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PROOF OF A BI-SYMMETRIC SEPTUPLE EQUIDISTRIBUTION ON ASCENT SEQUENCES

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Abstract. It is well known since the seminal work by Bousquet-Mélou, Claesson, Dukes and Kitaev (2010) that certain refinements of the ascent sequences with respect to several natural statistics are in bijection with corresponding refinements of $(\mathbf{2}+\mathbf{2})$-free posets and permutations that avoid a bi-vincular pattern. Different multiply-refined enumerations of ascent sequences and other bijectively equivalent structures have subsequently been extensively studied by various authors.

In this paper, our main contributions are

- a bijective proof of a bi-symmetric septuple equidistribution of Euler–Stirling statistics on ascent sequences, involving the number of ascents (asc), the number of repeated entries (rep), the number of zeros (zero), the number of maximal entries (max), the number of right-to-left minima (rmin) and two auxiliary statistics;
- a new transformation formula for non-terminating basic hypergeometric ${}_4\phi_3$ series expanded as an analytic function in base q around $q = 1$, which is utilized to prove two (bi)-symmetric quadruple equidistributions on ascent sequences.

A by-product of our findings includes the affirmation of a conjecture about the bi-symmetric equidistribution between the quadruples of Euler–Stirling statistics $(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})$ and $(\mathsf{rep}, \mathsf{asc}, \mathsf{max}, \mathsf{zero})$ on ascent sequences, that was motivated by a double Eulerian equidistribution due to Foata (1977) and recently proposed by Fu, Lin, Yan, Zhou and the first author (2018).

Keywords. Ascent sequences, equidistributions, Euler–Stirling statistics, Fishburn numbers, basic hypergeometric series

Mathematics Subject Classifications. 05A15, 05A19

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1. Introduction and main results

In the seminal paper [BMCDK10] by Bousquet-Mélou, Claesson, Dukes and Kitaev, ascent sequences were introduced, as they are in bijection with several different combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets, certain bivincular pattern-avoiding permutations, Stoimenow's involution and regular linearized chord diagrams [Sto98, Zag01]. Several natural statistics on posets, permutations and sequences are also kept track of by a sequence of bijections established by these authors. Since then, various joint distributions of classical statistics on ascent sequences and many other bijectively equivalent structures including Fishburn matrices [Fis70, Fis85] and $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences have been intensively explored [CL11, DKRS11, DP10, Jel12, Jel15, KR11, KR17, Lev13].

Recently in [FJL+20], a new decomposition of ascent sequences was discovered, which contributes to a systematic study of Eulerian and Stirling statistics on ascent sequences, certain pattern-avoiding permutations and $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences. In particular, their work led to conjecture the bi-symmetry of a quadruple Euler–Stirling statistics on ascent sequences (see Conjecture 1.4) that is motivated by a double Eulerian equidistribution due to Foata [Foa77]. However, it appears that the use of the new decomposition from [FJL+20] is not sufficient to prove the bi-symmetry conjecture.

In the present paper, we affirm this conjecture in two different ways: one by developing a second new decomposition of ascent sequences; and the other one by identifying the generating function of the quadruple statistics as a basic hypergeometric series to which a new transformation formula (that is derived in this paper) is applied. Let us start with some necessary definitions and then state the consequences of our results.

An *inversion sequence* $(s_1, s_2, \ldots, s_n)$ is a sequence of non-negative integers such that for all i, $0 \leqslant s_i < i$. We denote by $\mathcal{I}_n$ the set of inversion sequences of length n, which is in one-to-one correspondence with the set $\mathfrak{S}_n$ of permutations of $[n] := \{1, 2, \ldots, n\}$ via the well known Lehmer code σ (see for instance [Foa77, Leh60]). That is, for $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the map $\sigma : \mathfrak{S}_n \to \mathcal{I}_n$ is defined as

$$\sigma(\pi) = (s_1, s_2, \ldots, s_n), \qquad \text{where } s_i := |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

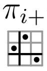
Some restrictions set up on permutations and inversion sequences could produce new sets of equal cardinality, but not necessarily through the Lehmer code. For instance, ascent sequences and ()-avoiding permutations (defined as below) are equinumerous.

Definition 1.1 (Ascent sequence)**.** For any sequence $s \in \mathcal{I}_n$, let

$$\mathsf{asc}(s) := |\{i \in [n-1] : s_i < s_{i+1}\}| \tag{1.1}$$

be the number of **asc**ents of s. An inversion sequence $s \in \mathcal{I}_n$ is an *ascent sequence* if for all $2 \leqslant i \leqslant n$, the s_i satisfy

$$s_i \leqslant \mathsf{asc}(s_1, s_2, \ldots, s_{i-1}) + 1.$$

Definition 1.2 (()-avoiding permutation)**.** We say that a permutation $\pi \in \mathfrak{S}_n$ *avoids the pattern*  if there is no subsequence $\pi_i\pi_{i+1}\pi_j$ of π satisfying both $\pi_i - 1 = \pi_j$ and $\pi_i < \pi_{i+1}$. Otherwise we say π contains the pattern . Sometimes the pattern  is written as $2|3\bar{1}$.

The ($\boxplus$)-avoiding permutations, more generally, permutations that avoid a specific bivincular pattern, were introduced and studied by Bousquet-Mélou, Claesson, Dukes and Kitaev [BMCDK10] as both of them are surprisingly in bijection with other classical combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets [Fis70, Fis85] and regular linearized chord diagrams [Sto98, Zag01].

Let $\mathcal{A}_n$ and $\mathfrak{S}_n(\boxplus)$ be the sets respectively of ascent sequences and ($\boxplus$)-avoiding permutations of length n. Bousquet-Mélou, Claesson, Dukes and Kitaev [BMCDK10] proved that

$$|\mathcal{A}_n| = \left|\mathfrak{S}_n(\boxplus)\right| = [t^n]\sum_{k=1}^{\infty}\prod_{i=1}^{k}(1-(1-t)^i), \tag{1.2}$$

and thus, as a consequence of a result by Zagier [Zag01] (who discovered that the series on the right-hand side of (1.2) is the generating functions of the Fishburn numbers), $|\mathcal{A}_n|$ is equal to the n-th Fishburn number (see A022493 of the OEIS [Inc11]). Their first explicit values are given as

$$(|\mathcal{A}_n|)_{n\geqslant 1} = (1,2,5,15,53,217,1014,5335,31240,201608,\ldots),$$

for which no closed form is known. The study of Fishburn numbers and their generalizations has remarkably led to many interesting results, including congruences [AS16, Gar15], asymptotic formulas [BLR14, HJ21, HJS23, Zag01], intriguing connections to transformations of hypergeometric series [AJ14], modular forms [BLR14, Zag01] and a variety of bijections [CL11, DKRS11, DP10, Jel12, Jel15, KR11, KR17, Lev13]. In particular, various members of the Fishburn family can be viewed as supersets of corresponding members of the Catalan family. Here the Fishburn (resp. Catalan) family refers to classes of combinatorial objects enumerated by the Fishburn (resp. Catalan) numbers.

This paper is devoted to new bijective and basic hypergeometric aspects of Fishburn structures, for which we review some classical statistics on ascent sequences and ($\boxplus$)-avoiding permutations. For any sequence $s \in \mathcal{I}_n$, $\mathsf{asc}(s)$ is defined in (1.1). Let

$$\begin{aligned}
\mathsf{rep}(s) &:= n - |\{s_1, s_2, \ldots, s_n\}|,\\
\mathsf{zero}(s) &:= |\{i \in [n] : s_i = 0\}|,\\
\mathsf{max}(s) &:= |\{i \in [n] : s_i = i-1\}|, \quad \text{and}\\
\mathsf{rmin}(s) &:= |\{s_i : s_i < s_j \text{ for all } j > i\}|,
\end{aligned}$$

be the respective numbers of **repe**ated entries, **zero**s, **maxi**mal entries (or maximals for short) and **r**ight-to-left **min**ima of s. For instance, when $s = (0,1,2,0,1,3,5) \in \mathcal{I}_7$, then $\mathsf{asc}(s) = 5$, $\mathsf{rep}(s) = 2$, $\mathsf{zero}(s) = 2$, $\mathsf{max}(s) = 3$ and $\mathsf{rmin}(s) = 4$. For any permutation $\pi \in \mathfrak{S}_n$, let

$$\begin{aligned}
\mathsf{des}(\pi) &:= |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|,\\
\mathsf{iasc}(\pi) &:= \mathsf{asc}(\pi^{-1}) = |\{i \in [n-1] : \pi_i + 1 \text{ appears to the right of } \pi_i\}|,
\end{aligned}$$

be the number of **des**ents and **i**nverse **asc**ents of π, respectively. Similar to rmin, the statistics lmin, lmax and rmax represent the numbers of **l**eft-to-right **min**ima, **l**eft-to-right **max**ima and **r**ight-to-left **max**ima, respectively.

Previous bijections developed in [BMCDK10, DP10, FJL+20] preserve natural statistics on posets, permutations, sequences and matrices. As examples, we list below five pairs of equidistributed statistics that were established in those papers.

$(\mathsf{asc},\mathsf{zero})$ on ascent sequences $\overset{1-1}{\longleftrightarrow}$ $(\mathsf{des},\mathsf{lmax})$ on ()-avoiding permutations,
$\overset{1-1}{\longleftrightarrow}$ $(\mathsf{mag}-1,\mathsf{min})$ on $(\mathbf{2}+\mathbf{2})$-free posets,
$\overset{1-1}{\longleftrightarrow}$ $(\mathsf{dim}-1,\mathsf{rowsum}_1)$ on Fishburn matrices,
$\overset{1-1}{\longleftrightarrow}$ $(\mathsf{rep},\mathsf{max})$ on $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences.

Remark 1.3. The statistics mag, min are abbreviations for **mag**nitude and the number of **min**imal elements of a poset; the statistics dim and rowsum_1 refer to **dim**ension and the **sum** of entries in the **first row** of a matrix.

In a recent paper [FJL+20], a joint *symmetric* distribution of statistics asc and rep over ascent sequences was discovered. The motivation came from a symmetric distribution of $(\mathsf{asc},\mathsf{rep})$ on inversion sequences

$$\sum_{s\in\mathcal{I}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)}=\sum_{s\in\mathcal{I}_n} u^{\mathsf{rep}(s)}x^{\mathsf{asc}(s)}. \tag{1.3}$$

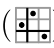
This is a direct consequence of a double Eulerian equidistribution due to Foata [Foa77]:

$$\sum_{s\in\mathcal{I}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)}=\sum_{\pi\in\mathfrak{S}_n} u^{\mathsf{des}(\pi)}x^{\mathsf{iasc}(\pi)}. \tag{1.4}$$

It turns out that not only (1.3) and (1.4) are true if $\mathcal{I}_n$ and $\mathfrak{S}_n$ are replaced by the corresponding subsets $\mathcal{A}_n$ and $\mathfrak{S}_n$(), but an even stronger result on a bi-symmetric equidistribution of Euler–Stirling statistics [1] over ascent sequences was conjectured.

Conjecture 1.4. [FJL+20] For each $n\geqslant 1$, the following bi-symmetric quadruple equidistribution holds:

$$\sum_{s\in\mathcal{A}_n} u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)}z^{\mathsf{zero}(s)}y^{\mathsf{max}(s)}=\sum_{s\in\mathcal{A}_n} u^{\mathsf{rep}(s)}x^{\mathsf{asc}(s)}z^{\mathsf{max}(s)}y^{\mathsf{zero}(s)}.$$

Remark 1.5. Conjecture 1.4 is equivalent to a bi-symmetric equidistribution between the quadruples $(\mathsf{des},\mathsf{iasc},\mathsf{lmax},\mathsf{lmin})$ and $(\mathsf{iasc},\mathsf{des},\mathsf{lmin},\mathsf{lmax})$ on ()-avoiding permutations, according to Theorem 12 of [FJL+20].

Two results in approaching this conjecture were presented in [FJL+20]: one is a generating function formula of ascent sequences with respect to the statistics asc, rep, zero, max (see Theorem 1.6); and the other one is a quadruple equidistribution between $(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max})$ and $(\mathsf{rep},\mathsf{asc},\mathsf{rmin},\mathsf{zero})$ on ascent sequences (see Theorem 1.7).

[1] We adopt the classification of statistics from [FJL+20]: any statistic whose distribution over a member of the Fishburn family equals the distribution of asc (resp. zero) on ascent sequences is called *an Eulerian* (resp. a Stirling) statistic. So according to Theorem 1.7, asc, rep are Eulerian statistics and zero, max, rmin are Stirling statistics.

Let $\mathcal{G}(t;x,y,u,z)$ denote the generating function of ascent sequences counted by the length (variable t), asc (variable u), rep (variable x), max (variable y) and zero (variable z). That is,

$$\mathcal{G}(t;x,y,u,z) := \sum_{n=1}^{\infty} t^n \sum_{s\in\mathcal{A}_n} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} u^{\mathsf{asc}(s)} z^{\mathsf{zero}(s)}. \tag{1.5}$$

Theorem 1.6. *[FJL+20] The generating function $\mathcal{G}(t;x,y,u,z)$ of ascent sequences is*

$$\mathcal{G}(t;x,y,u,z) = \sum_{m=0}^{\infty} \frac{zyrx^m(1-yr)(1-r)^m(x+u-xu)}{[x(1-u)+u(1-yr)(1-r)^m][x+u(1-x)(1-yr)(1-r)^m]} \times \prod_{i=0}^{m-1} \frac{1+(zr-1)(1-yr)(1-r)^i}{x+u(1-x)(1-yr)(1-r)^i}, \tag{1.6}$$

where $r = t\,(x+u-xu)$.

Theorem 1.7. *[FJL+20] There is a bijection $\Upsilon : \mathcal{A}_n \to \mathcal{A}_n$ which transforms the quadruple*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}) \quad \textit{to} \quad (\mathsf{rep}, \mathsf{asc}, \mathsf{rmin}, \mathsf{zero}).$$

Conjecture 1.4 can be settled, with the help of Theorems 1.6 and 1.7, by showing either (I) or (II), described as follows.

(I) $\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y)$;

(II) the quadruple $(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})$ has the same distribution as $(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{rmin})$ over ascent sequences.

In this paper, we settle Conjecture 1.4 independently in both ways, (I) and (II).

Our first main result (Theorem 1.8) is a bijective proof of a bi-symmetric septuple equidistribution on ascent sequences, which significantly generalizes (II) and consequently affirms Conjecture 1.4.

Theorem 1.8. *There is a bijection $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ such that for all $s \in \mathcal{A}_n$,*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos})s = (\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{rmin}, \mathsf{rpos}, \mathsf{max}, \mathsf{ealm})\Phi(s). \tag{1.7}$$

We postpone the definitions of the two auxiliary statistics ealm, rpos to Sections 3 and 4. The main idea to prove Theorem 1.8 relies on two *parallel* decompositions of ascent sequences that are in close relation to the two respective auxiliary statistics ealm and rpos. The former decomposition was discovered in [FJL+20]. However, using this decomposition alone appears to be not enough to prove Conjecture 1.4, which motivates us to develop the latter new decomposition in this paper, providing a crucial piece of the puzzle solved here.

Our second main result (Theorem 1.9) is a new transformation formula of non-terminating basic hypergeometric ${}_4\phi_3$ series, valid as an identity expanded in base $q = 1-r$ around $q = 1$, or, equivalently, $r = 0$. We define ${}_\alpha\phi_\beta$ series before stating Theorem 1.9. Other relevant definitions, in particular, for the q-shifted factorials and their products (see (7.1) and (7.2)) are deferred to Section 7 to keep the exposition short. An ${}_\alpha\phi_\beta$ basic hypergeometric series with α upper parameters $a_1,\dots,a_\alpha$, and β lower parameter $b_1,\dots,b_\beta$, base q and argument z is defined as

$$ {}_\alpha\phi_\beta\left[\begin{matrix}a_1,\dots,a_\alpha\\ b_1,\dots,b_\beta\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_\alpha;q)_k}{(q,b_1,\dots,b_\beta;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+\beta-\alpha}z^k. \tag{1.8}$$

Theorem 1.9. *Let a,b,c,d,e,r be complex variables, j be a non-negative integer. Then, assuming that none of the denominator factors in (1.9) have vanishing constant term in r, we have the following transformation of convergent power series in a and r:*

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}(1-r)^j,1-a,b,c\\ d,e,(1-r)^{j+1}(1-a)bc/de\end{matrix};1-r,1-r\right]\\ &=\frac{((1-r)/e,(1-r)(1-a)bc/de;1-r)_j}{((1-r)(1-a)/e,(1-r)bc/de;1-r)_j}\\ &\quad\times{}_4\phi_3\left[\begin{matrix}(1-r)^j,1-a,d/b,d/c\\ d,de/bc,(1-r)^{j+1}(1-a)/e\end{matrix};1-r,1-r\right].\end{aligned}\tag{1.9}$$

We utilize special cases of Theorem 1.9 to give analytic proofs of two different quadruple (bi)-symmetric equidistributions of Euler–Stirling statistics on ascent sequences, collected in Theorem 1.10. The first application of Theorem 1.9 is a proof of (I) by making use of the explicit form of the generating function in Theorem 1.6, and thus constitutes a non-combinatorial proof of the bi-symmetric equidistribution in Conjecture 1.4, while the second application establishes a symmetric equidistribution by employing a new explicit generating function obtained by a refined recursive construction of ascent sequences from [FJL+20].

Theorem 1.10. *For the generating function defined in (1.5), we have the bi-symmetry*

$$\mathcal{G}(t;x,y,u,z)=\mathcal{G}(t;u,z,x,y).\tag{1.10}$$

Furthermore, define

$$\mathfrak{G}(t;x,y,u,v):=\sum_{n=1}^{\infty}t^n\sum_{s\in\mathcal{A}_n}x^{\mathsf{rep}(s)}y^{\mathsf{max}(s)}u^{\mathsf{asc}(s)}v^{\mathsf{rmin}(s)},\tag{1.11}$$

then we have, with $r=t(x+u-xu)$,

$$\begin{aligned}\mathfrak{G}(t;x,y,u,v)&=\frac{vyt}{1-vytu}+\sum_{m=0}^{\infty}\frac{rv(1-yr)(1-r)^m}{(x-xu+u(1-yr)(1-r)^m)(1-tuvy)}\\ &\times\prod_{i=0}^{m}\frac{x(1-(1-yr)(1-r)^i)(x-xu+u(1-yr)(1-r)^i)}{(x-u(x-1)(1-yr)(1-r)^i)(x-xu+u(1-rv)(1-yr)(1-r)^i)},\end{aligned}\tag{1.12}$$

and the symmetry

$$\mathfrak{G}(t;x,y,u,v)=\mathfrak{G}(t;x,v,u,y),\tag{1.13}$$

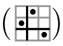
Remark 1.11. In the language of bijections, the (bi)-symmetric equidistributions in Theorem 1.10 mean that for any ascent sequence $s \in \mathcal{A}_n$,

$$\begin{aligned}(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})s &= (\mathsf{rep}, \mathsf{asc}, \mathsf{max}, \mathsf{zero})\Upsilon^{-1}(\Phi(s)),\\ (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin})s &= (\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{max})\Phi(s),\\ (\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{rmin})s &= (\mathsf{rep}, \mathsf{asc}, \mathsf{rmin}, \mathsf{zero})\Upsilon(\Phi(s)),\end{aligned}$$

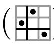
where Υ and Φ are the bijections respectively in Theorems 1.7 and 1.8.

Remark 1.12. We are not the first ones to study equivalent forms for generating functions of objects of the Fishburn family using tools from basic hypergeometric series. Initiating with work of Zagier [Zag01] who established the basic hypergeometric series in (1.2) as a concrete form of the generating function $\mathcal{G}(t;1,1,1,1)$ for the Fishburn numbers, Andrews and Jelínek [AJ14] subsequently proved three equivalent forms of $\mathcal{G}(t;1,1,1,z)$ by applying the Rogers–Fine identity. However, to the best of our knowledge, no algebraic or analytic arguments to determine equivalent forms of the generating functions $\mathcal{G}(t;x,y,u,z)$ or $\mathfrak{G}(t;x,y,u,v)$ were known, not even, say, for the special case $\mathcal{G}(t;1,1,u,z)$. Our analytic proofs of $\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y)$ and $\mathfrak{G}(t;x,y,u,v) = \mathfrak{G}(t;x,v,u,y)$ strengthen the already known existing ties between (refined) generating functions of objects of the Fishburn family with basic hypergeometric series that are expanded in base $q = 1 - r$ around $r = 0$. At the same time it demonstrates the benefit of having equivalent forms of generating functions, and the power of basic hypergeometric machinery.

All aforementioned (bi)-symmetric distributions on ascent sequences have counterparts over other members of the Fishburn family.

Corollary 1.13. *There are three bijections between* $\mathfrak{S}_n($$)$ *and itself such that the following three (bi)-symmetric equidistributions hold, respectively:*

$$\begin{aligned}(\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin}, \mathsf{rmax})\pi &= (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{rmax}, \mathsf{lmin})(\Psi^{-1}\circ\Phi\circ\Psi)(\pi),\\ (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin})\pi &= (\mathsf{iasc}, \mathsf{des}, \mathsf{lmin}, \mathsf{lmax})(\Psi^{-1}\circ\Upsilon^{-1}\circ\Phi\circ\Psi)(\pi),\\ (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{rmax})\pi &= (\mathsf{iasc}, \mathsf{des}, \mathsf{rmax}, \mathsf{lmax})(\Psi^{-1}\circ\Upsilon\circ\Phi\circ\Psi)(\pi),\end{aligned}$$

where Υ, Φ *are the bijections respectively in Theorems 1.7 and 1.8, and* $\Psi : \mathfrak{S}_n($$) \to \mathcal{A}_n$ *is the bijection from Theorem 12 of [FJL+20].*

Let us recall the definition of Fishburn matrices and associated three Stirling statistics.

Any cell (i,j) of a matrix M is called a *weakly north-east cell* if $M_{i,j} \neq 0$ and $M_{s,t} = 0$ for all other $s \leqslant i$ and $t \geqslant j$. A matrix is a *Fishburn* matrix if all of its entries are nonnegative integers such that neither row nor column contains only zero entries. Let $\mathcal{F}_n$ be the set of Fishburn matrices whose sum of entries equals n, then for any $M \in \mathcal{F}_n$, let

$$\begin{aligned}\mathsf{rowsum}_1(M) &:= \text{the sum of entries in the first row of } M,\\ \mathsf{ne}(M) &:= \text{the number of weakly north-east cells of } M,\\ \mathsf{mtr}(M) &:= \text{the largest index } i \text{ with } 1 \leqslant i \leqslant \dim(M) \text{ for which the submatrix}\end{aligned}$$

$$(M_{s,t})_{s\leqslant i-1,t\leqslant i-1} \text{ is an empty or an identity matrix.}$$

The first two statistics and the statistic

$$\mathsf{tr}(M) := \text{the number of non-zero } M_{i,i} \text{ for all } 1 \leqslant i \leqslant \dim(M)$$

were studied in [CYZ19, Jel15]. Clearly $\mathsf{mtr}(M) \leqslant \mathsf{tr}(M)$ holds for any Fishburn matrix M and the statistic mtr (short name for **m**odified **tr**ace) is introduced because of a bijection established by Chen, Yan and Zhou in [CYZ19, Theorem 16]: There is a bijection $\phi : \mathcal{A}_n \to \mathcal{F}_n$ with the property

$$(\mathsf{zero}, \mathsf{rmin}, \mathsf{maxasc})s = (\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{tr})\phi(s),$$

where $\mathsf{maxasc}(s) := |\{i \in [1, n] : s_i = \mathsf{asc}(s_1, \dots, s_{i-1}) + 1\}|$ counts the number of **max**imal **asc**ents. Through the bijection ϕ, it is not hard to find that

$$(\mathsf{zero}, \mathsf{rmin}, \mathsf{max})s = (\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{mtr})\phi(s).$$

Consequently, it follows directly from Theorem 1.8 that

Corollary 1.14. *There is a bijection between $\mathcal{F}_n$ and itself such that the following symmetric distribution holds:*

$$(\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{mtr})M = (\mathsf{rowsum}_1, \mathsf{mtr}, \mathsf{ne})(\phi \circ \Phi \circ \phi^{-1})M,$$

where Φ is the bijection in Theorem 1.8 and $\phi : \mathcal{A}_n \to \mathcal{F}_n$ is given in [CYZ19, Theorem 16].

Remark 1.15. The three Stirling statistics $\mathsf{rowsum}_1, \mathsf{ne}, \mathsf{mtr}$ are pairwise symmetric on $\mathcal{F}_n$. The fact that the pair $(\mathsf{ne}, \mathsf{mtr})$ is symmetric on $\mathcal{F}_n$ is a direct consequence of Corollary 1.14 and it is known from [CYZ19, FJL+20, Jel15] that the other two pairs $(\mathsf{rowsum}_1, \mathsf{ne})$ and $(\mathsf{rowsum}_1, \mathsf{mtr})$ are also symmetric.

Interestingly, the concept of maximal ascents also appears in the new decomposition of ascent sequences; see Definition 3.3 and Section 3.

The paper is organized as follows. We provide in the next section a brief road map of the sophisticated bijective proof of Theorem 1.8. Two key ingredients of the proof, including a new decomposition of ascent sequences and a sequence of transformations on ascent sequences, are presented in Sections 3 and 4. In Section 5 we complete the proof of Theorem 1.8 and put technical proofs of some lemmas and propositions in Section 8. A refined generating function of ascent sequences is derived in Section 6 and its amenability to transformations of basic hypergeometric series is demonstrated in Section 7. We end the paper in Section 9 with some final remarks; in particular we pose an open problem there and state a conjecture.

2. Road map of the bijective proof

The purpose of this section is to present a brief idea of the proof of Theorem 1.8. Some related definitions will be postponed to Sections 3 to 4.

For the trivial case $s = (0, 1, 2, \dots, |s| - 1)$, it is easily seen that $\Phi(s) = s$ satisfies (1.7), so it suffices to prove Theorem 1.8 for the remaining ascent sequences. Let $\mathcal{A}^*$ denote the set of ascent sequences except $s = (0, 1, 2, \dots, |s| - 1)$.

In the first step, we describe a new partition of $\mathcal{A}^*$ into five disjoint subsets denoted by $\mathcal{T}_i$ for $1 \leqslant i \leqslant 5$ in Section 3. Subsequently we review a different partition of $\mathcal{A}^*$ into five subsets from [FJL$^+$20]. These subsets are denoted by $\mathcal{D}_i$ for $1 \leqslant i \leqslant 5$ and will be defined in Section 5.

In the second step, we establish a bijection

$$\Phi : \mathcal{D}_i \cap \mathcal{A}_n \to \mathcal{T}_i \cap \mathcal{A}_n, \tag{2.1}$$

that satisfies (1.7) for every i ($1 \leqslant i \leqslant 5$) in order to prove Theorem 1.8. The bijection Φ is defined recursively, starting with the simplest one between $\mathcal{D}_1 \cap \mathcal{A}_n$ and $\mathcal{T}_1 \cap \mathcal{A}_n$, and then using induction to construct more difficult ones for other subsets that can be transformed into simpler subsets for which the bijection is already known.

More precisely, we begin with the bijection Φ in (2.1) for the simplest case $i = 1$. That is,

$$\Phi : \{s \in \mathcal{A}_n : |s| - \mathsf{max}(s) = 1\} \to \{s \in \mathcal{A}_n : |s| - \mathsf{rmin}(s) = 1\} \tag{2.2}$$

is explicitly defined, which forms an inductive basis to construct Φ for other subsets of ascent sequences s with larger value of $|s| - \mathsf{max}(s)$ or $|s| - \mathsf{rmin}(s)$.

For each $i \in \{2, 3, 4\}$, a bijection that maps $\mathcal{D}_i \cap \mathcal{A}_n$ (resp. $\mathcal{T}_i \cap \mathcal{A}_n$) to a subset of ascent sequences with reduced value of $|s| - \mathsf{max}(s)$ (resp. $|s| - \mathsf{rmin}(s)$) is described in Section 4 (resp. Section 5). These bijections combined with the basis (2.2) enable us to recursively define Φ between the subsets $\mathcal{D}_i \cap \mathcal{A}_n$ and $\mathcal{T}_i \cap \mathcal{A}_n$ for $i \in \{2, 3, 4\}$.

For the case $i = 5$, the construction of Φ instead employs the already defined bijection $\Phi : \mathcal{D}_4 \cap \mathcal{A}_n \to \mathcal{T}_4 \cap \mathcal{A}_n$ and a bijection that transforms $\mathcal{D}_5 \cap \mathcal{A}_n$ (resp. $\mathcal{T}_5 \cap \mathcal{A}_n$) into a subset of ascent sequences with smaller $\mathsf{max}(s) - \mathsf{ealm}(s)$ (resp. $\mathsf{rmin}(s) - \mathsf{rpos}(s)$). It proceeds as follows: We prove in Lemma 5.5 and Proposition 4.8 the following two bijections:

$$\begin{aligned} h_5 &: \mathcal{D}_5 \cap \mathcal{A}_n \to (\mathcal{D}_3 \dot\cup \mathcal{D}_4 \dot\cup \mathcal{D}_5) \cap \{s \in \mathcal{A}_n : \mathsf{ealm}(s) \neq 0\}, \\ f_5 &: \mathcal{T}_5 \cap \mathcal{A}_n \to (\mathcal{T}_3 \dot\cup \mathcal{T}_4 \dot\cup \mathcal{T}_5) \cap \{s \in \mathcal{A}_n : \mathsf{rpos}(s) \neq 0\}. \end{aligned}$$

Through these two bijections the values of $\mathsf{max}(s) - \mathsf{ealm}(s)$ and $\mathsf{rmin}(s) - \mathsf{rpos}(s)$ are decreased by one, respectively. In particular,

$$\begin{aligned} h_5 &: \{s \in \mathcal{D}_5 \cap \mathcal{A}_n : \mathsf{max}(s) - \mathsf{ealm}(s) = 2\} \to \{s \in \mathcal{D}_4 \cap \mathcal{A}_n : \mathsf{ealm}(s) \neq 0\}, \\ f_5 &: \{s \in \mathcal{T}_5 \cap \mathcal{A}_n : \mathsf{rmin}(s) - \mathsf{rpos}(s) = 2\} \to \{s \in \mathcal{T}_4 \cap \mathcal{A}_n : \mathsf{rpos}(s) \neq 0\}. \end{aligned}$$

Using then the already known bijection $\Phi : \mathcal{D}_4 \cap \mathcal{A}_n \to \mathcal{T}_4 \cap \mathcal{A}_n$ as a basis to recursively define $\Phi = f_5^{-1} \circ \Phi \circ h_5$ yields the desired bijection between $\mathcal{D}_5 \cap \mathcal{A}_n$ and $\mathcal{T}_5 \cap \mathcal{A}_n$. This will complete the bijective proof of Theorem 1.8.

3. A new decomposition of ascent sequences

The new decomposition is largely inspired by the new auxiliary statistic rpos, which together with some relevant statistics will be defined as follows.

- Let Rmin be the corresponding set-valued statistic of rmin, that is, for any $s \in \mathcal{I}_n$,

$$\mathsf{Rmin}(s) = \{s_i : s_i < s_j \text{ for all } j > i\}.$$

  For convenience, we index all right-to-left minima from left to right starting from 0 (rather than from 1). That is, right-to-left minima of s are indexed by $0, 1, \ldots, \mathsf{rmin}(s) - 1$ from left to right. Let $\mathsf{Rmin}(s)_j$ denote the j-th smallest element of $\mathsf{Rmin}(s)$ where $0 \leqslant j < \mathsf{rmin}(s)$.

- Let $\mathsf{Prm}(s)_j$ be the j-th smallest element of $\mathsf{Prm}(s)$, where $0 \leqslant j < \mathsf{rmin}(s)$ and

$$\mathsf{Prm}(s) := \{i : s_i \text{ is a right-to-left minimum of } s\}$$

  is the set of **p**ositions of **r**ight-to-left **m**inima of $s = (s_1, s_2, \ldots, s_{|s|})$.

Definition 3.1 (statistics rpos)**.** For any ascent sequence s with $\mathsf{rmin}(s) \neq |s|$, define $\mathsf{rpos}(s) = m$ if m is the maximal index such that the m-th right-to-left minimum appears at least twice after the $(m-1)$-th right-to-left minimum. If no such m exists or $\mathsf{rmin}(s) = |s|$, set $\mathsf{rpos}(s) = 0$.

For example, $\mathsf{rpos}(\mathbf{0}, \mathbf{0}, 1, 2, 3, 4) = 0$ and $\mathsf{rpos}(0, 0, 1, 2, 0, 1, 2, 1, \mathbf{3}, \mathbf{3}, 4) = 2$.

Definition 3.2 (statistics sebr)**.** Given any ascent sequence $s \in \mathcal{A}^*$, define $\mathsf{sebr}(s)$ to be the **s**mallest **e**ntry **b**etween the two **r**ightmost entries $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$, and assume $\mathsf{sebr}(s) = 0$ if the two rightmost entries $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ are next to each other.

For example, let $s = (0, 0, 1, 2, 0, 1, 2, 1, \mathbf{3}, 4, 5, \mathbf{3}, 4)$, then $\mathsf{rpos}(s) = 2$, $\mathsf{Rmin}(s)_2 = 3$ and $\mathsf{sebr}(s) = 4$ where the two rightmost entries 3 are in bold.

Let

$$\mathcal{T}_1 := \{s \in \mathcal{A}^* : |s| = \mathsf{rmin}(s) + 1\},$$
$$\mathcal{T}_2 := \{s \in \mathcal{A}^* - \mathcal{T}_1 : \mathsf{sebr}(s) = 0\}.$$

Then the complement of $\mathcal{T}_1 \dot{\cup} \mathcal{T}_2$ in $\mathcal{A}^*$ contains all ascent sequences $s \in \mathcal{A}^* - \mathcal{T}_1$ with $\mathsf{sebr}(s) \neq 0$. We next divide the remaining set $\mathcal{A}^* - \mathcal{T}_1 \dot{\cup} \mathcal{T}_2$ into the following two disjoint subsets $\mathcal{A}^1$ and $\mathcal{A}^2$ by comparing $\mathsf{sebr}(s)$ and $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}$. When $\mathsf{rpos}(s) = \mathsf{rmin}(s) - 1$, we assume that $\mathsf{sebr}(s) < \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}$. Define

$$\mathcal{A}^1 := \{s \in \mathcal{A}^* - \mathcal{T}_1 : \mathsf{sebr}(s) \geqslant \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}\},$$
$$\mathcal{A}^2 := \{s \in \mathcal{A}^* - \mathcal{T}_1 : 0 \neq \mathsf{sebr}(s) < \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}\}.$$

Now we refine the sets $\mathcal{A}^1$ and $\mathcal{A}^2$ through the concept of *maximal ascents*:

Definition 3.3. (Masc) For any $s = (s_1, s_2, \ldots, s_n) \in \mathcal{I}_n$, we say that s_i is a Maximal **asc**ent (M*asc*) of s if

$$s_i = \mathsf{asc}(s_1, s_2, \ldots, s_{i-1}) + 1.$$

In particular, the last entry s_n of s is an Masc if $s_{n-1} < s_n = \mathsf{asc}(s)$. All maximal entries are Masc's. For instance, given $s = (\mathbf{0}, \mathbf{1}, \mathbf{2}, 0, \mathbf{3}, 2)$, the entries in bold are all Masc's of s.

Set (see also Figure 3.1)

$$\mathcal{T}_3 := \{s \in \mathcal{A}^1 : \mathsf{sebr}(s) = \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}, \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} = \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1.$$
$$\text{and no Masc appears after the position } \mathsf{Prm}(s)_{\mathsf{rpos}(s)}\}.$$
$$\mathcal{T}_4 := \{s \in \mathcal{A}^2 : s_{|s|} \text{ is not an M}asc\}.$$
$$\mathcal{T}_5 := (\mathcal{A}^1 - \mathcal{T}_3) \,\dot{\cup}\, (\mathcal{A}^2 - \mathcal{T}_4).$$

Thus $\mathcal{A}^*$ is the disjoint union of subsets $\mathcal{T}_i$ for $1 \leqslant i \leqslant 5$.

For example, let $s = (0,0,1,2,0,1,2,1,3,4,5,3,4)$. Then $\mathsf{rpos}(s) = 2$ and $s \in \mathcal{T}_3$ because $\mathsf{sebr}(s) = \mathsf{Rmin}(s)_3 = 4$, $\mathsf{Prm}(s)_2 + 1 = \mathsf{Prm}(s)_3 = 13$ and no Masc appears after the 12-th entry. Let $s = (0,0,1,2,0,1,2,1,3,4,5,3,5)$. Then $s \in \mathcal{T}_4$ because $\mathsf{sebr}(s) = 4 < \mathsf{Rmin}(s)_3 = 5$ and the last entry is not an Masc.

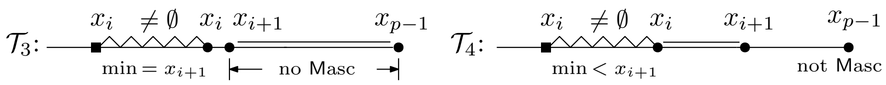


Figure 3.1: Two subsets of ascent sequences $s \in \mathcal{A}^*$ with $\mathsf{rpos}(s) = i$ and $\mathsf{rmin}(s) = p$, where $x_i = \mathsf{Rmin}(s)_i$ denotes the i-th right-to-left minimum of s; black dots and squares represent the rightmost and the second rightmost entry respectively.

4. A sequence of bijections on ascent sequences

In this section, we present a sequence of bijections that map each $\mathcal{T}_i$ for $1 \leqslant i \leqslant 5$ to a subset of ascent sequences s either with smaller $|s| - \mathsf{rmin}(s)$ or with smaller $\mathsf{rmin}(s) - \mathsf{rpos}(s)$.

Let us recall the statistic ealm introduced in [FJL+20]:

Definition 4.1 (statistic ealm)**.** Let s be an ascent sequence satisfying $\mathsf{max}(s) \neq |s|$. Then $\mathsf{ealm}(s) = s_{\mathsf{max}(s)+1}$, i.e., the entry right after the last maximal. For the ascent sequence $s = (0, 1, \ldots, |s| - 1)$ that has $\mathsf{max}(s) = |s|$, we set $\mathsf{ealm}(s) = 0$.

For example, $\mathsf{ealm}(0, 1, \mathbf{0}, 1, 3, 0, 2) = 0$.

Throughout the paper, define $\chi(a) = 1$ if the statement a is true; and $\chi(a) = 0$ otherwise.

Lemma 4.2. *There is a bijection*

$$f_2 : \mathcal{T}_2 \cap \mathcal{A}_n \to \{(i, s) : s \in \mathcal{A}^* \cap \mathcal{A}_{n-1}, \mathsf{rpos}(s) \leqslant i < \mathsf{rmin}(s)\}$$

that sends s to a pair $f_2(s) = (\mathsf{rpos}(s), s^*)$ satisfying

$$(\mathsf{asc}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin})s = (\mathsf{asc}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin})s^*,$$

$$\mathsf{zero}(s) = \mathsf{zero}(s^*) + \chi(\mathsf{rpos}(s) = 0), \quad \textit{and} \quad \mathsf{rep}(s) = \mathsf{rep}(s^*) + 1.$$

Proof. For any ascent sequence $s \in \mathcal{T}_2$ with $\mathsf{rpos}(s) = i < \mathsf{rmin}(s)$, the two rightmost $\mathsf{Rmin}(s)_i$ are next to each other. Removing one of them leads to an ascent sequence $s^* \in \mathcal{A}^*$ with $\mathsf{rpos}(s^*) \leqslant i$. We set $f_2(s) = (\mathsf{rpos}(s), s^*)$ and it is easily seen that f_2 is a bijection satisfying $|s^*| = |s| - 1$, $\mathsf{asc}(s^*) = \mathsf{asc}(s)$, $\mathsf{rep}(s^*) = \mathsf{rep}(s) - 1$, $\mathsf{zero}(s^*) = \mathsf{zero}(s) - \chi(\mathsf{rpos}(s) = 0)$, $\mathsf{max}(s^*) = \mathsf{max}(s)$, $\mathsf{ealm}(s^*) = \mathsf{ealm}(s)$ and $\mathsf{rmin}(s^*) = \mathsf{rmin}(s)$. □

Example 4.3. For $s = (0, 0, 1, 2, 0, 1, 2, 1, \mathbf{3}, \mathbf{3}, 4)$, according to Lemma 4.2, $f_2(s) = (2, s^*)$ where $s^* = (0, 0, 1, 2, 0, \mathbf{1}, 2, \mathbf{1}, 3, 4)$ is an ascent sequence with $\mathsf{rpos}(s^*) = 1$.

Let $\mathcal{P}_1$ be the set of ascent sequences $s \in \mathcal{A}^*$ whose last entry is an Masc, that is, $s_{|s|-1} < s_{|s|} = \mathsf{asc}(s)$. Denote by $\mathcal{P}_1^c$ the complement of $\mathcal{P}_1$ in $\mathcal{A}^*$.

Lemma 4.4. *There is a bijection*

$$\phi_1 : \mathcal{A}_n \cap \mathcal{P}_1 \to \mathcal{A}_{n-1} \cap \mathcal{A}^*$$

that transforms the septuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc} + 1, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin} + 1, \mathsf{rpos}).$$

Proof. For any ascent sequence $s \in \mathcal{P}_1$, remove the last entry and define the resulting sequence as $\phi_1(s)$. It is easy to examine the corresponding statistics. □

Lemma 4.5. *There is a bijection*

$$f_3 : \mathcal{T}_3 \cap \mathcal{A}_n \to \{s \in \mathcal{A}_n \cap \mathcal{P}_1 : \mathsf{rpos}(s) \neq 0\}$$

that transforms the quintuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc}, \mathsf{rep} + 1, \mathsf{max}, \mathsf{rmin} - 1, \mathsf{rpos} - 1),$$

and satisfies

$$\begin{aligned} \mathsf{zero}(s) &= \mathsf{zero}(f_3(s)) + \chi(\mathsf{rpos}(s) = 0), \\ \mathsf{ealm}(s) &= \mathsf{ealm}(f_3(s)) - \chi(\mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s) + 1). \end{aligned}$$

Proof. For any ascent sequence $s \in \mathcal{T}_3$ with $\mathsf{rpos}(s) = i$, remove the rightmost $\mathsf{Rmin}(s)_i$ and add the integer $\mathsf{asc}(s)$ at the end. Let $f_3(s)$ be the resulting sequence and the map f_3 is clearly a bijection (see Figure 4.1). Only when the entry $\mathsf{ealm}(s)$ on the $(\mathsf{max}(s) + 1)$-th position of s is also the $\mathsf{rpos}(s)$-th right-to-left minimum, we have $\mathsf{ealm}(f_3(s)) = \mathsf{ealm}(s) + 1$. It is not hard to verify the other statistics. □

Lemma 4.6. *There is a bijection*

$$f_4 : \mathcal{T}_4 \cap \mathcal{A}_n \to \{s \in \mathcal{A}_n \cap \mathcal{P}_1^c : \mathsf{rpos}(s) \neq 0\}$$

that transforms the quintuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin} - 1, \mathsf{rpos} - 1),$$

and satisfies

$$\begin{aligned} \mathsf{zero}(s) &= \mathsf{zero}(f_4(s)) + \chi(\mathsf{rpos}(s) = 0), \\ \mathsf{ealm}(s) &= \mathsf{ealm}(f_4(s)) - \chi(\mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s) + 1). \end{aligned}$$

Proof. For any ascent sequence $s \in \mathcal{T}_4 \cap \mathcal{A}_n$ with $\mathsf{rpos}(s) = i$, replacing the rightmost $\mathsf{Rmin}(s)_i$ by the integer $\mathsf{sebr}(s)$ yields an ascent sequence $f_4(s) \in \mathcal{P}_1^c$. The map f_4 is invertible and therefore bijective (see Figure 4.1). Similar to Lemma 4.5, it is straightforward to check the corresponding statistics. $\square$

Example 4.7. For $s = (0,0,1,2,0,\mathbf{1},2,\mathbf{1},2,4,3,5)$, then $f_3(s) = (0,0,1,2,0,1,\mathbf{2},\mathbf{2},4,3,5,7)$, which, according to Lemma 4.5, is an ascent sequence with $\mathsf{rpos}(f_3(s)) = 2$ and the last entry 7 is an $\mathcal{M}asc$. For $\tilde{s} = (0,0,1,2,0,\mathbf{1},2,\mathbf{1},4,3,5)$, then $f_4(\tilde{s}) = (0,0,1,2,0,1,\mathbf{2},\mathbf{2},4,3,5)$. By Lemma 4.6, it is an ascent sequence with $\mathsf{rpos}(f_4(\tilde{s})) = 2$ and the last entry 5 is not an $\mathcal{M}asc$.

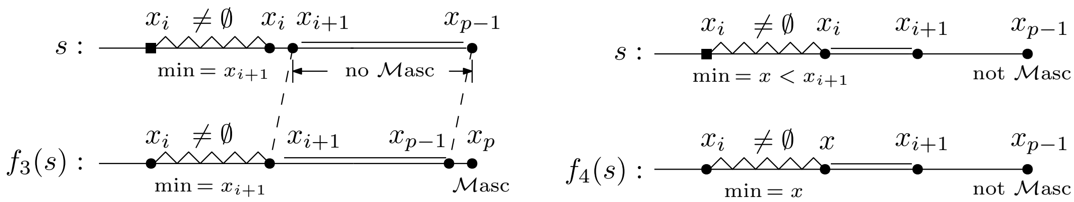


Figure 4.1: Two bijections f_3 and f_4 on ascent sequences $s \in \mathcal{T}_3 \dot\cup \mathcal{T}_4$ with $\mathsf{rpos}(s) = i$ and $\mathsf{rmin}(s) = p$. Here x_i always denotes the entry of the i-th right-to-left minimum, so x_{p-1} is the last entry of s.

Now we divide the subset $\mathcal{T}_5$ according to the change of statistics $\mathsf{max}, \mathsf{ealm}$.

Let $\mathsf{M}_{5,1}$ be the set of ascent sequences $s \in \mathcal{T}_5$ whose second rightmost entry $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ is not a maximal of s or s belongs to the set

$$\{s \in \mathcal{A}^1 : \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} \neq \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1 \text{ and } \mathsf{Prm}(s)_{\mathsf{rpos}(s)} \neq \mathsf{max}(s) + 1\}.$$

Furthermore, let

$$\mathsf{M}_{5,2} := \{s \in \mathcal{A}^1 : \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} \neq \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1 \text{ and } \mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s) + 1\}$$

and $\mathsf{M}_{5,3} := \mathcal{T}_5 - \mathsf{M}_{5,1} - \mathsf{M}_{5,2}$.

Proposition 4.8. *There is a bijection*

$$f_5 : \mathcal{T}_5 \cap \mathcal{A}_n \to (\mathcal{T}_3 \dot\cup \mathcal{T}_4 \dot\cup \mathcal{T}_5) \cap \{s \in \mathcal{A}_n : \mathsf{rpos}(s) \neq 0\}$$

that satisfies $\mathsf{zero}(s) = \mathsf{zero}(f_5(s)) + \chi(\mathsf{rpos}(s) = 0)$ *and transforms*

$$\begin{aligned}
(\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos}) \textit{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos} - 1);&\\
(\mathsf{max}, \mathsf{ealm}) \textit{ to } (\mathsf{max}, \mathsf{ealm}),\ \textit{if } s \in \mathsf{M}_{5,1},&\\
(\mathsf{max}, \mathsf{ealm}) \textit{ to } (\mathsf{max}, \mathsf{ealm} - 1),\ \textit{if } s \in \mathsf{M}_{5,2},&\\
(\mathsf{max}, \mathsf{ealm}) \textit{ to } (\mathsf{max} - 1, \mathsf{ealm} - 1),\ \textit{if } s \in \mathsf{M}_{5,3}.&
\end{aligned}$$

The proof of Proposition 4.8 is quite involved and therefore we put it in Section 8.

5. Bijective proof of the septuple equidistribution

This section is devoted to complete the bijective proof of Theorem 1.8. Before we proceed to prove Theorem 1.8, we review the last ingredient of the proof: a decomposition of ascent sequences from [FJL+20], which is associated with the statistic ealm.

The decomposition is formulated slightly different from [FJL+20]. The set $\mathcal{A}^*$ is partitioned into the following disjoint subsets:

$$\mathcal{D}_1 := \{s \in \mathcal{A}^* : |s| = \mathsf{max}(s) + 1\},$$
$$\mathcal{D}_2 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} \leqslant \mathsf{ealm}(s)\},$$
$$\mathcal{D}_3 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} = \mathsf{ealm}(s) + 1, \mathsf{max}(s) \notin \{s_i : \mathsf{max}(s) + 2 \leqslant i \leqslant |s|\}\},$$
$$\mathcal{D}_4 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} = \mathsf{ealm}(s) + 1, \mathsf{max}(s) \in \{s_i : \mathsf{max}(s) + 2 \leqslant i \leqslant |s|\}\},$$
$$\mathcal{D}_5 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} \geqslant \mathsf{ealm}(s) + 2\}.$$

For example, let $s = (0, 1, 2, 0, 1, 4, 3, 5)$, then $s \in \mathcal{D}_4$ because $\mathsf{max}(s) = 3$, $s_5 = \mathsf{ealm}(s) + 1 = 1$ and 3 appears after s_4. Let $s = (0, 1, 2, 0, 3, 4, 3, 5)$, then $s \in \mathcal{D}_5$ since $s_5 = 3 > \mathsf{ealm}(s) + 2$.

A sequence of transformations on $\mathcal{D}_i$ for $1 \leqslant i \leqslant 5$ from [FJL+20] is in parallel with the ones in Section 4. Here we provide these bijections explicitly in the proofs, but omit other details since they are very straightforward.

Lemma 5.1 (Lemma 8 of [FJL+20])**.** *There is a bijection*

$$h_2 : \mathcal{D}_2 \cap \mathcal{A}_n \to \{(i, s) : s \in \mathcal{A}^* \cap \mathcal{A}_{n-1}, \mathsf{ealm}(s) \leqslant i < \mathsf{max}(s)\}$$

that sends s to a pair $h_2(s) = (\mathsf{ealm}(s), s^*)$ satisfying

$$(\mathsf{asc}, \mathsf{rmin}, \mathsf{rpos}, \mathsf{max})s = (\mathsf{asc}, \mathsf{rmin}, \mathsf{rpos}, \mathsf{max})s^*,$$

$$\mathsf{zero}(s) = \mathsf{zero}(s^*) + \chi(\mathsf{ealm}(s) = 0) \quad \textit{and} \quad \mathsf{rep}(s) = \mathsf{rep}(s^*) + 1.$$

Proof. For any $s \in \mathcal{D}_2$, remove the entry $\mathsf{ealm}(s)$ at the $(\mathsf{max}(s) + 1)$-th position of s. Let the resulting sequence be s^* and define $h_2(s) = (\mathsf{ealm}(s), s^*)$. □

Let $\mathcal{P}_2$ be the set of ascent sequences $s \in \mathcal{A}^*$ such that the integer $\mathsf{max}(s) - 1$ appears exactly once in s. Denote by $\mathcal{P}_2^c$ the complement of $\mathcal{P}_2$ in $\mathcal{A}^*$.

Lemma 5.2 (Lemma 10 of [FJL+20])**.** *There is a bijection*

$$\phi_2 : \mathcal{A}_n \cap \mathcal{P}_2 \to \mathcal{A}_{n-1} \cap \mathcal{A}^*$$

that transforms the septuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos}) \textit{ to } (\mathsf{asc} + 1, \mathsf{rep}, \mathsf{zero}, \mathsf{max} + 1, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos}).$$

Proof. For any $s \in \mathcal{P}_2$, remove the unique entry $\mathsf{max}(s) - 1$ and replace all entries y by $y - 1$ if $y \geqslant \mathsf{max}(s)$. Let $\phi_2(s)$ be the resulting sequence. □

Lemma 5.3 (Lemma 9 of [FJL+20]). *There is a bijection*

$$h_3 : \mathcal{D}_3 \cap \mathcal{A}_n \to \{s \in \mathcal{A}_n \cap \mathcal{P}_2 : \mathsf{ealm}(s) \neq 0\}$$

that transforms the quintuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{max}, \mathsf{ealm}) \text{ to } (\mathsf{asc}, \mathsf{rep}+1, \mathsf{rmin}, \mathsf{max}-1, \mathsf{ealm}-1),$$

and satisfies

$$\begin{aligned}\mathsf{zero}(s) &= \mathsf{zero}(h_3(s)) + \chi(\mathsf{ealm}(s) = 0),\\ \mathsf{rpos}(s) &= \mathsf{rpos}(h_3(s)) - \chi(\mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s)+1).\end{aligned}$$

Proof. For any $s \in \mathcal{D}_3$, replace the entry $\mathsf{ealm}(s)$ on the $(\mathsf{max}(s)+1)$-th position by $\mathsf{max}(s)$. Define $h_3(s)$ as the resulting sequence. □

Lemma 5.4 (Lemma 11 of [FJL+20]). *There is a bijection*

$$h_4 : \mathcal{D}_4 \cap \mathcal{A}_n \to \{s \in \mathcal{A}_n \cap \mathcal{P}_2^c : \mathsf{ealm}(s) \neq 0\}$$

that transforms the quintuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{max}, \mathsf{ealm}) \text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{max}-1, \mathsf{ealm}-1),$$

and satisfies

$$\begin{aligned}\mathsf{zero}(s) &= \mathsf{zero}(h_4(s)) + \chi(\mathsf{ealm}(s) = 0),\\ \mathsf{rpos}(s) &= \mathsf{rpos}(h_4(s)) - \chi(\mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s)+1).\end{aligned}$$

Proof. For any $s \in \mathcal{D}_4$, replace the entry $\mathsf{ealm}(s)$ on the $(\mathsf{max}(s)+1)$-th position by $\mathsf{max}(s)$. Define $h_4(s)$ as the resulting sequence. □

By taking the change of statistics into account, we further divide the subset $\mathcal{D}_5$ into three disjoint subsets, i.e., $D_5 = \mathcal{D}_{5,1} \dot\cup \mathcal{D}_{5,2} \dot\cup \mathcal{D}_{5,3}$ where

$$\begin{aligned}\mathcal{D}_{5,1} &= \{s \in \mathcal{D}_5 : \min\{s_i, \mathsf{max}(s)+2 \leqslant i \leqslant |s|\} \leqslant \mathsf{ealm}(s)\},\\ &\qquad \dot\cup \{s \in \mathcal{D}_5 : \min\{s_i, \mathsf{max}(s)+2 \leqslant i \leqslant |s|\} = \mathsf{ealm}(s)+1, \mathsf{rpos}(s) \geqslant \mathsf{ealm}(s)+1\},\\ \mathcal{D}_{5,2} &= \{s \in \mathcal{D}_5 : \min\{s_i, \mathsf{max}(s)+2 \leqslant i \leqslant |s|\} = \mathsf{ealm}(s)+1, \mathsf{rpos}(s) = \mathsf{ealm}(s)\},\\ \mathcal{D}_{5,3} &= \{s \in \mathcal{D}_5 : \min\{s_i, \mathsf{max}(s)+2 \leqslant i \leqslant |s|\} \geqslant \mathsf{ealm}(s)+2\}.\end{aligned}$$

Note that by definition $\mathcal{D}_{5,2} = \mathcal{M}_{5,2}$.

Lemma 5.5. *There is a bijection*

$$h_5 : \mathcal{D}_5 \cap \mathcal{A}_n \to (\mathcal{D}_3 \dot\cup \mathcal{D}_4 \dot\cup \mathcal{D}_5) \cap \{s \in \mathcal{A}_n : \mathsf{ealm}(s) \neq 0\}$$

that satisfies $\mathsf{zero}(s) = \mathsf{zero}(h_5(s)) + \chi(\mathsf{ealm}(s) = 0)$ *and transforms*

$$\begin{aligned}(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{ealm}) &\text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{ealm}-1),\\ (\mathsf{rmin}, \mathsf{rpos}) &\text{ to } (\mathsf{rmin}, \mathsf{rpos}) \text{ if } s \in \mathcal{D}_{5,1},\\ (\mathsf{rmin}, \mathsf{rpos}) &\text{ to } (\mathsf{rmin}, \mathsf{rpos}-1) \text{ if } s \in \mathcal{D}_{5,2},\\ (\mathsf{rmin}, \mathsf{rpos}) &\text{ to } (\mathsf{rmin}-1, \mathsf{rpos}-1) \text{ if } s \in \mathcal{D}_{5,3}.\end{aligned}$$

Proof. For any ascent sequence $s \in \mathcal{D}_5$, define $h_5(s)$ to be the sequence after increasing the entry on the $(\mathsf{max}(s)+1)$th position by one. $\square$

We are now in a position to prove Theorem 1.8.

Proof. We prove this by induction on the numbers $|s|-\mathsf{max}(s)$ for all ascent sequences $s \in \mathcal{A}_n$. For the trivial case $|s| = \mathsf{max}(s) = n$, that is, $s = (0,1,\ldots,n-1)$, we have $\Phi(s) = s$.

For $s \in \mathcal{D}_1 \cap \mathcal{A}_n$, that is, $|s| = \mathsf{max}(s)+1 = n$. Assume that $\mathsf{ealm}(s) = i$ and $\mathsf{max}(s) = p$, then s has the form $(0,1,\ldots,p-1,i)$. Take $\Phi(s)$ to be the sequence after moving the second i to the right of the first i of s, i.e., $\Phi(s) = (0,1,\ldots,i-1,i,i,\ldots,p-1) \in \mathcal{T}_1 \cap \mathcal{A}_n$ and (1.7) clearly holds.

Suppose that the septuple $(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max},\mathsf{ealm},\mathsf{rmin},\mathsf{rpos})$ on ascent sequences $s \in \mathcal{A}_n$ with $|s|-\mathsf{max}(s) = N-1$ is equidistributed to $(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{rmin},\mathsf{rpos},\mathsf{max},\mathsf{ealm})$ on ascent sequences $s \in \mathcal{A}_n$ with $|s|-\mathsf{rmin}(s) = N-1$ under the bijection Φ, we next show it also holds when $N-1$ is replaced by N.

For any ascent sequence $s \in \mathcal{A}^* \cap \mathcal{A}_n$ with $|s|-\mathsf{max}(s) = N$, we are going to define Φ.

If $s \in \mathcal{D}_2 \cap \mathcal{A}_n$, then according to Lemma 5.1, $h_2(s) = (\mathsf{ealm}(s), s^*)$ with $s^* \in \mathcal{A}^* \cap \mathcal{A}_{n-1}$ and $|s^*|-\mathsf{max}(s^*) = N-1$. By induction hypothesis and Lemma 4.2, define

$$\Phi(s) = f_2^{-1}(\mathsf{ealm}(s), \Phi(s^*)) \in \mathcal{T}_2 \cap \mathcal{A}_n,$$

which is a bijection between the sets $\mathcal{D}_2 \cap \mathcal{A}_n$ and $\mathcal{T}_2 \cap \mathcal{A}_n$ such that $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N$. Furthermore, it follows from Lemma 4.2 and 5.1 that (1.7) is true between the subsets $\mathcal{D}_2 \cap \mathcal{A}_n$ and $\mathcal{T}_2 \cap \mathcal{A}_n$.

If $s \in \mathcal{D}_3 \cap \mathcal{A}_n$, then by Lemma 5.2 and 5.3, let $\tilde{s} = (\phi_2 \circ h_3)(s) \in \mathcal{A}_{n-1} \cap \mathcal{A}^*$ such that $|\tilde{s}|-\mathsf{max}(\tilde{s}) = N-1$. As a result, by induction hypothesis, Lemma 4.4 and 4.5, define

$$\Phi(s) = (f_3^{-1} \circ \phi_1^{-1} \circ \Phi \circ \phi_2 \circ h_3)(s) \in \mathcal{T}_3 \cap \mathcal{A}_n, \tag{5.1}$$

which is a bijection between the sets $\mathcal{D}_3 \cap \mathcal{A}_n$ and $\mathcal{T}_3 \cap \mathcal{A}_n$ such that $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N$. In addition, Φ also satisfies (1.7) because of Lemma 4.4, 4.5, 5.2 and 5.3.

If $s \in \mathcal{D}_4 \cap \mathcal{A}_n$, then according to Lemma 5.4, $h_4(s) \in \mathcal{A}_n \cap \mathcal{P}_2^c$. Furthermore by induction hypothesis, there is a bijection Φ between $\mathcal{A}_{n-1} \cap \mathcal{A}^*$ and itself with $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N-1$. Together with Lemma 4.4 and 5.2, we find that $\phi_1^{-1} \circ \Phi \circ \phi_2$ is the bijection between the set $\mathcal{A}_n \cap \mathcal{P}_2$ and $\mathcal{A}_n \cap \mathcal{P}_1$ with $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N-1$. In view of the induction hypothesis on the set $\mathcal{A}_n$, it follows that the complement $\mathcal{A}_n \cap \mathcal{P}_2^c$ is in bijection with $\mathcal{A}_n \cap \mathcal{P}_1^c$ via Φ such that $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N-1$. Define

$$\Phi(s) = (f_4^{-1} \circ \Phi \circ h_4)(s) \in \mathcal{T}_4 \cap \mathcal{A}_n, \tag{5.2}$$

which is a bijection between the sets $\mathcal{D}_4 \cap \mathcal{A}_n$ and $\mathcal{T}_4 \cap \mathcal{A}_n$ such that $|s|-\mathsf{max}(s) = |\Phi(s)| - \mathsf{rmin}(\Phi(s)) = N$. The bijection Φ satisfies (1.7) according to Lemma 4.6 and 5.4.

If $s \in \mathcal{D}_5 \cap \mathcal{A}_n$, we will define $\Phi : \mathcal{D}_5 \cap \mathcal{A}_n \to \mathcal{T}_5 \cap \mathcal{A}_n$ by the already known bijection (5.2).

If $\mathsf{max}(s) - \mathsf{ealm}(s) = 2$, then $h_5(s) \in \{s \in \mathcal{D}_4 \cap \mathcal{A}_n : \mathsf{ealm}(s) \neq 0\}$. In view of (5.2) for the case when $s \in \mathcal{D}_4$ with $|s|-\mathsf{max}(s) = N$, we know that $(\Phi \circ h_5)(s) \in \{s \in \mathcal{T}_4 \cap \mathcal{A}_n : \mathsf{rpos}(s) \neq 0\}$. As a result, we take

$$\Phi(s) = (f_5^{-1} \circ \Phi \circ h_5)(s) \in \mathcal{T}_5 \cap \mathcal{A}_n, \tag{5.3}$$

which is a bijection for the case $\mathsf{max}(s) - \mathsf{ealm}(s) = 2$ and $|s| - \mathsf{max}(s) = N$. Now with this known bijection, we can repeatedly use (5.3) to recursively define the bijection $\Phi : \mathcal{D}_5 \cap \mathcal{A}_n \to \mathcal{T}_5 \cap \mathcal{A}_n$ for other ascent sequences $s \in \mathcal{D}_5 \cap \mathcal{A}_n$ with $\mathsf{max}(s) - \mathsf{ealm}(s) > 2$.

In addition, by combining Proposition 4.8 and Lemma 5.5, we can recursively verify that for $1 \leqslant i \leqslant 3$, $s \in \mathcal{D}_{5,i}$ if and only if $(\Phi \circ h_5)(s) \in f_5(\mathsf{M}_{5,i})$, i.e., according to (5.3), $(f_5^{-1} \circ \Phi \circ h_5)(s) = \Phi(s) \in \mathsf{M}_{5,i}$. This implies that Φ satisfies (1.7).

To sum it up, for all $1 \leqslant i \leqslant 5$, the bijection $\Phi : \mathcal{D}_i \cap \mathcal{A}_n \to \mathcal{T}_i \cap \mathcal{A}_n$ satisfying (1.7) for $s \in \mathcal{D}_i \cap \mathcal{A}_n$ and $|s| - \mathsf{max}(s) = N$ is constructed, under the assumption that (1.7) is true when $s \in \mathcal{D}_i \cap \mathcal{A}_n$ and $|s| - \mathsf{max}(s) = N - 1$. It follows by induction that (1.7) holds, which finishes the proof. □

6. Refined generating functions

This section deals with refined enumerations of ascent sequences with respect to the Euler–Stirling statistics asc, rep, max and rmin, with the purpose to establish bi-symmetric distributions directly from the generating function.

Since the new decomposition of ascent sequences in Section 3 is parallel to the one from [FJL+20], it makes no real difference which one we choose to derive the refined generating functions, the decomposition from [FJL+20] or the new one in Section 3 of this paper. For convenience, we use the decomposition from [FJL+20] because some explicit computations were already done there; we only need to point out the differences when the statistic rmin is included.

We adopt the notations from [FJL+20]. Let $\mathcal{A}$ be the set of all ascent sequences, i.e., $\mathcal{A} = \mathcal{A}^* \cup \{s : s = (0, 1, \ldots, |s| - 1)\}$ and define

$$
\begin{aligned}
F(t; x, y, w, u, z, v) &:= \sum_{\substack{s \in \mathcal{A} \\ |s| > \mathsf{max}(s)}} t^{|s|} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} w^{\mathsf{ealm}(s)} u^{\mathsf{asc}(s)} z^{\mathsf{zero}(s)} v^{\mathsf{rmin}(s)}, \\
G(t; x, y, w, u, z, v) &:= \sum_{s \in \mathcal{A}} t^{|s|} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} w^{\mathsf{ealm}(s)} u^{\mathsf{asc}(s)} z^{\mathsf{zero}(s)} v^{\mathsf{rmin}(s)} \\
&= vtyz(1 - vtuy)^{-1} + F(t; x, y, w, u, z, v).
\end{aligned}
$$

Furthermore, let $a_p(t; x, w, u, z, v) := [y^p]F(t; x, y, w, u, z, v)$.

Here is the partition of the set $\mathcal{A}^*$ into disjoint subsets from [FJL+20]: the first two subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined in Section 5.

$$
\begin{aligned}
&\mathcal{D}_1 = \{s \in \mathcal{A}^* : |s| = \mathsf{max}(s) + 1\}, \\
&\mathcal{D}_2 = \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} \leqslant \mathsf{ealm}(s)\}, \\
&\mathcal{S}_3 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} > \mathsf{ealm}(s), \mathsf{max}(s) \notin \{s_i : \mathsf{max}(s) + 2 \leqslant i \leqslant |s|\}\}, \\
&\mathcal{S}_4 := \{s \in \mathcal{A}^* - \mathcal{D}_1 : s_{\mathsf{max}(s)+2} > \mathsf{ealm}(s), \mathsf{max}(s) \in \{s_i : \mathsf{max}(s) + 2 \leqslant i \leqslant |s|\}\}.
\end{aligned}
$$

By definition $\mathcal{D}_3 \dot\cup \mathcal{D}_4 \dot\cup \mathcal{D}_5 = \mathcal{S}_3 \dot\cup \mathcal{S}_4$. For each above subset, we will calculate the corresponding generating function in order to formulate a functional equation of $F(t; x, y, w, u, z, v)$ as below.

Proposition 6.1. *The generating function* $F(t;x,y,w,u,z,v)$ *satisfies*

$$\begin{aligned}
&\left(1-\frac{ry-1}{y(1-w)}\right)F(t;x,y,w,u,z,v)\\
&=\frac{xyzvt^2(y^2tuwv(1-z)+z(y-yr+1))}{(1-ytu)(1-ytuvw)(y-yzr+z)}-\frac{tx}{1-w}F(t;x,wy,1,u,z,v)\\
&\quad+(tux+y^{-1}-tu)\left(\frac{wy(1-z)+z(y-yr+1)}{(1-w)(y-yzr+z)}\right)F(t;x,y,1,u,z,v)\\
&\quad+\frac{y^2u^2vt^2z(1-v)(tux+y^{-1}-tu)}{1-ytu}\left(\frac{y^2tuvw(1-z)+z(y-yr+1)}{(1-ytuvw)(y-yzr+z)}\right)F(t;x,y,1,u,1,v),
\end{aligned} \tag{6.1}$$

where $r=t(u+x-xu)$.

Proof. We omit the proofs of the generating function formulas for each subset as they are direct extensions of the ones from [FJL+20]. For the first two subsets $\mathcal{D}_1$ and $\mathcal{D}_2$, the generating functions are respectively:

$$\sum_{s\in\mathcal{D}_1} t^{|s|}x^{\mathsf{rep}(s)}y^{\mathsf{max}(s)}w^{\mathsf{ealm}(s)}u^{\mathsf{asc}(s)}z^{\mathsf{zero}(s)}v^{\mathsf{rmin}(s)}=\frac{xyzvt^2(z+ytuwv-ytuzwv)}{(1-ytu)(1-ytuwv)}, \tag{6.2}$$

and

$$\begin{aligned}
&\sum_{s\in\mathcal{D}_2} t^{|s|}x^{\mathsf{rep}(s)}y^{\mathsf{max}(s)}w^{\mathsf{ealm}(s)}u^{\mathsf{asc}(s)}z^{\mathsf{zero}(s)}v^{\mathsf{rmin}(s)}\\
&=\frac{tx}{1-w}(F(t;x,y,w,u,z,v)-F(t;x,yw,1,u,z,v))+tx(z-1)F(t;x,y,0,u,z,v).
\end{aligned} \tag{6.3}$$

For the second two subsets $\mathcal{S}_3$ and $\mathcal{S}_4$, the generating functions are respectively:

$$\begin{aligned}
&\sum_{s\in\mathcal{S}_3} t^{|s|}x^{\mathsf{rep}(s)}y^{\mathsf{max}(s)}w^{\mathsf{ealm}(s)}u^{\mathsf{asc}(s)}z^{\mathsf{zero}(s)}v^{\mathsf{rmin}(s)}\\
&=(tux)\left(\frac{w+z-wz}{1-w}\right)F(t;x,y,1,u,z,v)-\frac{tux}{1-w}F(t;x,y,w,u,z,v)\\
&\quad-tux(z-1)F(t;x,y,0,u,z,v)\\
&\quad+\frac{y^2u^3t^3vxz(1-v)(z(1-tuywv)+tuywv)}{(1-tuywv)(1-tuy)}F(t;x,y,1,u,1,v),
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
&\sum_{s\in\mathcal{S}_4} t^{|s|}x^{\mathsf{rep}(s)}y^{\mathsf{max}(s)}w^{\mathsf{ealm}(s)}u^{\mathsf{asc}(s)}z^{\mathrm{zero}(s)}v^{\mathsf{rmin}(s)}\\
&=\frac{(w+z-wz)(1-ytu)}{(1-w)y}F(t;x,y,1,u,z,v)
\end{aligned}$$

$$+\left(\frac{ytuvw(1-v)}{1-ytuvw}+z-zv\right)yvu^2t^2zF(t;x,y,1,u,1,v)$$
$$-\frac{(1-tuy)}{(1-w)y}F(t;x,y,w,u,z,v)-\frac{(z-1)(1-tuy)}{y}F(t;x,y,0,u,z,v). \tag{6.5}$$

The sum of all generating functions (6.2)–(6.5) equals $F(t;x,y,w,u,z,v)$, which leads to

$$\begin{aligned}
&\left(1-\frac{yt(x+u-ux)-1}{y(1-w)}\right)F(t;x,y,w,u,z,v)\\
&=\frac{xyzvt^2(z+ytuwv-ytuzwv)}{(1-ytu)(1-ytuwv)}-\frac{tx}{1-w}F(t;x,yw,1,u,z,v)\\
&\quad+(z-1)(t(x+u-xu)-y^{-1})F(t;x,y,0,u,z,v)\\
&\quad+\frac{w+z-wz}{1-w}(uxt+y^{-1}-ut)F(t;x,y,1,u,z,v),\\
&\quad+\frac{yu^2vt^2z(1-v)(z-ztuywv+tuywv)}{1-ytuvw}\left(1+\frac{yutx}{1-yut}\right)F(t;x,y,1,u,1,v).
\end{aligned} \tag{6.6}$$

We next set $w=0$ and $r=t(x+u-xu)$ on both sides, yielding

$$\begin{aligned}
F(t;x,y,0,u,z,v)&=\frac{y^2xz^2vt^2}{(1-ytu)(y-yzr+z)}+\frac{z(ytux-ytu+1)}{y-yzr+z}F(t;x,y,1,u,z,v)\\
&\quad+\frac{y^2u^2vt^2z^2(1-v)(ytux-ytu+1)}{(1-ytu)(y-yzr+z)}F(t;x,y,1,u,1,v).
\end{aligned}$$

Substituting the above expression for $F(t;x,y,0,u,z,v)$ in (6.6), we arrive at (6.1). □

By solving (6.1) for the case $z=1$, we deduce the generating function for the quadruple $(\mathsf{asc},\mathsf{rep},\mathsf{max},\mathsf{rmin})$ of statistics on ascent sequences, which is part of Theorem 1.10.

Theorem 6.2. *The generating function $\mathfrak{G}(t;x,y,u,v)$ defined in (1.11) is given by (1.12).*

Proof. We apply the kernel method to (6.1). Choose

$$1-\frac{yr-1}{y(1-w)}=0, \text{ that is, } w=1+y^{-1}-r$$

so that the left-hand-side of (6.1) becomes zero. Consequently the functional equation (6.1) is simplified to

$$\begin{aligned}
F(t;x,y,1,u,z,v)&=\frac{xzvt^2(1-yr)(ytuv(1-z)+z)}{(1-ytu)(1-tuv(y-yr+1))(tux+y^{-1}-tu)}\\
&\quad+\frac{tx(y-yzr+z)}{(y-yr+1)(tux+y^{-1}-tu)}F(t;x,y-yr+1,1,u,z,v)\\
&\quad-\frac{yu^2vt^2z(1-v)(yr-1)(ytuv(1-z)+z)}{(1-ytu)(1-tuv(y-yr+1))}F(t;x,y,1,u,1,v).
\end{aligned} \tag{6.7}$$

We set $z = 1$ on both sides, leading to

$$\begin{aligned}F(t;x,y,1,u,1,v) &= \frac{xvt^2(1-yr)}{(1-ytu)(1-tuv(y-yr+1))(tux+y^{-1}-tu)}\\&\quad+\frac{tx}{(tux+y^{-1}-tu)}F(t;x,y-yr+1,1,u,1,v)\\&\quad-\frac{yu^2vt^2(1-v)(yr-1)}{(1-ytu)(1-tuv(y-yr+1))}F(t;x,y,1,u,1,v),\end{aligned}$$

which can be simplified as

$$\begin{aligned}&F(t;x,y,1,u,1,v)\\&=\frac{xvt^2(1-yr)}{(1-ytu+tuv(yr-1))(1-ytuv)(tux+y^{-1}-tu)}\\&\quad+\frac{tx(1-ytu)(1-tuv(y-yr+1))}{(tux+y^{-1}-tu)(1-ytu+tuv(yr-1))(1-ytuv)}F(t;x,y-yr+1,1,u,1,v).\end{aligned}$$

Define $\delta_m := r^{-1} - r^{-1}(1-yr)(1-r)^m$ so that $\delta_1 = yw = y+1-yr$. By iterating the above equation, we conclude that

$$\begin{aligned}&F(t;x,y,1,u,1,v)\\&=\sum_{m=0}^{\infty}\frac{rv(1-yr)(1-r)^m}{(x-xu+u(1-yr)(1-r)^m)(1-ytuv)}\\&\quad\times\prod_{i=0}^{m}\frac{x(1-(1-yr)(1-r)^i)(x-xu+u(1-yr)(1-r)^i)}{(x-u(x-1)(1-yr)(1-r)^i)(x-xu+u(1-rv)(1-yr)(1-r)^i)},\end{aligned}\tag{6.8}$$

which is equivalent to (1.12). □

7. Transformations of basic hypergeometric series

For convenience, we recall some standard notions from the theory of basic hypergeometric series, cf. [GR04].

For indeterminates a and q (the latter is referred to as the base), and non-negative integer k, the basic shifted factorial (or q-shifted factorial) is defined as

$$(a;q)_k := \prod_{j=1}^{k}(1-aq^{j-1}).\tag{7.1}$$

This also makes sense for $k = \infty$, where the infinite product is viewed as a formal power series in q (whereas, viewed as an analytic expression in q, we would need to insist on $|q| < 1$, for convergence). When dealing with products of q-shifted factorials, it is convenient to use the following short notation,

$$(a_1,\ldots,a_m;q)_k := (a_1;q)_k\cdots(a_m;q)_k,\tag{7.2}$$

where again k is a non-negative integer or ∞.

The ${}_\alpha\phi_\beta$ series defined in (1.8) (where the lower parameters are assumed to be chosen such that no poles occur in the summands of the series) terminates if one of the upper parameters, say a_1, is of the form q^{-n}. Since $(q^{-n};q)_k = 0$ for $k > n$, the series in that case contains only finitely many non-vanishing terms. If the series does not terminate, one usually imposes $|q| < 1$. See [GR04, Sec. 1.2] for conditions under which the series converges.

One of the most important identities in the theory of basic hypergeometric series is the Sears transformation [GR04, (III.15)],

$${}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right] = \frac{(e/a, de/bc; q)_n}{(e, de/abc; q)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, de/bc \end{matrix}; q, q\right]. \tag{7.3}$$

In (7.3), a, b, c, d, e and q are indeterminates and n is a non-negative integer (which is responsible that both ${}_4\phi_3$ series are actually finite sums and each contains only $n+1$ non-vanishing terms).

While for non-terminating basic hypergeometric series in base q we usually consider expansions around $q = 0$, in this paper (and more generally, when dealing with generating functions of members of the Fishburn family) we are dealing with power series in r, which can be written as basic hypergeometric series in base $q = 1 - r$, thus can be viewed as functions analytic around $q = 1$. We need to be cautious when we resort to non-terminating identities for basic hypergeometric series. The first part of the argument in the proof of Theorem 1.9, as our main result in this section, is similar to that used by Andrews and Jelínek in [AJ14] for establishing q-series identities around $q = 1$.

Proof of Theorem 1.9. For each $m \geqslant 0$ the expansion of $(1-a; 1-r)_m$ in monomials $a^i r^l$ only involves terms with $i+l \geqslant m$ and each factor in the denominator of the series has a non-vanishing constant term. Thus, in the expansion of the series in the variables a and r the contribution of coefficients for each monomial $a^i r^l$ is finite. It follows that both sides of the identity are formal power series in the monomials $a^i r^l$, thus are analytic functions in a.

Now both sides of (1.9) agree for $a = 1 - (1-r)^{-n}$ where $n = 0, 1, \ldots$ by the $(q, a, b, c, d, e) \mapsto (1-r, (1-r)^j, b, c, d, e)$ special case of the transformation in (7.3). Since we have shown (1.9) for infinitely many values of a accumulating at $a = -\infty$, i.e. $1 - a = \infty$ (the transformation (7.3) itself is valid in the limiting case $n \to \infty$ (i.e. $q^{-n} \to \infty$)!), by the identity theorem in complex analysis the transformation (1.9) is true for all a in its domain of analyticity. □

Remark 7.1. It is interesting to notice that while the classical Sears transformation in (7.3) concerns a transformation between two terminating ${}_4\phi_3$ series in base q, valid as an identity around $q = 0$, the identity in Theorem 1.9 concerns a transformation between two non-terminating ${}_4\phi_3$ series in base $q = 1 - r$, valid as an identity around $r = 0$ or, equivalently, $q = 1$.

We give two noteworthy specializations as immediate corollaries. The first one is obtained by letting $a \to 1$ in (1.9).

Corollary 7.2. *Let b, c, d, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominators in (7.4) have vanishing constant term in r, we have the following* *transformation of convergent power series in r:*

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}(1-r)^j, b, c\\ d, e\end{matrix}; 1-r, 1-r\right]\\
&\quad = \frac{((1-r)/e; 1-r)_j}{((1-r)bc/de; 1-r)_j}\, {}_3\phi_2\left[\begin{matrix}(1-r)^j, d/b, d/c\\ d, de/bc\end{matrix}; 1-r, 1-r\right].
\end{aligned}
\tag{7.4}
$$

The second one is obtained by replacing c by d/c in (1.9) and letting $d \to 0$.

Corollary 7.3. *Let a, b, c, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominators in (7.5) have vanishing constant term in r, we have the following transformation of convergent power series in a and r:*

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}(1-r)^j, 1-a, b\\ e, (1-r)^{j+1}(1-a)b/ce\end{matrix}; 1-r, 1-r\right]\\
&= \frac{((1-r)/e, (1-r)(1-a)b/ce; 1-r)_j}{((1-r)(1-a)/e, (1-r)b/ce; 1-r)_j}\, {}_3\phi_2\left[\begin{matrix}(1-r)^j, 1-a, c\\ ce/b, (1-r)^{j+1}(1-a)/e\end{matrix}; 1-r, 1-r\right].
\end{aligned}
\tag{7.5}
$$

The here obtained non-terminating basic hypergeometric transformations of base $q = 1-r$ (expanded around $r = 0$) are indeed powerful for proving equidistribution results for the Euler–Stirling statistics.

Proof of Theorem 1.10. Note that Theorem 6.2 as part of Theorem 1.10 is already proved in Section 6. It remains to establish (1.10) and (1.13).

To show the symmetry $\mathfrak{G}(t; x, y, u, v) = \mathfrak{G}(t; x, v, u, y)$ is equivalent to showing the identity

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\left((1-yr)(1-r), \frac{u(1-yr)}{x(u-1)}; 1-r\right)_k (1-r)^k}{\left(\frac{u(x-1)(1-yr)(1-r)}{x}, \frac{u(1-vr)(1-yr)(1-r)}{x(u-1)}; 1-r\right)_k}\\
&= \frac{\left(1 - \frac{x}{u(x-1)(1-yr)}\right)}{\left(1 - \frac{x}{u(x-1)(1-vr)}\right)} \sum_{k=0}^{\infty} \frac{\left((1-vr)(1-r), \frac{u(1-vr)}{x(u-1)}; 1-r\right)_k (1-r)^k}{\left(\frac{u(x-1)(1-vr)(1-r)}{x}, \frac{u(1-vr)(1-yr)(1-r)}{x(u-1)}; 1-r\right)_k}.
\end{aligned}
\tag{7.6}
$$

Identity (7.6) is readily verified by virtue of the $j = 1$ and

$$
\begin{aligned}
b &= (1-yr)(1-r), & c &= \frac{u(1-yr)}{x(u-1)},\\
d &= \frac{u(1-vr)(1-yr)(1-r)}{x(u-1)}, & e &= \frac{u(x-1)(1-yr)(1-r)}{x}
\end{aligned}
$$

special case of Corollary 7.2.

On the other hand, to show the bi-symmetry $\mathcal{G}(t;x,y,u,z) = \mathcal{G}(t;u,z,x,y)$, in view of Theorem 1.6, is equivalent to showing the identity

$$\sum_{k=0}^{\infty} \frac{((1-zr)(1-yr);1-r)_k \left(1-\frac{u(1-yr)}{x(u-1)}\right)}{\left(\frac{u(x-1)(1-yr)(1-r)}{x};1-r\right)_k \left(1-\frac{u(1-yr)}{x(u-1)}(1-r)^k\right)}(1-r)^k$$
$$= \frac{\left(1-\frac{u(1-yr)}{x(u-1)}\right)\left(1-\frac{x}{u(x-1)(1-yr)}\right)\Big)}{\left(1-\frac{x(1-zr)}{u(x-1)}\right)\left(1-\frac{u}{x(u-1)(1-zr)}\right)\Big)}$$
$$\times \sum_{k=0}^{\infty} \frac{((1-zr)(1-yr);1-r)_k \left(1-\frac{x(1-zr)}{u(x-1)}\right)}{\left(\frac{z(u-1)(1-zr)(1-r)}{u};1-r\right)_k \left(1-\frac{x(1-zr)}{u(x-1)}(1-r)^k\right)}(1-r)^k. \tag{7.7}$$

Now, identity (7.7) is readily verified by virtue of the $j=1$ and

$$a = r(z+y-zyr), \qquad b = \frac{u(1-yr)}{x(u-1)},$$
$$c = \frac{x(1-zr)}{u(x-1)}, \qquad e = \frac{u(1-yr)(1-r)}{x(u-1)}$$

special case of Corollary 7.3. $\square$

8. Technical Lemmas and Propositions

The purpose of this section is to prove Proposition 4.8. We take the 'divide-and-conquer' strategy. That is, we first divide the set $\mathcal{T}_5$ into four subsets $\mathcal{T}_{5,i}$, $1 \leqslant i \leqslant 4$, then establish Proposition 4.8 for each of them, and finally collect the subsets $\mathcal{T}_{5,i}$ according to the change of statistics.

By definition, $\mathcal{T}_5$ can be divided into the following four disjoint subsets: (see Figure 8.1)

$$\begin{aligned}
\mathcal{T}_{5,1} &:= \{s \in \mathcal{A}^1 : \mathsf{sebr}(s) > \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1} \text{ and } \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} = \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1\},\\
\mathcal{T}_{5,2} &:= \{s \in \mathcal{A}^1 : \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} \neq \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1\},\\
\mathcal{T}_{5,3} &:= \mathcal{A}^2 - \mathcal{T}_4,\\
\mathcal{T}_{5,4} &:= \{s \in \mathcal{A}^1 : \mathsf{sebr}(s) = \mathsf{Rmin}(s)_{\mathsf{rpos}(s)+1}, \mathsf{Prm}(s)_{\mathsf{rpos}(s)+1} = \mathsf{Prm}(s)_{\mathsf{rpos}(s)} + 1\} - \mathcal{T}_3.
\end{aligned}$$

Since $\mathcal{T}_3 \dot\cup \mathcal{T}_{5,1} \dot\cup \mathcal{T}_{5,2} \dot\cup \mathcal{T}_{5,4} = \mathcal{A}^1$, $\mathcal{T}_4 \dot\cup \mathcal{T}_{5,3} = \mathcal{A}^2$ and $\mathcal{T}_1 \dot\cup \mathcal{T}_2 = \mathcal{A}^* - (\mathcal{A}^1 \dot\cup \mathcal{A}^2)$, it is clear that $\mathcal{A}^*$ is the disjoint union of subsets $\mathcal{T}_i$, $\mathcal{T}_{5,i}$ for $1 \leqslant i \leqslant 4$.

8.1. Bijections on the first two subsets

Here we are going to introduce two bijections on the first two subsets $\mathcal{T}_{5,1}$ and $\mathcal{T}_{5,2}$ respectively. A new statistic $\min \mathrm{M}asc$ is defined in order to describe the image sets of $\mathcal{T}_{5,1}$.

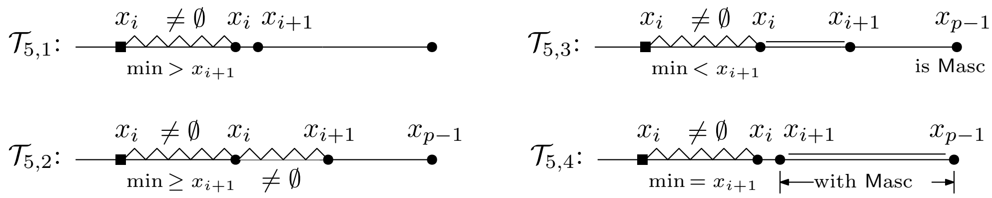


Figure 8.1: A partition of the set $\mathcal{T}_5$: For any $s \in \mathcal{T}_5$ with $\mathsf{rpos}(s) = i$ and $\mathsf{rmin}(s) = p$, $x_i = \mathsf{Rmin}(s)_i$ denotes the i-th right-to-left minimum of s; black dots and squares represent the rightmost and the second rightmost entry respectively.

Definition 8.1. (statistic $\min \mathsf{M}asc$) For any ascent sequence s, define $\min \mathsf{M}asc(s)$ to be the **min**imal $\mathsf{M}asc$ (see Definition 3.3) between the two rightmost entries $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$. If no such $\mathsf{M}asc$ exists, then we assume $\min \mathsf{M}asc(s) = 0$.

For example, given $s = (0, 1, 2, 1, 3, 4, 4, 3, 5)$, we have $\mathsf{rpos}(s) = 2$ and $\min \mathsf{M}asc(s) = 4$ because 4 is the minimal $\mathsf{M}asc$ between the two rightmost entries $\mathsf{Rmin}(s)_2 = 3$.

Lemma 8.2. *There is a bijection $f_{5,1}$ between the set $\mathcal{T}_{5,1} \cap \mathcal{A}_n$ and the set of ascent sequences $s \in \mathcal{A}_n$ such that*

- $\mathsf{rpos}(s) \neq 0$ *and* $\min \mathsf{M}asc(s) = 0$*;*
- *the rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)-1}$ *is next to the second rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$*;*
- *the two rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ *are not next to each other.*

In addition, the bijection $f_{5,1}$ sends the septuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{ealm}, \mathsf{rmin}, \mathsf{rpos} - 1),$$

and satisfies

$$\mathsf{zero}(s) = \mathsf{zero}(f_{5,1}(s)) + \chi(\mathsf{rpos}(s) = 0).$$

Proof. For any ascent sequence $s \in \mathcal{T}_{5,1}$ with $\mathsf{rpos}(s) = i$, insert $\mathsf{Rmin}(s)_{i+1}$ right after the second rightmost $\mathsf{Rmin}(s)_i$ and remove the rightmost $\mathsf{Rmin}(s)_i$; see Figure 8.2. Define the resulting sequence as $f_{5,1}(s)$. It is easily seen that $f_{5,1}$ is a bijection and it fulfills all properties listed in this lemma. $\square$

Lemma 8.3. *There is a bijection $f_{5,2}$ between the set $\mathcal{T}_{5,2} \cap \mathcal{A}_n$ and the set of ascent sequences $s \in \mathcal{A}_n$ such that* $\mathsf{rpos}(s) \neq 0$ *and*

- *the rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)-1}$ *is not next to the second rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$*;*
- *the two rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ *are not next to each other.*

In addition, the bijection $f_{5,2}$ sends the quintuple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos} - 1),$$

and satisfies

$$\begin{aligned} \mathsf{zero}(s) &= \mathsf{zero}(f_{5,2}(s)) + \chi(\mathsf{rpos}(s) = 0), \\ \mathsf{ealm}(s) &= \mathsf{ealm}(f_{5,2}(s)) - \chi(\mathsf{Prm}(s)_{\mathsf{rpos}(s)} = \mathsf{max}(s) + 1). \end{aligned}$$

Proof. For any ascent sequence $s \in \mathcal{T}_{5,2}$ with $\mathsf{rpos}(s) = i$, replace the rightmost $\mathsf{Rmin}(s)_i$ by $\mathsf{Rmin}(s)_{i+1}$; see Figure 8.2. Define the resulting sequence to be $f_{5,2}(s)$. It is straightforward to verify the change of statistics. □

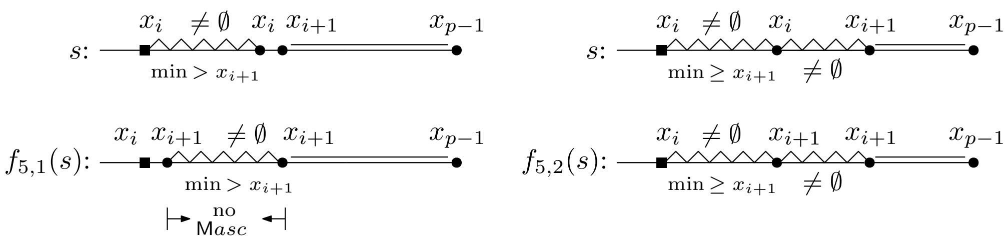


Figure 8.2: The bijections $f_{5,1}$ and $f_{5,2}$ in Lemma 8.2 and 8.3. Here $x_i = \mathsf{Rmin}(s)_i$ and $i = \mathsf{rpos}(s)$.

8.2. Bijections on the second two subsets

Now we turn to introduce a bijection on the other two subsets $\mathcal{T}_{5,3}$ and $\mathcal{T}_{5,4}$.

Proposition 8.4. *Let $\mathcal{B}$ be a set of ascent sequences $s \in \mathcal{A}^* \cap \mathcal{A}_n$ with the following properties:*

- $\mathsf{rpos}(s) \neq 0$ *and* $\min \mathsf{M}asc(s) \neq 0$*;*
- *the rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)-1}$ *is next to the second rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$*;*

Then, there is a bijection f_5^* between the set $(\mathcal{T}_{5,3} \dot\cup \mathcal{T}_{5,4}) \cap \mathcal{A}_n$ and $\mathcal{B}$ and it transforms the quadruple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos} - 1),$$

and satisfies

$$\mathsf{zero}(s) = \mathsf{zero}(f_5^*(s)) + \chi(\mathsf{rpos}(s) = 0).$$

If the second rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ is a maximal of s, then f_5^* transforms the pair

$$(\mathsf{max}, \mathsf{ealm}) \text{ to } (\mathsf{max} - 1, \mathsf{ealm} - 1);$$

otherwise it transforms the pair

$$(\mathsf{max}, \mathsf{ealm}) \text{ to } (\mathsf{max}, \mathsf{ealm}).$$

We divide Proposition 8.4 into two Lemmas (Lemma 8.5 and 8.13) and prove them in subsection 8.3 and 8.4 separately as the proofs employ different substitution/insertion rules.

Before we proceed with the proof of Proposition 8.4, we show how Proposition 8.4 contributes to complete the proof of Proposition 4.8.

Proof of Proposition 4.8. Note that the disjoint union of all image sets of $f_{5,1}$, $f_{5,2}$ and f_5^* is the set $(\mathcal{A}^1\dot{\cup}\mathcal{A}^2)\cap\mathcal{A}_n = (\mathcal{T}_3\dot{\cup}\mathcal{T}_4\dot{\cup}\mathcal{T}_5)\cap\mathcal{A}_n$ of ascent sequences s with $\mathsf{rpos}(s)\neq 0$. Consequently, we take $f_5(s) = f_{5,i}(s)$ when $s\in\mathcal{T}_{5,i}$ for $i=1,2$ and set $f_5(s) = f_5^*(s)$ when $s\in\mathcal{T}_{5,3}\dot{\cup}\mathcal{T}_{5,4}$. Furthermore, it is not hard to see that f_5 satisfies all desired properties after combining Lemma 8.2, 8.3 and Proposition 8.4 (Lemma 8.5 and 8.13). $\square$

8.3. Two substitution rules

Lemma 8.5. *Let $\mathcal{B}_1$ denote a subset of $\mathcal{B}$ (defined in Proposition 8.4) such that the non-zero integer $\min \mathsf{M}asc(s)$ does not appear after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$. Then, Proposition 8.4 is true when f_5^* is restricted between $\mathcal{T}_{5,3}\cap\mathcal{A}_n$ and $\mathcal{B}_1$.*

Two substitution rules $\mathcal{R}_1$ and $\mathcal{R}_2$ are of central importance in the construction of this bijection, so we introduce them before proving Lemma 8.5.

The key observation is that all the following substitutions (1)–(3) in $\mathcal{R}_1$ and (4)–(7) in $\mathcal{R}_2$ are reversible and they preserve all five Euler–Stirling statistics asc, rep, max, zero and rmin.

For convenience, given an ascent sequence s, let $x_j = \mathsf{Rmin}(s)_j$ for all $0\leqslant j<\mathsf{rmin}(s)$.

Rule $\mathcal{R}_1$: For any ascent sequence s such that for some i the entry x_i appears at least twice after the rightmost x_{i-1}, we will replace each non-rightmost x_i by an $\mathsf{M}asc$ m of s as long as

(i) $x_i<m$;

(ii) x_i is located after the Masc m and the rightmost x_{i-1};

(iii) all entries between this x_i and the rightmost x_i are different from m.

This substitution procedure starts with the *first* x_i (also the leftmost x_i) that satisfies (i)–(iii), and then proceeds with other non-rightmost x_i's from left to right.

Let k_1 and k_2 be the left and right neighbors of a given non-rightmost x_i respectively, then ($k_1>x_i$ or $k_1=x_{i-1}$) and $m\neq k_2\geqslant x_i$, so there are only three possible scenarios:

1. If one of the following is true (see Figure 8.3),
   - $x_i<k_1<m$ and $x_i<k_2<m$;
   - ($k_1\geqslant m$ or $k_1=x_{i-1}$) and ($k_2>m$ or $k_2=x_i$),

   then replace the entry x_i directly by m;
2. otherwise if (see Figure 8.4),
   - $x_i<k_1<m$ and ($k_2>m$ or $k_2=x_i$);

then insert m right before the leftmost entry that is to the left of x_i and all entries between it and k_1 inclusive are greater than x_i and smaller than m; afterwards remove x_i;

3. otherwise (see Figure 8.4),

   - ($k_1 \geqslant m$ or $k_1 = x_{i-1}$) and $x_i < k_2 < m$,

then insert m right after the rightmost entry that is to the right of x_i and all entries between it and k_2 inclusive are greater than x_i and smaller than m; afterwards remove x_i.

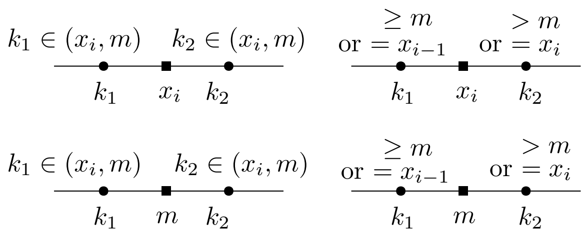


Figure 8.3: Substitution (1) in rule $\mathcal{R}_1$. Here $x_i = \mathsf{Rmin}(s)_i$ and $i = \mathsf{rpos}(s)$.

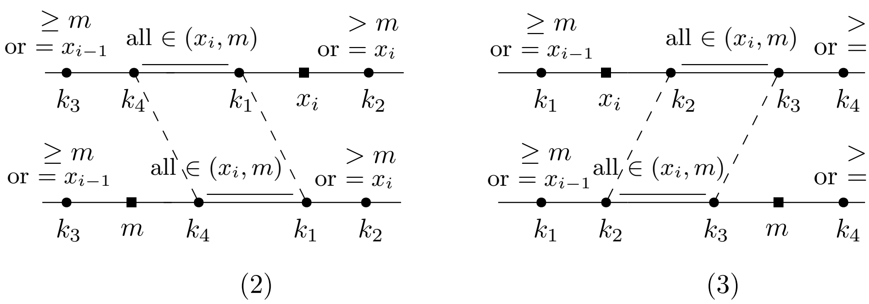


Figure 8.4: Substitutions (2) in rule $\mathcal{R}_1$ and (3) in rules $\mathcal{R}_1, \mathcal{R}_2$. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$.

Example 8.6. Given an ascent sequence $s = (0,1,2,0,1,4,1,2,1,1)$ where $x_1 = \mathsf{Rmin}(s)_1 = 1$ appears at least twice after the rightmost $x_0 = \mathsf{Rmin}(s)_0 = 0$ and $m = 4$ is an Masc, we will replace all non-rightmost 1's that are located after the Masc 4 by integers 4 according to Rule $\mathcal{R}_1$:

$$
\begin{aligned}
s = \ &(0,1,2,0,1,4,\mathbf{1},2,1,1) \quad \text{by substitution (3) of } \mathcal{R}_1,\\
\to\ &(0,1,2,0,1,4,2,4,\mathbf{1},1) \quad \text{by substitution (1) of } \mathcal{R}_1,\\
\to\ &(0,1,2,0,1,4,2,4,4,1) \in \mathcal{B}_1 \text{ (defined in Lemma 8.5)}.
\end{aligned}
$$

It is easy to verify that asc, rep, zero, max, rmin are preserved under the rule $\mathcal{R}_1$.

Rule $\mathcal{R}_2$: in addition to the conditions (i), (ii) and (iii) listed in $\mathcal{R}_1$, here we require that

(iv) the two rightmost x_i are not next to each other;

Like $\mathcal{R}_1$, the procedure starts with the *first* x_i (also the leftmost x_i) that satisfies (i)–(iii), and proceed with other non-rightmost x_i's from left to right.

Let k_1 and k_2 be the left and right neighbours of a given non-rightmost x_i respectively, then ($k_1 > x_i$ or $k_1 = x_{i-1}$) and $m \neq k_2 \geqslant x_i$, so there are four possible scenarios:

4. If $k_2 = x_i$, then k_2 is not a right-to-left minimum (because of (iv)). Assume that k_2 is followed by exactly k identical entries x_i that are not right-to-left minima, then remove k_2 and its k immediate followers, substitute x_i by m according to (5)–(7) below and finally add $(k+1)$ identical entries m after the newly inserted m;

5. otherwise $k_2 \neq x_i$, if one of the following is true (see Figure 8.5),

   - $x_i < k_1 < m$ and $x_i < k_2 < m$,
   - ($k_1 > m$ or $k_1 = x_{i-1}$) and $k_2 > m$,

   then replace the entry x_i by m;

6. otherwise if (see Figure 8.5)

   - $x_i < k_1 \leqslant m$ and $k_2 > m$,

   then insert m right after the rightmost entry that is to the right of x_i and all entries between it and k_2 inclusive are greater than m; afterwards remove x_i;

7. otherwise, do (3) of $\mathcal{R}_1$ (see Figure 8.4).

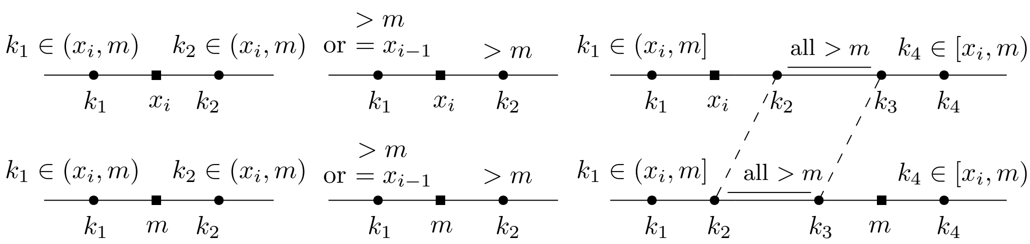


Figure 8.5: Substitution (5) and (6) in rule $\mathcal{R}_2$. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$.

Example 8.7. Given an ascent sequence $s = (0,1,2,0,1,4,4,1,5,2,1,3,1)$ where $x_1 = 1$ appears at least twice after the rightmost $x_0 = 0$ and $m = 4$ is an $\mathsf{M}asc$, we will replace all non-rightmost 1's that are located after the leftmost 4 by integers 4 according to Rule $\mathcal{R}_2$:

$$\begin{aligned} s = \ & (0,1,2,0,1,4,4,\mathbf{1},5,2,1,3,1) \quad \text{by substitution } (6) \text{ of } \mathcal{R}_2, \\ \to \ & (0,1,2,0,1,4,4,5,4,2,\mathbf{1},3,1) \quad \text{by substitution } (5) \text{ of } \mathcal{R}_2, \\ \to \ & (0,1,2,0,1,4,4,5,4,2,4,3,1) \in \mathcal{B}_1 \text{ (defined in Lemma 8.5)}. \end{aligned}$$

It is easy to verify that $\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max}, \mathsf{rmin}$ are preserved under the rule $\mathcal{R}_2$.

Remark 8.8. The reason to define two different substitution rules $\mathcal{R}_1, \mathcal{R}_2$ is that Case 1,2 and Case 3,4 in the proof of Lemma 8.5 have to be treated differently.

We are now in a position to complete the proof of Lemma 8.5.

Proof. We start with showing the bijection

$$g : \mathcal{T}_{5,3} \cap \mathcal{A}_n \to \{s \in \mathcal{A}_{n-1} : \mathsf{rpos}(s) \neq 0\}.$$

For any ascent sequence $s \in \mathcal{T}_{5,3} \cap \mathcal{A}_n$ with $\mathsf{rpos}(s) = i$, replacing the rightmost $\mathsf{Rmin}(s)_i$ by $\mathsf{sebr}(s)$ and removing the last entry leads to an ascent sequence s^* with $\mathsf{rpos}(s^*) = i + 1$. Define $g(s) = s^*$ and clearly g is invertible, so g is a bijection. Similar to Lemma 4.6, it is straightforward to verify that g transforms the quadruple

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos}) \text{ to } (\mathsf{asc} + 1, \mathsf{rep}, \mathsf{max}, \mathsf{rmin}, \mathsf{rpos} - 1),$$

and satisfies $\mathsf{zero}(g(s)) = \mathsf{zero}(s) - \chi(\mathsf{rpos}(s) = 0)$. If $\mathsf{Prm}_{\mathsf{rpos}(s)} \neq \mathsf{max}(s) + 1$, then $\mathsf{ealm}(s) = \mathsf{ealm}(g(s))$; otherwise $\mathsf{ealm}(s) = \mathsf{ealm}(g(s)) - 1$.

We next define the map

$$g_{5,3} : \{s \in \mathcal{A}_{n-1} : \mathsf{rpos}(s) \neq 0\} \to \mathcal{B}_1 \tag{8.1}$$

and then prove $g_{5,3}$ is a bijection so that

$$f_5^* := g_{5,3} \circ g : \mathcal{T}_{5,3} \cap \mathcal{A}_n \to \mathcal{B}_1$$

is the desired bijection for Lemma 8.5.

For any ascent sequence $s \in \mathcal{A}_{n-1}$ with $\mathsf{rpos}(s) = i \neq 0$, we discuss four possible scenarios and define the resulting sequence to be $g_{5,3}(s)$ in each case.

Case 1 (see Figure 8.6): if the rightmost $\mathsf{Rmin}(s)_{i-1}$ is next to the entry $\mathsf{Rmin}(s)_i$ and there is at least one Masc between the first two $\mathsf{Rmin}(s)_i$ that are located after the rightmost $\mathsf{Rmin}(s)_{i-1}$, let the smallest one be m, then

- insert $m + 1$ right after the Masc m;
- replace all entries y after the inserted $m + 1$ by $y + 1$ if $y \geqslant m$;
- if there are only two $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$, then stop; otherwise, replace each $\mathsf{Rmin}(s)_i$ that appears between the leftmost m and the rightmost $\mathsf{Rmin}(s)_i$ by an m according to rule $\mathcal{R}_1$.

Example 8.9. For $s = (0, 1, 2, 0, 1, 2, 5, 5, 2, 6, 3, 2, 1, 3, 7, 9) \in \mathcal{T}_{5,3} \cap \mathcal{A}_{16}$, then after applying the bijection g, we have $g(s) = (0, 1, 2, 0, 1, 2, 5, 5, 2, 6, 3, 2, 2, 3, 7)$ which belongs to Case 1. Then according to the steps in Case 1, $m = 5$ and

$$\begin{aligned} g(s) &\to (0, 1, 2, 0, 1, \mathbf{2}, 5, 6, 6, 2, 7, 3, 2, \mathbf{2}, 3, 8) \\ &\to (0, 1, 2, 0, 1, \mathbf{2}, 5, 6, 6, 5, 7, 5, 3, \mathbf{2}, 3, 8) = f_5^*(s). \end{aligned}$$

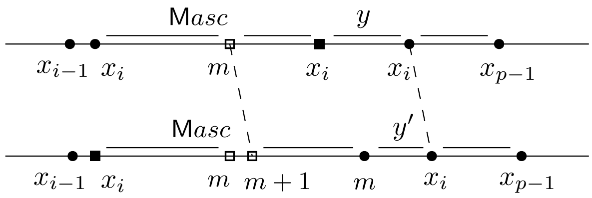


Figure 8.6: Case 1: the rightmost x_{i-1} is next to x_i and m is the smallest Masc between the first two x_i's that are after x_{i-1}. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$ and $y' = y + 1$ if $y \geqslant m$; otherwise $y' = y$.

Case 2 (see Figure 8.7): if the rightmost $\mathsf{Rmin}(s)_{i-1}$ is next to the entry $\mathsf{Rmin}(s)_i$ and *no* Masc appears between the first two $\mathsf{Rmin}(s)_i$ that appear after the rightmost $\mathsf{Rmin}(s)_{i-1}$, let $m-1$ be the number of ascents from the beginning s_1 to the second $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$, then

- insert m right before the second $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$;
- replace any entry y after the inserted m by $y + 1$ if $y \geqslant m$;
- if there are only two $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$, then stop; otherwise, replace each $\mathsf{Rmin}(s)_i$ that is between the leftmost m and the rightmost $\mathsf{Rmin}(s)_i$ by an m according to rule $\mathcal{R}_1$.

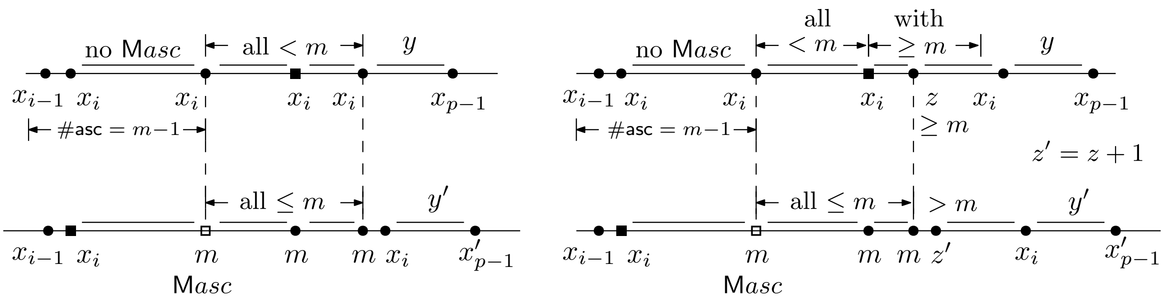


Figure 8.7: Case 2: the rightmost x_{i-1} is next to x_i and no Masc appears between the first two x_i's that are after x_{i-1}. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$ and $y' = y + 1$ if $y \geqslant m$; otherwise $y' = y$.

Example 8.10. For $s = (0, 1, 2, 0, 1, 2, 4, 5, 2, 6, 3, 2, 1, 3, 7, 10) \in \mathcal{T}_{5,3} \cap \mathcal{A}_{16}$, then after applying the bijection g, we have $g(s) = (0, 1, 2, 0, 1, 2, 4, 5, 2, 6, 3, 2, 2, 3, 7)$ which belongs to Case 2. Then according to the steps in Case 2, $m = 7$ and

$$\begin{aligned} g(s) &\to (0, 1, 2, 0, 1, \mathbf{2}, 4, 5, 7, 2, 6, 3, 2, \mathbf{2}, 3, 8) \\ &\to (0, 1, 2, 0, 1, \mathbf{2}, 4, 5, 7, 6, 3, 7, 7, \mathbf{2}, 3, 8) = f_5^*(s). \end{aligned}$$

Case 3 (see Figure 8.8): if the rightmost $\mathsf{Rmin}(s)_{i-1}$ is *not* next to the entry $\mathsf{Rmin}(s)_i$ and the two rightmost $\mathsf{Rmin}(s)_i$'s are *not* next to each other, let $m-2$ be the number of ascents from the beginning s_1 to the first $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$, then

- insert $\mathsf{Rmin}(s)_i$ immediately after the rightmost $\mathsf{Rmin}(s)_{i-1}$;
- if the second $\mathsf{Rmin}(s)_i$ after the rightmost $\mathsf{Rmin}(s)_{i-1}$ is followed by exactly k non-rightmost $\mathsf{Rmin}(s)_i$ (k could be zero), then replace these $(k+1)$ identical entries $\mathsf{Rmin}(s)_i$ by $(k+1)$ identical m;
- replace all entries y after the rightmost inserted m by $y+1$ if $y \geqslant m$;
- substitute each $\mathsf{Rmin}(s)_i$ that is between the leftmost m and the rightmost $\mathsf{Rmin}(s)_i$ by an m according to rule $\mathcal{R}_2$.

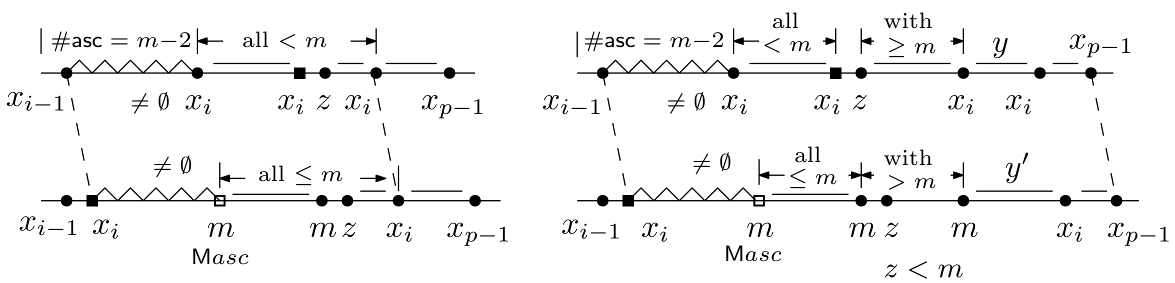


Figure 8.8: Case 3: the rightmost x_{i-1} is not next to x_i and the two rightmost x_i's are not next to each other. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$, $z < m$ and $y' = y+1$ if $y \geqslant m$; otherwise $y' = y$.

Example 8.11. For $s = (0,1,2,0,1,3,2,5,5,2,7,3,1,3,8) \in \mathcal{T}_{5,3} \cap \mathcal{A}_{15}$, then by applying bijection g, we obtain $g(s) = (0,1,2,0,1,3,2,5,5,2,7,3,2,3)$ which belongs to Case 3. According to the construction of $g_{5,3}$ for Case 3, we have $m=6$ and

$$\begin{aligned} g(s) &\to (0,1,2,0,1,\mathbf{2},3,\mathbf{2},5,5,\mathbf{2},8,3,\mathbf{2},3) \\ &\to (0,1,2,0,1,\mathbf{2},3,6,5,5,8,6,3,\mathbf{2},3) = f_5^*(s). \end{aligned}$$

Case 4 (see Figure 8.9): if the rightmost $\mathsf{Rmin}(s)_{i-1}$ is *not* next to the entry $\mathsf{Rmin}(s)_i$ and the two rightmost $\mathsf{Rmin}(s)_i$ are next to each other, let $m-2$ be the number of ascents from the beginning to the rightmost $\mathsf{Rmin}(s)_{i-1}$, then, assuming that exactly $(k+1)$ rightmost $\mathsf{Rmin}(s)_i$ are next to each other ($k \geqslant 1$), we

- remove k rightmost $\mathsf{Rmin}(s)_i$;
- insert two integers $\mathsf{Rmin}(s)_i m$ immediately after the rightmost $\mathsf{Rmin}(s)_{i-1}$;
- replace all entries y after the inserted m by $y+1$ if $y \geqslant m$;

- substitute each non-rightmost $\mathsf{Rmin}(s)_i$ that are between the leftmost m and the rightmost $\mathsf{Rmin}(s)_i$ by an m according to rule $\mathcal{R}_2$;
- insert $(k-1)$ m's immediately after the leftmost m.

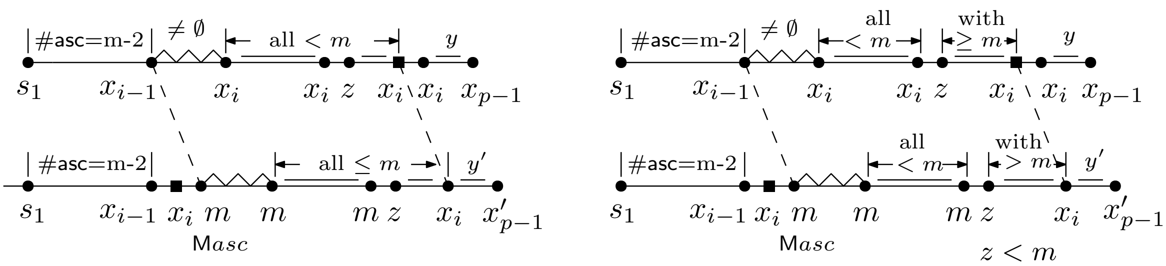


Figure 8.9: Case 4: the rightmost x_{i-1} is not next to x_i and the two rightmost x_i's are next to each other. Here $x_i = \mathsf{Rmin}(s)_i$ with $i = \mathsf{rpos}(s)$, $z < m$ and $y' = y + 1$ if $y \geqslant m$; otherwise $y' = y$.

Example 8.12. For $s = (0,1,2,0,1,3,2,5,5,2,7,3,2,1,3,8) \in \mathcal{T}_{5,3} \cap \mathcal{A}_{16}$, then by applying bijection g, we obtain $g(s) = (0,1,2,0,1,3,2,5,5,2,7,3,2,2,3)$ which belongs to Case 4. According to the construction of $g_{5,3}$ for Case 4, we have $m = 5$ and

$$\begin{aligned} g(s) &\to (0,1,2,0,1,\mathbf{2},5,3,2,6,6,2,8,3,\mathbf{2},3) \\ &\to (0,1,2,0,1,\mathbf{2},5,3,6,6,5,5,8,3,\mathbf{2},3) = f_5^*(s). \end{aligned}$$

By the construction of $g_{5,3}(s)$ (see (8.1)), one can readily see that $g_{5,3}(s) \in \mathfrak{B}_1$. It remains to show that $g_{5,3}$ is a bijection.

For any ascent sequence $\hat{s} \in \mathfrak{B}_1$ with $\mathsf{rpos}(\hat{s}) = i \neq 0$ and $\min \mathsf{M}asc(\hat{s}) = \hat{m}$, if all entries between the two rightmost $\mathsf{Rmin}(\hat{s})_i$ are less than or equal to $\hat{m}$, then $\hat{s}$ is produced from

- Case 2 if the last $\hat{m}$ is next to the rightmost $\mathsf{Rmin}(\hat{s})_i$ (see the left one of Figure 8.7);
- Case 3 otherwise if the first $\hat{m}$ is not next to $\mathsf{Rmin}(\hat{s})_i$ (see the left one of Figure 8.8);
- Case 4 otherwise (see the left one of Figure 8.9).

If there exists an entry that is greater than $\hat{m}$ and appears between the two rightmost $\mathsf{Rmin}(\hat{s})_i$, then $\hat{s}$ comes from

- Case 1 if the leftmost $\hat{m}$ is followed by $\hat{m}+1$ (see Figure 8.6);
- Case 2 otherwise if the first entry that is greater than $\hat{m}$ appears immediately after a non-leftmost $\hat{m}$ (see the right one of Figure 8.7);
- Case 3 otherwise if the first $\hat{m}$ is not next to $\mathsf{Rmin}(\hat{s})_i$ (see the right one of Figure 8.8);
- Case 4 otherwise (see the right one of Figure 8.9).

This implies that $g_{5,3}$ is surjective. Since all steps in all cases including the substitution rules $\mathcal{R}_1$ and $\mathcal{R}_2$ are reversible, the map $g_{5,3}$ is therefore injective. In consequence, $g_{5,3}$ is a bijection, implying the composition $g_{5,3} \circ g$ is the desired bijection f_5^* when restricted to the set $\mathcal{T}_{5,3} \cap \mathcal{A}_n$.

Regarding the statistics, the bijection $g_{5,3}$ sends $(\mathsf{asc}, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos})$ to $(\mathsf{asc} - 1, \mathsf{rep}, \mathsf{rmin}, \mathsf{rpos})$. Only when $\mathsf{rpos}(s) = 0$, $\mathsf{zero}(f_{5,3}(s)) = \mathsf{zero}(s) + 1$. In analogy to Lemma 4.6, one can examine the change of statistics max and ealm. $\square$

We next turn to introduce the bijection for the subset $\mathcal{T}_{5,4}$ where two insertion rules $\mathcal{R}_3, \mathcal{R}_4$ to modify the set of right-to-left minima are needed.

8.4. Two insertion rules

Lemma 8.13. *Proposition 8.4 is true when f_5^* is restricted between $\mathcal{T}_{5,4} \cap \mathcal{A}_n$ and the set $\mathcal{B} - \mathcal{B}_1$ of ascent sequences s from $\mathcal{B}$ (defined in Proposition 8.4) where the non-zero integer* $\min \mathsf{M}asc(s)$ *also appears after the rightmost* $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$.

We prove Lemma 8.13 right after the rules $\mathcal{R}_3$ and $\mathcal{R}_4$ are defined.

Rule $\mathcal{R}_3$: For any ascent sequence s, let m be an $\mathsf{M}asc$ of s that appears only once and it is not a right-to-left minimum, set

$$\kappa := \mathsf{max}\{l : \mathsf{Rmin}(s)_l \leqslant m - 1\}, \tag{8.2}$$

we will insert an m to s so that m becomes a new right-to-left minimum.

- If $\kappa = \mathsf{rmin}(s) - 1$, i.e., the last right-to-left minimum $\mathsf{Rmin}(s)_\kappa$ (or equivalently the last entry) is smaller than m, then we add m at the end of s; otherwise, we replace the rightmost $\mathsf{Rmin}(s)_{\kappa+1}$ by m, replace the rightmost $\mathsf{Rmin}(s)_{r+1}$ by $\mathsf{Rmin}(s)_r$ for $\kappa + 1 \leqslant r \leqslant \mathsf{rmin}(s) - 2$ and add $\mathsf{Rmin}(s)_{\mathsf{rmin}(s)-1}$ at the end.

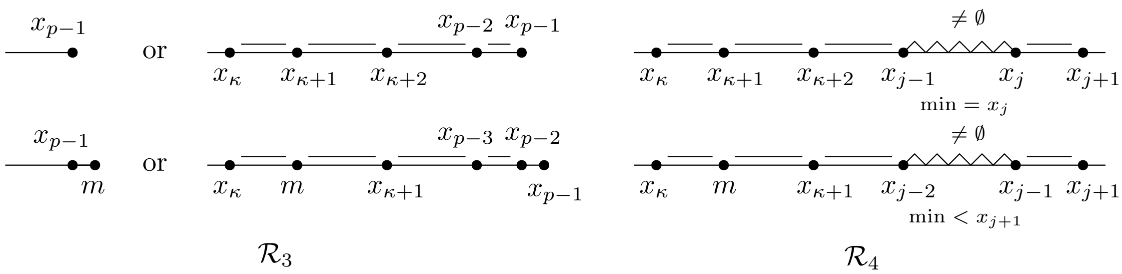


Figure 8.10: The insertion rules $\mathcal{R}_3$ and $\mathcal{R}_4$ where $x_i = \mathsf{Rmin}(s)_i$, $\mathsf{rmin}(s) = p$, $\mathsf{rpos}(s) = j$ and κ is the maximal index such that $x_\kappa \leqslant m - 1$.

Rule $\mathcal{R}_4$: in addition to the conditions of Rule $\mathcal{R}_3$, here we also required that $\kappa < \mathsf{rpos}(s)$. We insert an m and remove the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ so that m is a new right-to-left minimum. This is achieved by replacing the rightmost $\mathsf{Rmin}(s)_{\kappa+1}$ by m, replacing the rightmost $\mathsf{Rmin}(s)_{r+1}$ by $\mathsf{Rmin}(s)_r$ for $\kappa + 1 \leqslant r \leqslant \mathsf{rpos}(s) - 1$.

We are now ready to prove Lemma 8.13.

Proof. For any ascent sequence $s \in \mathcal{T}_{5,4}$, we distinguish three cases according to the location of the first Masc after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$. For the first two cases, the map $s \mapsto f_{5,4}(s)$ is explicitly defined, based on which the map $s \mapsto f_{5,4}(s)$ for the remaining case is recursively constructed.

Case 1 (see Figure 8.11): if the first Masc after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ is a right-to-left minimum, we then implement the following Step 1 on the pair $(s, \mathsf{rpos}(s))$ to construct a new sequence $f_{5,4}(s) \in \mathcal{B} - \mathcal{B}_1$.

Step 1 (see Figure 8.11): For any pair (s, u) where $s \in \mathcal{T}_{5,4}$ and $u \leqslant \mathsf{rpos}(s)$, assume that the rightmost $\mathsf{Rmin}(s)_j$ is the first Masc after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$, then

- remove all entries after the rightmost $\mathsf{Rmin}(s)_{j-1}$;
  (All removed entries form an increasing sequence of Masc's of s.)
- remove the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$;
  (The removal increases the value of the rpos-statistic by one, guaranteeing that the application of $g_{5,3}$ in the following operation is permissible.)
- if $u = \mathsf{rpos}(s)$, apply the bijection $g_{5,3}$ (see (8.1));
- let m equal the minimal Masc between the two rightmost $\mathsf{Rmin}(s)_{u+1}$, and insert m according to rule $\mathcal{R}_3$;
  (This operation inserts m after the rightmost $\mathsf{Rmin}(s)_{u+1}$, yielding a sequence belonging to the image set $\mathcal{B} - \mathcal{B}_1$.)
- for all t such that $u + 2 \leqslant t \leqslant \kappa$ (defined in (8.2)), replace each non-rightmost $\mathsf{Rmin}(s)_t$ entry that is located after $\mathsf{Rmin}(s)_{t-1}$ by an m according to rule $\mathcal{R}_1$;
  (This substitution ensures that the value of the rpos-statistic is always $u + 1$.)
- add $(\mathsf{rmin}(s) - j - 1)$ Masc's at the end
  (in order to preserve the statistics asc, rep, rmin).

Define $f_{5,4}(s)$ to be the resulting sequence after applying Step 1 to the pair $(s, \mathsf{rpos}(s))$.

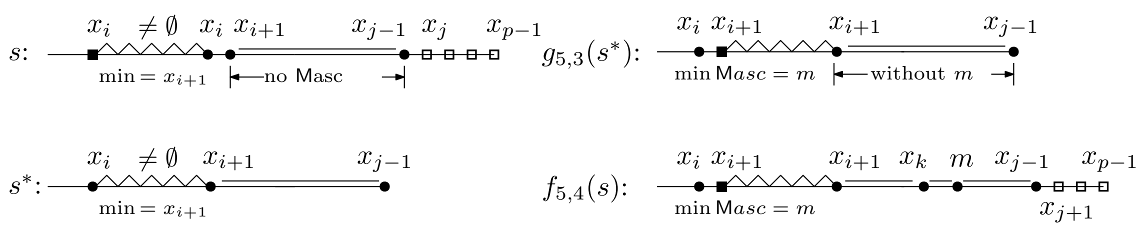


Figure 8.11: The construction of $s \mapsto f_{5,4}(s)$ for Case 1 when the first Masc after the rightmost x_i is a right-to-left minimum. Here $x_l = \mathsf{Rmin}(s)_l$, $i = \mathsf{rpos}(s)$ and $\mathsf{rmin}(s) = p$.

Example 8.14. For $s = (0,1,2,0,1,2,4,5,2,1,2,4,3,9,10) \in \mathcal{T}_{5,4} \cap \mathcal{A}_{15}$, $\mathsf{rpos}(s) = 1$ and the first Masc after the rightmost 1 is 9. We are going to apply Step 1 on the pair $(s, 1)$:

$$s \to (0,1,2,0,1,2,4,5,2,1,2,4,3) \to (0,1,2,0,1,2,4,5,2,2,4,3) \xrightarrow{g_{5,3}} (0,1,2,0,1,2,4,5,7,7,2,4,3)$$

and $m = 7$. Then apply the rule $\mathcal{R}_3$, leading to $(0,1,2,0,1,2,4,5,7,7,2,4,3,7)$. Finally add one M$asc$ at the end and yield

$$f_{5,4}(s) = (0,1,2,0,1,2,4,5,7,7,2,4,3,7,10).$$

Case 2 (see Figure 8.12): if the first Masc after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ appears exclusively between two right-to-left minima, then we implement Step 2 on the pair $(s, \mathsf{rpos}(s))$ to construct a new sequence $f_{5,4}(s) \in \mathcal{B} - \mathcal{B}_1$.

Step 2 (see Figure 8.12): For any pair (s, u) where $s \in \mathcal{T}_{5,4}$ and $u \leqslant \mathsf{rpos}(s)$, assume the first M$asc$ after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ appears exclusively between the rightmost $\mathsf{Rmin}(s)_{j-1}$ and $\mathsf{Rmin}(s)_j$, then

- remove the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$;
- add $\mathsf{Rmin}(s)_j$ right after the rightmost $\mathsf{Rmin}(s)_{j-1}$;
  (The first two operations allow us to separate the sequence into two parts and apply the bijections $g_{5,3}$ and $g_{5,3}^{-1}$ both locally and globally in the following steps.)
- if $u = \mathsf{rpos}(s)$, then separate the sequence right after the rightmost $\mathsf{Rmin}(s)_{j-1}$; apply $g_{5,3}$ (see (8.1)) to the left part, then let m be the minimal Masc of the resulting left part; replace all entries y from the right part by $y+1$ if $y \geqslant m$; afterwards put these two parts back together. Otherwise if $u \neq \mathsf{rpos}(s)$, then let m be the minimal Masc between the two rightmost $\mathsf{Rmin}(s)_{u+1}$.
- apply $g_{5,3}^{-1}$ to the entire sequence;
  (This produces a sequence without entry m after the rightmost $\mathsf{Rmin}(s)_{u+1}$ and the next two operations will insert entry m after it, leading to a sequence from the image set $\mathcal{B} - \mathcal{B}_1$.)
- If $\kappa < j$ (κ is defined in (8.2)), then insert m according to rule $\mathcal{R}_4$;
- for all t such that $u + 2 \leqslant t \leqslant \kappa$, replace every non-rightmost $\mathsf{Rmin}(s)_t$ that is located after the rightmost $\mathsf{Rmin}(s)_{t-1}$ by an m according to rule $\mathcal{R}_1$.

Define $f_{5,4}(s)$ to be the resulting sequence after applying Step 2 to the pair $(s, \mathsf{rpos}(s))$.

Example 8.15. For $s = (0,1,2,0,1,2,4,5,2,1,2,4,3,9,4) \in \mathcal{T}_{5,4} \cap \mathcal{A}_{15}$, $\mathsf{rpos}(s) = 1$ and the first Masc after the rightmost 1 is 9. It is located between two right-to-left minima 3 and 4. We are going to apply Step 2 on the pair $(s, 1)$:

$$s \to (0,1,2,0,1,2,4,5,2,2,4,3,4,9,4).$$

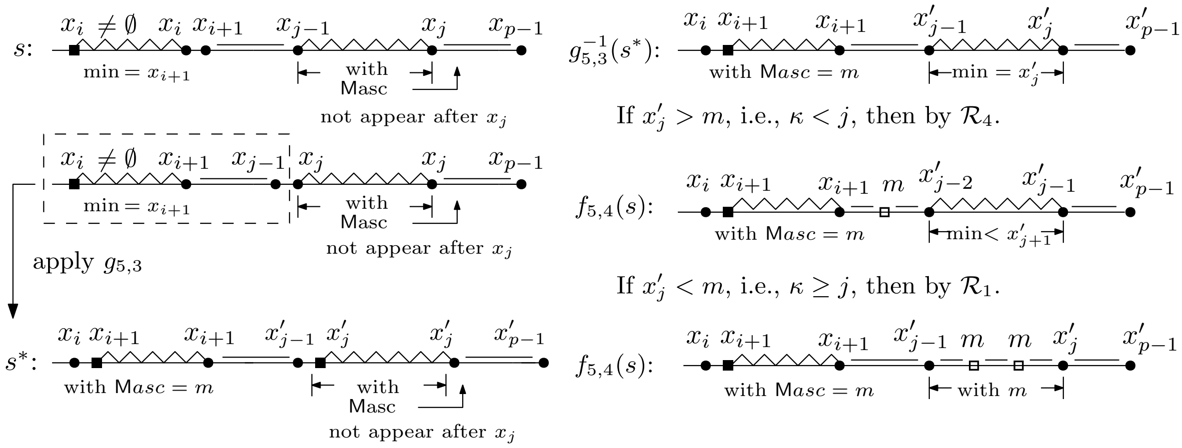


Figure 8.12: The construction $s \mapsto f_{5,4}(s)$ for Case 2. Here $\mathsf{rmin}(s) = p$, $x_l = \mathsf{Rmin}(s)_l$ with $i = \mathsf{rpos}(s)$ and $y' = y + 1$ if $y \geqslant m$; otherwise $y = y'$.

We split this sequence after the rightmost 3. Applying the bijection $g_{5,3}$ on the left part leads to a sequence $(0, 1, 2, 0, 1, 2, 4, 5, 7, 7, 2, 4, 3)$ and $m = 7$. Then the right part $(4, 9, 4)$ becomes $(4, 10, 4)$ because every element is increased by 1 if it is at least 7. Combining these two parts again yield

$$(0, 1, 2, 0, 1, 2, 4, 5, 7, 7, 2, 4, 3, 4, 10, 4).$$

Then apply $g_{5,3}^{-1}$ on the entire sequence, we get $(0, 1, 2, 0, 1, 2, 4, 5, 7, 7, 2, 4, 3, 4, 4)$. Finally apply the rule $\mathcal{R}_1$ to replace non-rightmost 4 by 7 and result an ascent sequence

$$f_{5,4}(s) = (0, 1, 2, 0, 1, 2, 4, 5, 7, 7, 2, 4, 3, 7, 4).$$

We next show the image sets of $f_{5,4}$ for Case 1 and Case 2 are disjoint.

For any $\hat{s} \in \mathcal{B} - \mathcal{B}_1$ with $m = \min \mathsf{M}asc(\hat{s})$, we divide $\mathcal{B} - \mathcal{B}_1$ into two disjoint subsets $\mathcal{C}_1$ and $\mathcal{C}_2$: $\mathcal{C}_1$ contains all ascent sequence $\hat{s} \in \mathcal{B} - \mathcal{B}_1$ satisfying the following conditions:

- m is a right-to-left minimum, say the $(k + 1)$th right-to-left minimum;
- either rightmost $\mathsf{Rmin}(\hat{s})_{t-1}$ and $\mathsf{Rmin}(\hat{s})_t$ are next to each other or the minimal entry in between is greater than or equal to $\mathsf{Rmin}(\hat{s})_{t+1}$ for all $k + 1 \leqslant t \leqslant \mathsf{rmin}(\hat{s}) - 1$.

Let $\mathcal{C}_2 := \mathcal{B} - \mathcal{B}_1 - \mathcal{C}_1$.

By the construction of $f_{5,4}(s)$ in Cases 1,2, it is clear that the image set of $f_{5,4}(s)$ for Case 1 is a subset of $\mathcal{C}_1$, while the one for Case 2 is a subset of $\mathcal{C}_2$. Together with the fact that all steps are reversible, it follows that $f_{5,4}$ is injective for these two cases, from which we will recursively define the map $f_{5,4}$ for the remaining case.

First note that for any $s \in \mathcal{T}_{5,4}$, $\mathsf{rmin}(s) - \mathsf{rpos}(s) \geqslant 3$. For the starting case $\mathsf{rmin}(s) - \mathsf{rpos}(s) = 3$, s belongs to Case 1 or 2. Since the image set of $f_{5,4}(s)$ when $\mathsf{rmin}(s) - \mathsf{rpos}(s) = 3$ is exactly $\mathcal{C}_1 \dot\cup \mathcal{C}_2$ and $f_{5,4}$ is injective for these two cases, $f_{5,4}$ is a bijection when $\mathsf{rmin}(s) - \mathsf{rpos}(s) = 3$.

Next assuming that there is a bijection $f_{5,4} : \mathcal{T}_{5,4} \cap \mathcal{A}_n \to \mathcal{C}_1 \dot{\cup} \mathcal{C}_2$ for all ascent sequences s with $\mathsf{rmin}(s) - \mathsf{rpos}(s) \leqslant N$, we will construct the map $f_{5,4}$ for the ones with $\mathsf{rmin}(s) - \mathsf{rpos}(s) = N + 1$ and prove it is a bijection.

For any ascent sequence $s \in \mathcal{T}_{5,4}$ with $\mathsf{rmin}(s) - \mathsf{rpos}(s) = N + 1$, if s belongs to Case 1 or 2, then $f_{5,4}(s)$ is already given and we stop; otherwise s must belong to the following case and a new sequence $f_{5,4}(s) \in \mathcal{C}_1 \dot{\cup} \mathcal{C}_2$ will be defined.

Case 3 (see Figure 8.13): if the first $\mathsf{M}asc$ after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ appears not only between two right-to-left minima, but also afterwards, then we implement the following step on the pair $(s, \mathsf{rpos}(s))$ to produce a new sequence $f_{5,4}(s)$.

Step 3 (see Figure 8.13): For any pair (s, u) where $s \in \mathcal{T}_{5,4}$ and $u \leqslant \mathsf{rpos}(s)$, assume that the first $\mathsf{M}asc$ after the rightmost $\mathsf{Rmin}(s)_{\mathsf{rpos}(s)}$ appears between the rightmost $\mathsf{Rmin}(s)_{j-1}$ and $\mathsf{Rmin}(s)_j$, as well as after the rightmost $\mathsf{Rmin}(s)_j$, then

- do the first three sub-steps (the first three black points) of Step 2;
- apply $f_{5,4}^{-1}$ according to induction hypothesis and let $s^\bullet$ denote the resulting sequence;
- if $s^\bullet$ belongs to Case 1, do Step 1 on the pair $(s^\bullet, \mathsf{rpos}(s))$ and then stop;
- if $s^\bullet$ belongs to Case 2, do Step 2 on the pair $(s^\bullet, \mathsf{rpos}(s))$ and then stop;
- otherwise repeat Step 3 on the pair $(s^\bullet, \mathsf{rpos}(s))$.

Define $f_{5,4}(s)$ to be the resulting sequence after applying Step 3 to the pair $(s, \mathsf{rpos}(s))$.

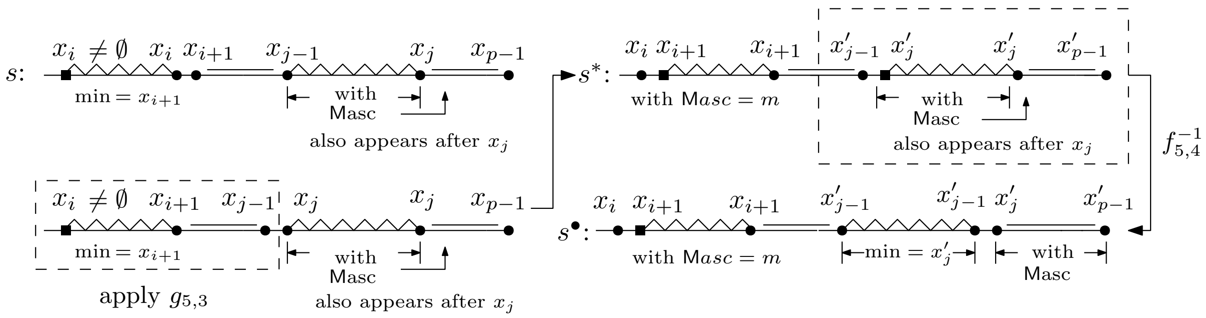


Figure 8.13: The construction of $s \mapsto s^\bullet$ for Case 3 and we repeat Steps 1–3 on the pair $(s^\bullet, \mathsf{rpos}(s))$ where $\mathsf{rpos}(s) = i$.

According to the construction of $f_{5,4}$ for Case 3, it is clear that $f_{5,4}(s) \in \mathcal{C}_1 \dot{\cup} \mathcal{C}_2$. According to the induction hypothesis, it remains to prove that the map $f_{5,4}$ is a bijection for all $s \in \mathcal{T}_{5,4}$ such that $\mathsf{rmin}(s) - \mathsf{rpos}(s) = N + 1$.

For any $\hat{s} \in \mathcal{B} - \mathcal{B}_1 = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2$ with $\min \mathsf{M}asc(\hat{s}) = m$ and $\mathsf{rmin}(\hat{s}) - \mathsf{rpos}(\hat{s}) = N$, then the sequence $\hat{s}$ is generated from

- Step 1 if $\hat{s} \in \mathcal{C}_1$;
- Step 2 if $\hat{s} \in \mathcal{C}_2$;

Since all Steps 1–3 including rules $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4$ are recursively reversible, we apply Step i in reverse order to $\hat{s}$ if $\hat{s} \in \mathcal{C}_i$ and obtain a pair $(s^\bullet, u)$ with $s^\bullet \in \mathcal{T}_{5,4}$ and $u = \mathsf{rpos}(\hat{s}) - 1$. If $u = \mathsf{rpos}(s^\bullet)$, then we stop and $s^\bullet = f_{5,4}^{-1}(\hat{s})$ with $\mathsf{rmin}(s^\bullet) - \mathsf{rpos}(s^\bullet) = N + 1$; otherwise $u < \mathsf{rpos}(s^\bullet)$, we implement Step 3 in reverse order on $s^\bullet$ (allowed by induction hypothesis) until a pair $(s, \mathsf{rpos}(s))$ is produced with $s = f_{5,4}^{-1}(\hat{s})$ satisfying $\mathsf{rmin}(s) - \mathsf{rpos}(s) = N + 1$. This implies that the map $f_{5,4}$ is surjective and injective, that is $f_{5,4}$ is the desired bijection f_5^* (defined in Proposition 8.4) when restricted to the set $\mathcal{T}_{5,4} \cap \mathcal{A}_n$. This completes the proof of Lemma 8.13. □

Example 8.16. Given $s^\bullet = (0,1,2,0,1,2,5,\mathbf{2},3,\mathbf{2},3,8,8,4) \in \mathcal{T}_{5,4}$ with $\mathsf{rpos}(s^\bullet) = 2$, $s^\bullet$ belongs to Case 2, so we implement Step 2 on the pair $(s^\bullet, 2)$ as follows:

$$\begin{aligned} s^\bullet &= (0,1,2,0,1,2,5,\mathbf{2},3,\mathbf{2},3,8,8,4) \\ &\to (0,1,2,0,1,2,5,2,3,3,\mathbf{4},8,8,\mathbf{4}), \end{aligned}$$

then apply the bijection $g_{5,3}$ from (8.1) to the prefix $(0,1,2,0,1,2,5,2,3,3)$ and obtain

$$g_{5,3}((0,1,2,0,1,2,5,2,\mathbf{3},\mathbf{3})) = (0,1,2,0,1,2,5,2,\mathbf{3},7,\mathbf{3}).$$

Add the subsequence $(4,9,9,4)$ at the end, where $(4,9,9,4)$ is obtained by increasing each entry of $(4,8,8,4)$ by one if it is larger than or equal to $m = 7$. This leads to the ascent sequence

$$(0,1,2,0,1,2,5,2,3,7,3,\mathbf{4},9,9,\mathbf{4}) \in \mathcal{B}_1.$$

Next after applying the inverse bijection $g_{5,3}^{-1}$, it becomes

$$g_{5,3}^{-1}((0,1,2,0,1,2,5,2,3,7,3,\mathbf{4},9,9,\mathbf{4})) = (0,1,2,0,1,2,5,2,3,7,3,4,\mathbf{4},\mathbf{4}).$$

Finally substitute non-rightmost entries 4 by 7 according to $\mathcal{R}_1$ and

$$f_5^*(s^\bullet) = f_{5,4}(s^\bullet) = (0,1,2,0,1,2,5,2,\mathbf{3},7,\mathbf{3},7,7,4) \in \mathcal{B} - \mathcal{B}_1.$$

Example 8.17. Given $s = (0,1,2,0,\mathbf{1},2,\mathbf{1},2,6,3,6,6,4) \in \mathcal{T}_{5,4}$ with $\mathsf{rpos}(s) = 1$ and $\mathsf{rmin}(s) = 5$. Since s belongs to Case 3, we implement Step 3 on the pair $(s,1)$ as follows:

$$\begin{aligned} s &= (0,1,2,0,\mathbf{1},2,\mathbf{1},2,6,3,6,6,4) \\ &\to (0,1,2,0,1,2,2,\mathbf{3},6,\mathbf{3},6,6,4); \end{aligned}$$

then apply the bijection $g_{5,3}$ from (8.1) to the subsequence $(0,1,2,0,1,2,2)$, yielding

$$g_{5,3}((0,1,2,0,1,2,2)) = (0,1,2,0,1,2,5,2);$$

attach the subsequence $(\mathbf{3},7,\mathbf{3},7,7,4)$ at the end, where $(\mathbf{3},7,\mathbf{3},7,7,4)$ comes from replacing each entry y of $(\mathbf{3},6,\mathbf{3},6,6,4)$ by $y+1$ if $y \geqslant m = 5$. Now the ascent sequence becomes

$$(0,1,2,0,1,2,5,2,\mathbf{3},7,\mathbf{3},7,7,4) \in \mathcal{B} - \mathcal{B}_1;$$

next apply the bijection $f_{5,4}^{-1}$ (by induction hypothesis) and it is known from Example 8.16 that

$$f_{5,4}^{-1}(0,1,2,0,1,2,5,2,\mathbf{3},7,\mathbf{3},7,7,4) = (0,1,2,0,1,2,5,\mathbf{2},3,\mathbf{2},3,8,8,4) = s^{\bullet}.$$

Since $s^{\bullet}$ belongs to Case 2, we implement Step 2 on the pair $(s^{\bullet},1)$ and get

$$\begin{aligned}(0,1,2,0,1,2,5,\mathbf{2},3,\mathbf{2},3,8,8,\mathbf{4}) &\to (0,1,2,0,1,2,5,2,3,3,\mathbf{4},8,8,\mathbf{4})\\ &\to (0,1,2,0,1,2,5,2,3,3,\mathbf{4},4,\mathbf{4})\\ &\to (0,1,2,0,1,2,5,2,5,3,\mathbf{5},5,\mathbf{4}) = f_{5,4}(s) = f_5^*(s).\end{aligned}$$

9. Final remarks

It is worthwhile to mention that an explicit formula for the refined generating function of the five Euler–Stirling statistics asc, rep, zero, max, rmin on ascent sequences can be derived from (6.7) and (6.8). We have the following result:

Theorem 9.1. *Let $r = t(x+u-xu)$. Then the refined generating function for the quintuple* $(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max},\mathsf{rmin})$ *of Euler–Stirling statistics on ascent sequences is*

$$\begin{aligned}
G(t;x,y,u,z,v) &:= \sum_{n=1}^{\infty} t^n \sum_{s\in\mathcal{A}_n} x^{\mathsf{rep}(s)} y^{\mathsf{max}(s)} u^{\mathsf{asc}(s)} z^{\mathsf{zero}(s)} v^{\mathsf{rmin}(s)} = \frac{vytz}{1-vytu}\\
&+ \sum_{k=0}^{\infty} \frac{yr^2vxz(tuv+z(r-tuv)-tuv(1-z)(1-yr)(1-r)^k)}{(x-ux+u(1-yr)(1-r)^k)(r-tuv+tuv(1-yr)(1-r)^{k+1})}\\
&\quad\times \frac{(1-yr)(1-r)^k}{x-u(x-1)(1-yr)(1-r)^k} \prod_{i=0}^{k-1} \frac{x-x(1-rz)(1-yr)(1-r)^i}{x-u(x-1)(1-yr)(1-r)^i}\\
&+ \sum_{k=0}^{\infty} \frac{yr^2u^2vtz(1-v)(tuv+z(r-tuv)-tuv(1-z)(1-yr)(1-r)^k)}{(r-tuv+tuv(1-yr)(1-r)^{k+1})(r-tuv+tuv(1-yr)(1-r)^k)}\\
&\quad\times \frac{(1-yr)(1-r)^k}{(x-xu+u(1-yr)(1-r)^k)} \sum_{m=k}^{\infty} \frac{rv(1-yr)(1-r)^m}{(x-xu+u(1-yr)(1-r)^m)}\\
&\quad\times \prod_{i=k}^{m} \frac{x(1-(1-yr)(1-r)^i)(x-xu+u(1-yr)(1-r)^i)}{(x-u(x-1)(1-yr)(1-r)^i)(x-xu+u(1-rv)(1-yr)(1-r)^i)}\\
&\quad\times \prod_{j=0}^{k-1} \frac{x-x(1-rz)(1-yr)(1-r)^j}{x-u(x-1)(1-yr)(1-r)^j}.
\end{aligned} \tag{9.1}$$

Proof. Note that an equivalent form of (6.7) is

$$\begin{aligned}F(t;x,y,1,u,z,v)&=\frac{tx(y-yzr+z)F(t;x,y-yr+1,1,u,z,v)}{(tux+y^{-1}-tu)(y-yr+1)}\\&\quad-\frac{txz(ytuv(1-z)+z)}{tux+y^{-1}-tu}F(t;x,y-yr+1,1,u,1,v)\\&\quad+z(ytuv(1-z)+z)F(t;x,y,1,u,1,v).\end{aligned}$$

Since the last two items contain a common factor $z(ytuv(1-z)+z)$, let

$$H(t;x,y,1,u,z,v):=F(t;x,y,1,u,z,v)-\frac{tx(y-yzr+z)F(t;x,y-yr+1,1,u,z,v)}{(tux+y^{-1}-tu)(y-yr+1)}.$$

Then, the previous equation becomes

$$\begin{aligned}H(t;x,y,1,u,z,v)&=z(ytuv(1-z)+z)H(t;x,y,1,u,1,v).\\F(t;x,y,1,u,z,v)&=\frac{tx(y-yzr+z)F(t;x,y-yr+1,1,u,z,v)}{(tux+y^{-1}-tu)(y-yr+1)}\\&\quad+z(ytuv(1-z)+z)H(t;x,y,1,u,1,v).\end{aligned}$$

Consequently (6.7) can be rewritten as

$$\begin{aligned}H(t;x,y,1,u,1,v)&=\frac{xvt^2(1-yr)}{(1-ytu)(1-tuv(y-yr+1))(tux+y^{-1}-tu)}\\&\quad+\frac{yu^2vt^2(1-v)(1-yr)}{(1-ytu)(1-tuv(y-yr+1))}F(t;x,y,1,u,1,v).\end{aligned}\tag{9.2}$$

By iterating the above equation, we find that, with $\delta_m=r^{-1}-r^{-1}(1-yr)(1-r)^m$,

$$\begin{aligned}&F(t;x,y,1,u,z,v)\\&=\sum_{k=0}^{\infty}z(\delta_k tuv(1-z)+z)H(t;x,\delta_k,1,u,1,v)\prod_{i=0}^{k-1}\frac{tx(\delta_i-\delta_i zr+z)}{(tux+\delta_i^{-1}-tu)(\delta_i-\delta_i r+1)}.\end{aligned}$$

Substituting $H(t;x,\delta_k,1,u,1,v)$ by the right-hand-side of (9.2) and then plugging (6.8) into the equation (after setting $y=\delta_k$), we obtain the formula for the generating function in (9.1). $\square$

Remark 9.2. Neither of the generating function formulas in (1.6) or in (1.12) is a direct specialization of the formula (9.1), although equivalent forms of the two former formulas can be obtained by setting $v=1$, respectively $z=1$, in the latter one.

The formula (9.1) for the generating function $G(t;x,y,u,z,v)$ is of theoretical interest; it is explicit but unfortunately rather complicated. It seems very difficult to apply this formula in order to prove equidistribution results by pure algebraic means (i.e., manipulations of series), although we know that $G(t;x,y,u,z,v)=G(t;x,v,u,z,y)$ holds, as a consequence of Theorem 1.8.

Open Problem 1. *Find a simpler form of the generating function $G(t;x,y,u,z,v)$ so that both $\mathcal{G}(t;x,y,u,z)$ and $\mathfrak{G}(t;x,y,u,v)$ (given in Theorems 1.6 and 1.10) are direct specializations of $G(t;x,y,u,z,v)$ at $v=1$ and $z=1$, respectively. Furthermore, prove that $G(t;x,y,u,z,v)=G(t;x,v,u,z,y)$ by transformations of basic hypergeometric series.*

We finally pose a conjecture on a symmetric equidistribution of Euler–Stirling statistics on inversion sequences, which is analogous to Theorem 1.8 but with the two statistics $\mathsf{ealm}, \mathsf{rpos}$ being removed, and $\mathcal{A}_n$ (the set of ascent sequences) being replaced by $\mathcal{I}_n$ (the set of inversion sequences).

Conjecture 9.3. There is a bijection $\Omega:\mathcal{I}_n\to\mathcal{I}_n$ such that for all $s\in\mathcal{I}_n$,

$$(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max},\mathsf{rmin})s=(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{rmin},\mathsf{max})\Omega(s).$$

Consequently for all $\pi\in\mathfrak{S}_n$,

$$(\mathsf{des},\mathsf{iasc},\mathsf{lmax},\mathsf{lmin},\mathsf{rmax})\pi=(\mathsf{des},\mathsf{iasc},\mathsf{lmax},\mathsf{rmax},\mathsf{lmin})(b^{-1}\circ\Omega\circ b)(\pi),$$

where $b:\mathfrak{S}_n\to\mathcal{I}_n$ is a bijection due to Baril and Vajnovszki (see Theorem 1 of [BV17]).

This has been verified by Maple up to $n=10$. Different from ascent sequences, a generating function formula for the quadruple $(\mathsf{asc},\mathsf{rep},\mathsf{zero},\mathsf{max})$ of Euler–Stirling statistics on inversion sequences remains unknown, but one for the pair $(\mathsf{asc},\mathsf{rep})$ of Eulerian statistics was established by Garsia and Gessel [GG79]: In view of (1.4), let

$$\begin{aligned}B_n(u,x)&:=\sum_{s\in\mathcal{I}_n}u^{\mathsf{asc}(s)}x^{\mathsf{rep}(s)}=\sum_{\pi\in\mathfrak{S}_n}u^{\mathsf{des}(\pi)}x^{\mathsf{iasc}(\pi)},\\ H_n(u,x)&:=\sum_{s\in\mathcal{I}_n}u^{\mathsf{asc}(s)}x^{n-1-\mathsf{rep}(s)}=\sum_{\pi\in\mathfrak{S}_n}u^{\mathsf{des}(\pi)}x^{n-1-\mathsf{iasc}(\pi)}.\end{aligned}$$

Then

$$B_n(u,x)=x^{n-1}H_n(u,x^{-1}),$$

and there holds

$$\sum_{n\geqslant0}\frac{H_n(u,x)\,t^n}{(1-u)^{n+1}(1-x)^{n+1}}=\sum_{k\geqslant1}\sum_{m\geqslant1}\frac{u^{k-1}x^{m-1}}{(1-t)^{km}},\tag{9.3}$$

which implies $H_n(u,x)=H_n(x,u)$, or equivalently, $B_n(u,x)=B_n(x,u)$ (see also (1.3)). One possible approach to solve Conjecture 9.3 is to deduce an extension of (9.3) by including the Stirling statistics $\mathsf{zero}, \mathsf{max}, \mathsf{rmin}$ and to read the symmetry directly from the extended generating function formula.

While Theorem 1.7 holds if $\mathcal{A}_n$ is replaced by $\mathcal{I}_n$ (see the following Proposition 9.4 which is a direct result of a bijection due to Baril and Vajnovszki [BV17]), it currently seems that the proof of Theorem 1.8 cannot be modified to affirm Conjecture 9.3.

Proposition 9.4. *There is a bijection $\varrho : \mathcal{I}_n \to \mathcal{I}_n$ such that for any $s \in \mathcal{I}_n$,*

$$(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})s = (\mathsf{rep}, \mathsf{asc}, \mathsf{rmin}, \mathsf{zero})\varrho(s).$$

Proof. Baril and Vajnovszki (see Theorem 1 of [BV17]) constructed a bijection $b : \mathfrak{S}_n \to \mathcal{I}_n$ satisfying that for any $\tau \in \mathfrak{S}_n$,

$$(\mathsf{des}, \mathsf{iasc}, \mathsf{lmin}, \mathsf{lmax}, \mathsf{rmax})\tau = (\mathsf{asc}, \mathsf{rep}, \mathsf{max}, \mathsf{zero}, \mathsf{rmin})b(\tau).$$

Let $\tau^c = (n+1-\tau_1)(n+1-\tau_2)\cdots(n+1-\tau_n)$ be the complement of τ, then for any $s \in \mathcal{I}_n$, let $\tau = b^{-1}(s)$ and we have

$$\begin{aligned}(\mathsf{asc}, \mathsf{rep}, \mathsf{zero}, \mathsf{max})s &= (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin})b^{-1}(s)\\ &= (\mathsf{des}, \mathsf{iasc}, \mathsf{lmax}, \mathsf{lmin})\tau\\ &= (\mathsf{iasc}, \mathsf{des}, \mathsf{rmax}, \mathsf{lmax})(\tau^{-1})^c\\ &= (\mathsf{rep}, \mathsf{asc}, \mathsf{rmin}, \mathsf{zero})\, b((\tau^{-1})^c),\end{aligned}$$

that is, by defining $\varrho(s) = b((\tau^{-1})^c)$ the proof is complete. □

If Conjecture 9.3 is true, then it follows from Proposition 9.4 that Conjecture 1.4 also holds if $\mathcal{A}_n$ is replaced by $\mathcal{I}_n$.

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