UNIVERSITY OF CALIFORNIA
Los Angeles

The Holographic Geometry of Conformal Blocks

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

River Curry Snively

2018

ABSTRACT OF THE DISSERTATION

The Holographic Geometry of Conformal Blocks

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Doctor of Philosophy in Physics
University of California, Los Angeles, 2018
Professor Per J. Kraus, Chair

Conformal blocks are the building blocks of correlation functions in conformal field theory. They figure prominently in the study of quantum gravity in light of the AdS/CFT correspondence which identifies conformal field theory as a holographic representation of quantum gravity in Anti de-Sitter space. Thus while conformal blocks are fundamentally CFT objects, it promises to be both conceptually and computationally useful to obtain for them a description on the AdS side. We lay out that description here, extending the holographic dictionary relating CFT objects to AdS objects by adding to it an entry for the conformal blocks themselves. The holographic dual to a global conformal block is a Feynman diagram-like object we call a geodesic Witten diagram, and a similar picture applies to semiclassical Virasoro blocks as well. We discuss ways in which this new picture for an old tool can help to further our understanding of how spacetime emerges from the quantum information structure of conformal field theory.

The dissertation of River Curry Snively is approved.

Eric D'Hoker

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University of California, Los Angeles

2018

This work is dedicated to my parents, who introduced me to blocks.

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ACKNOWLEDGMENTS

I am deeply indebted to my advisor Per Kraus for sharing his knowledge and time, for always being willing to spend an hour listening to mediocre ideas from me, and for his readiness to meet seemingly any question with a perfectly well-formed pedagogical answer from first principles.

It is also my pleasure to thank my coauthors on the work included in this thesis, Eliot Hijano and Eric Perlmutter, both for their contributions to the research and for being exemplars of graduate school success. Though it is not included in this thesis, I am very proud of the non-block research I have undertaken with Allic Sivaramakrishnan and would like to thank him for his contributions and his uncanny enthusiasm for physics, which kept me going through the many phases of that endeavor. Also, to all the people I have shared an office with, I am grateful for the thought-provoking conversations about grammar, comparative linguistics, and occasionally conformal field theory. I would also like to thank David Bauer for all the inspiration he has provided over the years in the form of absurd excerpts from the arXiv.

The introduction to this thesis is informed by all my attempts at explaining the context of my research to my parents and parents-in-law. I want to thank all of them for always showing an interest and for their understanding and support throughout my graduate career.

Finally, I wish to thank the most important physicist in my life, my wife Emma. Bringing us together is the greatest thing physics has done for me, and that includes holding me to the planet's surface. Attached to each small quantum of knowledge in the work described below there are a thousand happy memories of our life together while the work was in progress.

Co-author contributions. Chapter 2 of this thesis was published as [1] and written in collaboration with Eliot Hijano and Per Kraus. Chapter 3 was published as [3] and chapter 4 as [2], both in collaboration with Eliot Hijano, Per Kraus, and Eric Perlmutter. Each has undergone minor adjustments in the interest of cohesion, and final sections have been expanded to mention interesting follow-up work.

Figure permission. Figure 1.4 (a) originally appeared in [5] and has been reproduced with permission from Kyriakos Papadodimas. The question answered in this thesis – what is the bulk dual of a conformal block? – I first saw posed in this figure, and I could not resist including it.

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PUBLICATIONS

[1] Eliot Hijano, Per Kraus, and River Snively. Worldline approach to semi-classical conformal blocks. *Journal of High Energy Physics*, 07:131, 2015.

[2] Eliot Hijano, Per Kraus, Eric Perlmutter, and River Snively. Semiclassical Virasoro blocks from AdS_3 gravity. *Journal of High Energy Physics*, 12:077, 2015.

[3] Eliot Hijano, Per Kraus, Eric Perlmutter, and River Snively. Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks. *Journal of High Energy Physics*, 01:146, 2016.

[4] Per Kraus, Allic Sivaramakrishnan, and River Snively. Black holes from CFT: Universality of correlators at large c. *Journal of High Energy Physics*, 08:084, 2017.

CHAPTER 1

Introduction

Black hole formation introduces an upper bound on the amount of information a region of space can contain, a bound proportional to the area of the region's boundary [6]. This seems to indicate that perhaps the fundamental degrees of freedom describing a volume of quantum space live not in the space's volume but rather on a lower-dimensional surface [7], a tantalizing hint toward the correct description of quantum gravity that has come to be known as the holographic principle.

The holographic principle is spectacularly realized in the AdS/CFT correspondence [8–10], which posits that quantum gravity in $(d+1)$-dimensional anti-de Sitter space (the bulk) is described by d-dimensional conformally invariant quantum field theory. The correspondence provides a concrete definition of quantum gravity, one in which quantum-mechanical AdS spacetime is described by a theory localized to its lower-dimensional boundary.

From a certain point of view, e.g. [11, 12], any conformal field theory furnishes, via the AdS/CFT correspondence, a theory of quantum gravity in one-higher dimension, with however a generic CFT corresponding to a bulk theory in which gravity is strongly quantum and the semiclassical notion of spacetime loses its meaning, leaving the conformal field theory as the system's only sensible description. One day we may be able to understand the gravitational side of such a system, but for now to make progress one studies those conformal field theories whose bulk duals are described, like our universe, by perturbative quantum field theory at low energies. For such systems one can get a handle on both sides of the correspondence. A highly fruitful course of study has been to carve out the space of conformal field theories with perturbative semiclassical bulk duals, e.g. [5, 13–19], and then to use their properties to prove general conclusions about the bulk, e.g. that bulk scattering is local above the AdS horizon scale [20] and governed by a unitary S-matrix [21], that gravity in the bulk obeys a law analogous to Newton's $1/r^2$ at long distances [22] and that the graviton couples to itself [23] and to matter [24,25] in a way consistent with Einstein gravity. Pertaining in particular to black holes, it has been shown from very general assumptions about CFT_2 that the bulk spectrum contains the appropriate number of high-mass states – black hole microstates – to account for AdS_3 black hole entropy [26], that probe particles orbit within them in the expected way [27], and furthermore that low-energy physics in the vicinity of a black hole is governed by standard quantum field theory in curved spacetime [4] independent of the details of the microstate.

One extremely powerful technique for proving constraints on conformal field theories, and thereby on quantum gravity, is the conformal bootstrap program [28, 29], about which we will have more to say below. The conformal bootstrap operates in terms of the conformal block decomposition of CFT correlators. Conformal blocks – about which we will have *much* to say below – are the building blocks of conformal field theory correlation functions. They have been important in the study of conformal field theory since their inception [30–32] in the 1970s, nearly three decades before the advent of the AdS/CFT correspondence. Until recently, conformal blocks were thought to exist purely in the CFT. This thesis reviews recent developments bringing conformal blocks into the bulk.

1.1 Conformal field theory and conformal blocks

We proceed with a brief review of background material relevant to conformal blocks. This section is not intended as a complete pedagogical introduction to conformal field theory. The reader looking for such may consult one of the classic [33, 34] or modern [35] references cited here. Our objective is to introduce those concepts necessary to define conformal blocks and motivate their utility, assuming a baseline familiarity with quantum field theory. We restrict attention throughout to unitary CFTs in Euclidean signature.

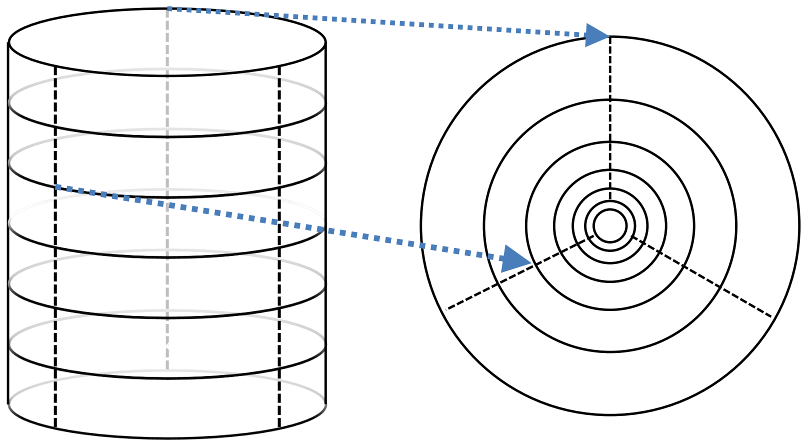

Figure 1.1: The conformal map from the cylinder to the plane, with $d = 2$ for ease of illustration. Solid lines map to solid lines and dashed lines to dashed lines. Note that because the transformation is conformal it preserves the angle of intersection (90°) between dashed and solid lines.

1.1.1 Conformal symmetry

Conformal field theory is the study of quantum field theories invariant under the conformal group, consisting of all coordinate transformations that preserve angles between vectors but not necessarily their lengths. The conformal group is generated by the translations, $x^\mu \mapsto x^\mu + a^\mu$, and rotations, $x^\mu \mapsto L^{\mu\nu} x_\nu$ (for L an orthogonal matrix) familiar from the Poincare group, supplemented by scaling, $x^\mu \mapsto \lambda x^\mu$, and inversion, $x^\mu \mapsto -\frac{x^\mu}{|x|^2}$. A conformal field theory living on the plane $\mathbb{R}^d$ is equivalent to one on the cylinder $S^{d-1} \times \mathbb{R}$ because the latter maps into the former via the conformal transformation $(t, n^\mu) \mapsto e^t n^\mu$, where t is the cylinder's axial coordinate and n^μ a unit vector on S^{d-1}. The transformation is illustrated in figure 1.1.

The conformal group's Lie algebra is spanned by the generators P_i of translations and

R_{ij} of rotations, the generator D of scale transformations, and the generators K_i of special conformal transformations, the latter implemented by an inversion, followed by the action of P_i and then another inversion: $K_i = IP_iI$. These together realize an SO$(d+1,1)$ algebra. We will not state all the commutators here (see e.g. [36]), but three relations important for present purposes are

$$[H, P_i] = P_i, \quad [H, K_i] = -K_i, \quad [P_i, K_j] = -2\left(\delta_{ij} H + R_{ij}\right) \ . \tag{1.1}$$

The two generators $H = -iD$ and R_{ij} are Hermitian while the remaining two, P_i and K_j, are Hermitian conjugates of one another.

Motivated by the map to the cylinder, it is standard to quantize CFT on $\mathbb{R}^d$ using spheres S^{d-1} centered at the origin as one's time slices (the circles on the right side of figure 1.1). Then D generates translations in Euclidean time and the system's Hamiltonian is therefore $H = -iD$.

Just as ordinary rotation-invariant quantum field theory admits operators with spin, operators in CFT need not be invariant under scaling. Indeed a generic operator will scale nontrivially. However, one can always choose one's basis of operators ϕ_i to be a scaling eigenbasis, i.e. such that $[D, \phi_i(x)] = \Delta_i \phi_i(x)$. Δ_i is the scaling dimension of the operator ϕ_i. Any operator lacking definite scaling dimension is simply a linear combination of operators that do.

Conformal covariance fixes the expectation value of any product of two local scalar operators of definite scaling dimensions in terms of the distance $x_{12} := |x_1 - x_2|$ between their positions.

$$\langle \phi_i(x_1) \phi_j(x_2) \rangle = \frac{N_{ij}}{x_{12}^{2\Delta_1}} \quad \text{if } \Delta_1 = \Delta_2, \quad \text{otherwise zero} \tag{1.2}$$

for some constant N_{ij}. The expression on the right hand side is the only combination of x_1, x_2 that transforms under translation and scaling consistent with the left hand side. For operators to have nonzero two-point function their scaling dimensions must match. N_{ij} is real and symmetric, and in practice one always chooses an orthonormal basis of primaries such that the two-point function is simply

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \frac{\delta_{ij}}{x_{12}^{2\Delta_i}} \ . \tag{1.3}$$

1.1.2 The state/operator correspondence

Here we introduce a very powerful feature of conformal field theory, the presence of a one-to-one correspondence between states in the CFT Hilbert space and local operators in the CFT path integral. (This correspondence operates entirely in CFT and is not to be confused with the AdS/CFT correspondence!)

As discussed above, time slices in the CFT are taken to be spheres centered at the origin. To declare that the system has state ψ_0 at initial "time" r_0, one cuts out the circle $r < r_0$ and enforces the condition $\psi = \psi_0$ at $r = r_0$ in the path integral. If one does so with r_0 approaching zero, there is no change to the path integral except very near the origin, and in the limit $r_0 \to 0$ the effect is the same as that of a local operator ϕ inserted at the origin. The correspondence between states ψ and local operators ϕ induced in this way is one-to-one because two local operators have the same effect in every path integral only if they are the same.

The essential role of conformal symmetry in establishing the state/operator correspondence is to guarantee that the Hilbert spaces of states on spheres of different radii are isomorphic. In a theory without scale invariance one would naturally expect a smaller sphere to host a smaller Hilbert space. But in a conformal field theory, the procedure of taking $r_0 \to 0$ does not make the Hilbert space any smaller, and we establish that states on a sphere *of any radius* are one-to-one equivalent to local operator insertions at the center of the sphere.

Finally, we mention here that just as ket states $|\phi\rangle$ are in one-to-one correspondence with operators inserted at the origin, bra states $\langle\phi|$ are likewise in correspondence with local operators inserted at the point at infinity on the plane. One inserts an operator of dimension Δ at the point at infinity using the limit $x_\infty \to \infty$ of the insertion $|x_\infty|^{2\Delta}\phi(x_\infty)$. The power law factor ensures that $\langle\phi|\phi\rangle = 1$ by equation (1.3).

1.1.3 Primaries and the OPE

The Hilbert space of any CFT admits a decomposition into irreducible representations of the conformal group with H-eigenvalues bounded from below. Any such irrep contains exactly one state $|p\rangle$ annihilated by all special conformal generators K_i, and the representation is spanned by all possible ways to act with strings of momentum generators P_i on that state. These statements follow from the algebra (1.1); for a proof see [33]. $|p\rangle$ is called a primary state, and the irrep it generates is called its conformal family. States in the conformal family are referred to as "descendants" of $|p\rangle$. One primary state of special significance is the vacuum, the state with lowest energy. It is assumed to be translation-invariant, which condition, stronger than conformal *covariance*, means it is annihilated by P_i as well as by K_i.

Meanwhile, a local operator ϕ is called primary if its insertion at the origin creates a primary state. Equivalently, the condition is that the operator has definite scaling dimension and is annihilated by the adjoint action of any special conformal generator K_i, i.e. that $[K_i, \phi(0)] = 0$. A third equivalent definition of a primary operator is that under a conformal transformation $x \mapsto x'(x)$ it transforms as a density of weight Δ:

$$\phi(x) \mapsto \phi'(x') = \left(\det \tfrac{\partial x'}{\partial x}\right)^{\Delta/d} \phi(x') \ . \tag{1.4}$$

The above equation applies to scalar primaries. Its generalization to tensors is immediate, differing only by the inclusion of transformations acting on the indices in the usual manner.

One operator appearing in any conformal field theory is the energy-momentum tensor $T_{\mu\nu}$, defined as usual in quantum field theory as the Noether current associated with spacetime translation. It is a spin-2 primary operator with dimension $\Delta = d$, as can be checked directly by applying a conformal transformation and comparing with the spin-2 version of the primary transformation law (1.4).

When two primary operators are inserted inside a correlation function at nearby points the result admits an operator product expansion (OPE) in powers of the separation x^μ.

$$\phi_1(x)\phi_2(0) = \sum_p C_{12p} \sum_{\mathbf{P}^k} \beta^{12p}_{\mathbf{P}^k} \, |x|^{\Delta_p - \Delta_1 - \Delta_2} x^k \mathbf{P}^k \phi_p(0) \tag{1.5}$$

p runs over the labels of all primaries in the CFT spectrum and $\mathbf{P}^k$ over all combinations of momentum generators. The states $\mathbf{P}^k|p\rangle$ span the Hilbert space so the existence of an expansion of the left hand side into the operators on the right hand side is automatic. We have used scaling symmetry to fix the power of x that appears in each term. The combination $x^k\mathbf{P}^k$ schematically represents an unspecified contraction of the $2k$ indices in the expression $x^{\mu_1}...x^{\mu_k}P_{\nu_1}...P_{\nu_k}$. When tensor primaries ϕ_p appear in the sum their indices are contracted with powers of x appearing in the schematic expression $|x|^{\Delta_p-\Delta_1-\Delta_2}$. We have separated out an overall factor C_{12p} for each conformal family, defining these by demanding that the primary ϕ_p itself enters the sum with coefficient precisely C_{12p}, i.e. that $\beta^{12p}_{\mathbf{P}^0} = 1$. The operator product expansion (1.5) converges inside a correlation function if and only if there exists a co-dimension-one sphere separating the two operators from all others present.

Conformal symmetry fixes the three-point functions of primaries ϕ_i with dimensions Δ_i up to an overall constant.

$$\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{C_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}x_{23}^{\Delta_j+\Delta_k-\Delta_i}x_{31}^{\Delta_k+\Delta_i-\Delta_j}} \, . \tag{1.6}$$

All three operators have been taken to be scalars for simplicity. The right hand side of equation (1.6) is, up to multiplication, the only expression with the correct conformal transformation properties to match the left hand side. Three-point functions involving descendant operators are fixed in terms of the corresponding primary three-point functions simply by taking derivatives.

The constant C_{ijk} in the numerator of (1.6) has been identified with the OPE coefficient by comparing that equation with (1.5) at $x_1 = x, x_2 = 0, x_3 = \infty$. In principle all coefficients $\beta^{12p}_{\mathbf{P}^k}$ can be similarly determined by comparing the two equations at general operator positions. This is important; it means the only data in the OPE not fixed by conformal symmetry is the so-called "conformal data": the OPE coefficients $\{C_{12p}\}$ and the spectrum of primary dimensions and spins $\{(\Delta_p, \ell_p)\}$. Because any n-point function can be evaluated by successive applications of the OPE, a CFT's observables are completely specified by its conformal data.

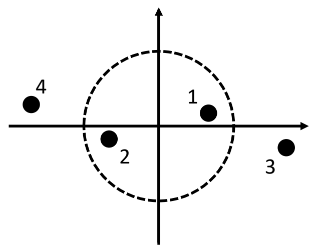

Figure 1.2: Positions of the operators on the z-plane in the four-point function 1.7. The dashed circle is $|z| = 1$.

1.1.4 Conformal blocks

That three-point functions of local operators take a universal form, (1.6), is a consequence of the fact that any three points in $\mathbb{R}^d$ can be mapped to any other three by a conformal transformation. The same cannot be said for four or more points, and so there is no generalization of equation (1.6) to four- and higher-point correlation functions; they are not fixed by conformal symmetry. However their structure is severely constrained, as we will now discuss in the case of four-point functions[1].

For conceptual simplicity and with no loss of generality let us consider the four operators to lie at positions $x_1, ..., x_4 \in \mathbb{R}^d$ all on the same two-dimensional plane, parameterized by a complex coordinate z, such that $|z_1|, |z_2| < 1$ and $|z_3|, |z_4| \geq 1$. Any choice of the four operator positions is conformally equivalent to one of this form [37]. We will study the four-point function

$$F_4(x_4, x_3, x_2, x_1) = \langle 0|\phi_4(x_4)\phi_3(x_3)\phi_2(x_2)\phi_1(x_1)|0\rangle \ . \tag{1.7}$$

Consider inserting into this correlation function a complete set of states $\sum_\alpha |\alpha\rangle\langle\alpha|$ at the dashed circle $|z| = 1$ in figure 1.2. We can label the states $|\alpha\rangle = |p, k\rangle$ where p runs over

[1] Higher-point functions admit a similar treatment and one can defined higher-point conformal blocks, but we focus here on four external points.

all primaries in the Hilbert space and k indexes an orthonormal basis of descendants of each primary.

$$F_4(x_4,x_3,x_2,x_1) = \sum_p \sum_k \langle 0|\phi_4(x_4)\phi_3(x_3)|p,k\rangle \langle p,k|\phi_2(x_2)\phi_1(x_1)|0\rangle \ . \tag{1.8}$$

Each term in the sum is a product of two three-point functions. These are proportional to C_{34p} and C_{12p} but otherwise fixed by symmetry. Thus if we pull out those factors and write the correlator as

$$F_4(x_4,x_3,x_2,x_1) = \sum_p C_{12p}C_{34p} \sum_k \frac{\langle 0|\phi_4(x_4)\phi_3(x_3)|p,k\rangle}{C_{34p}} \frac{\langle p,k|\phi_2(x_2)\phi_1(x_1)|0\rangle}{C_{12p}} \tag{1.9}$$

then the sum over k is fixed by conformal symmetry. It depends on the five dimensions Δ_i but not on OPE coefficients. The function

$$W_p^{43;21}(x_4,x_3,x_2,x_1) = \sum_k \frac{\langle 0|\phi_4(x_4)\phi_3(x_3)|p,k\rangle}{C_{34p}} \frac{\langle p,k|\phi_2(x_2)\phi_1(x_1)|0\rangle}{C_{12p}} \tag{1.10}$$

is known as a conformal partial wave. It captures the full contribution to the four-point function from the conformal family of p. A perhaps more illuminating expression is

$$W_p^{43;21}(x_4,x_3,x_2,x_1) = \tfrac{1}{C_{12p}C_{34p}} \langle \phi_4(x_4)\phi_3(x_3)\mathcal{P}_p\phi_2(x_2)\phi_1(x_1)\rangle \tag{1.11}$$

where $\mathcal{P}_p$ is the projector onto that conformal family.

Any four-point function is a linear combination of conformal partial waves.

$$\langle \phi_4(x_4)\phi_3(x_3)\phi_2(x_2)\phi_1(x_1)\rangle = \sum_p C_{12p}C_{34p}W_p^{43;21}(x_4,x_3,x_2,x_1) \ . \tag{1.12}$$

Conformal partial waves are not conformally invariant; they transform covariantly in the same way as the four-point function itself. For applications such as the conformal bootstrap it is convenient to multiply the partial wave by powers of position to trivialize the transformation law and promote them to conformal invariants. The resulting object,

$$G_p^{43;21}(x_4,x_3,x_2,x_1) = \frac{\left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2}\left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4}}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}} W_p^{43;21}(x_4,x_3,x_2,x_1) \ , \tag{1.13}$$

is called a conformal block. In this thesis we will be concerned entirely with conformal partial waves, and we will often refer to CPWs interchangeably as conformal blocks with the understanding that conformal blocks are actually related to CPWs by the inclusion of the prefactor displayed above.

In this discussion we have defined the blocks implicitly. It is quite cumbersome to compute blocks from the definitions stated here but it can be done in some cases [38]. A more efficient method for computing blocks takes advantage of the fact that they are eigenfunctions of the conformal Casimir operator [39]. Explicit expressions for conformal blocks can be found in section 3.2.1.

Conformal blocks are fundamental to conformal field theory because they capture all nontrivial position dependence of correlation functions. The conformal block decomposition (1.12) effectively disentangles the theory-dependent information (dimensions and OPE coefficients) contained in a correlator from the purely kinematic, but very complicated, information pertaining to the contributions from conformal descendants. We will see the utility of this splitting in the next subsection.

The discussion in this introductory chapter has taken place in general dimension d and we have focused on the global conformal symmetry that exist in any dimension. In two dimensions, global conformal symmetry is enhanced to Virasoro symmetry and some of the concepts discussed here become enhanced as well. We relegate discussion of Virasoro symmetry and Virasoro conformal blocks to later chapters.

1.1.5 The conformal bootstrap

A conformal field theory is defined by the dimensions of its primary operators and their OPE coefficients, i.e. by the conformal data mentioned above. While the conformal data can be viewed as the defining parameters of a CFT, those parameters cannot be tuned freely. Indeed only very special points in parameter space lead to consistent theories. The key constraint arises from the simple fact that the same correlator admits conformal block decompositions in multiple channels.

$$\begin{aligned}\langle \phi_4(x_4)\phi_3(x_3)\phi_2(x_2)\phi_1(x_1)\rangle &= \sum_p C_{43p}C_{12p}W_p^{43;21}(x_4,x_3,x_2,x_1)\\ \langle \phi_4(x_4)\phi_2(x_2)\phi_3(x_3)\phi_1(x_1)\rangle &= \sum_p C_{42p}C_{13p}W_p^{42;31}(x_4,x_2,x_3,x_1)\end{aligned} \tag{1.14}$$

The left hand sides above are equal. Equating the right hand sides leads to the crossing equation

$$\sum_p C_{43p}C_{12p}W_p^{43;21}(x_4,x_3,x_2,x_1) = \sum_p C_{42p}C_{13p}W_p^{42;31}(x_4,x_2,x_3,x_1)\ , \tag{1.15}$$

an equation that must be satisfied by the conformal data of any consistent theory. It packages an infinite number of constraints on the conformal data, even for fixed external operators, since both sides are functions of position.

The conformal bootstrap is an extremely well-studied and fruitful program (see [40] and references therein) in which one uses the infinite set of crossing equations to identify or exclude possible conformal field theories. Aside from the obvious AdS/CFT motivation, the conformal bootstrap is a powerful method for discovering and probing theories beyond the reach of perturbative quantum field theory. New analytical results from the conformal bootstrap often arise from new understanding of the analytical properties of conformal blocks, e.g. [20, 41].

1.2 Generalized Free CFT

In this section we review the conformal field theory structure that underlies perturbative quantum field theory in the bulk. The special conformal field theories admitting such a structure are called large-N CFTS, generalized free CFTS, or sometimes holographic CFTS. The canonical examples of holographic CFTs with perturbative bulk duals are gauge theories with gauge group SU(N) (or some other classical group) with $N \gg 1$ – hence the name large-N – and we discuss these in section 1.2.1. Gauge symmetry itself is not responsible for the emergence of perturbative field theory in the bulk. Rather, it is due to the manner in which the correlation functions organize in powers of $1/N$, factorizing into two-point functions at leading order and with $1/N$ corrections entering in a controlled way. In fact one can forget about specific examples and lay down instead the axioms of a conformal field theory realizing the desired properties [15, 42]. We review those axioms in section 1.2.2, primarily following the very thorough and excellent discussion in [5].

1.2.1 Large-N gauge theory

The purpose of this section is to motivate the axioms and nomenclature of generalized free CFT. Its contents will not be used in what follows and it can be skipped without consequence.

Consider a field X that transforms in the adjoint representation of an SU(N) gauge theory at large N. That is, X is a traceless $N \times N$ Hermitian matrix in color space. Let the gauge coupling scale with N as $g = \lambda/\sqrt{N}$ with λ, the 't Hooft coupling, fixed. We can form local U(N) singlet operators by taking traces of matrix products of X fields.

$$\tilde{\mathcal{O}}_k(x) \;=\; :\mathrm{Tr}(X(x)^k):, \quad k \geq 2 \tag{1.16}$$

In this expression the k operators X at the same position x are normal-ordered as indicated by the enclosing dots.

At zeroth order in weak coupling ($\lambda \to 0$) any correlation function involving the fields $\mathcal{O}_k$ can be computed via Wick contraction, or equivalently by interactionless Feynman diagrams drawn with propagators for the field X. The propagator $\langle X_a^b(x) X_c^d(y)\rangle$ is proportional to $\delta_a^d \delta_b^c$ so to keep track of the indices one can draw each X propagator as a pair of lines [43]. Meanwhile, we implement matrix multiplication of the external operators by connecting their indices in a chain. A Feynman diagram picks up one factor of N for every closed index loop. It is not hard to see, for example, that $\langle \tilde{\mathcal{O}}_k \tilde{\mathcal{O}}_k \rangle$ is proportional to N^k.

Away from weak coupling, we must also add gluon propagators, and a graph picks up two factors of g from the coupling to X each time we do so. The additional propagator – itself a pair of lines – may also split a single index loop into two, and if so the diagram picks up a factor of N, but we can never pick up more than one such factor per gluon. The choice to take N large with $\lambda = g^2 N$ fixed means adding more gluons to a diagram increases its power of λ but never N. Thus, even at finite coupling λ, one can determine the large-N

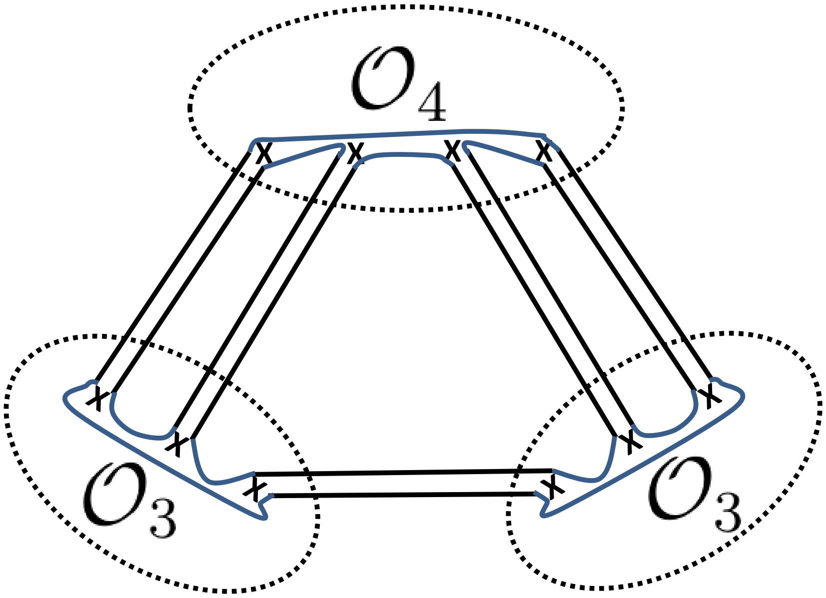

Figure 1.3: Example diagram for finding the large-N scaling of a three-point function of single trace operators $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_4 \rangle$. Insertions of X are marked. Black double lines originate from X propagators, while curved blue lines implement the index contractions dictated by the traces in the external operators. The diagram has four index loops and each contributes a factor of N. Including the operators' normalization factors $N^{-3/2}, N^{-3/2}, N^{-2}$ the net result is $1/N$.

scaling of a diagram without considering internal gluons. It is instructive to work out that

$$\langle \tilde{\mathcal{O}}_j \tilde{\mathcal{O}}_k \tilde{\mathcal{O}}_\ell \rangle \sim N^{(j+k+\ell-2)/2} \tag{1.17}$$

assuming $j+k+\ell$ is even and that the largest of j, k, ℓ is no bigger than the sum of the other two.

It is standard to normalize operators so that their two-point function has unit coefficient. Doing so, we define a set of operators $\mathcal{O}_k = N^{-k/2} \tilde{\mathcal{O}}_k$. Their two and three-point functions are

$$\langle \mathcal{O}_k \mathcal{O}_k \rangle \sim 1, \quad \langle \mathcal{O}_j \mathcal{O}_k \mathcal{O}_\ell \rangle \sim \frac{1}{N} \tag{1.18}$$

A very important and useful property of large-N gauge theories is factorization. Correlators of single-trace operators factorize into two-point functions at leading order in N. Consider, for instance, the four-point function $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_3\mathcal{O}_3\rangle$. It gets an order-one contribution from the disconnected Feynman diagram in which two X propagators from one $\mathcal{O}_2$ connect to the other $\mathcal{O}_2$ and likewise for the other pair, and all connected diagrams are suppressed by $1/N$. It follows that

$$\langle \mathcal{O}_2(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\mathcal{O}_3(w)\rangle = \langle \mathcal{O}_2(x)\mathcal{O}_2(y)\rangle\langle \mathcal{O}_3(z)\mathcal{O}_3(w)\rangle + O(1/N)\ . \tag{1.19}$$

Similar statements hold for higher-point functions. If there are more than two of a certain type of single-trace then one sums over all possible contractions.

The operators $\mathcal{O}_k$ are not the only gauge-invariant local operators one can build out of the field X. There are also "multi-trace" operators consisting of normal-ordered products of the $\mathcal{O}_k$. Correlators involving these normal-ordered operators are computed with Wick contractions in the usual way, where one simply avoids contractions between components of the same normal-ordered operators. One finds, for instance, that

$$\langle \mathcal{O}_k\mathcal{O}_j\ :\!\mathcal{O}_k\mathcal{O}_j\!:\rangle \sim 1, \quad \text{where } :\!\mathcal{O}_k\mathcal{O}_j\!:(x)\ =\ :\!\mathrm{Tr}(X(x)^k)\,\mathrm{Tr}(X(x)^j)\!: \tag{1.20}$$

In addition to $:\!\mathcal{O}_k\mathcal{O}_j\!:$ there is a spectrum of primary operators with conformal dimensions

$$h = h_k + h_j + n, \quad \bar{h} = \bar{h}_k + \bar{h}_j + \bar{n}, \quad \text{for all } n, \bar{n} \in \mathbb{Z}_{\geq 0} \tag{1.21}$$

built from normal-ordered products of $\mathcal{O}_k, \mathcal{O}_j$ with $n + \bar{n}$ derivatives inserted in specific combinations so as to build a primary operator.

1.2.2 The axioms of generalized free conformal field theory

A generalized free conformal field theory begins with a collection of operators, the single-trace operators, with the property that their correlation functions factorize into two-point functions at zeroth order in $1/N$, where N need not be the size of a gauge group; it is simply a large parameter. For example, the four-point function of two distinct pairs of scalar single-trace operators is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_1(x_3)\mathcal{O}_2(x_4)\rangle = \frac{1}{x_{13}^{2\Delta_1}}\frac{1}{x_{24}^{2\Delta_2}} + O(1/N)\ . \tag{1.22}$$

Three-point functions of single-trace operators are taken to be proportional to $1/N$, and one further assumes that the connected part of an n-point function of single trace operators is proportional to N^{2-n}. Other variants of this axiom may also be viable.

These axioms lead to incorrect thermodynamics at very high temperatures [5] so a generalized free CFT is not a complete consistent CFT on its own. It should rather be viewed as a decoupled sector of a larger CFT. We assume, as an axiom, that the dimension of the lightest new state in the expanded spectrum grows large with N. The additional states are usually interpreted in AdS/CFT as dual to black hole microstates, a conclusion supported in two dimensions by degeneracy-counting [26] and by constraining their correlation functions [4].

Because the energy-momentum tensor $T_{\mu\nu}$ has low scaling dimension it must be a single-trace operator in any generalized free theory. In general, after the stress tensor has been normalized so that $\langle T_{\mu\nu}(x)T^{\mu\nu}(0)\rangle = \frac{1}{|x|^{2d}}$, its three-point functions with all other operators are determined up to a single unknown constant $\sqrt{C_T}$.

$$\langle \phi_i(\infty)T_{\mu\nu}(x)\phi_j(\infty)\rangle = I_{\mu\nu}(x)\frac{\delta_{ij}\Delta_i}{\sqrt{C_T}} \tag{1.23}$$

where $I_{\mu\nu}(x)$ is an unimportant kinematic object. The constant C_T, which roughly speaking measures the number of local degrees of freedom, should be proportional to N^2.

1.2.3 Multi-trace operators from the bootstrap

We now endeavor to determine the operator product expansion of two distinct single-trace scalar primaries $\mathcal{O}_1, \mathcal{O}_2$ in a generalized free CFT, to leading order in $1/N$. For definiteness take all four operators to lie on a two-dimensional plane and place three of them at $0, 1, \infty$. The spectrum of operators appearing in the OPE can be read off from the conformal partial wave expansion

$$\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_2(z,\bar{z})\mathcal{O}_1(0)\rangle = \sum_p |C_{12p}|^2 W_p(z,\bar{z}) \tag{1.24}$$

As usual, the sum over p runs over all primaries in the spectrum, and C_{12p} is the coefficient with which primary p appears in the $\mathcal{O}_1\mathcal{O}_2$ OPE. The conformal partial waves $W_p(z,\bar{z})$ are fixed by symmetry. The details of their functional form depend on the spacetime dimension, but in any dimension they take the general form of a power series

$$W_p(z,\bar{z}) = z^{h_p-h_1-h_2}\bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2}\sum_{k,\bar{k}=0}^{\infty} a_{k,\bar{k}} z^k \bar{z}^{\bar{k}} \tag{1.25}$$

where all the coefficients $a_{k,\bar{k}}$ are nonzero. The chiral dimensions $h,\bar{h}$ of an operator are related to its scaling dimension Δ and spin ℓ by $\Delta = h+\bar{h}$, $\ell = \left|h-\bar{h}\right|$.

Now, in generalized free CFT the four-point function on the left hand side of (1.24) is the product of two two-point functions, up to $1/N$ corrections. That is

$$(1-z)^{-2h_2}(1-\bar{z})^{-2\bar{h}_2} + O(1/N) = \sum_p |C_{12p}|^2 W_p(z,\bar{z}) \tag{1.26}$$

Let us compare the two sides of this equation as power series in $z,\bar{z}$. Matching at leading order demands that the sum over p contain, as its lowest-dimension operator, one with dimensions $h_p = h_1 + h_2$, $\bar{h}_p = \bar{h}_1 + \bar{h}_2$ and unit OPE coefficient. Continuing to match the power series at higher orders in $z,\bar{z}$, we find that the right hand side contains a spectrum of operators with dimensions

$$h_p = h_1 + h_2 + n, \quad \bar{h}_p = \bar{h}_1 + \bar{h}_2 + \bar{n}, \quad \text{for all } n,\bar{n} \in \mathbb{Z}_{\geq 0} \tag{1.27}$$

all of which must enter into the $\mathcal{O}_1\mathcal{O}_2$ OPE with order-one coefficients. The operators we have discovered in this way are called double-trace by analogy with large-N gauge theory as discussed above. A double-trace operator with dimension $\Delta_1 + \Delta_2 + 2n + \ell$ and spin ℓ is often denoted $(\mathcal{O}_1\mathcal{O}_2)_{n,\ell}$. Similar considerations of crossing symmetry applied to correlators of multi-trace operators lead one to conclude that the spectrum also contains triple-trace operators and so on. For correlation functions involving multi-trace operators the Wick contraction rules of generalized free field theory still apply provided one treats the multi-trace operators as normal-ordered products of single-traces.

1.3 AdS and Holography

In this section we attempt a quick review of AdS field theory and holography. For more detail and better pedagogy the reader is directed to [44, 45].

1.3.1 Introducing Anti-de Sitter space and its conformal boundary

The symmetry group of d-dimensional Euclidean CFT is SO$(d+1,1)$. Suppose one was told there exists a duality relating conformal field theory to quantum gravity in a certain unspecified space. One would naturally look for a space with the same symmetry group. That space is Euclidean Anti-de Sitter space, AdS$_{d+1}$, defined as the homogeneous, isotropic space with negative curvature. Its metric in so-called global coordinates is

$$ds^2 = \frac{1}{(\cos\rho)^2}\left(d\rho^2 + dt^2 + (\sin\rho)^2\, d\Omega^2\right), \quad 0 \le \rho < \pi/2 \tag{1.28}$$

where $d\Omega^2$ is the metric on the sphere S^{d-1}.

The surface $\rho = \pi/2$ is referred to as the boundary of AdS, though strictly speaking AdS does not have a boundary[2] because that surface is infinitely far from all points in the space. To give the space an actual boundary one may impose a cutoff at $\cos\rho = \epsilon$ for some small number ϵ. While this choice of cutoff is quite natural in the coordinates (ρ, t, ϕ) in which we have chosen to work, there is no fundamental reason to prefer it over a more general choice of cutoff surface defined by

$$f(t, \hat{n}) \cos\rho = \epsilon \tag{1.29}$$

for some function f of the time coordinate t and angular coordinate $\hat{n} \in S^{d-1}$.

The induced metric on the cutoff surface is

$$ds_B^2 = \frac{f(\Omega, t)^2}{\epsilon^2}\left(dt^2 + d\Omega^2\right) \tag{1.30}$$

which for $f = 1$ we recognize as the metric of a cylinder $\mathbb{R} \times S^{d-1}$ of radius $1/\epsilon$. For generic f the metric is related to that of a cylinder by a conformal transformation, because f simply

[2] Even more strictly speaking, I mean to say that the *closure* of AdS does not have a boundary. That AdS itself does not is trivial by virtue of it being an open set.

rescales the boundary metric $h_{\mu\nu}$ by a local factor and this preserves the angles between vectors in tangent space (but not necessarily their lengths).

It is standard to strip off the factor of $1/\epsilon^2$ from the boundary metric to make it independent of ϵ. Afterward, taking the limit $\epsilon \to 0$ defines the conformal boundary of AdS, so named because it is defined only up to conformal transformations induced by changing the choice of cutoff function f.

1.3.2 Quantum fields in AdS and holography

Suppose $\phi(y)$ is a mass-m scalar quantum field living in AdS. Correlation functions involving ϕ and perhaps other bulk fields can be computed with standard Feynman diagram techniques, although one must work solely in position space[3] because curvature causes Fourier transforms along different directions not to commute.

The position-space propagator for ϕ between two points y, y' in the bulk is a hypergeometric function of the geodesic distance σ between the points,

$$G(y,y') = (2\cosh\sigma)^{-\Delta}{}_2F_1\left(\tfrac{\Delta}{2}, \tfrac{\Delta+1}{2}; \Delta - \tfrac{d-2}{2}; \operatorname{sech}^2\sigma\right) \ . \tag{1.31}$$

A number of different expressions for this same function exist [44] and can often be useful. The positive parameter Δ appearing above is related to the bulk field ϕ's mass by

$$\Delta(\Delta - d) = m^2 \ . \tag{1.32}$$

The long distance limit of the propagator (1.31) is $G(y,y') \to e^{-\Delta\sigma}$, and it follows that if one of the fields in a bulk position-space correlation function approaches $\rho \to \pi/2$ that function goes to zero proportional to $(\cos\rho)^{\Delta}$. By the same token, the operator

$$\mathcal{O}(t,\hat{n}) = \lim_{\rho\to\pi/2} (\cos\rho)^{-\Delta}\phi(\rho,t,\hat{n}) \tag{1.33}$$

is finite within any bulk correlation function. We interpret $(t,\hat{n})$ as coordinates on the cylindrical cutoff surface rescaled to have unit radius. Equivalent to the definition above,

[3]Or in Mellin space [46].

one may state that $\mathcal{O}(t, \hat{n})$ is the bulk field ϕ evaluated at the point $(t, \hat{n})$ on the cutoff surface and multiplied by $\epsilon^{-\Delta}$ to make the limit $\epsilon \to 0$ regular. This definition has the benefit of generalizing naturally to arbitrary cutoff surfaces (1.29) as follows.

$$\mathcal{O}(t, \hat{n}) = \epsilon^{-\Delta} \phi(\rho, \hat{n}, t) \quad \text{with} \quad \cos\rho = f(\hat{n}, t)^{-1}\epsilon \quad \text{and} \quad \epsilon \to 0 \,. \tag{1.34}$$

Under a redefinition of the cutoff surface the operator $\mathcal{O}(t, \hat{n})$ transforms as a conformal scalar field with dimension Δ. For example under a rescaling $f \to \lambda f$ it follows from the fact that ϕ is proportional to $(\cos\rho)^\Delta$ that

$$\mathcal{O}(t, \hat{n}) \to \lambda^{-\Delta} \mathcal{O}(t, \hat{n}) \tag{1.35}$$

as required for a scalar density of dimension Δ. A consequence is that the two-point function of a boundary operator with dimension Δ is

$$\langle \mathcal{O}(t, \hat{n}) \mathcal{O}(t', \hat{n}') \rangle = \frac{e^{\Delta(t+t')}}{|e^t \hat{n} - e^{t'} \hat{n}'|^{2\Delta}} \tag{1.36}$$

which matches the CFT two point on the cylinder obtained from applying the inverse of the map shown in figure 1.1 to the plane two-point function (1.3). Likewise conformal symmetry fixes the three-point function of boundary operators up to an overall constant, essentially the bulk fields' three-point coupling in the Lagrangian, up to loop corrections.

More generally, n-point functions of boundary operators $\langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle$ transform exactly like expectation values in conformal field theory. In fact, when bulk interactions are perturbative – suppressed by a factor $1/N$ for some large number N – they have precisely the structure of generalized free CFT, with one single-trace operator for every fundamental field in the bulk.

As mentioned in section 1.2.2, generalized free CFT is incomplete on its own and must be viewed as a sector of a larger CFT. At the same time, quantum field theory in the bulk is incomplete as well and should be viewed as a low-energy approximation to string theory or perhaps another quantum theory of matter and gravity. The AdS/CFT correspondence proposes to identify the completions on the two sides.

With one interesting exception, the AdS/CFT correspondence is agnostic about the fields and interactions governing low-energy physics in the bulk; any choice of bulk Lagrangian can be accommodated by some generalized free field theory [15]. The exception is gravity. Every generalized free CFT has as one of its single trace operators the stress tensor, with spin 2 and dimension d. Any bulk theory arising from AdS/CFT therefore has a fundamental field with spin 2 and mass zero, i.e. has a graviton. This illustrates a very appealing feature of the correspondence, that it seems not only to offer a possible quantum description for gravity but for its own consistency to *demand* the presence of gravity.

1.3.3 Witten diagrams and conformal blocks

One typically computes n-point functions of boundary operators in AdS using Feynman diagrams, called Witten diagrams in this context, where there is the additional feature that propagators may extend to the boundary. Such bulk-boundary propagators are defined by a limit analogous to (1.33),

$$G_{b\partial}(y;x) = \lim_{\rho \to \pi/2} (\cos\rho)^{-\Delta} G(y,\, (\rho, x)) \ . \tag{1.37}$$

That the result of the limit is a nontrivial function of the bulk position y is a feature of the geometry[4] of AdS.

The evaluation of Witten diagrams, especially with loops, is a difficult and interesting challenge that has received much recent attention [47–51]. These provide a window into perturbative quantum gravity in the bulk, to be compared with $1/N$ corrections to generalized free conformal field theory. The most efficient way to make such comparisons are in terms of conformal block expansions of correlators, because these isolate the data of interest, i.e. scaling dimensions and OPE coefficients, from kinematic data.

An oft-considered Witten diagram is the tree-level exchange diagram with four external

[4]It is a manifestation of the underappreciated fact that if one varies one of a geodesic's endpoints the variation in its length L,

$$\Delta L = v_\mu \Delta x^\mu + (g_{\mu\nu} - \tfrac{3}{2} v_\mu v_\nu) \Delta x^\mu \Delta x^\nu + O((\Delta x)^3), \tag{1.38}$$

does not vanish for $L \gg |\Delta x|$ even when Δx^μ is perpendicular to the tangent vector v_μ. This is in contrast to flat space, where perpendicular translation of a string's endpoint leads to negligible change in length in the limit of a long string. Of course, it is the presence of an additional length scale in curved space that makes the phenomenon possible.

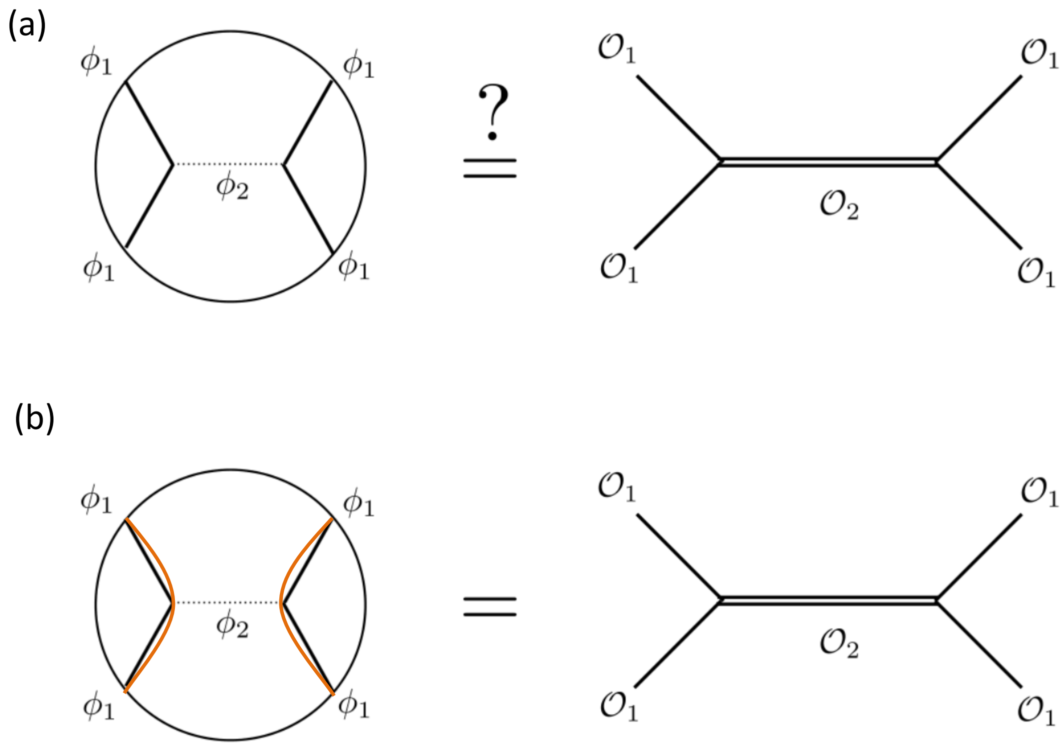

Figure 1.4: (a) A Witten diagram representing exchange of a bulk field ϕ_2, left, versus, right, a conformal block for exchange of the dual operator $\mathcal{O}_2$. The two objects are functions of the same data, and one might expect them to be equal, but they are not. (b) Answering the question implicitly posed above, the bulk object that computes a conformal block is a Geodesic Witten diagram, a main subject of this thesis.
Image (a) originally appeared in [5]. It has been reproduced with permission.

points, shown on the left hand side of figure 1.4 (a). The circle in that figure represents the boundary of AdS, where the four external operators (in this case identical scalars) are located. Bold lines are bulk-to-boundary propagators and the dotted line is a bulk-to-bulk propagator. This Witten diagram would contribute to the four-point function $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle$ in a bulk theory with a cubic interaction $\lambda \phi_1^2 \phi_2$ in its Lagrangian. Its decomposition into conformal blocks was undertaken in the very early days of the AdS/CFT correspondence [52]. As perhaps expected, the diagram contains a contribution from the block with exchanged operator $\mathcal{O}_2$ dual to ϕ_2, i.e. the block on the right hand side of the equation in figure 1.4 (a), but the equation is spoiled by additional contributions to the left hand side from conformal blocks associated with exchange of the double-trace operators $(\mathcal{O}_2\mathcal{O}_2)_{n,0}$.

The double-trace contribution can be understood from the fact at an intermediate position between the two pairs of operators the Witten diagram is associated to two possible intermediate states: one in which there is only a ϕ_2 particle, and a second in which there are still (or again) two ϕ_1 particles. This intermediate state is naturally represented by a double-trace state $(\mathcal{O}\mathcal{O})_{n,\ell}$, and indeed one with $\ell = 0$ because it must couple to the scalar field ϕ_2. The parameter n is associated with the energy of relative motion between the particles. It takes integer values as a consequence of the geometry of AdS [44].

1.4 Holographic conformal blocks and outline of this thesis

The first instance of a holographic interepretation supplied to a conformal block appears in [22] where it was observed that the Virasoro vacuum block of two light and two heavy operators is proportional to the length of a geodesic connecting the light operators' positions in the background created by the heavy operators.

That observation was greatly expanded in [1] where we developed a systematic picture for conformal blocks in terms of geodesic segments stretching between points in AdS, work which appears in Chapter 2. It predates the full discovery of geodesic Witten diagrams, and the appearance of geodesics in the context of conformal blocks was at the time somewhat puzzling.

Chapter 3, based on [3], contains the most fundamental results in this thesis. It introduces geodesic Witten diagrams, figure 1.4 (b), the holographic duals of conformal blocks, and shows how they can be used to decompose Witten diagrams into their constituent blocks.

In Chapter 4 we return to Virasoro blocks and use geodesic Witten diagram technology to unify the descriptions of the previous two chapters to give a complete description of Virasoro blocks in the semiclassical limit, work that was published as [2].

CHAPTER 2

Worldline approach to semi-classical conformal blocks

We extend recent results on semi-classical conformal blocks in 2d CFT and their relation to 3D gravity via the AdS/CFT correspondence. We consider four-point functions with two heavy and two light external operators, along with the exchange of a light operator. By explicit computation, we establish precise agreement between these CFT objects and a simple picture of particle worldlines joined by cubic vertices propagating in asymptotically AdS_3 geometries (conical defects or BTZ black holes). We provide a simple argument that explains this agreement.

2.1 Introduction

Conformal field theories (CFTs) in two dimensions are specified by a central charge, a list of primary operators, and their OPE coefficients, this data being subject to the consistency requirements of modular invariance and crossing symmetry [33, 53]. The CFT data appears directly in the decomposition of correlation functions in terms of conformal blocks, where the coefficient of each conformal block is given in terms of the OPE coefficients among primary operators. The conformal blocks correspond to the virtual exchange of a given primary operator and all of its Virasoro descendants. The conformal blocks are completely fixed by conformal invariance, although no closed form expression for them is known, except in special cases. At the same time, efficient recursion relations exist allowing one to compute the conformal block to any desired order in the conformally invariant cross ratio [54].

While all of this is ancient history, conformal blocks have received renewed attention recently, both due to their central role in the revival of the conformal bootstrap program [28], and also as a useful way to think about how local physics can emerge in the bulk in examples of AdS/CFT duality [5, 15, 20, 22, 55, 56].[1] It is the latter perspective which is closest to our considerations here. Recent work in this general direction includes the study of time evolution of entanglement entropy in excited states created by local operators [57–59]; the issue of universality in 2d conformal field theories [26, 27]; entanglement entropy in higher spin theories [60] and the emergence of chaos in thermal systems [61].

We are interested in conformal blocks in the semi-classical limit, corresponding to taking the central charge and operator dimensions to infinity while keeping their ratios fixed. There is excellent evidence that in this limit the conformal blocks exponentiate [62,63] as $e^{-\frac{c}{6}f(\frac{h_i}{c};x)}$, although there exists no proof of this directly from the fundamental definition as a sum over Virasoro descendants. It is then natural to expect that the function $f(\frac{h_i}{c};x)$ can be computed from a saddle point analysis, and indeed several realizations of this are known, as we discuss.

Since conformal blocks are fixed by conformal symmetry, they can be computed in any theory with this symmetry, provided of course that the desired values of the central charge and operator dimensions are available. If we have a family of theories with a variable central charge, where $1/c$ plays the role of $\hbar$, then we can think of taking $c \to \infty$ and computing $f(\frac{h_i}{c};x)$ by solving some "classical equations". An example is provided by Liouville theory, and this motivates the monodromy approach to the computation of semi-classical conformal blocks [62, 63]. This approach boils down to solving an ordinary differential equation, $\psi''(z) + T(z)\psi(z) = 0$, with prescribed monodromy. The monodromy condition fixes certain "accessory parameters" in $T(z)$, and these can be used to reconstruct the conformal block by integration.

Another useful realization is in terms of gravity in AdS_3. As was emphasized in the context of holographic entanglement entropy [64,65], the monodromy approach can be recast in terms of the problem of finding a solution of Einstein's equations with specified boundary behavior. In this context, $T(z)$ is identified as the boundary stress tensor of AdS_3 gravity. The authors of [64, 65] were thereby able to derive from first principles the Ryu-Takayanagi

[1] Most of this work is in the context of $d > 2$ dimensional CFT, where the conformal group is finite dimensional, and explicit formulas for the conformal blocks are known [38, 39].

formula in this context, equating the entanglement entropy with the (regulated) length of a bulk geodesic.

Especially relevant for our purposes is the illuminating paper [22] which, among other things, gave an AdS_3 bulk interpretation of vacuum conformal blocks for the case in which two of the external operators are heavy and two are light, in a sense made precise below. The picture is that the heavy operators set up a classical asymptotically AdS_3 geometry corresponding to a conical defect or BTZ black hole, and the light operators are described by a geodesic probing this background solution.

Here we extend the results of [22] in several directions. Firstly, we consider the case of nonvacuum conformal blocks corresponding to an exchanged primary O_p, and also allow for the light operators to have distinct conformal dimensions. The bulk picture is that we now have three geodesic segments in the background geometry, one segment for each of the light operators, with the segments meeting at a cubic vertex; see figure 1. The segment corresponding to the operator O_p has one endpoint on the vertex, with the other ending at the conical defect or BTZ horizon. The resulting formula for the conformal block is more intricate than the case considered in [22], but is still quite compact. By expanding this result for small cross ratio, we are able to check against the corresponding result obtained directly from the CFT recursion relation, and we indeed find agreement. We also verify agreement with the monodromy approach.

We also provide a simple argument that explains the agreement between the bulk geodesic approach and the monodromy approach. This is done by thinking about the backreaction on the metric produced by the configuration of particle geodesics. The relation to the CFT monodromy approach is especially transparent in the Chern-Simons formulation of AdS_3 gravity [66, 67], and this lets us establish that the solution sourced by the geodesics is in direct correspondence with a solution of the monodromy problem.

The geodesic approximation is only valid to first order in the light operator dimensions, while the full conformal block of course receives contributions at all orders. In the bulk we can think about solving Einstein's equations order by order to compute these corrections.

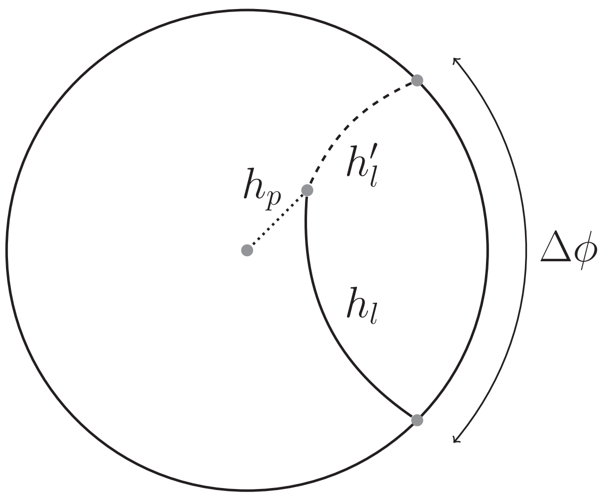

Figure 2.1: Geodesic configuration. The disk represents a slice of a conical defect or BTZ geometry, as supported by the heavy operator. Light operators are represented by geodesic segments as shown.

Equivalently, this can be phrased in terms of solving the monodromy problem at higher orders. We verify that the second order solution indeed yields a result in agreement with that obtained from the recursion relation.

Before proceeding, we would like to emphasize that a motivation for carrying out the work presented here is to eventually apply these results to computations that are not entirely dictated by symmetry. For instance, semi-classical correlation functions computed in the BTZ geometry display a specific form of information loss [68]. In the present context such correlation functions arise in a manner in which it is clear what effects have been thrown out, namely non-vacuum blocks and $1/c$ corrections, and this might be a useful way to think about what is needed to restore purity. For a calculation of one loop corrections of holographic entanglement entropy, see [69].

2.2 Conformal blocks

In this section we briefly review the definition of conformal blocks in 2D CFT, as well as some of the methods available to compute them, either in a series expansion or in the semi-classical limit.

2.2.1 Definitions

We follow the conventions of [33]. The correlation function of four primary operators is expanded as

$$\langle O_1(\infty,\infty)O_2(1,1)O_3(x,\overline{x})O_4(0,0)\rangle = \sum_p C_{34}^p C_{12}^p \mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\overline{x}) \ , \tag{2.1}$$

where $O_1(\infty,\infty) = \lim_{z_1,\overline{z}_1\to\infty} z_1^{2h_1}\overline{z}_1^{2\overline{h}_1} O_1(z_1,\overline{z}_1)$ inside the correlator. The expansion å is obtained by using the O_1O_2 and O_3O_4 OPEs, together with the fact that the OPE coefficients involving Virasoro descendants are related by conformal symmetry to those of the primaries. Each term in å corresponds to the virtual exchange of a primary O_p together with all of its Virasoro descendants.

The conformal block $\mathcal{F}_{34}^{21}(p|x)$ admits a series expansion,

$$\mathcal{F}_{34}^{21}(p|x) = x^{h_p-h_3-h_4}\tilde{\mathcal{F}}_{34}^{21}(p|x) \ , \quad \tilde{\mathcal{F}}_{34}^{21}(p|x) = \sum_{n=0}^{\infty}[\tilde{\mathcal{F}}_{34}^{21}]_n x^n \tag{2.2}$$

with $[\tilde{\mathcal{F}}^{21}_{34}]_0 = 1$. At the next two orders we have[2]

$$
\begin{aligned}
[\tilde{\mathcal{F}}^{21}_{34}]_1 &= \frac{(h_p + h_2 - h_1)(h_p + h_3 - h_4)}{2h_p} \\
[\tilde{\mathcal{F}}^{21}_{34}]_2 &= \frac{A + C}{B} \\
A &= (h_p + h_2 - h_1)(h_p + h_2 - h_1 + 1)\Big[(h_p + h_3 - h_4)(h_p + h_3 - h_4 + 1)(4h_p + \frac{c}{2}) \\
&\qquad - 6h_p(h_p + 2h_3 - h_4)\Big] \\
C &= (h_p + 2h_2 - h_1)\Big[4h_p(2h_p + 1)(h_p + 2h_3 - h_4) - 6h_p(h_p + h_3 - h_4)(h_p + h_3 - h_4 + 1\Big] \\
B &= 4h_p(2h_p + 1)(4h_p + \frac{c}{2}) - 36h_p^2
\end{aligned}
\tag{2.3}
$$

Higher order terms are readily computed using a convenient recursion relation [54].

2.2.2 Heavy-light correlators, and the semi-classical limit

It will now be convenient to define a rescaled stress tensor and rescaled conformal weights. If T_{CFT} and h denote the usual stress tensor and conformal weight, we now define T and ϵ as

$$T_{CFT}(z) = \frac{c}{6}T(z) \ , \quad h = \frac{c}{6}\epsilon, \tag{2.4}$$

In terms of which the OPE is

$$T(z)O(0) = \frac{\epsilon}{z^2}O(0) + \frac{6}{c}\frac{1}{z}\partial O(0) + \dots \ . \tag{2.5}$$

The semi-classical limit of the conformal blocks is defined by taking $c \to \infty$ at fixed ϵ, where ϵ refers to the external operators $O_{1,2,3,4}$ as well as the internal primary O_p. In this limit there is good evidence, though no direct proof, that the conformal blocks exponentiate

$$\tilde{\mathcal{F}}^{21}_{34}(p|x) = e^{-\frac{c}{6}\tilde{f}^{21}_{34}(\epsilon_i;x)} \ . \tag{2.6}$$

This can be verified directly to the first few orders in the x expansion using the recursion relation obtained in [54] and reviewed in appendix A. It has been verified to high order in [70].

[2] Note that 6.191 in [33] is incorrect; see the errata.

We will be interested in the case in which the four-point function involves two operators of equal dimension ϵ_h, with the other two operators having dimensions ϵ_l and ϵ_l'. Here the subscripts stand for "heavy" and "light", in the sense that we will be consider $\epsilon_l, \epsilon_l' \ll 1$ so that we can expand perturbatively in these quantities. Note though that our usage of "light" is somewhat nonstandard, since this label is often applied to operators whose dimension h is held fixed as $c \to \infty$, which is not the case here.

In particular, the correlator of interest is (suppressing henceforth the dependence on anti-holomorphic quantities)

$$\langle h|O_{l'}(1)O_l(x)|h\rangle = \langle O_h(\infty)O_{l'}(1)O_l(x)O_h(0)\rangle \ , \tag{2.7}$$

expanded in the $x \to 1$ OPE channel. As indicated, we can think of this correlator as the two-point function of light operators in the excited state created by the heavy operators. To expand in conformal blocks, first use invariance under $z \to 1-z$ to write

$$\langle O_h(\infty)O_{l'}(1)O_l(x)O_h(0)\rangle = \langle O_h(\infty)O_h(1)O_l(1-x)O_{l'}(0)\rangle \tag{2.8}$$

so that the expansion is

$$\langle h|O_{l'}(1)O_l(x)|h\rangle = \sum_p C^p_{ll'}C^p_{hh}\mathcal{F}^{hh}_{ll'}(p|1-x)\overline{\mathcal{F}}^{hh}_{ll'}(p|1-\overline{x}) \ . \tag{2.9}$$

Suppressing the labels, we then write, in the semi-classical limit,

$$\mathcal{F}(1-x) = (1-x)^{h_p-h_l-h_l'}\tilde{\mathcal{F}}(1-x) = (1-x)^{\frac{c}{6}(\epsilon_p-\epsilon_l-\epsilon_l')}e^{-\frac{c}{6}\tilde{f}(1-x)} \ . \tag{2.10}$$

As noted above, the recursion relation can be used to compute $\tilde{f}(x)$ in a power series, although the results rapidly get complicated. To simplify we further expand $\tilde{f}(x)$ in the light operator dimensions. More precisely, we replace

$$\epsilon_l \to \delta\epsilon_l \ , \quad \epsilon_l' \to \delta\epsilon_l' \ , \quad \epsilon_p \to \delta\epsilon_p \tag{2.11}$$

and then expand in δ. At linear order in δ we find

$$\begin{aligned}\tilde{f}_{\delta}(x) = &\frac{1}{2}(\epsilon_l' - \epsilon_l - \epsilon_p)x \\ &- \frac{1}{16}\frac{(1-4\epsilon_h)(\epsilon_l' - \epsilon_l)^2}{\epsilon_p}x^2 + \left[\frac{\epsilon_l' - \epsilon_l}{4} - \frac{(\epsilon_l' + \epsilon_l)\epsilon_h}{6}\right]x^2 - \left[\frac{\epsilon_h}{12} + \frac{3}{16}\right]\epsilon_p x^2 \\ &- \frac{1}{16}\frac{(1-4\epsilon_h)(\epsilon_l' - \epsilon_l)^2}{\epsilon_p}x^3 + \left[\frac{\epsilon_l' - \epsilon_l}{6} - \frac{(\epsilon_l' + \epsilon_l)\epsilon_h}{6}\right]x^3 - \left[\frac{\epsilon_h}{12} + \frac{5}{48}\right]\epsilon_p x^3 \\ &+ \dots\end{aligned} \tag{2.12}$$

while at quadratic order in δ we have

$$\begin{aligned}\tilde{f}_{\delta^2}(x) = &-\frac{1}{48}(16\epsilon_h - 3)(\epsilon_l' - \epsilon_l)^2 x^2 + \frac{1}{72}(16\epsilon_h - 3)(\epsilon_l' + \epsilon_l)\epsilon_p x^2 + \frac{1}{144}(16\epsilon_h - 3)\epsilon_p^2 x^2 \\ &- \frac{1}{48}(16\epsilon_h - 3)(\epsilon_l' - \epsilon_l)^2 x^3 + \frac{1}{72}(16\epsilon_h - 3)(\epsilon_l' + \epsilon_l)\epsilon_p x^3 + \frac{1}{144}(16\epsilon_h - 3)\epsilon_p^2 x^3 \\ &+ \dots .\end{aligned} \tag{2.13}$$

The expansions continue to higher orders in x and δ.

2.2.3 Monodromy method

A convenient method for computing the semi-classical conformal block is the monodromy method. This is well reviewed in [22,63], and so we will be brief. We consider the differential equation

$$\psi''(z) + T(z)\psi(z) = 0 \tag{2.14}$$

with

$$T(z) = \frac{\epsilon_h}{z^2} + \frac{\epsilon_l'}{(z-1)^2} + \frac{\epsilon_l}{(z-x)^2} + \frac{\epsilon_l + \epsilon_l'}{z(z-1)} + \frac{x(1-x)}{z(1-z)(z-x)}c_x(x) \tag{2.15}$$

$T(z)$ can be thought of as the stress tensor in the presence of the operators appearing in the four-point function (2.7). Up to the free parameter c_x, its form is fixed by demanding that it have the correct double poles and asymptotic behavior. Noting that the simple pole term at $z = x$ is $T(z) \sim \frac{c_x}{z-x}$, along with the OPE æ, we see that c_x is related to the x-derivative of the conformal block. We can then integrate as

$$f(1-x) = -\int c_x(x)dx \tag{2.16}$$

where $\mathcal{F} = e^{-\frac{c}{6}f}$ in the semi-classical limit.

$c_x(x)$ is determined by demanding that the two independent solutions of (2.14) undergo a specific monodromy as we go around a contour that encloses the singularities at $z = 1, x$, the monodromy being fixed by the dimension of the primary ϵ_p. Specifically, the monodromy matrix M is required to have eigenvalues

$$\lambda_\pm = e^{i\pi(1\pm\sqrt{1-4\epsilon_p})} \ . \tag{2.17}$$

The problem is tractable in perturbation theory, where we can expand in the dimensions of the light operators, or in $1-x$. For the former we write

$$\begin{aligned} \psi &= \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots \\ T(z) &= T^{(0)} + T^{(1)} + T^{(2)} + \dots \end{aligned} \tag{2.18}$$

with

$$\begin{aligned} T^{(0)} &= \frac{\epsilon_h}{z^2} \\ T^{(1)} &= \frac{\epsilon_l'}{(z-1)^2} + \frac{\epsilon_l}{(z-x)^2} + \frac{\epsilon_l+\epsilon_l'}{z(z-1)} + \frac{x(1-x)}{z(1-z)(z-x)} c_x^{(1)} \\ T^{(2)} &= \frac{x(1-x)}{z(1-z)(z-x)} c_x^{(2)} \end{aligned} \tag{2.19}$$

The equations are then

$$\begin{aligned} (\psi^{(0)})'' + T^{(0)}\psi^{(0)} &= 0 \\ (\psi^{(1)})'' + T^{(0)}\psi^{(1)} &= -T^{(1)}\psi^{(0)} \\ (\psi^{(2)})'' + T^{(0)}\psi^{(2)} &= -T^{(1)}\psi^{(1)} - T^{(2)}\psi^{(0)} \end{aligned} \tag{2.20}$$

The zeroth order solutions are

$$\psi_\pm^{(0)}(z) = z^{\frac{1\pm\alpha}{2}} \ , \quad \alpha = \sqrt{1-4\epsilon_h} \tag{2.21}$$

We then obtain solutions at the next two orders as

$$\begin{aligned} \psi_\pm^{(1)}(z) &= \left[-\frac{1}{\alpha}\int^z dz \psi_-^{(0)} T^{(1)} \psi_\pm^{(0)}\right] \psi_+^{(0)}(z) + \left[\frac{1}{\alpha}\int^z dz \frac{\psi_+^{(0)} T^{(1)} \psi_\pm^{(0)}}{W}\right] \psi_-^{(0)}(z) \\ \psi_\pm^{(2)}(z) &= \left[-\frac{1}{\alpha}\int^z \psi_-^{(0)} \big(T^{(1)}\psi_\pm^{(1)} + T^{(2)}\psi_\pm^{(0)}\big)\right] \psi_+^{(0)}(z) \\ &\quad + \left[\frac{1}{\alpha}\int^z \psi_+^{(0)} \big(T^{(1)}\psi_\pm^{(1)} + T^{(2)}\psi_\pm^{(0)}\big)\right] \psi_-^{(0)}(z) \end{aligned} \tag{2.22}$$

In this form it is easy to read off the monodromy matrix as we encircle $z = 1, x$. Writing $M = M^{(0)} + M^{(1)} + M^{(2)} + \ldots$ we have $M^{(0)} = \mathbb{I}$ and

$$\begin{aligned} M^{(1)}_{++} &= -\frac{1}{\alpha}\oint dz \psi^{(0)}_{-} T^{(1)} \psi^{(0)}_{+} \ , \quad M^{(1)}_{+-} = \frac{1}{\alpha}\oint dz \psi^{(0)}_{+} T^{(1)} \psi^{(0)}_{+} \\ M^{(1)}_{-+} &= -\frac{1}{\alpha}\oint dz \psi^{(0)}_{-} T^{(1)} \psi^{(0)}_{-} \ , \quad M^{(1)}_{--} = \frac{1}{\alpha}\oint dz \psi^{(0)}_{+} T^{(1)} \psi^{(0)}_{-} \end{aligned} \tag{2.23}$$

and similarly for $M^{(2)}$. The integrals are easily computed using residues. Now let us give a few examples.

- First order in $\epsilon_l' = \epsilon_l, \epsilon_p$

This case was consired in [22]. Using $M^{(1)}_{++} = M^{(1)}_{--} = 0$, the equation determining c_x is

$$M^{(1)}_{+-} M^{(1)}_{-+} = -4\pi^2 \epsilon_p^2 \ , \tag{2.24}$$

yielding

$$c_x = -\left[\frac{1}{x} + \frac{\alpha}{x}\left(\frac{x^{\frac{\alpha}{2}} + x^{-\frac{\alpha}{2}}}{x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}}}\right)\right]\epsilon_l + \frac{\alpha}{x(x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}})}\epsilon_p \tag{2.25}$$

This gives (choosing the integration constant so that $\tilde{f}(0) = 0$)

$$f_\delta(1-x) = \left[\ln x + 2\ln\left(\frac{x^{-\frac{\alpha}{2}} - x^{\frac{\alpha}{2}}}{\alpha}\right)\right]\epsilon_l - \ln\left(\frac{x^{-\frac{\alpha}{4}} - x^{\frac{\alpha}{4}}}{4\alpha(x^{-\frac{\alpha}{4}} + x^{\frac{\alpha}{4}})}\right)\epsilon_p \tag{2.26}$$

Using

$$\tilde{f}_\delta(x) = f(x) + (\epsilon_p - \epsilon_l - \epsilon_l')\ln x \tag{2.27}$$

and expanding in x we verify agreement with (2.12).

- First order in $\epsilon_l' \neq \epsilon_l, \epsilon_p$

We now allow for two independent light operators. The result turns out to agree precisely with (3.30), the latter being obtained from a bulk computation.

- Second order in $\epsilon_l' = \epsilon_l$ with $\epsilon_p = 0$

In this case we solve $M^{(2)} = 0$. In general this is quite complicated due to the complexity of the second order solution $\psi^{(2)}$. To give a very simple illustration, we choose the light operators as above, and further expand to lowest nontrivial order in $1-x$, which gives

$$c_x^{(2)} = \left[-\frac{2}{45}\epsilon_h + \frac{22}{135}\epsilon_h^2\right]\epsilon_l^2(1-x)^3 + \ldots \tag{2.28}$$

which yields

$$\tilde{f}_{\delta^2}(x) = f_{\delta^2}(x) = \left[-\frac{1}{90}\epsilon_h + \frac{11}{270}\epsilon_h^2\right]\epsilon_l^2 x^4 + \ldots \tag{2.29}$$

This matches the result from the recursion relation (the term one order beyond those given in (2.13)). It is simple to extend to higher orders in x, if desired.

2.3 Semi-classical conformal blocks from bulk geodesics

In [22] it was observed that the conformal block with $\epsilon_l' = \epsilon_l$ and $\epsilon_p = 0$ can be reproduced by computing the length of a geodesic in an asymptotically AdS$_3$ background. In this section we show how to extend this to reproduce the conformal block for general values of ϵ_l', ϵ_l, ϵ_p, assuming all are small.

2.3.1 Setup

It is conceptually easiest work in global coordinates corresponding to CFT on the cylinder. The geometry related to the operator O_h by the state-operator correspondence is

$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(\frac{1}{\alpha^2}d\rho^2 - dt^2 + \sin^2\rho d\phi^2\right) \tag{2.30}$$

which is obtained from global AdS$_3$ by $t \to \alpha t$, $\phi \to \alpha\phi$, although in (2.30) we take $\phi \cong \phi + 2\pi$. Here α is the same quantity as in (2.21). For $\alpha^2 > 0$ (2.30) represents a conical defect with a singularity at $\rho = 0$. We define $w = \phi + it_E$, with $t_E = it$.

For $\alpha^2 < 0$ we instead have a BTZ black hole with event horizon at $\rho = 0$, provided that we change the identifications to $t \cong t + 2\pi$. Since we take $\phi \cong \phi + 2\pi$, we will mainly restrict to the $\alpha^2 > 0$ case.

Now consider computing AdS correlation functions of the operators O_l and $O_{l'}$. The light operators are placed at:

$$O_{l'}(w = 0)\ , \quad O_l(w)\ . \tag{2.31}$$

We work on a fixed time slice of the conical defect, so that $w = \phi$. An operator of dimension (h, h) is dual to a bulk field of mass $m = 2\sqrt{h(h-1)}$. Since we are always assuming

$h \sim c \gg 1$, this is $m = 2h \gg 1$. In this regime, the scalar field is well approximated by a point particle, and correlation functions can be computed in terms of regulated geodesic lengths. To compute the conformal block involving the exchanged primary O_p we propose the following simple prescription, which we will subsequently verify in particular cases, and then justify on general grounds. Working on a fixed t slice, we take the external light operators to be inserted as in (2.31). Attached to the boundary point $w = 0$ is a segment of geodesic corresponding to a particle of mass $m_{l'} = 2h_{l'}$. Similarly, a $m_l = 2h_l$ geodesic segment is attached at w. The primary operator is represented by a geodesic segment attached at $\rho = 0$ in the background (2.30). The three geodesic segments meet at a cubic vertex, located at some point in the interior of (2.30). The worldline action is, after stripping off a factor of $\frac{c}{6}$,

$$S = \epsilon_l' L_{l'} + \epsilon_l L_l + \epsilon_p L_p \tag{2.32}$$

where the Ls denote the regulated lengths of the geodesics.[3] The location of the cubic vertex is obtained by minimizing S. This yields a contribution to the correlation function on the cylinder

$$G(w) = e^{-\frac{c}{6}S(w)} \ . \tag{2.33}$$

This is related to the conformal block on the plane by the conformal transformation $z = e^{iw}$,

$$\mathcal{F}(1-z) = z^{-h_l} G(w)\Big|_{w=-i\ln z} \ , \tag{2.34}$$

where the factor of z^{-h_l} takes into account the transformation of the operator O_l. In terms of the functions f and $\tilde{f}$,

$$\begin{aligned} f(1-z) &= \epsilon_l \ln z + S(w)\big|_{w=-i\ln z} \\ \tilde{f}(1-z) &= \epsilon_l \ln z + (\epsilon_p - \epsilon_l - \epsilon_l') \ln(1-z) + S(w)\big|_{w=-i\ln z} \end{aligned} \tag{2.35}$$

Each geodesic segment is obtained by extremizing the standard worldline action $I = \epsilon \int d\lambda \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$, where we choose λ to be proper length. Geodesics thus obey

$$\frac{1}{\cos^2\rho}\dot{\rho}^2 + \frac{p_\phi^2}{\alpha^2}\cot^2\rho = 1 \tag{2.36}$$

[3]Note that it is h that appears here rather than $m = 2h$, since we are computing the chiral half of the correlator.

where the conserved momentum conjugate to ϕ is

$$p_\phi = \alpha^2 \tan^2 \rho \, \dot{\phi} \ . \tag{2.37}$$

This gives

$$\cos\rho = \frac{1}{\sqrt{1 + p_\phi^2/\alpha^2}} \frac{1}{\cosh\lambda} \tag{2.38}$$

The regulated length is defined by imposing a cutoff near the boundary, at $\cos\rho = \Lambda^{-1}$.

As a simple example, consider the case $\epsilon_l' = \epsilon_l$ with $\epsilon_p \ll \epsilon_l$. In this regime we can first work out the geodesic connecting the two boundary points. Attached to this will be a geodesic connecting the midpoint with $\rho = 0$. To first order in ϵ_p/ϵ_l we can neglect the fact that the latter geodesic will "pull" on the former. A simple computation yields for the ϵ_l geodesic

$$\cos\rho = \frac{\sin\frac{\alpha w}{2}}{\cosh\lambda} \ . \tag{2.39}$$

Its regulated length is given by

$$2L_l = 2\lambda\Big|_{\cos\rho=\Lambda^{-1}} = 2\ln\left(\sin\frac{\alpha w}{2}\right) + 2\ln\left(\frac{\Lambda}{2}\right) \ . \tag{2.40}$$

The length of the ϵ_p geodesic is

$$L_p = \int_0^{\cos\rho=\sin\frac{\alpha w}{2}} \frac{d\rho}{\cos\rho} = -\ln\left(\tan\frac{\alpha w}{4}\right) \ . \tag{2.41}$$

The contribution to the correlator is then

$$G(w) = e^{-2h_l L_1 - h_p L_p} = \frac{\left(\tan\frac{\alpha w}{4}\right)^{h_p}}{\left(\sin\frac{\alpha w}{2}\right)^{2h_1}} \ , \tag{2.42}$$

where we dropped an uninteresting w-independent prefactor.[4] After a little bit of algebra, we then find

$$f(1-x) = \left[\ln x + 2\ln\left(\frac{x^{-\frac{\alpha}{2}} - x^{\frac{\alpha}{2}}}{\alpha}\right)\right]\epsilon_l - \ln\left(\frac{x^{-\frac{\alpha}{4}} - x^{\frac{\alpha}{4}}}{4\alpha(x^{-\frac{\alpha}{4}} + x^{\frac{\alpha}{4}})}\right)\epsilon_p \ , \tag{2.43}$$

in agreement with (2.26). Note that although we assumed $\epsilon_p \ll \epsilon_l$, since the result is linear it turns out to agree with (2.26), where no such assumption was made.

[4]This includes the dependence on the regulator Λ. Here and elsewhere we simply drop such regulator dependent terms, since they contribute no w-dependence.

Although we mainly focus on $\alpha > 0$, the result (2.43) makes sense in the $\alpha \to 0$ limit. In this case the bulk metric becomes Poincaré AdS after a coordinate rescaling. The ϵ_p geodesic now disappears down the infinite throat towards the Poincaré horizon.

In the above, the worldines were taken to extremize the action, which implies that the stress tensor of the particles is covariantly conserved. This is needed in order that the particles can consistently couple to gravity, a fact that we will need later when we explain the relation between the geodesic approach and the monodromy approach. In this regard, we also note that we only need to require that the worldlines extremize the action, and they need not furnish a global minimum.

2.3.2 Result for general light operators

Working in a fixed-t slice, our goal is to find the minimum value of the worldline action (2.32) as a function of the angular separation $\Delta\phi$ of the light operators O_l, $O_{l'}$. We are interested in the regime where the action is minimized by a configuration in which all three worldlines' lengths are nonzero. This requires each of the three weights $\epsilon_l, \epsilon_l', \epsilon_p$ to be less than the sum of the other two. In the complementary case, where one of the three weights is greater than or equal to the sum of the other two, it is simple to minimize the action but the relation to the CFT picture breaks down.

By varying the action with respect to the location of the cubic vertex, one finds that at the vertex

$$\epsilon_l \dot{x}_l^\mu + \epsilon_l' \dot{x}_{l'}^\mu + \epsilon_p \dot{x}_p^\mu = 0 \tag{2.44}$$

where the dots denote derivatives with respect to a proper length parameter that increases away from the vertex. The problem of computing the minimum value of S is identical to that of computing the energy in equilibrium of three elastic "rubber bands" whose tensions are $\epsilon_l, \epsilon_l', \epsilon_p$, independent of their lengths. Equation (2.44) is the equilibrium condition of vanishing net force on the cubic vertex.

The angular and radial components of equation (2.44) are

$$\epsilon_l p_\phi + \epsilon'_l p'_\phi = 0 \tag{2.45a}$$

$$\epsilon_l \sqrt{1 - \frac{p_\phi^2}{\alpha^2 \tan^2 \rho}} + \epsilon'_l \sqrt{1 - \frac{{p'_\phi}^2}{\alpha^2 \tan^2 \rho}} = \epsilon_p \tag{2.45b}$$

where ρ is the radial coordinate of the cubic vertex and

$$p_\phi = \alpha^2 (\tan^2 \rho_l) \dot{\phi}_l \tag{2.46}$$

is the conserved momentum along worldline l that comes from the metric's ϕ-translation isometry, with p'_ϕ defined similarly.

The sign of each square root in equation (2.45b) is positive if the corresponding worldline approaches the cubic vertex from radially outward and negative if it approaches from inward. When $|\epsilon_l^2 - \epsilon_l'^2| < \epsilon_p^2$ equations (2.45) can only be true if both square roots are positive, meaning both worldlines must approach the cubic vertex from the outward direction. On the other hand when $|\epsilon_l^2 - \epsilon_l'^2| > \epsilon_p^2$ equations (2.45) require the worldline corresponding to the smaller of ϵ_l, ϵ'_l to approach the vertex from the inward direction.

Equations (2.45) also imply

$$\frac{(p_\phi \epsilon_l)^2}{\alpha^2 \tan^2 \rho} = \frac{(p'_\phi \epsilon'_l)^2}{\alpha^2 \tan^2 \rho} = \mu^2 \tag{2.47}$$

where $\mu > 0$ is defined by

$$\mu^2 = \frac{\epsilon_l^2 + {\epsilon'_l}^2 - \epsilon_p^2/2}{2} - \frac{(\epsilon_l^2 - {\epsilon'_l}^2)^2}{4\epsilon_p^2}. \tag{2.48}$$

μ^2 is positive as a consequence of the assumption that each of the three weights $\epsilon_l, \epsilon'_l, \epsilon_p$ is less than the sum of the other two.

In light of equation (2.45a) it is useful to characterize the shapes of geodesics l, l' with a single parameter rather than the redundant set (p_ϕ, p'_ϕ). We define

$$P := \frac{\epsilon_l}{\alpha\mu} p_\phi = -\frac{\epsilon'_l}{\alpha\mu} p'_\phi \tag{2.49}$$

and assume without loss of generality that $P \geq 0$. Equation (2.47) implies that P is in fact the tangent of the ρ coordinate of the cubic vertex.

When both worldlines approach the vertex from outward, the angular separation $\Delta\phi$ between their endpoints is

$$\Delta\phi = \frac{\mu}{\alpha\epsilon_l}\int_1^\infty \frac{du}{u\sqrt{1+(Pu)^2}\sqrt{u^2-(\mu/\epsilon_l)^2}} + (\epsilon_l \to \epsilon_l') \ . \tag{2.50}$$

The integration variable u is related to the radial coordinate ρ by $u = (\tan\rho)/P$. When one of the worldlines approaches the vertex from inward instead, one must add to the right hand side of equation (2.50) the angle that worldline sweeps out while its radial coordinate is smaller than that of the cubic vertex. An expression for that angle is

$$\phi_{(\rho < \arctan P)} = \frac{2\mu}{\alpha\epsilon_s}\int_{\mu/\epsilon_s}^1 \frac{du}{u\sqrt{1+(Pu)^2}\sqrt{u^2-(\mu/\epsilon_s)^2}} \ , \tag{2.51}$$

where ϵ_s is the smaller of ϵ_l, ϵ_l'.

We will need to solve equation (2.50) for P as a function of $\Delta\phi$. One can put the equation in the form

$$e^{i\alpha\Delta\phi} = \left(\frac{\cos\gamma\cos\psi + i\sin\psi}{\cos\gamma + i\sin\gamma\sin\psi}\right)\left(\frac{\cos\gamma'\cos\psi + i\sin\psi}{\cos\gamma' + i\sin\gamma'\sin\psi}\right) \ , \tag{2.52}$$

where the angles ψ, γ, γ' are all between 0 and $\pi/2$ and are defined by $\cot\psi = P$, $\cos\gamma = \mu/\epsilon_l$, $\cos\gamma' = \mu/\epsilon_l'$.

We may trade out ψ for a new variable z defined by $\cos\psi = (z+z^{-1})/2$, $\sin\psi = (z-z^{-1})/2i$. Equation (2.52) is equivalent to

$$e^{i\alpha\Delta\phi} = \left(\frac{z + z\cos\gamma + \sin\gamma}{1+\cos\gamma + z\sin\gamma}\right)\left(\frac{z + z\cos\gamma' + \sin\gamma'}{1+\cos\gamma' + z\sin\gamma'}\right) \ . \tag{2.53}$$

This is a quadratic equation for z. It can be rewritten as a quadratic equation for P and solved to yield

$$P = \frac{\epsilon_l + \epsilon_l'}{2\mu}\cot\theta - \frac{\sqrt{\epsilon_p^2 - (\epsilon_l - \epsilon_l')^2\sin^2\theta}}{2\mu\sin\theta} \ . \tag{2.54}$$

We have introduced $\theta := \alpha\Delta\phi/2$.

The expression for P in equation (2.54) gives, via equation (2.49), the conserved momenta p_ϕ, p_ϕ' of worldlines l, l' in terms of the angular separation $\Delta\phi$ between their endpoints. In the case where one worldline approaches the cubic vertex from inward equation (2.54) continues to hold.

Now, the variation of the action (2.32) with respect to the locations x_l^μ, $x_{l'}^\mu$ of the worldlines' boundary endpoints is

$$dS = \epsilon_l g_{\mu\nu} \dot{x}_l^\mu dx_l^\nu + \epsilon_l' g_{\mu\nu} \dot{x}_{l'}^\mu dx_{l'}^\nu . \tag{2.55}$$

Here the dot denotes a derivative with respect to a proper length parameter that decreases away from the boundary. (In the picture where the worldlines are rubber bands and S is their energy, the right hand side of equation (2.55) is the work required to move the ends of the rubber bands.) It follows that the change in the action from a small increase in the angular separation $\Delta\phi$ of the wordlines' endpoints is

$$dS = \alpha\mu P d\Delta\phi \ . \tag{2.56}$$

The endpoints are at the locations of the light operators, (2.31). Their angular separation is w, and so

$$\frac{\partial S}{\partial w} = \alpha\mu P \tag{2.57}$$

where P is given by equation (2.54), and the variable θ is related to w by $\theta = \alpha w/2$. Equation (2.57) can be integrated to give

$$S(w) = (\epsilon_l + \epsilon_l') \ln \sin\theta + \epsilon_p \text{arctanh} \frac{\cos\theta}{\sqrt{1-\beta^2 \sin^2\theta}} - |\beta|\epsilon_p \ln\left(|\beta|\cos\theta + \sqrt{1-\beta^2\sin^2\theta}\right) \tag{2.58}$$

where $\beta := (\epsilon_l' - \epsilon_l)/\epsilon_p$.

Plugging this into (2.35), it yields

$$f(1-z) = \epsilon_l \ln z + (\epsilon_l + \epsilon_l') \ln \sin\theta + \epsilon_p \text{arctanh} \frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}} - |\beta|\epsilon_p \ln\left(|\beta|\cos\theta + \sqrt{1-\beta^2\sin^2\theta}\right) \tag{2.59}$$

with

$$\cos\theta = \frac{z^{\alpha/2} + z^{-\alpha/2}}{2} \ , \quad \sin\theta = \frac{z^{\alpha/2} - z^{-\alpha/2}}{2i} \ . \tag{2.60}$$

This result can be verified by checking that it agrees with the result from the monodromy approach, and also by expanding in z and verifying agreement with the result of the recursion relation. Also, it is straightforward to verify that upon setting $\epsilon_l' = \epsilon_l$ we recover (2.26).

The results in this section hold for the special case that the endpoints of the ϵ_l and ϵ_l' geodesics lie on a common time slice on the boundary. In appendix B we study the generalization to unequal times and explain how the correct conformal blocks emerge in this case as well.

2.4 Relation between geodesic and monodromy approaches

In this section we explain why computing the action for bulk geodesics gives answers for the semi-classical conformal blocks that agree with those of the monodromy approach. The argument is very simple. Given a geodesic configuration, we can work out the linearized metric perturbation sourced by the particles. This perturbation gives rise to a boundary stress tensor that can be identified with $T(z)$ appearing in the monodromy method. The monodromy conditions arise by requiring that Einstein's equations are obeyed at the geodesics. Each geodesic segment carries a conserved momentum p_ϕ, which can be identified with the accessory parameter c_x, since it appears as the residue of a simple pole in $T(z)$. Finally, $p_\phi = \frac{dS}{d\phi}$, just as c_x is related to the derivative of f. This establishes the relation between the geodesic action S and the function f appearing in the monodromy method.

We first show how to relate the simple pole in $T(z)$ to p_ϕ. We consider a metric with the usual Fefferman-Graham expansion near the boundary,

$$ds^2 = d\rho^2 + e^{2\rho} g^{(0)}_{\mu\nu} dx^\mu dx^\nu + g^{(2)}_{\mu\nu} dx^\mu dx^\nu + \dots \tag{2.61}$$

and take $g^{(0)}_{\mu\nu} dx^\mu dx^\nu = dw d\overline{w}$. We also include a particle with worldline action

$$S_h = 2h \int d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \ , \tag{2.62}$$

with λ equal to proper length. We consider the case that the worldline pierces the boundary at some location w_0, and we wish to consider the Einstein equations near this point. Expanding the Einstein equations for large ρ to first order, the only non vanishing equations read

$$
\begin{aligned}
g^{(2)}_{w\overline{w}} &= 2\pi h\delta^{(2)}(w-w_0) \\
\partial_{\overline{w}} g^{(2)}_{ww} - \partial_w g^{(2)}_{w\overline{w}} &= -4\pi p_w \delta^{(2)}(w-w_0) \\
\partial_w g^{(2)}_{\overline{w}\overline{w}} - \partial_{\overline{w}} g^{(2)}_{w\overline{w}} &= -4\pi p_{\overline{w}} \delta^{(2)}(w-w_0)
\end{aligned}
\tag{2.63}
$$

where the (rescaled) mass and canonical momentum are $\frac{c}{6}\epsilon = h$ and $\frac{c}{6}p^\mu = 2h\frac{dx^\mu}{d\lambda}$, c being the usual Brown-Henneaux central charge, $c = 3\ell/2G$. Now, the components of $g^{(2)}_{\mu\nu}$ are just the boundary stress tensor, $g^{(2)}_{\mu\nu} = T_{\mu\nu}$ (rescaled as in (2.4)). Thus,

$$
\begin{aligned}
\partial_{\overline{w}} T &= -2\pi\epsilon\partial_w\delta^{(2)}(w-w_0) + 4\pi p_w \delta^{(2)}(w-w_0) \\
\partial_w \overline{T} &= -2\pi\epsilon\partial_{\overline{w}}\delta^{(2)}(w-w_0) + 4\pi p_{\overline{w}} \delta^{(2)}(w-w_0)\ .
\end{aligned}
\tag{2.64}
$$

Using $\partial_{\overline{w}}\frac{1}{w} = 2\pi\delta^{(2)}(w)$, and $p_w = \frac{p_\phi}{2}$ we find

$$
\begin{aligned}
T(w) &= \frac{\epsilon}{(w-w_0)^2} + \frac{p_\phi}{w-w_0} + \dots \\
\overline{T}(\overline{w}) &= \frac{\epsilon}{(\overline{w}-\overline{w}_0)^2} + \frac{p_\phi}{\overline{w}-\overline{w}_0} + \dots
\end{aligned}
\tag{2.65}
$$

where ... denote non-singular terms. By the usual relation between canonical momentum and the variation of the action under a change of boundary conditions we have $\frac{c}{6}p_\phi = \frac{dS_h}{d\phi}$. The relation to the formulas appearing in monodromy approach is now clear: just as c_x appeared as the residue of the simple pole in the stress tensor and was related to derivative of f, the same is true of p_ϕ, now related to the derivative of the geodesic action.

The final step to demonstrate the equivalence of the two approaches is to to show how the monodromy conditions arise in the bulk. We first note that the full bulk metric will take the form

$$
ds^2 = d\rho^2 - Tdw^2 - \overline{T}d\overline{w}^2 + (e^{2\rho} + T\overline{T}e^{-2\rho})dwd\overline{w}\ . \tag{2.66}
$$

This is a solution of the source free Einstein's equations if $\partial_{\overline{w}}T = \partial_w\overline{T} = 0$. At the location of the particle worldlines these equations are corrected, as in (2.64).

We now pass to the Chern-Simons formulation of 2+1 gravity with negative cosmological constant. This was employed in a closely related context in [60]. The metric is replaced by an SL(2) × SL(2) connection,

$$A = \begin{pmatrix} \frac{1}{2}d\rho & e^{-\rho}T dw \\ -e^{\rho}dw & -\frac{1}{2}d\rho \end{pmatrix} \ , \quad \overline{A} = \begin{pmatrix} -\frac{1}{2}d\rho & e^{\rho}dw \\ -\overline{T}e^{-\rho}dw & \frac{1}{2}d\rho \end{pmatrix} \ . \tag{2.67}$$

Now consider the holonomy of this connection around a closed contour C, which we take to lie at fixed ρ. Focussing just on A,

$$\mathrm{H}ol[C] = Pe^{\oint A} \ . \tag{2.68}$$

In the absence of matter the holonomy would be trivial, since Einstein's equations are equivalent to flatness of the connections. But for a contour that encircles a particle worldline, the holonomy will pick up a contribution fixed by the mass of the particle. To relate this to the monodromy approach, we note that computing $\mathrm{H}ol[C]$ is equivalent to computing the monodromy of the system of differential equations

$$\frac{d\psi}{dw} = A\psi \tag{2.69}$$

where ψ is a two component vector. The bottom component obeys

$$\psi_2'' + T\psi_2 = 0 \ , \tag{2.70}$$

which we recognize as the ODE appearing in (2.14).

We focus on a contour that encircles the two operator insertion points at 0 and w. This contour encircles the worldline corresponding to the exchanged primary O_p. It is then clear that if T is such that the monodromy of the differential equation (2.70) is related to ϵ_p in the correct way, then the holonomy of the Chern-Simons connection will be such that we solve the Einstein equations in the presence of the particle worldline.

Summarizing, we see that given a linearized solution of Einstein's equations in the presence of particle worldlines we can find a solution of the monodromy problem. Further, the action of the bulk solution agrees with the function f appearing in the exponent of the semi-classical conformal block.

2.5 Discussion and subsequent developments

We have achieved a clean AdS_3 bulk interpretation of the semi-classical conformal blocks, extending the observations in [22]. To linear order in the light operator dimensions, we simply have to find the equilibrium configuration of three geodesic segments joined at cubic vertex, propagating in a geometry dual to the heavy operators. We close with a couple of observations and questions for the future.

It is straightforward in principle to go to higher orders in the light operator dimensions to reproduce the $O(\epsilon_L^2)$ correction [71] to the conformal block. From the bulk point of view, this just corresponds to solving Einstein's equations order by order in Newton's constant. Phrased in the language of the Chern-Simons formulation, the problem consists of finding locally flat connections with specified holonomies around contours representing the locations of the worldlines [72]. As at linear order, this is the same problem as in the monodromy approach.

While our work has focused on four-point conformal blocks, others have subsequently used geodesic networks to compute higher-point blocks [73–75]. In this case one has more worldlines attached with additional cubic vertices. Conformal blocks on higher genus Riemann surfaces have also been considered [76, 77], as well as conformal blocks with two pairs of heavy operators [78] which both create conical defects.

We have focused here on the semi-classical conformal block, which is the leading term in the large c expansion. More generally, we can think of writing

$$\tilde{\mathcal{F}}(x) = e^{-\frac{c}{6}(\tilde{f}^{(0)}(x)+\frac{1}{c}\tilde{f}^{(1)}(x)+\ldots)}\,. \tag{2.71}$$

In terms of the loop expansion in the bulk, we expect that $\tilde{f}^{(1)}$ is given by the effects of 1-loop fluctuations around the classical background. One can use the recursion relation to compute terms in $\tilde{f}^{(1)}(x)$ [79]. For example, setting $\epsilon_l' = \epsilon_l$ and $\epsilon_p = 0$, we find, at linear order in δ,

$$\tilde{f}^{(1)}(x) = \frac{3}{16}(4\epsilon_h - 1)x^2 + \frac{3}{16}(4\epsilon_h - 1)x^3 + \frac{1}{512}(4\epsilon_h - 1)(4\epsilon_h + 87)x^4 \\ - \frac{3}{4}(4\epsilon_h - 1)\epsilon_l x^2 - \frac{3}{4}(4\epsilon_h - 1)\epsilon_l x^3 - \frac{1}{14400}(5296\epsilon_h^2 + 37392\epsilon_h - 9675)\epsilon_l x^4 + \dots \tag{2.72}$$

where the $\dots$ denote terms higher order in ϵ_l and in x. It is important to note that to obtain this we first extract the large c asymptotics for general values of ϵ_l, ϵ_l' and ϵ_p, and only afterwards set $\epsilon_l' = \epsilon_l$ and $\epsilon_p = 0$. The terms in the first line are puzzling, as they are nonzero even upon setting $\epsilon_l = 0$. It will be interesting to understand their physical origin.

Once the bulk interpretation of the full conformal block is established, including the subleading $1/c$ effects, we can think of using this information to interpret specific CFT correlators. It will be interesting to apply this to the black hole context, where the problem of information loss can be phrassed in terms of such correlators. Very significant progress in these directions has been made [80–86].

Another interesting direction to consider is the extension to higher spin theories, where the Virasoro algebra is enhanced to a $\mathcal{W}$-algebra. In [60] it was established that the semi-classical vacuum block admits a bulk realization in terms of a Wilson line [87–89] embedded in an asymptotically AdS_3 background with higher spin fields excited. A natural question is how to extend this story beyond the vacuum block.

2.A Recursion relation

The series expansion of the conformal block can be computed using the recursion relation presented in [54]. We first write

$$\tilde{\mathcal{F}}_{21}^{34}(x) = \left(\frac{16q}{x}\right)^{h_p+\frac{1-c}{24}} (1-x)^{\frac{c-1}{24}-h_2-h_3}\theta_3(q)^{\frac{c-1}{2}-4\sum_i h_i} H(c,h_p,h_i,q) \tag{2.73}$$

where $H(c,h_p,h_i,q)$ is the quantity that will be computed by the recursion relation. q is related to x by

$$q = e^{i\pi\tau} \ , \quad \tau = i\frac{K(1-x)}{K(x)} \ , \quad K(x) = \frac{1}{2}\int_0^1 \frac{dt}{[t(1-t)(1-xt)]^{\frac{1}{2}}} \tag{2.74}$$

or equivalently

$$x = \left(\frac{\theta_2(q)}{\theta_3(q)}\right)^4 \tag{2.75}$$

This gives

$$16q = x + \frac{1}{2}x^2 + \frac{21}{64}x^3 + \frac{31}{128}x^4 + \frac{6257}{32768}x^5 + \dots \tag{2.76}$$

External conformal dimensions h_i are written in terms of λ_i as

$$h_i = \frac{c-1}{24} + \lambda_i^2 \tag{2.77}$$

We further define

$$\begin{aligned} \alpha_\pm &= \sqrt{\frac{1-c}{24}} \pm \sqrt{\frac{25-c}{24}} \\ \lambda_{pq} &= \alpha_+ p + \alpha_- q \\ \Delta_{mn}(c) &= \frac{c-1}{24} + \frac{(\alpha_+ m + \alpha_- n)^2}{4} \end{aligned} \tag{2.78}$$

The recursion relation is then

$$H(c,h_p,h_i,q) = 1 + \sum_{m>0,n>0} \frac{(16q)^{mn} R_{mn}(c,h_i) H(c,\Delta_{mn}+mn,h_i,q)}{h_p - \Delta_{mn}(c)} \tag{2.79}$$

with

$$R_{mn}(c,h_i) = -\frac{1}{2}\frac{\prod_{p,q}(\lambda_2+\lambda_1-\frac{\lambda_{pq}}{2})(\lambda_2-\lambda_1-\frac{\lambda_{pq}}{2})(\lambda_3+\lambda_4-\frac{\lambda_{pq}}{2})(\lambda_3-\lambda_4-\frac{\lambda_{pq}}{2})}{\prod'_{k,l}\lambda_{kl}} \tag{2.80}$$

The product in the numerator is taken over $p = -m+1, -m+3, \ldots, m-3, m-1$; $q = -n+1, -n+3, \ldots, n-1$. The product in the denominator is taken over $k = -m+1, -m+2, \ldots, m$; $l = -n+1, -n+2, \ldots n$, and the prime means that we omit $(k,l) = (0,0)$ and $(k,l) = (m,n)$.

2.B Extension to operators at different times

Equation (2.35) gives the relation between the value of the worldline action (2.32) and the corresponding conformal block when the two light operators lie on the same time slice. In this appendix we argue that the extension of (2.35) to operators at different times is

$$f(1-z) + \bar{f}(1-\bar{z}) - \epsilon_l \ln z\bar{z} = 2S(\phi,\tau) \tag{2.81}$$

where $S(\phi,\tau)$ is the worldline action as a function of the location (ϕ,τ) on the cylinder of the light operator O_l, with O_l' fixed at $(\phi,\tau) = (0,0)$. The relation between (ϕ,τ) and $(z,\bar{z})$ is $z = e^{iw}, \bar{z} = e^{i\bar{w}}$ with $w = \phi + i\tau$, $\bar{w} = \phi - i\tau$. Recall that f is defined in terms of the holomorphic conformal block $\mathcal{F}$ by $\mathcal{F} = e^{-\frac{c}{6}f}$. Similarly $\bar{f}$ is defined in terms of the antiholomorphic conformal block $\bar{\mathcal{F}}$ by $\bar{\mathcal{F}} = e^{-\frac{c}{6}\bar{f}}$.

To regularize the worldline action we place the boundary at $\cos\rho = \epsilon$, where the cutoff ϵ is independent of ϕ and τ.

Given that equation (2.35) holds for real w, the two sides of equation (2.81) agree when $\tau = 0$. Their derivatives with respect to τ agree as well; they both vanish by $\tau \to -\tau$ symmetry. The left hand side of equation (2.81) is the real part of a holomorphic function of $\phi + i\tau$ and therefore satisfies Laplace's equation. Thus if $S(\phi,\tau)$ satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\tau^2}\right) S(\phi,\tau) = 0 \tag{2.82}$$

then equation (2.81) must hold for all values of (ϕ,τ) for which $S(\phi,\tau)$ is defined.

We now want to show that $S(\phi,\tau)$ indeed satisfies (2.82). Let $\tilde{S}(x,y)$ be the worldline action as a function of the location x of operator O_l and the location y in AdS_3 of the cubic vertex. Let x_0 be given and let y_0 be the y that minimizes $\tilde{S}(x_0,y)$. Under the displacement

$(x_0, y_0) \to (x_0 + dx, y_0 + dy)$ the action $\tilde{S}$ becomes, to quadratic order in the displacements,

$$\begin{aligned}\tilde{S}(x_0 + dx, y_0 + dy) =& S(x_0) - \epsilon_l v_\mu dx^\mu + \frac{1}{2} K_{\mu\nu} dy^\mu dy^\nu \\ &+ \epsilon_l \left(dx^\mu dx^\nu \left(e^{-2L_l} g_{\mu\nu} + \frac{1}{2} v_\mu v_\nu \right) - 2e^{-L_l} \left(g_{\mu\nu} - v_\mu v_\nu \right) dx^\mu dy^\nu \right).\end{aligned} \tag{2.83}$$

Every tensor on the right hand side lives at the point x_0. In particular, dy^μ is the parallel transport of dy to the point x_0 and $g_{\mu\nu}$ is the metric at x_0. v_μ is the unit vector pointing from x_0 down the geodesic toward y_0, and L_l is the length of that geodesic. Because x_0 is on the boundary e^{-L_l} is proportional to the cutoff, ϵ. Equation (2.83) is true to zeroth order in the cutoff.

The term $\frac{1}{2} K_{\mu\nu} dy^\mu dy^\nu$ captures the change in the worldline action from the changes in lengths of geodesics l' and p and also the part of the length change of geodesic l that is independent of dx. The explicit form of K is

$$K_{\mu\nu} = \epsilon_l (2g_{\mu\nu} - 3v_\mu v_\nu) + \epsilon_l' (2g_{\mu\nu} - 3v_\mu' v_\nu') + \epsilon_p \sin \rho_0 v_\mu^p v_\nu^p \tag{2.84}$$

where the unit vectors v', v^p point from the cubic vertex down the corresponding geodesics and have been parallel transported to x_0, and ρ_0 is the ρ coordinate of the vertex.

Given a particular dx, the function $\tilde{S}(x_0 + dx, y_0 + dy)$ is minimized for some particular value of dy, call it dy_*, which is the solution to the linear equation

$$K_{\mu\nu} dy_*^\nu = 2\epsilon_l e^{-L_l} \left(g_{\mu\nu} - v_\mu v_\nu \right) dx^\nu. \tag{2.85}$$

Substituting for dy_* in $\tilde{S}(x_0 + dx, y_0 + dy_*)$ gives the minimized worldline action at $x = x_0 + dx$ to second order in dx:

$$\begin{aligned}(x_0 + dx) =& S(x_0) - \epsilon_l v_\mu dx^\mu + \epsilon_l dx^\mu dx^\nu \left(e^{-2L_l} g_{\mu\nu} + \frac{1}{2} v_\mu v_\nu \right) \\ & - \epsilon_l e^{-2L_l} dx^\mu \left(g_{\mu\rho} - v_\mu v_\rho \right) (K^{-1})^{\rho\sigma} (g_{\sigma\nu} - v_\sigma v_\nu) dx^\nu\end{aligned} \tag{2.86}$$

where $(K^{-1})^{\mu\sigma} K_{\sigma\nu} = \delta^\mu_\nu$. From equation (2.86) one can read off

$$\nabla_\mu \nabla_\nu S(x) = \epsilon_l \left(2e^{-2L_l} g_{\mu\nu} + v_\mu v_\nu \right) - 2\epsilon_l e^{-2L_l} \left(g_{\mu\rho} - v_\mu v_\rho \right) (K^{-1})^{\rho\sigma} (g_{\sigma\nu} - v_\sigma v_\nu). \tag{2.87}$$

The point x_0 is on the boundary cylinder, and so $g_{\mu\nu}$, $v_\mu v_\nu$, $K_{\mu\nu}$ are all of order ϵ^{-2}, in the sense that their components in the (ρ, ϕ, τ) coordinate system are of order ϵ^{-2}. $g^{\mu\nu}$ and $(K^{-1})^{\mu\nu}$ are both of order ϵ^2. Keeping only the most divergent terms in equation (2.87) we find

$$\nabla_\mu \nabla_\nu S(x) = \epsilon_l v_\mu v_\nu + O(\epsilon^{-1}). \tag{2.88}$$

Finally, we restrict to displacements dx that keep x on the boundary cylinder. Letting n^μ be the unit inward-pointing normal vector at x, the two-dimensional Laplacian of x is

$$\nabla^2 S = (g^{\mu\nu} - n^\mu n^\nu)\, \nabla_\mu \nabla_\nu S(x) = \epsilon_l \left(1 - (n \cdot v)^2\right) + O(\epsilon). \tag{2.89}$$

To lowest order in ϵ the quantity $n \cdot v$ is unity, and so

$$\nabla^2 S = 0 + O(\epsilon). \tag{2.90}$$

Thus the regularized worldline action satisfies Laplace's equation, which concludes the proof of equation (2.81).

CHAPTER 3

Geodesic Witten diagrams

We develop a new method for decomposing Witten diagrams into conformal blocks. The steps involved are elementary, requiring no explicit integration, and operate directly in position space. Central to this construction is an appealingly simple answer to the question: what object in AdS computes a conformal block? The answer is a "geodesic Witten diagram," which is essentially an ordinary exchange Witten diagram, except that the cubic vertices are not integrated over all of AdS, but only over bulk geodesics connecting the boundary operators. In particular, we consider the case of four-point functions of scalar operators, and show how to easily reproduce existing results for the relevant conformal blocks in arbitrary dimension.

3.1 Introduction

The conformal block decomposition of correlation functions in conformal field theory is a powerful way of disentangling the universal information dictated by conformal symmetry from the "dynamical" information that depends on the particular theory under study; see e.g. [30–32, 38, 39, 90, 91]. The latter is expressed as a list of primary operators and the OPE coefficients amongst them. The use of conformal blocks in the study of CFT correlation functions therefore eliminates redundancy, as heavily utilized, for instance, in recent progress made in the conformal bootstrap program, e.g. [28, 92].

In the AdS/CFT correspondence [8–10], the role of conformal blocks has been somewhat neglected. The extraction of spectral and OPE data of the dual CFT from a holographic correlation function, as computed by Witten diagrams [10], was addressed early on in the development of the subject [52, 93–99], and has been refined in recent years through the introduction of Mellin space technology [21, 46, 100–104]. In examining this body of work, however, one sees that a systematic method of decomposing Witten diagrams into conformal blocks is missing. A rather natural question appears to have gone unanswered: namely, what object in AdS computes a conformal block? A geometric bulk description of a conformal block would greatly aid in the comparison of correlators between AdS and CFT, and presumably allow for a more efficient implementation of the dual conformal block decomposition, as it would remove the necessity of actually computing the full Witten diagram explicitly. The absence of such a simpler method would indicate a surprising failure of our understanding of AdS/CFT: after all, conformal blocks are determined by conformal symmetry, the matching of which is literally the most basic element in the holographic dictionary.

In this paper we present an appealingly simple answer to the above question, and demonstrate its utility via streamlined computations of Witten diagrams. More precisely, we will answer this question in the case of four-point correlation functions of scalar operators, but we expect a similar story to hold in general. The answer is that conformal blocks are computed by "geodesic Witten diagrams." The main feature of a geodesic Witten diagram that distinguishes it from a standard exchange Witten diagram is that in the former, the bulk vertices are not integrated over all of AdS, but only over geodesics connecting points on the boundary hosting the external operators. This representation of conformal blocks in terms of geodesic Witten diagrams is valid in all spacetime dimensions, and holds for all conformal blocks that arise in four-point functions of scalar operators belonging to arbitrary CFTs (and probably more generally).

To be explicit, consider four scalar operators $\mathcal{O}_i$ with respective conformal dimensions Δ_i. The conformal blocks that appear in their correlators correspond to the exchange of primaries carrying dimension Δ and transforming as symmetric traceless tensors of rank ℓ; we refer to these as spin-ℓ operators. Up to normalization, the conformal partial wave[1] in

[1]Conformal partial waves and conformal blocks are related by simple overall factors as we review below.

CFT_d is given by the following object in AdS_{d+1}:

$$\int_{\gamma_{12}} d\lambda \int_{\gamma_{34}} d\lambda' G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) \times G_{bb}(y(\lambda), y(\lambda'); \Delta, \ell) \times G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4) \tag{3.1}$$

γ_{ij} denotes the bulk geodesic connecting boundary points x_i and x_j, with λ and λ' denoting the corresponding proper length parameters. $G_{b\partial}(y, x)$ are standard scalar bulk-to-boundary propagators connecting a bulk point y to a boundary point x. $G_{bb}(y(\ell), y(\ell'); \Delta, \ell)$ is the bulk-to-bulk propagator for a spin-ℓ field, whose mass squared in AdS units is $m^2 = \Delta(\Delta - d) - \ell$, pulled back to the geodesics. The above computes the s-channel partial wave, corresponding to using the OPE on the pairs of operators $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. As noted earlier, the expression (3.1) looks essentially like an exchange Witten diagram composed of two cubic vertices, except that the vertices are only integrated over geodesics. See figure 4.2. Note that although geodesics sometimes appear as an approximation used in the case of high dimension operators, here there is no approximation: the geodesic Witten diagram computes the exact conformal block for any operator dimension.

As we will show, geodesic Witten diagrams arise very naturally upon dismantling a full Witten diagram into constituents, and this leads to an efficient implementation of the conformal block decomposition. Mellin space techniques also provide powerful methods, but it is useful to have an approach that can be carried out directly in position space, and that provides an explicit and intuitive picture for the individual conformal blocks.

For the cases that we consider, the conformal blocks are already known, and so one of our tasks is to demonstrate that (3.1) reproduces these results. One route is by explicit computation. Here, the most direct comparison to existing results is to the original work of Ferrara, Gatto, Grillo, and Parisi [30–32], who provided integral representations for conformal blocks. In hindsight, these integral expressions can be recognized as geodesic Witten diagrams. Later work by Dolan and Osborn [38, 39, 90] provided closed-form expressions for some even-d blocks in terms of hypergeometric functions. Dolan and Osborn employed the very useful fact that conformal partial waves are eigenfunctions of the conformal Casimir operator. The most efficient way to prove that geodesic Witten diagrams compute conformal

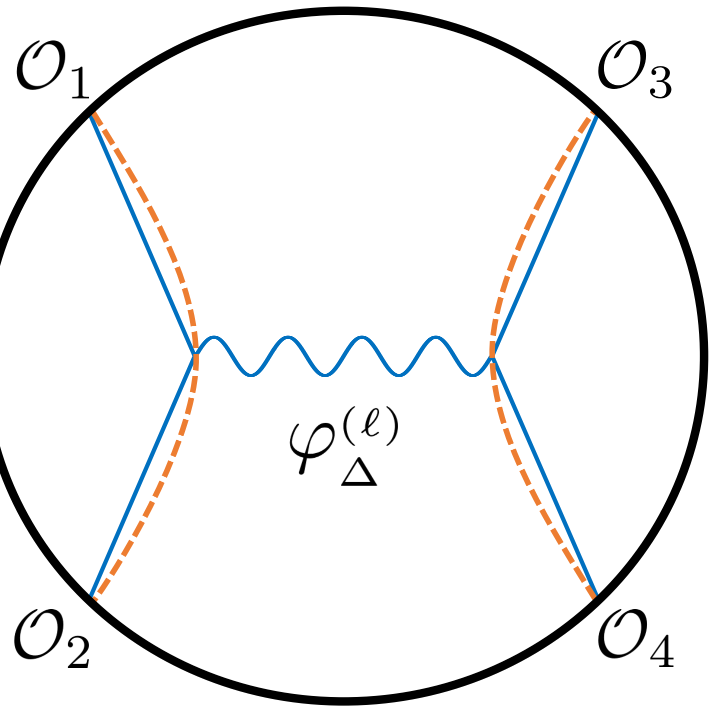

Figure 3.1: This is a geodesic Witten diagram in AdS_{d+1}, for the exchange of a symmetric traceless spin-ℓ tensor with $m^2 = \Delta(\Delta - d) - \ell$ in AdS units. Its main feature is that the vertices are integrated over the geodesics connecting the two pairs of boundary points, here drawn as dashed orange lines. This computes the conformal partial wave for the exchange of a CFT_d primary operator of spin ℓ and dimension Δ.

partial waves is to establish that they are the correct eigenfunctions. This turns out to be quite easy using embedding space techniques, as we will discuss.

Having established that geodesic Witten diagrams compute conformal partial waves, we turn to showing how to decompose a Witten diagram into geodesic Witten diagrams. We do not attempt an exhaustive demonstration here, mostly focusing on tree-level contact and exchange diagrams with four external lines. The procedure turns out to be quite economical and elegant; in particular, we do not need to carry out the technically complicated step of integrating bulk vertices over AdS. Indeed, the method requires no integration at all, as all integrals are transmuted into the definition of the conformal partial waves. The steps that are required are all elementary. We carry out this decomposition completely explicitly for scalar contact and exchange diagrams, verifying that we recover known results. These include certain hallmark features, such as the presence of logarithmic singularities due to anomalous dimensions of double-trace operators. We also treat the vector exchange diagram, again recovering the correct structure of CFT exchanges.

Let us briefly mention how the analysis goes. The key step is to use a formula expressing the product of two bulk-to-boundary propagators sharing a common bulk point as a sum of bulk solutions sourced on a geodesic connecting the two boundary points. The fields appearing in the sum turn out to be dual to the double-trace operators appearing in the OPE of the corresponding external operators, and the coefficients in the sum are closely related to the OPE coefficients. See equation (3.81). With this result in hand, all that is needed are a few elementary properties of AdS propagators to arrive at the conformal block decomposition. This procedure reveals the generalized free field nature of the dual CFT.

The results presented here hopefully lay the foundation for further exploration of the use of geodesic Witten diagrams. We believe they will prove to be very useful, both conceptually and computationally, in AdS/CFT and in CFT more generally.

The remainder of this paper is organized as follows. In section 2 we review relevant aspects of conformal blocks, Witten diagrams, and their relation. Geodesic Witten diagrams for scalar exchange are introduced in section 3, and we show by direct calculation and via the conformal Casimir equation that they compute conformal blocks. In section 4 we turn to the conformal block decomposition of Witten diagrams involving just scalar fields. We describe in detail how single and double trace operator exchanges arise in this framework. Section 5 is devoted to generalizing all of this to the case of spinning exchange processes. We conclude in section 6 with a discussion of some open problems and future prospects.

The ideas developed in this paper originated by thinking about the bulk representation of Virasoro conformal blocks in AdS_3/CFT_2, based on recent results in this direction [1, 22, 59, 64, 105, 106]. The extra feature associated with a bulk representation of Virasoro blocks is that the bulk metric is deformed in a nontrivial way; essentially, the geodesics backreact on the geometry. In this paper we focus on global conformal blocks (Virasoro blocks are of course special to CFT_2), deferring the Virasoro case to a companion paper [2].

3.2 Conformal blocks, holographic CFTs and Witten diagrams

Let us first establish some basic facts about four-point correlation functions in conformal field theories, and their computation in $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$. Both subjects are immense, of course; the reader is referred to [35, 107] and references therein for foundational material.

3.2.1 CFT four-point functions and holography

We consider vacuum four-point functions of local scalar operators $\mathcal{O}(x)$ living in d Euclidean dimensions. Conformal invariance constrains these to take the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} \frac{g(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}\ , \quad (3.2)$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$ and $x_{ij} \equiv x_i - x_j$. $g(u,v)$ is a function of the two independent conformal cross-ratios,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\ . \quad (3.3)$$

One can also define complex coordinates $z, \overline{z}$, which obey

$$u = z\overline{z}\ , \quad v = (1-z)(1-\overline{z})\ . \quad (3.4)$$

These may be viewed as complex coordinates on a two-plane common to all four operators after using conformal invariance to fix three positions at $0, 1, \infty$.

$g(u,v)$ can be decomposed into conformal blocks, $G_{\Delta,\ell}(u,v)$, as

$$g(u,v) = \sum_{\mathcal{O}} C_{12\mathcal{O}}\, C^{\mathcal{O}}{}_{34} G_{\Delta,\ell}(u,v) \quad (3.5)$$

where $\mathcal{O}$ is a primary operator of dimension Δ and spin ℓ.[2] Accordingly, the correlator can

[2]In this paper we only consider scalar correlators, in which only symmetric, traceless tensor exchanges can appear. More generally, ℓ would stand for the full set of angular momenta under the d-dimensional little group.

be written compactly as a sum of conformal partial waves, $W_{\Delta,\ell}(x_i)$:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}} C_{12\mathcal{O}}\, C^{\mathcal{O}}{}_{34} W_{\Delta,\ell}(x_i) \tag{3.6}$$

where

$$W_{\Delta,\ell}(x_i) \equiv \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} \frac{G_{\Delta,\ell}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \ . \tag{3.7}$$

Each conformal partial wave is fixed by conformal invariance: it contains the contribution to the correlator of any conformal family whose highest weight state has quantum numbers (Δ, ℓ), up to overall multiplication by OPE coefficients. It is useful to think of $W_{\Delta,\ell}(x_i)$ as the insertion of a projector onto the conformal family of $\mathcal{O}$, normalized by the OPE coefficients:

$$W_{\Delta,\ell}(x_i) = \frac{1}{C_{12\mathcal{O}}C^{\mathcal{O}}{}_{34}} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\, P_{\Delta,\ell}\, \mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \tag{3.8}$$

where

$$P_{\Delta,\ell} \equiv \sum_n |P^n\mathcal{O}\rangle\langle P^n\mathcal{O}| \tag{3.9}$$

and $P^n\mathcal{O}$ is shorthand for all descendants of $\mathcal{O}$ made from n raising operators P_μ. We will sometimes refer to conformal blocks and conformal partial waves interchangeably, with the understanding that they differ by the power law prefactor in (3.7).

Conformal blocks admit double power series expansions in u and $1-v$, in any spacetime dimension [38]; for $\ell = 0$, for instance,

$$G_{\Delta,0}(u,v) = u^{\Delta/2} \sum_{m,n=0}^{\infty} \frac{\left(\frac{\Delta+\Delta_{12}}{2}\right)_m \left(\frac{\Delta-\Delta_{34}}{2}\right)_m \left(\frac{\Delta-\Delta_{12}}{2}\right)_{m+n} \left(\frac{\Delta+\Delta_{34}}{2}\right)_{m+n}}{m!n!\left(\Delta+1-\frac{d}{2}\right)_m (\Delta)_{2m+n}} u^m(1-v)^n \ . \tag{3.10}$$

Higher ℓ blocks can be obtained from this one by the use of various closed-form recursion relations [90, 92]. Especially relevant for our purposes are integral representations of the conformal blocks [30–32]. For $\ell = 0$,

$$\begin{aligned} G_{\Delta,0}(u,v) = \frac{1}{2\beta_{\Delta 34}}\, u^{\Delta/2} \int_0^1 d\sigma\, \sigma^{\frac{\Delta+\Delta_{34}-2}{2}}(1-\sigma)^{\frac{\Delta-\Delta_{34}-2}{2}}(1-(1-v)\sigma)^{\frac{-\Delta+\Delta_{12}}{2}} \\ \times\, {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}, \Delta-\frac{d-2}{2}, \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma}\right) \end{aligned} \tag{3.11}$$

where we have defined a coefficient

$$\beta_{\Delta 34} \equiv \frac{\Gamma\left(\frac{\Delta+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}}{2}\right)}{2\Gamma(\Delta)} \ . \tag{3.12}$$

The blocks can also be expressed as infinite sums over poles in Δ associated with null states of $SO(d,2)$, in analogy with Zamolodchikov's recursion relations in $d = 2$ [54, 108, 109]; these provide excellent rational approximations to the blocks that are used in numerical work. Finally, as we revisit later, in even d the conformal blocks can be written in terms of hypergeometric functions.

Conformal field theories with weakly coupled AdS duals obey further necessary conditions on their spectra.[3] In addition to having a large number of degrees of freedom, which we will label[4] N^2, there must be a finite density of states below any fixed energy as $N \to \infty$; e.g. [5, 15, 26, 55]. For theories with Einstein-like gravity duals, this set of parametrically light operators must consist entirely of primaries of spins $\ell \leq 2$ and their descendants.

The "single-trace" operators populating the gap are generalized free fields: given any set of such primaries $\mathcal{O}_i$, there necessarily exist "multi-trace" primaries comprised of conglomerations of these with some number of derivatives (distributed appropriately to make a primary). Altogether, the single-trace operators and their multi-trace composites comprise the full set of primary fields dual to non-black hole states in the bulk. In a four-point function of $\mathcal{O}_i$, all multi-trace composites necessarily run in the intermediate channel at some order in $1/N$.

Focusing on the double-trace operators, these are schematically of the form

$$[\mathcal{O}_i\mathcal{O}_j]_{n,\ell} \approx \mathcal{O}_i\partial^{2n}\partial_{\mu_1}\dots\partial_{\mu_\ell}\mathcal{O}_j \ . \tag{3.13}$$

These have spin-ℓ and conformal dimensions

$$\Delta^{(ij)}(n,\ell) = \Delta_i + \Delta_j + 2n + \ell + \gamma^{(ij)}(n,\ell) \ , \tag{3.14}$$

[3] Finding a set of sufficient conditions for a CFT to have a weakly coupled holographic dual remains an unsolved problem. More recent work has related holographic behavior to polynomial boundedness of Mellin amplitudes [110, 111], and to the onset of chaos in thermal quantum systems [112].

[4] We are agnostic about the precise exponent: vector models and 6d CFTs are welcome here. More generally, we refer to the scaling of C_T, the stress tensor two-point function normalization, for instance.

where $\gamma^{(ij)}(n,\ell)$ is an anomalous dimension. The expansion of a correlator in the s-channel includes the double-trace terms[5]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \supset \sum_{m,\ell} P^{(12)}(m,\ell) W_{\Delta^{(12)}(m,\ell),\ell}(x_i) + \sum_{n,\ell} P^{(34)}(n,\ell) W_{\Delta^{(34)}(n,\ell),\ell}(x_i) \tag{3.15}$$

Following [15, 21], we have defined a notation for squared OPE coefficients,

$$P^{(ij)}(n,\ell) \equiv C_{12\mathcal{O}} C^{\mathcal{O}}{}_{34} \ , \quad \text{where} \quad \mathcal{O} = [\mathcal{O}_i \mathcal{O}_j]_{n,\ell} \ . \tag{3.16}$$

The $1/N$ expansion of the OPE data,

$$\begin{aligned} P^{(ij)}(n,\ell) &= \sum_{r=0}^{\infty} N^{-2r} P_r^{(ij)}(n,\ell) \ , \\ \gamma^{(ij)}(n,\ell) &= \sum_{r=1}^{\infty} N^{-2r} \gamma_r^{(ij)}(n,\ell) \ , \end{aligned} \tag{3.17}$$

induces a $1/N$ expansion of the four-point function. Order-by-order in $1/N$, the generalized free fields and their composites must furnish crossing-symmetric correlators. This is precisely the physical content captured by the loop expansion of Witten diagrams in AdS, to which we now turn.

3.2.2 A Witten diagrams primer

See [107] for background. We work in Euclidean AdS_{d+1}, with $R_{AdS} \equiv 1$. In Poincaré coordinates $y^\mu = \{u, x^i\}$, the metric is

$$ds^2 = \frac{du^2 + dx^i dx^i}{u^2} \ . \tag{3.18}$$

The ingredients for computing Witten diagrams are the set of bulk vertices, which are read off from a Lagrangian, and the AdS propagators for the bulk fields. A scalar field of mass $m^2 = \Delta(\Delta - d)$ in AdS_{d+1} has bulk-to-bulk propagator

$$G_{bb}(y,y';\Delta) = e^{-\Delta\sigma(y,y')} {}_2F_1\left(\Delta, \frac{d}{2}; \Delta + 1 - \frac{d}{2}; e^{-2\sigma(y,y')}\right) \tag{3.19}$$

[5] Unless otherwise noted, all sums over m, n and ℓ run from 0 to ∞ henceforth.

where $\sigma(y,y')$ is the geodesic distance between points y,y'. In Poincaré AdS,

$$\sigma(y,y') = \log\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right) \quad , \quad \xi = \frac{2uu'}{u^2+u'^2+|x-x'|^2} \ . \tag{3.20}$$

$G_{bb}(y,y';\Delta)$ is a normalizable solution of the AdS wave equation with a delta-function source,[6]

$$(\nabla^2 - m^2)G_{bb}(y,y';\Delta) = -\frac{2\pi^{d/2}\Gamma(\Delta-\frac{d-2}{2})}{\Gamma(\Delta)}\frac{1}{\sqrt{g}}\delta^{(d+1)}(y-y') \ . \tag{3.21}$$

The bulk-to-boundary propagator is

$$G_{b\partial}(y,x_i) = \left(\frac{u}{u^2+|x-x_i|^2}\right)^{\Delta} \ . \tag{3.22}$$

We will introduce higher spin propagators in due course.

A holographic CFT n-point function, which we denote $\mathcal{A}_n$, receives contributions from all possible n-point Witten diagrams. The loop-counting parameter is $G_N \sim 1/N^2$. At $O(1/N^2)$, only tree-level diagrams contribute. The simplest such diagrams are contact diagrams, which integrate over a single n-point vertex. Every local vertex in the bulk Lagrangian gives rise to a contact diagram: schematically,

$$\mathcal{L} \supset \prod_{i=1}^{n} \partial^{p_i}\phi_{\Delta_i,\ell_i} \quad \Rightarrow \quad \mathcal{A}_n^{\text{Contact}}(x_i) = \int_y \prod_{i=1}^{n} \partial^{p_i} G_{b\partial}(y,x_i) \tag{3.23}$$

where $G_{b\partial}(y,x_i)$ are bulk-to-boundary propagators for fields with quantum numbers (Δ_i,ℓ_i), and p_i count derivatives. We abbreviate

$$\int_y \equiv \int d^{d+1}y\,\sqrt{g(y)} \ . \tag{3.24}$$

There are also exchange-type diagrams, which involve "virtual" fields propagating between points in the interior of AdS. The simplest tree-level Witten diagrams are shown in figure 3.2.

We focus henceforth on tree-level four-point functions of scalar fields ϕ_i dual to scalar CFT operators $\mathcal{O}_i$. For a non-derivative interaction $\phi_1\phi_2\phi_3\phi_4$, and up to an overall quartic

[6] We use this normalization for later convenience. Our propagator is $2\pi^{d/2}\Gamma(\Delta-\frac{d-2}{2})/\Gamma(\Delta)$ times the common normalization found in, e.g., equation 6.12 of [107].

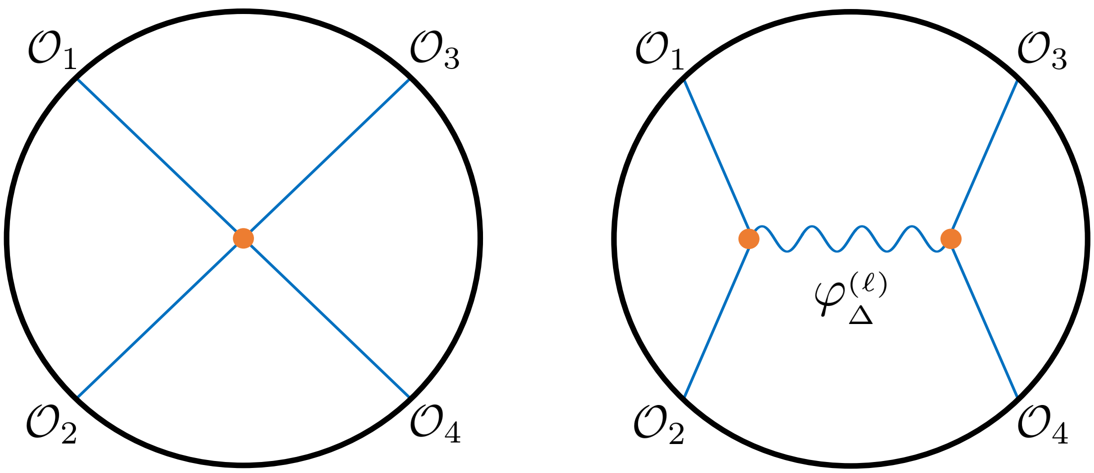

Figure 3.2: Tree-level four-point Witten diagrams for external scalar operators. On the left is a contact diagram. On the right is an exchange diagram for a symmetric traceless spin-ℓ tensor field of dual conformal dimension Δ. Here and throughout this work, orange dots denote vertices integrated over all of AdS.

coupling that we set to one, the contact diagram equals

$$\mathcal{A}_4^{\text{Contact}}(x_i) = D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \int_y G_{b\partial}(y,x_1)G_{b\partial}(y,x_2)G_{b\partial}(y,x_3)G_{b\partial}(y,x_4) \ . \tag{3.25}$$

$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i)$ is the D-function, which is defined by the above integral. For generic Δ_i, this integral cannot be performed for arbitrary x_i. There exists a bevy of identities relating various $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i)$ via permutations of the Δ_i, spatial derivatives, and/or shifts in the Δ_i [97,113]. Derivative vertices, which appear in the axio-dilaton sector of type IIB supergravity, for instance, define D-functions with shifted parameters. When $\Delta_i = 1$ for all i,

$$\frac{2x_{13}^2x_{24}^2}{\Gamma\left(2-\frac{d}{2}\right)\pi^{d/2}}D_{1111}(x_i) = \frac{1}{z-\overline{z}}\left(2\text{Li}_2(z) - 2\text{Li}_2(\overline{z}) + \log(z\overline{z})\log\frac{1-z}{1-\overline{z}}\right) \ . \tag{3.26}$$

This actually defines the D-bar function, $\overline{D}_{1111}(z,\overline{z})$. For various sets of $\Delta_i \in \mathbb{Z}$, combining (3.26) with efficient use of D-function identities leads to polylogarithmic representations of contact diagrams.

The other class of tree-level diagrams consists of exchange diagrams. For external scalars, one can consider exchanges of symmetric, traceless tensor fields of arbitrary spin ℓ. These are computed, roughly, as

$$\mathcal{A}_4^{\text{Exch}}(x_i) = \int_y \int_{y'} G_{b\partial}(y,x_1) G_{b\partial}(y,x_2) \times G_{bb}(y,y';\Delta,\ell) \times G_{b\partial}(y',x_3) G_{b\partial}(y',x_4) \ . \tag{3.27}$$

$G_{bb}(y,y';\Delta,\ell)$ is shorthand for the bulk-to-bulk propagator for the spin-ℓ field of dimension Δ, which is really a bitensor $[G_{bb}(y,y';\Delta)]_{\mu_1...\mu_\ell;\nu_1...\nu_\ell}$. We have likewise suppressed all derivatives acting on the external scalar propagators, whose indices are contracted with those of $[G_{bb}(y,y';\Delta)]_{\mu_1...\mu_\ell;\nu_1...\nu_\ell}$. Due to the double integral, brute force methods of simplifying exchange diagrams are quite challenging, even for $\ell = 0$, without employing some form of asymptotic expansion.

The key fact about these tree-level Witten diagrams relevant for a dual CFT interpretation is as follows. For contact diagrams (3.25), their decomposition into conformal blocks contains the infinite towers of double-trace operators in (3.15), and only these. This is true in any channel. For exchange diagrams (3.27), the s-channel decomposition includes a single-trace contribution from the operator dual to the exchanged bulk field, in addition to infinite towers of double-trace exchanges (3.15). In the t- and u-channels, only double-trace exchanges are present. The precise set of double-trace operators that appears is determined by the spin associated to the bulk vertices.

Higher-loop Witten diagrams are formed similarly, although the degree of difficulty increases rapidly with the loop order. No systematic method has been developed to compute these.

3.2.2.1 Logarithmic singularities and anomalous dimensions

When the external operator dimensions are non-generic, logarithms can appear in tree-level Witten diagrams [52, 94, 97]. These signify the presence of perturbatively small anomalous dimensions, of order $1/N^2$, for intermediate states appearing in the CFT correlator. Let us review some basic facts about this.

In general, if any operator of free dimension Δ_0 develops an anomalous dimension γ, so that its full dimension is $\Delta = \Delta_0 + \gamma$, a small-$\gamma$ expansion of its contribution to correlators yields an infinite series of logs:

$$G_{\Delta_0+\gamma,\ell}(u,v) \approx u^{\frac{\Delta_0-\ell}{2}} \left(1 + \frac{\gamma}{2}\log u + \dots\right) . \tag{3.28}$$

In the holographic context, the double-trace composites $[\mathcal{O}_i\mathcal{O}_j]_{n,\ell}$ have anomalous dimensions at $O(1/N^2)$. Combining (3.14), (3.15) and (3.17) leads to double-trace contributions to holographic four-point functions of the form

$$\begin{aligned} \mathcal{A}_4(x_i)\Big|_{1/N^2} \supset &\sum_{m,\ell} \left(P_1^{(12)}(m,\ell) + \frac{1}{2}P_0^{(12)}(m,\ell)\gamma_1^{(12)}(m,\ell)\partial_m\right) W_{\Delta_1+\Delta_2+2m+\ell,\ell}(x_i) \\ &+ \sum_{n,\ell} \left(P_1^{(34)}(n,\ell) + \frac{1}{2}P_0^{(34)}(n,\ell)\gamma_1^{(34)}(n,\ell)\partial_n\right) W_{\Delta_3+\Delta_4+2n+\ell,\ell}(x_i) \end{aligned} \tag{3.29}$$

where $\partial_m W_{\Delta_1+\Delta_2+2m+\ell,\ell} \propto \log u$ and likewise for the (34) terms. These logarithmic singularities should therefore be visible in tree-level Witten diagrams. In top-down examples of AdS/CFT, the supergravity fields are dual to protected operators, so the $\gamma^{(ij)}(n,\ell)$ are the only (perturbative) anomalous dimensions that appear, and hence are responsible for all logs.

For generic operator dimensions, the double-trace operators do not appear in both the $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$ OPEs at $O(N^0)$, so $P_0^{(ij)}(n,\ell) = 0$. On the other hand, when the Δ_i are related by the integrality condition $\Delta_1+\Delta_2-\Delta_3-\Delta_4 \in 2\mathbb{Z}$, one has $P_0^{(ij)}(n,\ell) \neq 0$ [52].

3.2.2.2 What has been computed?

In a foundational series of papers [93–97,114–117], methods of direct computation were developed for scalar four-point functions, in particular for scalar, vector and graviton exchanges. Much of the focus was on the axio-dilaton sector of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ in the context of duality with $\mathbb{N} = 4$ super-Yang Mills (SYM), but the methods were gradually generalized to arbitrary operator and spacetime dimensions.

This effort largely culminated in [96, 117] and [97]. [96] collected the results from all channels contributing to axio-dilaton correlators in $\mathbb{N} = 4$ SYM, yielding the full correlator at $O(1/N^2)$. In [117], a more efficient method of computation was developed for exchange diagrams. It was shown that scalar, vector and graviton exchange diagrams can generically be written as infinite sums over contact diagrams for external fields of variable dimensions. These truncate to finite sums if certain relations among the dimensions are obeyed.[7] These calculations were translated in [97] into CFT data, where it was established that logarithmic singularities appear precisely at the order determined by the analysis of the previous subsection. This laid the foundation for the modern perspective on generalized free fields. Further analysis of implications of four-point Witten diagrammatics for holographic CFTs (e.g. crossing symmetry, non-renormalization), and for $\mathbb{N} = 4$ SYM in particular, was performed in [38, 113, 118–128]. A momentum space-based approach can be found in [129, 130].

More recent work has computed Witten diagrams for higher spin exchanges [131, 132]. These works develop the split representation of massive spin-ℓ symmetric traceless tensor fields, for arbitrary integer ℓ. There is a considerable jump in technical difficulty, but the results are all consistent with AdS/CFT.

3.2.3 Mellin space

An elegant alternative approach to computing correlators, especially holographic ones, has been developed in Mellin space [100, 133]. The analytic structure of Mellin amplitudes neatly encodes the CFT data and follows a close analogy with the momentum space representation of flat space scattering amplitudes. We will not make further use of Mellin space in this paper, but it should be included in any discussion on Witten diagrams; we only briefly review its main properties with respect to holographic four-point functions, and further aspects and details may be found in e.g. [21, 46, 101–104, 134–136].

Given a four-point function as in (3.2), its Mellin representation may be defined by the integral transform

$$\begin{aligned} g(u,v) = \int_{-i\infty}^{i\infty} ds\, dt\, M(s,t) u^{t/2} v^{-(s+t)/2} \Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right) \\ \Gamma\left(\frac{\Delta_{34}-s}{2}\right)\Gamma\left(\frac{-\Delta_{12}-s}{2}\right)\Gamma\left(\frac{s+t}{2}\right)\Gamma\left(\frac{s+t+\Delta_{12}-\Delta_{34}}{2}\right)\ . \end{aligned} \tag{3.30}$$

[7] For instance, an s-channel scalar exchange is written as a finite linear combination of D-functions if $\Delta_1 + \Delta_2 - \Delta$ is a positive even integer [117].

The integration runs parallel to the imaginary axis and to one side of all poles of the integrand. The Mellin amplitude is $M(s,t)$. Assuming it formally exists, $M(s,t)$ can be defined for any correlator, holographic [100] or otherwise [137, 138]. $M(s,t)$ is believed to be meromorphic in any compact CFT. Written as a sum over poles in t, each pole sits at a fixed twist $\tau = \Delta - \ell$, capturing the exchange of twist-τ operators in the intermediate channel. Given rhe exchange of a primary $\mathcal{O}$ of twist $\tau_{\mathcal{O}}$, its descendants of twist $\tau = \tau_{\mathcal{O}} + 2m$ contribute a pole

$$M(s,t) \supset C_{12\mathcal{O}} C^{\mathcal{O}}{}_{34} \frac{\mathcal{Q}_{\ell,m}(s)}{t - \tau_{\mathcal{O}} - 2m} \tag{3.31}$$

where $m = 0, 1, 2, \ldots$. $\mathcal{Q}_{\ell,m}(s)$ is a certain degree-ℓ (Mack) polynomial that can be found in [103]. Note that an infinite number of descendants contributes at a given m. n-point Mellin amplitudes may be likewise defined in terms of $n(n-3)/2$ parameters, and are known to factorize onto lower-point amplitudes [104].

Specifying now to holographic correlators at tree-level,[8] the convention of including explicit Gamma functions in (3.30) has particular appeal: their poles encode the double-trace exchanges of $[\mathcal{O}_1\mathcal{O}_2]_{m,\ell}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,\ell}$. Poles in $M(s,t)$ only capture the single-trace exchanges, if any, associated with a Witten diagram. In particular, all local AdS interactions give rise to contact diagrams whose Mellin amplitudes are mere polynomials in the Mellin variables. In this language, the counting of solutions to crossing symmetry in sparse large N CFTs performed in [15] becomes manifestly identical on both sides of the duality. Exchange Witten diagrams have meromorphic Mellin amplitudes that capture the lone single-trace exchange: they take the form[9]

$$M(s,t) = C_{12\mathcal{O}} C^{\mathcal{O}}{}_{34} \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{\ell,m}(s)}{t - \tau_{\mathcal{O}} - 2m} + \text{Pol}(s,t) \ . \tag{3.32}$$

[8]This is the setting that is known to be especially amenable to a Mellin treatment. Like other approaches to Witten diagrams, the Mellin program has not been systematically extended to loop level (except for certain classes of diagrams; see Section 3.6). Because higher-trace operators appear at higher orders in $1/N$, some of the elegance of the tree-level story is likely to disappear. The addition of arbitrary external spin in a manner which retains the original simplicity has also not been done, although see [101].

[9]For certain non-generic operator dimensions, the sum over poles actually truncates [46, 100]. The precise mechanism for this is not fully understood from a CFT perspective. We thank Liam Fitzpatrick and Joao Penedones for discussions on this topic.

Pol(s,t) stands for a possible polynomial in s,t. The polynomial boundedness is a signature of local AdS dynamics [103,110]. Anomalous dimensions appear when poles of the integrand collide to make double poles.

A considerable amount of work has led to a quantitative understanding of the above picture. These include formulas for extraction of the one-loop OPE coefficients $P_1^{(ij)}(n,\ell)$ and anomalous dimensions $\gamma_1^{(ij)}(n,\ell)$ from a given Mellin amplitude (Section 2.3 of [21]); and the graviton exchange amplitude between pairwise identical operators in arbitrary spacetime dimension (Section 6 of [131]).

3.2.4 Looking ahead

Having reviewed much of what has been accomplished, let us highlight some of what has not.

First, we note that no approach to computing holographic correlators has systematically deconstructed loop diagrams, nor have arbitrary external spins been efficiently incorporated. Save for some concrete proposals in Section 3.6, we will not address these issues here.

While Mellin space is home to a fruitful approach to studying holographic CFTs in particular, it comes with a fair amount of technical complication. Nor does it answer the natural question of how to represent a single conformal block in the bulk. One is, in any case, left to wonder whether a truly efficient approach exists in position space.

Examining the position space computations reviewed in subsection 3.2.2, one is led to wonder: where are the conformal blocks? In particular, the extraction of dual CFT spectral data and OPE coefficients in the many works cited earlier utilized a double OPE expansion. More recent computations of exchange diagrams [131, 132] using the split representation do make the conformal block decomposition manifest, in a contour integral form [139]: integration runs over the imaginary axis in the space of complexified conformal dimensions, and the residues of poles in the integrand contain the OPE data. This is closely related to the shadow formalism. However, this approach is technically quite involved, does not apply to contact diagrams, and does not answer the question of what bulk object computes a single conformal block.

Let us turn to this latter question now, as a segue to our computations of Witten diagrams.

3.3 The holographic dual of a scalar conformal block

What is the holographic dual of a conformal block? This is to say, what is the geometric representation of a conformal block in AdS? In this section we answer this question for the case of scalar exchanges between scalar operators, for generic operator and spacetime dimensions. In Section 3.5, we will tackle higher spin exchanges. At this stage, these operators need not belong to a holographic CFT, since the form of a conformal block is fixed solely by symmetry. What follows may seem an inspired guess, but as we show in the next section, it emerges very naturally as an ingredient in the computation of Witten diagrams.

Let us state the main result. We want to compute the scalar conformal partial wave $W_{\Delta,0}(x_i)$, defined in (3.7), corresponding to exchange of an operator $\mathcal{O}$ of dimension Δ between two pairs of external operators $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3, \mathcal{O}_4$. Let us think of the external operators as sitting on the boundary of AdS_{d+1} at positions $x_{1,2,3,4}$, respectively. Denote the geodesic running between two boundary points x_i and x_j as γ_{ij}. Now consider the scalar *geodesic Witten diagram*, which we denote $\mathcal{W}_{\Delta,0}(x_i)$, first introduced in Section 3.1 and drawn in Figure 4.2:

$$\mathcal{W}_{\Delta,0}(x_i) \equiv \int_{\gamma_{12}} \int_{\gamma_{34}} G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) \times G_{bb}(y(\lambda), y(\lambda'); \Delta) \times G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4) \ , \tag{3.33}$$

where

$$\int_{\gamma_{12}} \equiv \int_{-\infty}^{\infty} d\lambda \ , \quad \int_{\gamma_{34}} \equiv \int_{-\infty}^{\infty} d\lambda' \tag{3.34}$$

denote integration over proper time coordinates λ and λ' along the respective geodesics.

Then $\mathcal{W}_{\Delta,0}(x_i)$ is related to the conformal partial wave $W_{\Delta,0}(x_i)$ by

$$\mathcal{W}_{\Delta,0}(x_i) = \beta_{\Delta 12}\beta_{\Delta 34}W_{\Delta,0}(x_i) \ . \tag{3.35}$$

The proportionality constant $\beta_{\Delta 34}$ is defined in equation (3.12) and $\beta_{\Delta 12}$ is defined analogously.

The object $\mathcal{W}_{\Delta,0}(x_i)$ looks quite like the expression for a scalar exchange Witten diagram for the bulk field dual to $\mathcal{O}$. Indeed, the form is identical, except that the bulk vertices are not integrated over all of AdS, but rather over the geodesics connecting the pairs of boundary points. This explains our nomenclature. Looking ahead to conformal partial waves for exchanged operators with spin, it is useful to think of the bulk-to-bulk propagator in (3.33) as pulled back to the two geodesics.

Equation (3.35) is a rigorous equality. We now prove it in two ways.

3.3.1 Proof by direct computation

Consider the piece of (3.33) that depends on the geodesic γ_{12}, which we denote φ^{12}_Δ:

$$\varphi^{12}_\Delta(y(\lambda')) \equiv \int_{\gamma_{12}} G_{b\partial}(y(\lambda), x_1)G_{b\partial}(y(\lambda), x_2)G_{bb}(y(\lambda), y(\lambda'); \Delta) \ . \tag{3.36}$$

In terms of $\varphi^{12}_\Delta(y(\lambda'))$, the formula for $\mathcal{W}_{\Delta,0}$ becomes

$$\mathcal{W}_{\Delta,0}(x_i) = \int_{\gamma_{34}} \varphi^{12}_\Delta(y(\lambda'))G_{b\partial}(y(\lambda'), x_3)G_{b\partial}(y(\lambda'), x_4) \ . \tag{3.37}$$

It is useful to think of $\varphi^{12}_\Delta(y)$, for general y, as a cubic vertex along γ_{12} between a bulk field at y and two boundary fields anchored at x_1 and x_2. We may then solve for $\varphi^{12}_\Delta(y)$ as a normalizable solution of the Klein-Gordon equation with a source concentrated on γ_{12}. The symmetries of the problem turn out to specify this function uniquely, up to a multiplicative constant. We then pull this back to γ_{34}, which reduces $\mathcal{W}_{\Delta,0}(x_i)$ to a single one-dimensional integral along γ_{34}. This integral can then be compared to the well-known integral representation for $G_{\Delta,0}(u, v)$ in (3.11), which establishes (3.35).

To make life simpler, we will use conformal symmetry to compute the geodesic Witten diagram with operators at the following positions:

$$\begin{aligned}\mathcal{W}_{\Delta,0}(u,v) &\equiv \frac{1}{C_{12\mathcal{O}}C^{\mathcal{O}}{}_{34}}\,\langle \mathcal{O}_1(\infty)\mathcal{O}_2(0)\,P_{\Delta,0}\,\mathcal{O}_3(1-z)\mathcal{O}_4(1)\rangle \\ &= u^{\frac{-\Delta_3-\Delta_4}{2}}G_{\Delta,0}(u,v)\end{aligned} \tag{3.38}$$

where, as usual, $\mathcal{O}_1(\infty) \equiv \lim_{x_1\to\infty} x_1^{2\Delta_1}\mathcal{O}(x_1)$. We implement the above strategy by solving the wave equation first in global AdS, then moving to Poincaré coordinates and comparing with CFT. We work with the global AdS metric

$$ds^2 = \frac{1}{\cos^2\rho}(d\rho^2 + dt^2 + \sin^2\rho d\Omega_{d-1}^2)\ . \tag{3.39}$$

The relation between mass and conformal dimension is

$$m^2 = \Delta(\Delta - d) \tag{3.40}$$

and so the wave equation for $\varphi_\Delta^{12}(y)$ away from γ_{12} is

$$\big(\nabla^2 - \Delta(\Delta-d)\big)\varphi_\Delta^{12}(y) = 0\ , \tag{3.41}$$

or

$$\left(\cos^2\rho\partial_\rho^2 + (d-1)\cot\rho\partial_\rho + \cos^2\rho\partial_t^2 - \Delta(\Delta-d)\right)\varphi_\Delta^{12}(y) = 0\ . \tag{3.42}$$

At γ_{12}, there is a source. To compute (3.38), we take $t_1 \to -\infty, t_2 \to \infty$, in which limit the geodesic becomes a line at $\rho = 0$, the center of AdS. This simplifies matters because this source is rotationally symmetric. Its time-dependence is found by evaluating the product of bulk-to-boundary propagators on a fixed spatial slice,

$$G_{b\partial}(t,t_1)G_{b\partial}(t,t_2) \propto e^{-\Delta_{12}t}\ . \tag{3.43}$$

Therefore, we are looking for a rotationally-symmetric, normalizable solution to the following radial equation:

$$\left(\cos^2\rho\partial_\rho^2 + (d-1)\cot\rho\partial_\rho + \cos^2\rho\Delta_{12}^2 - \Delta(\Delta-d)\right)\varphi_\Delta^{12} = 0\ . \tag{3.44}$$

The full solution, including the time-dependence and with the normalization fixed by equation (3.36), is

$$\varphi^{12}_{\Delta}(\rho,t) = \beta_{\Delta 12} \times e^{\Delta_1 t_1 - \Delta_2 t_2} {}_2F_1\Big(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \cos^2\rho\Big) \cos^{\Delta}\rho \; e^{-\Delta_{12} t} \ . \tag{3.45}$$

Now we need to transform this to Poincaré coordinates, and pull it back to γ_{34}. The relation between coordinates is

$$e^{-2t} = u^2 + |x|^2 \ , \quad \cos^2\rho = \frac{u^2}{u^2+|x|^2} \ . \tag{3.46}$$

Although the field φ^{12}_{Δ} is a scalar, it transforms nontrivially under the map from global to Poincaré AdS because the bulk-to-boundary propagators in its definition (3.36) transform as

$$G_{b\partial}(t_1, y) = |x_1|^{\Delta_1} G_{b\partial}(y, x_1) \ , \quad G_{b\partial}(t_2, y) = |x_2|^{\Delta_2} G_{b\partial}(y, x_2) \ . \tag{3.47}$$

After stripping off the power of x_1 needed to define the operator at infinity, the field in Poincaré coordinates is

$$\varphi^{12}_{\Delta}(u, x^i) = \beta_{\Delta 12} \times {}_2F_1\Big(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \cos^2\rho\Big) \cos^{\Delta}\rho \; e^{-\Delta_{12} t} \tag{3.48}$$

where on the right hand side ρ and t are to be viewed as functions of the Poincaré coordinates u, x^i via (3.46).

Now we want to evaluate our Poincaré coordinates on the geodesic γ_{34}. With our choice of positions in (3.38), γ_{34} is a geodesic in an AdS_3 slice through AdS_{d+1}. Geodesics in Poincaré AdS_3 are semi-circles: for general boundary points z_3, z_4,

$$2u^2 + (z - z_4)(\overline{z} - \overline{z}_3) + (\overline{z} - \overline{z}_4)(z - z_3) = 0 \ . \tag{3.49}$$

In terms of the proper length parameter λ',

$$\begin{aligned} z(\lambda') &= \frac{z_3+z_4}{2} + \frac{z_3-z_4}{2}\tanh\lambda' \ , \\ \overline{z}(\lambda') &= \frac{\overline{z}_3+\overline{z}_4}{2} + \frac{\overline{z}_3-\overline{z}_4}{2}\tanh\lambda' \ , \\ u(\lambda') &= \frac{|z_3-z_4|}{2\cosh\lambda'} \ . \end{aligned} \tag{3.50}$$

Plugging in $z_3 = 1 - z, \overline{z}_3 = 1 - \overline{z}, z_4 = \overline{z}_4 = 1$ into (3.50) and then (3.46), we find

$$\begin{aligned} \cos^2\rho\Big|_{\gamma_{34}} &= \frac{1}{2\cosh\lambda'}\frac{|z|^2}{e^{-\lambda'}+|1-z|^2 e^{\lambda'}} \ , \\ e^{-2t}\Big|_{\gamma_{34}} &= \frac{e^{-\lambda'}+|1-z|^2 e^{\lambda'}}{2\cosh\lambda'} \ . \end{aligned} \tag{3.51}$$

Therefore, the pullback of φ^{12}_{Δ} to γ_{34} is

$$\begin{aligned} \varphi^{12}_{\Delta}(u(\lambda'), x^i(\lambda')) = \beta_{\Delta 12}(2\cosh\lambda')^{\frac{-\Delta_{12}-\Delta}{2}}(e^{-\lambda'}+|1-z|^2 e^{\lambda'})^{\frac{\Delta_{12}-\Delta}{2}}|z|^{\Delta} \\ \times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \frac{1}{2\cosh\lambda'}\frac{|z|^2}{e^{-\lambda'}+|1-z|^2 e^{\lambda'}}\right) \ . \end{aligned} \tag{3.52}$$

Finally, we need to evaluate $G_{b\partial}(y(\lambda'), 1-z)G_{b\partial}(y(\lambda'), 1)$ in (3.37). Plugging (3.50) into the propagator (3.22), we get

$$G_{b\partial}(y(\lambda'), 1-z)G_{b\partial}(y(\lambda'), 1) = \frac{e^{\Delta_{34}\lambda'}}{|z|^{\Delta_3+\Delta_4}} \ . \tag{3.53}$$

We may now assemble all pieces of (3.37)–(3.38): plugging (3.52) and (3.53) into (3.37) gives the geodesic Witten diagram. Trading $z, \overline{z}$ for u, v we find, in the parameterization (3.38),

$$\begin{aligned} \mathcal{W}_{\Delta,0}(u,v) = \beta_{\Delta 12}\, u^{\frac{\Delta-\Delta_3-\Delta_4}{2}} \int_{-\infty}^{\infty} d\lambda' (2\cosh\lambda')^{\frac{-\Delta_{12}-\Delta}{2}}(e^{-\lambda'}+v e^{\lambda'})^{\frac{\Delta_{12}-\Delta}{2}} e^{\Delta_{34}\lambda'} \\ \times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \frac{1}{2\cosh\lambda'}\frac{u}{e^{-\lambda'}+v e^{\lambda'}}\right) \ . \end{aligned} \tag{3.54}$$

Defining a new integration variable,

$$\sigma = \frac{e^{2\lambda'}}{1+e^{2\lambda'}} \ , \tag{3.55}$$

we have

$$\begin{aligned} \mathcal{W}_{\Delta,0}(u,v) = \frac{\beta_{\Delta 12}}{2}\, u^{\frac{\Delta-\Delta_3-\Delta_4}{2}} \int_0^1 d\sigma\, \sigma^{\frac{\Delta+\Delta_{34}-2}{2}}(1-\sigma)^{\frac{\Delta-\Delta_{34}-2}{2}}(1-(1-v)\sigma)^{\frac{-\Delta+\Delta_{12}}{2}} \\ \times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma}\right) \ . \end{aligned} \tag{3.56}$$

Comparing (3.56) to the integral representation (3.11), one recovers (3.35) evaluated at the appropriate values of $x_1, ..., x_4$. Validity of the relation (3.35) for general operator positions is then assured by the identical conformal transformation properties of the two sides.

3.3.2 Proof by conformal Casimir equation

In the previous section we evaluated a geodesic Witten diagram $\mathcal{W}_{\Delta,0}$ and matched the result to a known integral expression for the corresponding conformal partial wave $W_{\Delta,0}$, thereby showing that $\mathcal{W}_{\Delta,0} = W_{\Delta,0}$ up to a multiplicative constant. In this section we give a direct argument for the equivalence, starting from the definition of conformal partial waves.

Section 3.3.2.1 reviews the definition of conformal partial waves $W_{\Delta,\ell}$ as eigenfunctions of the conformal Casimir operator. We prove this definition to be satisfied by geodesic Witten diagrams $\mathcal{W}_{\Delta,0}$ in section 3.3.2.3, after a small detour (section 3.3.2.2) to define the basic embedding space language used in the proof.

The arguments of the present section are generalized in section 3.5.5 to show that $\mathcal{W}_{\Delta,\ell} = W_{\Delta,\ell}$ (again, up to a factor) for arbitrary exchanged spin ℓ.

3.3.2.1 The Casimir equation

The generators of the d-dimensional conformal group $\mathrm{SO}(d+1,1)$ can be taken to be the Lorentz generators L_{AB} of $d+2$ dimensional Minkowski space (with L_{AB} antisymmetric in A and B as usual). The quadratic combination $L^2 \equiv \frac{1}{2}L_{AB}L^{AB}$ is a Casimir of the algebra, i.e. it commutes with all the generators L_{AB}. As a result, L^2 takes a constant value on any irreducible representation of the conformal group, which means all states $|P^n\mathcal{O}\rangle$ in the conformal family of a primary state $|\mathcal{O}\rangle$ are eigenstates of L^2 with the same eigenvalue. The eigenvalue depends on the dimension Δ and spin ℓ of $|\mathcal{O}\rangle$, and can be shown to be [38]

$$C_2(\Delta,\ell) = -\Delta(\Delta-d) - \ell(\ell+d-2) \ . \tag{3.57}$$

The $\mathrm{SO}(d+1,1)$ generators are represented on conformal fields by

$$[L_{AB}, \mathcal{O}_1(x_1)] = L^1_{AB}\mathcal{O}(x_1) \ , \tag{3.58}$$

where L^1_{AB} is a differential operator built out of the position x_1 of $\mathcal{O}_1$ and derivatives with respect to that position. The form of the L^1_{AB} depends on the conformal quantum numbers of $\mathcal{O}_1$. Equation (3.58) together with conformal invariance of the vacuum imply the following identity, which holds for any state $|\alpha\rangle$:

$$(L^1_{AB} + L^2_{AB})^2 \langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\alpha\rangle = \langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)L^2|\alpha\rangle \ . \tag{3.59}$$

Consistent with the notation for L^2, we have defined

$$(L^1_{AB} + L^2_{AB})^2 \equiv \tfrac{1}{2}(L^1_{AB} + L^2_{AB})(L^{1\,AB} + L^{2\,AB}) \ . \tag{3.60}$$

As discussed in section 3.2.1, one obtains a conformal partial wave $W_{\Delta,\ell}$ by inserting into a four-point function the projection operator $P_{\Delta,\ell}$ onto the conformal family of a primary $\mathcal{O}$ with quantum numbers Δ, ℓ:

$$W_{\Delta,\ell}(x_i) = \frac{1}{C_{12\mathcal{O}}C^{\mathcal{O}}{}_{34}} \sum_n \langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|P^n\mathcal{O}\rangle\langle P^n\mathcal{O}|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|0\rangle \ . \tag{3.61}$$

Applying the identity (3.59) to the equation above and recalling that each state $|P^n\mathcal{O}\rangle$ is an eigenstate of L^2 with the same eigenvalue $C_2(\Delta,\ell)$, we arrive at the Casimir equation

$$(L^1_{AB} + L^2_{AB})^2 W_{\Delta,\ell}(x_i) = C_2(\Delta,\ell)W_{\Delta,\ell}(x_i) \ . \tag{3.62}$$

One can take this second-order differential equation, plus the corresponding one with $1,2 \leftrightarrow 3,4$, supplemented with appropriate boundary conditions, as one's definition of $W_{\Delta,\ell}$ [140]. Regarding boundary conditions, it is sufficient to require that $W_{\Delta,\ell}$ have the correct leading behavior in the $x_2 \to x_1$ and $x_4 \to x_3$ limits. The correct behavior in both limits is dictated by the fact that the contribution to $W_{\Delta,\ell}$ of the primary $\mathcal{O}$ dominates that of its descendants since those enter the OPE with higher powers of x_{12} and x_{34}.

We will prove that geodesic Witten diagrams $\mathcal{W}_{\Delta,0}$ are indeed proportional to conformal partial waves $W_{\Delta,0}$ by showing that $\mathcal{W}_{\Delta,0}$ satisfies the Casimir equation (3.62) and has the correct behavior in the $x_2 \to x_1$ and $x_4 \to x_3$ limits. The proof is very transparent in the embedding space formalism, which we proceed now to introduce.

3.3.2.2 Embedding space

The embedding space formalism has been reviewed in e.g. [35,131,140]. The idea is to embed the d-dimensional CFT and the $d+1$ dimensional AdS on which lives the geodesic Witten diagram both into $d+2$ dimensional Minkowski space. We give this embedding space the metric

$$ds^2 = -(dY^{-1})^2 + (dY^0)^2 + \sum_{i=1}^{d}(dY^i)^2 \ . \tag{3.63}$$

The CFT will live on the projective null cone of embedding space, which is the Lorentz-invariant d-dimensional space defined as the set of nonzero null vectors X with scalar multiples identified: $X \equiv aX$. We will use null vectors X to represent points in the projective null cone with the understanding that X and aX signify the same point. The plane $\mathbb{R}^d$ can be mapped into the projective null cone via

$$X^+(x) = a|x|^2, \quad X^-(x) = a, \quad X^i(x) = ax^i \tag{3.64}$$

where we have introduced light cone coordinates $X^\pm = X^{-1} \pm X^0$. Of course, any nonzero choice of the parameter a defines the same map.

Conformal transformations on the plane are implemented by Lorentz transformations in embedding space. As a specific example, we may consider a boost in the 0 direction with rapidity λ. This leaves the X^i coordinates unchanged, and transforms $X^\pm$ according to

$$X^+ \to e^\lambda X^+, \quad X^- \to e^{-\lambda} X^- \ . \tag{3.65}$$

A point $X(x) = (|x|^2, 1, x^i)$ gets mapped to $(e^\lambda |x^2|, e^{-\lambda}, x^i)$ which is projectively equivalent to $X(e^\lambda x^i)$. Thus boosts in the 0 direction of embedding space induce dilatations in the plane.

Any field $\hat{\mathcal{O}}$ on the null cone defines a field $\mathcal{O}$ on the plane via restriction: $\mathcal{O}(x) \equiv \hat{\mathcal{O}}(X(x))$. Since $\hat{\mathcal{O}}$ is a scalar field in embedding space, the SO$(d+1,1)$ generators act on it as

$$[L_{AB}, \hat{\mathcal{O}}(X)] = (X_A \partial_B - X_B \partial_A)\hat{\mathcal{O}}(X) \ . \tag{3.66}$$

The induced transformation law for $\mathcal{O}$ is the correct one for a primary of dimension Δ if and only if $\hat{\mathcal{O}}$ satisfies the homogeneity condition

$$\hat{\mathcal{O}}(aX) = a^{-\Delta}\hat{\mathcal{O}}(X) \ . \tag{3.67}$$

Thus in the embedding space formalism a primary scalar field $\mathcal{O}(x)$ of dimension Δ is represented by a field $\hat{\mathcal{O}}(X)$ satisfying (3.67). Below, we drop the hats on embedding space fields. It should be clear from a field's argument whether it lives on the null cone (as $\mathcal{O}(X)$) or on the plane (as $\mathcal{O}(x)$). Capital letters will always denote points in embedding space.

Meanwhile, AdS_{d+1} admits an embedding into $d+2$ dimensional Minkowski space, as the hyperboloid $Y^2 = -1$. Poincare coordinates (u, x^i) can be defined on AdS via

$$Y^+ = \frac{u^2 + |x|^2}{u}, \quad Y^- = \frac{1}{u}, \quad Y^i = \frac{x^i}{u} \ . \tag{3.68}$$

The induced metric for these coordinates is the standard one, (3.18).

The AdS hyperboloid sits inside the null cone and asymptotes toward it. As one takes $u \to 0$, the image of a point (u, x^i) in AdS approaches $(Y^+, Y^-, Y^i) = u^{-1}(|x^2|, 1, x^i)$ which is projectively equivalent to $X(x^i)$. In this way, the image on the projective null cone of the point $x^i \in \mathbb{R}^d$ marks the limit $u \to 0$ of the embedding space image of a bulk point (u, x^i).

Isometries of AdS are implemented by embedding space Lorentz transformations, and so are generated by

$$L_{AB} = Y_A \partial_B - Y_B \partial_A \ . \tag{3.69}$$

The Casimir operator $L^2 = \frac{1}{2}(Y_A\partial_B - Y_B\partial_A)(Y^A\partial^B - Y^B\partial^A)$ is interior to the AdS slice $Y \cdot Y = -1$. That is, for Y belonging to the AdS slice, $L^2 f(Y)$ depends only on the values of f on the slice. In fact, applied to scalar functions on AdS the operator L^2 is simply the negative of the Laplacian of AdS:

$$L^2 f(Y) = -\nabla_Y^2 f(Y) \tag{3.70}$$

as long as Y is on the AdS slice. This fact, which is not surprising given that L^2 is a second-order differential operator invariant under all the isometries of AdS, can be checked directly from (3.69).

3.3.2.3 Geodesic Witten diagrams satisfy the Casimir equation

The geodesic Witten diagram $\mathcal{W}_{\Delta,0}(x_i)$ lifts to a function $\mathcal{W}_{\Delta,0}(X_i)$ on the null cone of embedding space via a lift of each of the four bulk-to-boundary propagators with the appropriate homogeneity condition

$$G_{b\partial}(y, aX_i) = a^{-\Delta_i} G_{b\partial}(y, X_i), \quad i = 1, 2, 3, 4 \ . \tag{3.71}$$

The geodesics in AdS connecting the boundary points X_1 to X_2 and X_3 to X_4 lift to curves in embedding space which can be parameterized by

$$\begin{aligned} Y(\lambda) &= \frac{e^{-\lambda} X_1 + e^{\lambda} X_2}{\sqrt{-2X_1 \cdot X_2}} \\ Y(\lambda') &= \frac{e^{-\lambda'} X_3 + e^{\lambda'} X_4}{\sqrt{-2X_3 \cdot X_4}} \end{aligned} \tag{3.72}$$

The geodesic Witten diagram is

$$\mathcal{W}_{\Delta,0}(X_i) = \int_{\gamma_{34}} F(X_1, X_2, Y(\lambda'); \Delta) G_{b\partial}(Y(\lambda'), X_3) G_{b\partial}(Y(\lambda'), X_4) \tag{3.73}$$

where we have isolated the part that depends on X_1, X_2:[10]

$$F(X_1, X_2, Y'; \Delta) = \int_{\gamma_{12}} G_{b\partial}(Y(\lambda), X_1) G_{b\partial}(Y(\lambda), X_2) G_{bb}(Y(\lambda), Y'; \Delta) \ . \tag{3.74}$$

$F(X_1, X_2, Y'; \Delta)$ is the lift to embedding space of $\varphi_{\Delta}^{12}(y)$ defined in (3.36). The bulk arguments of the bulk-to-boundary propagators have been promoted from points y in the bulk to points Y in embedding space. Although the propagators are defined only on the AdS slice, there is no ambiguity because $Y(\lambda)$ and $Y'(\lambda')$ always lie in the AdS slice.

The function $F(X_1, X_2, Y'; \Delta)$ is manifestly invariant under simultaneous SO$(d+1, 1)$ rotations of X_1, X_2, Y', and therefore it is annihilated by $(L^1 + L^2 + L^{Y'})_{AB}$. This means

$$(L^1_{AB} + L^2_{AB}) F(X_1, X_2, Y'; \Delta) = -L^{Y'}_{AB} F(X_1, X_2, Y'; \Delta) \tag{3.75}$$

and so (since of course L^1_{AB} commutes with $L^{Y'}_{AB}$)

$$(L^1_{AB} + L^2_{AB})^2 F(X_1, X_2, Y'; \Delta) = (L^{Y'})^2 F(X_1, X_2, Y'; \Delta) \ . \tag{3.76}$$

Recall that $(L^{Y'})^2$ is $-\nabla^2_{Y'}$. The function $F(X_1, X_2, Y'; \Delta)$, which depends on Y' via the bulk-to-bulk propagator $G_{bb}(Y(\lambda), Y'; \Delta)$, is an eigenfunction of $-\nabla^2_{Y'}$ with eigenvalue $-\Delta(\Delta - d)$.

[10]The fact that the bulk-to-bulk propagator satisfies the Laplace equation was used to similar effect in [117]. In particular, [117] defines a quantity $A(y', x_1, x_2)$ that is similar to $F(X_1, X_2, Y'; \Delta)$, except that the vertex is integrated over all of AdS instead of along a geodesic.

Thus we conclude that $F(X_1, X_2, Y'; \Delta)$ is an eigenfunction of $(L^1_{AB} + L^2_{AB})^2$ with eigenvalue $C_2(\Delta, 0)$, and therefore that

$$(L^1_{AB} + L^2_{AB})^2 \mathcal{W}_{\Delta,0}(X_i) = C_2(\Delta, 0)\mathcal{W}_{\Delta,0}(X_i) \ . \qquad (3.77)$$

Note that agreement does not hinge on what the actual eigenvalue is: it is guaranteed by the fact that the bulk-to-bulk propagator and the conformal partial wave furnish the same highest weight representation of $\mathrm{SO}(d+1, 1)$.

Furthermore, the behavior in the limit $x_2 \to x_1$ of the bulk-to-boundary and bulk-to-bulk propagators guarantees the geodesic Witten diagram to have the power-law behavior $\mathcal{W}_{\Delta,0}(x_i) \to (\text{constant}) \times |x_{12}|^{\Delta - \Delta_1 - \Delta_2}$ in that limit, and similarly in the $x_4 \to x_3$ limit. This proves $\mathcal{W}_{\Delta,0}$ is equal to the conformal partial wave $W_{\Delta,0}$ up to a constant factor.

Looking back at the proof, we can see why the bilocal function integrated between the geodesics had to be precisely the bulk-to-bulk propagator $G_{bb}(y, y'; \Delta)$. To get (3.77) we needed that function to be the appropriate eigenfunction of the Laplacian, and to get the correct limiting behavior we needed it to be the eigenfunction with normalizable boundary conditions at infinity. It also crucial that the vertices be integrated over geodesics rather than arbitrary curves or over all of AdS. A non-geodesic curve would introduce extra data to specify the curve, which would not be conformally invariant. Integrating the vertices over all of AdS (which would give the full Witten diagram) allows y and y' to collide, but the bulk-to-bulk propagator acted on by the wave operator picks up a source contribution when $y = y'$, hence the diagram would not be an eigenfunction of the Casimir operator in this case; indeed we know that it is a sum of eigenfunctions with different eigenvalues.

3.3.3 Comments

We close this section with a few comments.

3.3.3.1 Geodesic versus ordinary Witten diagrams

A natural question is *why*, intuitively, a relation like (3.35) is true. Let us offer two motivational remarks.

The first is that there are two ways to integrate a bulk point while preserving conformal invariance. One is over all of AdS, which defines a Witten diagram, while the other is over a geodesic. The latter is clearly over a smaller range, which makes it seem at least plausible that it represents a conformal partial wave rather than a full correlator. Indeed, the only obvious conformally invariant objects that appear in four-point functions are the correlator itself, and the conformal partial waves.

The second is a heuristic "derivation" starting from the exchange Witten diagram, $\mathcal{A}_4^{\rm Exch}$. Consider taking the following limit of heavy external operators,

$$\Delta_{1,2,3,4} \to \infty \ , \quad \Delta_{12}, \Delta_{34} \text{ fixed} \ . \tag{3.78}$$

As reviewed in Section 3.2 and computed in the next section, the full diagram equals a single trace exchange of $\mathcal{O}$, plus infinite towers of double trace exchanges of $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$. On the CFT side, the double-trace exchanges are exponentially smaller in this limit than that of the single-trace exchange, simply because the conformal partial waves decay exponentially as the internal operator dimension tends to infinity. So the Witten diagram reduces to the single-trace block in the limit. On the bulk side, the heavy limit restricts the cubic vertices to lie on geodesics, so $\mathcal{A}_4^{\rm Exch}$ reduces to $\mathcal{W}_{\Delta,0}$, the geodesic Witten diagram. This establishes equality between $\mathcal{W}_{\Delta,0}$ and $W_{\Delta,0}$ in the limit (3.78). To complete the argument we need to use the fact that the conformal block $G_{\Delta,0}$ only depends on Δ_i through Δ_{12} and Δ_{34}, as can be seen from the recursion relations in [90]. Furthermore, $G_{\Delta,0}$ and $W_{\Delta,0}$ only differ by a prefactor which has exponents linear in Δ_i (a form which is invariant as $\Delta_i \to \infty$); see (3.7). Using these two facts, it follows that if $\mathcal{W}_{\Delta,0}$ and $W_{\Delta,0}$ agree in the regime (3.78), then they agree for all values of Δ_i and Δ.

Note that the geodesic restriction ensures that a cut down the middle of the diagram crosses only the internal line, representing the CFT primary; contrast this with the exchange

Witten diagram, where integration over all of AdS ensures that the cut will sometimes cross two external lines, representing the (infinite towers of) double-trace operators.

3.3.3.2 Simplification of propagators and blocks

In even d, CFT$_d$ scalar conformal blocks can be resummed into hypergeometric functions. An apparently unrelated simplification occurs for AdS$_{d+1}$ scalar bulk-to-bulk propagators, which are rational functions of $S \equiv e^{-2\sigma(y,y')}$ rather than hypergeometric. From (3.19), the even d propagators are, at low d,

$$\begin{aligned}
d = 2: \quad & G_{bb}(y, y'; \Delta) = S^{\Delta/2} \frac{1}{1-S} \\
d = 4: \quad & G_{bb}(y, y'; \Delta) = S^{\Delta/2} \frac{(3-\Delta)S + (\Delta - 1)}{(\Delta-1)(1-S)^3} \\
d = 6: \quad & G_{bb}(y, y'; \Delta) = S^{\Delta/2} \frac{(5-\Delta)(\Delta-4)S^2 + 2(\Delta-5)(\Delta-1)S - (\Delta-2)(\Delta-1)}{(\Delta-2)(\Delta-1)(S-1)^5} \ .
\end{aligned} \tag{3.79}$$

The geodesic representation of the scalar conformal blocks reveals that these simplifications have a common origin. Conversely, the lack of simplification of the propagator in odd d gives a new perspective on why generic odd d conformal blocks cannot be reduced to special functions.

3.3.3.3 Relation to Mellin space

It is worth noting that the spin-ℓ conformal block has a Mellin representation with exponential dependence on the Mellin parameter: up to normalization [102],

$$G_{\Delta,\ell}(s,t) = e^{\pi i(\frac{d}{2}-\Delta)} \left(e^{\pi i(t+\Delta-d)} - 1\right) \frac{\Gamma\left(\frac{\Delta-\ell-t}{2}\right)\Gamma\left(\frac{2d-\Delta-\ell-t}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right)} P_{\Delta,\ell}(s,t) \tag{3.80}$$

where $P_{\Delta,\ell}(s,t)$ is a degree-ℓ Mack polynomial. (In the scalar case, $\ell = 0$.) It has been argued that for holographic CFTs with a gap, the Mellin amplitudes for their correlators are polynomially bounded at large s, t. It is interesting that despite its exponential growth at large t, the Mellin representation of a conformal block does have a semiclassical AdS description.

In [110], it was argued that starting with (3.80), one recovers the Mellin amplitude for the *full* spin-ℓ exchange Witten diagram by writing it as a sum over its poles and dropping all other contributions.[11] Evidently, this is the Mellin transform, so to speak, of the liberation of bulk vertices from the geodesics to all of AdS.

3.4 The conformal block decomposition of scalar Witten diagrams

We begin our treatment with the technically simplest case: tree-level four-point functions in AdS involving only scalar fields. All of the key steps will be visible in the decomposition of the four-point contact diagram, out of which the geometric representation of the scalar conformal block will naturally emerge. We then move on to the exchange diagram and, in the next section, to fields with spin.

3.4.1 An AdS propagator identity

The main technical tool that we will employ is an identity obeyed by AdS bulk-to-boundary propagators. Consider two scalar fields dual to gauge-invariant scalar operators $\mathcal{O}_1, \mathcal{O}_2$ of conformal dimensions Δ_1, Δ_2, respectively. Now consider a product of their bulk-to-boundary propagators, from points x_1 and x_2 on the boundary to the same point y in the bulk. Then the following identity holds:

$$G_{b\partial}(y, x_1)G_{b\partial}(y, x_2) = \sum_{m=0}^{\infty} a_m^{12}\, \varphi_{\Delta_m}^{12}(y) \tag{3.81}$$

where $\varphi_{\Delta_m}^{12}(y)$ is the field solution defined in (3.36). The bulk-to-bulk propagator $G_{bb}(y(\lambda), y; \Delta_m)$, running from the geodesic to the original bulk point y, is for a scalar field with mass $m_m^2 = \Delta_m(\Delta_m - d)$, where

$$\Delta_m = \Delta_1 + \Delta_2 + 2m\ . \tag{3.82}$$

[11] This is true up to polynomial contributions from contact diagrams.

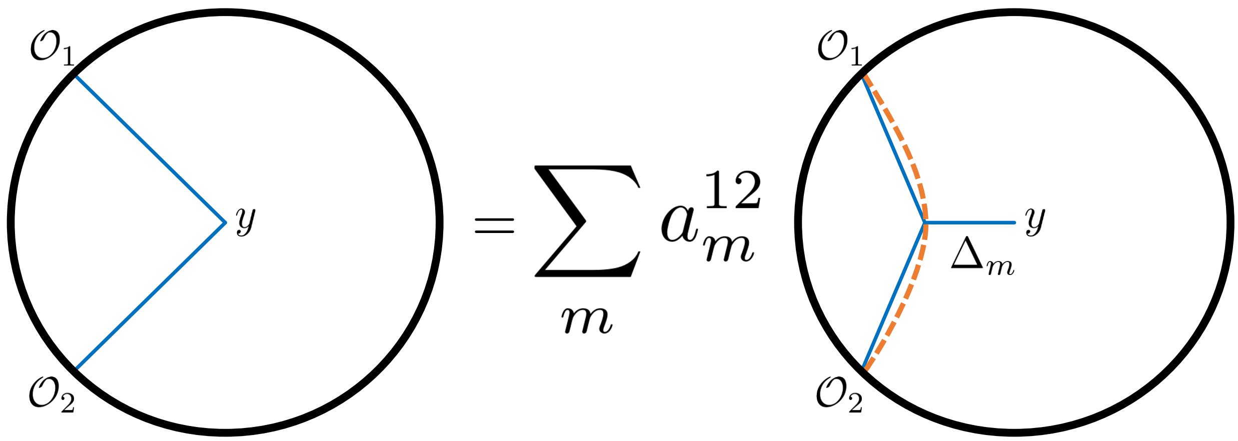

Figure 3.3: The identity (3.81) obeyed by AdS scalar propagators. The internal line represents bulk-to-bulk propagator for a scalar field of mass $m^2 = \Delta_m(\Delta_m - d)$. a_m^{12} and Δ_m are defined in (3.82) and (3.83), respectively.

The a_m^{12} are coefficient functions of Δ_1, Δ_2 and d:

$$a_m^{12} = \frac{1}{\beta_{\Delta_m 12}} \frac{(-1)^m}{m!} \frac{(\Delta_1)_m (\Delta_2)_m}{\left(\Delta_1 + \Delta_2 + m - \frac{d}{2}\right)_m} \ . \tag{3.83}$$

This identity is depicted in Figure 3.3.

In words, the original bilinear is equal to an infinite sum of three-point vertices integrated over the geodesic γ_{12}, for fields of varying masses $m_m^2 = \Delta_m(\Delta_m - d)$. To prove this, we work in global AdS with $t_1 \to -\infty, t_2 \to +\infty$, whereupon γ_{12} becomes a worldline at $\rho = 0$. We already solved for $\varphi^{12}_{\Delta_m}(y)$ in (3.45). Plugging that solution into (3.81), we must solve

$$(\cos\rho)^{\Delta_1+\Delta_2} = \sum_{m=0}^{\infty} a_m^{12} \beta_{\Delta_m 12} (\cos\rho)^{\Delta_m} {}_2F_1\left(\frac{\Delta_m + \Delta_{12}}{2}, \frac{\Delta_m - \Delta_{12}}{2}; \Delta_m - \frac{d-2}{2}; \cos^2\rho\right) \ . \tag{3.84}$$

Expanding as a power series in $\cos^2\rho$, the unique solution is given by Δ_m in (3.82) and a_m^{12} in (3.83).

The identity (3.81) is suggestive of a bulk operator product expansion, where the propagation of two boundary fields to the same bulk point is replaced by an infinite sum over field solutions. Note that the dimensions Δ_m are those of the scalar double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ at leading order in $1/N$. As we now show, this fact ensures that the decomposition of a given Witten diagram involving $G_{b\partial}(y,x_1)G_{b\partial}(y,x_2)$ includes the exchange of $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$, consistent with the generalized free field content of the dual CFT.

3.4.2 Four-point contact diagram

We want to compute the four-point scalar contact diagram (3.25), for all operator dimensions Δ_i generic. We reproduce the integral here:

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \int_y G_{b\partial}(y,x_1)\, G_{b\partial}(y,x_2)\, G_{b\partial}(y,x_3)\, G_{b\partial}(y,x_4)\ . \tag{3.85}$$

A helpful pictorial representation of the following calculation is given in Figure 3.4.

Using our geodesic toolkit, the evaluation of this diagram is essentially trivial. First, we use the identity (3.81) on the pairs (12) and (34). This yields

$$\begin{aligned} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \sum_{m,n} a_m^{12} a_n^{34} \int_{\gamma_{12}}\int_{\gamma_{34}} & G_{b\partial}(y(\lambda),x_1)\, G_{b\partial}(y(\lambda),x_2) \\ & \times \int_y G_{bb}(y(\lambda),y;\Delta_m)\, G_{bb}(y,y(\lambda');\Delta_n) \\ & \times G_{b\partial}(y(\lambda'),x_3)\, G_{b\partial}(y(\lambda'),x_4)\ . \end{aligned} \tag{3.86}$$

Next, we use

$$G_{bb}(y,y';\Delta) = \left\langle y \left| \frac{1}{\nabla^2 - m^2} \right| y' \right\rangle \tag{3.87}$$

to represent the product of bulk-to-bulk propagators integrated over the common bulk point y as

$$\int_y G_{bb}(y(\lambda),y;\Delta_m)\, G_{bb}(y,y(\lambda');\Delta_n) = \frac{G_{bb}(y(\lambda),y(\lambda');\Delta_m) - G_{bb}(y(\lambda),y(\lambda');\Delta_n)}{m_m^2 - m_n^2} \tag{3.88}$$

where we used completeness, $\int_y |y\rangle\langle y| = 1$. The integrated product is thus replaced by a difference of unintegrated propagators from γ_{12} to γ_{34}. This leaves us with

$$\begin{aligned} & D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \sum_{m,n} \frac{a_m^{12} a_n^{34}}{m_m^2 - m_n^2} \times \\ & \Bigg(\int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}(y(\lambda),x_1)G_{b\partial}(y(\lambda),x_2) \times G_{bb}(y(\lambda),y(\lambda');\Delta_m) \times\ G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4) \\ & - \int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}(y(\lambda),x_1)G_{b\partial}(y(\lambda),x_2) \times G_{bb}(y(\lambda),y(\lambda');\Delta_n) \times\ G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4) \Bigg)\ . \end{aligned} \tag{3.89}$$

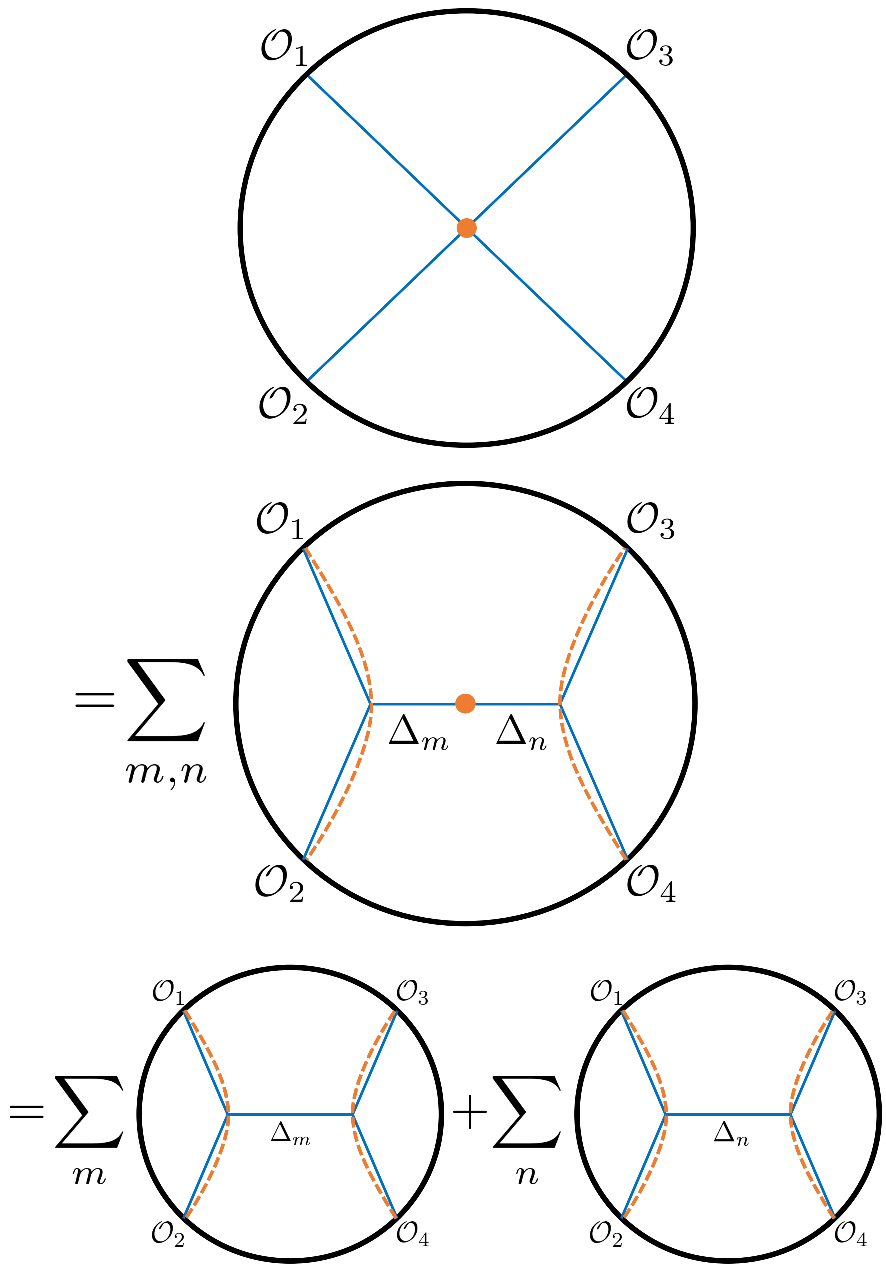

Figure 3.4: The decomposition of a four-point scalar contact diagram into conformal partial waves disguised as geodesic Witten diagrams. Passage to the second line uses (3.86), and passage to the last line uses (3.88). The last line captures the infinite set of CFT exchanges of the double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$. We have suppressed OPE coefficients; the exact result is in equation (3.90).

But from (3.33), we now recognize the last two lines as conformal partial waves! Thus, we have

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \sum_{m,n} \frac{a_m^{12} a_n^{34}}{m_m^2 - m_n^2} \left(\mathcal{W}_{\Delta_m,0}(x_i) - \mathcal{W}_{\Delta_n,0}(x_i)\right) \ . \tag{3.90}$$

This is the final result. In the CFT notation of section 3.2, we write this as a pair of single sums over double-trace conformal partial waves,

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \sum_m P_1^{(12)}(m,0)\, W_{\Delta_m,0}(x_i) + \sum_n P_1^{(34)}(n,0)\, W_{\Delta_n,0}(x_i) \tag{3.91}$$

with squared OPE coefficients

$$\begin{aligned} P_1^{(12)}(m,0) &= \left(\beta_{\Delta_m 12}\, a_m^{12}\right) \left(\beta_{\Delta_m 34} \sum_n \frac{a_n^{34}}{m_m^2 - m_n^2}\right) \\ P_1^{(34)}(n,0) &= \left(\beta_{\Delta_n 34}\, a_n^{34}\right) \left(\beta_{\Delta_n 12} \sum_m \frac{a_m^{12}}{m_n^2 - m_m^2}\right) \end{aligned} \tag{3.92}$$

where $m^2 = \Delta(\Delta - d)$ as always. The structure of the answer is manifestly consistent with CFT expectations: only double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$ are exchanged.

We will analyze this result more closely after computing the exchange diagram.

3.4.3 Four-point exchange diagram

Turning to the scalar exchange diagram, we reap the real benefits of this approach: unlike an approach based on brute force integration, this case is no harder than the contact diagram. A pictorial representation of the final result is given in Figure 3.5.

We take all external dimensions Δ_i, and the internal dimension Δ, to be generic. The diagram is computed as

$$\mathcal{A}_4^{\text{Exch}}(x_i) = \int_y \int_{y'} G_{b\partial}(y,x_1) G_{b\partial}(y,x_2) \times G_{bb}(y,y';\Delta) \times G_{b\partial}(y',x_3) G_{b\partial}(y',x_4) \ . \tag{3.93}$$

Expanding in the s-channel (12)-(34), the algorithm is the same as the contact case.

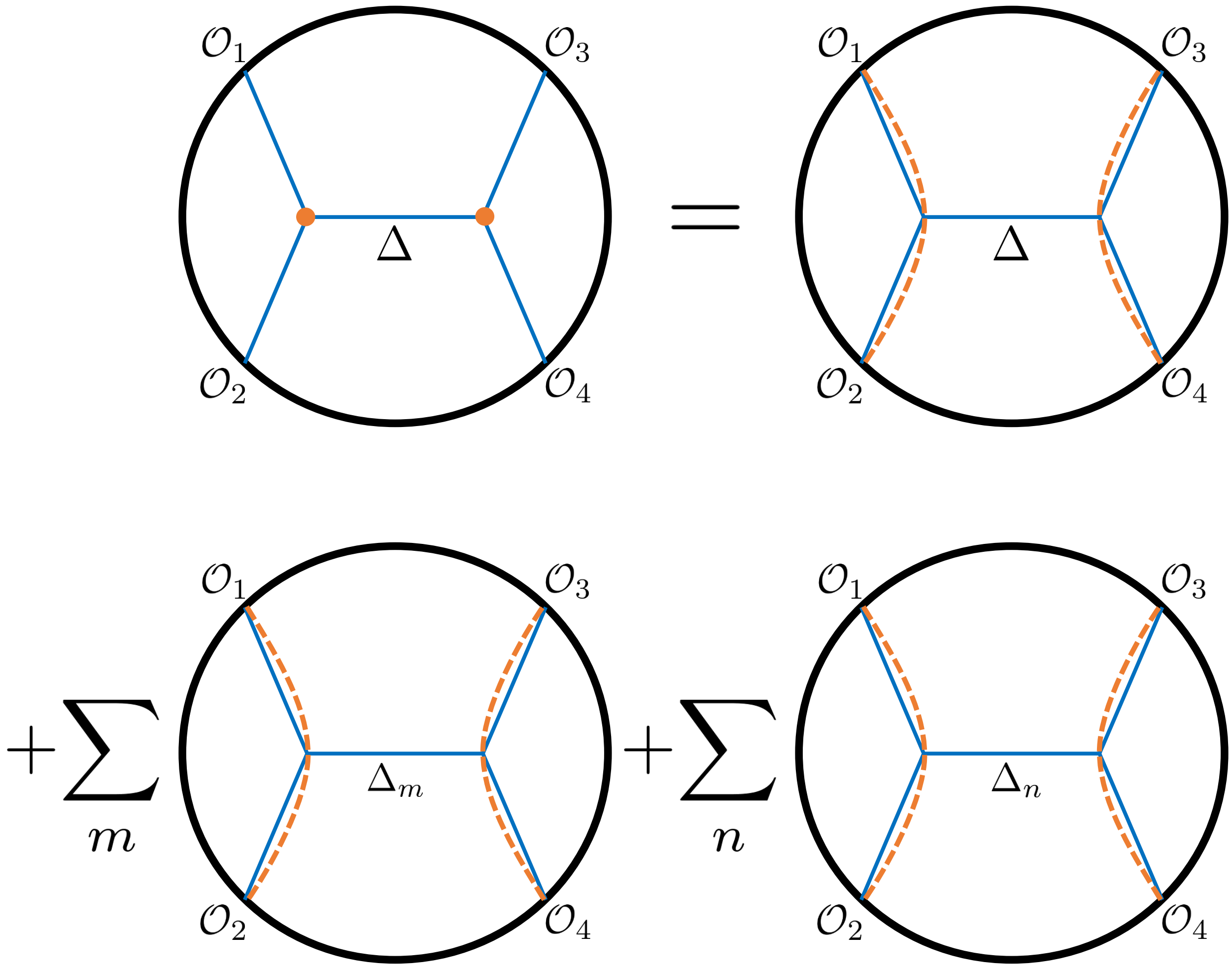

Figure 3.5: The decomposition of a four-point scalar exchange diagram (upper left) into conformal partial waves, for an exchanged scalar ϕ of mass $m^2 = \Delta(\Delta - d)$. We have skipped the intermediate steps, which are nearly identical to those of the contact diagram. The term in the upper right captures the single-trace exchange of the scalar operator dual to ϕ. The second line captures the infinite set of CFT exchanges of the double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$. We have suppressed OPE coefficients; the exact result is in equations (3.96)–(3.97).

First, use (3.81) twice to get

$$\begin{aligned}\mathcal{A}_4^{\text{Exch}}(x_i) = \sum_{m,n} a_m^{12} a_n^{34} \int_{\gamma_{12}} \int_{\gamma_{34}} & G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) \\ & \times \int_y \int_{y'} G_{bb}(y(\lambda), y; \Delta_m) G_{bb}(y, y'; \Delta) G_{bb}(y', y(\lambda'); \Delta_n) \\ & \times G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4) \ .\end{aligned} \tag{3.94}$$

This is of the same form as the contact diagram, only we have three bulk-to-bulk propagators and two integrations. We again use (3.87) to turn the second line into a sum over terms with a single bulk-to-bulk propagator:

$$
\begin{aligned}
&\int_y \int_{y'} G_{bb}(y(\lambda), y; \Delta_m) G_{bb}(y, y'; \Delta) G_{bb}(y', y(\lambda'); \Delta_n) \\
&= \frac{G_{bb}(y(\lambda), y(\lambda'); \Delta_m)}{(m_m^2 - m_\Delta^2)(m_m^2 - m_n^2)} + \frac{G_{bb}(y(\lambda), y(\lambda'); \Delta)}{(m_\Delta^2 - m_m^2)(m_\Delta^2 - m_n^2)} + \frac{G_{bb}(y(\lambda), y(\lambda'); \Delta_n)}{(m_n^2 - m_m^2)(m_n^2 - m_\Delta^2)} \, .
\end{aligned}
\tag{3.95}
$$

Recognizing the remaining integrals as conformal partial waves, we reach our final result:

$$
\mathcal{A}_4^{\text{Exch}}(x_i) = C_{12\Delta} C^{\Delta}{}_{34} W_{\Delta,0}(x_i) + \sum_m P_1^{(12)}(m,0)\, W_{\Delta_m,0}(x_i) + \sum_n P_1^{(34)}(n,0)\, W_{\Delta_n,0}(x_i)
\tag{3.96}
$$

where

$$
\begin{aligned}
C_{12\Delta} C^{\Delta}{}_{34} &= \left(\beta_{\Delta 12} \sum_m \frac{a_m^{12}}{m_\Delta^2 - m_m^2} \right) \left(\beta_{\Delta 34} \sum_n \frac{a_n^{34}}{m_\Delta^2 - m_n^2} \right) \\
P_1^{(12)}(m,0) &= \left(\beta_{\Delta_m 12} \frac{a_m^{12}}{m_m^2 - m_\Delta^2} \right) \left(\beta_{\Delta_m 34} \sum_n \frac{a_n^{34}}{m_m^2 - m_n^2} \right) \\
P_1^{(34)}(n,0) &= \left(\beta_{\Delta_n 34} \frac{a_n^{34}}{m_n^2 - m_\Delta^2} \right) \left(\beta_{\Delta_n 12} \sum_m \frac{a_m^{12}}{m_n^2 - m_m^2} \right) \, .
\end{aligned}
\tag{3.97}
$$

Its structure is precisely as required by AdS/CFT: in addition to the double-trace exchanges of $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$, there is a single-trace exchange of the operator dual to the exchanged field in the bulk of dimension Δ.

Comparing (3.97) to (3.92), we can immediately read off a new identity relating the double-trace OPE coefficients of the contact and exchange diagrams:

$$
\frac{P_1^{(12)}(m,0)\big|_{\text{Contact}}}{P_1^{(12)}(m,0)\big|_{\text{Exch}}} = m_m^2 - m_\Delta^2
\tag{3.98}
$$

and likewise for $P_1^{(34)}(n,0)$. This is quite simple. One can quickly check this against the $d=4$ example in Appendix B of [5].

3.4.4 Further analysis

3.4.4.1 OPE factorization

Notice that the squared OPE coefficients in (3.97) and (3.92) factorize naturally into terms associated with the (12) and (34) channels. To emphasize this, it is useful to define[12]

$$\alpha_s^{34} \equiv \sum_n \frac{a_n^{34}}{m_s^2 - m_n^2} \tag{3.99}$$

for some mass squared $m_s^2 = \Delta_s(\Delta_s - d)$, and similarly for α_s^{12}. This allows us to write the Witten diagrams in a tidy form as

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \sum_m a_m^{12}\alpha_m^{34}\,\mathcal{W}_{\Delta_m,0}(x_i) + \sum_n \alpha_n^{12}a_n^{34}\,\mathcal{W}_{\Delta_n,0}(x_i) \tag{3.100}$$

and

$$\mathcal{A}_4^{\text{Exch}}(x_i) = \alpha_\Delta^{12}\alpha_\Delta^{34}\,\mathcal{W}_{\Delta,0}(x_i) + \sum_m \frac{a_m^{12}\alpha_m^{34}}{m_m^2 - m_\Delta^2}\,\mathcal{W}_{\Delta_m,0}(x_i) + \sum_n \frac{\alpha_n^{12}a_n^{34}}{m_n^2 - m_\Delta^2}\,\mathcal{W}_{\Delta_n,0}(x_i)\ . \tag{3.101}$$

For compactness in the above equations we have used $\mathcal{W}_{\Delta,0}$ in place of $W_{\Delta,0}$. Recall that $\mathcal{W}_{\Delta,0}$ is a rescaling of the standard conformal partial wave, $\mathcal{W}_{\Delta,0}(x_i) = \beta_{\Delta 12}\beta_{\Delta 34}W_{\Delta,0}(x_i)$. The coefficient relating $\mathcal{W}_{\Delta,0}$ to $W_{\Delta,0}$ clearly factorizes.

Writing the OPE coefficients in terms of the coefficients a_m^{12}, a_n^{34} and masses m_m, m_n, m_Δ makes their origin transparent. But the sum defining α_s^{34} can actually be performed, yielding

$$\alpha_s^{34} = \frac{\Gamma(\Delta_3+\Delta_4)}{\Gamma(\Delta_3)\Gamma(\Delta_4)}\left(F(\Delta_s,\Delta_3,\Delta_4) + F(d-\Delta_s,\Delta_3,\Delta_4)\right) \tag{3.102}$$

where

$$\begin{aligned} F(\Delta_s,\Delta_3,\Delta_4) \equiv\ & \frac{1}{\left(\Delta_s - \frac{d}{2}\right)(\Delta_s - \Delta_3 - \Delta_4)} \\ & \times {}_4F_3\left(\begin{matrix} \frac{\Delta_3+\Delta_4}{2}, \frac{\Delta_3+\Delta_4+1}{2}, \frac{\Delta_3+\Delta_4-\Delta_s}{2}, \Delta_3+\Delta_4-\frac{d}{2} \\ \frac{\Delta_3+\Delta_4}{2}-\frac{d}{4}, \frac{\Delta_3+\Delta_4}{2}-\frac{d-2}{4}, \frac{\Delta_3+\Delta_4-\Delta_s+2}{2} \end{matrix}\ \middle|\ -1\right)\ . \end{aligned} \tag{3.103}$$

[12] We observe a likeness between α_s^{34} and calculations in [42] of $\ell = 0$ double-trace anomalous dimensions due to heavy operator exchange; see Section 4.3 therein. It is not immediately clear to us whether there is a deeper statement to be made.

3.4.4.2 Recovering logarithmic singularities

Recall from Section 3.2 that when the external operator dimensions obey $\Delta_1+\Delta_2-\Delta_3-\Delta_4 \in 2\mathbb{Z}$, logarithms appear in tree-level Witten diagrams due to anomalous dimensions of double-trace operators. In brute force calculation of the AdS integrals, these logarithms are extracted by isolating the relevant integration range. In Mellin space, they appear as double poles in the Mellin amplitude.

In the present approach, these logarithms fall out trivially as algebraic conditions. Considering the scalar four-point contact diagram written in the form (3.90), for instance, we see that terms for which $m_m^2 = m_n^2$ give rise to derivatives of conformal blocks, and hence to logarithms. This is equivalent to the condition $\Delta_m = \Delta_n$ or $\Delta_m = d - \Delta_n$. Since $d \in \mathbb{Z}$, both of these are equivalent to $\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 \in 2\mathbb{Z}$, which is precisely the integrality condition stated above. Identical structure is visible in (3.95): logarithms will appear when any of m_m^2, m_n^2, m_Δ^2 coincide.

As an explicit example, let us consider $D_{\Delta\Delta\Delta\Delta}(x_i)$. Then (3.91) can be split into $m \neq n$ and $m = n$ terms, the latter of which yield logarithms:

$$D_{\Delta\Delta\Delta\Delta}(x_i) = \sum_{n=0}^{\infty} 2a_n^{\Delta\Delta} \left(\sum_{m\neq n} \frac{a_m^{\Delta\Delta}}{m_n^2 - m_m^2} \right) \mathcal{W}_{2\Delta+2n,0}(x_i) + \left(\frac{(a_n^{\Delta\Delta})^2}{\partial_n m_n^2} \right) \partial_n \mathcal{W}_{2\Delta+2n,0}(x_i) \ . \tag{3.104}$$

This takes the form of the $\ell = 0$ terms in (3.29), with

$$P_1(n,0) = 2\beta^2_{(2\Delta+2n)\,\Delta\Delta} a_n^{\Delta\Delta} \left(\sum_{m\neq n} \frac{a_m^{\Delta\Delta}}{m_n^2 - m_m^2} \right) + \frac{(a_n^{\Delta\Delta})^2}{\partial_n m_n^2} \partial_n \left(\beta^2_{(2\Delta+2n)\,\Delta\Delta} \right) \tag{3.105}$$

and

$$\frac{1}{2} P_0(n,0)\gamma_1(n,0) = \frac{(a_n^{\Delta\Delta})^2}{\partial_n m_n^2} \beta^2_{(2\Delta+2n)\,\Delta\Delta} \ . \tag{3.106}$$

As an aside, we note the conjecture of [15], proven in [21], that

$$P_1(n,\ell) = \frac{1}{2} \partial_n (P_0(n,\ell)\gamma_1(n,\ell)) \ . \tag{3.107}$$

We have checked in several examples that this is obeyed by (3.105)–(3.106). It would be interesting to prove it using generalized hypergeometric identities.

3.4.5 Taking stock

We close this section with some perspective. Whereas traditional methods of computing Witten diagrams are technically involved and require explicit bulk integration [95] and/or solution of differential equations [117], the present method skips these steps with a minimum of technical complexity. It is remarkable that for neither the contact nor exchange diagrams have we performed any integration: the integrals have instead been absorbed into sums over, and definitions of, conformal partial waves.

For the contact diagram/D-function, we have presented an efficient algorithm for its decomposition into spin-0 conformal blocks in position space. Specific cases of such decompositions have appeared in previous works [5, 15], although no systematic treatment had been given. Moreover, perhaps the main virtue of our approach is that exchange diagrams are no more difficult to evaluate than contact diagrams.

D-functions also appear elsewhere in CFT, including in weak coupling perturbation theory. For example, the four-point function of the $\mathbf{20'}$ operator in planar $\mathbb{N} = 4$ SYM at weak coupling is given, at order λ, by [141]

$$\langle \mathcal{O}_{\mathbf{20'}}(x_1)\mathcal{O}_{\mathbf{20'}}(x_2)\mathcal{O}_{\mathbf{20'}}(x_3)\mathcal{O}_{\mathbf{20'}}(x_4)\rangle\big|_{\lambda} \propto \overline{D}_{1111}(z,\bar{z}) \tag{3.108}$$

where $\overline{D}_{1111}(z,\bar{z})$ was defined in (3.26). The ubiquity of D-functions at weak coupling may be related to constraints of crossing symmetry in the neighborhood of free fixed points [137].

3.5 Spinning exchanges and conformal blocks

The OPE of two scalar primary operators yields not just other scalar primaries but also primaries transforming in symmetric traceless tensor representations of the Lorentz group. We refer to such a rank-ℓ tensor as a spin-ℓ operator. Thus, for the full conformal block decomposition of a correlator of scalar primaries we need to include blocks describing spin-ℓ exchange. The expression for such blocks as geodesic Witten diagrams turns out to be the natural extension of the scalar exchange case. The exchanged operator is now described by a massive spin-ℓ field in the bulk, which couples via its pullback to the geodesics connecting the external operator insertion points. This was drawn in Figure 4.2.

In this section we do the following. We give a fairly complete account of the spin-1 case, showing how to decompose a Witten diagram involving the exchange of a massive vector field, and establishing that the geodesic diagrams reproduce known results for spin-1 conformal blocks. We also give an explicit treatment of the spin-2 geodesic diagram, again checking that we reproduce known results for the spin-2 conformal blocks. More generally, we use the conformal Casimir equation to prove that our construction yields the correct blocks for arbitrary ℓ.

3.5.1 Known results

Conformal blocks with external scalars and internal spin-ℓ operators were studied in the early work of Ferrara et. al. [30]. They obtained expressions for these blocks as double integrals. It is easy to verify that their form for the scalar exchange block precisely coincides with our geodesic Witten diagram expression (3.33). We thus recognize the double integrals as integrals over pairs of geodesics. Based on this, we expect agreement for general ℓ, although we have not so far succeeded in showing this due to the somewhat complicated form for the general spin-ℓ bulk-to-bulk propagator [131, 132]. Some more discussion is in section 3.5.6. We will instead use other arguments to establish the validity of our results.

Dolan and Osborn [39] studied these blocks using the conformal Casimir equation. Closed-form expressions in terms of hypergeometric functions were obtained in dimensions $d = 2, 4, 6$. For example, in $d = 2$ we have

$$\begin{aligned} G_{\Delta,\ell}(z,\bar{z}) = |z|^{\Delta-\ell}\times & \\ & \Big[z^{\ell}{}_2F_1\left(\frac{\Delta-\Delta_{12}+\ell}{2},\frac{\Delta+\Delta_{34}+\ell}{2},\Delta+\ell;z\right) \\ & \times {}_2F_1\left(\frac{\Delta-\Delta_{12}-\ell}{2},\frac{\Delta+\Delta_{34}-\ell}{2},\Delta-\ell;\bar{z}\right)+(z\leftrightarrow\bar{z})\Big] \end{aligned} \tag{3.109}$$

and in $d=4$ we have

$$G_{\Delta,\ell}(z,\overline{z}) = |z|^{\Delta-\ell}\frac{1}{z-\overline{z}}\times \Big[z^{\ell+1}{}_2F_1\left(\frac{\Delta-\Delta_{12}+\ell}{2},\frac{\Delta+\Delta_{34}+\ell}{2};\Delta+\ell;z\right) \times {}_2F_1\left(\frac{\Delta-\Delta_{12}-\ell}{2}-1,\frac{\Delta+\Delta_{34}-\ell}{2}-1;\Delta-\ell-2;\overline{z}\right)-(z\leftrightarrow\overline{z})\Big] \tag{3.110}$$

The $d=6$ result is also available, taking the same general form, but it is more complicated. Note that the $d=2$ result is actually a sum of two irreducible blocks, chosen so as to be even under parity. The irreducible $d=2$ blocks factorize holomorphically, since the global conformal algebra splits up as sl(2,$\mathbb{R}$) $\oplus$ sl(2,$\mathbb{R}$). An intriguing fact is that the $d=4$ block is expressed as a sum of two terms, each of which "almost" factorizes holomorphically. Results in arbitrary dimension are available in series form.

Since the results of Dolan and Osborn are obtained as solutions of the conformal Casimir equation, and we will show that our geodesic Witten diagrams are solutions of the same equation with the same boundary conditions, this will constitute exact agreement. Note, though, that the geodesic approach produces the solution in an integral representation. It is not obvious by inspection that these results agree with those in [39], but we will verify this in various cases to assuage any doubts that our general arguments are valid. As noted above, in principle a more direct comparison is to the formulas of Ferrara et. al. [30].

3.5.2 Geodesic Witten diagrams with spin-ℓ exchange: generalities

Consider a CFT$_d$ primary operator which carries scaling dimension Δ and transforms in the rank-ℓ symmetric traceless tensor representation of the (Euclidean) Lorentz group. The AdS$_{d+1}$ bulk dual to such an operator is a symmetric traceless tensor field $h_{\mu_1\ldots\mu_\ell}$ obeying the field equations

$$\begin{aligned}&\nabla^2 h_{\mu_1\ldots\mu_\ell} - [\Delta(\Delta-d)-\ell]h_{\mu_1\ldots\mu_\ell} = 0\ ,\\ &\nabla^{\mu_1}h_{\mu_1\ldots\mu_\ell} = 0\ .\end{aligned} \tag{3.111}$$

Our proposal is that the conformal partial wave $W_{\Delta,\ell}(x_i)$ is given by the same expression as in (3.33) except that now the bulk-to-bulk propagator is that of the spin-ℓ field pulled back to the geodesics. The latter defines the spin-ℓ version of the geodesic Witten diagram, $\mathcal{W}_{\Delta,\ell}(x_i)$: its precise definition is

$$\mathcal{W}_{\Delta,\ell}(x_i) \equiv \int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}(y(\lambda),x_1)G_{b\partial}(y(\lambda),x_2) \times G_{bb}(y(\lambda),y(\lambda');\Delta,\ell) \times G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4) \tag{3.112}$$

and $G_{bb}(y(\lambda),y(\lambda');\Delta,\ell)$ is the pulled-back spin-ℓ propagator,

$$G_{bb}(y(\lambda),y(\lambda');\Delta,\ell) \equiv [G_{bb}(y,y';\Delta)]_{\mu_1\ldots\mu_\ell,\nu_1\ldots\nu_\ell}\frac{dy^{\mu_1}}{d\lambda}\cdots\frac{dy^{\mu_\ell}}{d\lambda}\frac{dy'^{\nu_1}}{d\lambda'}\cdots\frac{dy'^{\nu_\ell}}{d\lambda'}\bigg|_{y=y(\lambda),\,y'=y(\lambda')} . \tag{3.113}$$

To explicitly evaluate this we will use the same technique as in section 3.3.1. Namely, the integration over one geodesic can be expressed as a normalizable spin-ℓ solution of the equations (3.111) with a geodesic source. Inserting this result, we obtain an expression for the geodesic Witten diagram as an integral over the remaining geodesic. If we call the above normalizable solution $h_{\nu_1\ldots\nu_\ell}$, then the analog of (3.37) is

$$\mathcal{W}_{\Delta,\ell}(x_i) = \int_{\gamma_{34}} h_{\nu_1\ldots\nu_\ell}(y(\lambda'))\frac{dy'^{\nu_1}}{d\lambda'}\cdots\frac{dy'^{\nu_\ell}}{d\lambda'}G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4) . \tag{3.114}$$

As in section 3.3.1, we will specifically compute

$$\begin{aligned}\mathcal{W}_{\Delta,\ell}(z,\bar{z}) &\equiv \frac{1}{C_{12\mathcal{O}}C^{\mathcal{O}}{}_{34}}\,\langle\mathcal{O}_1(\infty)\mathcal{O}_2(0)\,P_{\Delta,\ell}\,\mathcal{O}_3(1-z)\mathcal{O}_4(1)\rangle \\ &= |z|^{-\Delta_3-\Delta_4}G_{\Delta,\ell}(z,\bar{z})\end{aligned} \tag{3.115}$$

now written in terms of $(z,\bar{z})$ instead of (u,v) to facilitate easier comparison with (3.109) and (3.110). We recall that this reduces γ_{12} to a straight line at the origin of global AdS. The form of γ_{34} is given in (3.50), from which the pullback is computed using

$$\begin{aligned}\cos^2\rho\big|_{\gamma_{34}} &= \frac{1}{2\cosh\lambda'}\frac{|z|^2}{e^{-\lambda'}+|1-z|^2e^{\lambda'}} , \\ e^{2t}\big|_{\gamma_{34}} &= \frac{2\cosh\lambda'}{e^{-\lambda'}+|1-z|^2e^{\lambda'}} , \\ e^{2i\phi}\big|_{\gamma_{34}} &= \frac{(1-z)e^{\lambda'}+e^{-\lambda'}}{(1-\bar{z})e^{\lambda'}+e^{-\lambda'}} .\end{aligned} \tag{3.116}$$

We also recall

$$G_{b\partial}(y(\lambda'), 1-z)G_{b\partial}(y(\lambda'), 1) = \frac{e^{\Delta_{34}\lambda'}}{|z|^{\Delta_3+\Delta_4}} \ . \tag{3.117}$$

Carrying out this procedure for all dimensions d at once presents no particular complications. However, it does not seem easy to find the solution $h_{\nu_1\ldots\nu_\ell}$ for all ℓ at once. For this reason, below we just consider the two simplest cases of $\ell = 1, 2$, which suffice for illustrating the general procedure.

3.5.3 Evaluation of geodesic Witten diagram: spin-1

In the global AdS$_{d+1}$ metric

$$ds^2 = \frac{1}{\cos^2\rho}(d\rho^2 + dt^2 + \sin^2\rho d\Omega_{d-1}^2) \tag{3.118}$$

we seek a normalizable solution of

$$\nabla^2 A_\mu - [\Delta(\Delta-d)-1]A_\mu = 0 \ , \quad \nabla^\mu A_\mu = 0 \tag{3.119}$$

which is spherically symmetric and has time dependence $e^{-\Delta_{12}t}$. A suitable ansatz is

$$A_\mu dx^\mu = A_t(\rho,t)dt + A_\rho(\rho,t)d\rho \ . \tag{3.120}$$

Assuming the time dependence $e^{-\Delta_{12}t}$, the divergence free condition implies

$$\partial_\rho\left(\tan^{d-1}\rho A_\rho\right) - \Delta_{12}\tan^{d-1}\rho A_t = 0 \tag{3.121}$$

and the components of the wave equation are

$$\begin{aligned}
&\frac{\cos^{d-1}\rho}{\sin^{d-1}\rho}\partial_\rho\left(\frac{\sin^{d-1}\rho}{\cos^{d-3}\rho}\partial_\rho A_t\right) + \left(\Delta_{12}^2\cos^2\rho - (\Delta-1)(\Delta-d+1)\right)A_t \\
&\quad - 2\Delta_{12}\cos\rho\sin\rho A_\rho = 0 \\
&\frac{\cos^{d-1}\rho}{\sin^{d-1}\rho}\partial_\rho\left(\frac{\sin^{d-1}\rho}{\cos^{d-3}\rho}\partial_\rho A_\rho\right) + \left(\Delta_{12}^2\cos^2\rho - \frac{d-1}{\sin^2\rho} - (\Delta-1)(\Delta-d+1)\right)A_\rho \\
&\quad + 2\Delta_{12}\cos\rho\sin\rho A_t = 0 \ .
\end{aligned} \tag{3.122}$$

The normalizable solution is

$$
\begin{aligned}
A_\rho &= \Delta_{12}\sin\rho(\cos\rho)^{\Delta}{}_2F_1\left(\frac{\Delta+\Delta_{12}+1}{2},\frac{\Delta-\Delta_{12}+1}{2},\Delta-\frac{d-2}{2};\cos^2\rho\right)e^{-\Delta_{12}t} \\
A_t &= \frac{1}{\Delta_{12}\tan^{d-1}\rho}\partial_\rho(\tan^{d-1}\rho A_\rho)
\end{aligned}
\tag{3.123}
$$

where we have inserted a factor of Δ_{12} in A_ρ to ensure a smooth $\Delta_{12}\to 0$ limit. In particular, setting $\Delta_{12}=0$ we have $A_\rho=0$ and

$$
A_t = (\cos\rho)^{\Delta-1}{}_2F_1\left(\frac{\Delta+1}{2},\frac{\Delta-1}{2},\Delta-\frac{d-2}{2};\cos^2\rho\right) . \tag{3.124}
$$

It is now straightforward to plug into (3.114) to obtain an integral expression for the conformal block. Because the general formula is rather lengthy we will only write it out explicitly in the case $\Delta_{12}=0$. In this case we find (not paying attention to overall normalization factors)

$$
\begin{aligned}
\mathcal{W}_{\Delta,1}(z,\overline{z}) &= |z|^{\Delta-\Delta_3-\Delta_4-1}(1-|1-z|^2) \\
&\quad \int_0^1 d\sigma\sigma^{\frac{\Delta+\Delta_{34}-1}{2}}(1-\sigma)^{\frac{\Delta-\Delta_{34}-1}{2}}\left(1-(1-|1-z|^2)\sigma\right)^{-\frac{\Delta+1}{2}} \\
&\quad \times {}_2F_1\left(\frac{\Delta+1}{2},\frac{\Delta-1}{2},\Delta-\frac{d-2}{2};\frac{|z|^2\sigma(1-\sigma)}{1-(1-|1-z|^2)\sigma}\right) .
\end{aligned}
\tag{3.125}
$$

Setting $d=2,4$, it is straightforward to verify that the series expansion of this integral reproduces the known $d=2,4$ results in (3.109),(3.110) for $\Delta_{12}=0$. We have also verified agreement for $\Delta_{12}\neq 0$.

3.5.4 Evaluation of geodesic Witten diagram: spin-2

In this section we set $\Delta_{12}=0$ to simplify formulas a bit. We need to solve

$$
\nabla^2 h_{\mu\nu}-[\Delta(\Delta-d)-2]h_{\mu\nu}=0 , \quad \nabla^\mu h_{\mu\nu}=0 , \quad h^\mu_\mu=0 . \tag{3.126}
$$

$h_{\mu\nu}$ should be static and spherically symmetric, which implies the general ansatz

$$
h_{\mu\nu}dx^\mu dx^\nu = f_{\rho\rho}(\rho)g_{\rho\rho}d\rho^2+f_{tt}(\rho)g_{tt}dt^2+\frac{1}{d-1}f_{\phi\phi}(\rho)\tan^2\rho\, d\Omega^2_{d-1} . \tag{3.127}
$$

We first impose the divergence free and tracelessness conditions. We have

$$
h^\mu_\mu = f_{\rho\rho}+f_{tt}+f_{\phi\phi} . \tag{3.128}
$$

We use this to eliminate $f_{\phi\phi}$,

$$f_{\phi\phi} = -f_{\rho\rho} - f_{tt} \ . \tag{3.129}$$

Moving to the divergence, only the component $\nabla^\mu h_{\mu\rho}$ is not automatically zero. We find

$$\nabla^\mu h_{\mu\rho} = f'_{\rho\rho} + \frac{d+1}{\cos\rho\sin\rho} f_{\rho\rho} - \frac{\cos\rho}{\sin\rho} f_{\rho\rho} + \frac{\cos\rho}{\sin\rho} f_{tt} = 0 \tag{3.130}$$

which we solve as

$$f_{tt} = -\tan\rho f'_{\rho\rho} + \left(1 - \frac{d+1}{\cos^2\rho}\right) f_{\rho\rho} \ . \tag{3.131}$$

We then work out the $\rho\rho$ component of the field equation,

$$\nabla^2 h_{\rho\rho} - [\Delta(\Delta-d)-2] h_{\rho\rho} = f''_{\rho\rho} + \left(\frac{d+3}{\cos\rho\sin\rho} - 2\cot\rho\right) f'_{\rho\rho} - \frac{(\Delta+2)(\Delta-d-2)}{\cos^2\rho} f_{\rho\rho} \ . \tag{3.132}$$

Setting this to zero, the normalizable solution is

$$f_{\rho\rho} = (\cos\rho)^{\Delta+2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+2}{2}, \Delta - \frac{d-2}{2}; \cos^2\rho\right) \ . \tag{3.133}$$

This completely specifies the solution, and we now have all we need to plug into (3.114). We refrain from writing out the somewhat lengthy formulas. The series expansion of the result matches up with (3.109) and (3.110) as expected.

3.5.5 General spin: proof via conformal Casimir equation

As in the case of scalar exchange, the most efficient way to verify that a geodesic Witten diagram yields a conformal partial wave is to check that it is an eigenfunction of the conformal Casimir operator with the correct eigenvalue and asymptotics.

We start from the general expression (3.112). A rank-n tensor on AdS is related to a tensor on the embedding space via

$$T_{\mu_1\ldots\mu_n} = \frac{\partial Y^{M_1}}{\partial y^{\mu_1}} \cdots \frac{\partial Y^{M_n}}{\partial y^{\mu_n}} T_{M_1\ldots M_n} \ . \tag{3.134}$$

In particular, this holds for the bulk-to-bulk propagator of the spin-ℓ field, and so we can write

$$G_{bb}(y,y';\Delta,\ell) = [G_{bb}(Y,Y';\Delta)]_{M_1\ldots M_\ell, N_1\ldots N_\ell} \frac{dY^{M_1}}{d\lambda} \cdots \frac{dY^{M_\ell}}{d\lambda} \frac{dY'^{N_1}}{d\lambda'} \cdots \frac{dY'^{N_\ell}}{d\lambda'} \ . \tag{3.135}$$

Now, $[G_{bb}(Y,Y';\Delta)]_{M_1\ldots M_\ell,N_1\ldots N_\ell}$ only depends on Y and Y'. Since $Y^M\frac{dY^M}{d\lambda}=\frac{1}{2}\frac{d}{d\lambda}(Y\cdot Y)=0$, when pulled back to the geodesics the only contributing structure is

$$[G_{bb}(Y,Y';\Delta)]_{M_1\ldots M_\ell,N_1\ldots N_\ell}=f(Y\cdot Y')Y'_{M_1}\ldots Y'_{M_\ell}Y_{N_1}\ldots Y_{N_\ell}\ . \tag{3.136}$$

We also recall a few other useful facts. Lifted to the embedding space, the geodesic connecting boundary points X_1 and X_2 is

$$Y(\lambda)=\frac{e^{\lambda}X_1+e^{-\lambda}X_2}{\sqrt{-2X_1\cdot X_2}}\ . \tag{3.137}$$

The bulk-to-boundary propagator lifted to the embedding space is

$$G_{b\partial}(X_i,Y)\propto(X_i\cdot Y)^{-\Delta_i}\ . \tag{3.138}$$

We follow the same strategy as in the case of scalar exchange. We start by isolating the part of the diagram that contains all the dependence on $X_{1,2}$,

$$\begin{aligned}&F_{M_1\ldots M_\ell}(X_1,X_2,Y';\Delta)=\\&\int_{\gamma_{12}}G_{b\partial}(X_1,Y(\lambda))G_{b\partial}(X_2,Y(\lambda))[G_{bb}(Y(\lambda),Y';\Delta)]_{M_1\ldots M_\ell,N_1\ldots N_\ell}\frac{dY^{M_1}}{d\lambda}\ldots\frac{dY^{M_\ell}}{d\lambda}\ .\end{aligned} \tag{3.139}$$

Here, $Y(\lambda)$ lives on γ_{12}, but Y' is left arbitrary. This is the spin-ℓ generalization of $\varphi^{12}_{\Delta}(y)$ defined in (3.36), lifted to embedding space. We now argue that this is annihilated by the SO(d+1,1) generators $L^1_{AB}+L^2_{AB}+L^{Y'}_{AB}$. This generator is the sum of three generators in the scalar representation, plus a "spin" term acting on the free indices $N_1\ldots N_\ell$. This operator annihilates any expression of the form $g(X_1\cdot X_2,X_1\cdot Y',X_2\cdot Y')X_{N_1}\ldots X_{N_\ell}$, where each X stands for either X_1 or X_2. To show this, we just note the SO(d+1,1) invariance of the dot products, along with the fact that X_N is the normal vector to the $X^2=0$ surface and so is also SO(d+1,1) invariant. From (3.136)-(3.138) we see that $F_{N_1\ldots N_\ell}(X_1,X_2,Y';\Delta)$ is of this form, and so is annihilated by $L^1_{AB}+L^2_{AB}+L^{Y'}_{AB}$. We can therefore write

$$\begin{aligned}(L^1_{AB}+L^2_{AB})^2F_{N_1\ldots N_\ell}(X_1,X_2,Y';\Delta)&=(L^{Y'}_{AB})^2F_{N_1\ldots N_\ell}(X_1,X_2,Y';\Delta)\\&=C_2(\Delta,\ell)F_{N_1\ldots N_\ell}(X_1,X_2,Y';\Delta)\end{aligned} \tag{3.140}$$

where we used that $(L^{Y'}_{AB})^2$ is acting on the spin-ℓ bulk-to-bulk propagator, which is an eigenfunction of the conformal Casimir operator[13] with eigenvalue (3.59). The relation (3.140) holds for all Y', and hence holds upon integrating Y' over γ_{34} with any weight. Hence we arrive at the conclusion

$$(L^1_{AB} + L^2_{AB})^2 \mathcal{W}_{\Delta,\ell}(x_i) = C_2(\Delta,\ell)\mathcal{W}_{\Delta,\ell}(x_i) \tag{3.141}$$

which is the same eigenvalue equation obeyed by the spin-ℓ conformal partial wave, $W_{\Delta,\ell}(x_i)$. The short distance behavior as dictated by the OPE is easily seen to match in the two cases, establishing that we have the same eigenfunction. We conclude that the spin-ℓ geodesic Witten diagram is, up to normalization, equal to the spin-ℓ conformal partial wave.

3.5.6 Comparison to double integral expression of Ferrara et. al.

It is illuminating to compare our expression (3.112) to equation (50) in [31], which gives the general result (in $d = 4$) for the scalar conformal partial wave with spin-ℓ exchange, written as a double integral. We will rewrite the result in [31] in a form permitting easy comparison to our formulas. First, it will be useful to rewrite the scalar bulk-to-bulk propagator (3.19) by applying a quadratic transformation to the hypergeometric function,

$$G_{bb}(y,y';\Delta) = \xi^{\Delta}{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2},\Delta+1-\frac{d}{2};\xi^2\right) . \tag{3.142}$$

Next, recall that in embedding space the geodesics are given by (3.72), from which we compute the quantity ξ with one point on each geodesic

$$\xi^{-1} = -Y(\lambda)\cdot Y(\lambda') = \frac{1}{2}\frac{e^{\lambda+\lambda'}x_{13}^2 + e^{\lambda-\lambda'}x_{14}^2 + e^{-\lambda+\lambda'}x_{23}^2 + e^{-\lambda-\lambda'}x_{24}^2}{x_{12}x_{34}} . \tag{3.143}$$

We also define a modified version as

$$\xi_-^{-1} = -\frac{dY(\lambda)}{d\lambda}\cdot\frac{dY(\lambda')}{d\lambda'} = \frac{1}{2}\frac{e^{\lambda+\lambda'}x_{13}^2 - e^{\lambda-\lambda'}x_{14}^2 - e^{-\lambda+\lambda'}x_{23}^2 + e^{-\lambda-\lambda'}x_{24}^2}{x_{12}x_{34}} . \tag{3.144}$$

Comparing to [31], we have $\xi^{-1} = \lambda_+$ and $\xi_-^{-1} = \lambda_-$.

[13]Note that the conformal Casimir is equal to the spin-ℓ Laplacian up to a constant shift: $(L^{Y'}_{AB})^2 = \nabla^2_\ell + \ell(\ell+d-1)$ [142].

With these definitions in hand, it is not hard to show that the result of [31] takes the form

$$\begin{aligned} W_{\Delta,\ell}(x_i) = \int_{\gamma_{12}}\int_{\gamma_{34}} & G_{b\partial}(y(\lambda),x_1)G_{b\partial}(y(\lambda),x_2) \\ & \times C'_\ell(2\xi_-^{-1})G_{bb}(y(\lambda),y(\lambda');\Delta)\times G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4)\ . \end{aligned} \tag{3.145}$$

Here $G_{bb}(y(\lambda),y(\lambda');\Delta)$ is the scalar bulk-to-bulk propagator (3.142), and $C'_\ell(x)$ is a Gegenbauer polynomial. This obviously looks very similar to our expression (3.112), and indeed agrees with it for $\ell=0$. The two results must be equal (up to normalization) since they are both expressions for the same conformal partial wave. If we assume that equality holds for the integrand, then we find the interesting result that the pullback of the spin-ℓ propagator, as written in (3.113), is equal to $C'_\ell(2\xi_-^{-1})G_{bb}(y(\lambda),y(\lambda');\Delta)$. The general spin-$\ell$ propagator is very complicated (see [131, 132]), but apparently has a simple relation to the scalar propagator when pulled back to geodesics. It would be interesting to verify this.

3.5.7 Decomposition of spin-1 Witten diagram into conformal blocks

In the case of scalar exchange diagrams, we previously showed how to decompose a Witten diagram into a sum of geodesic Witten diagrams, the latter being identified with conformal partial waves of both single- and double-trace exchanges. We now wish to extend this to the case of higher spin exchange; we focus here on the case of spin-1 exchange for simplicity. A picture of the final result is given in Figure 3.6.

As discussed in section 3.2, given two scalar operators in a generalized free field theory, we can form scalar double trace primaries with schematic form $[\mathcal{O}_1\mathcal{O}_2]_{m,0}\sim\mathcal{O}_1\partial^{2m}\mathcal{O}_2$ and dimension $\Delta^{(12)}(m,0)=\Delta_1+\Delta_2+2m+O(1/N^2)$, and vector primaries $[\mathcal{O}_1\mathcal{O}_2]_{m,1}\sim\mathcal{O}_1\partial^{2m}\partial_\mu\mathcal{O}_2$ with dimension $\Delta^{(12)}(m,1)=\Delta_1+\Delta_2+1+2m+O(1/N^2)$. The analysis of [15], and later [21,131,132] demonstrated that these conformal blocks, and their cousins $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,1}$, should appear in the decomposition of the vector exchange Witten diagram, together with the exchange of a single-trace vector operator. The computations below will confirm this expectation.

The basic approach is the same as in the scalar case, although the details are more com-

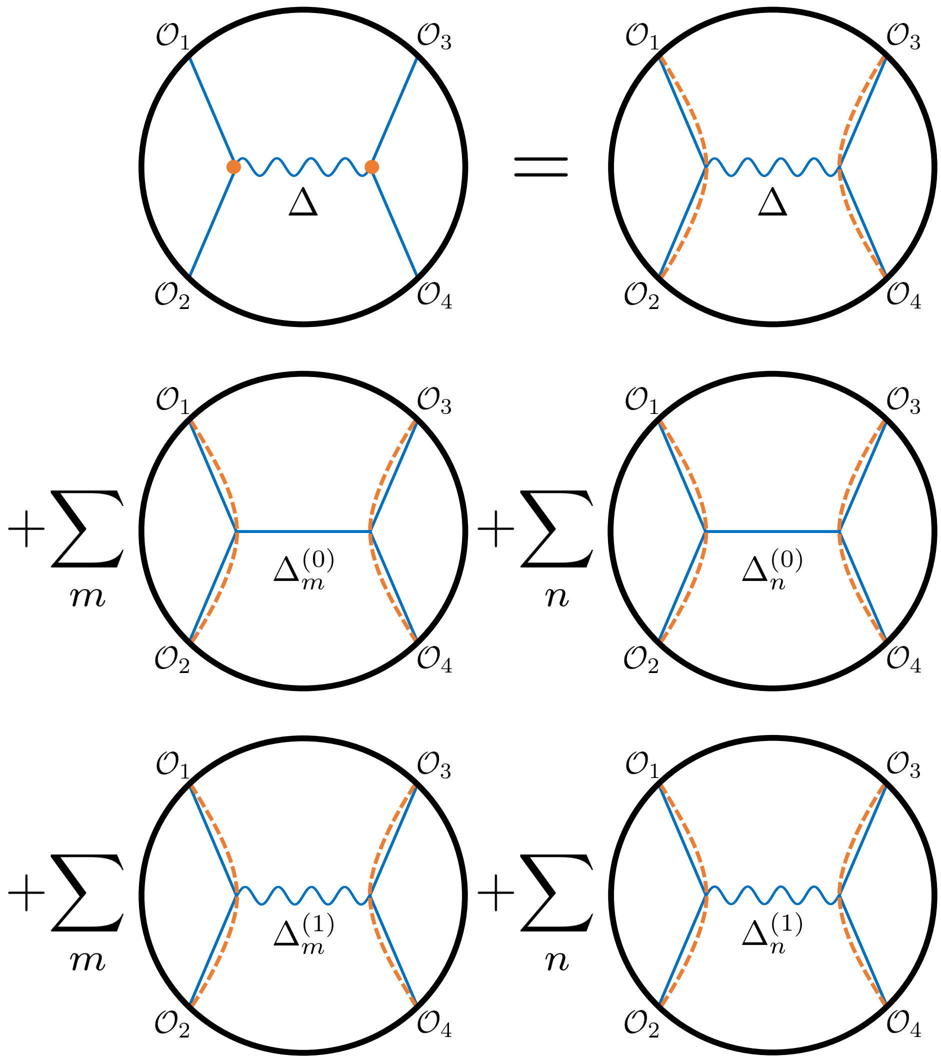

Figure 3.6: The decomposition of a four-point vector exchange diagram (upper left) into conformal partial waves. The term in the upper right captures the single-trace exchange of the dual vector operator. The second line captures the CFT exchanges of the $\ell = 0$ double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,0}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,0}$. Likewise, the final line captures the CFT exchanges of the $\ell = 1$ double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{m,1}$ and $[\mathcal{O}_3\mathcal{O}_4]_{n,1}$.

plicated. Before diving in, let us note the main new features. In the scalar case a basic step was to write, in (3.81), the product of two bulk-to-boundary propagators $G_{b\partial}(y, x_1)G_{b\partial}(y, x_2)$ as a sum over solutions $\varphi^{12}_{\Delta}(y)$ of the scalar wave equation sourced on the γ_{12} geodesic. Here, we will similarly need a decomposition of $G_{b\partial}(y, x_1)\nabla_\mu G_{b\partial}(y, x_2)$, where ∇_μ is a covariant derivative with respect to bulk coordinates y. It turns out that this can be expressed as a sum over massive spin-1 solutions and derivatives of massive scalar solutions. This translates into the statement that the spin-1 exchange Witten diagram decomposes as a sum of spin-1 and spin-0 conformal blocks, as noted above.

Now to the computation. We consider a theory of massive scalars coupled to a massive vector field via couplings $\phi_i \nabla_\mu \phi_j A^\mu$. The Witten diagram with vector exchange is then

$$\mathcal{A}_4^{\text{Vec}}(x_i) = \int_y \int_{y'} G_{b\partial}(y,x_1)\nabla_\mu G_{b\partial}(y,x_2) \times G_{bb}^{\mu\nu}(y,y';\Delta) \times G_{b\partial}(y',x_3)\nabla_\nu G_{b\partial}(y',x_4)\ . \quad (3.146)$$

Our first task is to establish the expansion

$$G_{b\partial}(y,x_1)\nabla_\mu G_{b\partial}(y,x_2) = \sum_m \left(c_m A_{m,\mu}(y) + b_m \nabla_\mu \varphi_m(y)\right) \quad (3.147)$$

where $A_{m,\mu}(y)$ and $\varphi_m(y)$ denote the solutions to the massive spin-1 and spin-0 equations sourced on γ_{12}, found earlier in sections 3.5.3 and 3.4.1, respectively.[14] m labels the masses of the bulk fields, to be determined shortly. We will not attempt to compute the coefficients c_m and b_m, which is straightforward but involved, contenting ourselves to determining the spectrum of conformal dimensions appearing in the expansion, and showing how the expansion coefficients can be obtained if desired.

Following the scalar case, we work in global AdS and send $t_1 \to -\infty$, $t_2 \to \infty$. Dropping normalizations, as we shall do throughout this section, we have

$$\begin{aligned} G_{b\partial}(y,x_1)\nabla_\rho G_{b\partial}(y,x_2) &= \sin\rho\,(\cos\rho)^{\Delta_1+\Delta_2-1} e^{-\Delta_{12}t} \\ G_{b\partial}(y,x_1)\nabla_t G_{b\partial}(y,x_2) &= (\cos\rho)^{\Delta_1+\Delta_2} e^{-\Delta_{12}t}\ . \end{aligned} \quad (3.148)$$

Letting $\Delta_m^{(\ell)}$ denote the dimension of the corresponding spin, we have, from (3.123) and

[14] φ_m is just φ_m^{12}, whose superscript we suppress for clarity, and likewise for φ_n and φ_n^{34}.

(3.45),

$$\varphi_m = {}_2F_1\Big(\frac{\Delta_m^{(0)}+\Delta_{12}}{2}, \frac{\Delta_m^{(0)}-\Delta_{12}}{2}; \Delta_m^{(0)} - \frac{d-2}{2}; \cos^2\rho\Big)(\cos\rho)^{\Delta_m^{(0)}}\ e^{-\Delta_{12}t}$$
$$A_{m,\rho} = \Delta_{12}\sin\rho\,(\cos\rho)^{\Delta_m^{(1)}}\,{}_2F_1\left(\frac{\Delta_m^{(1)}+\Delta_{12}+1}{2}, \frac{\Delta_m^{(1)}-\Delta_{12}+1}{2}, \Delta_m^{(1)} - \frac{d-2}{2}; \cos^2\rho\right) e^{-\Delta_{12}t}$$
$$A_{m,t} = \frac{1}{\Delta_{12}\tan^{d-1}\rho}\partial_\rho(\tan^{d-1}\rho A_{m,\rho}) \tag{3.149}$$

The various terms have the following powers $(\cos^2\rho)^k$ in an expansion in powers of $\cos^2\rho$,

$$\begin{aligned}
G_{b\partial}(y,x_1)\nabla_\rho G_{b\partial}(y,x_2): &\qquad k = \frac{\Delta_1+\Delta_2-1}{2}+q \\
A_{m,\rho}: &\qquad k = \frac{\Delta_m^{(1)}}{2}+q \\
\nabla_\rho\varphi_m: &\qquad k = \frac{\Delta_m^{(0)}-1}{2}+q \\
G_{b\partial}(y,x_1)\nabla_t G_{b\partial}(y,x_2): &\qquad k = \frac{\Delta_1+\Delta_2}{2} \\
A_{m,t}: &\qquad k = \frac{\Delta_m^{(1)}-1}{2}+q \\
\nabla_t\varphi_m: &\qquad k = \frac{\Delta_m^{(0)}}{2}+q
\end{aligned} \tag{3.150}$$

where $q = 0, 1, 2, \ldots$. Comparing, we see that we have the right number of free coefficients for (3.147) to hold, provided we have the following spectrum of dimensions appearing

$$\begin{aligned}
\Delta_m^{(0)} &= \Delta_1+\Delta_2+2m \\
\Delta_m^{(1)} &= \Delta_1+\Delta_2+1+2m
\end{aligned} \tag{3.151}$$

with $m = 0, 1, 2, \ldots$. The formulas above can be used to work out the explicit coefficients c_m and b_m. We noted at the beginning of this subsection that this spectrum of dimensions coincides with the expected spectrum of double-trace scalar and vector operators appearing in the OPE, at leading order in large N.

We may now rewrite (3.146) as[15]

$$\mathcal{A}_4^{\text{Vec}}(x_i) = \sum_{m,n} \int_y \int_{y'} \left(c_m A_{m,\mu}(y) + b_m \nabla_\mu \varphi_m(y)\right) G_{bb}^{\mu\nu}(y,y';\Delta) \left(c_n A_{n,\nu}(y') + b_n \nabla_\nu \varphi_n(y')\right) \ . \tag{3.152}$$

We expand this out in an obvious fashion as

$$\mathcal{A}_4^{\text{Vec}}(x_i) = \mathcal{A}_{AA}(x_i) + \mathcal{A}_{A\phi}(x_i) + \mathcal{A}_{\phi A}(x_i) + \mathcal{A}_{\phi\phi}(x_i) \ . \tag{3.153}$$

The next step is to relate each term to geodesic Witten diagrams, which we now do in turn.

3.5.7.1 $\mathcal{A}_{AA}$

We have

$$\mathcal{A}_{AA} = \sum_{m,n} c_m c_n \int_y \int_{y'} A_{m,\mu}(y) G_{bb}^{\mu\nu}(y,y';\Delta) A_{n,\nu}(y') \ . \tag{3.154}$$

The solution $A_{m,\mu}(y)$ can be expressed as

$$\begin{aligned} A_m^\mu(y) &= \int_{\gamma_{12}} G_{b\partial}(y(\lambda),x_1) \nabla_\nu G_{b\partial}(y(\lambda),x_2) G_{bb}^{\mu\nu}(y(\lambda),y;\Delta_m^{(1)}) \\ &= -\Delta_2 \int_{\gamma_{12}} G_{b\partial}(y(\lambda),x_1) G_{b\partial}(y(\lambda),x_2) \frac{dy_\nu(\lambda)}{d\lambda} G_{bb}^{\mu\nu}(y(\lambda),y;\Delta_m^{(1)}) \ . \end{aligned} \tag{3.155}$$

The second equality follows from the relation $\nabla_\mu G_{b\partial}(x,y(\lambda)) = -\Delta \frac{dy_\mu(\lambda)}{d\lambda} G_{b\partial}(x,y(\lambda))$, which is easily verified for a straight line geodesic at the center of global AdS, and hence is true in general. Using this we obtain (dropping the normalization, as usual)

$$\begin{aligned} \mathcal{A}_{AA} = \sum_{m,n} c_m c_n \int_y \int_{y'} \int_{\gamma_{12}} \int_{\gamma_{34}} &\left[G_{b\partial}(y(\lambda),x_1) G_{b\partial}(y(\lambda),x_2) \frac{dy_\mu(\lambda)}{d\lambda}\right] \\ &\times \left[G_{bb}^{\mu\nu}(y(\lambda),y;\Delta_m^{(1)}) G_{bb,\nu\alpha}(y,y';\Delta) G_{bb}^{\alpha\beta}(y',y(\lambda');\Delta_n^{(1)})\right] \\ &\times \left[G_{b\partial}(y(\lambda),x_3) G_{b\partial}(y(\lambda),x_4) \frac{dy'_\beta(\lambda')}{d\lambda'}\right] \ . \end{aligned} \tag{3.156}$$

The bulk-to-bulk propagator for the vector field obeys

$$\left(\nabla^2 - m^2\right) G_{bb}^{\mu\nu}(y,y';\Delta) = \delta^{\mu\nu}(y-y') \tag{3.157}$$

[15]Following the precedent of Section 3.4, all quantities with an m subscript refer to the double-trace operators appearing in the $\mathcal{O}_1\mathcal{O}_2$ OPE, and those with an n subscript refer to the double-trace operators appearing in the $\mathcal{O}_3\mathcal{O}_4$ OPE.

where $\delta^{\mu\nu}(y-y')$ denotes a linear combination of $g^{\mu\nu}\delta(y-y')$ and $\nabla^\mu\nabla^\nu\delta(y-y')$. Using this, and the fact that the propagator is divergence free at non-coincident points, we can verify the composition law

$$\int_{y'} G_{bb}^{\mu\nu}(y,y';\Delta)G_{bb,\nu\alpha}(y',y'';\Delta') = \frac{1}{m^2-(m')^2}\Big(G_{bb}{}^{\mu}{}_{\alpha}(y,y'';\Delta) - G_{bb}{}^{\mu}{}_{\alpha}(y,y'';\Delta')\Big) \ . \quad (3.158)$$

We use this relation twice within (3.156) to obtain a sum of three terms, each with a single vector bulk-to-bulk propagator. Note also that these propagators appear pulled back to the geodesics. Each term is thus a geodesic Witten diagram with an exchanged vector, that is, a spin-1 conformal partial wave. The spectrum of spin-1 operators that appears is

$$\Delta\ , \quad \Delta_1+\Delta_2+1+2m\ , \quad \Delta_3+\Delta_4+1+2n\ , \quad m,n=0,1,2,\ldots \quad (3.159)$$

So the contribution of $\mathcal{A}_{AA}$ is a sum of spin-1 conformal blocks with internal dimensions corresponding to the original exchanged field, along with the expected spin-1 double trace operators built out of the external scalars.

3.5.7.2 $\mathcal{A}_{A\phi}$ and $\mathcal{A}_{\phi A}$

We start with

$$\mathcal{A}_{\phi A} = \sum_{m,n} c_m b_n \int_y\int_{y'} \nabla_\mu\varphi_m(y)G_{bb}^{\mu\nu}(y,y';\Delta)A_{n,\nu}(y') \ . \quad (3.160)$$

Next we integrate by parts in y, use $\nabla_\mu G_{bb}^{\mu\nu}(y,y';\Delta)\propto\nabla^\nu\delta(y-y')$, and integrate by parts again, to get

$$\mathcal{A}_{\phi A} = \sum_{m,n} b_m c_n \int_{y'} \nabla_\mu\varphi_m(y')A_n^\mu(y') \ . \quad (3.161)$$

Now we write $A_n^\mu(y')$ as an integral over γ_{34} as in (3.155) and then again remove the bulk-to-bulk propagator by integrating by parts. This yields

$$\mathcal{A}_{\phi A} = \sum_{m,n} b_m c_n \int_{\gamma_{34}} G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4)\frac{dy^\mu(\lambda')}{d\lambda'}\nabla_\mu\varphi_m(y(\lambda')) \ . \quad (3.162)$$

Writing φ_m as an integral sourced on γ_{12} we obtain

$$\mathcal{A}_{\phi A} = \sum_{m,n} b_m c_n \times$$
$$\int_{\gamma_{12}} \int_{\gamma_{34}} G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) \frac{d}{d\lambda'} G_{bb}(y(\lambda), y(\lambda'); \Delta_m^{(0)}) G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4) \ . \tag{3.163}$$

Integrating by parts and using $\frac{d}{d\lambda'}\big(G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4)\big) \propto G_{b\partial}(y(\lambda'), x_3) G_{b\partial}(y(\lambda'), x_4)$ we see that $\mathcal{A}_{\phi A}$ decomposes into a sum of spin-0 exchange geodesic Witten diagrams. That is, $\mathcal{A}_{\phi A}$ contributes a sum of spin-0 blocks with conformal dimensions

$$\Delta_1 + \Delta_2 + 2m \ , \quad m = 0, 1, 2, \ldots \tag{3.164}$$

By the same token $\mathcal{A}_{A\phi}$ yields a sum of spin-0 blocks with conformal dimensions

$$\Delta_3 + \Delta_4 + 2n \ , \quad n = 0, 1, 2, \ldots \tag{3.165}$$

3.5.7.3 $\mathcal{A}_{\phi\phi}$

We have

$$\mathcal{A}_{\phi\phi} = \sum_{m,n} b_m b_n \int_y \int_{y'} \nabla_\mu \varphi_m(y) G_{bb}^{\mu\nu}(y, y'; \Delta) \nabla_\nu \varphi_n(y') \ . \tag{3.166}$$

Integration by parts reduces this to

$$\mathcal{A}_{\phi\phi} = \sum_{m,n} b_m b_n \int_y \nabla^\mu \varphi_m(y) \nabla_\mu \varphi_n(y) \ . \tag{3.167}$$

Now rewrite the scalar solutions as integrals over the respective geodesic sources,

$$\begin{aligned} \mathcal{A}_{\phi\phi} \ &= \ \sum_{m,n} b_m b_n \int_{y'} \int_{\gamma_{12}} \int_{\gamma_{34}} G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) G_{b\partial}(y(\lambda), x_3) G_{b\partial}(y(\lambda), x_4) \\ &\qquad \times \nabla^{\mu'} G_{bb}(y(\lambda), y'; \Delta_m^{(0)}) \nabla_{\mu'} G_{bb}(y', y(\lambda'); \Delta_n^{(0)}) \ . \end{aligned} \tag{3.168}$$

The composition law analogous to (3.158) is easily worked out to be

$$\begin{aligned} \int_{y'} \nabla^{\mu'} G_{bb}(y(\lambda), y'; \Delta_m^{(0)}) \nabla_{\mu'} G_{bb}(y', y(\lambda'); \Delta_n^{(0)}) &= c_{mn} G_{bb}(y(\lambda), y(\lambda'); \Delta_m^{(0)}) \\ &\quad + d_{mn} G_{bb}(y(\lambda), y(\lambda'); \Delta_n^{(0)}) \end{aligned} \tag{3.169}$$

with some coefficients c_{mn} and d_{mn} that we do not bother to display here. Inserting this in (3.168) we see that $\mathcal{A}_{\phi\phi}$ decomposes into a sum of scalar blocks with conformal dimensions

$$\Delta_1 + \Delta_2 + 2m \ , \quad \Delta_3 + \Delta_4 + 2n \ , \quad m, n = 0, 1, 2, \ldots \tag{3.170}$$

3.5.7.4 Summary

We have shown that the Witten diagram involving the exchange of a spin-1 field of dimension Δ decomposes into a sum of spin-1 and spin-0 conformal blocks. The full spectrum of conformal blocks appearing in the decomposition is

$$\begin{aligned} \text{scalar}: &\quad \Delta_1 + \Delta_2 + 2n \ , \quad \Delta_3 + \Delta_4 + 2n \\ \text{vector}: &\quad \Delta \ , \quad \Delta_1 + \Delta_2 + 1 + 2n \ , \quad \Delta_3 + \Delta_4 + 1 + 2n \end{aligned} \tag{3.171}$$

where $n = 0, 1, 2, \ldots$. This matches the spectrum expected from $1/N$ counting, including single- and double-trace operator contributions. With some patience, the formulas above can be used to extract the coefficient of each conformal block, but we have not carried this out in full detail here.

While we have not explored this in any detail, it seems likely that the above method can be directly generalized to the case of arbitrary spin-ℓ exchange. The split (3.153) will still be natural, and a higher spin version of (3.158) should hold.

3.6 Discussion and future work

In this paper, we have shed new light on the underlying structure of tree-level scattering amplitudes in AdS. Four-point scalar amplitudes naturally organize themselves into geodesic Witten diagrams; recognizing these as CFT conformal partial waves signals the end of the computation, and reveals a transparency between bulk and boundary with little technical effort required. We are optimistic that this reformulation extends, in some manner, to computations of generic holographic correlation functions in AdS/CFT. To that end, we close with some concrete observations and proposals, as well as a handful of future directions.

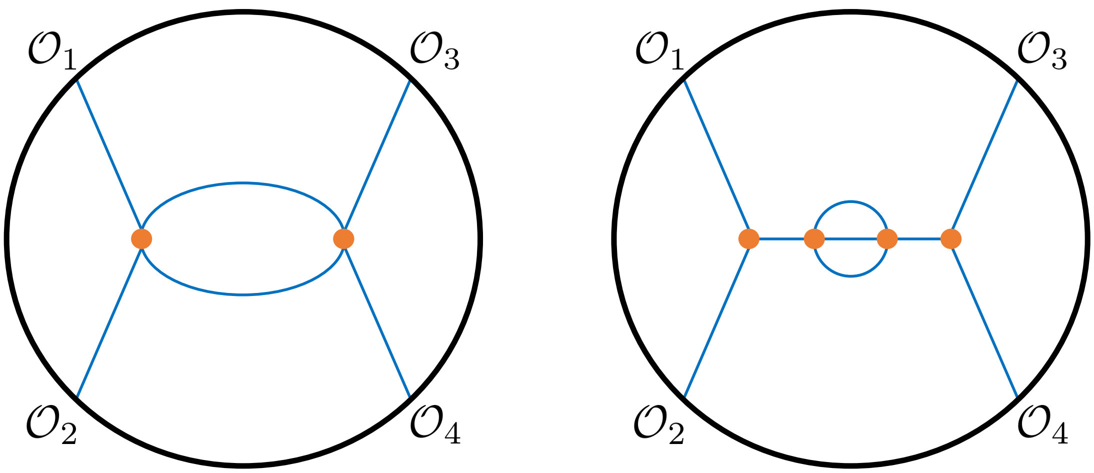

Figure 3.7: Some examples of loop diagrams that can be written as infinite sums over tree-level diagrams, and hence decomposed into conformal blocks using our methods.

Adding loops It is clearly of interest to try to generalize our techniques to loop level. We first note that there is a special class of loop diagrams that we can compute already using these methods: namely, those that can be written as an infinite sum of tree-level exchange diagrams [100]. For the same reason, this is the only class of loop diagrams whose Mellin amplitudes are known [100]. These diagrams only involve bulk-to-bulk propagators that all start and end at the same points; see Figure 3.7 for examples. Careful study of the resulting sums would be useful.

More generally, though, we do not yet know how to decompose generic diagrams into geodesic objects. This would seem to require a "geodesic identity" analogous to (3.81) that applies to a pair of bulk-to-bulk propagators, rather than bulk-to-boundary propagators. It would be very interesting to find these, if they exist. Such identities would also help to decompose an exchange Witten diagram in the crossed channel.

Decomposition of Witten diagrams in the crossed channel The present work studies the decomposition of an s-channel exchange Witten diagram into s-channel partial waves. It is clearly of interest to understand, using the language of geodesic Witten diagrams, how the same diagram decomposes into t-channel partial waves, which corresponds to using the

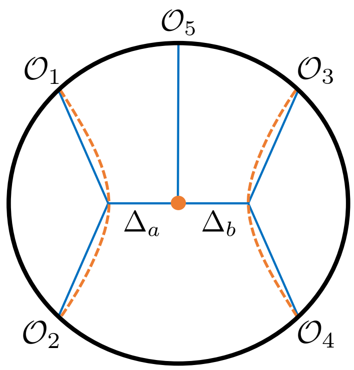

Figure 3.8: This is the basic constituent that emerges in applying our technology to the decomposition of a five-point tree-level Witten diagram. However, it is not equal to the five-point conformal partial wave, as discussed in the text.

OPE on the pairs of operators $\mathcal{O}_1\mathcal{O}_3$ and $\mathcal{O}_2\mathcal{O}_4$. As explored in [5], the statement is that the basic scalar exchange Witten diagram decomposes into t-channel partial waves involving only scalar double trace operators $[\mathcal{O}_1\mathcal{O}_3]_{n,0}$ and $[\mathcal{O}_2\mathcal{O}_4]_{n,0}$. As we mention above, this decomposition seems to require new geodesic identities involving bulk-to-bulk propagators.

Derivative interactions We have primarily focused on decomposing Witten diagrams with non-derivative interactions. For example, the contact diagram of figure 3.4 is based on the interaction $\phi_1\phi_2\phi_3\phi_4$. We would like to be able to efficiently decompose Witten diagrams with derivative interactions too, like $\phi_1\nabla^{\mu_1}\nabla^{\mu_2}...\nabla^{\mu_k}\phi_2\phi_3\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_k}\phi_4$; a precise version of the identity in equation (3.147) would be sufficient to treat the $k=1$ case, but we would like to generalize that to add more Lorentz indices. We anticipate that this identity exists and involves propagators of massive fields of spin $\ell \leq k$.

Adding legs Consider for example a five-point correlator of scalar operators $\langle\mathcal{O}_1(x_1)\ldots\mathcal{O}_5(x_5)\rangle$. We can define associated conformal partial waves by inserting projection operators as

$$W_{\Delta,\ell;\Delta',\ell'}(x_i) = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)P_{\Delta,\ell}\mathcal{O}_5(x_5)P_{\Delta',\ell'}\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\ . \tag{3.172}$$

Using the OPE on $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$, reduces this to three-point functions. The question is, can we represent $W_{\Delta,\ell;\Delta',\ell'}(x_i)$ as a geodesic Witten diagram?

Suppose we try to dismantle a tree level five-point Witten diagram. For definiteness, we take $\ell' = \ell = 0$. All tree level five-point diagrams will lead to the same structures upon using our geodesic identities: namely, they can be written as sums over geodesic-type diagrams, each as in Figure 3.8, which we label $\widehat{\mathcal{W}}_{\Delta_a,0;\Delta_b,0}(x_i)$. This is easiest to see starting from a ϕ^5 contact diagram, and using (3.81) on the pairs of propagators (12) and (34). In that case, $\Delta_a \in \{\Delta_m\}$ and $\Delta_b \in \{\Delta_n\}$. As an equation, Figure 3.8 reads

$$\begin{aligned}\widehat{\mathcal{W}}_{\Delta_a,0;\Delta_b,0}(x_i) = &\int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}(y(\lambda),x_1)G_{b\partial}(y(\lambda),x_2)\\ &\times \int_{y_5} G_{bb}(y(\lambda),y_5;\Delta_a)\, G_{b\partial}(y_5,x_5)\, G_{bb}(y_5,y(\lambda');\Delta_b)\\ &\times\, G_{b\partial}(y(\lambda'),x_3)G_{b\partial}(y(\lambda'),x_4)\ .\end{aligned} \tag{3.173}$$

Note that the vertex at y_5, indicated by the orange dot in the figure, must be integrated over all of AdS. Could these diagrams be computing $W_{\Delta_a,0;\Delta_b,0}(x_i)$ as defined above? The answer is no, as a simple argument shows. Suppose we set $\Delta_5 = 0$ in (3.173), which requires $\Delta_a = \Delta_b \equiv \Delta$. From (3.172) it is clear that we must recover the four-point conformal partial wave with the exchanged primary $(\Delta, 0)$. So we should ask whether (3.173) reduces to the expression for the four-point geodesic Witten diagram, $\mathcal{W}_{\Delta,0}$. Using $G_{b\partial}(y_5,x_5)|_{\Delta_5=0} \propto 1$, the integral over y_5 becomes

$$\int_{y_5} G_{bb}(y(\lambda),y_5;\Delta)G_{bb}(y_5,y(\lambda');\Delta) \propto \frac{\partial}{\partial m^2_\Delta} G_{bb}(y(\lambda),y(\lambda');\Delta)\ . \tag{3.174}$$

Therefore, the $\Delta_5 = 0$ limit of (3.173) does not give back the four-point partial wave, but rather its derivative with respect to m^2_Δ, which is a different object.

We conclude that although we can decompose a five-point Witten diagram into a sum of diagrams of the type in Figure 3.8, this is not the conformal block decomposition. This raises two questions: what is the meaning of this decomposition in CFT terms, and (our original question) what diagram computes the five-point partial wave?

External operators with spin Another obvious direction in which to generalize is to consider correlation functions of operators carrying spin. As far as the conformal blocks go, partial information is available. In particular, [91] obtained expressions for such blocks as differential operators acting on blocks with external scalars, but this approach is limited to the case in which the exchanged operator is a symmetric traceless tensor, since only such operators appear in the OPE of two scalar operators. The same approach was taken in [143]. Explicit examples of mixed symmetry exchange blocks were given in [144].

Our formulation in terms of geodesic Witten diagrams suggests an obvious proposal for the AdS computation of an arbitrary conformal partial wave: take our usual expression (3.1), now with the bulk-to-boundary and bulk-to-bulk propagators corresponding to the fields dual to the respective operators. Of course, there are many indices here which have to be contracted, and there will be inequivalent ways of doing so. But this is to be expected, as in the general case there are multiple conformal blocks for a given set of operators, corresponding to the multiplicity of ways in which one spinning primary can appear in the OPE of two other spinning primaries. It will be interesting to see whether this proposal turns out to be valid. As motivation, we note that it would be quite useful for bootstrap purposes to know all the conformal blocks that arise in the four-point function of stress tensors.

After the publication of this paper, the geodesic Witten diagram construction has indeed been extended to external operators with spin [145–147] as well as to antisymetric tensor exchange [148]. The natural extension to fermionic conformal blocks [149] has also been shown to hold [150].

A related pursuit would be to decompose all four-point scalar contact diagrams, including any number of derivatives at the vertices. This would involve a generalization of (3.147) to include more derivatives.

Virasoro blocks and $\mathbf{AdS_3/CFT_2}$ Our calculations give a new perspective on how to construct the dual of a generic Virasoro conformal block: starting with the geodesic Witten diagram, we dress it with gravitons. Because Virasoro blocks depend on c, a computation in semiclassical AdS gravity would utilize a perturbative $1/c \sim G_N$ expansion. In [2], we put the geodesic approach to use in constructing the holographic dual of the heavy-light Virasoro blocks of [105], where one geodesic essentially backreacts on AdS to generate a conical defect or black hole geometry. It would be worthwhile to pursue a $1/c$ expansion around the geodesic Witten diagrams more generally.

A closely related question is how to decompose an AdS_3 Witten diagram into Virasoro, rather than global, blocks. For a tree-level diagram involving light external operators like those considered here, there is no difference, because the large c Virasoro block with light external operators reduces to the global block [108]. It will be interesting to see whether loop diagrams in AdS_3 are easier to analyze using Virasoro symmetry.

Assorted comments The geodesic approach to conformal blocks should be useful in deriving various CFT results, not only mixed symmetry exchange conformal blocks. For example, the conformal blocks in the limits of large τ, ℓ or d [151–157], and subleading corrections to these, should be derivable using properties of AdS propagators. One can also ask whether there are similar structures present in bulk spacetimes besides AdS. For instance, an analog of the geodesic Witten diagram in a thermal spacetime would suggest a useful ingredient for parameterizing holographic thermal correlators[16]. Perhaps the existence of a dS/CFT correspondence suggests similar structures in de Sitter space as well.

Indeed, since this paper's publication Geodesic Witten diagram technology has been developed for conformal field theories with boundary [160, 161], for p-adic CFTs [162], and – under the name Geodesic Feynman Diagrams [163, 164] – for computing BMS blocks [165] with application to flat space holography .

It is natural to wonder whether there are analogous techniques to those presented here that are relevant for holographic correlators of nonlocal operators like Wilson loops or surface operators, perhaps involving bulk minimal surfaces.

Let us close by noting a basic fact of our construction: even though a conformal block is not a semiclassical object per se, we have given it a representation in terms of classical fields propagating in a smooth spacetime geometry. In a bulk theory of quantum gravity putatively dual to a finite N CFT, we do not yet know how to compute amplitudes. Whatever the prescription, there is, evidently, a way to write the answer using geodesic Witten diagrams.

[16]It turns out that holographic torus conformal blocks can be computed by Witten diagrams without geodesics [158, 159].

It would be interesting to understand how this structure emerges.

CHAPTER 4

Semiclassical Virasoro Blocks from AdS_3 Gravity

4.1 Introduction

Correlation functions in conformal field theories admit a decomposition in terms of conformal blocks, obtained by using the OPE to reduce products of local operators at distinct points to a sum of local operators at a single point, and collecting the contribution of operators lying in a single representation of the conformal algebra; see e.g. [30–33, 38, 39, 53, 91]. This yields a concrete algorithm to go from the basic CFT data — a list of primary operators and their OPE coefficients — to correlation functions. The conformal blocks are fully determined by conformal symmetry, and so are universal to all CFTs. They feature prominently in many applications of CFT, including in the conformal bootstrap program [28, 92] and in the study of the emergence of bulk locality from CFT [15, 55, 100].

Given a consistent theory of gravity in AdS, one can compute correlation functions that obey CFT axioms, and hence admit a decomposition into conformal blocks. A natural question is: what object in AdS gravity computes a CFT conformal block? In previous work [3] we answered this question for AdS_{d+1}/CFT_d for any d. There, an elegant prescription was found in the case of four-point conformal blocks with external scalar operators. The main result is that a conformal block — more precisely, a conformal partial wave — is obtained from a "geodesic Witten diagram." This is essentially an ordinary exchange Witten diagram but with vertices integrated over geodesics connecting the external operators, rather than over all of AdS. See Figure 4.2. The case of $d = 2$ is special because the global conformal algebra is enhanced to two copies of the Virasoro algebra. The corresponding Virasoro conformal blocks are much richer objects, containing an infinite number of global conformal blocks. In the present work we address the bulk construction of the Virasoro blocks.

Unlike the case of global blocks, Virasoro blocks depend on the central charge c of the theory. Because we will be working in the context of classical gravity, and $1/c$ plays the role of $\hbar$ in the bulk, we must restrict attention to the regime $c \to \infty$, corresponding to so-called semiclassical Virasoro blocks. There are various ways to take this limit, corresponding to the manner in which the operator dimensions behave as $c \to \infty$. Two natural choices bookend the spectrum of possibilities: either keep all operator dimensions fixed, or let all operator dimensions scale linearly with c. As we review in Section 4.2, analytical expressions for the Virasoro blocks have been derived at various points on this spectrum using CFT techniques. Indeed, with a few exceptions [53, 62, 166], these are some of the only analytical expressions for Virasoro blocks available.

In what follows, we will present a framework that computes all known semiclassical Virasoro blocks using 3D gravity. Partial results on bulk derivations of Virasoro blocks are already known [1, 22, 106], and we will incorporate and reproduce them here. One object whose bulk dual has not been constructed as yet is the elegant formula obtained recently by Fitzpatrick, Kaplan and Walters (FKW) [105], for the four-point conformal block in the case that two external operator dimensions grow linearly in c, while the rest remain fixed (see (4.2.10) and (4.2.11)). This is known as the "heavy-light limit." By combining the ideas of [1, 22, 106] with our other work on global blocks [3], we will indeed arrive at a more complete story for the holographic construction of semiclassical Virasoro blocks. We provide a diagrammatic overview in Figure 4.1.

Our main results can be summarized as follows. First, it is well-known that in the $c \to \infty$ limit with operator dimensions held fixed, the Virasoro block reduces to the global block [54, 108]. Therefore, the geodesic Witten diagram provides the bulk construction of the Virasoro block in this simple limit. More significantly, we will reproduce the FKW result in the heavy-light limit. The main idea is essentially to start with the geodesic Witten diagram for the global block, and allow one of the geodesics to backreact on AdS_3. This sets up a conical defect or BTZ geometry for the remaining part of the geodesic Witten diagram corresponding to the light operators; explicit computation leads quickly to the correct result. (The appearance of a defect or black hole depends on whether the heavy operator dimension

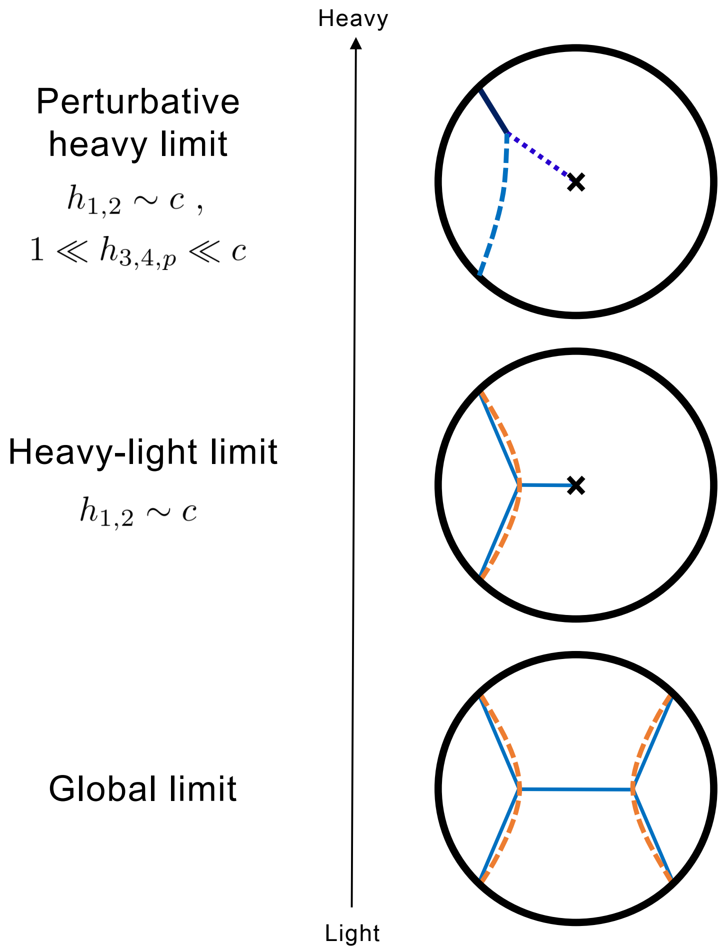

Figure 4.1: The spectrum of gravity duals of large c Virasoro blocks. Operator dimensions increase from bottom to top; h_i and h_p denote external and internal holomorphic operator dimensions, respectively. In the limit of fixed dimensions, the Virasoro block becomes the global block, represented by a geodesic Witten diagram. Upon ramping up two external dimensions to enter the heavy-light regime, the bulk dual becomes a geodesic Witten diagram evaluated in a conical defect geometry. Further taking the remaining dimensions to scale with c, albeit perturbatively, one minimizes the worldline action of a cubic vertex of geodesics in the presence of the defect. This is equivalent to making a saddle-point approximation to the heavy-light geodesic Witten diagram. Not shown is the fully non-perturbative Virasoro block for all heavy operators, whose form is unknown.

is above or below the black hole threshold, $h = c/24$.) Interpolation amongst the various semiclassical limits may be systematically computed, in principle, by treating backreaction effects, providing an intuitive bridge between the different regimes. We will explicitly demonstrate in Appendix 4.4, for instance, that the saddle-point approximation to the geodesic Witten diagram for the heavy-light block manifestly reduces to the worldline prescription for the perturbative heavy blocks given in [1]. This correctly interpolates between the two regimes. We have therefore provided a bulk construction for all known semiclassical Virasoro blocks.

4.2 Review of semiclassical Virasoro blocks

We consider a four-point function of Virasoro primary operators $\mathcal{O}_i(z_i, \bar{z}_i)$ on the plane,

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\mathcal{O}_3(z_3, \bar{z}_3)\mathcal{O}_4(z_4, \bar{z}_4)\rangle \ . \tag{4.2.1}$$

$\mathcal{O}_i$ has holomorphic and anti-holomorphic conformal weights $(h_i, \bar{h}_i)$, respectively. Using SL(2,$\mathbb{C}$) invariance, three of the operators can be taken to specified locations. It will be convenient to thereby consider

$$\langle \mathcal{O}_1(\infty, \infty)\mathcal{O}_2(0, 0)\mathcal{O}_3(z, \bar{z})\mathcal{O}_4(1, 1)\rangle \ , \tag{4.2.2}$$

where $\mathcal{O}_1(\infty, \infty) = \lim_{z_1, \bar{z}_1 \to \infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\mathcal{O}_1(z_1, \bar{z}_1)$ inside the correlator. A basis for the Hilbert space of the CFT consists of the set of primary states $|\mathcal{O}_p\rangle$ (equivalently, local primary operators $\mathcal{O}_p$) and their Virasoro descendants, i.e. the set of irreducible highest weight representations of the Virasoro algebra. This implies the existence of a Virasoro conformal block decomposition of the four-point function,

$$\langle \mathcal{O}_1(\infty, \infty)\mathcal{O}_2(0, 0)\mathcal{O}_3(z, \bar{z})\mathcal{O}_4(1, 1)\rangle = \sum_p C_{12p}C^p{}_{34}\mathcal{F}(h_i, h_p, c; z-1)\overline{\mathcal{F}}(\bar{h}_i, \bar{h}_p, c; \bar{z}-1) \ , \tag{4.2.3}$$

where the sum runs over all irreducible representations of the Hilbert space. We use h_i to stand for $h_{1,2,3,4}$. For simplicity, we have assumed equal left- and right-moving central charges. The fact that the holomorphic and anti-holomorphic Virasoro algebras commute with each other leads to holomorphic factorization for given p.

The Virasoro blocks can be conveniently defined using a projector, which we denote P_p, acting within the Hilbert space. The s-channel Virasoro block is obtained by inserting this projector between the operators $\mathcal{O}_2$ and $\mathcal{O}_3$:[1]

$$\langle \mathcal{O}_1(\infty,\infty)\mathcal{O}_2(0,0)\, P_p\, \mathcal{O}_3(z,\overline{z})\mathcal{O}_4(1,1)\rangle = \mathcal{F}(h_i,h_p,c;z-1)\overline{\mathcal{F}}(\overline{h}_i,\overline{h}_p,c;\overline{z}-1)\ . \qquad (4.2.4)$$

We refer to $\mathcal{F}(h_i,h_p,c;z-1)$ alone as the Virasoro block.[2]

Unlike for global conformal blocks, no closed-form expressions for Virasoro blocks are known, except in some very special cases [53, 62, 166]. We briefly mention what is known in general. OPE considerations reveal that $\mathcal{F}(h_i,h_p,c;z)$ has the structure $z^{h_p-h_3-h_4}f(z)$, where $f(z)$ is analytic in the unit disk. Zamolodchikov [54, 108] has provided recursion relations allowing one to efficiently compute terms in the power series expansion of $f(z)$ around the origin. These recursion relations can be solved [167, 168]. The expansion coefficients are rational functions of the conformal weights, which rapidly become extremely complicated. The coefficients have also been computed using combinatorial methods inspired by the AGT correspondence [169, 170].

Of greater relevance here is the semiclassical limit corresponding to taking $c\to\infty$. If h_i are all held fixed in the limit, the Virasoro block simply reduces to the global block, which is a hypergeometric function [32]:

$$\lim_{c\to\infty}\mathcal{F}(h_i,h_p,c;z-1) = (z-1)^{h_p-h_3-h_4}{}_2F_1(h_p-h_{12},h_p+h_{34};2h_p;z-1)\ . \qquad (4.2.5)$$

where $h_{ij}\equiv h_i-h_j$. Instead, we are interested in the case in which we hold fixed some ratios h_i/c. If all ratios h_i/c are held fixed in the limit, then one can apply Zamolodchikov's monodromy method (well reviewed in [22,63]) to determine the semiclassical Virasoro block. The equations resulting from this approach turn out to be equivalent to those of 3D gravity with negative cosmological constant; this becomes especially transparent in the Chern-Simons formulation (see e.g. [1]). However, this is still too complicated to admit an exact solution. Progress can be made in perturbation theory by taking $h_3/c, h_4/c, h_p/c \ll 1$, keeping h_1/c

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[1] We won't concern ourselves with the normalization of this function, which is fixed by matching its small z behavior to the $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$ OPEs, and throughout will freely discard any z-independent prefactors.

[2] In d-dimensional conventions, as in [3], this projection is better known as a conformal partial wave. However, in 2d CFT literature, one often finds the convention used here, in which the $z\to1$ expansion of the block itself starts at $(z-1)^{h_p-h_3-h_4}$, as opposed to $(z-1)^{h_p}$.

and h_2/c finite. Results obtained in this approach can be found in [1,22,106]. The 3D gravity picture in this case consists, at lowest order in the above small parameters, of particle worldlines moving in a background geometry of the "heavy" operators $h_{1,2}$. Higher orders in perturbation theory account for the backreaction of the particles on the geometry. We called this the perturbative heavy limit in Figure 4.1.

Let us give slightly more detail. To distinguish heavy and light operators we now write

$$h_1 = h_{H_1} \ , \quad h_2 = h_{H_2} \ , \quad h_3 = h_{L_1} \ , \quad h_4 = h_{L_2} \ . \tag{4.2.6}$$

The bulk prescription for computing the semiclassical Virasoro block to first order[3] in $h_{L_1}/c, h_{L_2}/c, h_p/c \ll 1$ was first explained in [22] in the simplified case of $h_{L_1} = h_{L_2}, h_{H_1} = h_{H_2}, h_p = 0$, which corresponds to the vacuum Virasoro block. The heavy operators backreact to generate the metric

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho}\left(\frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2 \rho \, d\phi^2\right) \ , \tag{4.2.7}$$

with $\phi \cong \phi + 2\pi$. For real $\alpha < 1$, this is a conical defect solution with a singularity at $\rho = 0$; for $\alpha^2 < 0$ it becomes a BTZ black hole after Wick rotation. This can be thought of as representing the geometry sourced by a particle of mass $m^2 = 4h_{H_1}(h_{H_1} - 1)$ sitting at the origin of global AdS_3, where

$$\alpha = \sqrt{1 - \frac{24 h_{H_1}}{c}} \ . \tag{4.2.8}$$

The "light" operators are incorporated by a geodesic in the background (4.2.7) connecting their locations on the boundary. The appearance of geodesics makes sense because these operators, while parametrically lighter than the heavy operators, still have $h_{L_1}/c, h_{L_2}/c$ fixed in the large c limit. The Virasoro vacuum block is then simply given by e^{-mL}, where $m^2 = 4h_{L_1}(h_{L_1} - 1)$, and L is the geodesic length, regulated with a near boundary cutoff. An elementary computation yields $e^{-mL} \propto \left|\sin \frac{\alpha w}{2}\right|^{-4h_{L_1}}$, which is the correct result derived from CFT [22].

[3] This regime can also be described as holding fixed $h_{L_1,L_2,p}$, and then working to first order in $1/h_{L_1,L_2,p}$. These two procedures turn out to agree, as discussed in [105].

In [1] this was further generalized to allow for $h_{L_1} \neq h_{L_2}$ and $h_p \neq 0$. The picture is now of three geodesic segments, living in the geometry (4.2.7), and joined at a cubic vertex. Two of the geodesics are anchored at the locations of $\mathcal{O}_{L_1}$ and $\mathcal{O}_{L_2}$, while the geodesic corresponding to $\mathcal{O}_p$ stretches between the cubic vertex and the singularity at $\rho = 0$. The location of the cubic vertex is found by extremizing the total geodesic action $S = m_p L_p + m_1 L_1 + m_2 L_2$, and then the Virasoro block in this regime is obtained from e^{-S}. In [1] it was explained why this prescription works, by thinking about the relationship between Zamolodchikov's monodromy method and the linearized backreaction produced by these worldlines.

4.2.1 The heavy-light semiclassical limit

The case considered in the present work corresponds to

$$c \to \infty \quad with \quad \frac{h_{H_1}}{c}, \quad \frac{h_{H_2}}{c}, \quad h_{H_1} - h_{H_2}, \quad h_{L_1}, \quad h_{L_2}, \quad h_p \quad fixed\ . \tag{4.2.9}$$

This so-called "heavy-light limit" was considered recently in [105]. By a clever use of conformal mappings, they were able to relate the Virasoro block in this limit to a global block, with a result

$$\langle \mathcal{O}_{H_1}(\infty,\infty)\mathcal{O}_{H_2}(0,0)\, P_p\, \mathcal{O}_{L_1}(z,\bar{z})\mathcal{O}_{L_2}(1,1)\rangle \to \mathcal{F}\left(h_i, h_p, c; z-1\right) \overline{\mathcal{F}}(\bar{h}_i, \bar{h}_p, c; \bar{z}-1)\ , \tag{4.2.10}$$

with

$$\mathcal{F}\left(h_i, h_p, c; z-1\right) = z^{(\alpha-1)h_{L_1}}(1-z^\alpha)^{h_p - h_{L_1} - h_{L_2}}{}_2F_1\Big(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1-z^\alpha\Big)\ , \tag{4.2.11}$$

where α was defined in (4.2.8), and

$$h_{12} \equiv h_{L_1} - h_{L_2}\ , \quad H_{12} \equiv h_{H_1} - h_{H_2}\ . \tag{4.2.12}$$

Note that in the definition of α it doesn't matter whether h_{H_1} or h_{H_2} appears, since we are taking $(h_{H_1} - h_{H_2})/c \to 0$ in the limit. Setting $\alpha = 1$ yields the global conformal block. The result (4.2.10) can be checked by expanding in $z - 1$ and matching to the series expansion (up to some finite order).

Our goal in the remainder of this paper is to show how to reproduce this result from AdS_3 gravity. To this end, it will also useful to rewrite the result on the cylinder, $z = e^{iw}$, with $w = \phi + i\tau$. Taking into account the usual transformation rule for primary operators, and dropping a constant multiplicative prefactor, we have, in the heavy-light limit,

$$\langle \mathcal{O}_{H_1}(\tau = -\infty)\mathcal{O}_{H_2}(\tau = \infty)\, P_p\, \mathcal{O}_{L_1}(w, \overline{w})\mathcal{O}_{L_2}(0,0)\rangle \to \mathcal{F}(h_i, h_p, c; w)\overline{\mathcal{F}}(\overline{h}_i, \overline{h}_p, c; \overline{w}) \ ,$$

$$\mathcal{F}(h_i, h_p, c; w) = \left(\sin\frac{\alpha w}{2}\right)^{-2h_{L_1}} \left(1 - e^{i\alpha w}\right)^{h_p + h_{12}} {}_2F_1\left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{i\alpha w}\right) \ . \tag{4.2.13}$$

4.3 Semiclassical Virasoro blocks from AdS_3 gravity

4.3.1 Bulk prescription

The bulk recipe for reproducing (4.2.13) is easy to motivate once we recall some previous results. As reviewed above and made clear in Figure 4.1, the heavy-light limit (4.2.9) sits halfway between two other large c limits: holding all h_i and h_p fixed, or holding ratios h_i/c and h_p/c fixed. While the bulk prescription for computing the Virasoro block in the latter limit was just described in the previous section, the prescription for the former limit may be extracted from more recent work [3], as we now discuss. We can thus obtain the prescription for computing the heavy-light block as a middle ground between those known results.

Consider setting $\alpha = 1$ in (4.2.13), which as noted above yields the global conformal block. This is equivalent to holding all h_i fixed as $c \to \infty$. In [3], a simple bulk setup for computing conformal partial waves for symmetric, traceless spin-ℓ exchange was proposed and proven in arbitrary spacetime dimension. The picture is that of a geodesic Witten diagram, as we now explain in the setting of $\text{AdS}_3/\text{CFT}_2$. Consider the global block corresponding to exchange of $\mathcal{O}_p$. For simplicity, we take $\mathcal{O}_p$ to be spinless, so $h_p = \overline{h}_p \equiv \Delta/2$. To define the geodesic Witten diagram, we begin with an ordinary exchange Witten diagram in AdS_3, where the exchanged field is a scalar of mass $m^2 = \Delta(\Delta - 2)$. In the full Witten diagram, the cubic vertices are integrated over all of AdS; to compute instead the geodesic Witten diagram, and hence the global conformal block, we restrict the integration to the bulk geodesics γ_{12} and

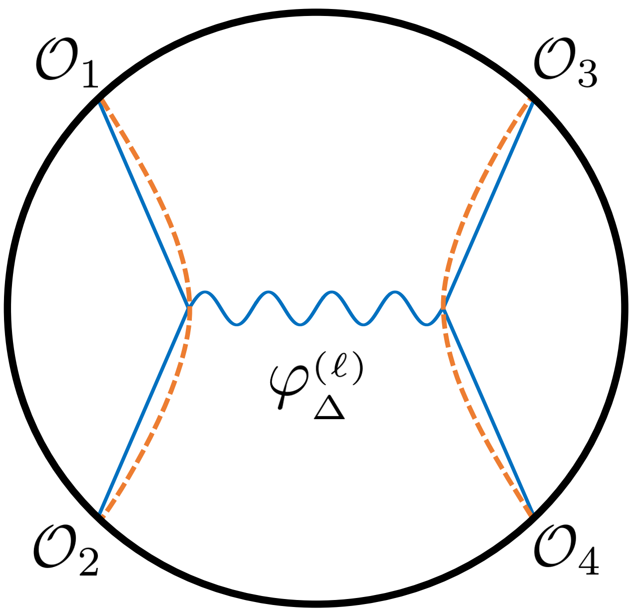


Figure 4.2: This is a geodesic Witten diagram in AdS_{d+1}, for the exchange of a symmetric traceless spin-ℓ tensor with $m^2 = \Delta(\Delta - d) - \ell$ in AdS units, introduced in [3]. The vertices are integrated over the geodesics connecting the two pairs of boundary points, here drawn as dashed orange lines. This computes the conformal partial wave for the exchange of a CFT_d primary operator of spin ℓ and dimension Δ. When $d = 2$, this yields the product of holomorphic and anti-holomorphic global conformal blocks. To compute the heavy-light Virasoro blocks instead, we allow one geodesic to backreact, creating a conical defect.

γ_{34} connecting the indicated boundary points. Then the geodesic Witten diagram for scalar exchange, denoted $\mathcal{W}_{\Delta,0}$, is

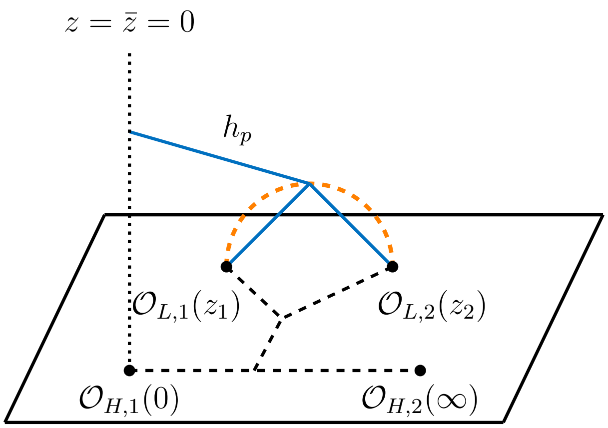
$$\mathcal{W}_{\Delta,0}(x_i) = \\ \int_{\gamma_{12}} d\lambda \int_{\gamma_{34}} d\lambda' G_{b\partial}(x_1, y(\lambda)) G_{b\partial}(x_2, y(\lambda)) \times G_{bb}(y(\lambda), y(\lambda'); \Delta) \times G_{b\partial}(x_3, y(\lambda')) G_{b\partial}(x_4, y(\lambda')) \ , \tag{4.3.1}$$

where λ and λ' denote proper length. See Figure 4.2. $G_{b\partial}$ and G_{bb} are bulk-to-boundary and bulk-to-bulk propagators, respectively. We use the convention that x denotes a point on the boundary, and y a point in the bulk. Up to normalization factors that can be found in [3], (4.3.1) is equal to the corresponding product of holomorphic and anti-holomorphic global blocks for $\mathcal{O}_p$ exchange. This generalizes nearly verbatim to $d > 2$. While it is familiar that geodesics can appear in Witten diagrams as an approximation in the case that the mass of the corresponding field is large, here there is no approximation: (4.3.1) is an exact expression for fields of any mass, i.e. any operator dimensions.

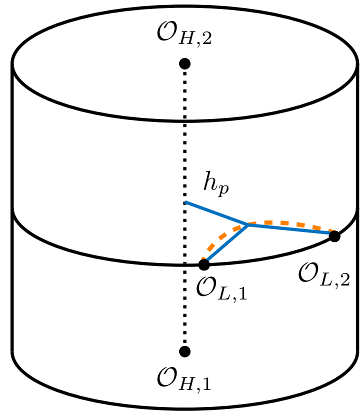
Now we need to generalize this to $\alpha \neq 1$. This is equivalent to taking the heavy-light limit (4.2.9) instead of keeping all h_i fixed. In the geodesic Witten diagram picture, we now want to "scale up" the dimensions h_1 and h_2 with large c. This suggests a rather natural proposal: let γ_{12} backreact, and evaluate (4.3.1) in the new spacetime. This is most naturally phrased if, as in (4.2.13), we take the heavy operators to be located at past and future infinity. Then the γ_{12} geodesic, which sits at $\rho = 0$, will backreact on the AdS_3 geometry to generate a conical defect or BTZ black hole, with metric (4.2.7). To obtain the heavy-light Virasoro block, we still compute (4.3.1), but now with the propagators for the light operators being defined in the conical defect metric (4.2.7) that is produced by the heavy operators.[4] We can think of this as the conical defect and light particle geodesic exchanging a bulk field corresponding to the primary $\mathcal{O}_p$. This provides a pleasingly intuitive picture for the heavy-light Virasoro block. We have drawn this setup in Figure 4.3, and in the middle frame of Figure 4.1.

We may also reason starting from the worldline picture described in Section 4.2, which computes the Virasoro block in the limit of fixed h_i/c and h_p/c. We can obtain the heavy-light block by "undoing" the saddle-point approximation for the propagation of the light fields $h_{L_1,L_2,p}$, while keeping the conical defect geometry sourced by the heavy fields. This again suggests the picture in terms of the geodesic Witten diagram in the conical defect background. Actually, at first glance there appears to be a mismatch between the worldline picture in [1] and the approach presented here. Namely, in [1] the worldlines of the light fields meet at a vertex whose location is found by minimizing the total worldline action. The location of this vertex typically does not lie on the geodesic connecting the external light operators. By contrast, here the interactions are constrained to occur on the geodesic. Despite this apparent difference, the results agree, as we explain in appendix 4.4.

[4] Actually, in the next section this statement will be refined slightly.



(a) Poincaré coordinates



(b) Global coordinates

Figure 4.3: Bulk setup for computing a heavy-light semiclassical Virasoro block. The heavy operators $\mathcal{O}_{H_{1,2}}$ set up a conical defect geometry centered at the dotted line in the bulk. The conical defect sources a bulk field dual to the exchanged primary operator $\mathcal{O}_p$. The external light operators $\mathcal{O}_{L_{1,2}}$ interact with the bulk field along a geodesic; in particular, the interaction vertex is to be integrated over the bulk geodesic (dashed orange line) connecting the light operator insertion points. In the Poincaré figure, the corresponding Virasoro block in the CFT is indicated by the dashed black lines.

In the remainder of this section we verify our prescription by direct computation, showing how the bulk diagram reproduces (4.2.13). We will restrict to the case that all operators are spinless, obeying $h = \overline{h}$ (in the next section we consider the case $h \neq 0$ and $\overline{h} = 0$.)

4.3.2 Evaluating the geodesic Witten diagram

We now reproduce (4.2.13) using the geodesic Witten diagram in the conical defect background. Let us use CFT$_2$ notation to denote this as $\mathcal{W}_{2h_p,0}$. With the operators at the specified configurations, we want to compute

$$\begin{aligned}\mathcal{W}_{2h_p,0}(w) \equiv & \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\partial}(\tau_1 = -\infty, \tau(\lambda)) G_{b\partial}(\tau_2 = \infty, \tau(\lambda)) \\ & \times G_{bb}^{(\alpha)}(y(\lambda), y(\lambda'); 2h_p) G_{b\partial}^{(\alpha)}(w_1 = 0, y(\lambda')) G_{b\partial}^{(\alpha)}(w_2 = w, y(\lambda')) \ ,\end{aligned} \tag{4.3.2}$$

where $G^{(\alpha)}$ is a propagator in the conical defect metric (4.2.7). In the first line, we have specifically highlighted the τ-dependence to make clear that these operators generate in- and out-states on the cylinder. We will assemble this integrand piece-by-piece.

We first recall a few facts. A bulk scalar field ϕ dual to a CFT operator $\mathcal{O}$ of dimension (h, h) has mass $m^2 = 4h(h-1)$ and obeys $(\nabla^2 - m^2)\phi = 0$ in the absence of interactions. The bulk-to-boundary propagator in global AdS (i.e. $\alpha = 1$) is

$$G_{b\partial}(x', y) = \left(\frac{\cos\rho}{\cosh(\tau - \tau') - \sin\rho\cos(\phi - \phi')} \right)^{2h} . \tag{4.3.3}$$

Similarly, the scalar bulk-to-bulk propagator in global AdS, which obeys the wave equation with a delta function source, is,

$$G_{bb}(y, y'; 2h) = \xi^{2h}{}_2F_1(h, h + \frac{1}{2}, 2h; \xi^2) = \frac{e^{-2h\sigma(y,y')}}{e^{-2\sigma(y,y')} - 1} , \tag{4.3.4}$$

where ξ is related to the chordal distance $\xi^{-1}-1$, and $\sigma(y, y')$ is the geodesic distance between the two bulk points:

$$\sigma(y, y') = \ln\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi} \right) , \quad \xi = \frac{\cos\rho\cos\rho'}{\cosh(\tau - \tau') - \sin\rho\sin\rho'\cos(\phi - \phi')} . \tag{4.3.5}$$

In (4.3.2), we need to evaluate the propagators for the light external and internal operators in the conical defect geometry. We will obtain these by taking the global AdS results (4.3.3)–(4.3.5) and making the replacements $\tau \to \alpha\tau$ and $\phi \to \alpha\phi$, which takes the metric to that of the conical defect. It should be noted that this does not in fact produce the proper bulk-to-boundary propagator for the conical defect, because the periodicity $\phi \cong \phi + 2\pi$ is not respected. Therefore, the Virasoro block computed using this propagator will not be single-valued under $\phi \cong \phi+2\pi$. However, this is in fact what we want, because the Virasoro blocks have a branch cut and are not single-valued. This branch cut will be correctly reproduced using these non-single-valued propagators.

Having established that, we begin our calculation. The product of heavy operator propagators, with endpoints anchored at past and future infinity, and evaluated at $\rho = 0$, is (dropping a prefactor)

$$G_{b\partial}(\tau_1 = -\infty, \tau)G_{b\partial}(\tau_2 = \infty, \tau) = e^{-2H_{12}\tau} . \tag{4.3.6}$$

Noting that $\lambda = \alpha\tau$ is proper time at the origin, (4.3.6) gives the first line of (4.3.2). For the light external and internal operators we use the bulk-to-boundary propagator in the conical defect geometry, as described above. To pull them back to the geodesics we need an expression for the geodesic itself, connecting the insertion points of the external light operators. Consider a geodesic beginning and ending at points w_1 and w_2 on the boundary, respectively. To simplify matters, we will take the two points to lie on a common time slice, so that $w_{12} = w_1 - w_2$ is real. We then have

$$\cos\rho(\lambda) = \frac{\sin\frac{\alpha w_{12}}{2}}{\cosh\lambda} \ , \quad e^{2i\alpha w(\lambda)} = \frac{\cosh(\lambda - \frac{i\alpha w_{12}}{2})}{\cosh(\lambda + \frac{i\alpha w_{12}}{2})} e^{i\alpha(w_1+w_2)} \ . \tag{4.3.7}$$

The bulk-to-boundary propagators for the light fields evaluated on the geodesic then work out to be

$$G_{b\partial}^{(\alpha)}(w_1, y(\lambda')) = \frac{e^{-2h_{L_1}\lambda'}}{(\sin\frac{\alpha w_{12}}{2})^{2h_{L_1}}} \ , \quad G_{b\partial}^{(\alpha)}(w_2, y(\lambda')) = \frac{e^{2h_{L_2}\lambda'}}{(\sin\frac{\alpha w_{12}}{2})^{2h_{L_2}}} \ . \tag{4.3.8}$$

Plugging into (4.3.2), we set $w_1 = 0, w_2 = w$. Finally, the bulk-to-bulk propagator for the field of dimension h_p evaluated with one endpoint at $\rho = 0$ at time τ, and the other on the geodesic at time $\tau' = 0$, is

$$G_{bb}^{(\alpha)}(y(\lambda), y(\lambda'); 2h_p) = \xi^{2h_p}{}_2F_1(h_p, h_p + \frac{1}{2}, 2h_p; \xi^2) \ , \quad \xi = \frac{\sin\frac{\alpha w_{12}}{2}}{\cosh\lambda\cosh\lambda'} \ . \tag{4.3.9}$$

Putting everything together, we get the following integral expression

$$\begin{aligned} \mathcal{W}_{2h_p,0}(w) = \left(\sin\tfrac{\alpha w}{2}\right)^{2h_p - 2h_{L_1} - 2h_{L_2}} &\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' e^{-\frac{2H_{12}}{\alpha}\lambda - 2h_{12}\lambda'} (\cosh\lambda\cosh\lambda')^{-2h_p} \\ &\times {}_2F_1\left(h_p, h_p + \frac{1}{2}, 2h_p; \frac{(\sin\frac{\alpha w}{2})^2}{(\cosh\lambda\cosh\lambda')^2}\right) \ . \end{aligned} \tag{4.3.10}$$

The integrals can be evaluated by writing the series expansion of the hypergeometric function and using some identities. This is carried out in subsection 4.3.3 and the result is

$$\begin{aligned} \mathcal{W}_{2h_p,0}(w) \propto \left(\sin\tfrac{\alpha w}{2}\right)^{2h_p - 2h_{L_1} - 2h_{L_2}} &\times {}_2F_1\left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{i\alpha w}\right) \\ &\times {}_2F_1\left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{-i\alpha w}\right) \ , \end{aligned} \tag{4.3.11}$$

which matches (4.2.13). (Recall that we have systematically dropped all normalization factors.) This is one of our main results.

It is also illuminating to reduce the expression for $\mathcal{W}_{2h_p,0}(w)$ to a single integral as follows. Consider the part of the integral depending on λ,[5]

$$\begin{aligned}\varphi_p(y') &= \int_{-\infty}^{\infty} d\lambda G_{b\partial}(\tau_1 = -\infty, \frac{\lambda}{\alpha}) G_{b\partial}(\tau_2 = \infty, \frac{\lambda}{\alpha}) G_{bb}(y(\lambda), y'; 2h_p) \\ &= \alpha \int_{-\infty}^{\infty} d\tau e^{-2H_{12}\tau} G_{bb}(\rho = 0, \tau; y'; 2h_p) \ . \end{aligned} \tag{4.3.12}$$

In (4.3.2), y' is pulled back to the light geodesic, but we leave it general here. $\varphi_p(y')$ obeys $(\nabla^2 - 4h_p(h_p-1))\varphi_p = 0$ away from a delta function source at $\rho = 0$; is rotationally invariant; has a time dependence $e^{-2H_{12}\tau}$; and has normalizable falloff at the AdS boundary. These properties uniquely fix φ_p, and by solving the field equation in the conical defect background we find

$$\varphi_p(y') = (\cos\rho')^{2h_p} {}_2F_1\Big(h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \cos^2\rho'\Big) e^{-2H_{12}\tau'} \ . \tag{4.3.13}$$

The geodesic corresponding to the external light operators thus propagates in the conical defect dressed by the scalar field solution $\varphi_p(y(\lambda'))$ corresponding to the primary $\mathcal{O}_p$:

$$\begin{aligned}\mathcal{W}_{2h_p,0}(w) &= \int_{-\infty}^{\infty} d\lambda' \, \varphi_p(y(\lambda')) \, G_{b\partial}(w_1 = 0, y(\lambda')) \, G_{b\partial}(w_2 = w, y(\lambda')) \\ &= \left(\sin\frac{\alpha w}{2}\right)^{2h_p - 2h_{L_1} - 2h_{L_2}} \int_{-\infty}^{\infty} d\lambda' e^{-2h_{12}\lambda' - 2\frac{H_{12}}{\alpha}\lambda'} (\cosh\lambda')^{-2h_p} \\ &\qquad \times {}_2F_1\left(h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \frac{\sin^2\frac{\alpha w}{2}}{\cosh^2\lambda'}\right) \ . \end{aligned} \tag{4.3.14}$$

This formula can also be seen to reproduce (4.2.13).

To summarize the results of this section, we verified a simple bulk prescription for reproducing the semiclassical heavy-light Virasoro block, involving a light particle geodesic interacting with a heavy particle worldline via the exhange of a light intermediate field. To be precise, our computation doesn't quite allow us to extract the individual factors $\mathcal{F}$ and $\overline{\mathcal{F}}$ in (4.2.13) because of our restriction to real w. This limitation will be overcome in the next section.

[5] This field solution was denoted $\varphi_{\Delta}^{12}(y')$ in [3].

4.3.3 Evaluating the geodesic integrals

Equation (4.3.10) above gives an integral expression which reproduces the Virasoro conformal block with an exchanged scalar of conformal dimensions (h_p, h_p). In this appendix we evaluate the integrals and put the result into a form that can be readily compared with the known formula (4.2.13) for the conformal blocks.

We begin with the integral expression

$$\begin{aligned} I = &\int_{-\infty}^{\infty} d\lambda e^{-\frac{2H_{12}}{\alpha}\lambda}(\cosh\lambda)^{-2h_p} \\ &\times \int_{-\infty}^{\infty} d\lambda' e^{-2h_{12}\lambda'}(\cosh\lambda')^{-2h_p}\,{}_2F_1\left(h_p, h_p+\frac{1}{2}, 2h_p; \frac{(\sin\frac{\alpha w}{2})^2}{(\cosh\lambda\cosh\lambda')^2}\right) . \end{aligned} \tag{4.3.15}$$

In terms of which equation (4.3.10) reads

$$\mathcal{W}_{2h_p,0}(w) = \left(\sin\tfrac{\alpha w}{2}\right)^{2h_p-2h_{L_1}-2h_{L_2}} \times I. \tag{4.3.16}$$

Notice that I receives divergent contributions from large λ or λ' unless

$$\left|\tfrac{H_{12}}{\alpha}\right| < h_p \quad \text{and} \quad |h_{12}| < h_p. \tag{4.3.17}$$

In what follows we assume that these conditions are met. A similar assumption was necessary in [1].

We expand the hypergeometric function in powers of $x \equiv \sin^2\frac{\alpha w}{2}$ to find

$$I = \sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty} d\lambda\, e^{-\frac{2H_{12}}{\alpha}\lambda}(\cosh\lambda)^{-2n-2h_p}\right)\left(\int_{-\infty}^{\infty} d\lambda'\, e^{-2h_{12}\lambda'}(\cosh\lambda')^{-2n-2h_p}\right)\frac{(h_p)_n(h_p+\frac{1}{2})_n}{(2h_p)_n n!}x^n , \tag{4.3.18}$$

where $(h)_n = \frac{\Gamma(h+n)}{\Gamma(h)}$ is the Pochhammer symbol. Condition (4.3.17) ensures that both integrals above are finite. They are given by

$$\begin{aligned} \int_{-\infty}^{\infty} d\lambda\, e^{-\frac{2H_{12}}{\alpha}\lambda}(\cosh\lambda)^{-2n-2h_p} &= 2^{2m-1}B\left(m-\tfrac{H_{12}}{\alpha}, m+\tfrac{H_{12}}{\alpha}\right) , \\ \int_{-\infty}^{\infty} d\lambda'\, e^{-2h_{12}\lambda'}(\cosh\lambda')^{-2n-2h_p} &= 2^{2m-1}B\left(m-h_{12}, m+h_{12}\right) , \end{aligned} \tag{4.3.19}$$

where B is the beta function $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Substituting for the integrals in equation (4.3.18) and then using twice the identity

$$\Gamma(2h_p+2n) = 2^{2n}\Gamma(2h_p)(h_p)_n(h_p+\tfrac{1}{2})_n , \tag{4.3.20}$$

which follows from the Legendre duplication formula, we find

$$\begin{aligned} I =& \frac{2^{4h_p-2}\Gamma(h_p+\frac{H_{12}}{\alpha})\Gamma(h_p-\frac{H_{12}}{\alpha})\Gamma(h_p+h_{12})\Gamma(h_p-h_{12})}{\Gamma(2h_p)\Gamma(2h_p)} \\ &\times \sum_{n=0}^{\infty} \frac{(h_p+\frac{H_{12}}{\alpha})_n(h_p-\frac{H_{12}}{\alpha})_n(h_p+h_{12})_n(h_p-h_{12})_n}{(2h_p)_n(h_p)_n(h_p+\frac{1}{2})_n n!} x^n \ . \end{aligned} \tag{4.3.21}$$

We recognize the sum on the second line as the power series of a ${}_4F_3$ hypergeometric function. Let $\mathcal{N}$ stand for the factor in the top line multiplying this function. Then

$$\mathcal{W}_{2h_p,0}(w) = \mathcal{N}\left(\sin\tfrac{\alpha w}{2}\right)^{2h_p-2h_{L_1}-2h_{L_2}} {}_4F_3\left(\begin{matrix} h_p+\frac{H_{12}}{\alpha}, h_p-\frac{H_{12}}{\alpha}, h_p+h_{12}, h_p-h_{12} \\ 2h_p,\ h_p,\ h_p+\frac{1}{2}\end{matrix}\ \Big|\sin^2\tfrac{\alpha w}{2}\right) \ . \tag{4.3.22}$$

To facilitate comparison with the result (4.2.13), we would like to write this ${}_4F_3$ hypergeometric function as a product of ${}_2F_1$ functions. To that end we employ the identity

$${}_4F_3\left(\begin{matrix} a,\ b-a,\ a',\ b-a' \\ \frac{b}{2},\ \frac{b+1}{2},\ b\end{matrix}\ \Big|\frac{z^2}{4(z-1)}\right) = {}_2F_1\Big(a,a',b;z\Big)\,{}_2F_1\Big(a,a',b;\frac{z}{z-1}\Big) \ , \tag{4.3.23}$$

which is valid when $z \notin \{1,\infty\}$. Using this identity with

$$z = 1-e^{i\alpha w} \ , \qquad a = h_p+h_{12} \ , \qquad a' = h_p - \tfrac{H_{12}}{\alpha} \ , \qquad b = 2h_p \ , \tag{4.3.24}$$

one finds

$$\begin{aligned} \mathcal{W}_{2h_p,0}(w) = \mathcal{N}\left(\sin\tfrac{\alpha w}{2}\right)^{2h_p-2h_{L_1}-2h_{L_2}} &\times {}_2F_1\left(h_p+h_{12}, h_p-\frac{H_{12}}{\alpha}, 2h_p; 1-e^{i\alpha w}\right) \\ &\times {}_2F_1\left(h_p+h_{12}, h_p-\frac{H_{12}}{\alpha}, 2h_p; 1-e^{-i\alpha w}\right) \ , \end{aligned} \tag{4.3.25}$$

which matches (4.2.13). This is the result (4.3.11) quoted above.

4.4 Recovering the worldline approach

In previous work [1] we presented a bulk construction for conformal blocks in a special case of the heavy-light limit (4.2.9) considered here. Here we show how that construction arises as a saddle point approximation to the present, more general one.

Specifically, we worked to first order in the limit where h_{L_1}, h_{L_2}, h_p are large, and in addition assumed $h_{H_1} = h_{H_2}$. In that case we showed the Virasoro conformal partial wave, W, to be $W \propto e^{-2S_{\text{free}}}$, where S_{free} is found by minimizing the action

$$S = h_{L_1} L_{L_1} + h_{L_2} L_{L_2} + h_p L_p \tag{4.4.1}$$

of a configuration of worldlines in the conical defect background. The worldlines L_i originate at the external light operators' positions, worldline p originates at the conical defect, and all three meet at a cubic vertex in the bulk. Here L_p is the length of worldline p and L_{L_i} is the length of worldline L_i regularized by putting the boundary points at large but finite distance from the origin.

The subscript "free" on S_{free} is meant to emphasize that the vertex joining worldlines L_1, L_2, p is unconstrained: it will go wherever in the bulk it needs to go in order to make S as small as possible, and in particular it need not lie on the geodesic connecting the light operators' positions.

Meanwhile, in the present approach, setting $h_{H_1} = h_{H_2}$ and taking h_{L_1}, h_{L_2}, h_p large, the geodesic Witten diagram (4.3.1) becomes

$$\mathcal{W} \propto \int d\lambda \int d\lambda' e^{-2S(y(\lambda), y(\lambda'))} , \tag{4.4.2}$$

where $S(y(\lambda), y(\lambda'))$ is the action of the worldline configuration in which the vertex joining L_1, L_2, p is located at $y(\lambda)$ and the one joining p to the defect is located at $y(\lambda')$. With the light operator dimensions large, S is large, and the leading behavior of the integral in (4.4.2) is dominated by the immediate neighborhood of the point (λ, λ') that minimizes S. Therefore

$$\mathcal{W} \propto e^{-2S_{\text{geo}}[h_{L_1}, h_{L_2}, h_p]} , \tag{4.4.3}$$

with S_{geo} found by minimizing the worldline action (4.4.1) with respect to the positions of the two cubic vertices, but now with both vertices constrained to lie on their respective geodesics.

Clearly S_{geo} and S_{free} are different (and $S_{\text{geo}} > S_{\text{free}}$). Nevertheless, the two prescriptions $W \propto e^{-S_{\text{free}}}$ and $W \propto e^{-S_{\text{geo}}}$ are in fact the same up to overall normalization, because the difference between S_{free} and S_{geo} is a constant, independent of the operator locations, as we will now show.

4.4.1 Equivalence of the minimization prescriptions

Agreement between the prescriptions follows from the following two observations. Here h is some positive number:

(a) $S_{\text{free}}[h_{L_1},\, h_{L_2},\, h_p] = S_{\text{free}}[h_{L_1}+h,\, h_{L_2}+h,\, h_p] - S_{\text{free}}[h,\, h,\, 0]$, up to a constant.

(b) $S_{\text{geo}}[h_{L_1},\, h_{L_2},\, h_p] = S_{\text{free}}[h_{L_1}+h,\, h_{L_2}+h,\, h_p] - S_{\text{free}}[h,\, h,\, 0]$ in the limit $h \gg h_{L_1}, h_{L_2}, h_p$.

It is easy to see that (a) and (b) together imply $S_{\text{free}} = S_{geo}$ up to a constant, as desired.

We work on the cylinder. Property (a) can be read off from the expression obtained in [1] for S_{free} as a function of the separation w_{12} between the light external operators:

$$\begin{aligned} S_{\text{free}}[h_{L_1},\, h_{L_2},\, h_p] = {} & (h_{L_1}+h_{L_2}) \log \sin \frac{\alpha w_{12}}{2} + h_p \operatorname{arctanh} \frac{h_p \cos \frac{\alpha w_{12}}{2}}{\sqrt{h_p^2 - (h_{L_2}-h_{L_1})^2 \sin^2 \frac{\alpha w_{12}}{2}}} \\ & - |h_{L_2}-h_{L_1}| \log\left(|h_{L_2}-h_{L_1}| \cos \tfrac{\alpha w_{12}}{2} + \sqrt{h_p^2 - (h_{L_2}-h_{L_1})^2 \sin^2 \tfrac{\alpha w_{12}}{2}} \right) \\ & + \text{constant}\ . \end{aligned} \tag{4.4.4}$$

Only the first term and the constant change upon substituting $h_{L_{1,2}} \to h_{L_{1,2}} + h$ and the change in the first term is precisely $S_{\text{free}}[h,\, h,\, 0]$.

Proceeding now to prove (b), we start from the fact that when h is much larger than h_{L_1}, h_{L_2}, h_p the function

$$S = (h_{L_1}+h)L_{L_1} + (h_{L_2}+h)L_{L_2} + h_p L_p \tag{4.4.5}$$

is minimized when the total length of worldlines L_1 and L_2 is as small as possible, i.e. when their union is a geodesic. The location of the vertex is then found by minimizing S subject to that constraint. Therefore in the limit $h \gg h_{L_1}, h_{L_2}, h_p$

$$S_{\text{free}}[h_{L_1}+h,\, h_{L_2}+h,\, h_p] = S_{\text{geo}}[h_{L_1}+h,\, h_{L_2}+h,\, h_p]\ . \tag{4.4.6}$$

Now, the position of the intersection vertex that gives S_{geo} depends on the light operator dimensions only through their difference, and a shift of both dimensions by the same amount h merely shifts S_{geo} by $h(L_{L_1} + L_{L_2})$. Thus equation (4.4.6) is equivalent to

$$S_{\text{free}}[h_{L_1} + h,\, h_{L_2} + h,\, h_p] = S_{\text{geo}}[h_{L_1},\, h_{L_2},\, h_p] + h(L_{L_1} + L_{L_2}) \ , \tag{4.4.7}$$

and (b) follows from the fact that $S_{\text{free}}[h,\ h,\ 0] = h(L_{L_1} + L_{L_2})$.

4.5 Final comments

We conclude with a few remarks.

At a purely technical level, one aspect of our scalar field computation that could be improved would be to relax the reality condition on w. This would allow us to cleanly separate the individual chiral blocks from their product. This is straightforward in principle, but it turns out to be technically challenging to evaluate the resulting integrals in this case. We also mentioned some technical subtleties with our higher spin calculation in the main text.

Moving into more novel territory, our techniques may be combined with gravitational perturbation theory to derive new results away from the strict limits considered so far. For instance, the semiclassical heavy-light Virasoro block is the leading term in a $1/c$ expansion of the exact Virsaoro block expanded around the limit (4.2.9). These $1/c$ corrections can be worked out explicitly in a power series expansion in $1 - z$ using Zamolodchikov's recursion relation, or the more efficient recursion relation of [105] adapted to the heavy-light limit specifically. See [1] for some explicit results, and [167] for closed-form, albeit complicated, expressions for coefficients at any order in $1/c$. These results should correspond to quantum fluctuations of the background geometry. It would be interesting to try to reproduce these from a bulk analysis.

Similarly, it would also be interesting to see how the simple relation between the global and Virasoro blocks is modified at subleading orders in $1/c$, in the global limit of large c with dimensions fixed. This may be computed in the bulk by incorporating graviton loop corrections to the AdS_3 geodesic Witten diagram.

It would be natural to generalize the heavy-light limit to CFTs with W-symmetry. Semi-classical W_N conformal blocks for vacuum exchange have been computed in [60] with all charges scaling with c in some manner; it would be useful to loosen that requirement.

An important open question in the world of Virasoro blocks is whether there is a compact form for the semiclassical Virasoro block where all operator dimensions scale linearly with c. This is the limit usually considered in the context of Liouville theory. Whatever the answer for this block, the expectation is that its bulk dual involves a spacetime with interacting conical defects, not unlike a multi-centered black hole solution. This connection can be seen via the correspondence between Zamolodchikov's monodromy equations and the Einstein equations expressed in Chern-Simons form; see e.g. [1, 60]. Understanding this picture in detail, and what it implies for various questions in CFT – e.g. two-interval Rényi entropies [64, 65] – would be very interesting.

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