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Journal
Physics of Fluids, 28(8)

ISSN
00319171

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Publication Date
1985

DOI
10.1063/1.865262

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Magnetic damping of the $\ell = 1$ diocotron mode

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(Received 29 October 1984; accepted 8 May 1985)

Previous work has treated the diocotron mode for a low-energy beam and has found that any dissipative effect will cause this mode to grow. However, a treatment that considers the axial velocity of the beam shows that this mode will damp if the beam is relatively thin and the axial velocity of the beam is comparable with c. This damping is attributed to the perturbed magnetic field and is similar to the stabilization of sausage and kink modes in neutral plasmas. The damping rate for this case is compared with the damping observed in the University of California, Irvine (UCI) modified Betatron. Additionally, beam temperature effects are considered and these results are compared with the diocotron mode damping in the collective focusing ion accelerator.

I. INTRODUCTION

The $l = 1$ (see Fig. 1) diocotron mode describes the oscillations of a beam in a cylindrical chamber. The modes with $l > 1$ are not observed, probably because of Landau damping,[1] so they will not be considered. Previous treatments of the diocotron mode have shown that it has negative energy.[1,2] If the mode energy is negative any power dissipation will cause the energy to decrease, and therefore the wave amplitude will grow. These treatments, though, are only valid for beams with negligible kinetic energy. In this paper the effects of axial beam velocity (v_z) and beam rotational velocity (v_θ) are examined. When the axial beam velocity (v_z) is considered, it can be shown that the mode will damp if $(v_z/c)^2 > (a/b)^2$, where a is the beam radius and b is the radius of the chamber. For a reasonably thin and axially accelerated beam, damping should take place. The damping is attributed to a magnetic field created by the axial beam motion (see Fig. 2) which can make the mode energy positive. When the mode energy is positive, dissipative effects cause damping of the mode. For a good but not perfect conducting boundary, the magnetic field will also cause a dissipation since it induces currents in the wall and creates an I^2R power loss. The damping rate when $(v_z/c)^2 \gg (a/b)^2$ is shown to be $2\omega(a/b)^2(\delta/b)$, where $\delta = (c^2/2\pi\sigma\omega)^{1/2}$ is the skin depth. In the modified Betatron[3] electrostatic probes show results consistent with this damping.

The diocotron mode is also examined for a rotating beam. It is expected that for a finite length beam the beam rotation may create an axial magnetic field (see Fig. 3) that is sufficient to also cause magnetic damping. To explore the possibility of magnetic damping of the diocotron mode because of beam rotation, two equilibria will be examined. A "warm" equilibrium is examined in Sec. III. This analysis is done based on a rigid rotor Vlasov equilibrium with a Gibbs distribution. This equilibrium is expected to be the equilibrium into which a warm beam will evolve. With this equilibrium it is shown that the magnetic field cannot cause the mode to damp. Additionally, an equilibrium with velocity shear is postulated in Sec. IV, and it is shown that with this equilibrium the diocotron mode can be magnetically damped. Both of these results will be shown to be consistent with the electrostatic probe measurements in the collective focusing ion accelerator (CFIA).[4–6]

II. DAMPING OF THE DIOCOTRON MODE FOR AN AXIALLY MOVING BEAM

It has been shown that the $l = 1$ diocotron mode energy of a low-energy beam is negative, so any dissipation will cause growth. This calculation is based on only the electrostatic fields. However, when a beam is accelerated axially there will be a poloidal magnetic field (see Fig. 2). If the beam moves off center it will compress the magnetic field and create a perturbed field energy. This magnetic field energy can make the energy positive, and the result can be shown quickly. It is expected that the axially moving beam will have a perturbed magnetic field given by $\delta \mathbf{B} = \mathbf{v} \times \delta \mathbf{E}/c$ so $|\delta \mathbf{B}| \approx |(v_z/c)\delta \mathbf{E}|$. Since the electrostatic energy of the diocotron mode is[1,2]

$$W_{es} \cong -\frac{1}{16\pi}\left(\frac{a}{b}\right)^2 \int r\,dr\,d\theta\,\delta \mathbf{E}^2, \tag{1}$$

where a is the beam radius, b is the radius of the chamber, and n is the beam density; if $\int r\,dr\,\delta \mathbf{B}^2 > (a/b)^2 \int r\,dr\,\delta \mathbf{E}^2$, it is expected that the diocotron mode energy will be positive. Since $\delta \mathbf{B}^2 \approx (v_z/c)^2\,\delta \mathbf{E}^2$, it is expected that if $(v_z/c)^2 > (a/b)^2$, then the mode will have positive energy and any dissipation will cause damping. Additionally, the magnetic field also creates a dissipation since it induces currents in the wall. Thus it is expected that for a beam where $v_z \cong c$, a small ratio of beam to chamber radius (a/b) will cause the diocotron mode to damp.

To show this effect rigorously, consider the equilibrium magnetohydrodynamic (MHD) equations for a beam:

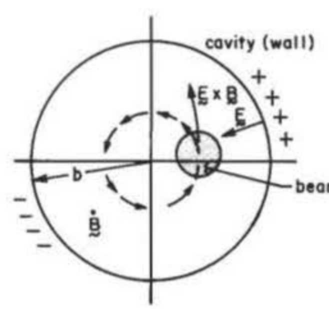

FIG. 1. The $l = 1$ diocotron mode. This mode is caused by the $\mathbf{E} \times \mathbf{B}$ drift from the action of the charge induced on the wall and the toroidal magnetic field.

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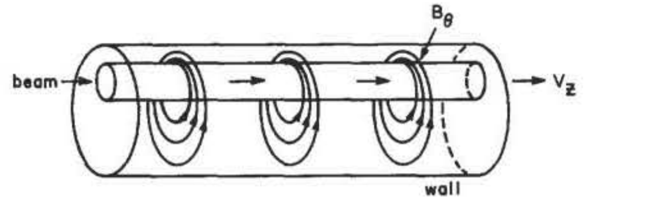

FIG. 2. Magnetic field caused by axial beam motion. This magnetic field will be perturbed when the beam moves and is expected to increase the diocotron mode energy.

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(\gamma\mathbf{v}) + v\cdot\nabla(\gamma\mathbf{v}) + \frac{\nabla p}{mn} = \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right). \tag{3}$$

Also consider a general set of perturbed MHD equations that will be used for all the calculations in this paper. In these equations assume $\mathbf{v} = \omega_0(r)r\hat{\theta} + v_z\hat{z}$, $p = p(r)$, $n = n(r)$, all perturbed and equilibrium quantities are independent of z, $\gamma = [1 - (v_z/c)^2]^{-1/2}$, all perturbations are of the form $\exp(-i\omega t + il\theta)$, the coordinates are defined in Fig. 4, and $k_z = 0$.

$$-i(\omega - l\omega_0)\delta n + \delta v_r \frac{\partial n}{\partial r} + n\nabla\cdot\delta\mathbf{v} = 0, \tag{4}$$

$$-i(\omega - l\omega_0)\delta v_r - 2\omega_0\delta v_\theta + \frac{1}{\gamma mn}\frac{\partial\delta p}{\partial r} - \frac{1}{\gamma mn^2}\frac{\partial p}{\partial r}\delta n = (e/\gamma m)\delta[\mathbf{E} + (\mathbf{v}\times\mathbf{B})/c]\cdot\hat{r}, \tag{5}$$

$$-i(\omega - l\omega_0)\delta v_\theta + \left(2\omega_0 + r\frac{\partial\omega_0}{\partial r}\right)\delta v_r + \frac{il\delta p}{\gamma mnr} = (e/\gamma m)\delta[\mathbf{E} + (\mathbf{v}\times\mathbf{B})/c]\cdot\hat{\theta}, \tag{6}$$

$$-i(\omega - l\omega_0)\delta v_z = (e/m\gamma^3)\delta[\mathbf{E} + (\mathbf{v}\times\mathbf{B})/c]\cdot\hat{z}. \tag{7}$$

These equations are augmented by the perturbed Maxwell's equations:

$$\delta B_r = (lc/\omega r)\delta E_z, \tag{8}$$

$$\delta B_\theta = \frac{ic}{\omega}\frac{\partial}{\partial r}\delta E_z, \tag{9}$$

$$\delta B_z = -\frac{ic}{\omega r}\frac{\partial}{\partial r}(r\delta E_\theta) - \frac{lc}{\omega r}\delta E_r, \tag{10}$$

$$\frac{i\omega}{c}\delta E_r = -\frac{il}{r}\delta B_z + \frac{4\pi ne}{c}\delta v_r, \tag{11}$$

$$\frac{i\omega}{c}\delta E_\theta = \frac{\partial}{\partial r}(\delta B_z) + \frac{4\pi e}{c}v_\theta\delta n + \frac{4\pi ne}{c}\delta v_\theta, \tag{12}$$

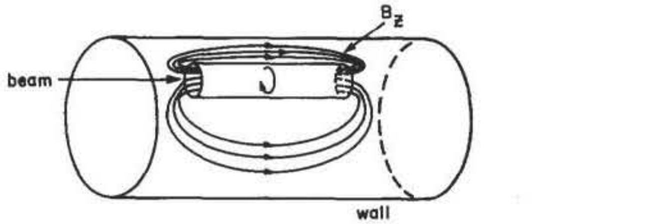

FIG. 3. Magnetic field caused by beam rotation. This magnetic field will be perturbed when the beam moves and is expected to influence the diocotron mode energy.

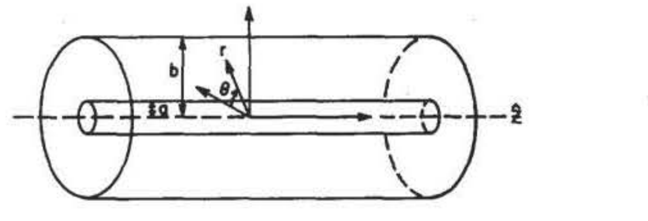

FIG. 4. Coordinate system used for all the calculations.

$$\frac{i\omega}{c}\delta E_z = -\frac{1}{r}\frac{\partial}{\partial r}(r\delta B_\theta) + \frac{il}{r}\delta B_r + \frac{4\pi ne}{c}\delta v_z + \frac{4\pi e}{c}v_z\delta n, \tag{13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\delta E_r) + \frac{il}{r}\delta E_\theta = 4\pi e\delta n, \tag{14}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\delta B_r) + \frac{il}{r}\delta B_\theta = 0. \tag{15}$$

For the axially moving beam case, assume a cold, constant density, cylindrical beam in a cylindrical chamber with a large axial magnetic field. In this case it can be shown that $v_\theta = \omega_0 r = -(\omega_p^2/2\Omega_z)[1 - (v_z/c)^2]\,r$, $\mathbf{E} = 2\pi ner\hat{r}$, $\mathbf{B} = B_z\hat{z} + 2\pi ner(v_z/c)\hat{\theta}$, $p = 0$, $\delta p = 0$, $\omega_p^2 = 4\pi ne^2/(\gamma m)$, and $\Omega_z = eB_z/(\gamma mc)$. Furthermore, assume $\omega_p^2 \ll \Omega_z^2$ so $\omega_0^2 \ll \Omega_z^2$. Assuming that $\omega_0\omega r^2/c^2 \ll 1$, Eq. (10) indicates

$$\delta E_r \cong -\frac{i}{l}\frac{\partial}{\partial r}(r\delta E_\theta). \tag{16}$$

Using the above, Eqs. (13) and (14) will provide two coupled differential equations for δE_z and δE_θ:

$$\nabla^2\delta E_z \cong -\frac{i4\pi ne\omega}{c^2}\delta v_z - \frac{4\pi e\omega}{(\omega - l\omega_0)c}\left(\frac{v_z}{c}\right)\frac{\partial n}{\partial r}\delta v_r, \tag{17}$$

$$\nabla^2(r\delta E_\theta) \cong -[il^2\omega_0/c(\omega - l\omega_0)]4\pi ne(v_z/c)\delta v_z + \frac{4\pi le}{(\omega - l\omega_0)}\frac{\partial n}{\partial r}\delta v_r. \tag{18}$$

Assuming $\omega_p^2 r^2[(a/b)^2 + (l-1)]/l \ll c^2$, then the above equations indicate that

$$\delta E_z \cong -(\omega/c)(v_z/c)(r/l)\delta E_\theta. \tag{19}$$

This allows $\delta\mathbf{B}$, $\delta\mathbf{v}$, and δn to be determined:

$$\delta B_r \cong -(v_z/c)\delta E_\theta, \tag{20}$$

$$\delta B_\theta \cong (v_z/c)\delta E_r, \tag{21}$$

$$\delta B_z \approx (\omega_0 r/c)\delta E_r, \tag{22}$$

$$\delta v_r \cong (e\delta E_\theta/m\Omega_z)[1 - (v_z/c)^2], \tag{23}$$

$$\delta v_\theta \cong -\frac{e}{m}\frac{\delta E_r}{\Omega_z}\left[1 - \left(\frac{v_z}{c}\right)^2\right], \tag{24}$$

$$\delta v_z \cong -\frac{ie}{l\gamma^3(\omega - l\omega_0)m}\left(\frac{\omega r}{c}\right)\left(\frac{v_z}{c}\right)\delta E_\theta, \tag{25}$$

$$\delta n \cong -\frac{i\delta v_r}{(\omega - l\omega_0)}\frac{\partial n}{\partial r}. \tag{26}$$

Utilizing the previous expansion, $\omega_p^2 r^2[(a/b)^2 + (l-1)]/l \ll c^2$, the following equivalent form of Eq. (18) is obtained:

$$\nabla^2(r\delta E_\theta) \cong -\frac{2l\omega_0}{(\omega - l\omega_0)}\frac{1}{rn}\frac{\partial n}{\partial r}(r\delta E_\theta). \tag{27}$$

Outside the beam $\nabla^2(r\delta E_\theta)=0$. The above eigenvalue equation can be solved by using the boundary conditions $r\delta E_\theta$ is finite when $r\to 0$, $r\delta E_\theta=0$ when $r=b$, and $(1/n)(\partial n/\partial r)=-\delta(r-a)$. This yields the following dispersion relation:

$$-\frac{2\omega_0}{(\omega-l\omega_0)}=\frac{2}{[1-(a/b)^2]}$$

or

$$\omega=\omega_0(l-1)+\omega_0\left(\frac{a}{b}\right)^2=-\frac{2Nec}{B_z b^2}\left[1-\left(\frac{v_z}{c}\right)^2\right], \tag{28}$$

where $l=1$ has been used and $N=n\pi a^2$ is the line density of the beam. The fields can also be determined:

$$\delta E_\theta=\begin{cases}A, & r\leqslant a,\\ A\dfrac{[1-(b/r)^2]}{[1-(b/a)^2]}, & r\geqslant a,\end{cases} \tag{29}$$

where A is a constant. Using the above quantities the energy can be obtained. The sign of the energy will determine if dissipative effects will cause the diocotron mode to grow or damp. The wave energy density (w) is given by the following expression:

$$\frac{\partial w}{\partial t}=\frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right). \tag{30}$$

Assuming that the magnetic permittivity is unity, then Fourier decomposition of the above expression will yield the following[7]:

$$W=\frac{1}{16\pi}\int r\,dr\,d\theta\,(\delta\mathbf{B}^*\cdot\delta\mathbf{B})+\frac{1}{16\pi}\int r\,dr\,d\theta\left(\delta\mathbf{E}^*\cdot\frac{\partial}{\partial\omega}(\omega\boldsymbol{\epsilon})\cdot\delta\mathbf{E}\right), \tag{31}$$

where $\boldsymbol{\epsilon}$ is the dielectric tensor. The dielectric tensor is given by the following[8]:

$$\boldsymbol{\epsilon}=\mathbf{I}-\frac{4\pi}{i\omega}\boldsymbol{\sigma},$$

where

$$\delta\mathbf{J}=\boldsymbol{\sigma}\cdot\delta\mathbf{E}. \tag{32}$$

Since $\delta E_z \ll \delta E_\theta$ and δE_r, only the perpendicular dielectric tensor needs to be determined for the energy calculation. Using the above equations $\boldsymbol{\sigma}$ can be determined:

$$-\frac{4\pi}{i\omega}\boldsymbol{\sigma}=\begin{bmatrix}0, & -\dfrac{2i\omega_0}{\omega}\\ \dfrac{2i\omega_0}{\omega}, & -\dfrac{2\omega_0^2\, r(\partial n/\partial r)}{\omega(\omega-l\omega_0)n}\end{bmatrix}. \tag{33}$$

The electric field energy can be determined from Poisson's equation by multiplying it by $r^2\,\delta E_\theta^*$ and integrating over θ and r:

$$\frac{1}{16\pi}\int r\,dr\,d\theta\,(\delta E_r^2+\delta E_\theta^2)=\frac{1}{8\pi}\int r\,dr\,d\theta\,\frac{\omega_0 r\delta E_\theta^2}{l(\omega-l\omega_0)n}\frac{\partial n}{\partial r}. \tag{34}$$

The magnetic field energy can be found in a similar way:

$$\frac{1}{16\pi}\int r\,dr\,d\theta\,(\delta B_r^2+\delta B_\theta^2)=\frac{1}{8\pi}\left(\frac{v_z}{c}\right)^2\int r\,dr\,d\theta\frac{\omega_0 r\delta E_\theta^2}{l(\omega-l\omega_0)n}\frac{\partial n}{\partial r}. \tag{35}$$

Using the above equations the wave energy can be determined:

$$W\cong-\frac{1}{8\pi}\left[\left(\frac{v_z}{c}\right)^2-\left(\frac{a}{b}\right)^2\right]\int r\,dr\,d\theta\,\frac{r}{n}\frac{\partial n}{\partial r}\delta E_\theta^2, \tag{36}$$

where a is the beam radius, b is the radius of the beam chamber, and n is the equilibrium density. Since the beam density decreases with r, then $\partial n/\partial r<0$, the above equation shows that if $(v_z/c)^2>(a/b)^2$ the diocotron mode has a positive energy. It is easy to see that in betatron-type accelerators the diocotron mode will have positive energy once the beam has reasonably been accelerated. The growth (or decay) rate of the diocotron mode can be determined from the power loss by using the following:

$$\Gamma=-P/2W, \tag{37}$$

where W is the wave energy and P is the power loss.

If the walls of the cavity have resistance, then a magnetic field tangential to the wall will induce wall currents and create a power loss. This power loss is well known and is the following[9]:

$$P=\frac{\omega\delta}{16\pi}\int d\theta\, b\delta B_\theta^2\bigg|_{r=b}=\frac{\omega\delta b}{8}\delta B_\theta^2\bigg|_{r=b}, \tag{38}$$

where $\delta=(c^2/2\pi\sigma\omega)^{1/2}$ is the resistive skin depth. Using the above equations the growth rate resulting from resistive losses can be shown to be the following:

$$\Gamma\cong-\frac{2\omega(a/b)^2(\delta/b)}{[1-(a/b)^2(c/v_z)^2]}. \tag{39}$$

Other types of power dissipation will create growth or damping in accord with Eq. (37).[2] The above growth rate shows that, if the ratio of beam to wall radius (a/b) is such that $(a/b)^2<(v_z/c)^2$, then there will be damping of the diocotron mode. Otherwise, growth is expected.

As stated previously, electrostatic probes show evidence for the decay of the diocotron mode in the modified betatron.[3] In Fig. 5 a cutaway view of the UCI modified betatron accelerator is shown. The modified betatron is an electron beam accelerator that is basically a betatron with a large

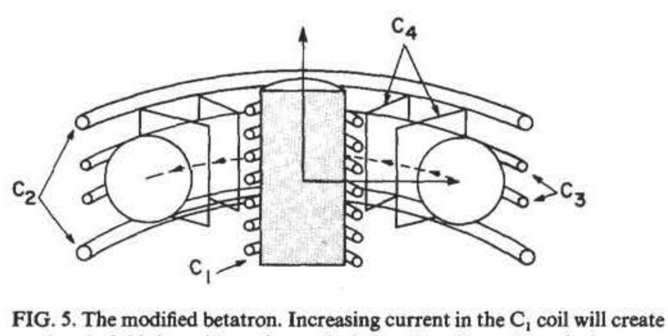

FIG. 5. The modified betatron. Increasing current in the C_1 coil will create an electric field that will accelerate the beam. C_2 coils produce the betatron field which, in a conventional betatron, will balance the beam's centrifugal force; C_3 coils can trim the betatron field; C_4 coils produce the toroidal magnetic field.

toroidal magnetic field. In this experiment and CFIA, electrostatic probes were used to monitor the beam oscillations. The electrostatic probes measure the electric potential between the probes' plate and the chamber wall. A reproduction of a typical electrostatic probe measurement is shown in Fig. 6. To interpret the electrostatic probe measurement, the method of injection must be considered. The beam is injected by the method of inductive charging.[3,10] When the injector is fired it is believed that this causes a noise spike that shows up in the probe measurements. The beam is believed to be initially hollow[11,12] and therefore subject to the hollow beam diocotron instability.[13,14] This instability probably causes the initial growth in the diocotron mode. The damping of the mode can be attributed to the above calculated magnetic damping. The damping appears after the beam is accelerated and $(v_z/c)^2 > (a/b)^2$. Although not all the quantities needed to calculate the damping rate are known, an approximate damping rate can be found by assuming values that the experiment indicates are reasonable. Specifically, Eq. (39) predicts a damping rate of about 20 μsec assuming $\omega = 6\times 10^6$ sec^{-1}, $\sigma = 10^{15}$ sec^{-1}, and $(a/b)^2 = \frac{1}{8}$. This damping rate is consistent with the observed damping.

In the modified betatron, the diocotron beam motion can only be loosely related to the probe measurements since during the measurements the beam equilibrium moves toward the outer wall[15] and therefore the probe and xrays indicate that some electrons are lost throughout the measurement. Calculations show that effects from the rising axial magnetic field and electron loss should not be large enough to account for the observed decay in amplitude, especially since the measured probe amplitude should experience some increase when the beam approaches and eventually collides with the probe wall. Also, there is evidence to indicate the existence of a background of nonaccelerated electrons in addition to the accelerated beam.[15] These background electrons may cause other damping effects.[16]

III. DIOCOTRON MODE ENERGY IN A WARM ROTATING BEAM

Before proceeding with the calculation of the diocotron mode energy for a warm rotating beam, it is advantageous to describe the collective focusing ion accelerator (CFIA) since several approximations in the calculation are motivated by it. Experiments were made with CFIA[6] where unneutralized electrons were injected into a mirror and accelerated by increasing the mirror field. Figure 7 shows a diagram of the experiment. Since only one cell of the accelerator was used, the experiment was equivalent to a mirror. In this experiment electrostatic probes were also used to measure the beam oscillations. During the beam lifetime the mirror (toroidal) magnetic field was increased by a factor of about 50, thereby causing an increase in v_θ and a compression of the beam radius. Like the betatron, the method of injection was inductive charging. Electrostatic probe measurements (see Fig. 8) indicate that the diocotron mode initially grew, then damped after acceleration.[6] Like the betatron the initial growth is probably caused by the hollow beam instability. Unlike the betatron the damping cannot be attributed to the magnetic field from the axial beam motion since $(v_z/c)^2 \approx 0$. However, the poloidal acceleration will increase the beam's rotational velocity which, in a similar manner, may cause magnetic damping. It is easy to see in Fig. 3 that an axial

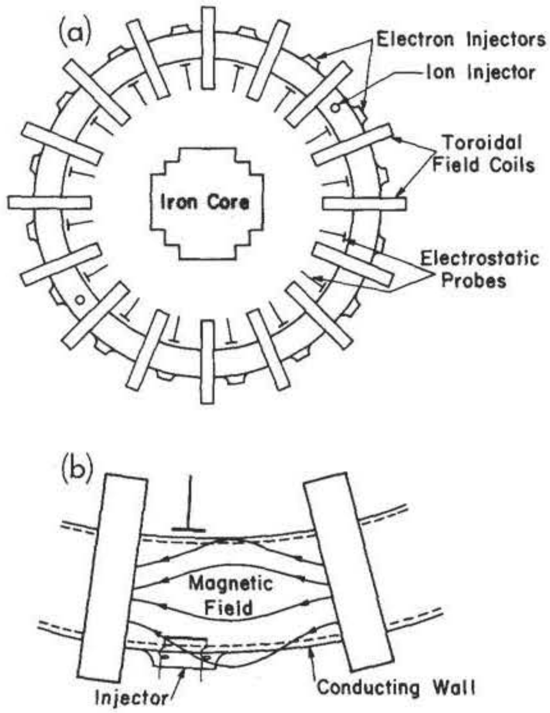

FIG. 7. The collective focusing ion accelerator: (a) shows a top view of the entire accelerator: (b) shows a cell of the accelerator, which is equivalent to a mirror. In the experiment indicated in this paper, only one cell was used and no ions were introduced.

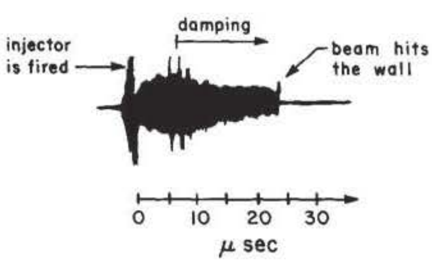

FIG. 6. Envelope of a typical electrostatic probe measurement in the modified betatron.

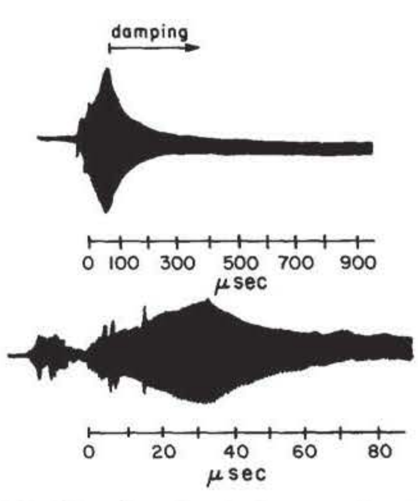

FIG. 8. Envelope of a typical electrostatic probe measurement in CFIA.

magnetic field from the beam rotation will act like the previously described poloidal magnetic field by causing a compression of the magnetic field lines as the beam moves away from the center of the chamber. Like the axial motion case, the magnetic field from the beam rotation should oppose off-center motion of the beam and add energy to the diocotron mode. However, an in depth calculation must be used to demonstrate that the magnetic field will cause damping in the presence of pressure/temperature effects.

First, consider a cold "rigid rotor." In a cold beam without angular velocity shear, the angular velocity must be close to either $\omega_- \equiv -\omega_p^2/2\Omega_z$ or $\omega_+ \equiv -\Omega_z$. These accelerators would be expected to have an angular velocity ω_-. In this case it can be shown that $|\delta \mathbf{B}| \approx |(\omega_- r/c)\delta\mathbf{E}|$. As was shown in Sec. II, if this perturbed magnetic field is large enough to make the diocotron mode energy positive, the mode should damp. Since the energy without a perturbed magnetic field is

$$W_{es} = -\frac{1}{16\pi}\left(\frac{a}{b}\right)^2 \int r\,dr\,d\theta\,\delta\mathbf{E}^2, \tag{40}$$

then if $(\omega_- a/c)^2 > (a/b)^2$ it would be expected that the energy would be positive. In CFIA, $\omega_- \approx 10^8\ \mathrm{sec}^{-1}$, $b^2 = 25\ \mathrm{cm}^2$, and $a^2 \approx 1\ \mathrm{cm}^2$, so the above is not satisfied. However, in this experiment the beam is accelerated so it is not expected that the cold beam treatment will apply since the particles have a large transverse energy.

To indicate how the beam rotational velocity is changed by temperature consider the rigid rotor Vlasov equilibrium for a relatively cold constant density beam[17,18]:

$$f_0 = (mn_0/2\pi)\delta(H - \omega_0 P_\theta - \tfrac{1}{2} m v_\perp^2), \tag{41}$$

where H is the energy and P_θ is the canonical angular momentum, which are both constants of motion,

$$H = \tfrac{1}{2} m [v_r^2 + v_\theta^2 + (\omega_p^2/2) r^2], \tag{42}$$

$$P_\theta = rm(v_\theta + r\Omega_z/2). \tag{43}$$

Using the above the pressure can be found[17,18]:

$$\tfrac{1}{2} mnv_\perp^2\ [1 - (r/a)^2]. \tag{44}$$

Using the pressure, the equilibrium MHD equation [Eq. (3)], and $\omega_0^2 \ll \Omega_z^2$, an expression for ω_0 is then obtained:

$$\omega_0 \cong \omega_- - v_\perp^2/a^2\Omega_z. \tag{45}$$

If the effect of the perpendicular energy in the above is small, then $\omega_0 \cong \omega_-$ and the beam can be considered cold. However, in CFIA, $a \approx 1$ cm, $v_\perp^2 \approx \frac{1}{2}c^2$, $B_z \approx 10^4$ G, and $n \approx 10^{10}$ cm^{-3}, so $v_\perp^2/a^2\Omega_z\omega_- \approx 30$ and ω_0 is not close to ω_-. Specifically, since

$$v_\perp^2 \gg a^2\,\omega_p^2, \tag{46}$$

the beam must be considered warm in a rigid rotor treatment. To account for the beam temperature, consider the following Vlasov equilibrium[17,18]:

$$f_0 = \frac{n_0 m}{2\pi\Pi} \exp\left(-\frac{(H - \omega_0 P_\theta)}{\Pi}\right). \tag{47}$$

This equilibrium is consistent with a warm beam without velocity shear. The equilibrium density is given by the following:

$$n(r) = n_0 \exp\{-(m/2\Pi)[r^2(\omega_0\Omega_z + \omega_0^2) + 2e\phi/m]\}. \tag{48}$$

Since the above is a function of the electrostatic potential ϕ and the equilibrium fields must be solved by using Poisson's equation, the fields will not have a closed form solution.[18] However, by examining the inequality in Eq. (46), it is expected that the quantity ϕ appearing in the exponential can be treated as a small perturbation, since

$$|4e\phi/r^2 m| < \omega_p^2 \ll \omega_0\Omega_z \cong v_\perp^2/a^2, \tag{49}$$

where $\omega_p^2 \equiv 4\pi n_0 e^2/m$. Approximating the small quantity ϕ as $\pi n_0 e r^2$, the above relation can then be verified since the line density $N \equiv n_0\pi a^2$ shows

$$N = \int_0^{2\pi} d\theta \int_0^\infty r\,dr\,n(r) \equiv n_0\pi a^2$$
$$\cong -\frac{n_0 2\pi\Pi}{m(\Omega_z\omega_0 + \omega_0^2 + \omega_p^2/2)}. \tag{50}$$

Since $\Pi = \frac{1}{2} m v_\perp^2$ and assuming $\Omega_z^2 \gg \omega_p^2$, the above indicates

$$\omega_0 \cong -v_\perp^2/\Omega_z a^2 + \omega_-, \tag{51}$$

as was anticipated. In this equilibrium the pressure can also be shown to be the following:

$$p = m\int dv_r \int dv_\theta\,[v_r^2 + (v_\theta - \omega_0 r)^2] f_0 = \Pi n(r). \tag{52}$$

An equation of state relating p and n must also be found before proceeding with an MHD calculation. Equation (52) indicates $p/n = \text{const} = \Pi$. Also, the ratio of δp and δn can be found by determining the velocity dependence of δf. This can be found from integrating the Vlasov equation:

$$\delta f = \frac{e}{m}\int_{-\infty}^0 d\tau\, e^{-i\omega\tau} e^{-il(\theta' - \theta)} \delta\mathbf{E}' \cdot \frac{\partial f_0}{\partial \mathbf{v}'}, \tag{53}$$

where $\tau \equiv t' - t$ and all primed variables are evaluated at t'. For a rigid rotor the above equation leads to the following perturbed distribution[19]:

$$\delta f = e\frac{\partial f}{\partial Q} I(r), \tag{54}$$

where

$$I(r) = ir\delta E_\theta + (\omega - l\omega_0)\int_{-\infty}^0 d\tau$$
$$\times e^{-i\omega\tau} e^{-il(\theta' - \theta)} r' \delta E'_\theta,$$
$$Q = (H - \omega_0 P_\theta).$$

The velocity dependence of δf is set by the above equation. Using the above it can be shown that

$$\frac{\delta p}{\delta n} \equiv \frac{m \int d\mathbf{v}\,[v_r^2 + (v_\theta - \omega_0 r)^2](\partial f/\partial Q)}{\int d\mathbf{v}(\partial f/\partial Q)} = \Pi. \tag{55}$$

Therefore, $p/n = \text{const}$ is a valid equation of state. This is expected since the magnetic moment is expected to be constant. Assuming this equation of state, $\delta B_r = \delta B_\theta = \delta v_z = 0$, and $\omega_0^2 \ll \Omega_z^2$, Eqs. (4)–(7) show that δn is the following:

$$\delta n \cong -\frac{(ie/m)\delta E_\theta(\partial n/\partial r)}{\Omega_z[(\omega - l\omega_-) - (\omega - l\omega_0)\beta]}, \tag{56}$$

where $\beta = 4\pi p/B_z^2$ is the plasma beta. For CFIA, β is $\approx 10^{-4}$, so δn is approximately the following:

$$\delta n \cong \frac{-(ie/m)\delta E_\theta(\partial n/\partial r)}{\Omega_z(\omega - l\omega_-)}. \tag{57}$$

Also, $\delta\,\mathbf{J}$ is the following:

$$\delta\,\mathbf{J} = \left\{e\omega_0 r\delta n - \frac{\omega_p^2}{4\pi\Omega_z}\left[\delta E_r - \frac{\Pi}{e}\frac{\partial}{\partial r}\left(\frac{\delta n}{n}\right)\right]\right\}\hat{\theta} + \frac{\omega_p^2}{4\pi\Omega_z}\left(\delta E_\theta - \frac{il\Pi}{e}\frac{\delta n}{nr}\right)\hat{r}, \tag{58}$$

since Eqs. (57) and (16) show

$$\frac{\partial}{\partial r}\left(\frac{\delta n}{n}\right) \cong \frac{2el\delta E_r}{m\Omega_z(\omega - l\omega_-)a^2}, \tag{59}$$

then $\delta\,\mathbf{J}$ can be expressed as the following:

$$\delta\mathbf{J} = \left(-\frac{i\omega_-\omega_0 r\delta E_\theta}{(\omega - \omega_-)2\pi n}\frac{\partial n}{\partial r} + A\delta E_r\right)\hat{\theta} - A\delta E_\theta\hat{r}, \tag{60}$$

where

$$A = (\omega_-/4\pi)[1 - lv_\perp^2/\Omega_z(\omega - l\omega_-)a^2].$$

The plasma dielectric can be found from the above. This leads to the following energy:

$$W = -\frac{1}{16\pi}\left(\frac{\omega_0}{\omega_-}\right)\int d\theta\, r\, dr\, \delta\mathbf{E}^2 + \frac{1}{8\pi}\int d\theta\, r\, dr\, \delta\,\mathbf{B}_z^2, \tag{61}$$

where the expression for $\delta\,\mathbf{E}^2$ was obtained from integrating Poisson's equation. In the above it should be noted that the electrostatic energy is negative. Here $|\delta\,\mathbf{B}_z|$ can be obtained from Eq. (11) and is $\approx |(\omega_0 a/c)\delta\mathbf{E}|$. In order for the perturbed magnetic field to cause the above energy to be positive,

$$\left(\frac{\omega_0 a}{c}\right)^2 \approx \left(\frac{v_\perp}{c}\frac{a_c}{a}\right)^2 > \left(\frac{\omega_0}{\omega_-}\right),$$

where $a_c = v_\perp/\Omega_z$. Clearly $|v_\perp/c| < 1$, $|a_c/a| < 1$, and $\omega_0 > \omega_-$, so this is not true and the energy is negative. The growth rate is expected to be given by the following:

$$\Gamma = -P/2W, \tag{62}$$

where P is the power loss from the axial magnetic field. This power loss is given by the following[9]:

$$P = \frac{\omega\delta}{16\pi}\int d\theta\, b\delta B_z^2\bigg|_{r=b} = \frac{\omega\delta b}{8}\delta B_z^2\bigg|_{r=b}, \tag{63}$$

where $\delta = (c^2/2\pi\sigma\omega)^{1/2}$ is the skin depth of the conducting wall. Assuming the wall will not allow the magnetic field to penetrate, the perturbed magnetic field between the wall and the beam can be related to the flux in the beam:

$$\left|\int_a^b r\, dr\, \delta B_z\right| = \left|\int_0^a r\, dr\, \delta B_z\right| \approx \left|\left(\frac{v_\perp}{c}\right)\left(\frac{a_c}{a}\right)\int_0^b r\, dr\, \delta E_r\right|. \tag{64}$$

Assuming the beam length is greater than b, then outside the beam δB_z should be essentially constant in r so that the above equation shows

$$b^4\delta B_{z_{\rm out}}^2 \approx a^2\left(\frac{v_\perp}{c}\right)^2\left(\frac{a_c}{a}\right)^2\frac{1}{2\pi}\int_0^b r\, dr\int_0^{2\pi} d\theta\,\delta\mathbf{E}^2. \tag{65}$$

Therefore, the growth rate is approximately the following:

$$\Gamma \approx \frac{\omega_-}{\omega_0}\left(\frac{\omega\delta}{b}\right)\left(\frac{v_\perp}{c}\right)^2\left(\frac{a_c}{a}\right)^2\left(\frac{a}{b}\right)^2. \tag{66}$$

Thus in the warm beam limit there is growth given by the above. It can be seen that this growth increases with $v_\perp$, decreases for small ratios of beam to wall radius (a/b), and decreases for small ratios of beam radius to Debye radius $[\omega_-/\omega_0 \approx (a/\lambda_d)^2]$. In CFIA it is reasonable to assume $\omega_-/\omega_0 \approx 1/30$, $\omega \approx 6\times 10^6$ sec^{-1}, $\sigma \approx 3\times 10^{13}$ sec^{-1}, and $a_c/a \approx \frac{1}{4}$, $a/b \approx \frac{1}{3}$, so the anticipated growth time is about 10 msec, which is an unobservably large time in CFIA. It should be noted that this equilibrium is expected to be the final equilibrium but it is probably not the equilibrium when damping is observed. Since Fig. 8 shows that the damping rate decreases with time, the beam may be evolving into this nondamping distribution. Section IV will consider an equilibrium with angular velocity shear and show that the initial damping can be explained by assuming such an equilibrium. It is expected that this velocity shear would eventually disappear and evolve into the equilibrium analyzed in this section.

IV. MAGNETIC DAMPING OF A ROTATING BEAM WITH ANGULAR VELOCITY SHEAR

In CFIA electrostatic probes indicate that the initial damping had a characteristic damping time of about 20 μsec; then the damping appeared to taper off and stop (see Fig. 8). The equilibrium analyzed in Sec. III was a Gibbs potential without any velocity shear. It is expected to be the final equilibrium for CFIA and predicts unobservable growth in the diocotron mode which is consistent with the experiment. An equilibrium that has velocity shear can explain the initial damping.

As a representative equilibrium, consider an equilibrium where the beam particles execute orbits like those seen in Fig. 9. In Fig. 9(a) the $\mathbf{E}\times\mathbf{B}$ drift radius is larger than the cyclotron radius (a_c) and the average v_θ is $\omega_- r \equiv -(\omega_p^2/2\Omega_z)r$. However, particles in the center of the beam have orbits like that in Fig. 9(b) where the cyclotron radius is larger than the $\mathbf{E}\times\mathbf{B}$ drift radius and $v_\theta \approx \omega_+ r \equiv -\Omega_z r$. In such a beam the angular velocity is expected to be the following:

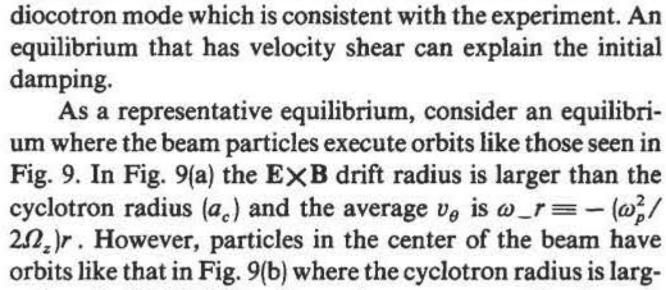

FIG. 9. Particle motion for the cases where (a) twice the cyclotron radius (a_c) is larger than the $\mathbf{E}\times\mathbf{B}$ drift radius, and (b) twice the cyclotron radius is smaller than the $\mathbf{E}\times\mathbf{B}$ drift radius. In case (a) the average v_θ corresponds to the $\mathbf{E}\times\mathbf{B}$ drift. However, in case (b) the average v_θ depends on the cyclotron frequency.

$$\omega_0 \approx \begin{cases} \omega_+, & r<2a_c\,, \\ \omega_-, & r>2a_c\,. \end{cases} \tag{67}$$

Since there is a large velocity shear it is expected that the equilibrium will eventually evolve into one where the angular velocity is constant and the poloidal velocity becomes the average poloidal velocity, which is $\approx \omega_+(a_c/a)^2 r \approx (v_\perp^2/a^2\Omega_z) r$. This is consistent with the equilibrium in Sec. III. Consider a beam of constant density. Since $\omega = \omega_+$ or ω_- this equilibrium is consistent with a constant density beam without pressure.[18,19] Using Eqs. (4)–(6) the perturbed density can be shown to be the following:

$$\delta n \cong -\frac{ie}{m}\delta E_\theta \frac{(\Omega_z + 2\omega_0)(\partial n/\partial r)}{(\omega - l\omega_0)v^2} - \frac{(ie/m)\delta E_\theta n}{(\omega - l\omega_0)}\frac{\partial}{\partial r}\left(\frac{\Omega_z + 2\omega_0}{v^2}\right), \tag{68}$$

$$v^2 = \left[(\Omega_z + 2\omega_0)\left(\Omega_z + 2\omega_0 + r\frac{\partial \omega_0}{\partial r}\right) - (\omega - l\omega_0)^2\right].$$

Since $\partial\omega_0/\partial r \cong -(\omega_+ - \omega_-)\delta(r-2a_c)$, the second term can then be shown to contribute terms such as the following:

$$\frac{\delta(r-2a_c)}{K+\delta(r-2a_c)}, \quad \frac{\delta'(r-2a_c)}{[K+\delta(r-2a_c)]^2}. \tag{69}$$

These terms will have negligible effects on δn since they are only finite in the infinitesimal interval $2a_c - \epsilon < r < 2a_c + \epsilon (\epsilon \ll a)$ and are therefore negligible. Using this δn can be expressed as the following:

$$\delta n \cong -\frac{ie}{m}\frac{\delta E(\partial n/\partial r)}{\Omega_z(\omega - l\omega_-)}, \tag{70}$$

which is the typical result for a cold beam where $\omega_0 = \omega_-$ throughout the beam. Additionally, the perturbed current can be calculated and the energy can be determined as was done in Secs. II and III. When this is done the electrostatic energy can be shown to be the following:

$$W_{es} \cong -(1/16\pi)(a/b)^2 \int r \quad dr\, d\theta\, \delta\mathbf{E}^2\,. \tag{71}$$

However, it is expected that the beam rotation will create a perturbed axial magnetic field given by the following:

$$\delta B_z \approx \begin{cases} (v_\perp/c)\delta E_r\,, & r<2a_c\,, \\ (\omega_- r/c)\delta E_r\,, & r>2a_c\,. \end{cases} \tag{72}$$

Therefore, using this and the results for the electrostatic energy, the electromagnetic energy is expected to be positive if

$$(v_\perp/c)^2(2a_c/a)^2 \gtrsim (a/b)^2\,,$$

in which case the mode will damp. In CFIA it is reasonable to assume that $(v_\perp/c)^2 \approx \frac{1}{2}$, $(2a_c/a) \approx \frac{1}{2}$, and $(a/b) \approx \frac{1}{3}$, so it is expected that the diamagnetic field will make the mode energy positive. The growth rate can be calculated from the power loss via the following equation:

$$\Gamma = -P/2W\,, \tag{73}$$

where P is the power loss. The power loss from the diamagnetic field inducing currents in the conducting boundary is given by the following:

$$P = \frac{\omega\delta}{16\pi}\int d\theta\, b\delta B_z^2\Big|_{r=b} = \frac{\omega\delta b}{8}\delta B_z^2\Big|_{r=b}\,, \tag{74}$$

where $\delta = (c^2/2\pi\sigma\omega)^{1/2}$ is the skin depth. In order to find the diamagnetic field at the boundary, consider that, for a good conducting wall, the magnetic flux inside the beam will be the same as the return flux which is in the area between the beam and the wall. Therefore, the ratio

$$\frac{(1/\pi a^2)\left[2\pi \int_0^a r\, dr\,(\delta B_z^2)\right]}{\delta B_z^2|_{r=b}} \tag{75}$$

is expected to be about $(b/a)^4$. Using this and assuming that the energy is primarily given by just the magnetic field energy then the damping rate predicted by Eq. (73) is the following:

$$\Gamma \approx \omega(\delta/b)(a/b)^2\,. \tag{76}$$

In the above it can be seen that the damping rate is proportional to the ratio of beam to chamber radius (a/b) and is similar to the result obtained in Sec. II in the limit $(v_z/c)^2 \gg (a/b)^2$. The above can explain the initial damping in CFIA. Like the modified betatron, some of the above quantities are not known exactly, but assuming $\omega = 6\times10^6\ \text{sec}^{-1}$, $\sigma = 3\times10^{13}\ \text{sec}^{-1}$ and $(a/b) = \frac{1}{3}$; the anticipated decay time is about 20 μsec. Figure 8 shows a typical electrostatic probe measurement. It can be seen that the predicted decay rate is roughly the initial observed decay rate.

Additional evidence to support the damping of the diocotron mode in this experiment is that x-ray measurements do not indicate that sufficient quantities of electrons were lost to account for the observed decay. It has also been observed that after hundreds of microseconds the shape of the probe oscillations changes to a shape consistent with much smaller beam oscillations.[6]

ACKNOWLEDGMENTS

We would also like to acknowledge the help of the experimentalists who worked on the modified betatron and CFIA: H. Ishizuka, A. Fisher, B. Mandelbaum, and F. Goldin.

This work was supported by the Office of Naval Research and the U. S. Department of Energy and is based on a Ph.D. thesis submitted to the University of California, Irvine.

[1] R. Briggs, J. Daugherty, and R. Levy, Phys. Fluids **13**, 421 (1970).

[2] W. White, J. Malmberg, and C. Driscoll, Phys. Rev. Lett. **49**, 1822 (1982).

[3] H. Ishizuka, G. Lindley, B. Mandelbaum, A. Fisher, and N. Rostoker, Phys. Rev. Lett. 53, 266 (1984); A Mandelbaum, H. Ishizuka, and N. Rostoker, in the *Proceedings of the 5th International Conference on High Power Electron and Ion Beam Research and Technology*, San Francisco, California, 1983 (Lawrence Livermore National Laboratories, Livermore, CA, 1983), p. 476.

[4] A. Fisher, P. Gilad, F. Goldin, and N. Rostoker, IEEE Trans. Nucl. Sci. **28**, 3383 (1981).

[5] A. Fisher, P. Gilad, F. Goldin, and N. Rostoker, Appl. Phys. Lett. **36**, 264 (1980).

[6] F. Goldin, A. Fisher, and N. Rostoker, Bull. Am. Phys. Soc. **29**, 1279 (1984).

[7] S. Ichimaru, *Basic Principles of Plasma Physics* (Benjamin, New York, 1973), pp. 216–219.

[8] S. Ichimaru, *Basic Principles of Plasma Physics* (Benjamin, New York, 1973), pp. 36–39.

[9] J. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), pp. 338.

[10]J. Daugherty, J. Eninger, and G. Janes, Phys. Fluids **12**, 2677 (1969).
[11]S. Eckhouse, A. Fisher, and N. Rostoker, Phys. Rev. Lett. **42**, 94 (1979).
[12]C. Kapetanakos, D. Hammer, C. Striggler, and R. Davidson, Phys. Rev. Lett. **30**, 1303 (1973).
[13]R. Levy, Phys. Fluids **8**, 1288 (1965).
[14]R. Davidson, *Theory of Nonneutral Plasmas* (Benjamin, New York, 1974), pp. 69–78.
[15]G. Roberts and N. Rostoker, submitted to Phys. Fluids.
[16]G. Roberts, Ph.D. thesis, University of California, Irvine, 1984, p. 89.
[17]R. Davidson and N. Krall, Phys. Rev. Lett. **22**, 833 (1969).
[18]R. Davidson, *Theory of Nonneutral Plasmas* (Benjamin, New York, 1974), pp. 100–109.
[19]H. Uhm and R. Davidson, Phys. Fluids **23**, 813 (1980).