_2 vortex of $\mathrm{SO}(3)_m$ under the assumed symmetry breaking) is decorated with a spinor of $\mathrm{SO}(3)_e$. To explicitly visualize the effect of the "decorated vortex", one can follow the procedure of Ref. [77], and create a vortex of (n_4, n_5). Then the physics in the vortex core becomes a zero-dimensional quantum mechanics problem, whose exact solution reveals that there is a spin-1/2 carried by each vortex.

4.4 Field theory and Renormalization group analysis

Eq. 4.3 is a topological term in the low energy effective field theory that describes the physics of the ordered phases. But a complete field theory which reduces to the WZW term in the infrared is still demanded. For example, the O(5) nonlinear sigma model with a WZW term at level-1 can be derived as the low energy effective field theory of the $N = 2$ QCD with SU(2) gauge field, which has an explicit SO(5) global symmetry [134].

The WZW term in Eq. 4.3 can be derived in the same manner, by coupling the matrix field $\mathcal{P}$ to the Dirac fermions of the $N_f = 4$ QED:

$$\mathcal{L} = \sum_{j=1}^{4} \bar{\psi}_j \gamma \cdot (\partial - ia)\psi_j + m \sum_{i,j} \bar{\psi}_i \psi_j \mathcal{P}_{ij}. \quad (4.8)$$

The WZW term of $\mathcal{P}$ is generated after integrating out the fermions using the same method as Ref. [149], and the PSU(4) global symmetry becomes explicit in $N_f = 4$ QED [2].

Our goal is to demonstrate that the $N_f = 4$ QED corresponds to an unfine-tuned dQCP between the noncollinear magnetic order and the VBS order, or in our notation a "SO(3)-to-SO(3)" transition (as the dQCP is sandwiched between two ordered phases both with GSM SO(3)). The PSU(4) global symmetry of $N_f = 4$ QED must be explicitly broken down to the physical symmetry. The most natural terms that beak this PSU(4) global symmetry down to $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$ are four-fermion interaction terms. It turns out that there are *only two* such linearly independent four-fermion interaction terms that beak the PSU(4) global symmetry down to $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$ [3]:

$$\mathcal{L}_1 = \left(\bar{\psi}\vec{\sigma}\psi\right) \cdot \left(\bar{\psi}\vec{\sigma}\psi\right), \quad \mathcal{L}_2 = \left(\bar{\psi}\vec{\tau}\psi\right) \cdot \left(\bar{\psi}\vec{\tau}\psi\right), \quad (4.9)$$

where ψ carries both indices from the Pauli matrices $\vec{\sigma}$ and $\vec{\tau}$, so that ψ is a vector representation $(\frac{1}{2}, \frac{1}{2})$ of $\mathrm{SO}(4) \sim \mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$.

One can think of some other four fermion terms, for example $\mathcal{L}' = \sum_\mu \left(\bar{\psi}\vec{\sigma}\gamma_\mu\psi\right) \cdot \left(\bar{\psi}\vec{\sigma}\gamma_\mu\psi\right)$, but we can repeatedly use the Fiez identity, and reduce these terms to a linear combination of $\mathcal{L}_1$ and $\mathcal{L}_2$, as well as SU(4) invariant terms: $\mathcal{L}' = -2\mathcal{L}_2 - \mathcal{L}_1 + \cdots$ (for

[2] the global symmetry of the $N_f = 4$ QED is PSU(4) instead of SU(4) because the Z_4 center of the SU(4) flavor symmetry group is also part of the U(1) gauge group.

[3] This is true under the assumption of an emergent Lorentz invariance, which often happens at quantum critical points and algebraic spin liquids (such as the original dQCP on the square lattice).

more details please refer to the appendix). The ellipses are SU(4) invariant terms, which according to Ref. [150, 151, 152] are irrelevant at the $N_f = 4$ QED.

The renormalization group (RG) flow of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be most conveniently calculated by generalizing the two dimensional space of Pauli matrices $\vec{\tau}$ to an N-dimensional space, *i.e.* we generalize the QED$_3$ to an $N_f = 2N$ QED$_3$ with SU(2)× SU(N) symmetry. And we consider the following two independent four fermion terms:

$$g\mathcal{L} = g\left(\bar{\psi}\vec{\sigma}\psi\right)\cdot\left(\bar{\psi}\vec{\sigma}\psi\right), \quad g'\mathcal{L}' = g'\left(\bar{\psi}\vec{\sigma}\gamma_\mu\psi\right)\cdot\left(\bar{\psi}\vec{\sigma}\gamma_\mu\psi\right). \tag{4.10}$$

One can check that all SU(2)× SU(N) four fermion interations in this QED$_3$ can be written in terms of the linear combinations of these two terms above up to SU($2N$) invariant terms which according to Ref. [150, 151, 152] are irrelevant under RG even for small N. At the first order of $1/N$ expansion, the RG equation reads

$$\begin{aligned}\beta(g) &= \left(-1 + \frac{128}{3(2N)\pi^2}\right)g + \frac{64}{(2N)\pi^2}g', \\ \beta(g') &= -g' + \frac{64}{3(2N)\pi^2}g. \end{aligned} \tag{4.11}$$

There are two RG flow eigenvectors: $(1, -1)$ with RG flow eigenvalue $-1-64/(3(2N)\pi^2)$, and $(3, 1)$ with eigenvalue $-1 + 64/((2N)\pi^2)$. This means that when $N = 2$ there is one irrelevant eigenvector with

$$\mathcal{L} - \mathcal{L}' = 2(\mathcal{L}_1 + \mathcal{L}_2) + \cdots, \tag{4.12}$$

and a relevant eigenvector with

$$3\mathcal{L} + \mathcal{L}' = 2(\mathcal{L}_1 - \mathcal{L}_2) + \cdots. \tag{4.13}$$

Again the ellipses are SU(4) invariant terms that are irrelevant. In fact, $\mathcal{L}_1+\mathcal{L}_2$ preserves the exchange symmetry (duality) between $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$, in other words $\mathcal{L}_1+\mathcal{L}_2$ preserves the O(4) symmetry that contains an extra improper rotation in addition to SO(4), while $\mathcal{L}_1-\mathcal{L}_2$ breaks the O(4) symmetry down to SO(4). Thus $\mathcal{L}_1+\mathcal{L}_2$ and $\mathcal{L}_1-\mathcal{L}_2$ both *must be* eigenvectors under RG. The RG flow is sketched in Fig. 4.1.

To make a complete story, we should also discuss other perturbations on the $N_f=4$ QED. The fermion bilinear terms are forbidden either by the flavor symmetry, or discrete space-time symmetries, while higher order fermion interactions (such as eight fermion interactions) are very likely to be irrelevant. The monopoles of a_μ were ignored in this RG calculation. According to Ref. [153], monopoles of QED carry nontrivial quantum numbers. A multiple-monopole could be a singlet under the global symmetry, and hence allowed in the action, but it will have a higher scaling dimension than the single monopole. It is known that with large$-N_f$ all the monopoles are irrelevant, but the scaling dimension of the multiple-monopole for the current case with $N_f=4$ needs further study.

Since $u(\mathcal{L}_1-\mathcal{L}_2)$ is relevant, then when the coefficient $u>0$, a simple mean field theory implies that this term leads to a nonzero expectation value for $\langle\bar{\psi}\vec{\sigma}\psi\rangle$. It appears that this order parameter is a three component vector, and so the GSM should be S^2. However, using the "Senthil-Fisher" mechanism of Ref. [14], the actual GSM is enlarged to SO(3) due to the gauge fluctuation of a_μ (for a review of the "Senthil-Fisher" mechanism, please refer to the appendix). When $u<0$, the condensed order parameter is $\langle\bar{\psi}\vec{\tau}\psi\rangle$, and the "Senthil-Fisher" mechanism again enlarges the GSM to SO(3). Based on our calculation, because $u(\mathcal{L}_1-\mathcal{L}_2)$ is the *only* relevant perturbation allowed by symmetry, u drives a direct unfine-tuned continuous SO(3)-to-SO(3) transition, which is consistent with a transition between the $\sqrt{3}\times\sqrt{3}$ noncollinear magnetic order and the $\sqrt{12}\times\sqrt{12}$ VBS order. And our theory predicts that at the critical point, there is an emergent PSU(4) symmetry.

Now let us investigate the perturbation $\mathcal{L}_1 + \mathcal{L}_2$. First of all, let us think of a seemingly different term: $\mathcal{L}_3 = \sum_{a,b} \left(\bar{\psi}\sigma^a\tau^b\psi\right)\left(\bar{\psi}\sigma^a\tau^b\psi\right)$. This term also preserves the O(4) symmetry, and after some algebra we can show that $\mathcal{L}_3 = -(\mathcal{L}_1+\mathcal{L}_2)+\cdots$. Another very useful way to rewrite $\mathcal{L}_3$ is that:

$$\mathcal{L}_3 = -\left(\bar{\psi}^t J\epsilon\bar{\psi}\right)\left(\psi^t J\epsilon\psi\right) + \cdots = -\hat{\Delta}^\dagger\hat{\Delta} + \cdots \tag{4.14}$$

where $\hat{\Delta} = \psi^t J\epsilon\psi$, $J = \sigma^2 \otimes \tau^2$. ϵ is the antisymmetric tensor acting on the Dirac indices.

Thus although the O(4) invariant deformation in our system (at low energy it corresponds to $\mathcal{L}_1 + \mathcal{L}_2$) is perturbatively irrelevant at the $N_f = 4$ QED fixed point, when it is strong and nonperturbative, the standard Hubbard-Stratonovich transformation and mean field theory suggests that, depending on its sign, it may lead to either a condensate of $\hat{\Delta}$, or condensate of $\left(\bar{\psi}\sigma^a\tau^b\psi\right)$ over certain critical strength of $\mathcal{L}_3$. The condensate of $\left(\bar{\psi}\sigma^a\tau^b\psi\right)$ has GSM $[S^2 \times S^2]/Z_2$, and is identical to the submanifold of $\mathcal{P}$ when $\vec{M}_e = \vec{M}_m = 0$ in Eq. 4.4. The Z_2 in the quotient is due to the fact that $\mathcal{P}$ is unaffected when both $\vec{N}_e$ and $\vec{N}_m$ change sign simultaneously.

Now we show that the condensate of $\hat{\Delta}$ is precisely the self-dual Z_2 topological order described by Eq. 4.1. First of all, in the superconductor phase with $\hat{\Delta}$ condensate, there will obviously be a Bogoliubov fermion. This Bogoliubov fermion carries the $(1/2, 1/2)$ representation under $\mathrm{SO}(3)_e\times\mathrm{SO}(3)_m$. The deconfined $\pi-$flux of the gauge field a_μ is bound to a $2\pi-$vortex of the complex order parameter $\hat{\Delta}$, which then traps 4 Majorana zero modes. The 4 Majorana zero modes transform as a vector under the SO(4) action that acts on the flavor indices. The 4 Majorana zero modes define 4 different states that can be separated into two groups of states depending on their fermion parities. In fact, the two groups should be identified as the $(1/2, 0)$ doublet and the $(0, 1/2)$ doublet of $\mathrm{SO}(3)_e\times\mathrm{SO}(3)_m$. Therefore, the $\pi-$flux with two different types of doublets should be viewed as two different topological excitations. Let us denote the $(1/2, 0)$ doublet as e and the $(0, 1/2)$ doublet as m. Both e and m have bosonic topological spins. And they differ by a Bogoliubov fermion. Therefore, their mutual statistics is semionic (which rises from the braiding between the fermion and the $\pi-$flux). At this point, we can identify the topological order of the $\hat{\Delta}$ condensate as the Z_2 topological order described by Eq. 4.1.

4.5 Interpretation of the dQCP as the boundary of a $3d$ system

Decorating quantum numbers to topological defects is also a key physical picture of constructing symmetry protected topological (SPT) states. The analogy between the dQCP on the square lattice and a $3d$ bulk SPT state with an SO(5) symmetry was discussed in Ref. [134]. Many $3d$ SPT states can be constructed by decorating the defects in the system with a lower dimensional SPT state, and then proliferating the defects [53, 48].

The physics around the dQCP discussed in this work is equivalent to the boundary state of a $3d$ bosonic SPT state with $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$ symmetry, once we view both SO(3) groups as onsite symmetries. We have already mentioned that the topological WZW term Eq. 4.3 is identical to the boundary theory of a $3d$ SPT state with PSU(4) symmetry [148], which comes from a $\Theta-$term in the $3d$ bulk. And by breaking the symmetry down to either $\mathrm{SO}(3)_e \times \mathrm{SO}(2)_m$ or $\mathrm{SO}(2)_e \times \mathrm{SO}(3)_m$, the bulk SPT state is reduced to a $\mathrm{SO}(3) \times \mathrm{SO}(2)$ SPT state, which can be interpreted as the decorated vortex line construction [48], namely one can decorate the SO(2) vortex line with the Haldane phase with the SO(3) symmetry, and then proliferate the vortex lines. In our case, the bulk SPT state with $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$ symmetry can be interpreted as a similar decorated vortex line construction, *i.e.* we can decorate the Z_2 vortex line of one of the SO(3) manifolds with the Haldane phase of the other SO(3) symmetry, then proliferating the vortex lines. The Z_2 classification of the Haldane phase is perfectly compatible with the Z_2 nature of the vortex line of a SO(3) manifold. Using the method in Ref. [134], one can also derive that the $(3+1)d$ bulk SPT state must have a topological response action $\mathcal{S} = i\pi \int w_2[\mathcal{A}_e] \cup w_2[\mathcal{A}_m]$ in the presence of background $\mathrm{SO}(3)_e$ gauge field $\mathcal{A}_e$ and $\mathrm{SO}(3)_m$ gauge field $\mathcal{A}_m$ (w_2 represents the second Stiefel–Whitney class). This topological response theory also matches exactly with the decorated vortex line construction.

We have shown that the physics around the critical point has the same effective field theory as the boundary of a $3d$ SPT state [148]. The anomaly (once we view all the symmetries as onsite symmetries) of the large-N generalizations of our theory will be analyzed in the future, and a Lieb-Shultz-Mattis theorem for SU(N) and SO(N) spin systems on the triangular and Kagome lattice can potentially be developed in the same way as Ref. [154, 155].

4.6 Summary

In summary, we proposed a theory for a potentially direct unfine-tuned continuous quantum phase transition between the noncollinear magnetic order and VBS order on the triangular lattice, and at the critical point the system has an emergent PSU(4) global symmetry. Our proposed dQCP is fundamentally different from the original example on the square lattice due to the very different structure of both the magnetic and VBS orders compared with the unfrustrated square lattice. Our conclusion is based on a controlled RG calculation, and an effective nonlinear sigma model with a topological WZW term.

Similar structure of noncollinear magnetic order and VBS orders can be found on the Kagome lattice. For example, it was shown in Ref. [156] that the vison band structure could have symmetry protected four degenerate minima just like the triangular lattice (although the emergence of O(4) symmetry in the infrared is less likely). Indeed, algebraic spin liquids with $N_f = 4$ QED as their low energy description have been discussed extensively on both the triangular and the Kagome lattice [157, 158, 146, 159]. Ref. [146] also made the observation that the noncollinear magnetic order, the VBS order, and the Z_2 spin liquid are all nearby a $N_f = 4$ QED (the so-called $\pi-$flux state from microscopic construction). The Z_2 spin liquid was shown to be equivalent to the one constructed from Schwinger boson [145], which can evolve into the $\sqrt{3} \times \sqrt{3}$ magnetic order, and the $\sqrt{12} \times \sqrt{12}$ VBS order through an $\mathrm{O}(4)^*$ transition. But we should stress that in this work we only focus on the field theory for the "SO(3)-to-SO(3)" dQCP, without fully determining the relation between the field theory and the microscopic degrees of freedom.

It is a challenge to find an antiferromagnetic spin model on a frustrated lattice without sign problem. But we note that in Ref. [141] spin nematic phases with GSM S^N/Z_2 (analogous to the spin-1/2 $\sqrt{3} \times \sqrt{3}$ state with GSM SO(3)= S^3/Z_2) and the $\sqrt{12} \times \sqrt{12}$ VBS order are found in a series of sign-problem free models on the triangular lattice. Thus it is possible to design a modified version of the models discussed in Ref. [141] to bring together the spin nematic order and VBS order, and then access the dQCP that we are proposing.

The ordered phases and the "Senthil-Fisher" mechanism

Here we reproduce the discussion in Ref. [14], and demonstrate how the GSM of the order of $\bar{\psi}\vec{\sigma}\psi$ (and similarly $\bar{\psi}\vec{\tau}\psi$) is enlarged from S^2 to SO(3). First we couple the $N_f = 4$ QED to a three component dynamical unit vector field $\boldsymbol{N}(x, \tau)$:

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - i a_\mu)\psi + m\bar{\psi}\boldsymbol{\sigma}\psi \cdot \boldsymbol{N}. \tag{4.15}$$

The flavor indices are hidden in the equation above for simplicity. Now following the standard $1/m$ expansion of Ref. [149], we obtain the following action after integrating out the fermion ψ_j:

$$\mathcal{L}_{eff} = \frac{1}{g}(\partial_\mu \boldsymbol{N})^2 + i2\pi \mathrm{Hopf}[\boldsymbol{N}] + i2a_\mu J^T_\mu + \frac{1}{e^2} f^2_{\mu\nu}, \tag{4.16}$$

where $1/g \sim m$. $J^T_0 = \frac{1}{4\pi}\epsilon_{abc} N^a \partial_x N^b \partial_y N^c$ is the Skyrmion density of $\boldsymbol{N}$, thus J^T_μ is the Skyrmion current. The second term of Eq. 4.16 is the Hopf term of $\boldsymbol{N}$ which comes from the fact that $\pi_3[S^2] = \mathbb{Z}$.

Now if we introduce the CP^1 field $z_\alpha = (z_1, z_2)^t = (n_1 + in_2, n_3 + in_4)^t$ for $\boldsymbol{N}$ as $\boldsymbol{N} = z^\dagger \boldsymbol{\sigma} z$, the Hopf term becomes precisely the $\Theta-$term for the O(4) *unit* vector $\boldsymbol{n}$ with $\Theta = 2\pi$:

$$i2\pi \mathrm{Hopf}[\boldsymbol{N}] = \frac{i2\pi}{2\pi^2} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d. \tag{4.17}$$

In the CP^1 formalism, the Skyrmion current $J^T_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu \alpha_\rho$, where α_μ is the gauge field that the CP^1 field z_α couples to. The coupling between a_μ and α_μ

$$2ia_\mu J^T_\mu = \frac{i2}{2\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu \alpha_\mu \tag{4.18}$$

takes precisely the form of the mutual CS theory of a Z_2 topological order, and it implies that the gauge charge z_α is an anyon of a Z_2 topological order, and the condensate of z_α (equivalently the order of $\boldsymbol{N}$) has a GSM = SO(3) = S^3/Z_2, where S^3 is the manifold of the unit vector $\vec{n}$.

Deriving the WZW term

Let us consider a theory of QED$_3$ with $N_f = 4$ flavors of Dirac fermions coupled to a matrix order parameter field $\mathcal{P}$:

$$\mathcal{L} = \sum_{i,j} \bar{\psi}_i(\gamma_\mu(\partial_\mu - ia_\mu)\delta_{ij} + m\mathcal{P}_{ij})\psi_j. \tag{4.19}$$

$\mathcal{P}$ takes values in the target manifold $\mathcal{P} \in \mathcal{M} = \frac{U(4)}{U(2)\times U(2)}$. We can parametrize the matrix field $\mathcal{P} = U^\dagger \Omega U$, where $U \in SU(4)$ and $\Omega = \sigma^z \otimes \mathbf{1}_{2\times 2}$. $\mathcal{P}$ satisfies $\mathcal{P}^2 = \mathbf{1}_{4\times 4}$ and $\mathrm{tr}\mathcal{P} = 0$.

The effective action after integrating over the fermion fields formally reads

$$\begin{aligned}\mathcal{S}_{eff}[a_\mu, \mathcal{P}] &= -\ln \int D\bar{\psi}D\psi \exp\left[-\int d^3x \mathcal{L}(\psi, a_\mu, \mathcal{P})\right] \\ &= -\ln\det[\mathcal{D}(a_\mu, \mathcal{P})] = -\mathrm{Tr}\ln[\mathcal{D}(a_\mu, \mathcal{P})].\end{aligned} \tag{4.20}$$

The expansion of $\mathcal{S}_{eff}$ has the following structure

$$\mathcal{S}_{eff}[a_\mu, \mathcal{P}] = \mathcal{S}_{eff}[a_\mu = 0, \mathcal{P}] + O(a) \tag{4.21}$$

and we will look at the first term in the expansion. In general, all terms that respect the symmetry of the original action will appear in the expansion of the fermion determinant. Here we want to derive the topological term of $\mathcal{P}$. One way to obtain the effective action is the perturbative method developed in Ref. [149]. Let us vary the action over the matrix field $\mathcal{P}$

$$\delta\mathcal{S}_{eff} = -\mathrm{Tr}(m\delta\mathcal{P}(\mathcal{D}^\dagger\mathcal{D})^{-1}\mathcal{D}^\dagger) \tag{4.22}$$

and then expand $(\mathcal{D}^\dagger\mathcal{D})^{-1}$ in gradients of $\mathcal{P}$.

$$
\begin{aligned}
(\mathcal{D}^\dagger\mathcal{D})^{-1} &= (-\partial^2 + m^2 - m\gamma_\mu\partial_\mu\mathcal{P})^{-1} \\
&= (-\partial^2 + m^2)^{-1} \\
&\times \ (\sum_{n=0}^{\infty}((-\partial^2 + m^2)^{-1}m\gamma_\mu\partial_\mu\mathcal{P})^n)
\end{aligned}
$$

Since the coefficient of the WZW term is dimensionless, we will look at the following term in the expansion

$$
\begin{aligned}
\delta W(\mathcal{P}) &= -\mathrm{Tr}[m^2\delta\mathcal{P}(-\partial^2 + m^2)^{-1} \\
& \quad ((-\partial^2 + m^2)^{-1}m\gamma_\mu\partial_\mu\mathcal{P})^3\mathcal{P}] \\
&= -K\int d^3x \ \mathrm{Tr}[\delta\mathcal{P}(\gamma_\mu\partial_\mu\mathcal{P})^3\mathcal{P}]
\end{aligned}
$$

where $K = \int \frac{d^3p}{(2\pi)^3}\frac{m^5}{(p^2+m^2)^4} = \frac{1}{64\pi}$ is a dimensionless number, and “Tr" is the trace over the Dirac and flavor indices. After tracing over the Dirac indices,

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 2i\epsilon_{\mu\nu\rho} \tag{4.23}$$

we obtain the following term for the variation

$$\delta W(\mathcal{P}) = -\frac{2\pi i}{64\pi^2}\epsilon_{\mu\nu\rho}\int d^3x \ \mathrm{tr}[\delta\mathcal{P}\partial_\mu\mathcal{P}\partial_\nu\mathcal{P}\partial_\rho\mathcal{P}\mathcal{P}], \tag{4.24}$$

where “tr" is the trace for the flavor indices only.

We can restore the topological term of the nonlinear σ-model by the standard method of introducing an auxiliary coordinate u. The field $\tilde{\mathcal{P}}(x,u)$ interpolates between $\tilde{\mathcal{P}}(x,u=0)=\Omega$ and $\tilde{\mathcal{P}}(x,u=1)=\mathcal{P}(x)$. The topological term reads

$$W(\tilde{\mathcal{P}}) = -\frac{2\pi i}{256\pi^2}\epsilon_{\mu\nu\rho\delta}\int_0^1 du \int d^3x \mathrm{tr}[\tilde{\mathcal{P}}\partial_\mu\tilde{\mathcal{P}}\partial_\nu\tilde{\mathcal{P}}\partial_\rho\tilde{\mathcal{P}}\partial_\delta\tilde{\mathcal{P}}] \tag{4.25}$$

(the extra factor of 1/4 comes from the anti-symmetrization of the u coordinate with other indices).

Linear Dependence of four-fermion interactions in $N_f = 2N$ QED$_3$

In this section, we study all the SU(2)×SU(N) symmetric four-fermion interactions in the $N_f = 2N$ QED$_3$ and their linear dependence up to SU($2N$)-invariant terms .

First of all, we can write down all the SU(2)×SU(N) symmetric four-fermion terms:

$$(\bar{\psi}\psi)(\bar{\psi}\psi), \qquad (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi), \tag{4.26}$$

$$(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi), \qquad (\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi), \tag{4.27}$$

$$(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi), \qquad (\bar{\psi}\gamma^\mu T^a\psi)(\bar{\psi}\gamma^\mu T^a\psi), \tag{4.28}$$

$$(\bar{\psi}\vec{\sigma}T^a\psi)(\bar{\psi}\vec{\sigma}T^a\psi), \qquad (\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi), \tag{4.29}$$

where $\vec{\sigma}$ is the generator of the SU(2) symmetry and T^a (with $a = 1, 2, ..., N^2-1$) is the generator of the SU(N) symmetry. Here, we've also implicitly assumed the summation over repeated indices in these expressions. The two terms on the second line are exactly the terms introduced in Eq. 9 of the main text.

Since all the SU($2N$) invariant four-fermion interaction are irrelevant under RG [150, 151, 152], we are only concerned with the linear dependence of all the SU(2)×SU(N) symmetric four-fermion interactions up to SU($2N$) invariant ones. First, we notice that the terms in Eq. 4.26 are SU($2N$) invariant. Therefore, we can ignore them for this analysis. Notice that we can rewrite the two terms in Eq. 4.28 as

$$\begin{aligned}&(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi)\\&=-\frac{N}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)-\frac{N}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi),\\&\quad-\frac{N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)-\frac{N+4}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)\end{aligned}\tag{4.30}$$

$$\begin{aligned}&(\bar{\psi}T^a\gamma^\mu\psi)(\bar{\psi}T^a\gamma^\mu\psi)\\&=\frac{N}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)-\frac{3N}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi),\\&\quad+\frac{N-4}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)-\frac{3N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi).\end{aligned}\tag{4.31}$$

Therefore, up to SU($2N$) invariant terms, the two terms in Eq. 4.28 can be written as linear combination of the two terms in Eq. 4.27. In the rewriting given above, we've used the Fierz identity $\sum_a T^a_{ij}T^a_{kl}=N\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}$ for the SU(N) group as well as the Fierz identities $\vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd}=2\delta_{ad}\delta_{bc}-\delta_{ab}\delta_{cd}$ for the Pauli matrices $\vec{\sigma}$ and $\gamma^\mu_{\alpha\beta}\gamma^\mu_{\eta\rho}=2\delta_{\alpha\rho}\delta_{\beta\eta}-\delta_{\alpha\beta}\delta_{\eta\rho}$ for the Gamma matrices γ^μ. Similarly, we can rewrite the two terms in Eq. 4.29 as

$$\begin{aligned}&(\bar{\psi}\vec{\sigma}T^a\psi)(\bar{\psi}\vec{\sigma}T^a\psi)\\&=\frac{N}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)+\frac{N-4}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi),\\&\quad-\frac{3N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)-\frac{3N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)\end{aligned}\tag{4.32}$$

$$\begin{aligned}&(\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi)\\&=-\frac{N+4}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)+\frac{3N}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi),\\&\quad+\frac{3N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)-\frac{9N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)\end{aligned}\tag{4.33}$$

Therefore, all the SU(2)×SU(N) symmetric four-fermion interactions can be written as linear combinations of $(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi)$ and $(\bar{\psi}\gamma^{\mu}\vec{\sigma}\psi)(\bar{\psi}\gamma^{\mu}\vec{\sigma}\psi)$, namely the two terms in Eq. 4.27 (as well as Eq. 9 in the main text).

Chapter 5

Stable Gapless Phases

5.1 Spin Liquids

Spin liquids are another important class of exotic quantum system. While a precise definition is difficult to formulate due to edge cases, heuristically a spin liquid phase has no magnetic order, an odd number of electrons per unit cell, and is distinct from the classical disordered phase[15]. Unsatisfactory as this definition may be, we can nevertheless group many similar systems together as spin liquids. Resonating valence bond (RVB) phases and dimer models are two prominent examples. There are also bosonic analogues of spin liquids, which carry over most of the physics despite not being built from electrons.

Previous work has established the existence and stability of gapless abelian bose liquid (ABL) phases as the low-energy effective theory for 3+1d lattice models[19, 160]. The low-energy physics is dominated by an emergent gauge structure reminiscent of electromagnetism or gravity, and stability is guaranteed by a combination of self-duality and gauge invariance. However, it is possible to generalize these models and find an infinite family of similar ABL phases, all of which are stable and gapless.

5.2 Stable Gapless phases without Symmetry

It is now well-known that quantum disordered states of many-body systems can be fundamentally different from classical disordered states. Without assuming any symmetry, there is simply one type of trivial classical state, but there can be many stable quantum disordered states. Many of these nontrivial quantum disordered phases have a gapped spectrum and topological degeneracy on a manifold with nontrivial topology [25, 26, 27], such as fractional quantum Hall states. In this paper we consider another kind of stable quantum disordered phases without assuming any symmetry. These states are characterized by their bulk gapless bosonic modes that *cannot* be interpreted as Goldstone modes. Furthermore, physical quantities have power-law (or algebraic) correlations instead of short-range correlations found in gapped systems.

Although such gapless states are not rare at all in condensed matter systems, they usually occur at quantum critical points and are protected by certain symmetries. Generically, we would expect there to be relevant perturbations that will open the gap in these critical states. But the examples we will discuss in this paper all have very stable gapless bosonic modes, which are invulnerable to any weak perturbations. Thus, to establish that an algebraic Bose liquid (ABL) phase is stable, we must show that *all* potential gap-opening perturbations are irrelevant at the IR fixed point of the ABL phase. Drawing intuition from the $(2+1)d$ compact lattice U(1) gauge field, we must demonstrate not only a direct mass term of these gapless modes are forbidden, but also that the space-time topological defects in the dual picture must also be suppressed (or irrelevant).

A few examples of this type of states are already known. In Ref [19, 161, 162] a stable ABL phase with photon like excitations were proposed, and it has attracted great interests [163, 164, 165]. So far compelling experimental evidences for such liquid states have been found [166, 167, 168, 169]. Later, a different type of ABL phase with graviton like excitation was studied in Ref. [170, 171, 160]. It turns out that the graviton-ABL state has a close cousin with a different dispersion relation [172, 173]. So far these are the only three types of known stable ABL states with emergent gapless bosonic excitations without assuming any symmetry. The Bose metal phase proposed in Ref. [174, 175] rely on a special quasi one dimensional conservation, which is different from the scenarios we will focus on.

In this work, we expand these ideas even further, demonstrating that there are an infinite number of gapless phases that fit into this class of states. We provide several examples of these so-called “higher-rank" ABL theories. We also investigate the topological properties of these models, showing that they are “topologically ordered" in the same sense as the photon and graviton theories, even though they are gapless in the bulk. At finite system size L, the emergent gauge bosons will lead to an energy splitting between different sectors that scales as a power law of $1/L$.

5.3 Review of Rank-1 and -2 Theories

5.3.1 The Rank-1 Case

We first review the essential facts about the well-known $U(1)$ photon ABL phase in $3+1d$. In order to connect to the more general construction, we will address the problem from a somewhat different (but physically equivalent) viewpoint than the original works [19, 161, 162]. The gauge structure (and its duality) is of paramount importance, so we will omit some details in favor of a more easily generalizable procedure. For simplicity, we will consider the cubic lattice, where spins are defined on the links, i.e. the corner-sharing octahedra. The most important term of the Hamiltonian is simply an

Ising antiferromagnetic interaction on each octahedron:

$$H = \frac{J}{2}\sum_{oct}(S^z_{oct})^2, \tag{5.1}$$

With a very large J, this term will give rise to a locally conserved z-component of spin, which we will enforce as a constraint on the low energy Hilbert space:

$$\sum_{i\in oct} S^z_i = 0. \tag{5.2}$$

There are certainly other terms on the lattice that involve $S^{\pm}$, but their specific forms are not important, as long as they are all dominated by the J term. Under the standard change of variables $S^z \sim n - 1/2$ and $S^{\pm} \sim e^{\pm i\theta}$, this model becomes a boson rotor model on the links of a cubic lattice. Noting that the locally conserved integer S^z_{oct} generates a $U(1)$ gauge symmetry, after we change variables again to $E_{rr'} \sim (-1)^r n_{rr'}$ and $A_{rr'} \sim (-1)^r \theta_{rr'}$. Note that $A_{rr'}$ is only defined modulo 2π.

The operators E and A are defined on the links of the lattice, and this endows them with a vector structure. We can thus identify a vector of operators $\mathbf{E}(\vec{x})$ and $\mathbf{A}(\vec{x})$ at each site of the cubic lattice, along with lattice derivatives $\partial_i E_j(\vec{x}) = E_j(\vec{x}+\hat{i}) - E_j(\vec{x})$. They satisfy the normal commutation relations $[A_j(\vec{x}),\ E_k(\vec{y})] = i\delta_{jk}\delta^3(\vec{x}-\vec{y})$. When phrased in terms of these new variables, the low energy effective Hamiltonian is bound to take the following form:

$$H = U\sum_r \mathbf{E}(r)^2 - K\sum_{\square}\cos[\mathrm{curl}(\mathbf{A})_{\square}] \tag{5.3}$$

Once we project all the physics down to the low energy subspace of the Hilbert space that obeys the constraint imposed by the J term, the low energy effective Hamiltonian must have the gauge symmetry $\mathbf{A} \to \mathbf{A} + \nabla f$, which is generated by the local constraint which can now be written as

$$\partial_i E_i = 0. \tag{5.4}$$

Tentatively ignoring the fact that $\mathbf{A}$ is compactly defined, we can expand the low energy effective Hamiltonian at the minimum of the cosine function (spin wave expansion):

$$H = U \sum_r E_i^2 + \frac{K}{2} \sum_r (\epsilon_{ijk} \partial_j A_k)^2 \tag{5.5}$$

Equation (5.5) is the effective low energy Hamiltonian for $(3+1)d$ quantum electrodynamics (QED) in its deconfined phase. Solving the equation of motion of Eq. 5.5 from the Heisenberg equation directly, we will obtain a gapless photon excitation with linear dispersion relation $\omega \sim c|\vec{k}|$, where the speed of light $c \sim \sqrt{UK}$. However, we know that in $2+1d$, the compact QED suffers from the instanton effect: proliferation of magnetic monopoles in the space time opens up the photon gap, but that effect is only made clear in terms of the dual variables. Thus, we will also consider the dual theory to ascertain if there is a similar gap-opening effect.

We see that the solution of E_i to the local constraint Eq. 5.4 can be written as the curl of another vector field h_i, $E_i = \epsilon_{ijk}\partial_j h_k$. This new field h_i is defined on each plaquette center and it is canonically conjugate to the magnetic field B_i. We can now rewrite the Hamiltonian Eq. 5.5 as

$$H = U \sum_r (\epsilon_{ijk} \partial_j h_k)^2 + \frac{K}{2} \sum_r B_i^2 \tag{5.6}$$

In contrast to the $2+1d$ case, this new Hamiltonian has the same form as the original

Eq. (5.5), and formally h_i has the same gauge symmetry as A_i:

$$h_i \to h_i + \nabla_i f. \tag{5.7}$$

We thus say that the system (at least in the photon phase) is *self-dual*. This is an emergent feature in the infrared.

In the dual theory, we might expect relevant "vertex operators" $\alpha \cos(2\pi N h_i)$, whose analogue in $(2+1)d$ plays the role of the flux creation. In $(3+1)d$, this vertex operator corresponds to hopping of the magnetic monopole of the compact U(1) gauge field. Whether this vertex operator is important or not, can be determined by evaluating its correlation function in the limit where $\alpha = 0$. However, in $(3+1)d$, the correlation function between two such terms in the limit $\alpha = 0$ is

$$\langle \cos(2\pi N h_i(\vec{x})) \ \cos(2\pi N h_j(\vec{y})) \rangle_0 \sim \delta_{ij} \delta^3(\vec{x} - \vec{y}), \tag{5.8}$$

because this is not a gauge invariant correlation function under gauge transformation Eq. 5.7. Thus at the Gaussian fixed point, the vertex operators $\cos(2\pi N h_i)$ are irrelevant - at least perturbatively. When the vertex operator is strong enough, it will induce magnetic monopole condensation and drive the system into the confined phase. Thus, the gapless photon is perturbatively protected by the gauge symmetry of both the original and the dual theory, i.e. the self-duality protects the stability of the photon phase.

Our review in this subsection is no more than restating the known fact that the $(3+1)d$ compact U(1) gauge field has a deconfined phase, which corresponds to the phase where neither the charge nor the magnetic monopole condenses. In this subsection, we identified the photon phase where the magnetic monopole is gapped as the phase where the vertex operator is irrelevant. The language and logic used in this subsection can be conveniently generalized to other ABL phases.

5.3.2 The Rank-2 Case

In this section we will review the construction of another ABL phase with rank-2 tensor gapless bosonic excitations that are analogous to gravitons. We will omit the exact details of the microscopic derivation; interested readers are referred to the original papers Ref. [170, 160].

The $3+1d$ microscopics in this system give rise to a symmetric rank-2 tensor field, in contrast to the rank-1 tensor field in the previous example. Once again, the system can be simply phrased in terms of the boson number n_{ij} and its canonical conjugate phase variables θ_{ij}. We can again define gauge field variables $E_{ij} \sim n_{ij}$ $(i \neq j)$, $E_{ii} \sim 2n_{ii}$, and $A_{ij} \sim \theta_{ij}$, noting that A_{ij} is compactly defined with modulo 2π.

The low-energy subspace has the local constraint

$$\partial_i E_{ij} = 0, \tag{5.9}$$

which is imposed by a large local term similar to Eq. 5.1. This constraint generates the gauge transformation

$$A_{ij} \to A_{ij} + \frac{1}{2}\left(\partial_i \lambda_j + \partial_j \lambda_i\right) \tag{5.10}$$

This gauge transformation is the same as that of linearized gravity, if we were to treat A_{ij} as the fluctuation of a background metric η_{ij}. Hence, we term the gauge boson a "graviton". The original works [170, 160] use the language of general relativity to write the Hamiltonian in terms of the curvature tensor for A_{ij}. We will avoid that notation here while noting that it has very nice connections to the Lifshitz gravity proposed independently in Ref. [176, 177, 178].

We want to establish the simplest Hamiltonian possible that is gauge-invariant. To do this, we will again write down a gauge-invariant quantity B_{ij} which should be thought of as the "2-curl" of A_{ij}:

$$B_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c A_{bd} \tag{5.11}$$

The low energy effective Hamiltonian, or the Hamiltonian after the "spin-wave expansion", then takes the simple form

$$H = U\sum_r E_{ij}^2 + K\sum_r B_{ij}^2, \tag{5.12}$$

where U and K may, in general, take different values for the $\sum_{ij} X_{ij}^2$ and $\sum_i X_{ii}^2$ terms, if an ordinary cubic lattice symmetry is assumed; however, lattice symmetry is not essential to our work here.

The spin-wave expanded Hamiltonian above already gave us a gapless "graviton-like" bosonic mode with a quadratic dispersion. In order to guarantee that this gapless mode is not ruined by the compactness of the gauge field, we must once again consider the dual theory. The dual variables solve the constraint equation (5.9), and we can write E as the 2-curl of a new field h:

$$E_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c h_{bd}. \tag{5.13}$$

We see that h transforms under the same gauge transformation as tensor A and is canonically conjugate to the tensor field B [170, 160]. The vertex operators will take the form $\cos(2\pi N h_{ij})$, and just as before the gauge-dependence makes them irrelevant at the infrared Gaussian fixed point because it violates the gauge symmetry of the Gaussian fixed point field theory. This will once again guarantee the gaplessness of the graviton mode, and we see from the above Hamiltonian that $\omega \sim k^2$.

5.3.3 Additional constraints

For $n \geq 2$, the rank-n theories have additional structure because they can accommodate several types of local constraints. Interestingly, we can also enforce more than one local constraint simultaneously. For example, we can take the above theory and additionally require that

$$E = \sum_i E_{ii} = 0 \tag{5.14}$$

This generates the gauge transformation

$$A_{ij} \rightarrow A_{ij} + \delta_{ij}\lambda \tag{5.15}$$

We now ask a modified question - what is the simplest theory that is invariant under the gauge transformations generated by constraints (5.9) and (5.14) simultaneously? We see that our definition of B_{ij} in Eq. (5.11) is not good enough. However, we can use the quantity $B = \sum_i B_{ii}$ to define a new tensor:

$$Q_{ij} = \epsilon_{ikl}\partial_k \left(B_{jl} - \frac{1}{2}\delta_{jl}B \right), \tag{5.16}$$

which is invariant under both gauge transformations. The new effective low energy Hamiltonian is now [173]:

$$H = U\sum_r E_{ij}^2 + K\sum_r Q_{ij}^2. \tag{5.17}$$

The new dual fields are defined in the same way, where E and h have the same functional relation as Q and A [173]. Thus this theory is again "self-dual" with identical gauge symmetries on the two sides of the duality. This theory (Eq. 5.17) is again gapless, though it has a different dispersion: because there are now three spatial derivatives of A in the Q tensor Eq. 5.16, the dispersion of the low energy excitation is $\omega \sim k^3$.

5.4 General Procedure

To generalize these arguments to higher rank tensor fields, we need to first establish which types of gauge transformations will be allowed. To simplify our discussion, we want the field theory to be rotationally symmetric, though it is possible that the lattice regularization may possess irrelevant rotation-breaking terms. Additionally, the gauge constraint should depend only on $E_{ijk...}$ and no other locally defined tensor fields.

These two requirements restrict the constraints that we will consider to higher-dimensional versions of the Gauss law and traceless conditions. These constraints are "rotationally" symmetric in the correct way to respect lattice symmetries (again, we stress that the states we construct should be insensitive to weak lattice symmetry breaking). We enumerate the allowed gauge transformations in Table 1 for rank one through three. To simplify notation, we denote the symmetrizing operation

$$T_{(ijk)} = \frac{1}{3!}\left(T_{ijk} + T_{jik} + sym\right) \tag{5.18}$$

An important generic question is the number of gapless modes in the system. This is determined by switching to a Lagrangian formulation and thinking of the λ tensor as a Lagrange multiplier. Each degree of freedom of λ will reduce the number of gapless modes

Rank of theory	Local constraint	Gauge transformation
$n = 1$	$\partial_i E_i = 0$	$A_i \to A_i + \partial_i \lambda$
$n = 2$	$\partial_i E_{ij} = 0$ $\partial_i \partial_j E_{ij} = 0$ $E_{ii} = 0$	$A_{ij} \to A_{ij} + \partial_{(i}\lambda_{j)}$ $A_{ij} \to A_{ij} + \partial_i \partial_j \lambda$ $A_{ij} \to A_{ij} + \delta_{ij}\lambda$
$n = 3$	$\partial_i E_{ijk} = 0$ $\partial_i \partial_j E_{ijk} = 0$ $\partial_i \partial_j \partial_k E_{ijk} = 0$ $\delta_{ij} E_{ijk} = 0$ $\delta_{ij}\partial_k E_{ijk} = 0$	$A_{ijk} \to A_{ijk} + \partial_{(i}\lambda_{jk)}{}^*$ $A_{ijk} \to A_{ijk} + \partial_{(i}\partial_j \lambda_{k)}$ $A_{ijk} \to A_{ijk} + \partial_i \partial_j \partial_k \lambda$ $A_{ijk} \to A_{ijk} + \delta_{(ij}\lambda_{k)}{}^*$ $A_{ijk} \to A_{ijk} + \delta_{(ij}\partial_{k)}\lambda$

Table 5.1: Allowed gauge transformations which are rotationally invariant and do not depend on an auxiliary tensor field.
* These gauge transformations are not totally independent - λ_{jk} should be made traceless.

by one (though there is a subtlety to this counting, which is detailed in the appendix). For example, for the familiar photon phase,

$$L_1 = E^2 - B^2 + \lambda(\partial_i E_i). \tag{5.19}$$

E_i has three components initially, so the one free component of a scalar λ reduces the number of gapless modes to the familiar two of the photon. For higher rank cases, though it quickly becomes tedious to count the number of free components of an arbitrary rank symmetric tensor, the idea is straightforward. Indeed, it is also possible to diagonalize the Hamiltonian directly, and this reproduces the previous results.

The essential component of many ABL theories is the process by which gap-opening perturbations are prohibited. Generically, any relevant term in the Lagrangian should open a gap, and so to eliminate *all* such terms places strict requirements on the theory. In the theories we consider in this paper, we use gauge-invariance and self-duality to protect the photon gap from perturbations at a Gaussian IR fixed point, just like the examples reviewed in the previous section.

The gauge structure in all of the theories we consider is emergent in the IR. Indeed, it is due to a constraint on the low-energy Hilbert space of the microscopic model. This means that the gapless phase is not stable to arbitrarily strong perturbations, since moving out of the constrained subspace generically destroys the gauge structure. As an example, consider the gauge charge excitation in the rank-1 theory. The low-energy subspace is that of the charge vacuum, but if we tune the charge gap to zero, the gauge charges condense and gap out the gauge boson through the Higgs mechanism.

Additionally, the gauge structure will constrain the form of the Hamiltonian. As we have seen above, we want to use A to construct two gauge-invariant tensors E and B which play the usual roles in electromagnetism. Given that there is a direct relation between the gauge transformations on A and the constraints on E, it is a straightforward task to build the most relevant terms. In this case, "most relevant" means that B has the fewest number of spatial derivatives of A, but it must be gauge invariant still.

Just to limit the variety of states, we require rotational invariance in this paper, which also constrains the form of the Hamiltonian (as does gauge invariance). But we want to stress that weakly breaking the rotational invariance will not destroy the states we construct, namely it will not gap out the bosonic modes of the ABL phase. For example, the low energy photon excitations of the ABL phase studied in [19, 161, 162] have a rotational invariant dispersion at low energy, but we know that breaking the rotational invariance will not destroy the photon excitations. The local gauge constraints are similarly influenced by the requirement of rotational invariance, as was noted above. We can then consider tensor representations of rotational group $SO(3)$, and it turns out that we will only be interested in the symmetric pieces.

For example, to construct the gauge invariant rank-3 magnetic field Q_{ijk} with both a derivative constraint and a trace constraint on E_{ijk}, the resulting theories will involve B_{ijk} which is a 3-curl of A and $B_k = B_{iik}$. Because Q_{ijk} carries three vector indices, it can be constructed with three vector representation of $SO(3)$. The standard expansion of a tensor defined over three copies of the fundamental representation of $SO(3)$ is

$$1 \otimes 1 \otimes 1 = 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 0$$

The spin-2 and spin-0 pieces here are antisymmetric in at least two indices. Requiring overall symmetrization will reduce the expansion to a fully symmetric spin-3 part $T_{(ijk)}$ and a symmetric spin-1 part $T'_{ijk} = \delta_{(ij}T_{k)}$. Thus, we can understand connection between the allowed constraints and how they will involve traces of the curls of A by considering which parts of the tensor representation are symmetric.

However, as was noted previously, gauge structure is not enough to guarantee the gaplessness of the photon. In 2+1d this manifests as the so-called "instanton effect", which is to say that the magnetic flux insertion operator is always relevant at the Gaussian fixed point. Thus, the instantons proliferate and open a gap for the photons. Thus in general in our $(3+1)d$ ABLs, we need to argue that all of the vertex operators that generically take the form $\cos(2\pi N h_{\alpha\beta\ldots})$ for the dual gauge field h are irrelevant.

5.5 Examples

In this section we will discuss a few examples of new ABL phases. The first example is similar to the graviton theories detailed in the previous section, except that it has a different local constraint. This is an interesting property of rank-n theories for $n \geq 2$ which greatly enhances the variety of gapless gauge theories. There are roughly n different constraints involving only derivatives for a given rank-n theory in addition to the various types of traceless conditions.

The original graviton model had as its local constraint Eq. 5.9. We can instead contract another derivative on E_{ij} to get a different theory:

$$\partial_i \partial_j E_{ij} = 0 \tag{5.20}$$

Compared to the theory governed by Eq. 5.9, this theory has a scalar (as opposed to vector) charge and has five total degrees of freedom (up from three). Even before determining the simplest possible Hamiltonian, we see that the gapless excitations are distinct in character from the original gravitons:

$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \lambda \tag{5.21}$$

We can construct the Hamiltonian of this ABL state using the following symmetrized gauge invariant tensor field B:

$$B_{ij} = \frac{1}{2} \left(\epsilon_{iab} \partial_a A_{bj} + \epsilon_{jcd} \partial_c A_{id} \right). \tag{5.22}$$

The corresponding low energy Hamiltonian again takes the schematic form of $E^2 + B^2$, as before, and this theory is again self-dual, but it now has a linear dispersion $\omega \sim k$.

We can also consider enforcing the constraint Eq. (5.14) in addition to Eq. (5.20). However, in this case, B_{ij} given by Eq. (5.22) is already invariant under both gauge transformations. In fact, in conjunction with the graviton theory discussed previously, we have now characterized *all* rank-2 symmetric gauge theories whose gauge transformations satisfy our criteria above.

While the rank-1 and rank-2 systems have nice interpretations as “photons" and “gravitons" due to the familiarity with known systems, there is no such nice identification for the rank-3 case. We cannot leverage any analogy to linearized gravity nor electromagnetism, and instead we will proceed using our general method.

To illustrate this case, we consider two canonically conjugate symmetric rank-3 tensor fields A_{ijk} and E_{ijk} where A is defined modulo 2π. We then impose the local constraint

$$\partial_i E_{ijk} = 0 \tag{5.23}$$

which generates the gauge transformation

$$A_{ijk} \to A_{ijk} + \partial_{(i}\lambda_{jk)} \tag{5.24}$$

The corresponding lattice system is given in the appendix. As before, we seek a "magnetic" field that is gauge-invariant and of lowest number of derivatives of A. Additionally, it should be symmetric. We see that

$$B_{ijk} = \epsilon_{iab}\epsilon_{jcd}\epsilon_{kef}\partial_a\partial_c\partial_e A_{bdf} \tag{5.25}$$

is the simplest tensor that fits the requirements. From this tensor we can construct a state with the following low energy effective Hamiltonian

$$H = U\sum_r E^2 + K\sum_r B^2, \tag{5.26}$$

where the coefficients of the $\sum X_{iii}^2$, $\sum X_{iij}^2$, and $\sum X_{ijk}^2$ terms may in general be different. This system is self-dual in the same way as before, by defining the dual variable h_{ijk} as

$$E_{ijk} = \epsilon_{iab}\epsilon_{jcd}\epsilon_{kef}\partial_a\partial_c\partial_e h_{bdf} \tag{5.27}$$

and requiring that h_{ijk} transform in the same way as A_{ijk} under a change of gauge. The vertex operators of the dual variables $\cos(2\pi N h_{ijk})$ are easily seen to be gauge dependent, and thus irrelevant in the same way as before. This is a new gapless Bose liquid with $\omega \sim k^3$ with four independent modes for each momentum $\vec{k}$.

We can then ask what happens when another local constraint is imposed.

$$\delta_{ij} E_{ijk} = 0. \tag{5.28}$$

This constraint gives rise to the gauge transformation

$$A_{ijk} \to A_{ijk} + \delta_{(ij}\lambda_{k)}, \tag{5.29}$$

which provides a nice example of the "mode overcounting" discussed in the appendix. In particular, the above constraint gives rise to new physics only when the 1-form field λ_k is not exact, *i.e.* $\lambda_k \neq \partial_k \Gamma$. If λ_k is a total derivative of some scalar function, then this constraint Eq. 5.28 is not independent of the transformation Eq. 5.24 and the system as described by B_{ijk} given before in Eq. 5.25 is invariant under both.

If $\lambda_k \neq \partial_k \Gamma$, then we have to construct a new "magnetic field" that is invariant under both gauge transformations. To do so, we need to define two quantities:

$$D_{ij} = \delta_{ij}\partial^2 - \partial_i \partial_j \tag{5.30}$$

$$B_k = B_{iik}. \tag{5.31}$$

Using these quantities, the new low energy effective Hamiltonian is schematically $E^2 + Q^2$ where we have defined

$$Q_{ijk} = \partial^2 B_{ijk} - \frac{3}{4} D_{(ij} B_{k)}. \tag{5.32}$$

This theory has a rather soft dispersion $\omega \sim k^5$, and only one single mode at each momentum $\vec{k}$. And just like all the examples before, this theory is also self-dual.

Continuing this procedure to higher rank theories generates an entire infinite family of ABL phases. The procedure is exactly the same, though the precise enumeration of possible gauge transformations (and, indeed, even the number of degrees of freedom) becomes tedious quickly. However, by leveraging the gauge structure in addition to the self-duality at the IR fixed point, we are able to in all cases derive the appropriate low-energy effective Hamiltonian for a given local constraint.

5.6 Topological Order

The $U(1)$ spin liquid in $3+1d$, in addition to its stability, also possesses a curious type of topological order, which was discussed in detail in Ref. [19]. When the system is put on a three dimensional torus T^3 with size L^3, it is possible to thread electric flux around each of the noncontractible loops. The flux integrals each commute with the low-energy Hamiltonian and each other, so they constitute constants of motion. When the flux spreads out over the whole thermodynamically large system, the energy cost goes to zero as $1/L$. An identical picture holds for the magnetic flux, so topological order is characterized by six integers. The system is stable due to the gap in both electric and magnetic charges, which makes it exponentially unlikely for a "particle-hole" pair to be created and propagate all the way around the torus to change the flux.

There is a similar construction for the graviton ABL discussed in Ref. [170, 160]. However, one must be more careful in the selection of which fluxes of E_{ij} are used. In principle, there are twenty seven different fluxes - three orientations of the flux surface and nine components of E_{ij}. Upon calculation, one can show that fifteen of these are zero, and of the remaining twelve only nine are independent. The same result holds for the magnetic fluxes, meaning that the graviton ABL has topological order characterized by eighteen integers. It is similar to the $U(1)$ case in that the ground states are split by $1/L$ and are exponentially unlikely to mix.

We claim that similar arguments hold for the whole infinite family of theories constructed in the previous sections with only derivative constraints. The topological order is characterized by $6k$ integers corresponding to the electric and magnetic fluxes, where k the number of independent components of the charge tensor. Since the Hamiltonian densities for these theories are generically E^2+B^2, we expect that in all cases the ground state degeneracy closes as $1/L$ in the thermodynamic limit.

To understand the origin of the $6k$, we consider a generic local constraint written as

$$(\partial_i \hat{E}_{ijk\ldots} - \hat{\rho}_{jk\ldots})|Phys\rangle = 0 \tag{5.33}$$

It is natural to interpret the violations of the local constraints as "charges." The particular choice of contraint endows the charges with a tensor structure, and the underlying symmetry of $E_{ijk\ldots}$ is reflected in that structure. Going back to the lattice model, these charges can also be thought of as the open ends of strings. The constraint is then interpreted as the condition that strings do not end on sites. Due to the all-important electromagnetic duality protecting these phases, there is a corresponding magnetic charge tensor with exactly the same structure as the electric charge.

Using these charge tensors, we can then create "particle-hole" pairs of a given type of charge and wind them around a noncontractible loop of T^3. Three dimensions times two species of charge gives the factor of six, and there are k independent charges depending on the particular constraint. We see that $k=1$ for the ordinary QED, while $k=3$ for the graviton ABL which has a vector charge.

To extend these ideas to constraints with more derivatives, we see that the constraint $\partial_i\partial_j E_{ij}=0$ can be rewritten as $\partial_i F_i = 0$ for a vector field $F_i = \partial_j E_{ij}$. This constraint has a scalar charge, and it can be shown easily that the fluxes of F_i commute with each other and the Hamiltonian. This extra step is needed to invoke the divergence theorem, since we need to work with the divergence of a vector field. Importantly, the characterization is still the same - since the underlying charge is a scalar, this theory is characterized by six winding numbers.

Finally, we consider the second type of constraint detailed above, such as $\delta_{ij}E_{ij} = 0$. In the rank-2 case, we imagine threading a flux of E_{xx} around the noncontractible loop in the x-direction while simultaneously threading a flux of E_{yy} in the y-direction. The two "strings" involved in this threading process need not intersect, but in the ground state the fluxes spread out over the whole system. Once this occurs, we see that the traceless constraint fixes the flux of E_{zz} through the z-direction so that the three integers sum to zero. This new phase is characterized by 16 integers. Extending these constraints to higher-rank theories is straightforward but tedious, and simply removes topological degrees of freedom from the diagonal fluxes.

5.7 Summary and Discussion

In this work we have demonstrated that there is an infinite family of strongly-correlated gapless boson systems whose low-energy Hilbert space does not break any symmetries with gapless excitations stable with respect to small arbitrary perturbations. The gaplessness is protected by a combination of emergent gauge invariance (enforced by a local constraint on the low-energy Hilbert space) and a generalized electromagnetic duality. Within some limitations, the dispersion and representation of the emergent gauge boson can be tuned. Additionally, these theories have an interesting type of topological order characterized by $6k$ integers, depending on the exact underlying local constraint.

Although we have made heavy use of the gauge structure in constructing these ABL phases, we have not made a careful analysis of the associated gauge groups. Apart from the simplest $U(1)$ spin liquid, the higher-rank gauge fields are not algebra-valued 1-forms, and therefore do not fit into the standard Yang-Mills architecture.

Our rank-2 model with constraint $\partial_i E_{ij} = 0$ can potentially be thought of as $U(1) \times U(1) \times U(1)$ gauge theory after an (unusual) symmetrization between the space index and the flavor index; this provides a possible realization of our rank-2 theory starting with three copies of the photon phase. We also note that this connection between linearized gravity and the Yang-Mills gauge theory was already observed in the loop quantum gravity literature [179, 180]. Further study of these models will hopefully elucidate these connections.

Appendix A - Rank-3 Lattice Hamiltonian

For concreteness, we will present a lattice Hamiltonian for one of the rank-3 cases. In line with the previous work by Xu, this Hamiltonian has two pieces: a generic boson hopping term and a density-density repulsion term.

The unit cell for this lattice consists of a face-centered cubic lattice that also has a site at the center (see Figure 1). The boson occupation at the corners of the fcc unit cell are three-fold degenerate, labeled $n_{xxx,\vec{r}}$, $n_{yyy,\vec{r}}$, and $n_{zzz,\vec{r}}$. The faces are two-fold degenerate with labels $n_{xxy,\vec{r}+\hat{x}/2+\hat{y}/2}$, $n_{xyy,\vec{r}+\hat{x}/2+\hat{y}/2}$, and so on, and the center is labeled $n_{xyz,\vec{r}+\hat{x}/2+\hat{y}/2+\hat{z}/2}$.

The hopping term of the Hamiltonian $H = H_t + H_v$ is generic, and in principle contains all 45 exchanges. The potential takes the form for average boson density $\bar{n}$

$$H_v = H_{xx} + H_{yy} + H_{zz} + H_{xy} + H_{yz} + H_{xz} \tag{5.34}$$

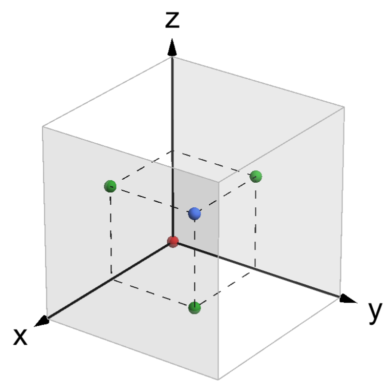

Figure 5.1: The unit cell for the simplest rank-3 model. The red site is three-fold degenerate (n_{xxx}), the green sites are two-fold degenerate (n_{xxy}) and the blue site is nondegenerate (n_{xyz}).

$$H_{xx} = V(n_{xxx,\vec{r}} + n_{xxx,\vec{r}+\hat{x}} + n_{xyx,,\vec{r}+\hat{x}/2+\hat{y}/2}+ \\ n_{xyx,\vec{r}+\hat{x}/2-\hat{y}/2} + n_{xzx,\vec{r}+\hat{x}/2+\hat{z}/2}+ \\ + n_{xzx,,\vec{r}+\hat{x}/2-\hat{z}/2} - 6\bar{n})^2 \quad (5.35)$$

$$H_{xy} = V(n_{xyx,\vec{r}+\hat{x}/2+\hat{y}/2} + n_{xyx,\vec{r}+3\hat{x}/2+\hat{y}/2}+ \\ + n_{xyy,\vec{r}+\hat{x}/2+\hat{y}/2} + n_{xyy,\vec{r}+\hat{x}/2+3\hat{y}/2}+ \\ + n_{xyz,\vec{r}+\hat{x}/2+\hat{y}/2+\hat{z}/2} + n_{xyz,\vec{r}+\hat{x}/2+\hat{y}/2+3\hat{z}/2} - 6\bar{n})^2 \quad (5.36)$$

with similar expressions for the other four terms. We define $E_{ijk} = (-1)^{\vec{r}}(n_{ijk} - \bar{n})$ and use the usual lattice derivative to see that the low-energy subspace of this Hamiltonian has the local constraint $\partial_i E_{ijk} = 0$.

Appendix B - Mode Overcounting

There is a subtle point that needs to be addressed about the definitions of the above gauge transformations for rank greater than or equal to three. In particular, the two gauge transformations below are not totally independent:

$$A_{ijk} \rightarrow A_{ijk} + \partial_{(i}\lambda_{jk)} \tag{5.37}$$

$$A_{ijk} \rightarrow A_{ijk} + \delta_{(ij}\partial_{k)}\lambda \tag{5.38}$$

The first contains the second as a special case. This is understood in terms of tensor representations of $SO(3)$ by noting that a symmetric rank-2 tensor (six degrees of freedom) has a single scalar trace mode in addition to the five spin-2 modes. As such, a more correct accounting would require tracelessness of λ_{ij}, which is achieved via

$$\tilde{\lambda}_{ij} = \lambda_{ij} - \frac{1}{3}\delta_{ij}\lambda_{kk} \tag{5.39}$$

This type of overcounting of trace modes persists into higher rank, and becomes increasingly complicated as the number of trace modes increases.

Chapter 6

Gapless Topological Order

6.1 Pyrochlores to Black Holes

In the previous chapter, we investigated several models very similar to electromagnetism and gravity. These models supported a curious type of topological order, where they are degenerate on a torus but the ground states only close as $1/L$ with the "true" ground state. Moreover, since there are gauge bosons in the system with arbitrarily small energies (also $1/L$), there has been long-standing debate as to whether or not these models are topologically ordered.

A very similar type of degeneracy was identified in the high-energy literature, in the context of asymptotic symmetries of (real) electromagnetism and gravity at conformal infinity. These symmetries were shown to be intimately related to Weinberg's soft theorems, and give rise to so-called "soft hair" on black holes. This is due to a particular set of boundary conditions at conformal infinity which mimic the periodic boundary conditions in condensed matter systems. By considering the similarities (along with new work on higher-form symmetries), we can use insights from black holes to explain questions in the pyrochlores and vice-versa.

6.2 Gapless Topological Order

The black hole information paradox, which calls into question the fate of information falling into a black hole, has led to considerable work on the entanglement structure of quantum gravity theories. Central to this paradox is the statement that black holes have no hair, which is known to hold classically and was initially thought to hold quantum mechanically as well [181]. However, recent work has shown that both the electromagnetic field and the gravitational field contain "soft" hair [182]. This hair comes in the form of soft (zero-energy) bosons which have long been known to exist in the zero-k limit of these theories [183].

A key step in the identification of soft bosons with information-carrying soft hair is finding the corresponding large-scale time-dependent symmetry, as classical electromagnetic and gravitational theories obey the no-hair theorem in the steady state [184]. In the case of gravitation the classical symmetry group is known to be that of Bondi, Metzner, and Sachs, and the corresponding electromagnetic symmetry is similar in structure [185, 182].

In addition, the soft photon and graviton theorems have played an important role in the relation between symmetries and quantum memories. This results in the so-called "triangle" that relates the soft boson theorems to large gauge symmetries and memories, with deep connections to the Ward identities for those gauge theories[186, 187, 188, 189, 190]

Of particular interest is the connection between these ideas and the notion topoligcal order. Ordinarily, topological order manifests by the existence of global modes which 'wrap' around the system and which are only accessible by means of gapped excitations. In this way such topological modes encode protected quantum information. Recently it has come to light that gauge theories with similar structure to electromagnetism, gravitation, and higher-order equivalents generically exhibit a peculiar variant of this phenomenon [19, 160, 191, 170, 192]. The peculiarity stems from the fact that these gauge theories have a stable, deconfined IR Gaussian fixed point, and thus have exactly gapless gauge bosons in the spectrum. Nevertheless, their ground states are degenerate on a torus and indistinguishable by local operators. As such they exhibit protected topological charges which, in contrast to more typical systems, are protected by large-scale gauge symmetries rather than by an energy gap.

In this work we show that these two observations are intimately related: the soft hair which [182] discovered corresponds to topological zero-modes which live on the boundaries of our low-energy phase of spacetime, be they at infinity or at the horizon of a black hole. Equivalently, states with different numbers of soft bosons correspond to different topological sectors. As a result, the degeneracy of the gravitational vacuum is really a reflection of the underlying "gapless topological order" of gravitation and electromagnetism. Notably this result holds even though spacetime at a glance is simply connected, and we show that this is a direct consequence of the metric signature and gapless nature of soft modes.

This work also yields a possible resolution to the firewall paradox of [193]. Outgoing Hawking radiation can be entangled initially with its infalling counterpart, but upon interaction with the soft sector at the horizon loses this entanglement. This interaction is required by the correspondence between the soft sector and flux integrals that can be performed around the black hole. As a result it is equivalence, postulate (4) of [193], which is violated. Interestingly this violation is purely quantum mechanical, as it relies on scattering with the soft sector, which cannot be detected except via entanglement measurements. In this way the classical equivalence principle is preserved.

This paper is organized as follows. In section II, we review the constructions of lattice QED and lattice linearized gravity. In section III, we analyze the topological winding procedure and provide a much more concrete description of topological degeneracy in gapless systems. In section IV, we demonstrate how matter falling into a black hole can be seen as changing topological sectors. The remainder of the paper analyzes some subtleties of these phenomena and speculates on the implications of the structure of spacetime and the information paradox.

6.3 Gapless Topological Order and Deconfined Gauge Theories

The first system exhibiting what we call gapless topological order[1] was the $U(1)$ spin liquid on the pyrochlore lattice[19]. The gapless excitation is a "photon" for an emergent $U(1)$ gauge symmetry, and the spinons carry electric charges. The ground state degeneracy was shown to be manifold-dependent, and argued to be stable in the presence of a spinon gap and infinite system size. However, the ground state degeneracy only closed with the ground state as $1/L$, which have the same energy as the lowest-lying photon states. Later works[160, 191, 170, 192] found similar topological degeneracy in stable gapless phases – all of which are gauge theories.

As we will see below, these degenerate ground states should instead be identified with the "soft" gauge bosons. The operator that inserts soft bosons will be shown to be the same that moves between degenerate ground states.

Importantly, not all stable gapless systems inherit this structure. Systems with spontaneously broken 0-form[2] symmetries, such as superfluids, lack the gauge structure necessary for constructing the topological sectors. Systems with gapless matter, such as Weyl semimetals, are also excluded even if there is a gauge structure. In this second

[1] Note that this is a distinct phenomenon from "quasi-topological order" [30].
[2] I.e. the symmetry acts on point-like objects

case, the photon may still be stable but the topological sectors will not be protected by the charge gap.

Furthermore systems without the appropriate gauge structure may exhibit power-law splitting and local indistinguishability but have no low-lying modes which are sensitive to the topology of the system. Thus, for instance, modes which are localized on a scale $L^{1/2}$ may be gapless in this sense and may be locally indistinguishable yet not be global and hence not provide topological charge.

In section III we discuss the idea of spontaneously broken higher-form symmetries, which connect the gauge structure to topological sectors and Wilson lines. This provides a unified way to understand topological order in gauge theories with both discrete and continuous gauge groups, but may not be sufficiently general to characterize all topologically ordered phases.

To explicitly draw a connection between the deconfined gauge theories and the gravitational ground state, we first review the constructions of two relevant lattice systems - electromagnetism and linearized gravity. In particular, we stress that these models can be built from local bosonic degrees of freedom on a lattice, and that the corresponding emergent gauge theories exist at exactly stable IR (continuum) fixed points.

By emergent gauge theory, we mean that the theory has a low-energy Hilbert space with local constraints $\hat{Q}(x)$, all of which commute with the Hamiltonian and each other. For example, this can happen for an easy-axis Heisenberg model on the cubic lattice when typical energies are smaller than the exchange coupling[19]. Physical states *in this reduced Hilbert space*, i.e. that with such low energies, are closed under these operators, which is to say that

$$\hat{Q}(x) \left|\mathrm{Phys}\right\rangle = 0. \tag{6.1}$$

Closure under $\hat{Q}(x)$ generates corresponding local conservation laws. The model then

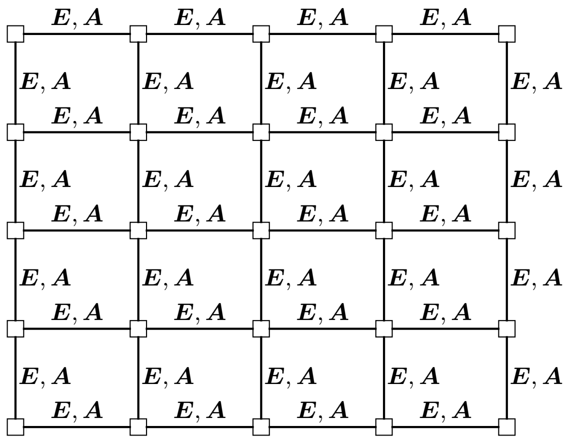

Figure 6.1: A two-dimensional slice of the lattice version of our system is shown. Only a 5×5 sublattice is shown, but the system may arbitrarily large in each dimension. Likewise the system shown may have open (as shown) or topologically non-trivial boundaries. Note that the conjugate vector fields $\boldsymbol{E}$ and $\boldsymbol{A}$ live on the links in the lattice, and for clarity these are only shown in the open boundary case.

becomes a gauge theory when we identify the physical low-energy states that differ only by a gauge transformation. We can then write an effective low-energy field theory in terms of the gauge field for these constraints at the IR fixed point.

6.3.1 Electromagnetism

First, we review the simplest case of gapless topological order – ordinary QED in $3+1d$ Following [19], this quantum theory has a compact $U(1)$ gauge group with two canonically conjugate vector fields $\boldsymbol{E}_i \in \mathbb{Z}$ and $\boldsymbol{A}_i \in [0, 2\pi)$ (along with the corresponding operators $\hat{E}_i$ and $\hat{A}_i$) that live on the links of a cubic lattice as shown in Figure 6.1. This model can be derived starting from the Heisenberg model and introducing easy-axis anisotropy. Because charged excitations are gapped, the low-energy Hilbert space has a local conservation law

$$\partial_i \hat{E}_i \left|\text{Phys}\right\rangle = 0, \tag{6.2}$$

which just follows from Gauss's law. By considering the commutator $[\hat{A}_i(x), \hat{E}_j(z)] = i\delta_{ij}\delta(x-z)$ and a local phase rotation $\exp \int d^d x \lambda(x) \partial_i \hat{E}_i(x)$, we see that $\hat{A}$ is shifted by

$$\hat{A}_i \to \hat{A}_i + \partial_i \lambda, \tag{6.3}$$

where the derivative operator acts as a finite difference on the lattice. After taking the spin wave limit[19], we see that the most relevant terms in the low-energy effective Hamiltonian (acting on this reduced Hilbert space) are

$$\hat{H} = \frac{U}{2} \sum_i \hat{E}_i^2 + K \sum_i \hat{B}_i^2 \tag{6.4}$$

where we have defined $\hat{B}_i = \epsilon_{ijk} \partial_j \hat{A}_k$ as the usual curvature, and U and K depend on the microscopic couplings.

In the following sections, it will be useful to think of the local constraint (and its accompanying gauge transformation) as the essential component of the field theory. It is argued by [19] that the IR fixed point defined by this gauge theory is completely stable, and that all other terms are irrelevant in the renormalization group sense. Thus the actual lattice realization of this theory is not enormously important, provided that this local constraint is enforced in the IR.

Gauge-charged matter in the theory show up as defects of this conservation law. This follows from the Gauss constraint

$$\left(\partial_i \hat{E}_i - \hat{\rho}\right) |\text{Phys}\rangle = 0 \tag{6.5}$$

which enlarges the original gauge constraint to include charged matter. We note that the tensor form of $\hat{\rho}$ is determined by the constraint. Furthermore, the energy gap of the charged matter (i.e., the mass of the spinons) has to be large compared to other scales to enforce the constraint.

A simple, but insufficiently general, argument to demonstrate a topological degeneracy starts by putting the system on the three dimensional torus T^3. Then, we create a charge-anticharge pair and propagate them around a non-contractible loop of the torus. This threads a single electric flux, which can spread out over the whole system uniformly and thus has total energy that goes to zero as $1/L$ (for system size $\sim L$). Without loss of generality we can assume the winding direction (and thus flux) is perpendicular to a surface Σ with normal vector in the i direction. Then, the topological sector is determined by flux integrals

$$\hat{\Phi}_i = \int_\Sigma dS_i \hat{E}_i = \left(\int_\Sigma \star F \right)_i , \tag{6.6}$$

where the index i is not summed over. These integrals compute the electric flux through a surface Σ perpendicular to the components of E_i. Since the charges of E_i are labeled by integers, the fluxes are also integers. The flux integrals commute with each other and the Hamiltonian

$$\left[\hat{\Phi}_i, \hat{H} \right] = 0 \tag{6.7}$$

Thus, the ground states are labeled by three integers, corresponding to the eigenvalues of these electric flux integrals. Because $\hat{A}$ is compact, we should also include monopoles in the spectrum on the lattice [19], which have a corresponding interpretation in the continuum [189]. However, this only expands the number of topological sectors by adding three integers corresponding to the magnetic flux winding, and is not essential to our results.

This argument holds when the system lives on T^3, but runs into several problems when the system is put on, say, a solid torus (one periodic dimension and two open dimensions). This difficulty is addressed in Section III.

6.3.2 Linearized Gravity

To discuss gravity as a gauge theory requires leaving Yang-Mills behind. First, the gravitational gauge group is non-compact due to the four translations, and we must be careful about gauging the local rotations of the frame fields. Second, the gauge field is no longer an algebra-valued 1-form field, but instead a symmetric 2-tensor. This means that the gauge-charged matter carries a Lorentz index instead of a color index:

$$\left(\partial_\mu \hat{T}_{\mu\nu} - \hat{\rho}_\nu\right) |\text{Phys}\rangle = 0. \tag{6.8}$$

Since the stress-energy tensor $\hat{T}_{\mu\nu}$ is the generator of translations, we identify the gauge charge as the momentum carried by an excitation. This identification is valid in the linear regime where gravitons do not couple to one another and hence cannot themselves carry gauge charge.

We want to draw an explicit connection to a lattice model, so first we will do a partial gauge fixing and then a linear approximation. The first step is to foliate spacetime in a timelike direction using the ADM formalism[194]. This is a partial gauge fixing of the full gauge group (in particular, we use the synchronous gauge), and the new dynamical variables are the symmetric 2-tensor spatial metric $A_{ij} \in [0, 2\pi)$ on each slice and its conjugate $E_{ij} \in \mathbb{Z}$ (the stress tensor), along with the corresponding operators $\hat{A}_{ij}$ and $\hat{E}_{ij}$. We can then linearize this theory, considering only small fluctuations around the background metric.

The lattice bosonic rotor model we consider [171] reproduces these variables with $\hat{E}_{xx}$, $\hat{E}_{yy}$, and $\hat{E}_{zz}$ (along with their conjugate $\hat{A}$) living on each vertex of a cubic lattice, while $\hat{E}_{xy}$ and similar living on the faces. Then we set up the Hamiltonian to enforce the following constraints in the low-energy:

$$\partial_i \hat{E}_{ij} \left|\text{Phys}\right\rangle = 0 \tag{6.9a}$$

$$\left(\delta_{ij}\partial^2 - \partial_i\partial_j\right) \hat{A}_{ij} \left|\text{Phys}\right\rangle = 0 \tag{6.9b}$$

$$\hat{A}_{ij} \to \hat{A}_{ij} + \partial_{(i}\lambda_{j)} \tag{6.10a}$$

$$\hat{E}_{ij} \to \hat{E}_{ij} + \left(\delta_{ij}\partial^2 - \partial_i\partial_j\right)\phi \tag{6.10b}$$

where Latin indices run over space while Greek run over spacetime, and $S_{(ij)}$ denotes symmetrization.

The local constraint Eq. 6.9a is actually three constraints, one for each of the three components labeled by j. These are the zero-momentum constraints on the ground state. Starting instead from the local $SU(2)$ invariance of the frame fields and linearizing, one might conclude that it is a $U(1)\times U(1)\times U(1)$ gauge theory [179]. This is correct, but the three $U(1)$'s are not independent - they rotate into each other under a spatial rotation. This follows from the fact that the charge ρ_j is a vector, though care must be taken when making this identification[3]. This holds even if the background is not flat because the gauge constraint is by definition local.

We see that the curvature tensor is just

$$\hat{R}_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c\hat{A}_{bd} \tag{6.11}$$

and so we are able to write the low-energy effective Lagrangian (after enforcing the

[3]Formally this amounts to promoting the crystal symmetries to the rotation group in three dimensions.

constraints) as

$$\mathcal{L} = E_{ij}\dot{A}_{ij} - \frac{J}{2}\left(E_{ij}^2 - \frac{1}{2}E_{ii}^2\right) - \frac{g}{2}A_{ij}R_{ij}. \tag{6.12}$$

This is the Lagrangian for a spin-2 linearly-dispersing excitation, which we will call a graviton. It can be shown to arise from a purely local bosonic lattice Hamiltonian [171], and the couplings J and g depend on the microscopics. Much like the lattice model and corresponding field theory for electromagnetism, this model for linearized ADM gravity exists at an exactly stable IR fixed point, provided that the low-energy subspace enforces the gauge constraints. We note in passing that this model appears to have a Chern–Simons-like term $\hat{A}_{ij}\hat{R}_{ij}$ which is only gauge-invariant up to boundary terms, but it will not modify the $3+1d$ topological properties of the model.

Gapless topological order is present in linearized ADM gravity, and the argument follows precisely as in QED. If the system exists on T^3, one can thread a charge $\hat{\rho}_j$ around a non-contractible loop and annihilate it with an anticharge. This leaves a flux of $\hat{E}_{ij}$ around the loop, which has energy density scaling as $1/L$. As before, the flux is perpendicular to the surface Σ and in the direction of i. The new flux integrals are

$$\hat{\Phi}_{ij} = \int_{\Sigma} dS_i \hat{E}_{ij}, \tag{6.13}$$

where once more there is no summation over i. These commute with each other and with the Hamiltonian, so we see that they characterize the gapless topological order of the ground state. Moreover, there are three such fluxes for each surface - in the definition above, the surface is defined by the vector index i while the index j is free. Thus, there are nine integers that characterize the ground state in the “electric” sector.

There is also a contribution from the “magnetic” sector due to the compactness of the gauge field $\hat{A}$; however, it is unimportant to the analysis as the electric sector already guarantees a degeneracy. Moreover, the physicality of such linearized metric monopoles is difficult to justify in the full continuum theory.

6.3.3 Entanglement Entropy

A final point worth noting pertains to the entanglement structure of these theories. In gapped systems, there is a universal constant term in the entanglement entropy across an arbitrary cut through the system. This term characterizes the topological order [106], and is constant because it reflects the charge-winding freedom and is hence independent of system size.

In gapless systems by contrast there is instead a universal coefficient of a (subleading) logarithmic term [195, 196, 197, 195] due to both the (gapless) topological order and the photon. This topological piece can be derived using the Bisognano-Wichmann theorem and charge conservation on the entanglement cut. Intuitively the logarithmic scaling in system size arises because the spectrum near the ground state sector consists of a power-law of states, and so below any given cutoff the number of accessible states is a power-law. The entropy is just the logarithm of that and hence is logarithmic in system size.

In the electromagnetic case the entanglement entropy, including the non-universal area-law part, is

$$S^{U(1)} = \alpha L^{d-1} + \left(\gamma_{top}^{U(1)} + \gamma_{photon}^{U(1)}\right) \log L \tag{6.14}$$

where $\gamma_{top}^{U(1)} = (d-1)/2$ for space dimension d. Likewise in the gravitational case the arguments in [195] permit us to calculate the universal coefficient of the $\log L$ term coming from topological order. Since the charge is a d-dimensional vector and each component is independently conserved, one finds that

$$\gamma_{top}^{LG} = \frac{d(d-1)}{2} \tag{6.15}$$

which gives $\gamma_{top}^{LG} = 3$ in $3+1d$.

The similarity in entanglement entropy between the electromagnetic and gravitational cases is striking, as is their difference from the case of gapped topological order. This makes it clear that the phenomenon of gapless topological order is universal in the systems where it appears and simultaneously quite distinct from the more common notion of gapped topological order.

6.4 Topological Degeneracy, Winding Operations, and Soft Bosons

Now that we have stable lattice gauge theories with exactly gapless bosons, we want to consider the continuum limit. We argue that the fundamental objects in these theories – local constraints, gauge transformations, and global flux integrals – carry over into the full continuum theory of electromagnetism and linearized gravity. It is important to note that these connections are all made in the IR, where we expect the gauge constraints to hold - this is *not* an attempt to build a full quantum theory of gravity.

The IR stability of these gauge theories follows from the local constraint on the low-energy Hilbert space. For both of these systems (and the infinite family described in [191]), this constraint is the conservation of some tensor-valued gauge charge.

In the previous section we followed the standard arguments to construct the degenerate ground states of QED and linearized gravity on the torus by starting with the lattice models and explicitly calculating the flux integrals. Importantly, the states with nonzero flux are only degenerate in the limit of infinite system size, as the energies only go to zero as $1/L$.

However, the argument in that section depends on the topology in an awkward way, by relying on the periodicity of the perpendicular directions to the flux. For concreteness, we calculate the commutator of $\hat{\Phi}_z$ with the (continuum) QED Hamiltonian along the surface $z = 0$, which gives

$$\left[\hat{\Phi}_z, \hat{H}\right] = 2i \int dxdy\ \epsilon_{zij}\partial_i \hat{B}_j = 2i \int dxdy \left(\nabla \times \hat{\boldsymbol{B}}\right)_z. \tag{6.16}$$

Provided that $\hat{\boldsymbol{B}}$ satisfies the periodic boundary conditions

$$\hat{\boldsymbol{B}}\left(\frac{L}{2}, y, 0\right) = \hat{\boldsymbol{B}}\left(\frac{-L}{2}, y, 0\right) \tag{6.17}$$

and similarly for y, then the integral vanishes for even finite L. The generalization to linearized gravity is straightforward.

Due to the reliance on the periodic boundary conditions in the perpendicular directions, this method is unsuited to showing the existence of the topological degeneracy in the more general case with just one periodic direction. We are still able to construct the appropriate degenerate ground states however by using a winding construction adapted from the Minkowski spacetime arguments in [187, 182, 188] and similar arguments about lattice $SU(3)$ in [198].

6.4.1 Winding

We begin with open boundary conditions and consider a point charge e located at $\boldsymbol{r}_0$. This produces an electric field

$$\boldsymbol{E} = \frac{e}{4\pi}\frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}. \tag{6.18}$$

If we partition the space with a planar surface Σ as shown in Figure 6.2, the integral over this surface is easy to evaluate and yields

$$\int_\Sigma \boldsymbol{E}\cdot d\boldsymbol{\Sigma} = \pm\frac{e}{2}\int_0^\infty \frac{rh}{(h^2+r^2)^{3/2}}dr = \pm\frac{e}{2}, \tag{6.19}$$

where h is the distance from the charge to the surface, r is the distance along the surface, and the sign of the integral depends on the sense of orientation of the surface. If we now place a second charge $-e$ at $\boldsymbol{r}_1$ on the opposite side of the surface we find

$$\int_\Sigma \boldsymbol{E}\cdot d\boldsymbol{\Sigma} = \pm e. \tag{6.20}$$

Note that this result is independent of where we place the second charge, and so this integral only tells us about the total partition of charges across Σ. As such we are free to move both charges as far away from the surface as we wish, leaving a field which is asymptotically constant, as shown in Figure 6.3.

Now suppose that we wish to impose periodic boundary conditions. This may be done by "unfolding" the space and inserting periodically spaced copies of all charges, as shown in Figure 6.4. Of course this must be regularised when the system is finite, but in the limit as the system becomes infinite this procedure is correct. If we place N such

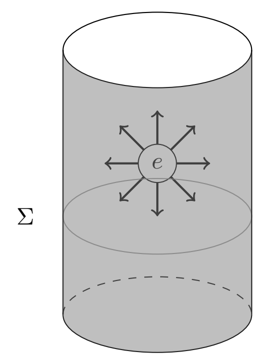

Figure 6.2: A cylindrical system is shown with one possible cut surface Σ. The charges of interest are integrals over this surface of the normal component of the electric field.

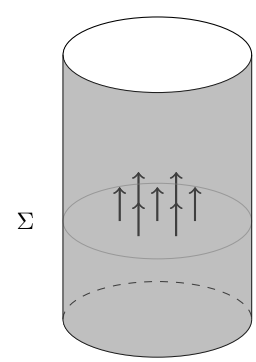

Figure 6.3: A cylindrical system is shown with one possible cut surface Σ. The charges of interest are integrals over this surface of the normal component of the electric field. In this case the hard charges $\pm e$ (not shown) have been placed at distant mirrored positions on either side of the surface such that the field here is uniform and normal to the surface.

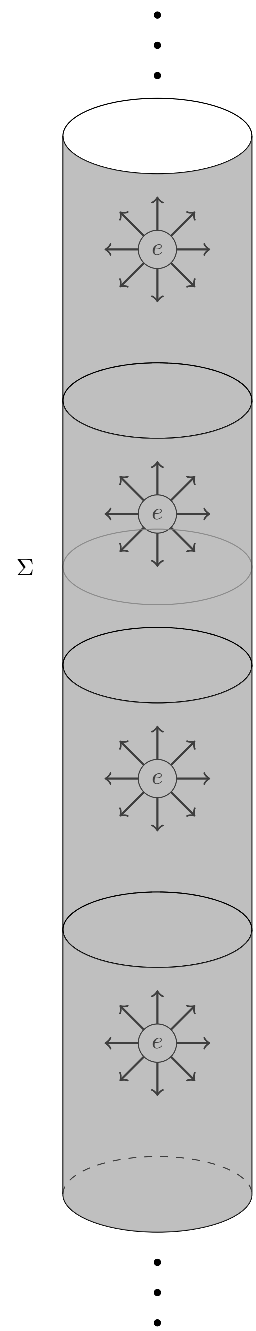

Figure 6.4: A periodic unfolding of the system shown in Figure 6.2. The system is tiled a total of N times but only the four closest to the surface Σ are shown.

pairs of charges, one from each pair on each side of the surface, the flux integral reads

$$\int_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{\Sigma} = \pm Ne. \tag{6.21}$$

By contrast suppose we begin by placing a pair of charges at their periodic locations, but both on one side of the surface. The integral will now vanish. No matter how many times we do this, the integral still vanishes. In the limit as the space becomes infinite this unfolding procedure remains perfectly well-defined, but the value of this global integral may be made to be any even integer simply by appropriate choice of the order in which it occurs. Thus while local observable like the electric field converge by this process, the integral is sensitive to the order in which we place charges and hence the physical manner in which the periodic limit is reached.

The dependence of flux integrals on the manner in which we unfold the space corresponds precisely to the topological degeneracy in the theory. This is because altering the order of placement corresponds in the periodic case to creating a dipole and winding it around the periodic dimension before destroying it. To show this, we consider Poisson's equation for our pair of point charges:

$$\nabla^2 \phi = e\left(\delta(\boldsymbol{r} - \boldsymbol{r}_0) - \delta(\boldsymbol{r} - \boldsymbol{r}_1)\right). \tag{6.22}$$

In momentum space this is

$$-k^2 \tilde{\phi} = e\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}_0} - e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1}\right). \tag{6.23}$$

As a result

$$\tilde{\phi} = -\frac{e}{k^2}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}_0} - e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1}\right), \tag{6.24}$$

so

$$\tilde{\boldsymbol{E}} = ie\frac{\boldsymbol{k}}{k^2}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}_0} - e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1}\right) \tag{6.25}$$

The flux integral in momentum space is then

$$\int_\Sigma \boldsymbol{E}\cdot d\boldsymbol{\Sigma} = -\int_\Sigma\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left(-i\boldsymbol{k}\cdot\hat{n}\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\tilde{\phi}d^2\boldsymbol{x}, \tag{6.26}$$

where $\hat{n}$ is the unit vector normal to Σ. For simplicity we may take the two dimensions parallel to Σ to be infinite, in which case

$$\int_\Sigma \boldsymbol{E}\cdot d\boldsymbol{\Sigma} = -\frac{ie}{2\pi}\int\frac{dk_n}{k_n}\left(e^{ik_n(r_{0,n}-r_n)} - e^{ik_n(r_{1,n}-r_n)}\right), \tag{6.27}$$

where the subscript n denotes the component normal to Σ. Now if the remaining direction is periodic with finite size then the integral is actually a sum:

$$\begin{aligned}\int_\Sigma \boldsymbol{E}\cdot d\boldsymbol{\Sigma} &= -e\sum_{l=1}^{\infty}\frac{i}{2\pi l}\left(e^{2\pi il(r_{0,n}-r_n)/L} - e^{2\pi il(r_{1,n}-r_n)/L}\right)\\ &\quad - i\frac{e}{L}\lim_{k\to 0}\frac{e^{ik(r_{0,n}-r_n)} - e^{ik(r_{1,n}-r_n)}}{k} \qquad (6.28)\\ &= -e\sum_{l=1}^{\infty}\frac{i}{2\pi l}\left(e^{2\pi il(r_{0,n}-r_n)/L} - e^{2\pi il(r_{1,n}-r_n)/L}\right)\\ &\quad + e\left(\frac{r_{0,n}-r_{1,n}}{L}\right). \qquad (6.29)\end{aligned}$$

The special case-handling for the $l = 0$ mode is necessary because this mode is degenerate

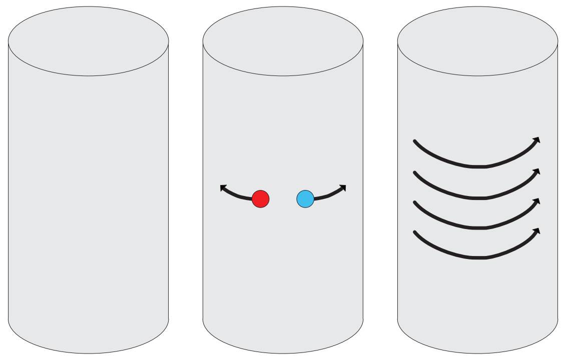

Figure 6.5: Winding charges around a cylinder leaves a static uniform electric field pointing along the winding direction. For clarity the third spatial dimension is not shown.

in equation (6.25). While any value will satisfy this component when $l = 0$ (and hence $k = 0$), we choose the value which is consistent with the limit as $k \to 0$, such that it remains well-defined and consistent with the integral formulation in the the limit as $L \to \infty$. Now if we pick $r_{0,n} = -r_{1,n} = h$ and $r_n = 0$, which we can do just by choice of Σ, then

$$\int_\Sigma \boldsymbol{E} \cdot d\boldsymbol{\Sigma} = e \left[2\frac{h}{L} + \sum_{l=1}^{\infty} \frac{1}{\pi l} \sin\left(2\pi l \frac{h}{L}\right) \right]. \tag{6.30}$$

This may be evaluated as

$$\int_\Sigma \boldsymbol{E} \cdot d\boldsymbol{\Sigma} = e \left[2\frac{h}{L} + \frac{i}{\pi} \log \frac{1 - e^{2i\pi h/L}}{1 - e^{-2i\pi h/L}} \right] = e \left[1 + 2\lfloor \frac{h}{L} \rfloor \right]. \tag{6.31}$$

As h increases the charges wind around the torus, and as this happens the flux integral increments. The offset of 1 just comes from our choice of coordinates. Note that the same argument holds when the remaining dimensions are finite.

The flux increment we see is associated with a mode with $k = 0$, which is the soft photon sector. In the high-energy context soft photons really are the vanishing-energy analogues of photons, but in the condensed matter language this is a bit of a misnomer, as it is not a photon mode but rather a large gauge transformation of the electric field. To see that the mode really is soft note that the field associated with it is a static one which, when integrated over a surface of size L^2 yields a constant value. That means the field scales as L^{-2} and so the field energy density scales as L^{-4}. Integrating over the volume gives energy scaling as L^{-1}. As $L \to \infty$ this vanishes and so the modes associated with this winding procedure are actually soft.

This winding argument clearly holds for each periodic direction, and so on T^3 we find three independent integer-valued topological charges. In a more complicated topology the number may vary. For instance, consider a sphere with a hollowed-out center, and identify points on the outer edge with points at the same angular coordinates on the inner edge. In this case the number of periodic directions scales as L^2, normalized by the UV lattice spacing. These directions may be distinguished by the flux integral

$$Q_\epsilon = \int_\Sigma \epsilon(\boldsymbol{r}) \boldsymbol{E} \cdot d\boldsymbol{\Sigma}. \tag{6.32}$$

By appropriate choice of ϵ this charge may be made sensitive to different winding directions $\hat{n}$. This just alters the modes which are selected in integrating over the surface. In this way we can decode the precise direction of each winding which has occurred.

This curious physics is strongly dimension-dependent. To understand this note that in general a theory with d spatial dimensions obeying a local flux constraint has field quanta with amplitude

$$\psi \sim \frac{1}{L^{d-1}}. \tag{6.33}$$

This is just because the flux integral over a hypersurface of area L^{d-1} must be independent of L. The energy density is then

$$\frac{dE}{dV} \sim \psi^2 \sim \frac{1}{L^{2d-2}}. \tag{6.34}$$

As a result the energy of the mode is

$$E \sim L^d \frac{dE}{dV} \sim \frac{1}{L^{d-2}}. \tag{6.35}$$

In our universe, where $d = 3$, this yields soft modes with energy scaling as $1/L$. More generally $d = 2$ is the critical dimension where the modes take on a constant energy independent of L. Below this the modes are infinitely gapped in the thermodynamic limit and so are irrelevant.

6.4.2 Soft Bosons and Topological Sectors

We now explicitly connect the soft theorems to topological ground state degeneracy. This has already been done in Minkowski spacetime [187, 182, 188] by proving that the Ward identities for the operators that detect topological sectors are equivalent to the soft photon and graviton theorems, though the degeneracy was not noted as topological in these works. As a result we only need to show that these arguments continue to hold in the condensed matter language.

First we note that the equivalence described in [187] follows from calculating the LiÃĺnard-âĂŞWiechert fields for a massive particle-antiparticle pair and examining the field behavior near null infinity. Due to the periodic boundary conditions placed on the gauge fields at null infinity, this process can be viewed as the analogue of the winding procedures described above. However, instead of leaving the massive particles at $\mathcal{I}^{\pm}_{\mp}$, in the condensed matter system these particles are annihilated.

Since the particles have annihilated, they no longer contribute to the electric flux integrals that distinguish the topological degeneracy. In the language of [187], there is no "hard" charge, and the only remaining piece is the "soft" charge left over from the winding procedure. However, this "soft" charge encodes the history of the winding process for a given periodic direction, and is identified with the threaded electric flux. This is the analogue of $\boldsymbol{\beta}$ being detectable in [182].

To make the connection between this winding process and the soft theorems explicit we must quantize the electric field, construct the soft photon operator corresponding to this winding procedure, demonstrate that its flux through Σ matches that above, and show that it commutes with the Hamiltonian. We begin by writing

$$\hat{\boldsymbol{A}} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \hat{a}_{i,\boldsymbol{k}} \boldsymbol{e}_i e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} + \text{h.c.}, \tag{6.36}$$

where i is summed over, $\boldsymbol{e}_i$ form a basis of unit vectors and $\omega = k$ with the appropriate choice of units. This allows us to write

$$\hat{\boldsymbol{E}} = \partial_t \hat{\boldsymbol{A}} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} i\omega \hat{a}_{i,\boldsymbol{k}} \boldsymbol{e}_i e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} + \text{h.c.} \tag{6.37}$$

and

$$\hat{\boldsymbol{B}} = \nabla \times \hat{\boldsymbol{A}} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \hat{a}_{i,\boldsymbol{k}} \boldsymbol{k} \times \boldsymbol{e}_i e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} + \text{h.c.}. \tag{6.38}$$

We now wish to construct the soft photon operator $W_{\hat{n}}$ which produces the field associated with winding a pair of charges around a periodic dimension of the system. As this is a static field it is described by a coherent state. This means that the operator which creates it is a displacement operator, so

$$\hat{W}^\dagger_{\hat{n}}(h) = \exp\left(e\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot\hat{n}h} - e^{-i\boldsymbol{k}\cdot\hat{n}h}}{k^2}\hat{a}^\dagger_{\hat{k},\boldsymbol{k}} + \text{h.c.}\right). \tag{6.39}$$

This indeed produces the field in equation (6.25), as

$$\langle 0|\hat{W}_{\hat{n}}(h)\hat{\boldsymbol{E}}\hat{W}^\dagger_{\hat{n}}(h)|0\rangle = e\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}i\omega\frac{\boldsymbol{k}}{k}\frac{e^{i\boldsymbol{k}\cdot\hat{n}h} - e^{-i\boldsymbol{k}\cdot\hat{n}h}}{k^2} \tag{6.40}$$

$$= ie\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\boldsymbol{k}\frac{e^{i\boldsymbol{k}\cdot\hat{n}h} - e^{-i\boldsymbol{k}\cdot\hat{n}h}}{k^2}, \tag{6.41}$$

where $|0\rangle$ is the vacuum state annihilated by $|a_{i,\boldsymbol{k}}\rangle$. It follows that the flux $\hat{W}_{\hat{n}}$ carries across Σ is the same as the classical flux in equation (6.26) when $h = L$, so this operator does in fact correspond to the winding process.

Finally to see that $\hat{W}_{\hat{n}}(L)$ is indeed a soft operator note that the Hamiltonian is given by

$$\hat{H} = \int d^3\boldsymbol{r}|\hat{\boldsymbol{E}}|^2 + |\hat{\boldsymbol{B}}|^2. \tag{6.42}$$

The magnetic component vanishes because $\boldsymbol{A} \parallel \boldsymbol{k}$ and $\boldsymbol{B} \propto \boldsymbol{k}\times\boldsymbol{A}$. The electric component may be resolved in Fourier space as

$$\int d^3\boldsymbol{r}|\hat{\boldsymbol{E}}|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}k^2\sum_{\hat{m}}\hat{a}^\dagger_{\hat{m},\boldsymbol{k}}\hat{a}_{\hat{m},\boldsymbol{k}}, \tag{6.43}$$

where $\hat{m}$ range over an orthonormal basis. Now note that

$$\hat{a}e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\alpha\hat{a}^\dagger + \alpha^*\hat{a}}(\hat{a} + \alpha) \tag{6.44}$$

so

$$\left[\hat{a}^\dagger \hat{a}, e^{-\alpha \hat{a}^\dagger + \alpha^* \hat{a}}\right] = e^{-\alpha \hat{a}^\dagger + \alpha^* \hat{a}} \left(|\alpha|^2 + \alpha \hat{a}^\dagger + \alpha^* \hat{a}\right). \tag{6.45}$$

As a result

$$\begin{aligned}\left[\hat{H}, \hat{W}_{\hat{n}}(L)\right] &= \hat{W}_{\hat{n}} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} k^2 e^2 \left|\frac{e^{i\boldsymbol{k}\cdot\hat{n}L} - e^{-i\boldsymbol{k}\cdot\hat{n}L}}{k^2}\right|^2 \\ &\quad + ek^2 \left(\frac{e^{i\boldsymbol{k}\cdot\hat{n}L} - e^{-i\boldsymbol{k}\cdot\hat{n}L}}{k^2}\left(a^\dagger_{\hat{k},\boldsymbol{k}} - a_{\hat{k},\boldsymbol{k}}\right)\right) \end{aligned} \tag{6.46}$$

$$\begin{aligned} &= \hat{W}_{\hat{n}} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} 4e^2 \frac{\sin^2(k_n L)}{k^2} \\ &\quad + 2ie\sin(k_n L)\left(\hat{a}^\dagger_{\hat{k},\boldsymbol{k}} - \hat{a}_{\hat{k},\boldsymbol{k}}\right). \end{aligned} \tag{6.47}$$

If the system is periodic along $\hat{n}$ then the integral in that dimension must be replaced by a sum so

$$\begin{aligned}\left[\hat{H}, \hat{W}_{\hat{n}}(L)\right] &= \frac{1}{L}\hat{W}_{\hat{n}} \sum_{l=0}^{\infty} \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3} 4e^2 \frac{\sin^2(2\pi l)}{k^2} \\ &\quad + 2ie\sin(2\pi l)\left(\hat{a}^\dagger_{\hat{k},\boldsymbol{k}} - \hat{a}_{\hat{k},\boldsymbol{k}}\right). \end{aligned} \tag{6.48}$$

In this form it is clear that all modes with $l \neq 0$ vanish. The second term vanishes when $l = 0$ but the first does not. In particular if $k_\perp = 0$ as well then the first term does not vanish. As a result the only contribution comes from the term with $k = 0$. This term must be written as a limit in order to ensure continuity in the vicinity of $h = L$, and this limit must be approached along the $\hat{n}$ direction, as the field is in this direction when

$h = L$. Thus we find

$$\left[\hat{H}, \hat{W}_{\hat{n}}(L)\right] = \frac{4e^2}{L^3}\hat{W}_{\hat{n}} \lim_{k\to 0} \frac{\sin^2(k_n L)}{k_n^2} \tag{6.49}$$

$$= \frac{4e^2}{L}\hat{W}_{\hat{n}} \lim_{u\to 0} \frac{\sin^2(u_n)}{u_n^2} \tag{6.50}$$

$$= \frac{4e^2}{L}\hat{W}_{\hat{n}}. \tag{6.51}$$

As promised this vanishes as L^{-1} and so the operator is indeed soft.

Thus we have shown that the operation which winds a particle-antiparticle pair around a large loop has support preferentially as $k \to 0$, and scales in such a way that it commutes with the Hamiltonian up to terms of order L^{-1}. As such we identify it as the soft photon operator in our system.

6.4.3 Local Indistinguishability and Generalizations

Now that we have shown the ground state degeneracy on a torus, we turn to another important characteristic of topological order - local degeneracy. Roughly speaking, this means that any local measurement should be unable to determine which topological sector the system occupies. We can see this heuristically by noting physical processes with typical length scale ΔX cannot resolve momenta more precisely than $\Delta P \sim \Delta X^{-1}$. Since the topological degeneracy comes from the $1/L$ modes, local measurements with $\Delta X \ll L$ cannot determine the topological sector.

Though a more complete field-theoretic treatment is left to future work, we can get some intuition about this result by considering the careful treatment of IR divergences first discussed in [183]. By considering scattering processes that both involve virtual infrared bosons and the emission/absorption of infrared bosons, the IR divergence goes away. The new transition rate is given in terms of a positive function C that depends on the details of the gauge theory, a positive function b, the UV cutoff Λ, the total energy of emitted soft modes E, and the original transition rate $\Gamma^0_{\alpha\beta}$:

$$\Gamma_{\alpha\beta} = (E/\Lambda)^C b(C)\Gamma^0_{\alpha\beta} \tag{6.52}$$

When considering the modes that change topological sectors, we see that the energy goes to zero as $1/L$ and thus the transition rate for local scattering processes to change topological sectors vanishes. In this light, the IR divergence in QED and linearized gravity is similar to IR divergences in spontaneously broken (0-form) symmetries. This is in agreement with the arguments of [199], though it does not mean as has been claimed [200] that such modes are unmeasurable or trivially decoupled, simply that modes at some asymptotic distance L require space and time proportional to L to measure.

This identification solves the longstanding question of the connection between $1/L$ photon modes and topological sectors. That is, $1/L$ photon modes carry the charges which identify different topological sectors. This is a significant point, but in retrospect is not entirely surprising, as the $1/L$ modes by definition are sensitive to global physics.

The arguments above do not rely on any details of QED other than charge conservation and the existence of gapless modes, which combined allows us to determine the scaling of the fields. The general nature of this construction then leads to the somewhat remarkable conjecture that any stable, deconfined, continuous gauge theory with a soft theorem should have some notion of topological degeneracy. The different topological sectors can be reached by winding gauge-charged matter around the large loops of the torus, which can be interpreted as threading (locally invisible) soft bosons. Importantly, one only expects stability provided that the matter is massive, so that there is an exponential cost to "unwind" the topological sectors.

Such behavior does not extend to gapless theories without a gauge structure (such as superfluids), since there are no electric fields or charges, nor large gauge transformations. From the viewpoint of [187], there are no hard charges and thus the Ward identity is not equivalent to a flux integral. It should be noted that while there is a notion of ground-state degeneracy in systems with spontaneously broken continuous symmetries, this degeneracy is not dependent on the topology of the manifold on which the system resides.

6.4.4 Higher Form Symmetries

The process of creating a charge-anticharge pair and moving them around a closed path C is a well-known object in gauge theories: Wilson loops. Confinement of the gauge theory can be determined by the area or perimeter law scaling of these objects, captured in the Wilson loop operator

$$W = e^{\int_C A} \tag{6.53}$$

This is manifestly invariant under the ordinary gauge transformation, since the integral of df vanishes. However, we could consider a more general gauge transformation:

$$A \to A + \lambda \tag{6.54}$$

If we require that $d\lambda = 0$ but $\lambda \neq df$ for any f, then the field strength $F = dA$ is left invariant but the Wilson loops change by a factor of $\exp \int_C \lambda$.

This symmetry, known as a 1-form symmetry, is one of the infinite family of generalized global symmetries with very interesting properties[22]. As opposed to 0-form symmetries, whose charges are point-like objects such as particles, the 1-form symmetries act line-like objects. The elements of the 1-form symmetries act on surfaces, which for

$U(1)$ are

$$U(\alpha, M^2) = \exp\left(i\frac{\alpha}{2g^2} \int_{M^2} \star F \right) \tag{6.55}$$

These operators form a 2-group due to the ways the manifolds can be stacked. However, for fixed M^2 (say, the xy-plane) this symmetry is just a $U(1)$ symmetry, and its irreduciple representations are labeled by integers. This symmetry can spontaneously break, giving rise to a Goldstone boson, namely the photon. Charged matter explicitly breaks this symmetry, but for energies much smaller than the charge gap the symmetry is restored. Whether or not the a symmetry spontaneously breaks determines whether or not the gauge theory is deconfined[22].

Our previous discussion can thus be rephrased in this language as identifying "topological" surfaces that live in the homology of the manifold, specifically closed surfaces that do not bound a volume, and noting that, in the thermodynamic limit, acting with the generators of the 1-form symmetry moves between ground states. This unifies gapped and gapless topological order in gauge theories, in that they both have spontaneously broken 1-form symmetries. In fact, the ground state degeneracy in $SU(3)$ noted in[198] is topological in the same way.

A more complete analysis of higher-form symmetries and their relation to topological phases is left to future work.

6.5 Topological Order in Open Systems

In the previous two sections we argued that the IR fixed point for both electromagnetism and linearized gravity should have a well-defined continuum gauge theory arising from the local constraints of charge and momentum conservation. These Gauss-law type constraints give rise to a ground-state degeneracy on a torus, which we have termed gapless topological order due to the presence of the gauge bosons.

More specifically, the generators of topological degeneracy in gravitation and electromagnetism are the charge operators of [182], written schematically as

$$\hat{Q}^{\pm}_{\epsilon,\mathrm{EM}} \sim \int_{\mathcal{I}^{\pm}_{\mp}} \epsilon \star F \tag{6.56}$$

for electromagnetism, with a similar integral over $\mathcal{I}$ or the equivalent boundary surface holding for gravitation.

These operators generate the ground-state degeneracy of the vacuum, such that

$$\langle 0|\hat{Q}|0\rangle = 0 \tag{6.57}$$

and

$$\left[\hat{H}, \hat{Q}\right] = 0. \tag{6.58}$$

Eq. (6.58) follows because $\hat{Q}$ represents a soft mode corresponding to an asymptotic symmetry. Eq. (6.57) simply represents the fact that the field configuration associated with a system containing a soft boson is distinct from that of a system not containing it. In a finite universe the commutator is of order L^{-1}, matching the condensed matter case.

This topological degeneracy is a surprising result of the metric signature which holds even in open spacetimes. To see this, note that a key result enabling equations (6.57) and (6.58) is the antipodal mapping, which associates antipodal points on the boundaries of the past and future [182]. This mapping emerges because the ground state involves only massless modes, which propagate through the spacetime bulk at c. More specifically, the antipodal mapping is possible because in the absence of charges massless fields are fully

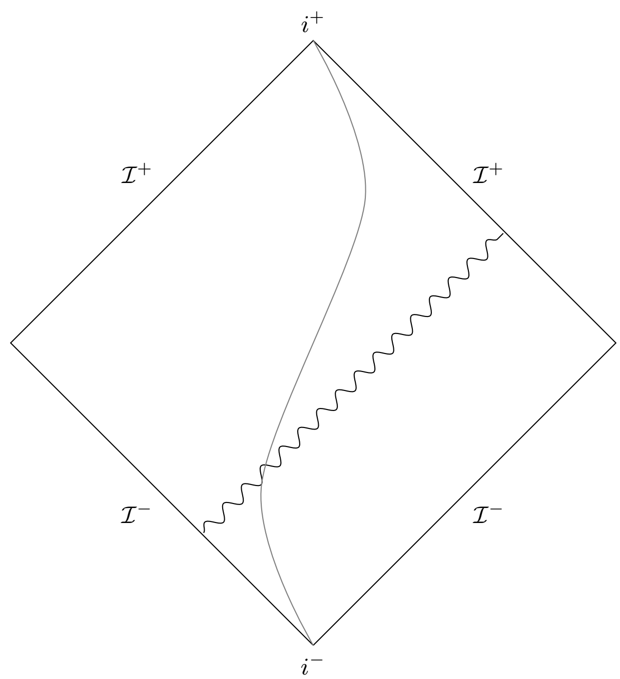

Figure 6.6: Massless particles (wiggling lines) travel between past and future null infinity while massive ones (regular lines) travel between i^- and i^+. This difference means that massive (gapped) charges accumulate at $i^\pm$ while soft charges appear at $\mathcal{I}^\pm_\pm$.

determined by their values on any one Cauchy surface, as the wave equation

$$\Box\phi = 0 \tag{6.59}$$

may be used to propagate them from that surface to the rest of spacetime. This establishes a correspondence between the values these fields take on at $\mathcal{I}^-$ and $\mathcal{I}^+$. As a result we may write as a slight rephrasing of [182]

$$\phi^-(u) = e^{i\alpha(u)}\phi^+(u) \tag{6.60}$$

where u is the relevant null coordinate, $\phi^\pm$ are evaluated at antipodal points on the corresponding null surfaces, and α is a function dependent on the gauge condition taken at these surfaces.

Upon threading a flux quantum through from one null surface to the other an overall factor of $e^{i\alpha}$ is accumulated. This factor may be set to unity by appropriate choice of gauge to yield periodic boundary conditions [187]. Even without doing this it is clear that Eq. (6.60) connects antipodal points on the space, and so in the asymptotic compactification gives it topological structure. This is shown in Fig. 6.6. This sidesteps the problem of the topology of the universe, since we need not specify the genus of spacetime.

In the presence of a more complicated topology or additional horizons (i.e. black holes) the identification is between modes on different horizons which overlap when propagated both forwards and backwards in time. For a simple example, consider a soft graviton with large angular quantum numbers, such that it is highly directed. This graviton propagates from a region on $\mathcal{I}^-$ until it encounters a black hole. The graviton becomes bound to the event horizon by scattering into one of the surface soft modes.

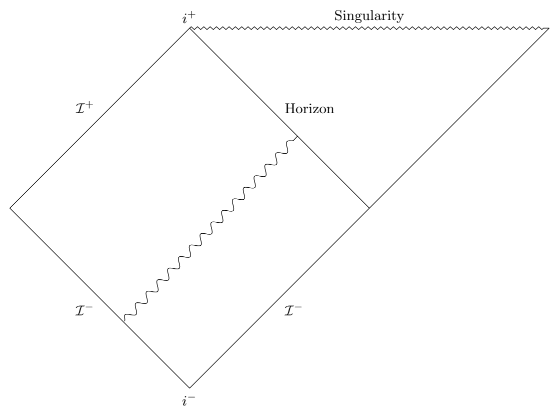

Figure 6.7: Massless soft graviton (wiggling line) propagates from $\mathcal{I}^-$ and impinges on the black hole (top-right), resulting in entanglement between the two horizons.

This process is shown in Fig. 6.7. An analogous equation to Eq. (6.60) relates the mode which arrives in this fashion at $\mathcal{I}^-$ to the mode which arrives on the black hole's horizon. More generally, if the mode did not fully scatter onto the black hole's horizon there would be a relation between the three boundaries, namely the black hole, $\mathcal{I}^-$, and $\mathcal{I}^+$, with at most a phase accumulation between each of them.

Massive charged excitations by definition propagate from i^- to i^+ and break this structure by introducing scattering processes, but this propagation is exponentially suppressed as e^{-mL} and hence does not break the ground state degeneracy in the thermodynamic limit. This is the analogue of the circumferential create-wind-destroy propagation process on a torus. A key difference is that topological winding in a $3+1d$ spacetime is complicated by its infinite nature, which makes it the case that winding soft flux through the universe requires an infinite amount of time, or at least the time required to reach an acceptable approximation of asymptotic infinity.

For a concrete example consider inserting a single $+e$ electric charge at i^- with velocity $\boldsymbol{\beta}$. The electromagnetic tensor at $\mathcal{I}^{\pm}_{\mp}$ is [182]

$$F_{\pm,rt} = \frac{e(1-\beta^2)}{4\pi r^2(1 \mp \boldsymbol{\beta}\cdot\hat{r})^2}, \tag{6.61}$$

where the subscript rt denotes the component which goes as $dr \wedge dt$. The soft charge at $\mathcal{I}^{\pm}_{\mp}$ is then

$$Q^{\pm}_{\epsilon} = \frac{1}{e^2} \int_{\mathcal{I}^{\pm}_{\mp}} \epsilon \star F_{\pm,rt} \tag{6.62}$$

$$= \frac{1}{e} \lim_{r\to\infty} r^2 \int_{S^2} \frac{\epsilon(1-\beta^2)}{4\pi(1-\boldsymbol{\beta}\cdot\hat{r})r^2} \tag{6.63}$$

$$= \frac{1}{e} \int_{S^2} \frac{\epsilon(1-\beta^2)}{4\pi(1-\boldsymbol{\beta}\cdot\hat{r})}. \tag{6.64}$$

Now suppose that we thread a negative charge $-e$ with velocity $\boldsymbol{\beta'} \neq \boldsymbol{\beta}$. The net charge is evidently

$$Q^{\pm}_{\epsilon} = \frac{1}{e} \int_{S^2} \frac{\epsilon}{4\pi} \left[\frac{(1-\beta^2)}{1-\boldsymbol{\beta}\cdot\hat{r}} - \frac{(1-\beta'^2)}{1-\boldsymbol{\beta'}\cdot\hat{r}} \right]. \tag{6.65}$$

This is nonzero though the net charge which has been threaded through $i^{\pm}$ is zero. Of course if $\beta \neq \beta'$ then the charges are always located far apart spatially at $i^{\pm}$, but in the compactified coordinates in which we identify i^- with i^+ this is fine. A similar argument holds for black hole event horizons and for supertranslations, but unfortunately the fields are much more difficult to write out explicitly. The key difference is that the charges must begin at i^-, enter the black hole, be emitted via Hawking radiation, and then head towards i^+. Other than that the argument is precisely the same, and the integral may be taken over the horizon of the black hole along with the necessary spatial cut to reach $\mathcal{I}^{\pm}_{\mp}$.

It is usually useful to think of topological winding as an operation which may be iterated. This is not the case for the universe due to the infinite time required to wind. It is worth asking then in what sense this order is topological. The answer is twofold. First, while we cannot iterate a winding process on a single universe, we can simulate the process given several spacetimes. To see this suppose we start with a universe with no soft charge. We can then wind a dipole from i^- to i^+ as described above and note the soft charge which appears. We can set up a second empty spacetime with this soft charge from the beginning. If we then wind a dipole through that spacetime the soft charge doubles, and so it is clear that the amount of soft charge is a quantity which changes with dipole winding, which is locally unobservable, and which permits us to indefinitely move from sector to sector via this process. These are the hallmarks of topological order.

The second argument is somewhat more direct: it is entirely valid to wind multiple dipoles simultaneously, as they can be separated spatially yet lead to the same soft charge so long as their asymptotic velocities are the same. As a result it is sensible to talk about iterated winding, just with the iteration occurring in space rather than time.

What both of these arguments fail to address is what happens to the hard charge at infinity. This likewise has two answers which differ just as a matter of interpretation. First, suppose we bring a dipole out of the vacuum at some point near i^- and then wind it to some point near i^+ before annihilating it. Far from the origin we may draw a surface and integrate over it to measure the resulting soft charge. This is the case so long as the creation and annihilation occur outside of the surface, and so the winding effectively encompasses a loop in spacetime between the creation and annihilation points. The limit may then be taken as this surface goes off to infinity, keeping these two points outside as it goes.

The alternative interpretation of this process is that we may 'glue' two spacetimes together as a result of the periodic boundary conditions at infinity. In this process, $\mathcal{I}^+$ and i^+ in one spacetime are identified with the antipodal $\mathcal{I}^-$ and i^- in the other spacetime, and vice-versa. As a result a charge wound from i^- to i^+ in one spacetime simply carries on to the next one, before wrapping back to the first spacetime once more. In this way we avoid formal accumulation of charge at infinity.

Regardless of interpretation, it is clear that there is topological order in these systems, both as a result of the odd boundary conditions associated with an open spacetime and as the continuum limit of the corresponding lattice systems. This order manifests via global operators that distinguish a charge which is not locally measurable, and which have a direct connection to the hard (local) charge wound through the system.

6.6 Limits

As the topological order discussed here is quite broad in nature it is worth discussing the limits in which it is applicable. In particular it relies primarily on two key assumptions; linearity and low-energy (IR).

Linearity in this context does not mean that the metric is a small perturbation against Minkowski space. Rather, it means that the quantum mechanical perturbations we consider correspond to small metric perturbations against whatever background metric we choose. This is equivalent to saying that all perturbing gravitational waves have small amplitude, or equivalently that gravitons are not so prevalent as to interact strongly with one another. In fact we do not even require that this be true universally, as we only need it to hold in the regions around which we perform flux integrals. The gravitational field may be perturbed in an arbitrarily nonlinear manner outside of these regions, and these nonlinear effects will appear simply as fluxes of the relevant conserved charges through the bounding surface.

Along similar lines, working in the low-energy (IR) limit means that we are considering gravitons with energies of order $1/L$, where L is a characteristic scale for the universe. This is true even in the presence of a black hole, where the scale of the universe and not that of the black hole remains the relevant parameter. This is because in an infinite universe, black holes support precise zero modes reflecting the BMS symmetry of relativity. The fact that we confine our discussion to these modes does not make our conclusions any weaker, however, as our claim is precisely that these modes give rise to topological order. The existence of higher-energy modes is irrelevant to this point.

6.7 Black Holes and Information

As mentioned previously, there are modes which exist on the event horizon of a black hole which are analogous to the modes on the horizons at infinity. These modes actually obey the same dispersion relation up to local horizon distortions, just with the expansion coordinate converted from $\xi = 1/r$ to $\xi = r - r_s$, r_s being the horizon radius [182].

Now consider the formation of a Hawking pair at the horizon. For simplicity, we consider QED, so the state is a charge singlet of charge-1 particles. The state is then one of the Bell states, given by

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right). \tag{6.66}$$

One particle falls into the interior while the other escapes to $\mathcal{I}^+$. By the preceding arguments there are flux integrals which can detect the fact that a particle has escaped. These integrals measure the soft charge on the horizon, and so the state of the outgoing particle must be entangled with the soft sector. If these integrals can detect the degree of freedom we have considered, the state must really be

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|+-+\rangle + |-+-\rangle\right) \tag{6.67}$$

up to a minus sign and overall phase factor, where the additional qubit describes the state of the flux integral that labels the soft sector. In this way it is possible to entangle the outgoing particle with the soft sector. Now Eq. (6.67) is the GHZ state for three particles, and so we know that if we trace out the soft sector there will be no remaining entanglement between the infalling and outgoing Hawking particles. This ought to occur

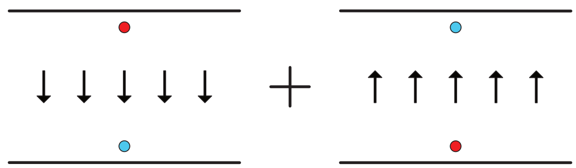

Figure 6.8: Separating charges in different directions generates a GHZ state between the charge positions and the gauge field.

for all portions of the state of the particle which may be read from soft flux integrals, and so if these indeed encode all of the information which falls in then there is no firewall paradox.

This resolution amounts to quantum mechanical violation of equivalence via the monogamy of entanglement, and is essentially a physical realization of the nonlocal gravitational modes proposed by [201]. Notably this exchange of entanglement is a purely quantum mechanical effect. The soft theorems guarantee that interactions with the soft sector are not classically measurable, so this resolution of the paradox represents a way to preserve the classical equivalence principle while minimally violating it quantum mechanically.

It is important to emphasize that this argument does not resolve the broader information paradox. To see this note that the Bell state is not recoverable from $|\phi\rangle$ after tracing out the particle which fell in. This is another way of saying that the information is not transferred from the particle to the horizon nor is it cloned, it is just entangled with the horizon.

Conclusion

We have argued that a peculiar type of gapless topological order exists in the lattice models of electromagnetism and linearized gravity, and that these models both flow to exactly stable IR fixed points with well-behaved continuum descriptions. Thus, we can use this topological order to characterize the IR behavior and ground state of the continuum theories, provided that the gauge constraints hold (i.e. the metric deviations are small).

While there is no natural way to impose periodic boundary conditions on the universe, we have used the Lorentzian signature of the metric to identify non-contractible loops of the gauge fields in spacetime, allowing for the construction of non-local operators which commute with the Hamiltonian and whose eigenvalues distinguish the various ground states. Finally, we have connected all of these objects to well-known results in the literature.

We have seen that gapless topological order, as described in this paper, shares many properties with ordinary topological order. Primarily, they both have a family of locally indistinguishable ground states, and are degenerate on a torus. However, we have made very general arguments for both the local indistinguishability and degeneracy, and thus expect the arguments to hold for other deconfined continuous gauge theories with soft boson theorems. A precise characterization and proof is left to future work.

Finally, we have discussed applications of this work to black holes, with the key insight that the firewall paradox may be resolved by reducing the equivalence principle to be purely classical, with violation at the level of entanglement. This is suggestive of a phase transition in the vacuum across the event horizon, but we leave a more detailed analysis of this phenomenon to a later work.

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