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Authors

Rhodes, John
Schilling, Anne

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UNIFIED THEORY FOR FINITE MARKOV CHAINS

JOHN RHODES AND ANNE SCHILLING

ABSTRACT. We provide a unified framework to compute the stationary distribution of any finite irreducible Markov chain or equivalently of any irreducible random walk on a finite semigroup S . Our methods use geometric finite semigroup theory via the Karnofsky–Rhodes and the McCammond expansions of finite semigroups with specified generators; this does not involve any linear algebra. The original Tsetlin library is obtained by applying the expansions to $P(n)$, the set of all subsets of an n element set. Our set-up generalizes previous groundbreaking work involving left-regular bands (or \mathcal{R} -trivial bands) by Brown and Diaconis, extensions to \mathcal{R} -trivial semigroups by Ayyer, Steinberg, Thiéry and the second author, and important recent work by Chung and Graham. The Karnofsky–Rhodes expansion of the right Cayley graph of S in terms of generators yields again a right Cayley graph. The McCammond expansion provides normal forms for elements in the expanded S . Using our previous results with Silva based on work by Berstel, Perrin, Reutenauer, we construct (infinite) semaphore codes on which we can define Markov chains. These semaphore codes can be lumped using geometric semigroup theory. Using normal forms and associated Kleene expressions, they yield formulas for the stationary distribution of the finite Markov chain of the expanded S and the original S . Analyzing the normal forms also provides an estimate on the mixing time.

1. INTRODUCTION

The Tsetlin library [Cet63] is a Markov chain, whose states are all permutations S_n of n books on a shelf. Given an arrangement of books $\sigma \in S_n$, construct $\sigma' \in S_n$ from σ by removing book a from the shelf and inserting it to the front. To each such transition $\sigma \xrightarrow{a} \sigma'$, we associate a probability x_a . If the probability x_a is large, it means that book a is popular, whereas if x_a is small, then book a is unpopular. Running this Markov chain for a while has the effect of accumulating the popular books in the front. The stationary distribution is the limiting distributing of the books, when one lets the Markov chain run for a long time. The precise formula was derived by Hendricks [Hen72, Hen73].

In the meantime, many generalizations of the Tsetlin library have been studied, such as walks on hyperplane arrangements [Bid97, BHR99], Brown’s significant generalization to left regular bands [Bro00] based on important work by Brown and Diaconis [BD98], hierarchies of libraries [Bjö08, Bjö09], edge flipping in graphs [CG12], random walks on linear extensions of a poset [AKS14a], random walks on general \mathcal{R} -trivial semigroups [ASST15b], and others [AS10, Ayy11, AS13, ASST15a, PS18]. The main technique, that made the analysis of all of these random walks possible, is the concept of reduced words of the elements in the underlying semigroup. As pointed out in [ASST15b, Section 4.4] and [MSS15, Remark 3.1] in the context of \mathcal{R} -trivial semigroups, this is the Karnofsky–Rhodes expansion of the support semilattice introduced in the seminal paper by Brown [Bro00]. This is an example, where concepts from semigroup theory were rediscovered in the setting of probability.

As is often the case in mathematics, once there is a toehold, an avalanche of results can follow by applying the results of the new field. The theory that we develop in this paper makes it possible to compute the stationary distribution for any irreducible finite Markov chain. It uses the power of geometric finite semigroup theory [MRS11] via the Karnofsky–Rhodes and the McCammond expansions of finite semigroups and does not use any linear algebra. From the theory of regular languages (that

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is, finite semigroup definable languages), we can define Markov chains on the expanded semigroup using semaphore codes [BPR10, RSS16]. The Karnofsky–Rhodes and McCammond expansions ensure the existence of normal forms for elements (or paths) in the semaphore code. Using Kleene expressions, Zimin words and elementary combinatorics, we are able to derive the stationary distribution of all irreducible random walks associated to a finite semigroup. Since there are only a finite number of states and a finite number of maps between the states, given a probability distribution there is a decidable algorithm giving the stationary distribution of the Markov chain, which is new.

This generalizes known stationary distributions of random walks and provides an abundance of new interesting examples of random walks and their stationary distributions. We obtain new examples by the $\bar{}$ construction from finite semigroup theory [LRS17] and the technology of the solution of the Burnside problem by McCammond and others [McC91]. In addition, we provide a standard interpretation of these constructions to understand how they apply to the real world.

The random walks that we deal with are in general not diagonalizable, unlike in the case of left regular bands [Bro00]. Our approach differs in that we start with an irreducible, infinitely countable random walk, namely the random walk on semaphore codes with the Bernoulli distribution as its stationary distribution. Using advanced finite semigroup theory, we find projections of these walks via lumping first in the case when the minimal ideal is left zero. The lumping is allowed thanks to the fact that the Karnofsky–Rhodes expansion of a right Cayley graph is itself a right Cayley graph of a finite semigroup. We obtain the general case from the case when the minimal ideal is left zero as a limiting case by applying the flat operator [LRS17] and then limiting the probability of the new introduced generator to zero. The resulting random walks on finite semigroups are in general not diagonalizable, but we can nonetheless compute the stationary distributions. Using the hitting time of semaphore codes [RSS16] and Kleene expressions, we can also estimate the mixing time of these walks via the techniques in [ASST15b, Lemma 3.6].

The paper is organized as follows. Section 2 is devoted to discrete Markov chains and their analysis using semigroup theory. We begin with a review of Cayley graphs of finite semigroups (Section 2.1), Markov chains in the language of semigroups (Section 2.2), and random walks on semaphore codes (Section 2.3). Ideals in semigroups are intimately related to (possibly infinite) semaphore codes. We proceed to explain the Karnofsky–Rhodes (Section 2.4) and McCammond expansion (Section 2.5) of the right Cayley graph (Section 2.1) of a finite semigroup S with generators A . The McCammond expansion guarantees normal forms of all elements in terms of the generators (Section 2.6). It is possible to lump the random walk on semaphore codes by reducing to simple paths without loops (Section 2.7) in the case when the minimal ideal in the semigroup is left zero. Using Kleene expressions and Zimin words, we provide explicit expressions for the stationary distributions (Section 2.8). Adding a zero to the semigroup, it is possible to obtain the stationary distribution for any finite semigroup from the case when the minimal ideal is left zero as a limiting case (Section 2.9). In Section 2.10 we provide bounds on the mixing time. In Section 3, we discuss many examples of semigroups and how our methods yield the stationary distributions of known and new Markov chains, such as the original Tsetlin library (Section 3.1), edge flipping on a line (Section 3.2), cyclic walks using the Rees matrix semigroup (Sections 3.3 and 3.4), and random walks on general \mathcal{R} -trivial semigroups (Section 3.5). The $\bar{}$ and flat \flat operations, introduced in Section 2.9, are then used in Sections 3.6 and 3.7 to produce many infinite families of examples of Markov chains. We conclude in Section 3.8 with examples in the Burnside class.

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2. RANDOM WALKS ON SEMIGROUPS

Let S be a finite semigroup. We are interested in considering S together with a choice of generators A , denoted by (S, A) . Every finite semigroup has a finite set of generators (for example, the elements of S itself, but possibly fewer). We will construct Markov chains for (S, A) using this set-up by associating a probability x_a to each generator $a \in A$.

2.1. Cayley graphs. Given a *finite semigroup* S and a set of *generators* A , we can view A as a finite, non-empty alphabet. Denote by A^+ the set of all words $a_1 \dots a_\ell$ of length $\ell \geq 1$ over A with multiplication given by concatenation. Thus (A^+, A) is the free semigroup with generators A . Furthermore, let $A^* = A^+ \cup \{1\}$, so that A^* is A^+ with the identity added; it is the free monoid generated by A . The elements of A^* are typically called words. A subset of A^* is called a *language*.

A semigroup S with multiplication \cdot generated by a subset $A \subseteq S$ determines a surmorphism

$$(2.1) \quad \varphi: (A^+, A) \rightarrow (S, A)$$

mapping $a_1 \dots a_\ell \in A^+$ to $a_1 \cdot a_2 \cdot \dots \cdot a_\ell \in S$. Given a word $w \in A^+$, we denote $[w]_S := \varphi(w)$ to avoid the reference to φ . The pair (S, A) is also sometimes called an A -semigroup (see [MRS11, Definition 2.15]).

Definition 2.1 (Graph). A *labeled directed graph* Γ (or *graph* for short) consists of a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$, and a labelling set A . An edge $e \in E(\Gamma)$ is a tuple $e = (v, a, w) \in V(\Gamma) \times A \times V(\Gamma)$. We often also write $e: v \xrightarrow{a} w$.

A *path* p from vertex v to vertex w in a graph Γ is a sequence of edges

$$p = \left(v = v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \dots \xrightarrow{a_\ell} v_\ell = w \right),$$

where each tuple $(v_i, a_{i+1}, v_{i+1}) \in E(\Gamma)$ for $0 \leq i < \ell$. The initial (resp. terminal) vertex v (resp. w) of p is denoted by $\iota(p)$ (resp. $\tau(p)$). The *length* of p is $\ell(p) := \ell$ and $a_1 \dots a_\ell$ is called the label of the path. If p and q are paths, $\ell(q) = k \leq \ell(p)$, and the first $k+1$ vertices and k edges of p and q agree, we say that q is an *initial segment* of p , written $q \subseteq p$.

We can define a preorder \prec on $V(\Gamma)$ by $v \prec w$ if there is a path from v to w in Γ . This induces an equivalence relation \sim on $V(\Gamma)$, where $v \sim w$ if $v \prec w$ and $w \prec v$. A *strongly connected component* of Γ is a \sim -equivalence class.

Definition 2.2 (Rooted graph). A *rooted graph* is a pair (Γ, r) , where Γ is a graph and $r \in V(\Gamma)$, such that $r \prec v$ for all $v \in V(\Gamma)$.

A path is called *simple* if it visits no vertex twice. Empty (or trivial) paths are considered simple. For a rooted graph (Γ, r) , let $\text{Simple}(\Gamma, r)$ be the set of simple paths of Γ starting at r (including the empty path).

We are now ready to apply this set-up to semigroups. If S is a semigroup, then S^1 denotes S with an adjoined identity $\mathbb{1}$ even if S already has an identity.

Definition 2.3 (Right and left Cayley graph). Let (S, A) be a finite semigroup S together with a set of generators A . The *right Cayley graph* $\text{RCay}(S, A)$ of S with respect to A is the rooted graph with vertex set $V(\text{RCay}(S, A)) = S^1$, root $r = \mathbb{1} \in S^1$, and edges $s \xrightarrow{a} s'$ for all $(s, a, s') \in S^1 \times A \times S^1$, where $s' = sa$ in S^1 . The *left Cayley graph* $\text{LCay}(S, A)$ is defined in the analogous fashion with the only difference that $s' = as$.

Remark 2.4. Since $(s, a) \in S^1 \times A$ uniquely determines the edge $s \xrightarrow{a} sa$ in $\text{RCay}(S, A)$, we sometimes also index the edge set as $E(\text{RCay}(S, A)) = S^1 \times A$ for right Cayley graphs and analogously for left Cayley graphs.

An example of a right Cayley graph is given in Figure 1.

For a semigroup S , two elements $s, s' \in S$ are in the same \mathcal{R} -class if the corresponding right ideals are equal, that is, $sS^1 = s'S^1$. The strongly connected components of $\text{RCay}(S, A)$ are precisely the

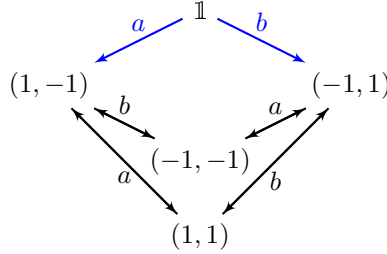


FIGURE 1. The right Cayley graph $\text{RCay}(Z_2 \times Z_2, \{a, b\})$ of the Klein 4-group and generators $a = (1, -1)$ and $b = (-1, 1)$. Transition edges are indicated in blue. Double edges mean that right multiplication by the label for either vertex yields the other vertex.

\mathcal{R} -classes of S^1 . In other words, the vertices of a strongly connected component are exactly the vertices that represent the elements in an \mathcal{R} -class of S^1 . Edges that go between distinct strongly connected components will turn out to play an important role in the Karnofsky–Rhodes expansion that we will need later.

Definition 2.5 (Transition edges). Let Γ be a graph. Then $e = (v, a, w) \in E(\Gamma)$ with $v, w \in V(\Gamma)$ and $a \in A$ is a *transition edge* if $v \not\sim w$. In other words, there is no path from w to v in Γ .

In Figure 1, the transition edges are indicated in blue. Note that the edges leaving 1 in the right Cayley graph are always transitional. Other edges might or might not be transitional.

2.2. Markov chains. A *Markov chain* \mathcal{M} consists of a finite or countable state space Ω together with transition probabilities $\mathcal{T}_{s',s}$ for the transition $s \rightarrow s'$ for $s, s' \in \Omega$. The matrix $\mathcal{T} = (\mathcal{T}_{s',s})_{s,s' \in \Omega}$ is called the *transition matrix*. In our convention, the column sums of \mathcal{T} are equal to one, or equivalently, that \mathcal{T} is a column-stochastic matrix.

A Markov chain is called *irreducible* if for any $s, s' \in \Omega$ there exists an integer m (possibly depending on s, s') such that $\mathcal{T}_{s',s}^m > 0$. In other words, one can get from any state s to any other state s' using only steps with positive probability. A state $s \in \Omega$ is called *recurrent* if the system returns to s in finitely many steps with probability one.

The *stationary distribution* of \mathcal{M} is a vector $\Psi = (\Psi_s)_{s \in \Omega}$ such that $\mathcal{T}\Psi = \Psi$ and $\sum_{s \in \Omega} \Psi_s = 1$. In other words, Ψ is a right-eigenvector of \mathcal{T} with eigenvalue one. If the Markov chain is irreducible, the stationary distribution is unique [LPW09].

Let us now partition the state space into $(\Omega_1, \dots, \Omega_\ell)$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\ell} \Omega_i = \Omega$. One may view such a partition also as an equivalence relation $s \sim s'$ if $s, s' \in \Omega_i$ for some $1 \leq i \leq \ell$. We say that \mathcal{M} can be *lumped* with respect to the partition $(\Omega_1, \dots, \Omega_\ell)$ if the transition matrix \mathcal{T} satisfies [LPW09, Lemma 2.5] [KS76] for all $1 \leq i, j \leq \ell$

$$(2.2) \quad \sum_{t \in \Omega_j} \mathcal{T}_{t,s} = \sum_{t \in \Omega_j} \mathcal{T}_{t,s'} \quad \text{for all } s, s' \in \Omega_i.$$

The lumped Markov chain is a random walk on the equivalence classes, whose stationary distribution labeled by w is $\sum_{s \sim w} \Psi_s$.

As explained in [LPW09, Proposition 1.5] and [ASST15b, Theorem 2.3], every finite state Markov chain \mathcal{M} has a random letter representation, that is, a representation of a semigroup S acting on the left on the state space Ω . In this setting, we transition $s \xrightarrow{a} s'$ with probability $0 \leq x_a \leq 1$, where $s, s' \in \Omega$, $a \in S$ and $s' = a.s$ is the action of a on the state s . Let $A = \{a \in S \mid x_a > 0\}$. We assume that A generates S ; if not, it suffices to consider the subsemigroup generated by A . Note

that $\sum_{a \in A} x_a = 1$. The *transition matrix* \mathcal{T} of \mathcal{M} is the $|\Omega| \times |\Omega|$ -matrix

$$(2.3) \quad \mathcal{T}_{s',s} = \sum_{\substack{a \in A \\ s \xrightarrow{a} s'}} x_a \quad \text{for } s, s' \in \Omega.$$

Note that we may assume that the action of S on Ω is faithful as this does not affect the random walk.

Definition 2.6 (Ideal). Let S be a semigroup. A two-sided *ideal* I (or ideal for short) is a subset $I \subseteq S$ such that $uIv \subseteq I$ for all $u, v \in S^1$. Similarly, a *left ideal* I is a subset $I \subseteq S^1$ such that $uI \subseteq I$ for all $u \in S^1$.

If I, J are ideals of S , then $IJ \subseteq I \cap J$, so that $I \cap J \neq \emptyset$. Hence every finite semigroup has a unique minimal ideal denoted $K(S)$. As shown in [CP61, KRT68], the minimal ideal $K(S)$ of a finite semigroup S is the disjoint union of all the minimal left ideals of S and the Rees Theorem applies. By [ASST15b, Remark 2.8] the faithful left action of S generated by A on Ω is isomorphic to the left action of S on $K(S)$.

For a finite A -semigroup (S, A) , let $\mathcal{M}(S, A)$ be the Markov chain, where the transition $s \xrightarrow{a} s'$ for $s, s' \in S$ and $a \in A$ is given by $s' = as$ in the left Cayley graph with probability $0 < x_a \leq 1$. Note that we are assuming that all probabilities x_a for $a \in A$ are nonzero. Then it was shown in [HM11] (see also [ASST15b, Proposition 3.2]) that the recurrent states of $\mathcal{M}(S, A)$ are the elements in $K(S)$. Furthermore, the connected components of the recurrent states in the random walk are the minimal left ideals of S . The restriction of the random walk to any minimal left ideal is irreducible. Moreover, the chain so obtained is independent of the chosen minimal left ideal. This random walk and the random walk with states a left ideal L of $K(S)$ and S acting on the left made faithful, that is $x \xrightarrow{a} y$ for $x \in L$ and $y = ax$, are essentially the same. So we may not distinguish the two cases.

In the following, we first treat the case when $K(S)$ is left zero (that is, $xy = x$ for all $x, y \in K(S)$) using semaphore codes, the Karnofsky–Rhodes and McCammond expansion of the right Cayley graph of (S, A) , and Kleene expressions. In Corollary 2.33, we add a zero to the semigroup and generators to deduce the case for general $K(S)$ from the case when $K(S)$ is left zero.

2.3. Semaphore codes. Ideals in a semigroup are related to semaphore codes [BPR10, RSS16]. They also give rise to Markov chains since they allow for a left action, as we will now explain. As before, let A be a finite, non-empty alphabet. The semigroup A^+ has three orders: “is a suffix”, “is a prefix”, and “is a factor”. In particular, for $u, v \in A^+$

$$\begin{aligned} u \text{ is a suffix of } v &\iff \exists w \in A^* \text{ such that } wu = v, \\ u \text{ is a prefix of } v &\iff \exists w \in A^* \text{ such that } uw = v, \\ u \text{ is a factor of } v &\iff \exists w_1, w_2 \in A^* \text{ such that } w_1uw_2 = v. \end{aligned}$$

A *prefix code* \mathcal{C} of A^+ (or over A) is a subset $\mathcal{C} \subseteq A^+$ so that all elements in \mathcal{C} are pairwise incomparable in the prefix order [BPR10].

Definition 2.7. [BPR10, Proposition 3.5.4] A prefix code $\mathcal{S} \subseteq A^+$ is a *semaphore code* if $A\mathcal{S} \subseteq \mathcal{S}A^*$.

In other words, a semaphore code is a prefix code \mathcal{S} over A for which there is a left action in the following sense:

$$(2.4) \quad \begin{aligned} &\text{If } u \in \mathcal{S} \subseteq A^+ \text{ and } a \in A, \text{ then } au \text{ has a prefix in } \mathcal{S} \text{ (and hence a unique prefix of } au\text{).} \\ &\text{The left action } a.u \text{ is the prefix of } au \text{ that is in } \mathcal{S}. \end{aligned}$$

Semaphore codes over A are inherently related to ideals of A^+ [RSS16, Proposition 4.3]. Given an ideal $I \subseteq A^+$ we construct a semaphore code as follows. Given $u = a_1a_2 \dots a_j \in A^+$, check whether u is in I . If $u \notin I$, ignore u . If $u \in I$, we find the (necessarily unique) index $1 \leq i \leq j$ such that $a_1 \dots a_{i-1} \notin I$, but $a_1 \dots a_i \in I$. Then $a_1 \dots a_i$ is a code word and the set of all such words

forms the semaphore code $\mathcal{S} =: I\beta_\ell$. Conversely, given a semaphore code \mathcal{S} , the corresponding ideal is obtained as $\mathcal{S}A^*$. This yields a bijection

$$(2.5) \quad I \longleftrightarrow I\beta_\ell$$

between ideals $I \subseteq A^+$ and semaphore codes \mathcal{S} over A .

Using the left action in (2.4), we can define a Markov chain $\mathcal{M}^{\mathcal{S}}$ on the semaphore code \mathcal{S} . We transition $s \xrightarrow{a} s'$ for $s, s' \in \mathcal{S}$ and $a \in A$ with probability x_a , where $s' = a.s$. The stationary distribution for the Markov chain $\mathcal{M}^{\mathcal{S}}$ was computed in [RSS16, Theorem 8.1].

Proposition 2.8. [RSS16] *The stationary distribution of the Markov chain $\mathcal{M}^{\mathcal{S}}$ is*

$$\Psi_s^{\mathcal{S}} = \prod_{a \in s} x_a \quad \text{for all } s \in \mathcal{S}.$$

Given a finite semigroup S with generators A , recall $\varphi: (A^+, A) \rightarrow (S, A)$ as defined in (2.1).

Definition 2.9. Let (S, A) be a finite semigroup S with generators A and $I \subseteq S$ an ideal. Define

$$\mathcal{I}(S, A, I) := \varphi^{-1}(I) \subseteq A^+.$$

In particular, let $\mathcal{I}(S, A) := \mathcal{I}(S, A, K(S))$, where recall that $K(S)$ is the minimal ideal in S .

Lemma 2.10. $\mathcal{I}(S, A, I)$ is an ideal in A^+ .

Proof. Let $u, v \in A^+$ and $w \in \mathcal{I}(S, A, I)$. Then $[uvw]_S = [u]_S[w]_S[v]_S \in I$ since $I \subseteq S$ is an ideal. \square

Definition 2.11. Let $\mathcal{S}(S, A)$ be the semaphore code associated to $\mathcal{I}(S, A)$ according to (2.5).

We may now define the Markov chain $\mathcal{M}^{\mathcal{S}}(S, A)$ for the finite A -semigroup (S, A) as the Markov chain on the semaphore code $\mathcal{S}(S, A)$. A semigroup S is *left zero* if $xy = x$ for all $x, y \in S$. When $K(S)$ is left zero, we can obtain the stationary distribution of $\mathcal{M}(S, A)$ from the stationary distribution $\mathcal{M}^{\mathcal{S}}(S, A)$ by lumping.

Theorem 2.12. *If $K(S)$ is left zero, then the finite Markov chain $\mathcal{M}(S, A)$ is a lumping of $\mathcal{M}^{\mathcal{S}}(S, A)$, where \mathcal{S} is the semaphore code associated to $K(S)$. Furthermore,*

$$\Psi_w = \sum_{\substack{s \in \mathcal{S} \\ [s]_S = w}} \Psi_s^{\mathcal{S}} \quad \text{for } w \in K(S).$$

Proof. We need to check that (2.2) holds, where \mathcal{T} is the transition matrix of $\mathcal{M}^{\mathcal{S}}$. In particular

$$(2.6) \quad \sum_{t, [t]_S = w} \mathcal{T}_{t,s} = \sum_{t, [t]_S = w} \mathcal{T}_{t,s'}$$

for all $w \in K(S)$ and $s, s' \in \mathcal{S}$ such that $[s]_S = [s']_S$. Recall from (2.3) that $\mathcal{T}_{t,s} = \sum_{a \in A, t=as} x_a$. Since S is a semigroup, we have $[as]_S = [as']_S$ for all $a \in A$. By the definition of a semaphore, as (resp. as') also has a prefix $a.s$ (resp. $a.s'$) in $K(S)$ under φ . Since $K(S)$ is left zero, this implies that $[a.s]_S = [as]_S$ and similarly $[a.s']_S = [as']_S$. Hence $[a.s]_S = [a.s']_S$, which implies (2.6). The stationary distribution of the lumped Markov chain is obtained from the stationary distribution of the unlumped Markov chain by summing over all states in an equivalence class. This proves the claim. \square

2.4. The Karnofsky–Rhodes expansion. To compute explicit expressions for the stationary distributions of Markov chains on finite semigroups, we need the *Karnofsky–Rhodes expansion* [Els99] of the right Cayley graph $\text{RCay}(S, A)$. See also [MRS11, Definition 4.15] and [MSS15, Section 3.4]. In addition, we will require the McCammond expansion [MRS11], which is discussed in the next section.

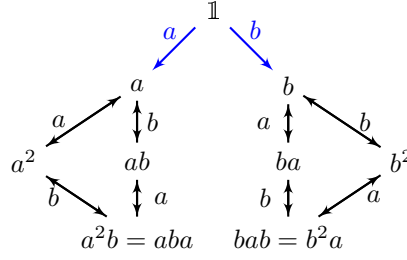


FIGURE 2. The Karnofsky–Rhodes expansion $\text{KR}(S, A)$ of the right Cayley graph of Figure 1.

Definition 2.13 (Karnofsky–Rhodes expansion). The *Karnofsky–Rhodes expansion* $\text{KR}(S, A)$ is obtained as follows. Start with the right Cayley graph $\text{RCay}(A^+, A)$. Identify two paths in $\text{RCay}(A^+, A)$

$$p := \left(\mathbb{1} \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_\ell} v_\ell \right) \quad \text{and} \quad p' := \left(\mathbb{1} \xrightarrow{a'_1} v'_1 \xrightarrow{a'_2} \cdots \xrightarrow{a'_{\ell'}} v'_{\ell'} \right)$$

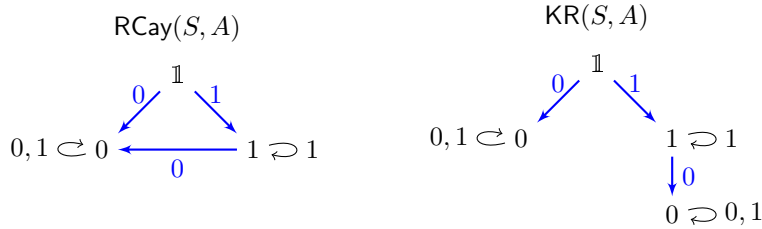
in $\text{KR}(S, A)$ if and only if the corresponding paths in $\text{RCay}(S, A)$

$$[p]_S := \left(\mathbb{1} \xrightarrow{a_1} [w_1]_S \xrightarrow{a_2} \cdots \xrightarrow{a_\ell} [w_\ell]_S \right) \quad \text{and} \quad [p']_S := \left(\mathbb{1} \xrightarrow{a'_1} [w'_1]_S \xrightarrow{a'_2} \cdots \xrightarrow{a'_{\ell'}} [w'_{\ell'}]_S \right),$$

where $w_i = a_1 a_2 \dots a_i$ and $w'_i = a'_1 a'_2 \dots a'_i$, end at the same vertex $[w_\ell]_S = [w'_{\ell'}]_S$ and in addition the set of transition edges of $[p]_S$ and $[p']_S$ in $\text{RCay}(S, A)$ is equal.

An example for $\text{KR}(S, A)$ is given in Figure 2. In this figure, the paths a^2b and aba are equal because they end in the same vertex when projected onto S and they share the same transition edge, which is the first a . On the other hand, the paths ab and ba are distinct because for the first path the transition edge is the first a and for the second path the transition edge is the first b .

Example 2.14. Take the semigroup $S = \{0, 1\}$ under multiplication with generators $A = \{0, 1\}$, where $0 \cdot x = x \cdot 0 = 0$ and $1 \cdot x = x \cdot 1 = x$ for all $x \in S$. The right Cayley graph and its Karnofsky–Rhodes expansion are given by



Proposition 2.15. $\text{KR}(S, A)$ is the right Cayley graph of a semigroup, also denoted by $\text{KR}(S, A)$.

Proof. Since the graph $\text{KR}(S, A)$ is constructed from the right Cayley graph $\text{RCay}(A^+, A)$, many of the properties of right Cayley graphs are automatically satisfied. That is, $\text{KR}(S, A)$ is deterministic and complete, the root $\mathbb{1}$ is not the endpoint of any edge, and every vertex is accessible from $\mathbb{1}$. In addition to this, we need to check that if two paths $p = p_1 p_2 \dots p_\ell$ and $q = q_1 q_2 \dots q_k$ (written in terms of their edge labels p_i and q_i) starting at $\mathbb{1}$ satisfy $\tau(p) = \tau(q)$, then $\tau(y p) = \tau(y q)$ for any path y in $\text{KR}(S, A)$. Here $y p$ stands for the path given by the concatenation of the edge labels of y with those of p . The condition $\tau(p) = \tau(q)$ by the definition of the Karnofsky–Rhodes expansion is equivalent to the conditions that $[p]_S = [q]_S$ and that the set of transition edges in p and q agree.

Since $\text{RCay}(S, A)$ is a right Cayley graph, we have $[y p]_S = [y q]_S$ for any path y . Now suppose by contradiction that the transition edges in $y p$ and $y q$ do not agree. Note that a non-transition

edge p_i in p cannot become a transition edge p_i in yp . Hence, without loss of generality, let us assume that there is a transition edge p_i in yp that is not a transition edge in yg and that all transition edges among p_1, \dots, p_{i-1} are also transition edges in yg . Since in p and q all transition edges agree by assumption, let q_j in q be the transition edge corresponding to p_i in p . In particular, this implies that $\tau(p_1 \dots p_{i-1}) = \tau(q_1 \dots q_{j-1})$ and $\tau(p_1 \dots p_i) = \tau(q_1 \dots q_j)$. But this in turn implies that $v = \tau(yp_1 \dots p_{i-1}) = \tau(yq_1 \dots q_{j-1})$ and $w = \tau(yp_1 \dots p_i) = \tau(yq_1 \dots q_j)$, meaning that the edge between vertex v and vertex w is the same in yp and yg , contradicting the assumption that the edge is a transition edge in yp , but not in yg . Hence $\text{KR}(S, A)$ is a right Cayley graph. \square

Since by Proposition 2.15 $\text{KR}(S, A)$ is the right Cayley graph of a semigroup, we can consider the corresponding Markov chain $\mathcal{M}(\text{KR}(S, A))$. The Markov chain $\mathcal{M}(S, A)$ can be obtained from $\mathcal{M}(\text{KR}(S, A))$ by the projection $w \mapsto [w]_S$ for $w \in \text{KR}(S, A)$ since both (S, A) and $\text{KR}(S, A)$ are semigroups.

Proposition 2.16. *The Karnofsky–Rhodes expansion of $\text{KR}(S, A)$ is stable, that is*

$$\text{KR}(\text{KR}(S, A), A) = \text{KR}(S, A).$$

Proof. This is clear from the definition, since the set of transition edges does not change from $\text{KR}(S, A)$ to $\text{KR}(\text{KR}(S, A), A)$. \square

For additional properties of the Karnofsky–Rhodes expansion, see [Els99, MRS11, MSS15].

2.5. The McCammond expansion. The McCammond expansion [MRS11] of a rooted graph is intimately related to the unique simple path property.

Definition 2.17 (Unique simple path property). A rooted graph (Γ, r) has the *unique simple path property* if for each vertex $v \in V(\Gamma)$ there is a unique simple path from the root r to v .

As proven in [MRS11, Proposition 2.32], the unique simple path property is equivalent to (Γ, r) admitting a unique directed spanning tree \mathbb{T} . Note that the unique simple path property not only depends on the graph Γ , but also on the chosen root r . In this paper, we always choose $r = \mathbb{1}$.

It was established in [MRS11, Section 2.7] that every rooted graph (Γ, r) has a universal simple cover, which has the unique simple path property.

Definition 2.18 (McCammond expansion). For a rooted graph (Γ, r) , define its *McCammond expansion* (Γ^{Mc}, r) as the graph with

$$\begin{aligned} V(\Gamma^{\text{Mc}}) &= \text{Simple}(\Gamma, r), \\ E(\Gamma^{\text{Mc}}) &= \{(p, a, q) \in V(\Gamma^{\text{Mc}}) \times A \times V(\Gamma^{\text{Mc}}) \mid (\tau(p), a, \tau(q)) \in E(\Gamma), \\ &\quad \ell(q) = \ell(p) + 1 \text{ or } (q \subseteq p \text{ and } \ell(q) \leq \ell(p))\}. \end{aligned}$$

Note that by definition there are two types of edges $(p, a, q) \in E(\Gamma^{\text{Mc}})$: either $\ell(q) = \ell(p) + 1$ or $\ell(q) \leq \ell(p)$ as paths in $\text{Simple}(\Gamma, r)$. The spanning tree \mathbb{T} has vertex set $V(\Gamma^{\text{Mc}})$ and only those edges $(p, a, q) \in E(\Gamma^{\text{Mc}})$ such that $\ell(q) = \ell(p) + 1$.

From now on choose $r = \mathbb{1}$. The simple path

$$\mathbb{1} \xrightarrow{a_1} v_1 \xrightarrow{a_2} \dots \xrightarrow{a_\ell} v_\ell$$

in $\text{Simple}(\Gamma, \mathbb{1})$ is naturally indexed by the word $a_1 a_2 \dots a_\ell$. We will use this labeling for the McCammond expansion of $\text{KR}(S, A)$. In particular, if $a_1 a_2 \dots a_\ell \in \text{Simple}(\Gamma, \mathbb{1})$ and $a_1 a_2 \dots a_\ell a \in \text{Simple}(\Gamma, \mathbb{1})$, then the edge $a_1 a_2 \dots a_\ell \xrightarrow{a} a_1 a_2 \dots a_\ell a$ is in the spanning tree \mathbb{T} . Otherwise we have $a_1 a_2 \dots a_\ell \xrightarrow{a} a_1 a_2 \dots a_k$ for some unique $1 \leq k < \ell$. Thus under the right action of $a \in A$ on $a_1 a_2 \dots a_\ell$, we either move forward in the spanning tree or fall backwards somewhere on the unique geodesic from $\mathbb{1}$ to $a_1 a_2 \dots a_\ell$, but staying in the same \mathcal{R} -class. An example of a McCammond expansion of a Karnofsky–Rhodes graph is given in Figure 3.

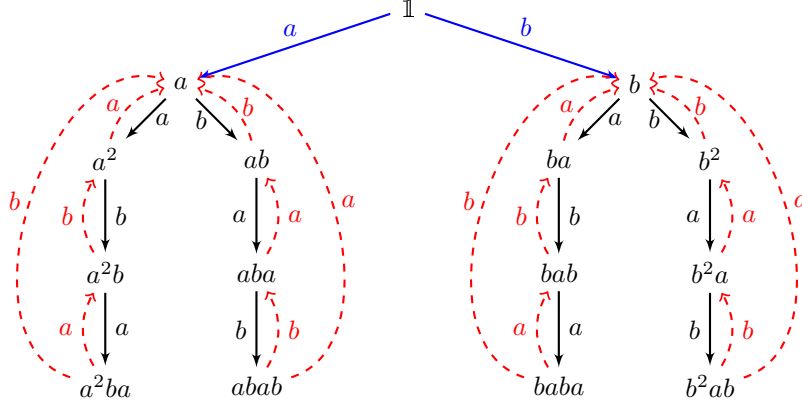


FIGURE 3. The McCammond expansion of $(\Gamma, \mathbb{1}) = \text{KR}(S, A)$ of Figure 2. The edges $(p, a, q) \in E(\Gamma^{\text{Mc}})$ with $\ell(q) = \ell(p) + 1$ are solid, whereas the edges with $\ell(q) \leq \ell(p)$ are dashed and red.

For a non-simple path in $(\Gamma^{\text{Mc}}, \mathbb{1})$, we can remove loops; it does not matter in which order these loops are removed. This is also known as the Church–Rosser property [CR36] or a Knuth–Bendix rewriting system. This is proved in [MRS11].

We denote the McCammond expansion of an A -semigroup (S, A) by $\text{Mc}(S, A)$, which is the McCammond expansion of its right Cayley graph. The McCammond expansion of the right Cayley graph is in general not a right Cayley graph, see Remark 2.19 below. This makes random walks on semigroups more difficult to understand. It is however true that the McCammond expansion is stable under repeated McCammond expansions if the root is unchanged.

Remark 2.19. The McCammond expansion of a right Cayley graph is not always a right Cayley graph itself. Let S be the semigroup generated by the elements in $A = \{a_1, a_2, a_3, c\}$ which act on the states $Q = \{0, 1, 2, 3, \square\}$ as follows:

$$\begin{array}{rclcl}
 & & & & 0 \mapsto 0 \\
 a_1: & 0 \mapsto 1 & , & a_2: & 1 \mapsto 2 & , & a_3: & 2 \mapsto 3 & , & c: & 1 \mapsto 0 \\
 & 3 \mapsto 2 & , & & 2 \mapsto 1 & , & & 1 \mapsto 3 & , & & 2 \mapsto 3 & , \\
 & & & & & & & & & & 3 \mapsto 0
 \end{array}$$

and otherwise $q \mapsto \square$. For part of the right Cayley graph of S , see Figure 4. Then $a_1a_2a_3$ and $a_1a_2a_3a_1a_2a_3$ end at the same vertex in both $\text{RCay}(S, A)$ and $\text{Mc} \circ \text{KR}(S, A)$, that is $\tau(a_1a_2a_3) = \tau(a_1a_2a_3a_1a_2a_3)$. However, multiplying on the left by c , we obtain

$$ca_1a_2a_3 = \tau(ca_1a_2a_3) \neq \tau(ca_1a_2a_3a_1a_2a_3) = ca_1a_3$$

in $\text{Mc} \circ \text{KR}(S, A)$, as can be seen from the right picture in Figure 4.

2.6. Normal forms. Consider $\Gamma(S, A) := \text{Mc} \circ \text{KR}(S, A)$. The non-empty paths in $\Gamma(S, A)$ starting at $\mathbb{1}$ are naturally labeled by elements in A^+ . The elements in the semaphore code $\mathcal{S}(S, A)$ of Definition 2.11 are in natural correspondence with the paths $p = a_1a_2 \cdots a_\ell \in A^+$ in the rooted graph $\Gamma(S, A)$ starting at $\mathbb{1}$ such that $[a_1a_2 \cdots a_\ell]_S \in K(S)$, but $[a_1a_2 \cdots a_{\ell-1}]_S \notin K(S)$. The paths in $\mathcal{S}(S, A)$, considered in $\Gamma(S, A)$, do not necessarily have to be simple, that is, they can contain loops.

Recall from Definition 2.18, that the vertices in $\Gamma(S, A)$ are simple paths (that is, paths without loops) in the Karnofsky–Rhodes expansion of the right Cayley graph of (S, A) . Hence given an arbitrary path p in $\Gamma(S, A)$, its endpoint $\tau(p)$ is a vertex in $\Gamma(S, A)$ and hence a simple path in $\text{KR}(S, A)$, which is p read in $\Gamma(S, A)$ with “loops stripped away”.

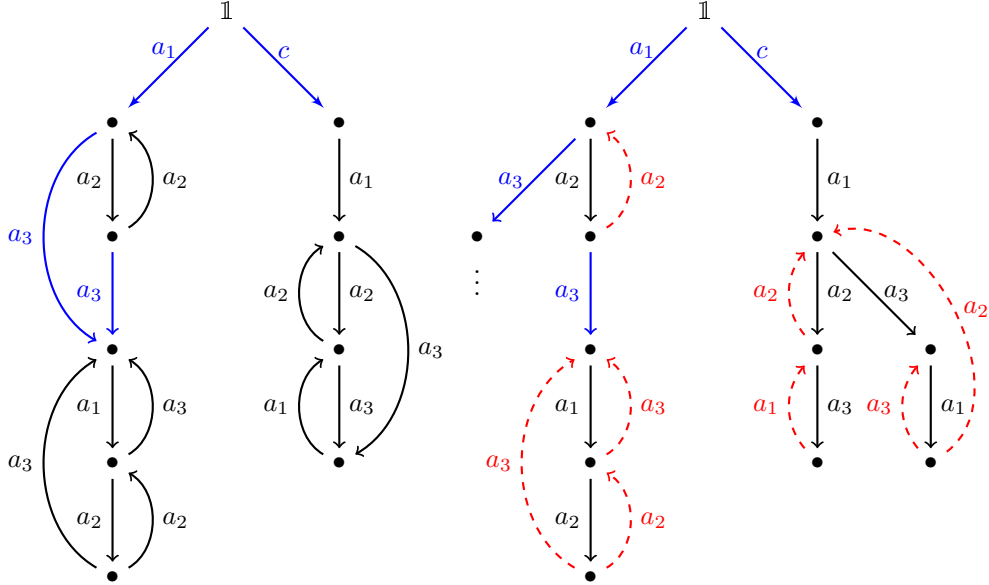


FIGURE 4. **Left:** Part of the right Cayley graph of the semigroup (S, A) in Remark 2.19; transition edges are blue; only one c edge is drawn. **Right:** Part of the expansion $Mc \circ KR(S, A)$.

Definition 2.20. Define the set of *normal forms*

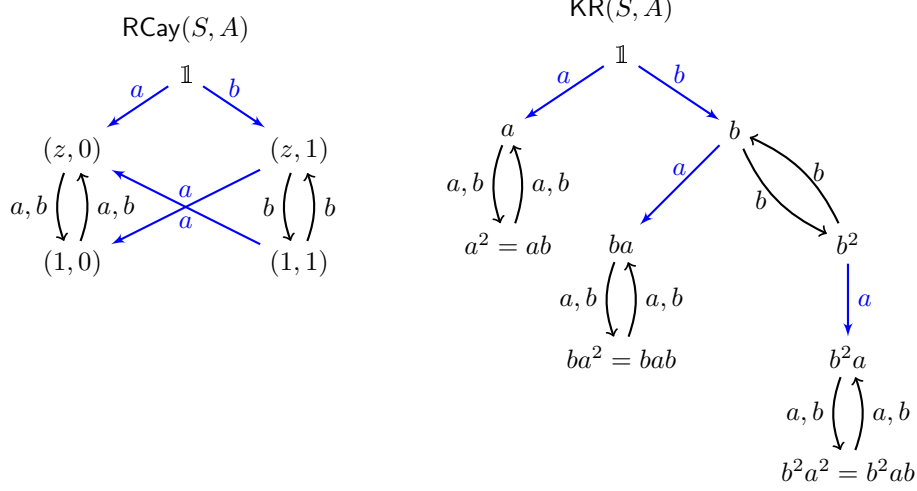
$$\mathcal{N}(S, A) = \{\tau(p) \mid p \in \mathcal{S}(S, A)\}.$$

Note that the normal forms in $\mathcal{N}(S, A)$ are precisely the elements in $\mathcal{S}(S, A)$ (as considered as elements in $\Gamma(S, A)$) without loops. In other words, they are the shortest simple paths in $KR(S, A)$ from the root $\mathbb{1}$ to the ideal.

Remark 2.21. We can construct a new graph from $\Gamma(S, A)$ by contracting each \mathcal{R} -class to a vertex. The remaining edges correspond to transition edges in $RCay(S, A)$. The resulting graph is a tree. Given an ideal $I \subseteq S$, the vertices v in $\Gamma(S, A)$ such that $[v]_S \in I$ project to a lower set in this tree.

As outlined in Section 2.3, there is a Markov chain $\mathcal{M}^{S(S, A)}$ associated to the semaphore code $S(S, A)$.

Example 2.22. Consider $S = Z_2 \times \{0, 1\}$ with generators $A = \{a, b\}$ with $a = (z, 0)$ and $b = (z, 1)$, where z is the generator of Z_2 with $z^2 = 1$ and the operation on $\{0, 1\}$ is multiplication. The right Cayley graph and its Karnofsky–Rhodes expansion are given by



$\text{KR}(S, A)$ is not stable under Mc since $\text{KR}(S, A)$ does not have the unique path property: there are several edges, where a and b act in the same way which is reflected in the fact that $a^2 = ab$, $ba^2 = bab$ and $b^2a^2 = b^2ab$. The minimal ideal is $K(S) = \{(z, 0), (1, 0)\}$.

Let us consider the semigroup $S' := S/K(S)$. Then

$$\mathcal{N}(S', A) = \{a, ba, b^2a\} \quad \text{and} \quad \mathcal{S}(S', A) = \{a, bb^{2k}a, b^2b^{2k}a \mid k \geq 0\}.$$

The Markov chain $\mathcal{M}^{S(S', A)}$ is given by the left action

$$\begin{aligned} a.a &= a, & a.bb^{2k}a &= a, & a.b^2b^{2k}a &= a, \\ b.a &= ba, & b.bb^{2k}a &= b^2b^{2k}a, & b.b^2b^{2k}a &= bb^{2(k+1)}a. \end{aligned}$$

2.7. Lumping. We are now ready to construct the two Markov chains $\mathcal{M}(S, A)$ and $\mathcal{M}(\text{KR}(S, A))$ as projections or lumpings of $\mathcal{M}^{S(S, A)}$ in the case when the minimal ideal $K(S)$ is left zero. The chain $\mathcal{M}^{S(S, A)}$ is called a *prelibrary* and $\mathcal{M}(S, A)$ and $\mathcal{M}(\text{KR}(S, A))$ are called *libraries* when $K(S)$ is left zero.

The lumped chain $\mathcal{M}(\text{KR}(S, A))$ is obtained from $\mathcal{M}^{S(S, A)}$ via the equivalence relation $s \sim s'$ if $[s]_{\text{KR}(S, A)} = [s']_{\text{KR}(S, A)}$ for $s, s' \in \mathcal{S}(S, A)$, whereas the lumped chain $\mathcal{M}(S, A)$ is obtained from $\mathcal{M}(\text{KR}(S, A))$ via the equivalence relation $w \sim w'$ if $[w]_{(S, A)} = [w']_{(S, A)}$ for $w, w' \in \text{KR}(S, A)$. In particular, for $w, w' \in \text{KR}(S, A)$ (resp. (S, A)) we transition $w \xrightarrow{a} w'$ with probability x_a , where $w' = aw$.

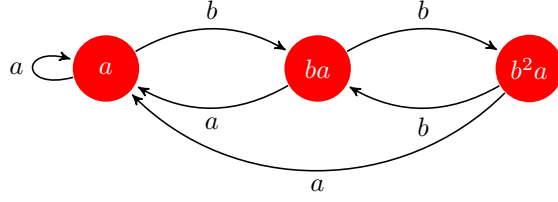
Corollary 2.23. *If $K(S)$ is left zero, the Markov chain $\mathcal{M}(\text{KR}(S, A))$ is the lumping of $\mathcal{M}^{S(S, A)}$. Furthermore, the stationary distribution of the library $\mathcal{M}(\text{KR}(S, A))$ is given by*

$$\Psi_w^{\text{KR}(S, A)} = \sum_{\substack{s \in \mathcal{S}(S, A) \\ [s]_{\text{KR}(S, A)} = w}} \Psi_s^{S(S, A)} = \sum_{\substack{s \in \mathcal{S}(S, A) \\ [s]_{\text{KR}(S, A)} = w}} \prod_{a \in s} x_a \quad \text{for all } w \in K(\text{KR}(S, A)).$$

Proof. The second equality follows directly from Proposition 2.8. The other statement follows in a similar fashion as in the proof of Theorem 2.12. Note also that since $K(S)$ is left zero, we have $K(\text{KR}(S, A)) = \{[v]_{\text{KR}(S, A)} \mid v \in \mathcal{N}(S, A)\}$. \square

Remark 2.24. Note that the states $w \in \text{KR}(S, A) \setminus K(\text{KR}(S, A))$ are not recurrent and hence $\Psi_w^{\text{KR}(S, A)} = 0$.

Remark 2.25. If the McCammond expansion of the Karnofsky–Rhodes expansion of the right Cayley graph of (S, A) is stable, then by Proposition 2.15 $\text{Mc} \circ \text{KR}(S, A) = \text{KR}(S, A)$ is a right Cayley graph. In this case Corollary 2.23 follows from the fact that $\text{Mc} \circ \text{KR}(S, A)$ is a right Cayley graph.

FIGURE 5. Markov chain $\mathcal{M}(\text{KR}(S', A))$ of Example 2.26.

Example 2.26. Let us continue Example 2.22. The minimal ideal $K(S')$ is trivial and therefore left zero. Hence by Corollary 2.23, the Markov chain $\mathcal{M}(\text{KR}(S', A))$ is obtained from $\mathcal{M}^{S(S', A)}$ by lumping and is depicted in Figure 5.

To compute the stationary distributions of Corollary 2.23 more explicitly, we will need the notion of Kleene expressions and Zimin words as discussed in the next section. To this end, we need to break the lumping into two steps. First we project each semaphore code word $s \in \mathcal{S}(S, A)$ (or path possibly with loops from $\mathbb{1}$ to $\mathcal{N}(S, A)$) to its normal form (or endpoint) in $\mathcal{N}(S, A)$ given by $\tau(s)$. In other words, we map each code word $s \in \mathcal{S}(S, A)$ to the word resulting from s by reading it in $\Gamma(S, A)$ with the “loops stripped away”. Next we identify any normal forms in $\mathcal{N}(S, A)$ that represent the same element in $\text{KR}(S, A)$.

For the stationary distribution of the projected Markov chain, we need the preimage of the normal forms of Definition 2.20.

Definition 2.27. For a normal form $w \in \mathcal{N}(S, A)$, define

$$\text{NF}^{-1}(w) = \{s \in \mathcal{S}(S, A) \mid \tau(s) = w\} \subseteq \mathcal{S}(S, A).$$

Furthermore, define

$$\text{Red}_{\text{KR}(S, A)}(w) = \{n \in \mathcal{N}(S, A) \mid [n]_{\text{KR}(S, A)} = w\} \quad \text{for } w \in K(\text{KR}(S, A)).$$

Note that, if the graph $\Gamma(S, A)$ has non-trivial connected components, then the paths in $\text{NF}^{-1}(w)$ from the root to the ideal can be unbounded in length. With Definition 2.27, we rewrite the stationary distributions of Corollary 2.23 as

$$(2.7) \quad \Psi_w^{\text{KR}(S, A)} = \sum_{v \in \text{Red}_{\text{KR}(S, A)}(w)} \sum_{s \in \text{NF}^{-1}(v)} \Psi_s^{\mathcal{S}(S, A)}.$$

Furthermore, if the McCammond expansion of the Karnofsky–Rhodes expansion of $\text{RCay}(S, A)$ is stable, that is $\text{Mc} \circ \text{KR}(S, A) = \text{KR}(S, A)$, then $\text{Red}_{\text{KR}(S, A)}(w) = \{w\}$ and (2.7) simplifies to

$$(2.8) \quad \Psi_w^{\text{KR}(S, A)} = \sum_{s \in \text{NF}^{-1}(w)} \Psi_s^{\mathcal{S}(S, A)}.$$

Finally, the Markov chain $\mathcal{M}(S, A)$ can be obtained from $\mathcal{M}(\text{KR}(S, A))$ by lumping as well.

Corollary 2.28. *The Markov chain $\mathcal{M}(S, A)$ is the lumping of $\mathcal{M}(\text{KR}(S, A))$ with stationary distribution*

$$\Psi_w^{(S, A)} = \sum_{\substack{v \in \text{KR}(S, A) \\ [v]_{(S, A)} = w}} \Psi_v^{\text{KR}(S, A)} \quad \text{for all } w \in (S, A).$$

Proof. The lumping condition (2.2) follows from the fact that both (S, A) and $\text{KR}(S, A)$ are semi-groups. \square

2.8. Kleene expressions and Zimin words. We are now going to discuss how to compute the stationary distribution $\Psi_w^{\text{KR}(S,A)}$ of Corollary 2.23 more explicitly using Kleene expressions and Zimin words.

Given a set L , set $L^0 = \{\varepsilon\}$ given by the empty string, $L^1 = L$, and recursively $L^{i+1} = \{wa \mid w \in L^i, a \in L\}$ for each integer $i > 0$. Then the *Kleene star* of L is

$$L^* = \bigcup_{i \geq 0} L^i.$$

The collection of *regular languages* over an alphabet A is the smallest collection of languages

- containing the empty language and the singletons $\{a\}$ for $a \in A$;
- closed under union, concatenation and Kleene star.

A *Kleene expression* is an expression involving letters in A , unions, \cdot , and \star and is a compact way to describe a regular language.

Zimin words allow us to rewrite the star of a union in terms of just products and star:

$$(2.9) \quad (\{a\} \cup \{b\})^* = \{a, b\}^* = a^*(ba^*)^*.$$

Iterating (2.9), we can also express the star of the union of three elements in terms of multiplication and star as

$$\{a, b, c\}^* = \{a, b\}^*(c\{a, b\}^*)^* = a^*(ba^*)^*(ca^*(ba^*)^*)^*,$$

or more generally

$$(2.10) \quad \{a_1, a_2, \dots, a_n\}^* = \{a_1, \dots, a_{n-1}\}^*(a_n\{a_1, \dots, a_{n-1}\}^*)^* \\ = \{a_1, \dots, a_{n-2}\}^*(a_{n-1}\{a_1, \dots, a_{n-2}\}^*)^*(a_n\{a_1, \dots, a_{n-2}\}^*(a_{n-1}\{a_1, \dots, a_{n-2}\}^*)^*)^* = \dots.$$

The following result was proven in [MRS11, Section 3.2].

Theorem 2.29. [MRS11] *For $w \in \mathcal{N}(S, A)$, $\text{NF}^{-1}(w)$ has a Kleene expression without union, that is, only using letters in A , \cdot and \star .*

We can now use a Kleene expression for $\text{NF}^{-1}(w)$ in (2.7) to compute the stationary distribution $\Psi_w^{\text{KR}(S,A)}$. The advantage in doing so is that one can immediately obtain rational expressions. Namely, using the geometric series, we find that

$$\sum_{s \in a^*} \Psi_s^S = \sum_{\ell=0}^{\infty} x_a^\ell = \frac{1}{1-x_a}.$$

Similarly, using the Zimin words (2.9)

$$\sum_{s \in \{a,b\}^*} \Psi_s^S = \sum_{s \in a^*(ba^*)^*} \Psi_s^S = \frac{1}{1-x_a} \cdot \frac{1}{1-\frac{x_b}{1-x_a}} = \frac{1}{1-x_a-x_b}.$$

In general, using the recursion (2.10) we derive by induction

$$(2.11) \quad \sum_{s \in \{a_1, a_2, \dots, a_n\}^*} \Psi_s^S = \frac{1}{1-x_{a_1}-x_{a_2}-\dots-x_{a_n}}.$$

To take advantage of this, it is important that every element in $\text{NF}^{-1}(v)$ appearing in (2.7) occurs exactly once in the Kleene expression. This is not necessarily true for any Kleene expression. For example, in a^*a^* the letter a appears more than once. The condition is, however, ensured if

$$(2.12) \quad \sum_{w \in K(\text{KR}(S,A))} \Psi_w^{\text{KR}(S,A)} = 1.$$

Conjecture 2.30. The Kleene expressions constructed in [MRS11] satisfy (2.12).

In all examples worked out in this paper, Conjecture 2.30 holds. We believe that the proof is a straightforward induction on the proof in [MRS11] of the existence of such Kleene expressions, but the details are beyond the goal of this paper.

Example 2.31. Continuing Examples 2.22 and 2.26, we find that for the elements in $\mathcal{N}(S', A)$

$$\mathbf{NF}^{-1}(a) = a, \quad \mathbf{NF}^{-1}(ba) = b(b^2)^*a, \quad \mathbf{NF}^{-1}(b^2a) = b^2(b^2)^*a.$$

Since $K(S')$ is left zero, we find by (2.8) that for $w \in \mathcal{N}(S', A)$

$$\Psi_w^{\mathbf{KR}(S', A)} = \sum_{s \in \mathbf{NF}^{-1}(w)} \Psi_s^{S(S', A)}$$

and specifically

$$\Psi_a^{\mathbf{KR}(S', A)} = x_a, \quad \Psi_{ba}^{\mathbf{KR}(S', A)} = \frac{x_a x_b}{1 - x_b^2}, \quad \Psi_{b^2a}^{\mathbf{KR}(S', A)} = \frac{x_a x_b^2}{1 - x_b^2}.$$

Note that since $x_a + x_b = 1$

$$\Psi_a^{\mathbf{KR}(S', A)} + \Psi_{ba}^{\mathbf{KR}(S', A)} + \Psi_{b^2a}^{\mathbf{KR}(S', A)} = x_a + \frac{x_a x_b (1 + x_b)}{1 - x_b^2} = x_a + \frac{x_a x_b}{1 - x_b} = x_a + x_b = 1,$$

verifying (2.12).

For another example, see Section 3.1.

2.9. The bar and flat operation. We will now discuss the *bar* and *flat operation* [LRS17], which will make it possible to extend Corollary 2.23 to any finite A -semigroup S , not just those whose minimal ideal $K(S)$ is left zero.

In [LRS17], $(S, A)^{\text{bar}}$ was defined by considering the right regular representation of s acting faithfully on S^1 and by adding all constant maps on S^1 ; the constant map onto $s \in S^1$ is denoted by \bar{s} . The semigroup multiplication of $(S, A)^{\text{bar}}$ is the composition of functions acting on the right. In this spirit, let $(S, A)^{\text{bar}} = (S^{\text{bar}}, A \cup \{\bar{1}\})$, where $S^{\text{bar}} = S \cup \bar{S} \cup \{\bar{1}\}$ and $\bar{S} = \{\bar{x} \mid x \in S\}$. The element $\bar{1}$ acts as a constant map $z \cdot \bar{1} = \bar{1}$ and in addition $\bar{1} \cdot z = \bar{z}$ for any $z \in S^{\text{bar}}$, where we interpret $\bar{\bar{x}} = \bar{x}$ if $z = \bar{x}$. Furthermore, for any $x, y \in S$

$$x \cdot \bar{y} = \bar{y}, \quad \bar{x} \cdot \bar{y} = \bar{y}, \quad \text{and} \quad \bar{x} \cdot y = \overline{x \cdot y}.$$

Given a semigroup S , let S^{op} denote the semigroup obtained by reversing the multiplication on S . Then $(S, A)^{\flat} = (((S, A)^{\text{op}})^{\text{bar}})^{\text{op}}$. The relations with respect to bar get reversed, that is, $\bar{\bar{1}} \cdot z = \bar{\bar{1}}$ and $z \cdot \bar{\bar{1}} = \bar{\bar{z}}$ for any $z \in S^{\flat}$. Furthermore, for any $x, y \in S$

$$\bar{\bar{y}} \cdot x = \bar{\bar{y}}, \quad \bar{\bar{y}} \cdot \bar{x} = \bar{\bar{y}}, \quad \text{and} \quad y \cdot \bar{\bar{x}} = \widetilde{y \cdot x}.$$

In particular, $K((S, A)^{\flat}) = \{\bar{\bar{z}} \mid z \in S\} \cup \{\bar{\bar{1}}\}$, which is left zero.

Remark 2.32. Note that up to the labeling of the vertices, $\mathbf{KR}(S \cup \{\square\}, A \cup \{\square\})$ is exactly $(\mathbf{KR}(S, A)^{\flat})^{\flat}$. This is true since right multiplication in $(\mathbf{KR}(S, A)^{\flat})^{\flat}$ by any $\bar{\bar{z}}$ with $z \in S$ or $\bar{\bar{1}}$ lands in $K((\mathbf{KR}(S, A)^{\flat})^{\flat})$ and similarly right multiplication by \square in $\mathbf{KR}(S \cup \{\square\}, A \cup \{\square\})$ also lands in $K(\mathbf{KR}(S \cup \{\square\}, A \cup \{\square\}))$. Hence the flat \flat operation can be interpreted as adding a new zero to the semigroup.

We may now generalize Corollary 2.23. Recall the hypothesis of Corollary 2.23 that $K(S)$ is left zero. In Corollary 2.33 below there is no restriction on $K(S)$. The proof uses the flat construction to reduce the general case to the previous case and then limiting the probability of the new variable to zero.

Corollary 2.33. *If $K(S)$ is not left zero, the Markov chain $\mathcal{M}(\mathbf{KR}(S, A))$ has stationary distribution*

$$(2.13) \quad \Psi_w^{\mathbf{KR}(S, A)} = \lim_{x_{\square} \rightarrow 0} \left(\sum_{\substack{s \in S(S \cup \{\square\}, A \cup \{\square\}) \\ [s]_{\mathbf{KR}(S \cup \{\square\}, A \cup \{\square\})} = w \square}} \prod_{a \in s} x_a \right) \quad \text{for all } w \in K(\mathbf{KR}(S, A)).$$

Proof. Consider $\text{KR}(S', A')$ where $S' = S \cup \{\square\}$ and $A' = A \cup \{\square\}$ (or equivalently $(\text{KR}(S, A))^b$). The minimal ideal of this semigroup consists of all elements $w\square$, where $w \in S$. Since $\square v = \square$ for all $v \in S$, the minimal ideal is left zero. Hence Corollary 2.23 applies to $\mathcal{M}(\text{KR}(S', A'))$ and

$$\Psi_{w\square}^{\text{KR}(S', A')} = \sum_{\substack{s \in S(S', A') \\ [s]_{\text{KR}(S', A')} = w\square}} \Psi_s^{S(S', A')} \quad \text{for all } w \in S.$$

Taking the limit x_\square to zero, the states with nonzero stationary probability will be in $K(\text{KR}(S, A))$ and we obtain (2.13) for the stationary distribution. \square

Example 2.34. Recall the semigroup $S = Z_2 \times \{0, 1\}$ with generators $A = \{a, b\}$ with $a = (z, 0)$ and $b = (z, 1)$, where z is the generator of Z_2 with $z^2 = 1$, from Example 2.22. If we add a generator \square which acts as zero, then the normal forms of $\Gamma(S', A') = \text{Mc} \circ \text{KR}(S', A')$ with $S' = S \cup \{\square\}$ and $A' = A \cup \{\square\}$ are given by

$$\mathcal{N}(S', A') = \{\square, a\square, a^2\square, ab\square, b\square, b^2\square, ba\square, ba^2\square, bab\square, b^2a\square, b^2a^2\square, b^2ab\square\}.$$

We have

$$\begin{aligned} \text{NF}^{-1}(\square) &= \square, & \text{NF}^{-1}(a\square) &= a(\{a, b\}^2)^* \square, \\ \text{NF}^{-1}(a^2\square) &= a(\{a, b\}^2)^* a\square, & \text{NF}^{-1}(ab\square) &= a(\{a, b\}^2)^* b\square, \\ \text{NF}^{-1}(b\square) &= b(b^2)^* \square, & \text{NF}^{-1}(b^2\square) &= b(b^2)^* b\square, \\ \text{NF}^{-1}(ba\square) &= b(b^2)^* a(\{a, b\}^2)^* \square, & \text{NF}^{-1}(ba^2\square) &= b(b^2)^* a(\{a, b\}^2)^* a\square, \\ \text{NF}^{-1}(bab\square) &= b(b^2)^* a(\{a, b\}^2)^* b\square, & \text{NF}^{-1}(b^2a\square) &= b(b^2)^* ba(\{a, b\}^2)^* \square, \\ \text{NF}^{-1}(b^2a^2\square) &= b(b^2)^* ba(\{a, b\}^2)^* a\square, & \text{NF}^{-1}(b^2ab\square) &= b(b^2)^* ba(\{a, b\}^2)^* b\square. \end{aligned}$$

Hence the stationary distribution for $\mathcal{M}(\text{KR}(S', A'))$ is

$$\begin{aligned} \Psi_{\square}^{\text{KR}(S', A')} &= x_\square, & \Psi_{a\square}^{\text{KR}(S', A')} &= \frac{x_a x_\square}{1 - (x_a + x_b)^2}, \\ \Psi_{a^2\square}^{\text{KR}(S', A')} &= \frac{x_a(x_a + x_b)x_\square}{1 - (x_a + x_b)^2}, & \Psi_{b\square}^{\text{KR}(S', A')} &= \frac{x_b x_\square}{1 - x_b^2}, \\ \Psi_{b^2\square}^{\text{KR}(S', A')} &= \frac{x_b^2 x_\square}{1 - x_b^2}, & \Psi_{ba\square}^{\text{KR}(S', A')} &= \frac{x_a x_b x_\square}{(1 - x_b^2)(1 - (x_a + x_b)^2)}, \\ \Psi_{ba^2\square}^{\text{KR}(S', A')} &= \frac{x_a(x_a + x_b)x_b x_\square}{(1 - x_b^2)(1 - (x_a + x_b)^2)}, & \Psi_{b^2a\square}^{\text{KR}(S', A')} &= \frac{x_a x_b^2 x_\square}{(1 - x_b^2)(1 - (x_a + x_b)^2)}, \\ \Psi_{b^2a^2\square}^{\text{KR}(S', A')} &= \frac{x_a(x_a + x_b)x_b^2 x_\square}{(1 - x_b^2)(1 - (x_a + x_b)^2)}. \end{aligned}$$

Note that $\Psi_{a\square}^{\text{KR}(S', A')} + \Psi_{a^2\square}^{\text{KR}(S', A')} = x_a$, $\Psi_{b\square}^{\text{KR}(S', A')} + \Psi_{b^2\square}^{\text{KR}(S', A')} = \frac{x_\square x_b}{1 - x_b}$, and

$$\Psi_{ba\square}^{\text{KR}(S', A')} + \Psi_{ba^2\square}^{\text{KR}(S', A')} + \Psi_{b^2a\square}^{\text{KR}(S', A')} + \Psi_{b^2a^2\square}^{\text{KR}(S', A')} = \frac{x_a x_b x_\square}{(1 - x_b)(1 - x_a - x_b)}.$$

Hence the total sum of all stationary states is $x_\square + x_a + x_b = 1$.

Taking the limit $x_\square \rightarrow 0$, we obtain the stationary distribution for $\mathcal{M}(\text{KR}(S, A))$. Note that using $x_\square + x_a + x_b = 1$, we have $1 - (x_a + x_b)^2 = 1 - (1 - x_\square)^2 = 2x_\square - x_\square^2$. Hence

$$\begin{aligned} \Psi_a^{\text{KR}(S,A)} &= \frac{x_a}{2}, & \Psi_{a^2}^{\text{KR}(S,A)} &= \frac{x_a}{2}, \\ \Psi_b^{\text{KR}(S,A)} &= 0, & \Psi_{b^2}^{\text{KR}(S,A)} &= 0, \\ \Psi_{ba}^{\text{KR}(S,A)} &= \frac{x_a x_b}{2(1 - x_b^2)}, & \Psi_{ba^2}^{\text{KR}(S,A)} &= \frac{x_a x_b}{2(1 - x_b^2)}, \\ \Psi_{b^2 a}^{\text{KR}(S,A)} &= \frac{x_a x_b^2}{2(1 - x_b^2)}, & \Psi_{b^2 a^2}^{\text{KR}(S,A)} &= \frac{x_a x_b^2}{2(1 - x_b^2)}. \end{aligned}$$

We can lump further to $\mathcal{M}(S, A)$ by using that $a = b^2 a = ba^2$ and $a^2 = ba = b^2 a^2$ in S , so that

$$\Psi_a^{(S,A)} = \Psi_{a^2}^{(S,A)} = \frac{x_a}{2} \left(1 + \frac{x_b^2}{1 - x_b^2} + \frac{x_b}{1 - x_b^2} \right) = \frac{x_a}{2} \left(1 + \frac{x_b}{1 - x_b} \right) = \frac{x_a}{2} + \frac{x_b}{2} = \frac{1}{2}.$$

2.10. Bounds on the mixing time. In this section we use the techniques developed in [ASST15b] combined with the normal forms coming from the McCammond expansion to provide an upper bound on the mixing time for $\mathcal{M}(\text{KR}(S, A))$ and $\mathcal{M}(S, A)$.

The total variation distance between two probability distributions ν and μ is defined by

$$\|\nu - \mu\|_{TV} = \max_{B \subseteq S} |\nu(B) - \mu(B)|,$$

where $\nu(B) = \sum_{s \in B} \nu(s)$. Let \mathcal{M} be a finite state irreducible Markov chain with stationary distribution Ψ . Let $d(k) = \sup_\nu \|\mathcal{T}^k \nu - \Psi\|$. Then, for $\varepsilon > 0$, the *mixing time* of \mathcal{M} is [LPW09]

$$t_{\text{mix}}(\varepsilon) = \min\{k \mid d(k) \leq \varepsilon\}.$$

Often authors choose $\varepsilon = e^{-1}$ or $\varepsilon = 1/4$ to define the mixing time. We bound for $c > 0$, when $\|\mathcal{T}^k \nu - \Psi\| \leq e^{-c}$.

Lemma 2.35. [ASST15b, Lemma 3.6] *Let \mathcal{M} be an irreducible Markov chain associated to the semigroup S and probability distribution $0 \leq p(s) \leq 1$ for $s \in S$. We assume that $\{s \in S \mid p(s) > 0\}$ generates S . Let Ψ be the stationary distribution and $f: S \rightarrow \mathbb{N}$ be a function, called a **statistic**, such that:*

- (1) $f(s's) \leq f(s)$ for all $s, s' \in S$;
- (2) if $f(s) > 0$, then there exists $s' \in S$ with $p(s') > 0$ such that $f(s's) < f(s)$;
- (3) $f(s) = 0$ if and only if $s \in K(S)$.

Then if $p = \min\{p(s) \mid s \in S, p(s) > 0\}$ and $n = f(\mathbb{1})$, we have that

$$\|\mathcal{T}^k \nu - \Psi\|_{TV} \leq \sum_{i=0}^{n-1} \binom{k}{i} p^i (1-p)^{k-i} \leq \exp\left(-\frac{(kp - (n-1))^2}{2kp}\right),$$

for any probability distribution ν on S , where the last inequality holds as long as $k \geq (n-1)/p$.

Now consider $\mathcal{M}(S, A)$ for an A -semigroup (S, A) with probabilities $0 < x_a \leq 1$ on the generators. Let n be the maximal length of a chain from $\mathbb{1}$ to a leaf in the McCammond expansion $\text{Mc} \circ \text{KR}(S, A)$. Let ℓ be the maximal distance between two transition edges in $\text{Mc} \circ \text{KR}(S, A)$ (which is well-defined since $\text{Mc} \circ \text{KR}(S, A)$ is a tree). Define the statistic $f(s)$ to be the maximum of the number of transition arrows in the unique path from v to a leaf in $\text{Mc} \circ \text{KR}(S, A)$, where v runs over all vertices in $\text{Mc} \circ \text{KR}(S, A)$ such that $[v]_{\text{KR}(S,A)} = s$. Note that this statistic is constant on \mathcal{R} -classes. Furthermore, it satisfies Conditions (1) and (3) of Lemma 2.35. Condition (2) is not necessarily satisfied. However, if we take ℓ steps at a time in the original Markov chain $\mathcal{M}(S, A)$, then with probability at least $p^\ell > 0$, where $p = \min\{x_a \mid a \in A\}$, the statistic strictly decreases. Hence, in

this random walk, where we take ℓ steps at a time as compared to $\mathcal{M}(S, A)$, Lemma 2.35 applies and we obtain

$$\|\mathcal{T}^{k\nu} - \Psi\|_{TV} \leq \exp\left(-\frac{(kp^\ell/\ell - (n-1)/\ell)^2}{2kp^\ell/\ell}\right) = \exp\left(-\frac{(kp^\ell - (n-1))^2}{2k\ell p^\ell}\right).$$

This shows that the mixing time is at most $2(n + \ell c - 1)/p^\ell$ (see also [AKS14b, Section 6]).

3. EXAMPLES

In this section we derive explicit Markov chains from the general theory of Section 2 for specific choices of A -semigroups (S, A) . We begin by treating known examples, such as the Tsetlin library, in this setting and then move on to new examples.

3.1. The Tsetlin library. The Tsetlin library [Cet63] is a Markov chain whose states are all permutations S_n of n books (on a shelf). Given $\pi \in S_n$, construct $\pi' \in S_n$ from π by removing book a from the shelf and inserting it to the front. In this case write $\pi \xrightarrow{a} \pi'$. Let $0 < x_a \leq 1$ be probabilities for each $1 \leq a \leq n$ such that $\sum_{a=1}^n x_a = 1$. In the Tsetlin library Markov chain, we transition $\pi \xrightarrow{a} \pi'$ with probability x_a . The stationary distribution for the Tsetlin library was derived by Hendricks [Hen72, Hen73]

$$(3.1) \quad \Psi_\pi = \prod_{i=1}^n \frac{x_{\pi_i}}{x_{\pi_{i+1}} + \cdots + x_{\pi_n}} \quad \text{for all } \pi \in S_n.$$

We are now going to derive the stationary distribution using the methods developed in Section 2.

Consider the semigroup $P(n)$, which consists of the set of all non-empty subsets of $\{1, 2, \dots, n\}$. Multiplication in $P(n)$ is union of sets. We pick as generators $A = [n] := \{1, 2, \dots, n\}$. Then the right Cayley graph $\text{RCay}(P(n), [n])$ is the Boolean poset with $\mathbb{1}$ as root. The right Cayley graph for $P(3)$ is depicted in Figure 6. Except for the loops at a given vertex, all edges are transitional. Hence $\Gamma(P(n), [n]) = \text{Mc} \circ \text{KR}(P(n), [n]) = \text{KR}(P(n), [n])$ is a tree with leaves given by the permutations S_n of $[n]$. The case $n = 3$ is given in Figure 7. The minimal ideal is $K(P(n)) = [n]$ and

$$\mathcal{N}(P(n), [n]) = S_n.$$

Let $\pi \in S_n$ be a permutation. Then $\text{NF}^{-1}(\pi)$ consists of all paths in $\Gamma(P(n), [n])$ starting at $\mathbb{1}$ and ending at $\pi = \pi_1 \pi_2 \dots \pi_n$. Such a path has to pass through the vertex $\pi_1 \dots \pi_i$ for each $1 \leq i \leq n$. At a given vertex $\pi_1 \dots \pi_i$, it can loop with π_1, \dots, π_i . Hence we can write the paths to π via Kleene expressions as

$$\text{NF}^{-1}(\pi) = \pi_1 \pi_1^* \pi_2 \{\pi_1, \pi_2\}^* \cdots \pi_i \{\pi_1, \dots, \pi_i\}^* \cdots \pi_n.$$

Using the Zimin words (2.10), this expression can be written entirely in terms of star and multiplication without the sets.

The states of the Markov chain $\mathcal{M}^{\mathcal{S}(P(n), [n])}$ consist of all words in the alphabet $[n]$ that end once all letters in $[n]$ are used. The state space of the lumped Markov chain $\mathcal{M}(\text{KR}(P(n), [n]))$ is S_n . We transition $\pi \xrightarrow{a} \pi'$ with probability x_a , where π' is obtained from π by prepending a to π and removing the letter a from π . Equivalently, this corresponds to moving the letter a in π to the front, which is exactly the transition in the Tsetlin library.

By (2.8), the stationary distribution associated to $\pi \in S_n$ is

$$\Psi_\pi^{\text{KR}(P(n), [n])} = \sum_{s \in \text{NF}^{-1}(\pi)} \prod_{a \in s} x_a.$$

Using (2.11), this can be rewritten as

$$\Psi_\pi^{\text{KR}(P(n), [n])} = \prod_{i=1}^n \frac{x_{\pi_i}}{1 - \sum_{j=1}^{i-1} x_{\pi_j}} = \prod_{i=1}^n \frac{x_{\pi_i}}{\sum_{j=i+1}^n x_{\pi_j}},$$

where we used that $\sum_{j=1}^n x_{\pi_j} = 1$. This agrees with (3.1).

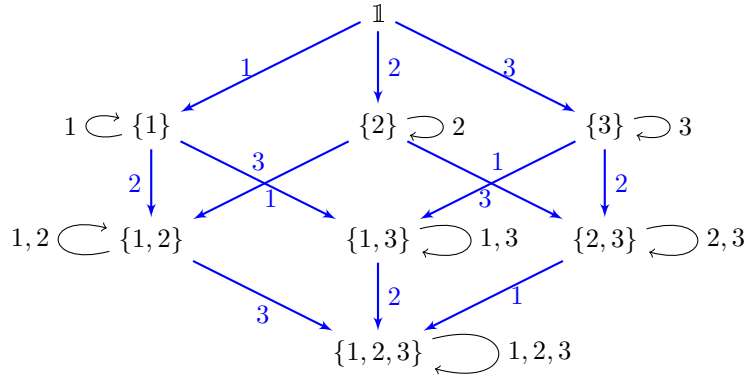


FIGURE 6. The right Cayley graph $\text{RCay}(S, A)$ with $S = P(3)$ and $A = \{1, 2, 3\}$. Transition edges are drawn in blue.

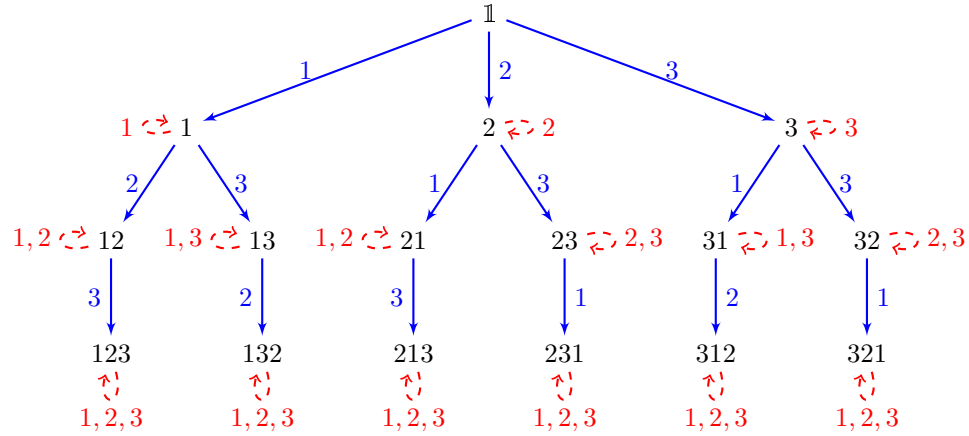


FIGURE 7. $\Gamma(P(3), [3]) = \text{Mc} \circ \text{KR}(P(3), [3]) = \text{KR}(P(3), [3])$, which is the Karnofsky–Rhodes expansion of the right Cayley graph of Figure 6.

3.2. Edge flipping on a line. The example of this section is a Markov chain obtained by edge flipping on a line and was suggested to us by Persi Diaconis. It is a Boolean arrangement [BHR99] for which stationary distributions were derived in [BD98] and which was also analyzed in [CG12]. Take a line with $n + 1$ vertices. Each vertex can either be 0 or 1. So the state space is $\mathcal{S} = \{0, 1\}^{n+1}$ of size 2^{n+1} . Pick edge i for $1 \leq i \leq n$ (between vertices i and $i + 1$) with probability x_i . Then with probability $\frac{1}{2}$ make the adjacent vertices both 0 (respectively both 1). Let us call this Markov chain \mathcal{M} .

In our setting, this Markov chain can be treated in a similar fashion to the Tsetlin library. Let $P^\pm(n)$ be the set of signed subsets of $[n]$, that is, take a subset of $[n]$ and in addition associate to each letter a sign $+$ or $-$. Right multiplication of such a subset X by a generator $x \in [\pm n] := \{\pm 1, \dots, \pm n\}$ is addition of x to X if neither x nor $-x$ are in X and otherwise return X . The minimal ideal in the Karnofsky–Rhodes expansion of this monoid is the set of signed permutations S_n^\pm . Here signed permutations are represented in one-line notation $\pi_1\pi_2 \dots \pi_n$, where $|\pi_1| \dots |\pi_n|$ is a permutation and $\pi_i \in [\pm n]$ for each $1 \leq i \leq n$.

The Kleene expression for $\pi \in S_n^\pm$ is very similar to the case of the Tsetlin library

$$\text{NF}^{-1}(\pi) = \pi_1 \{ \pm \pi_1 \}^* \pi_2 \{ \pm \pi_1, \pm \pi_2 \}^* \cdots \pi_i \{ \pm \pi_1, \dots, \pm \pi_i \}^* \cdots \pi_n.$$

The state space of the lumped Markov chain $\mathcal{M}(\text{KR}(P^\pm(n), [\pm n]))$ is S_n^\pm . We transition $\pi \xrightarrow{a} \pi'$ with probability y_a for $a \in [\pm n]$, where π' is obtained from π by prepending a to π and removing the letter a or $-a$ from π .

By (2.8), the stationary distribution associated to $\pi \in S_n^\pm$ is

$$\Psi_\pi^{\text{KR}(P^\pm(n), [\pm n])} = \sum_{s \in \text{NF}^{-1}(\pi)} \prod_{a \in s} y_a.$$

Using (2.11), this can be rewritten as

$$\Psi_\pi^{\text{KR}(P^\pm(n), [\pm n])} = \prod_{i=1}^n \frac{y_{\pi_i}}{1 - \sum_{j=1}^{i-1} (y_{\pi_j} + y_{-\pi_j})}.$$

The edge flipping Markov chain \mathcal{M} can be obtained from $\mathcal{M}(\text{KR}(P^\pm(n), [\pm n]))$ via the action of $P^\pm(n)$ on $\mathcal{S} = \{0, 1\}^{n+1}$. For $s \in \mathcal{S}$, the letter $a \in [n]$ acts on $s = s_1 \dots s_{n+1}$ by changing s_a and s_{a+1} to 0 and $-a$ acts by changing s_a and s_{a+1} to 1. A signed permutation $\pi \in S_n^\pm$ can be associated with a state s since $s := \pi \cdot s'$ is independent of $s' \in \mathcal{S}$ (since every letter appears once in π). Hence setting $y_a = y_{-a} = \frac{x_a}{2}$, we obtain the stationary distribution for s in \mathcal{M} by lumping

$$(3.2) \quad \Psi_s^{\mathcal{M}} = \sum_{\substack{\pi \in S_n^\pm \\ s = \pi \cdot 0^{n+1}}} \Psi_\pi^{\text{KR}(P^\pm(n), [\pm n])} = \frac{1}{2^n} \sum_{\substack{\pi \in S_n^\pm \\ s = \pi \cdot 0^{n+1}}} \prod_{i=1}^n \frac{x_i}{1 - \sum_{j=1}^{i-1} x_{|\pi_j|}}.$$

Note that more generally one could set $y_a = px_a$ and $y_{-a} = (1-p)x_a$ for $0 < p < 1$. The case above is $p = \frac{1}{2}$. Formula (3.2) has a similar structure as the stationary distributions in [BD98, Theorem 2] and [Den12, Eq. (2.5)].

Example 3.1. For $n = 2$, we have

$$\begin{aligned} \Psi_{12} = \Psi_{-12} = \Psi_{1-2} = \Psi_{-1-2} &= \frac{x_1 x_2}{4(1-x_1)} = \frac{x_1}{4}, \\ \Psi_{21} = \Psi_{-21} = \Psi_{2-1} = \Psi_{-2-1} &= \frac{x_1 x_2}{4(1-x_2)} = \frac{x_2}{4}, \end{aligned}$$

where we dropped the superscripts. Hence

$$\begin{aligned} \Psi_{000} = \Psi_{111} = \Psi_{12} + \Psi_{21} &= \frac{x_1 + x_2}{4} = \frac{1}{4}, \\ \Psi_{001} = \Psi_{110} = \Psi_{1-2} &= \frac{x_1}{4}, \\ \Psi_{010} = \Psi_{101} &= 0, \\ \Psi_{011} = \Psi_{100} = \Psi_{-21} &= \frac{x_2}{4}. \end{aligned}$$

Example 3.2. For $n = 3$, we have for example

$$\begin{aligned} \Psi_{0010} = \Psi_{1-23} &= \frac{x_1 x_2}{8(x_2 + x_3)}, \\ \Psi_{0001} = \Psi_{12-3} + \Psi_{21-3} + \Psi_{2-31} &= \frac{x_1 x_2 x_3}{8} \left(\frac{1}{x_3(x_2 + x_3)} + \frac{1}{x_3(x_1 + x_3)} + \frac{1}{x_1(x_1 + x_3)} \right). \end{aligned}$$

The most likely states are 0000 and 1111 with

$$\Psi_{0000} = \Psi_{1111} = \sum_{\pi \in S_3} \Psi_\pi + \Psi_{13-2} + \Psi_{31-2} = \frac{1}{8} \left(1 + \frac{x_1 x_3}{1-x_1} + \frac{x_1 x_3}{1-x_3} \right).$$

In general, the most likely states are 0^{n+1} and 1^{n+1} since the largest number of summands contribute in (3.2) (see also the important paper [CG12, Section 9]). In particular, all permutations in S_n contribute and since these terms are exactly the stationary distributions of the Tsetlin library, they sum to one. Hence $\Psi_{0^{n+1}}^{\mathcal{M}} \sim \frac{1}{2^n}$ plus lower order terms in the limit $n \rightarrow \infty$. The states 0101...

and 1010... appear with probability zero and are hence the least likely. One of the second least likely states is 001010... For this state, only one summand in (3.2) contributes and $\Psi_{001010\dots}^M \sim \frac{1}{2^{n-1}}$.

3.3. Cyclic walks – Rees matrix semigroup $B(n)$. The Rees matrix semigroup $B(n)$ consists of the elements $\{ij \mid 1 \leq i, j \leq n\} \cup \{\square\}$ with multiplication

$$ij \cdot k\ell = \begin{cases} i\ell & \text{if } j = k, \\ \square & \text{otherwise,} \end{cases}$$

and \square acts as zero. Let us choose as generators $A = \{a_i \mid 1 \leq i \leq n\}$, where $a_i = i(i+1)$ for $1 \leq i < n$ and $a_n = n1$.

Example 3.3. Let us consider the special case of $S = B(2)$ with generators $A = \{a, b\}$, where $a = 12$ and $b = 21$. The right Cayley graph and its Karnofsky–Rhodes/McCammond expansion $\text{Mc} \circ \text{KR}(S, A)$ are given in Figure 8. Note that in this example $\text{KR}(S, A)$ is stable under the McCammond expansion. The minimal ideal $K(S) = \{\square\}$ and the normal forms are given by

$$\mathcal{N}(B(2), A) = \{aa, abb, baa, bb\}.$$

The Markov chain $\mathcal{M}(\text{KR}(B(2), A))$ is depicted in Figure 9. To compute the stationary distribution, we first obtain

$$\begin{aligned} \text{NF}^{-1}(aa) &= a(ba)^*a, & \text{NF}^{-1}(abb) &= ab(ab)^*b, \\ \text{NF}^{-1}(bb) &= b(ab)^*b, & \text{NF}^{-1}(baa) &= ba(ba)^*a. \end{aligned}$$

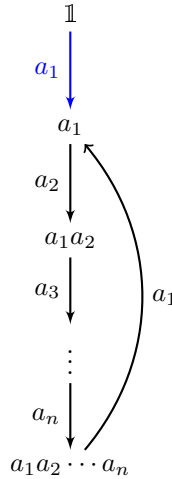
By (2.8), we obtain the stationary distribution

$$\begin{aligned} \Psi_{aa}^{\text{KR}(S,A)} &= \frac{x_a^2}{1 - x_a x_b}, & \Psi_{abb}^{\text{KR}(S,A)} &= \frac{x_a x_b^2}{1 - x_a x_b}, \\ \Psi_{bb}^{\text{KR}(S,A)} &= \frac{x_b^2}{1 - x_a x_b}, & \Psi_{baa}^{\text{KR}(S,A)} &= \frac{x_a^2 x_b}{1 - x_a x_b}. \end{aligned}$$

We indeed verify, using $x_a + x_b = 1$, that

$$\begin{aligned} \Psi_{aa}^{\text{KR}(S,A)} + \Psi_{bb}^{\text{KR}(S,A)} + \Psi_{abb}^{\text{KR}(S,A)} + \Psi_{baa}^{\text{KR}(S,A)} &= \frac{1}{1 - x_a x_b} (x_a^2 + x_b^2 + x_a x_b^2 + x_a^2 x_b) \\ &= \frac{1}{1 - x_a x_b} (x_a^2 + x_b^2 + x_a x_b) = \frac{1}{1 - x_a x_b} ((x_a + x_b)^2 - x_a x_b) = 1. \end{aligned}$$

In general, $\text{RCay}(B(n), A)$ contains a cycle of the form



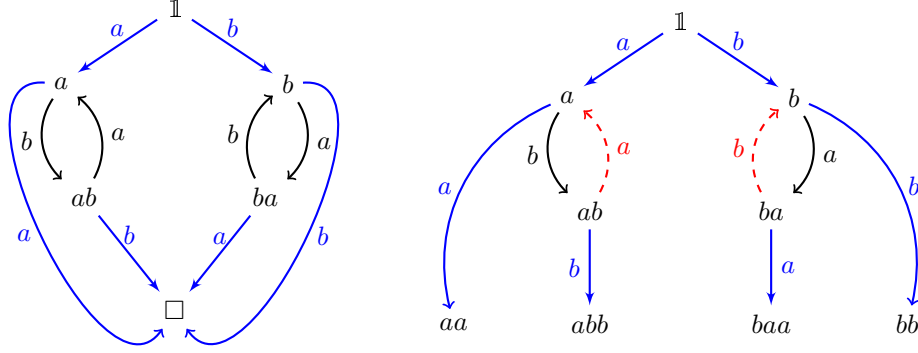


FIGURE 8. **Left:** The right Cayley graph $\text{RCay}(B(2), \{a, b\})$ with generators $a = 12$ and $b = 21$. **Right:** $\text{Mc} \circ \text{KR}(B(2), \{a, b\}) = \text{KR}(B(2), \{a, b\})$. Transition edges are indicated in blue.

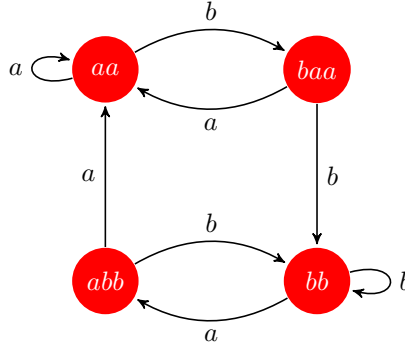


FIGURE 9. Markov chain $\mathcal{M}(\text{KR}(B(2), A))$ of Example 3.3.

For the vertex $a_1 a_2 \cdots a_{i-1}$, right multiplication by b with $b \neq a_i$ yields \square . There are similar cycles, where each subindex j is replaced by $k + j$ modulo n for a given $0 \leq k < n$. Hence the elements in $\mathcal{N}(B(n), A)$ are of the form

$$y_{j,i}^k := a_{k+1} a_{k+2} \cdots a_{k+j} a_i, \quad (0 \leq k < n, 1 \leq j \leq n)$$

where all indices are considered modulo n and $i \neq k + j + 1$. Note that the McCammond expansion of the Karnofsky–Rhodes expansion is again stable.

We have

$$\text{NF}^{-1}(y_{j,i}^k) := a_{k+1} (a_{k+2} a_{k+3} \cdots a_{k+n} a_{k+1})^* a_{k+2} a_{k+3} \cdots a_{k+j} a_i.$$

This allows us to compute the stationary distribution of the lumped Markov chain $\mathcal{M}(\text{KR}(B(n), A))$ by (2.8)

$$\Psi_{y_{j,i}^k}^{\text{KR}(B(n), A)} = \frac{x_{k+1} x_{k+2} \cdots x_{k+j} x_i}{1 - x_1 x_2 \cdots x_n}, \quad (0 \leq k < n, 1 \leq j \leq n, i \neq k + j + 1)$$

where for simplicity we have set $x_m := x_{a_m}$ for $1 \leq m \leq n$ and again all indices are considered modulo n .

3.4. Further cyclic walks – Rees matrix semigroups. Let S be a semigroup, I and I' be non-empty sets, and P a matrix indexed by I' and I with entries $p_{i',i}$ taken from S . Then the *Rees matrix semigroup* $(S; I, I'; P)$ is the set $I \times S \times I'$ together with the multiplication

$$(i, s, i')(j, t, j') = (i, s p_{i',j} t, j').$$

Similarly, define $(S; I, I'; P)^\square$ to be the *Rees matrix semigroup with zero* as the set $I \times S \times I' \cup \{\square\}$. Here P is an $I' \times I$ matrix with entries in $S \cup \{\square\}$ with multiplication

$$(i, s, i')(j, t, j') = \begin{cases} (i, s p_{i',j} t, j') & \text{if } p_{i',j} \neq \square, \\ \square & \text{else,} \end{cases}$$

and \square acts as zero.

Consider the special case of the Rees matrix semigroup $S = (Z_p; [n], [n]; \text{id})^\square$, which consists of the elements $\{(i, g, j) \mid 1 \leq i, j \leq n, g \in Z_p\} \cup \{\square\}$, where Z_p is the cyclic group with p elements. The multiplication in this case is given by

$$(i, g, j) \cdot (k, g', \ell) = \begin{cases} (i, gg', \ell) & \text{if } j = k, \\ \square & \text{otherwise,} \end{cases}$$

and \square acts as zero. Let us choose as generators $A = \{a_i \mid 1 \leq i \leq n\}$, where $a_i = (i, \text{id}, i+1)$ for $1 \leq i < n$ and $a_n = (n, (23 \dots p1), 1)$, where $(23 \dots p1)$ is the generator of Z_p shifting $i \mapsto i+1$ modulo p . The right Cayley graph and the Karnofsky–Rhodes/McCammond expansion are very similar to the case of $B(n)$ discussed in Section 3.3, except that now the cycles have length np . Again $\text{KR}(S, A)$ is stable under the McCammond expansion and the stationary contribution can easily be computed. We will demonstrate this in the next example.

Example 3.4. Consider the special case of $S = (Z_2; [2], [2]; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})^\square$ with generators $A = \{a, b\}$, where $a = (1, \text{id}, 2)$ and $b = (2, (21), 1)$. The right Cayley graph and $\text{Mc} \circ \text{KR}(S, A)$ are similar to those in Figure 8, except that the cycles now contain the elements $a, ab, aba, abab$ and $b, ba, bab, baba$, respectively. The minimal ideal is $K(S) = \{\square\}$ and the normal forms are given by

$$\mathcal{N}(S, A) = \{aa, abb, abaa, ababb, bb, baa, babb, babaa\}.$$

The Markov chain $\mathcal{M}(\text{KR}(S, A))$ is depicted in Figure 10. To compute the stationary distribution, we first obtain

$$\begin{aligned} \text{NF}^{-1}(aa) &= a(baba)^*a, & \text{NF}^{-1}(baa) &= b(abab)^*aa, \\ \text{NF}^{-1}(abaa) &= a(baba)^*baa, & \text{NF}^{-1}(babaa) &= b(abab)^*abaa, \end{aligned}$$

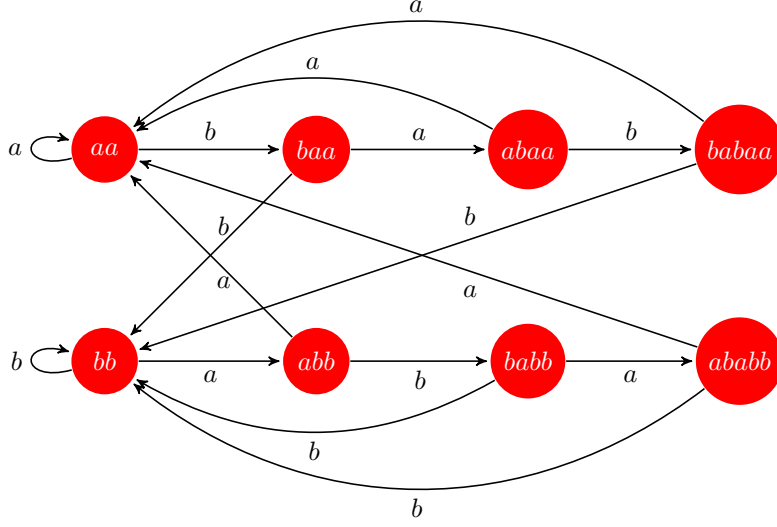
and similarly with the letters a and b interchanged everywhere. Hence by (2.8) we obtain the stationary distribution

$$\Psi_{aa}^{\text{KR}(S,A)} = \frac{x_a^2}{1 - x_a^2 x_b^2}, \quad \Psi_{baa}^{\text{KR}(S,A)} = \frac{x_a^2 x_b}{1 - x_a^2 x_b^2}, \quad \Psi_{abaa}^{\text{KR}(S,A)} = \frac{x_a^3 x_b}{1 - x_a^2 x_b^2}, \quad \Psi_{babaa}^{\text{KR}(S,A)} = \frac{x_a^3 x_b^2}{1 - x_a^2 x_b^2},$$

and similarly with the letters a and b interchanged everywhere. It can be verified that the stationary distributions add up to one, confirming (2.12).

Example 3.5. Consider the Rees matrix semigroup $S = (Z_2; [2], [2]; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix})$ with generators $A = \{a, b\}$ with $a = (1, 1, 2)$ and $b = (2, 1, 1)$. The Markov chain $\mathcal{M}(S, A)$ is not irreducible, see Figure 11. Since the minimal ideal $K(S)$ is not left zero, we need to apply Corollary 2.33. To this end, we consider $S' = S \cup \{\square\}$ and $A' = A \cup \{\square\}$. The normal forms are

$$\mathcal{N}(S', A') = \{\square, a\square, ab\square, aba\square, abab\square, b\square, ba\square, bab\square, baba\square\}.$$

FIGURE 10. Markov chain $\mathcal{M}(\text{KR}(S, A))$ of Example 3.4.

We have

$$\begin{aligned} \text{NF}^{-1}(\square) &= \square, \\ \text{NF}^{-1}(a\square) &= aa^*(bb^*aa^*bb^*aa^*)^*\square, \\ \text{NF}^{-1}(ab\square) &= aa^*(bb^*aa^*bb^*aa^*)^*bb^*\square, \\ \text{NF}^{-1}(aba\square) &= aa^*(bb^*aa^*bb^*aa^*)^*bb^*aa^*\square, \\ \text{NF}^{-1}(abab\square) &= aa^*(bb^*aa^*bb^*aa^*)^*bb^*aa^*bb^*\square, \end{aligned}$$

and similarly with a and b interchanged. We obtain

$$\begin{aligned} \Psi_{\square}^{(S', A')} &= x_{\square} & \xrightarrow{x_{\square} \rightarrow 0} & 0, \\ \Psi_{a\square}^{(S', A')} &= \frac{x_a x_{\square}}{(1-x_a)(1-\frac{x_a^2 x_b^2}{(1-x_a)^2(1-x_b)^2})} = \frac{x_a(1-x_a)(1-x_b)^2}{x_{\square} + 2x_a x_b} & \xrightarrow{x_{\square} \rightarrow 0} & \frac{1}{2}x_a^2, \\ \Psi_{ab\square}^{(S', A')} &= \frac{x_a x_b(1-x_a)(1-x_b)}{x_{\square} + 2x_a x_b} & \xrightarrow{x_{\square} \rightarrow 0} & \frac{1}{2}x_a x_b, \\ \Psi_{aba\square}^{(S', A')} &= \frac{x_a^2 x_b(1-x_b)}{x_{\square} + 2x_a x_b} & \xrightarrow{x_{\square} \rightarrow 0} & \frac{1}{2}x_a^2, \\ \Psi_{abab\square}^{(S', A')} &= \frac{x_a^2 x_b^2}{x_{\square} + 2x_a x_b} & \xrightarrow{x_{\square} \rightarrow 0} & \frac{1}{2}x_a x_b, \end{aligned}$$

and similarly with a and b interchanged.

3.5. Markov chains on \mathcal{B} -trivial monoids. Markov chains for \mathcal{B} -trivial monoids were studied in detail in [ASST15b]. In particular, the eigenvalues of the transition matrix and their multiplicities, the stationary distributions, and bounds on the mixing times were derived for general \mathcal{B} -trivial monoids. This class of Markov chains contains a vast number of previously studied Markov chains, such as the Tsetlin library [Cet63, Hen72, Hen73], walks on hyperplane arrangements [Bid97, BD98, BHR99], Brown's generalization to left regular bands [Bro00], edge flipping in graphs [CG12], random walks on linear extensions of a poset [AKS14a], and others [AS10, Ayy11, AS13, ASST15a, ASST15b].

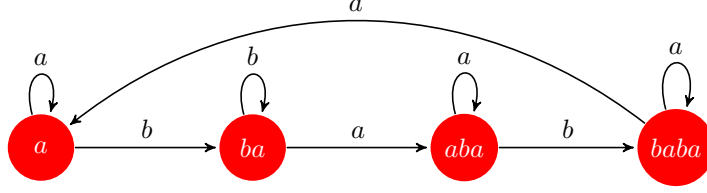


FIGURE 11. Part of the Markov chain $\mathcal{M}(\text{KR}(S, A))$ of Example 3.5. The other part is exactly the same with a and b interchanged everywhere. This chain is not irreducible.

We will now show how the stationary distribution of a Markov chain for an \mathcal{R} -trivial monoid as given in [ASST15b, Theorem 4.12] can be derived using the methods of this paper.

Recall that for a semigroup S , two elements $s, s' \in S$ are in the same \mathcal{R} -class if the corresponding right ideals are equal, that is, $sS = s'S$. We say that S is \mathcal{R} -trivial if all \mathcal{R} -classes are trivial, meaning that $sS = s'S$ implies that $s = s'$.

The Karnofsky–Rhodes expansion of the right Cayley graph of an \mathcal{R} -trivial semigroup (S, A) is a directed tree after the removal of loop edges and $\text{KR}(S, A)$ is stable under the McCammond expansion. If $w = w_1 w_2 \cdots w_\ell \in \mathcal{N}(S, A)$, then

$$\text{NF}^{-1}(w) = w_1 N_1^* w_2 N_2^* \cdots w_{\ell-1} N_{\ell-1}^* w_\ell,$$

where $N_i = N_i^w = \{a \in A \mid [w_1 \cdots w_i a]_S = [w_1 \cdots w_i]_S\}$ is the set of generators that stabilize the element $[w_1 \cdots w_i]_S$. We set $N_\ell = \emptyset$. Hence the stationary distribution of $\mathcal{M}(\text{KR}(S, A))$ is

$$\Psi_w^{\text{KR}(S, A)} = \prod_{i=1}^{\ell} \frac{x_{w_i}}{1 - \sum_{a \in N_i} x_a} \quad \text{for } w = w_1 \cdots w_\ell \in \mathcal{N}(S, A)$$

in agreement with [ASST15b, Corollary 4.13].

If the Karnofsky–Rhodes expansion is not the same as the original right Cayley graph, one can in fact lump the Markov chain $\mathcal{M}(\text{KR}(S, A))$ further to $\mathcal{M}(S, A)$ by applying $\varphi: (A^+, A) \rightarrow (S, A)$ of (2.1). Recall from Definition 2.27 that $\text{Red}(s)$ is the set of all elements $w \in \mathcal{N}(S, A)$ such that $[w]_S = s$. For this lumped Markov chain, we obtain the stationary distribution by (2.7)

$$\Psi_s = \sum_{w \in \text{Red}(s)} \prod_{i=1}^{\ell(w)} \frac{x_{w_i}}{1 - \sum_{a \in N_i^w} x_a} \quad \text{for } s \in S,$$

where $\ell(w)$ is the length of the word w , in agreement with [ASST15b, Theorem 4.12].

3.6. Adding constants: The bar operation. In this and the next section, we use the two operations bar and \flat introduced in Section 2.9 (see also [LRS17]) to produce new Markov chains from known examples.

We use two *stability conditions*. The first one is stability under the McCammond expansion:

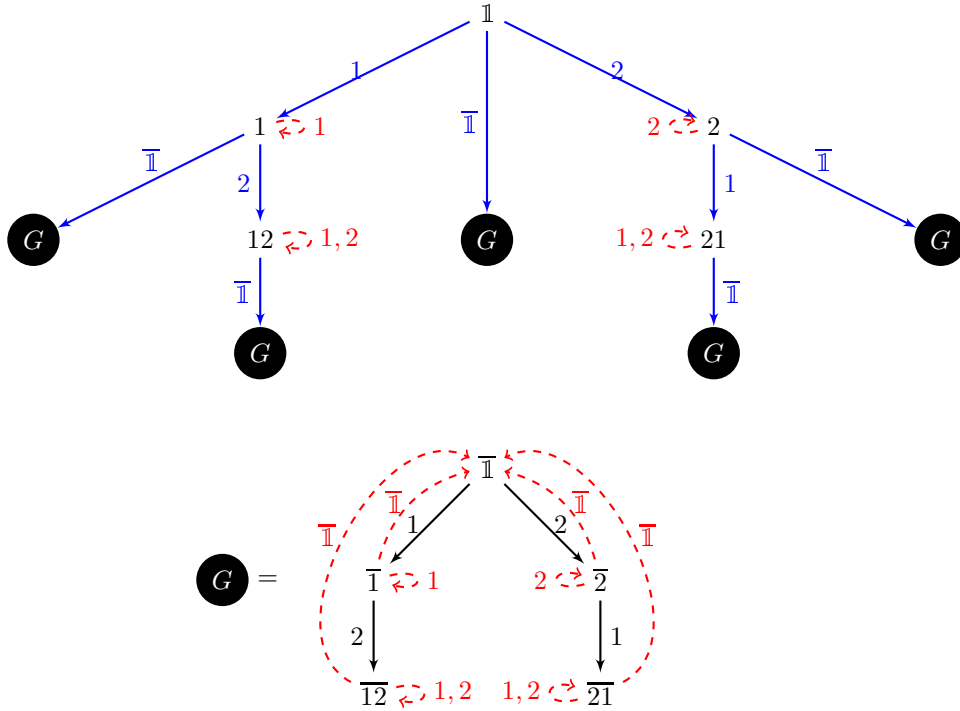
$$(3.3) \quad \text{Mc} \circ \text{KR}(S, A) = \text{KR}(S, A).$$

The second stability condition is stability under both the Karnofsky–Rhodes and McCammond expansion:

$$(3.4) \quad \text{Mc} \circ \text{KR}(S, A) = (S, A).$$

Since $\text{KR}^2 = \text{KR}$, stability condition (3.4) implies (3.3), but not vice versa. For example, the semigroup $S = \{0, 1\}$ with generators $A = \{0, 1\}$ of Example 2.14 satisfies (3.3), but not (3.4).

Conjecture 3.6. $\text{KR}(S, A)$ is stable under Mc if and only if $\text{Mc} \circ \text{KR}(S, A)$ is a right Cayley graph. In other words, if Mc changes any of the \mathcal{R} -classes of $\text{KR}(S, A)$, then $\text{Mc} \circ \text{KR}(S, A)$ cannot be a right Cayley graph.


 FIGURE 12. The Karnofsky–Rhodes expansion of $(\text{KR}(P(2), [2]))^{\text{bar}}$.

Proposition 3.7. *Suppose that (S, A) satisfies the stability condition (3.4). Then $(S, A)^{\text{bar}}$ satisfies the stability condition (3.3).*

Proof. By assumption (3.4), (S, A) is stable under both KR and Mc. Under the **bar** construction, $\text{RCay}((S, A)^{\text{bar}})$ is obtained from $\text{RCay}(S, A)$ by adding a new edge labeled $\bar{1}$ from each vertex to a new vertex labeled $\bar{1}$. Underneath the vertex $\bar{1}$, there is a copy of $\text{RCay}(S, A)$, where each vertex $x \in V(\text{RCay}(S, A))$ is replaced by \bar{x} thanks to the relation $\bar{x} \cdot a = \overline{x \cdot a}$ for any $a \in A$. In addition, each vertex below $\bar{1}$ has an edge labeled $\bar{1}$ looping back to vertex $\bar{1}$. Since the original $\text{RCay}(S, A)$ was stable under the Karnofsky–Rhodes and McCammond expansion, the effect of the Karnofsky–Rhodes expansion of $(S, A)^{\text{bar}}$ is to have a separate vertex $\bar{1}$ below each vertex, which is stable under the McCammond expansion, proving the claim. \square

Example 3.8. Let us consider $(S, A) = \text{KR}(P(2), [2])$, which satisfies the stability condition (3.4). Then the Karnofsky–Rhodes expansion of $(S, A)^{\text{bar}} = (\text{KR}(P(2), [2]))^{\text{bar}}$ is given in Figure 12. It can easily be checked that it is stable under the McCammond expansion, so that $(S, A)^{\text{bar}}$ satisfies (3.3) verifying Proposition 3.7.

Remark 3.9. Note that if (S, A) satisfies (3.3), then $(S \cup \{\square\}, A \cup \{\square\})$, where \square is a new zero element (that is $\square \cdot x = x \cdot \square = \square$ for all $x \in S$), also satisfies (3.3).

Example 3.10. Consider the semigroup $N_n = \langle a \mid a^n = 0 \rangle = \{0, a, a^2, \dots, a^{n-1}\}$ with generator $\{a\}$. Its right Cayley graph is a line with $n + 1$ vertices starting at 1 with intermediate vertices a^i ($1 \leq i < n$) and ending in $0 = a^n$, with a loop at a^n . Hence it satisfies the stability condition (3.4). The Karnofsky–Rhodes expansion of $(N_n, \{a\})^{\text{bar}}$ is the previous right Cayley graph with a string of length $n + 1$ attached to each previous vertex starting at $\bar{1}$ with intermediate vertices \bar{a}^i ($1 \leq i < n$) and ending in $\bar{a}^n = \bar{0} = 0$. In addition, there is an edge labeled $\bar{1}$ going from each vertex \bar{a}^i ($0 \leq$

$i \leq n$) to vertex $\bar{\mathbb{I}}$. This graph has the unique path property and hence $(N_n, \{a\})^{\text{bar}}$ satisfies (3.3) confirming Proposition 3.7.

Let us now add a new zero \square as a generator, so that $A = \{a, \bar{\mathbb{I}}, \square\}$ and $S = (N_n, \{a\})^{\text{bar}} \cup \{\square\}$. By Remark 3.9, (S, A) also satisfies (3.3). The minimal ideal is $K(S) = \{\square\}$. Then the normal forms are

$$\mathcal{N}(S, A) = \{a^i \square, a^i \bar{\mathbb{I}} a^j \square \mid 0 \leq i, j \leq n\}.$$

Furthermore

$$\begin{aligned} \text{NF}^{-1}(a^i \square) &= a^i \square, & (0 \leq i < n) \\ \text{NF}^{-1}(a^n \square) &= a^n a^* \square, \\ \text{NF}^{-1}(a^i \bar{\mathbb{I}} a^j \square) &= a^i \bar{\mathbb{I}} (a^* \bar{\mathbb{I}})^* a^j \square, & (0 \leq i, j < n) \\ \text{NF}^{-1}(a^n \bar{\mathbb{I}} a^j \square) &= a^n a^* \bar{\mathbb{I}} (a^* \bar{\mathbb{I}})^* a^j \square, & (0 \leq j < n) \\ \text{NF}^{-1}(a^i \bar{\mathbb{I}} a^n \square) &= a^i \bar{\mathbb{I}} (a^* \bar{\mathbb{I}})^* a^n a^* \square, & (0 \leq i < n) \\ \text{NF}^{-1}(a^n \bar{\mathbb{I}} a^n \square) &= a^n a^* \bar{\mathbb{I}} (a^* \bar{\mathbb{I}})^* a^n a^* \square. \end{aligned}$$

It is clear in this case, that each word in the Kleene expressions occurs at most once. Assigning probabilities $x_a, x_{\bar{\mathbb{I}}}, x_{\square}$ to the three generators in $A = \{a, \bar{\mathbb{I}}, \square\}$, we hence obtain the stationary distribution of the lumped Markov chain $\mathcal{M}(S, A)$

$$\begin{aligned} \Psi_{a^i \square}^{(S, A)} &= x_a^i x_{\square}, & (0 \leq i < n) \\ \Psi_{a^n \square}^{(S, A)} &= \frac{x_a^n x_{\square}}{1 - x_a}, \\ \Psi_{a^i \bar{\mathbb{I}} a^j \square}^{(S, A)} &= \frac{x_a^{i+j} x_{\bar{\mathbb{I}}} x_{\square}}{1 - \frac{x_{\bar{\mathbb{I}}}}{1 - x_a}}, & (0 \leq i, j < n) \\ \Psi_{a^n \bar{\mathbb{I}} a^j \square}^{(S, A)} &= \frac{x_a^{n+j} x_{\bar{\mathbb{I}}} x_{\square}}{1 - x_a - x_{\bar{\mathbb{I}}}}, & (0 \leq j < n) \\ \Psi_{a^i \bar{\mathbb{I}} a^n \square}^{(S, A)} &= \frac{x_a^{i+n} x_{\bar{\mathbb{I}}} x_{\square}}{1 - x_a - x_{\bar{\mathbb{I}}}}, & (0 \leq i < n) \\ \Psi_{a^n \bar{\mathbb{I}} a^n \square}^{(S, A)} &= \frac{x_a^{2n} x_{\bar{\mathbb{I}}} x_{\square}}{(1 - x_a)(1 - x_a - x_{\bar{\mathbb{I}}})}. \end{aligned}$$

Using $\sum_{i=0}^{n-1} x_a^i = \frac{1-x_a^n}{1-x_a}$ it can easily be verified that the stationary probabilities add up to one.

Now let us consider a general finite A -semigroup (S, A) that satisfies (3.4). By Proposition 3.7 and Remark 3.9, $(S', A') = ((S, A)^{\text{bar}} \cup \{\square\}, A \cup \{\bar{\mathbb{I}}, \square\})$ satisfies (3.3) and hence yields a Markov chain $\mathcal{M}(S', A')$ with minimal ideal $K(S') = \{\square\}$. The normal forms $\mathcal{N}(S', A')$ are given by

$$w \square, w \bar{\mathbb{I}} w' \quad \text{for } w, w' \in \mathcal{N},$$

where $\mathcal{N} := \mathcal{N}(S \cup \{\square\}, A \cup \{\square\}, \{\square\}) \setminus \{\square\}$, where the removal of \square means the normal forms in $\mathcal{N}(S \cup \{\square\}, A \cup \{\square\}, \{\square\})$ without the last \square . If the Kleene expressions for $\text{NF}^{-1}(w)$ for $w \in \mathcal{N}$ are known, then the Kleene expressions for the normal forms in $\mathcal{N}(S', A')$ can also be derived:

$$(3.5) \quad \begin{aligned} \text{NF}^{-1}(w \square) &= \text{NF}^{-1}(w) \square, \\ \text{NF}^{-1}(w \bar{\mathbb{I}} w' \square) &= \text{NF}^{-1}(w) \bar{\mathbb{I}} (\cup_{v \in \mathcal{N}} \text{NF}^{-1}(v) \bar{\mathbb{I}})^* \text{NF}^{-1}(w') \square. \end{aligned}$$

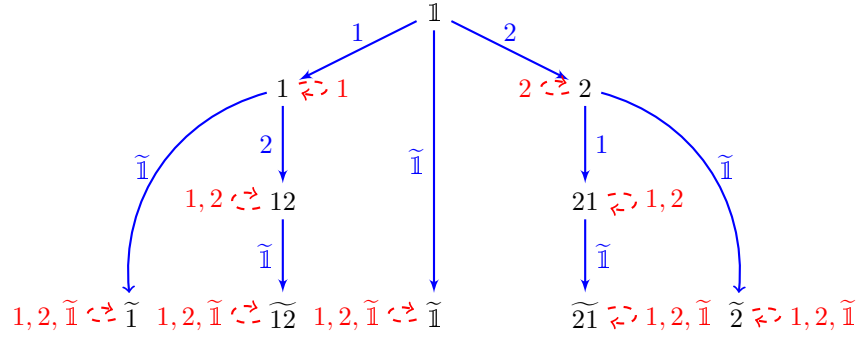
Remark 3.11. Intuitively, one can interpret the bar operation as reproduction: each cell (or vertex in the Cayley graph) produces a copy of itself (with edges back to its origin $\bar{\mathbb{I}}$). Equivalently, applying $\bar{\mathbb{I}}$ means to reset the library to the beginning. By Proposition 3.7, one can repeat the operation $\text{KR} \circ \text{bar}$ an arbitrary number of times. Repeating it n times and letting n tend to infinity, has the flavor of a fractal: in any portion of the graph, one can zoom in and find the original right Cayley graph, or in fact $(\text{KR} \circ \text{bar})^k(S, A)$ for any $k > 0$.

3.7. **The \flat operation.** Recall the flat operation \flat from Section 2.9.

Proposition 3.12. *Suppose that (S, A) satisfies the stability condition (3.3). Then $(S, A)^\flat$ also satisfies the stability condition (3.3).*

Proof. The right Cayley graph $\text{RCay}((S, A)^\flat)$ can be obtained from $\text{RCay}(S, A)$ by adding to each vertex x a new edge labeled $\tilde{\mathbb{1}}$ to \tilde{x} . Observe that this implies that KR commutes with \flat . By assumption, the Karnofsky–Rhodes expansion $\text{KR}(S, A)$ has the unique path property (since it is stable under the McCammond expansion by (3.3)). Since KR and \flat commute, $\text{KR}((S, A)^\flat)$ is obtained from $\text{KR}(S, A)$ by adding a new edge labeled $\tilde{\mathbb{1}}$ to each vertex x which goes to \tilde{x} , which is a trivial one point \mathcal{R} -class. Since $\text{KR}(S, A)$ has the unique path property so does $\text{KR}((S, A)^\flat)$, proving the claim. \square

Example 3.13. Consider $(P(2), [2])$. Recall that $(P(n), [n])$ yields the Tsetlin library (see Section 3.1) and satisfies the stability condition (3.3). The Karnofsky–Rhodes expansion of $(P(2), [2])^\flat$ is depicted below:



This expansion indeed has the unique path property, so that stability condition (3.3) holds. This verifies Proposition 3.12.

Propositions 3.7 and 3.12 allow us to construct an infinite tower of semigroups satisfying one of the stability conditions from a given semigroup.

Theorem 3.14. *Suppose (S, A) satisfies stability condition (3.3). Then*

$$\begin{aligned} (\text{bar} \circ \text{KR} \circ \flat)^n(S, A) & \text{ satisfies stability condition (3.3) for any } n \geq 0, \\ \flat \circ (\text{bar} \circ \text{KR} \circ \flat)^n(S, A) & \text{ satisfies stability condition (3.3) for any } n \geq 0, \\ \text{KR} \circ \flat \circ (\text{bar} \circ \text{KR} \circ \flat)^n(S, A) & \text{ satisfies stability condition (3.4) for any } n \geq 0. \end{aligned}$$

Similarly, if (S, A) satisfies stability condition (3.4), then

$$\begin{aligned} (\text{KR} \circ \flat \circ \text{bar})^n(S, A) & \text{ satisfies stability condition (3.4) for any } n \geq 0, \\ \text{bar} \circ (\text{KR} \circ \flat \circ \text{bar})^n(S, A) & \text{ satisfies stability condition (3.3) for any } n \geq 0, \\ \flat \circ \text{bar} \circ (\text{KR} \circ \flat \circ \text{bar})^n(S, A) & \text{ satisfies stability condition (3.3) for any } n \geq 0. \end{aligned}$$

Proof. This follows directly from Propositions 3.7 and 3.12 and the fact that $\text{KR}^2 = \text{KR}$. \square

Example 3.15. Let us now compute the normal forms and Kleene expressions for

$$(S, A) = \flat \circ \text{KR} \circ \text{bar} \circ \text{KR}(P(n), [n]).$$

Recall that KR and \flat commute. The Cayley graph for $\text{KR} \circ \text{bar} \circ \text{KR}(P(2), [2])$ is given in Figure 12. Let \mathcal{N} denote the set of normal forms of $\flat \circ \text{KR}(P(n), [n])$ with the last $\tilde{\mathbb{1}}$ removed. The normal forms of (S, A) are given by

$$w\tilde{\mathbb{1}}, \quad w\tilde{\mathbb{1}}w'\tilde{\mathbb{1}} \quad \text{for } w, w' \in \mathcal{N}.$$

Then $\text{NF}^{-1}(w\tilde{\mathbb{1}})$ and $\text{NF}^{-1}(w\tilde{\mathbb{1}}w'\tilde{\mathbb{1}})$ are given by the same formula as in (3.5) with \square replaced by $\tilde{\mathbb{1}}$. The normal forms in \mathcal{N} are all words without repeated letters in the alphabet $[n]$. For a word $w = w_1w_2 \dots w_k \in \mathcal{N}$, we have

$$\text{NF}^{-1}(w) = w_1w_1^*w_2\{w_1, w_2\}^* \cdots w_k\{w_1, \dots, w_k\}^*.$$

It is clear in this example that no words are repeated in the Kleene expressions. Hence the stationary distribution follows directly by applying (2.11).

3.8. Burnside examples. The Burnside semigroups with generators in A are the infinite semigroups of the form

$$\mathcal{B}(m, n) = \langle A \mid t^m = t^{m+n} \quad \forall t \in A^+ \rangle.$$

For $m \geq 6$ and $n \geq 1$, McCammond [McC91] showed that the Burnside semigroups are finite \mathcal{J} -above, have a decidable word problem, and their maximal subgroups are cyclic. These results were generalized by de Luca and Varricchio [dLV92], Guba [Gub93a, Gub93b], and do Lago [dL96] to $m \geq 3$ and $n \geq 1$. Recall that \mathcal{J} -order in a semigroup S is defined by $s \geq_{\mathcal{J}} s'$ if $s' = s$, $s' = xs$, $s' = sy$, or $s' = xsy$ for some $x, y \in S$. If for every $s \in S$, there are only finitely many elements \mathcal{J} -above s , then S is called finite \mathcal{J} -above.

In particular, the above results imply that for any $s \in \mathcal{B}(m, n)$, the elements in $\{s' \in \mathcal{B}(m, n) \mid s' \geq_{\mathcal{J}} s\}$ together with zero \square form a finite semigroup. Let us call this semigroup $S_s^{\mathcal{J}}$.

Conjecture 3.16. $(\text{KR}(S_s^{\mathcal{J}}), A)$ satisfies the stability condition (3.4).

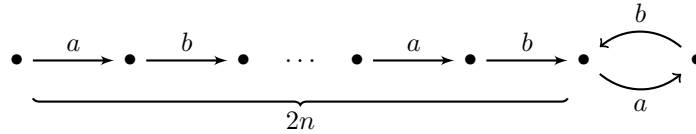
Conjecture 3.16 should follow from the results in [McC91, McC01]. Moreover, for each element $s \in \mathcal{B}(m, n)$, the language of words which represent s is regular and can be described by a single Kleene expression without unions [McC01, Theorem 8.11].

Definition 3.17. If S is a finite \mathcal{J} -above A -semigroup and $w \in A^+$, then the straight line automaton $\text{str}^S(w)$ is the path w together with the strong components (that is, \mathcal{R} -classes) of its prefixes.

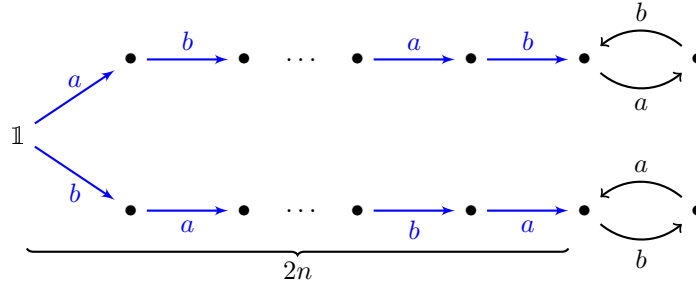
Theorem 3.18. [McC91, Gub93a, Gub93b, McC01] For $m \geq 3$, the set of all words equivalent to $w \in A^+$ under the relations $t^m = t^{m+n}$ is a regular language given by a straight line $\text{str}^{\mathcal{B}(m, n)}(w)$.

Given $\text{str}^{\mathcal{B}(m, n)}(w)$ for $w \in A^+$, we can construct a semigroup S_w as follows. Consider all factors of the accepted words of $\text{str}^{\mathcal{B}(m, n)}(w)$ including the empty word. These are the elements of S_w , under the equivalence relation $w_1 \equiv w_2$ if the relation $t^m = t^{m+n}$ can be used, in addition to the sink state \square . Multiplication in $\text{RCay}(\text{KR}(S_w), A)$ is given by $w \cdot a = wa$ for $a \in A$ if wa is another factor of an accepted word and \square otherwise.

Example 3.19. In $S = \mathcal{B}(n, 1)$ with $A = \{a, b\}$, consider $\text{str}^S(w)$ for $w = (ab)^n$:



Then $\text{RCay}(\text{KR}(S_w), A)$ is given by



with arrows going to \square omitted. This graph is stable under the McCammond expansion and satisfies (3.4). The normal forms are

$$\mathcal{N}(\text{KR}(S_w), A, \{\square\}) = \{(ab)^j b, (ab)^k aa, (ba)^j a, (ba)^k bb \mid 0 < j \leq n, 0 \leq k \leq n\}.$$

We have

$$\begin{aligned} \text{NF}^{-1}((ab)^j b) &= (ab)^j b && \text{for } 0 < j < n, \\ \text{NF}^{-1}((ab)^k aa) &= (ab)^k aa && \text{for } 0 \leq k < n, \\ \text{NF}^{-1}((ab)^n b) &= (ab)^n (ab)^* b, \\ \text{NF}^{-1}((ab)^n aa) &= (ab)^n (ab)^* aa, \end{aligned}$$

and similarly with a and b interchanged. Hence the stationary distribution is given by

$$\begin{aligned} \Psi_{(ab)^j b} &= x_a^j x_b^{j+1}, & \Psi_{(ab)^k aa} &= x_a^{k+2} x_b^k && (0 < j < n, 0 \leq k < n) \\ \Psi_{(ab)^n b} &= \frac{x_a^n x_b^{n+1}}{1 - x_a x_b}, & \Psi_{(ab)^n aa} &= \frac{x_a^{n+2} x_b^n}{1 - x_a x_b}, \end{aligned}$$

and similarly with a and b interchanged. Using

$$\sum_{j=1}^{n-1} x_a^j x_b^{j+1} = x_b \frac{1 - x_a^n x_b^n}{1 - x_a x_b} - x_b \quad \text{and} \quad \sum_{k=0}^{n-1} x_a^{k+2} x_b^k = x_a^2 \frac{1 - x_a^n x_b^n}{1 - x_a x_b}$$

it can be checked that the stationary distributions add to one.

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(J. Rhodes) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, U.S.A.
E-mail address: rhodes@math.berkeley.edu, blvdbastille@gmail.com

(A. Schilling) DEPARTMENT OF MATHEMATICS, UC DAVIS, ONE SHIELDS AVE., DAVIS, CA 95616-8633, U.S.A.
E-mail address: anne@math.ucdavis.edu