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On the quest for a hyperbolic effective-field model of disperse flows

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The cornerstone of multiphase flow applications in engineering practice is a scientific construct that translates the basic laws of fluid mechanics into a set of governing equations for effective interpenetrating continua, the effective-field (or two-fluid) model. Over more than half a century of development this model has taken many forms but all of them fail in a way that was known from the very beginning: mathematical ill-posedness. The aim of this paper is to refocus awareness of this problem from a unified fundamental perspective that clarifies the manner in which such failures took place and to suggest the means for a final closure.

Key words: multiphase and particle-laden flows, multiphase flow, particle/fluid flows

1. Introduction

When a disperse multiphase system is too complex to simulate in a frame that keeps track of all interfaces, one must resort to an approach that instead is focused on phase volume fractions together with the mean mass, momentum and energy of each phase. If the dispersion involves multiple length scales, fixed or allowed to vary by breakup and coalescence mechanisms, we also need the interfacial area concentration. Evolution equations must be provided for all these quantities. The overall procedure, based on methods of continuum mechanics, and established long ago, yields an ill-posed mathematical problem, which despite extensive efforts to remedy, and despite claims to the contrary, remains to this day. On basic scientific grounds, the significance of this issue goes far beyond a mere curiosity, for as noted by Drew & Passman (1998):

A model that is not properly formulated mathematically cannot describe physical phenomena correctly. While mathematical correctness does not imply physical validity, the latter cannot be obtained without the former.

On practical grounds, this has implications, which increase in significance just as advancements in numerical schemes and computing power allow unprecedented detail and fidelity in solving systems of hyperbolic conservation laws. In this paper we provide new results and a synthesis that aim to: (*a*) examine critically the status of this quest; (*b*) clarify the issues and dispel errors and misconceptions that seem to have

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crept into the literature over a 40-year period of pursuit; and (*c*) point the way for the final assault.

It is sufficient for our purposes here to address non-dissipative disperse systems of two compressible phases with a fixed particle length scale so that no interfacial area transport equation is needed. (This means we do not consider mass and heat exchange between the particles and the carrier fluid nor viscous stresses and heat fluxes inside the two phases. And because they need some specific modelling, we exclude from our analysis all mixtures in which the particles display permanent compressive or sliding contacts, such as granular porous media, foams or concentrated polymer solutions.) So the system is to be described in terms of the phase volume fractions, mass densities, velocities, entropies and pressures, all subject locally to the Euler equations and mass and entropy conservation. Volume or ensemble averaging of these equations yields the six effective-field evolution equations. The two equations of state relate pressures to densities and entropies, and the volume fractions sum up to unity, so we have a total of nine equations for the ten unknowns. To close the system we need a tenth equation that is independent of the above, or a supplementary assumption that reduces the number of unknowns. A common such assumption is that locally the average pressures of each phase are equal. This closure however produces a system whose eigenvalues are not real. On closer examination, it becomes evident that a better closure can be obtained by a relation between the two average pressures in terms of local flow properties. In fact one can enlarge the possibilities (and accommodate much more physics) by taking into account the velocity fluctuations of both phases and this is the point of view adopted in the present paper. Two more evolution equations are needed for the pseudo-turbulent kinetic energies so that one ends up with eleven equations for twelve unknowns. We demonstrate that the missing relation is bound to the conservation of the overall energy and that it provides a necessary link between the force and the stresses appearing in the two momentum balances on the one hand, and the two evolution equations for the pseudo-turbulent kinetic energies on the other. It is for that complete set of equations that we want to restore hyperbolicity.

The general framework for our discussion is laid out in § 2 and the analysis of the hyperbolicity issue is carried out in three steps: (*a*) *ad hoc* approaches that address various mechanisms but which neglect pseudo-turbulence (§ 3); (*b*) a consistent treatment of added-mass phenomena that ignores only particle velocity fluctuations (§ 4); and (*c*) a first-step treatment of the velocity fluctuations of both phases that ignores added-mass phenomena (§ 5). The main results are summarized in § 6 where we also point the way to a complete treatment.

2. The roots of ill-posedness

2.1. *The six main equations for dispersed mixtures*

We are interested in the flow of suspensions of particles and will characterize the carrier fluid and the dispersed phase with subscripts c and d respectively. Since our ultimate goal is hyperbolicity we focus on the non-dissipative equations of motion and eliminate external forces like gravity. The conservation equations for entropy and mass, and the momentum balances of the two phases appear in the generic form (Buyevich & Shchelchkova 1978; Nigmatulin 1979; Zhang & Prosperetti 1994; Jackson 1997)

$$\frac{\mathrm{d}_k s_k}{\mathrm{d}t} = 0, \quad \frac{\partial}{\partial t}(\alpha_k \rho_k) + \nabla \cdot (\alpha_k \rho_k \boldsymbol{u}_k) = 0 \quad (k = c, d), \tag{2.1}$$

$$\alpha_d \rho_d \frac{\mathrm{d}_d \boldsymbol{u}_d}{\mathrm{d}t} + \nabla \cdot \Pi_d + \alpha_d \nabla p_c = -\boldsymbol{F}, \quad \alpha_c \rho_c \frac{\mathrm{d}_c \boldsymbol{u}_c}{\mathrm{d}t} + \nabla \cdot \Pi_c + \alpha_c \nabla p_c = \boldsymbol{F}. \tag{2.2}$$

In these equations the material time-derivatives are defined as $\mathrm{d}_k/\mathrm{d}t = \partial/\partial t + \boldsymbol{u}_k \cdot \nabla$. Note the main role played by the Archimedes force $\alpha_d \nabla p_c$ on the particles. Note also that the particle pressure appears nowhere explicitly but that the pressure jump $p_d - p_c$ is hidden in the phasic stresses Π_d and Π_c (Lhuillier & Theofanous 2010), so that the above special form of two-fluid model is actually a two-pressure model. By summing up (2.2) we can see that the interfacial forces, $\boldsymbol{F}$, cancel as they should, while the divergence of the sum of the phasic stresses remains, and the two pressure gradient terms sum up to ∇p_c. This helps us understand the basic nature of these divergences in complementing the pressure gradient as a volumetric source of momentum. By Green's theorem, integration over any finite control volume of the two-phase mixture translates these terms to *forces applied to the surface of the control volume* – thus, it is absolutely essential that these stresses, or their divergences, are not confused with interfacial forces which are effective *within* every point of the effective medium. The ten unknowns $s_k, \rho_k, p_k, \boldsymbol{u}_k$ and α_k are the entropy, mass density, pressure, velocity and volume fraction of phase k ($k = c, d$). There are three relations between them: two equations of state and the volume sharing condition

$$\rho_d = \rho_d(p_d, s_d), \quad \rho_c = \rho_c(p_c, s_c), \quad \alpha_d + \alpha_c = 1. \tag{2.3}$$

There are thus nine equations or relations involving ten variables. For the above equations to form a closed set we lack one relation or one equation. This missing information will be expressed in the form of a conservation-of-energy constraint that incorporates the energetics of fluctuations.

2.2. *Conservation of energy*

We must consider energy conservation because entropy, and not energy, was taken as a basic unknown. The thermodynamic relation $\mathrm{d}e_k = T_k \mathrm{d}s_k - p_k \mathrm{d}(1/\rho_k)$ is supposed to relate the mean internal energy e_k of phase k to its mean entropy, mass density, pressure and temperature. As a consequence entropy conservation is equivalent to $\alpha_k \rho_k \mathrm{d}_k e_k/\mathrm{d}t = -p_k\,(\partial\alpha_k/\partial t + \nabla \cdot (\alpha_k \boldsymbol{u}_k))$ meaning that the internal energy of a non-compressible phase is conserved. The equations for the kinetic energies are obtained directly from the momentum balances. Then one can obtain the equations for the total energy $E_k = e_k + u_k^2/2 + K_k$ of phase k where the pseudo-turbulent kinetic energy K_k is defined as

$$K_k = \tfrac{1}{2}\langle(\boldsymbol{u}_k^0 - \boldsymbol{u}_k) \cdot (\boldsymbol{u}_k^0 - \boldsymbol{u}_k)\rangle, \tag{2.4}$$

with $\boldsymbol{u}_k^0$ the local-instantaneous velocity and the brackets indicating mean values. The overall energy of the mixture $\alpha_d \rho_d E_d + \alpha_c \rho_c E_c$ must obey a conservation equation and this conservation will be satisfied provided the following condition is fulfilled:

$$\begin{aligned} &\alpha_d \rho_d \frac{\mathrm{d}_d K_d}{\mathrm{d}t} + \alpha_c \rho_c \frac{\mathrm{d}_c K_c}{\mathrm{d}t} + (p_c - p_d)\left[\frac{\partial \alpha_d}{\partial t} + \nabla \cdot (\alpha_d \boldsymbol{u}_d)\right] \\ &\quad + (\boldsymbol{u}_c - \boldsymbol{u}_d) \cdot \boldsymbol{F} + \Pi_d : \nabla \boldsymbol{u}_d + \Pi_c : \nabla \boldsymbol{u}_c + \nabla \cdot \boldsymbol{Q} = 0 \end{aligned} \tag{2.5}$$

where $\boldsymbol{Q}$ is some Galilean-invariant energy flux to be determined later on. The above equality must be considered as a necessary condition to be satisfied between the (as yet unknown) transport equations for K_d, K_c and α_d and the non-dissipative forces and stresses that appear in the two momentum balances. Any model for disperse mixtures which does not fulfil (2.5) must be rejected. Note that the six equations of the generic model, the two evolution equations for K_c and K_d, the three relations (2.3) and the energy conservation condition (2.5) make a total of twelve equations for twelve unknowns.

2.3. *The one-pressure model*

Before examining the first attempts to restore hyperbolicity, it is wise to present the set of equations at the origin of the issue. The simplest (and intuitive) way to obtain closure is to assume the equality of the two mean pressures ($p_d = p_c = p$) and to neglect the two pseudo-turbulent kinetic energies ($K_c = K_d = 0$). The resulting momentum balances are a special case of the generic equations (2.2) with $\boldsymbol{F} = 0$, $\Pi_d = \Pi_c = 0$, and one can check that the necessary condition (2.5) is fulfilled provided $\boldsymbol{Q} = 0$. The one-pressure model is thus closed and it satisfies the energy conservation condition. Unfortunately, after investigating its mathematical character it was proved to be non-hyperbolic (Rakhmatulin 1956; Stewart & Wendroff 1984). This is the starting point for a long story.

3. The first attempts to restore hyperbolicity

3.1. *Interfacial pressure*

The first chronological attempt to restore hyperbolicity was based on the concept of mean interfacial pressure (Stuhmiller 1977). Discarding the velocity fluctuations ($K_d = K_c = 0$), assuming the equality of the two bulk pressures ($p_d = p_c = p$) and defining $p^\star$ as the mean pressure on the interfaces, the momentum balances appear as in (2.2) with

$$\boldsymbol{F} = (p - p^\star)\nabla\alpha_d, \quad \Pi_d = \Pi_c = 0. \tag{3.1}$$

The hyperbolicity of that set of equations was investigated in Stuhmiller (1977) for non-compressible mixtures and in Ndjinga (2007) and Chang *et al.* (2007) for compressible ones. For non-compressible mixtures Stuhmiller (1977) found the hyperbolicity condition

$$p - p^\star \geqslant \frac{\alpha_d\alpha_c\rho_d\rho_c}{\alpha_d\rho_c + \alpha_c\rho_d}(\boldsymbol{u}_d - \boldsymbol{u}_c)^2. \tag{3.2}$$

One can understand the above inequality after noticing that the interfacial pressure $p^\star$ has introduced in the momentum balances a kind of diffusion force involving the *gradient of volume fraction*. It is clear that, to play a stabilizing role, a diffusion force must be directed opposite to the gradient of volume fraction. Stuhmiller's result can be interpreted by saying that the 'diffusion coefficient' $p - p^\star$ must be positive in order to prevent non-physical accumulation of particles in some parts of the flow, and that it must be positive and large enough to restore hyperbolicity. That result is appealing; unfortunately the model equations themselves are not tenable. When the two bulk pressures are equal to p it is difficult to understand why the mean interfacial pressure should be different from p. Moreover, condition (2.5) for energy conservation cannot be fulfilled with the above set (3.1) of force and stresses. Consequently the above model has to be discarded, but we have learned that a $\nabla\alpha_d$ diffusion force with the right sign can be beneficial to our quest for hyperbolicity.

3.2. *Volume fraction transport equation*

A second proposal discards velocity fluctuations, introduces the mean interfacial pressure $p^\star$ but releases the equal-pressure assumption and writes the momentum balances as in the generic equations with

$$\boldsymbol{F} = (p_c - p^\star)\nabla\alpha_d, \quad \Pi_d = \alpha_d(p_d - p_c)\boldsymbol{I}, \quad \Pi_c = 0. \tag{3.3}$$

With these expressions the energy conservation condition (2.5) can be satisfied with $\boldsymbol{Q}=0$ and

$$\frac{\partial \alpha_d}{\partial t}+\boldsymbol{u}^\star\cdot\nabla\alpha_d=0,\quad \boldsymbol{u}^\star=\boldsymbol{u}_d-\frac{p^\star-p_c}{p_d-p_c}(\boldsymbol{u}_d-\boldsymbol{u}_c), \tag{3.4}$$

meaning that the volume fraction is transported with a velocity $\boldsymbol{u}^\star$ which depends on $p^\star$. The set of seven equations based on the six generic equations completed by (3.4) is very popular (Saurel & Abgrall 1999; Gallouët, Hérard & Seguin 2004) because it is hyperbolic with seven different (and surprisingly simple) eigenvalues

$$\lambda_1=u_d,\quad \lambda_2=u_c,\quad \lambda_3=u^\star,\quad \lambda_{4,5}=u_d\pm c_d,\quad \lambda_{6,7}=u_c\pm c_c, \tag{3.5}$$

where c_k is the sound speed of phase k. While the eigenvalues λ_1,λ_2 and λ_3 are a mere consequence of entropy conservation and (3.4), it is worth noting that the interfacial pressure has no influence at all on $\lambda_{4,5,6,7}$ which would have the same values if $\boldsymbol{F}=0$. This surprising feature can be traced to the special form (3.4) of the evolution equation for the volume fraction. Moreover the eigenvalues $\lambda_{4,5}$ are suspect because a sound wave or pressure disturbance propagates with velocity c_d inside the particles only, and when outside the particles will move with a different velocity. It is only when the particles display permanent contacts that a pressure wave is likely to propagate with velocity c_d throughout the suspension. Moreover, for dispersed mixtures one expects the mean interfacial velocity and mean interfacial pressure to be equal to the bulk velocity and bulk pressure of the dispersed phase, an intuitive result which is incompatible with relation (3.4) between $\boldsymbol{u}^\star$ and $p^\star$. As it happens, the evolution equation (3.4) first appeared in the Baer–Nunziato (BN) model (Baer & Nunziato 1986), but the BN model describes very special mixtures, namely fluid-saturated granular media with permanent contacts between the grains (Bdzil *et al.* 1999). When dealing with less concentrated suspensions of particles, not only does the contact pressure disappear but simultaneously the evolution equation of the volume fraction is qualitatively modified as shown in what follows.

For compressible particles which do not exchange mass with the carrier fluid the evolution of the volume fraction can be generally written as

$$\frac{\partial \alpha_d}{\partial t}+\nabla\cdot(\alpha_d\boldsymbol{u}_d)=\alpha_d\langle\nabla\cdot\boldsymbol{u}_d^0\rangle \tag{3.6}$$

where the right-hand side represents the rate of change of the particle volume. When the particles are well dispersed in the fluid there is no link between the mean divergence of their local-instantaneous velocity $\langle\nabla\cdot\boldsymbol{u}_d^0\rangle$ and the divergence of their mean velocity $\nabla\cdot\boldsymbol{u}_d$. The former is related to the rate of change of the volume while the latter is related to the rate of change of the number density. If a pressure difference exists between the two phases it will act on $\langle\nabla\cdot\boldsymbol{u}_d^0\rangle$ only. However, when the particles are so concentrated as to display permanent compressive contacts between neighbours any change of volume will have a direct influence on the number density. A pressure difference will now act simultaneously on $\langle\nabla\cdot\boldsymbol{u}_d^0\rangle$ and $\nabla\cdot\boldsymbol{u}_d$. Hence for volume fractions above some minimum value of the order of the random close packing ($\alpha_{RCP}\approx 0.65$ for spheres) the evolution of the volume fraction will be rewritten as

$$\frac{\partial \alpha_d}{\partial t}+\boldsymbol{u}_d\cdot\nabla\alpha_d=\alpha_d(1-\nu(\alpha_d))(\langle\nabla\cdot\boldsymbol{u}_d^0\rangle-\nabla\cdot\boldsymbol{u}_d)\quad(\alpha_d>\alpha_{RCP}), \tag{3.7}$$

where $\nu(\alpha_d)$, which represents the ratio between the surface of the contacts and the total surface of a particle, increases from zero at random close packing up to one for crushed particles when there are no more 'holes' in the granular material. Since the change of volume is now correlated with the change of number density, a pressure difference will play a role in the right-hand side of (3.7), which was the type of equation considered by Baer & Nunziato (1986). It is clear that (3.6) and (3.7) hold in two complementary ranges of volume fraction. The use of (3.7) (or its generalization (3.4)) when there are no permanent contacts between the particles is not physically tenable despite its ability to restore hyperbolicity.

3.3. *Added-mass*

Added-mass effects were early on considered to be involved in the hyperbolicity issue. In fact added-mass is one of the rare non-dissipative physical phenomena that two-fluid models have to take into account. The first attempts to introduce added-mass effects into the two-fluid model (Stuhmiller 1977; Voinov & Petrov 1977; Lahey *et al.* 1980; Lhuillier 1985) focused on the expression of the added-mass force but none of these attempts could restore hyperbolicity (Jones & Prosperetti 1985; Fitt 1993). In the next Section we present a more systematic approach that takes into account not only the added-mass force but also the added-mass stresses.

4. Added-mass as a particular case of pseudo-turbulence

A more systematic study of added-mass effects and their representation in a two-fluid model was performed within two very different approaches: a variational principle based on a Lagrangian (Geurst 1986), and an averaging of the equations of motion of particles moving in a Euler fluid (Wallis 1991; Lhuillier & Theofanous 2010). These two different approaches share the same two assumptions concerning the pseudo-turbulent kinetic energies:

$$K_d = 0, \quad K_c = E(\alpha_d)\frac{(\boldsymbol{u}_d - \boldsymbol{u}_c)^2}{2}, \tag{4.1}$$

meaning that all the particles are moving with the same velocity and that added-mass effects are responsible for extra kinetic energy of the carrier phase written in terms of the relative velocity and a function of the volume fraction which is known to be $\alpha_d/2$ in the dilute limit. In Wallis (1991) and Lhuillier & Theofanous (2010) the velocity fluctuations and the pressure field on the interfaces are deduced from a Bernoulli equation. The resulting momentum balances can be presented as in the generic equations (2.2) with the added-mass force

$$\boldsymbol{F} = \frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{u}_d \otimes \boldsymbol{J}) + (\boldsymbol{J} \cdot \nabla)\boldsymbol{u}_c + \alpha_c \rho_c \nabla K_c + \boldsymbol{J} \times \left(\nabla \times \frac{\boldsymbol{J}}{\alpha_c \rho_c}\right), \tag{4.2}$$

$$\boldsymbol{J} = \alpha_c \rho_c E(\alpha_d)(\boldsymbol{u}_d - \boldsymbol{u}_c), \tag{4.3}$$

and the added-mass stresses

$$\Pi_c = (\boldsymbol{u}_d - \boldsymbol{u}_c) \otimes \boldsymbol{J}, \quad \Pi_d = -\alpha_d \alpha_c \rho_c \frac{\partial E}{\partial \alpha_d} \frac{(\boldsymbol{u}_d - \boldsymbol{u}_c)^2}{2} \boldsymbol{I}. \tag{4.4}$$

Although the particle pressure never appeared in the above calculations, the particle momentum balance suggests that $p_d = p_c - \alpha_c \rho_c (\partial E/\partial \alpha_d)(\boldsymbol{u}_d - \boldsymbol{u}_c)^2/2$. It can be checked that condition (2.5) for energy conservation is fulfilled with assumptions (4.1) and expressions (4.2)–(4.4). We have studied the mathematical character of the above pair of momentum balances completed by mass and entropy conservation equations. That mathematical character is bound to $E(\alpha_d)$ exclusively (Lhuillier & Theofanous

2010). In case of rigid spheres moving in an incompressible fluid, the hyperbolic character is obtained when $e(\alpha_d) = E(\alpha_d)/\alpha_d$ and its first and second derivatives with respect to α_d (denoted by a prime and a double prime) satisfy the inequality

$$\left[1 + e + \alpha_c e'\right] \left[\rho_d/\rho_c + e - \alpha_d e'\right] + \frac{\alpha_d \alpha_c}{2} \left[\alpha_d + \alpha_c \rho_d/\rho_c + e\right] e'' \leqslant 0. \tag{4.5}$$

Note that this inequality is independent of the relative velocity and cannot be satisfied when $E = 0$ or $E = \alpha_d/2$. We looked for expressions of $E(\alpha_d)$ for which the above inequality is satisfied for arbitrary values of the density ratio ρ_d/ρ_c. We found that the quadratic expression

$$E = \frac{\alpha_d}{2}(1 + C\alpha_d) \quad \text{with } C \leqslant -3, \tag{4.6}$$

restores hyperbolicity for any value of the density ratio. That result was also obtained in Pauchon & Smereka (1992) for massless particles and it can be understood if we note that a force $-[\alpha_c \rho_c \nabla K_c + \nabla \cdot \Pi_d]$ acts on the particles and that the $\nabla \alpha_d$ component of that force allows us to write

$$\alpha_d \rho_d \frac{\mathrm{d}_d \boldsymbol{u}_d}{\mathrm{d}t} + \alpha_d \nabla p_c = \alpha_d \alpha_c \rho_c \frac{(u_d - u_c)^2}{2} \frac{\partial^2 E}{\partial \alpha_d^2} \nabla \alpha_d + \cdots. \tag{4.7}$$

For that force to have a stabilizing role requires $\partial^2 E/\partial \alpha_d^2 \leqslant 0$ and result (4.6) means that to restore hyperbolicity requires $\partial^2 E/\partial \alpha_d^2 \leqslant -3$. The analogy with Stuhmiller's result (3.2) is patent.

Unfortunately there is no known homogeneous configuration of the particles that leads to (4.6) with $C \leqslant -3$. For example one finds $C \approx 0.32$ for a random distribution (Biesheuvel & Spoelstra 1989), $C = 0$ for Zuber's periodic configuration (Zuber 1964) and $C \approx 0$ for cubic arrays. See Smereka & Milton (1991) for a synthesis of the results. We conclude that added-mass has a beneficial role in the quest for hyperbolicity but, used alone, the function $E(\alpha_d)$ is restricted to unphysical values. In fact the only case of a negative C we are aware of was obtained by van Wijngaarden (1976) upon applying equal forces on bubbles initially at rest. The final configuration has a non-trivial velocity distribution ($K_d \neq 0$) with $C \approx -0.24$. This suggests that the combined effects of particle pseudo-turbulence and added-mass is perhaps the clue to restore hyperbolicity.

5. Pseudo-turbulence without added-mass

In suspensions the particles have a chaotic motion because of fluid-mediated interactions or direct collisions. And their chaotic motion is transmitted to the fluid because of the boundary conditions obeyed by the fluid velocity on the surface of a particle. The simplest way to depict the velocity fluctuations of the particles and the carrier fluid is to introduce the kinetic energies K_d and K_c defined in (2.4). In a two-phase flow the evolution of pseudo-turbulence generally involves the four mechanisms of production, diffusion, exchange and dissipation. It can be guessed that the general transport equations for K_d and K_c take a complex form, even upon neglect of all dissipative phenomena. In what follows our aim will not be to consider the equations in full but only to work with over-simplified equations just to check their potentialities in the quest for hyperbolicity. Hence we place ourselves in a position similar to Stuhmiller when he tested the potentialities of a $\nabla \alpha_d$ force (see § 3.1).

If (*a*) dissipative and diffusive processes are neglected, (*b*) the distribution of velocity fluctuations is assumed to be isotropic, and (*c*) there is no coupling between the velocity fluctuations of the particles and those of the carrier fluid, then the transport equations of K_d and K_c are reduced to the simple form

$$\alpha_k \rho_k \frac{\mathrm{d}_k K_k}{\mathrm{d}t} + \frac{2}{3}\alpha_k \rho_k K_k \nabla \cdot \boldsymbol{u}_k = 0. \tag{5.1}$$

Such an equation was already considered in Saurel, Gavrilyuk & Renaud (2003) and Hérard (2003) and it can be presented in the condensed form

$$\frac{\mathrm{d}_k}{\mathrm{d}t}\left(K_k/(\alpha_k \rho_k)^{2/3}\right) = 0. \tag{5.2}$$

If one takes for granted the equal-pressure assumption ($p_d = p_c = p$) together with the evolution equations (5.1) then the necessary condition (2.5) is satisfied with $\boldsymbol{Q} = 0$ and

$$\boldsymbol{F} = 0, \quad \Pi_d = \tfrac{2}{3}\alpha_d \rho_d K_d \boldsymbol{I}, \quad \Pi_c = \tfrac{2}{3}\alpha_c \rho_c K_c \boldsymbol{I}. \tag{5.3}$$

The eight-equation model considered hereafter is made up of the six generic equations (taking expressions (5.3) into account) completed by the two equations (5.2). We have investigated the eigenvalues of that set of equations and concluded that four of them are real:

$$\lambda_1 = \lambda_2 = u_d, \quad \lambda_3 = \lambda_4 = u_c, \tag{5.4}$$

which is a direct consequence of entropy conservation and (5.2). The four remaining eigenvalues are the solutions of the quartic equation

$$\left[(\lambda - u_d)^2 - A_d c_m^2\right]\left[(\lambda - u_c)^2 - A_c c_m^2\right] - A c_m^4 = 0 \tag{5.5}$$

where c_m and ρ_m are the sound speed and mass density of the mixture:

$$\rho_m = \alpha_d \rho_d + \alpha_c \rho_c, \quad \frac{1}{\rho_m c_m^2} = \frac{\alpha_d}{\rho_d c_d^2} + \frac{\alpha_c}{\rho_c c_c^2}, \tag{5.6}$$

while A_d, A_c and A are three non-dimensional scalars, depending on the particle volume fraction, the density ratio and the pseudo-turbulent kinetic energies:

$$A_d = \frac{10}{9}\frac{K_d}{c_m^2} + \alpha_d \frac{\rho_m}{\rho_d}, \quad A_c = \frac{10}{9}\frac{K_c}{c_m^2} + \alpha_c \frac{\rho_m}{\rho_c}, \quad A = \alpha_d \alpha_c \frac{\rho_m}{\rho_d}\frac{\rho_m}{\rho_c}, \tag{5.7}$$

hence verifying the inequality $0 < A \leqslant A_c A_d$. For the one-pressure eight-equation model to be hyperbolic the quartic equation (5.5) must have four real roots and this is obtained provided

$$q^2(p^2 - 4q) \geqslant r(27r + 4p^3 - 18pq), \tag{5.8}$$

where

$$p = 2(A_d + A_c) + \Delta M^2, \quad r = (A_d - A_c)^2 \Delta M^2, \tag{5.9}$$

$$q = 4A + (A_d - A_c)^2 + 2(A_d + A_c)\Delta M^2, \tag{5.10}$$

and $\Delta M = \|\boldsymbol{u}_d - \boldsymbol{u}_c\|/c_m$ is the Mach number of the relative velocity. The only result that can be obtained easily concerns the case of no relative velocity for which $\Delta M = 0$. Definitions (5.9) and (5.10) imply $r = 0$ and $p^2 \geqslant 4q$, so that inequality (5.8) is always satisfied, meaning that the system of equations is hyperbolic for any (positive) values of K_c and K_d. For a non-zero relative velocity the problem is much harder and we performed a numerical study of the eigenvalues of (5.5) (or the domain where (5.8) holds) in the parameter space $[\sqrt{K_d}, \sqrt{K_c}, \boldsymbol{u}_d - \boldsymbol{u}_c]$ with the volume fraction α_d and the density ratio ρ_d/ρ_c as parameters. A good approximation of the hyperbolic domain is given by

$$\sqrt{K_d\left(1+\frac{\alpha_c\rho_d}{\alpha_d\rho_c}\right)}+\sqrt{K_c\left(1+\frac{\alpha_d\rho_c}{\alpha_c\rho_d}\right)} \geqslant 1.5\|\boldsymbol{u}_d-\boldsymbol{u}_c\|. \tag{5.11}$$

When $K_d = 0$ hyperbolicity is obtained with K_c much larger than the added-mass kinetic energy which is of order $\alpha_d(\boldsymbol{u}_d - \boldsymbol{u}_c)^2$. This should not be a surprise because in the set of equations considered in the present Section the fluctuating kinetic energy is involved in the divergence of a kinetic stress only, while the set of equations describing added-mass has a much richer list of stresses and forces involving K_c.

We note here that Hank, Saurel & Metayer (2011) have considered a modified form of the above equations in which the Archimedes force $\alpha_d \nabla p_c$ is ignored from the momentum equation of the particles while $\alpha_c \nabla p_c$ is replaced by ∇p_c in the momentum equation for the carrier phase. In that case one finds $A = 0$ and it is clear from (5.5) that there are four more real eigenvalues so that their modified set of equations is indeed hyperbolic. However there is no physical justification for suppressing the Archimedes force in a suspension, even if dilute. Hence the hyperbolicity obtained in Hank *et al.* (2011) is not physically tenable.

6. Conclusions

The only model of suspensions which is both hyperbolic and compatible with physics is the Baer–Nunziato model. That model however is specific to suspensions with permanent compressive contacts between the particles and it is unphysical to apply it to suspensions with concentrations below random close packing. Hence the hyperbolicity of effective-field models of disperse mixtures without permanent contacts is not resolved yet. After a critical appraisal of the existing models we conclude that two physical phenomena are necessary: (*a*) added-mass; and (*b*) pseudo-turbulent velocity fluctuations. When considered alone, each of these two physical phenomena does not lead to fully satisfactory results. Added-mass alone can restore hyperbolicity but with a curious dependence of the kinetic energy on the volume fraction. Velocity fluctuations without added-mass can restore hyperbolicity but only for unexpectedly high values of the pseudo-turbulent kinetic energies, of the order of the relative velocity squared.

A promising direction is to associate added-mass and the pseudo-turbulence of the particles. Preliminary results in that direction have already been obtained for massless bubbles (Yurkovetsky & Brady 1996; Khang *et al.* 1997; Spelt & Sangani 1998) but they must be extended to massive particles in a form which satisfies the energy conservation condition (2.5) before considering the hyperbolicity issue.

Ultimately, however, any model must be anchored in experimental evidence, as along the lines pursued for example by Chang *et al.* (2011) and Wagner *et al.* (2012).

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