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# The Maximal-Density Mass Function for Primordial Black Hole Dark Matter

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**Abstract.** The advent of gravitational wave astronomy has rekindled interest in primordial black holes (PBH) as a dark matter candidate. As there are many different observational probes of the PBH density across different masses, constraints on PBH models are dependent on the functional form of the PBH mass function. This complicates general statements about the mass functions allowed by current data, and, in particular, about the maximum total density of PBH. We use analytical arguments to show that the mass function which maximizes the fraction of the matter density in PBH subject to all constraints is a finite linear combination of monochromatic mass functions. We explicitly compute the maximum fraction of dark matter in PBH for different combinations of current constraints, and discuss implications for the viability of models predicting extended mass functions.

**Keywords:** dark matter theory, primordial black holes, gravitational waves / sources

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## 1 Introduction

The possibility that density fluctuations in the early universe collapsed into primordial black holes (PBH) has been studied for several decades. Apart from their potential utility as a probe of the primordial universe, PBH are an excellent candidate for cosmological dark matter, as sufficiently large black holes are stable and dynamically cold. Further, with simple formation mechanisms, they can be produced with a cosmological density matching the observed density of dark matter.

If PBH account for a significant fraction of dark matter, it is possible that observed gravitational wave signals have a primordial origin. Direct observations of binary black hole mergers thus far all involve black holes with masses of several times  $10M_{\odot}$  [1–5], in a range where microlensing constraints on the abundance of compact objects are ineffective. The observed merger rate is compatible with PBH as dark matter, and other constraints historically applied in the LIGO mass range are subject to large astrophysical uncertainties [6, 7]. This has led to renewed interest in primordial production mechanisms, and it remains possible that PBH in this mass window account for much or all of dark matter [8].

However, depending on the formation mechanism, PBH may exist today with masses as small as  $10^{-16}M_{\odot}$ , or as large as those of supermassive black holes. Thus, constraining the total density contained in PBH requires the combination of constraints that span this vast range of mass scales. Such observables include microlensing surveys [9–12], CMB data [13], and the statistics of wide binaries [14]. In general, constraints from these observables have been computed under the assumption of a single-valued (hereafter, monochromatic) PBH mass function. However, as realistic production mechanisms necessarily result in an extended mass function, it is essential to correctly combine constraints across all masses.

This problem has recently been studied by several authors [15–17]. In general, the constraints depend non-trivially on the functional form of the mass function, and statements about the implications of constraints for properties of the PBH population can be difficult to generalize. In particular, the total fraction  $f_{\text{PBH}}$  of dark matter that may be accounted for by PBH varies with the form of the mass function, so  $f_{\text{PBH}} = 1$  is ruled out for some forms of the mass function, and allowed for others. This has led to confusion regarding the observational viability of the PBH dark matter scenario.

Depending on the set of constraints considered, observational data may or may not already rule out  $f_{\text{PBH}} = 1$  for monochromatic mass functions. Since the many constraints span a wide mass range, and since several do not overlap significantly, some authors have argued that broadening the mass function might relax constraints on PBH [18, 19], possibly allowing for  $f_{\text{PBH}} = 1$  even if that possibility were excluded by constraints for monochromatic mass functions. However, [16, 17] have evaluated the constraints *numerically* for several forms of extended mass functions, and found that extended mass functions are typically subject to stronger constraints than monochromatic mass functions.

These findings motivate the question we now pose: what is the theoretical maximum density of PBH permitted by constraints for a fully general mass function? Our goal is ultimately to clarify the observational status of PBH dark matter, and to understand the circumstances under which extending the mass function can relax constraints. We also seek a procedure which is flexible and simple enough to allow us to compare results for different sets of constraints, and to elucidate the dependence of the maximal density on the form of the constraints themselves. To that end, we derive the form of the mass function which optimizes the density subject to all observational constraints. This allows us to obtain a general bound on the density of PBH with minimal numerical computation, independently of the true form of the PBH mass function. Our approach addresses a similar question to that of [17]. However, while the approach of [17] must be applied separately to each chosen functional form, here we consider the question from a mathematical standpoint, with no prior prejudice on the form of the “optimal” mass function.

This paper is organized as follows. In section 2, we establish conventions and notations, and review the application of constraints from the monochromatic case to extended mass functions. In section 3, we present a pedagogical derivation of our main results regarding the maximum density of PBH, and we apply them to current data. We discuss these results in section 4 and conclude in section 5. Finally, in appendix A we compare our analytical results with direct numerical techniques.

## 2 Interpreting constraints for extended mass functions

We follow [16] to convert constraints for monochromatic mass functions to constraints for extended mass functions. We denote the mass function by  $\psi$  and adopt their normalization and conventions, such that

$$\psi \propto M \frac{dn}{dM}, \quad \int dM \psi(M) = \frac{\Omega_{\text{PBH}}}{\Omega_{\text{DM}}} \equiv f_{\text{PBH}} \quad (2.1)$$

where  $n$  is the number density of PBH at fixed mass. Most observables that can constrain primordial black holes are determined by the properties of single black holes, with no need to consider relationships between them. In such a case, an observable quantity  $A$  receives a linear combination of contributions from each mass bin, and the contribution from black holes of mass  $M$  is proportional to  $\psi(M)$ . As such, the observable can be written as a functional of  $\psi$  in the form

$$A[\psi] = A_0 + \int dM \psi(M) K_1(M). \quad (2.2)$$

We note in passing that there are some constraints for which relationships between black holes are significant. For example, gravitational wave observations of mergers are dependent on the properties of pairs of black holes, and so one must combine contributions from pairs of

mass bins. In the simplest case, where the contributions scale linearly with number in each mass bin, such an observable can clearly be written in the form

$$A[\psi] = A_0 + \int dM \psi(M) K_1(M) + \int dM dM' \psi(M) \psi(M') K_2(M, M') \quad (2.3)$$

and one can always express a generic observable by including higher-order terms of this form. Note that higher-order terms also account for non-linear dependence of  $A$  on  $\psi$  at fixed mass. For example, an observable which scales as  $\psi(M)^2$  can be expressed exactly at second order by setting  $K_2(M, M') \propto \delta(M - M')$ .

All of the constraints that we consider in this work are of the simplest kind, and we will find eq. (2.2) sufficient. In this case, it is straightforward to relate constraints for a monochromatic mass function to constraints for a generic mass function, and we briefly review the argument given in [16]. Let  $\psi_{\text{mono}}(M_0; M) \equiv f_{\text{max}}(M_0) \delta(M - M_0)$ , where  $f_{\text{max}}(M_0)$  is the largest coefficient allowed by constraints for a mass function of this form. If we take  $\psi(M) = \psi_{\text{mono}}(M_0; M)$  in eq. (2.3), we obtain

$$K_1(M_0) = \frac{A[\psi_{\text{mono}}] - A_0}{f_{\text{max}}(M_0)} \quad (2.4)$$

Suppose that the difference  $A[\psi] - A_0$  is observable with the desired significance when  $A[\psi]$  crosses a threshold value  $A_{\text{obs}}$ . Then  $A[\psi_{\text{mono}}] = A_{\text{obs}}$  by definition of  $f_{\text{max}}$ , so eq. (2.4) gives  $K_1(M)$  independent of  $\psi$ . Substituting for  $K_1(M)$  in eq. (2.3) while leaving  $\psi$  generic gives the condition

$$\mathcal{C}[\psi] \equiv \int dM \frac{\psi(M)}{f_{\text{max}}(M)} \leq 1. \quad (2.5)$$

This expresses the constraint on a mass function  $\psi(M)$  when the constraint for a monochromatic mass function is  $\int dM \psi_{\text{mono}}(M_0; M) \leq f_{\text{max}}(M_0)$ .

### 3 The optimal mass function

#### 3.1 Single-constraint case

For pedagogical purposes, we first consider the case of a single constraining observable. For such situations, when all observables can be expressed in the form of eq. (2.2), the constraint on the mass function has the form  $\mathcal{C}[\psi] \leq 1$ , with  $\mathcal{C}[\psi]$  as defined in eq. (2.5). The problem is then to maximize  $\int dM \psi(M)$  subject to this constraint. The optimal mass function saturates the constraint, so it suffices to require  $\mathcal{C}[\psi] = 1$ .

Naively, this problem looks as though it can be solved using the method of Lagrange multipliers, by finding stationary points of the functional

$$\mathcal{S}[\psi, \lambda] = \int dM \left( \psi(M) - \lambda \frac{\psi(M)}{f_{\text{max}}(M)} \right). \quad (3.1)$$

However, the Euler-Lagrange equation in  $\psi$  admits no non-trivial solutions. This is because  $\int dM \psi(M)$  can be made arbitrarily large, even subject to  $\mathcal{C}[\psi] = 1$ , unless  $\psi(M) > 0$  is imposed. Positivity can be imposed by setting  $\psi = \phi^* \phi$  and performing an unconstrained optimization in  $\phi$ , but the corresponding Euler-Lagrange equation leads to the condition that  $\phi$  is, at every point, either zero or non-analytic.

The variational approach does not generalize to the case of multiple constraints, so we do not pursue it any further. Rather, we observe that since  $\mathcal{C}[\psi]$  is linear, we have  $\mathcal{C}[\mathcal{C}[\psi]^{-1}\psi] = 1$ . Thus, we can impose  $\mathcal{C}[\psi] = 1$  by rescaling  $\psi \rightarrow \mathcal{C}[\psi]^{-1}\psi$ , and then the problem is to maximize the functional

$$\mathcal{M}[\psi] \equiv \int dM (\mathcal{C}[\psi]^{-1}\psi(M)) = \frac{\int dM \psi(M)}{\int dM \frac{\psi(M)}{f_{\max}(M)}} \quad (3.2)$$

subject only to positivity. We call  $\mathcal{M}[\psi]$  the *normalized mass* of  $\psi$ .

It is now simple to show that  $\mathcal{M}[\psi]$  is maximized by taking  $\psi$  to be a monochromatic mass function. Let  $M_{\max} \equiv \operatorname{argmax} f_{\max}(M)$  and  $f_{\text{mono}} \equiv f_{\max}(M_{\max})$ , and define

$$\psi_0(M) \equiv f_{\text{mono}} \delta(M - M_{\max}) \quad (3.3)$$

so that  $\psi_0(M)$  is the monochromatic mass function which maximizes the PBH density, and  $f_{\text{mono}}$  is the maximum PBH density allowed for a monochromatic mass function. Choose any mass function  $\psi \equiv \psi_0 + \delta\psi$ . Since  $\psi_0$  vanishes everywhere except for  $M_{\max}$ , positivity of  $\psi$  requires that  $\delta\psi(M) \geq 0$  for all  $M \neq M_{\max}$ . Then we have

$$\mathcal{M}[\psi] = \frac{\int dM [\psi_0(M) + \delta\psi(M)]}{\int dM [\psi_0(M)/f_{\max}(M) + \delta\psi(M)/f_{\max}(M)]}. \quad (3.4)$$

Since  $\psi_0$  saturates the constraint of eq. (2.5), we must have  $\int dM [\psi_0(M)/f_{\max}(M)] = 1$  and  $\int dM \psi_0(M) = f_{\text{mono}}$ , so we write

$$\mathcal{M}[\psi] = \frac{f_{\text{mono}} + \int dM \delta\psi(M)}{1 + \int dM \delta\psi(M)/f_{\max}(M)} \quad (3.5)$$

but  $f_{\max}(M) \leq f_{\text{mono}}$  by definition, so we have

$$\mathcal{M}[\psi] = \frac{f_{\text{mono}} + \int dM \delta\psi(M)}{1 + \int dM \delta\psi(M)/f_{\max}(M)} \leq \frac{f_{\text{mono}} + \int dM \delta\psi(M)}{1 + \int dM \delta\psi(M)/f_{\text{mono}}} = f_{\text{mono}}. \quad (3.6)$$

Thus we have shown that  $\mathcal{M}[\psi] \leq f_{\text{mono}} \equiv \mathcal{M}[\psi_0]$ , so no functional form allows a higher total PBH density than does the Dirac delta. In particular, for fixed PBH density, we conclude that an extended mass function is always more strongly constrained than the optimal monochromatic mass function. While this will not hold for the case of multiple constraints, it remains an excellent approximation if the constraints are weakest by far in a mass range where a single observable dominates.

### 3.2 Combining constraints

Realistically, the single-constraint case is too simplistic. In general, a mass function is ruled out on the basis of a  $\chi^2$  test statistic. If PBH are constrained by multiple observables  $A_j$ , then the test statistic is found by adding the individual  $\chi^2$  statistics in quadrature. That is,

$$\chi^2[\psi] = \sum_{j=1}^N \chi_j^2 = \sum_{j=1}^N \left( \frac{A_j[\psi] - A_{\text{obs},j}}{\sigma_j} \right)^2. \quad (3.7)$$

To fail to reject  $\psi$  at some significance level requires that  $\chi^2[\psi] \leq \gamma^2$  for some threshold value  $\gamma^2$ , i.e.,

$$\sum_{j=1}^N \left( \int dM \psi(M) \frac{K_{1,j}(M)}{\gamma \sigma_j} \right)^2 \leq 1. \quad (3.8)$$

If we set  $N = 1$ , this reduces to

$$\int dM \psi(M) \frac{K_{1,1}(M)}{\gamma \sigma_1} \leq 1 \quad (3.9)$$

so matching with eq. (2.5) gives  $K_{1,j}(M)/(\gamma \sigma_j) = 1/f_{\max,j}(M)$ , where  $f_{\max,j}(M)$  is the analogue of  $f_{\max}(M)$  for the  $j$ th constraint alone. For general  $N$ , [16] show that the constraint takes the form

$$\sum_{j=1}^N \left( \int dM \frac{\psi(M)}{f_{\max,j}(M)} \right)^2 \leq 1. \quad (3.10)$$

Since the individual constraints are added in quadrature, the argument applied to the single-constraint case does not extend to the case of multiple constraints, and indeed, there are cases in which the density is not maximized by a monochromatic mass function. However, we will show that the maximizer is in general a linear combination of  $N$  monochromatic mass functions.

### 3.3 The general problem

For the case of several constraining observables, one has  $N$  constraint functions  $f_{\max,1}, \dots, f_{\max,N}$ . For brevity, we define  $g_j(M) \equiv 1/f_{\max,j}(M)$ , and by analogy with eq. (2.5), we define

$$\mathcal{C}_j[\psi] \equiv \int dM \psi(M) g_j(M). \quad (3.11)$$

Then the problem is to find  $\psi$  to maximize

$$\mathcal{M}[\psi] \equiv \frac{\int dM \psi(M)}{\left( \sum_{j=1}^N \mathcal{C}_j[\psi]^2 \right)^{1/2}} = \frac{\int dM \psi(M)}{\|\mathcal{C}[\psi]\|} \quad (3.12)$$

where  $\mathcal{C}[\psi]$  denotes the vector with components  $\mathcal{C}_j[\psi]$ . We define  $f_{\max,\text{all}} = \max \mathcal{M}[\psi]$ .

Since rescaling  $\psi$  does not change  $\mathcal{M}[\psi]$ , we can always set  $\int dM \psi(M) = 1$ , and then the problem is equivalent to minimizing  $\|\mathcal{C}[\psi]\|$  subject to this constraint. When solving this problem with real data, the integrals are arbitrarily well approximated by a discrete sum of the form

$$\|\mathcal{C}[\psi_Q]\|^2 = \sum_{j=1}^N \left( \sum_{k=1}^Q a_k g_j(M_k) \right)^2 = \left\| \sum_{k=1}^Q a_k \mathbf{g}(M_k) \right\|^2. \quad (3.13)$$

Thus, the problem is to minimize the norm of a sum of  $a_k \mathbf{g}(M_k)$  for some  $\{M_k\}_{k=1,\dots,Q}$ , subject to our normalization condition, which now takes the form  $\sum_{k=1}^Q a_k = 1$ . Geometrically, this is the same as minimizing the norm over the convex hull of the  $\mathbf{g}(M)$ , i.e., to compute

$$\min \{ \|\mathbf{x}\| \mid \mathbf{x} \in \text{conv} \{ \mathbf{g}(M) \mid M \in U \} \}. \quad (3.14)$$

We henceforth denote  $\text{conv} \{ \mathbf{g}(M) \mid M \in U \}$  by  $\text{conv}(\mathbf{g})$ . Since the minimizer is the projection of the origin onto a convex set, it is unique in the sense that any optimal mass function

$\psi$  must have the same  $\mathcal{C}[\psi]$ . This does not require that the minimizing mass function is itself unique.

Such a geometric formulation simplifies the interpretation of the problem. In particular, the result for the case of a single constraint is now immediate: the convex hull is 1-dimensional, so the point with minimum norm is simply the minimum value of  $g(M)$ . The corresponding mass function is monochromatic, with a peak at  $\text{argmin } g(M)$ . It is also clear that the monochromatic mass function is not generally the minimizer of the norm in the case of multiple constraints: we have no guarantee that  $\|\mathbf{g}(M)\|$  attains the minimum of the norm on  $\text{conv}(\mathbf{g})$  for any single  $M$ .

Still, minimizing the norm over the convex hull of a discretization of  $\mathbf{g}(M)$  is a simple computational problem, and it is easy to validate the result. We find an optimal mass function in three steps:

1. Choose a discretization of  $\mathbf{g}(M)$  of the form  $G = \{\mathbf{g}(M_1), \dots, \mathbf{g}(M_R)\}$ . We choose the  $M_k$  using adaptive sampling to capture features of the constraint functions as precisely as possible. The convex hull of  $G$  is now a polytope  $A$ .
2. Find the point  $p_{\min} \in A$  with minimum norm. We implement the algorithm of [20], which requires only the extreme points of  $A$  as inputs. To avoid computing the convex hull in a high-dimensional space, we supply all of the points of  $G$ , of which the extreme points of  $A$  form a subset. The algorithm determines the facet  $S$  of  $A$  which contains  $p_{\min}$ , and gives the barycentric coordinates of  $p_{\min}$  in  $S$  as a vector  $\mathbf{w}$ .
3. Define a mass function

$$\psi_{\text{opt}}(M) = \sum_{k=1}^{|\mathbf{w}|} w_k \delta(M - M_k) \quad (3.15)$$

where  $\mathbf{g}(M_k)$  is the  $k$ th point of  $S$ . Note that  $\mathbf{g}(M_k) \in G$  for each  $M_k$  since  $S \subset A$ .

Observe that  $\mathcal{C}[\psi_{\text{opt}}] = \sum_{k=1}^{|\mathbf{w}|} w_k \mathbf{g}(M_k) \equiv p_{\min}$ . Thus,  $\psi_{\text{opt}}$  is a mass function which attains the maximum total dark matter fraction. In particular, for any mass function  $\psi$ , we have  $\mathcal{M}[\psi] \leq \mathcal{M}[\psi_{\text{opt}}] = \|p_{\min}\|^{-1}$ , so  $f_{\text{max,all}} = \|p_{\min}\|^{-1}$  is an upper bound on the fraction of dark matter in PBH irrespective of the functional form of the mass function. We will refer to  $\psi_{\text{opt}}$  as the *semi-analytical optimum* mass function.

We can now explain geometrically why the maximizing mass function is a linear combination of no more than  $N$  monochromatic mass functions. Observe that for any  $\mathbf{g}(M)$ , the minimum of the norm must lie on the boundary of the convex hull  $\text{conv}(\mathbf{g})$ , and since  $\mathbf{g}(M_k) \in \mathbb{R}^N$ , this boundary has dimension at most  $N - 1$ . One can construct an arbitrarily refined triangulation of this boundary formed from  $(N - 1)$ -simplices, each with  $N$  points of  $G$  as vertices. The minimizer of the norm is a linear combination of these vertices, each of which is one of the original  $\mathbf{g}(M_k)$ , corresponding to a monochromatic mass function.

### 3.4 Results

We perform the maximization explicitly for several sets of constraints. Set **A** includes robust constraints from evaporation [21]; GRB lensing [22]; microlensing from HSC [9], Kepler [10], EROS [11], and MACHO [12]; and CMB limits from Planck [13]. Set **B** includes dynamical constraints from Segue I [23], Eridanus II [24], and non-disruption of wide binaries [14]. Set **C** includes a constraint from white dwarf explosions [25], a constraint from neutron star



	$f_{\text{mono}}$	$f_{\text{max,all}}$	$\langle M/M_{\odot} \rangle$	$\sigma[\psi]/M_{\odot}$
<b>A</b>	27.17	27.25	31.09	2.259
<b>AB</b>	1.372	1.965	0.009	0.162
<b>AC</b>	1.371	1.443	1.807	7.294
<b>ABC</b>	1.371	1.402	0.015	0.220
$\bar{\mathbf{A}}$	0.991	1.502	1.492	4.827
$\bar{\mathbf{A}}\mathbf{B}$	0.991	1.437	0.017	0.221
$\bar{\mathbf{A}}\mathbf{C}$	0.330	0.484	5.430	7.963
$\bar{\mathbf{A}}\mathbf{B}\mathbf{C}$	0.330	0.405	0.182	0.741

**Table 1.** Optimal mass function properties for each of several sets of constraints. The column  $f_{\text{mono}}$  gives the maximum DM fraction allowed for a monochromatic mass function, and the column  $f_{\text{max,all}}$  gives the maximum DM fraction across all functional forms. Also given here are the mean PBH mass and the standard deviation for the semi-analytical optimum mass function.

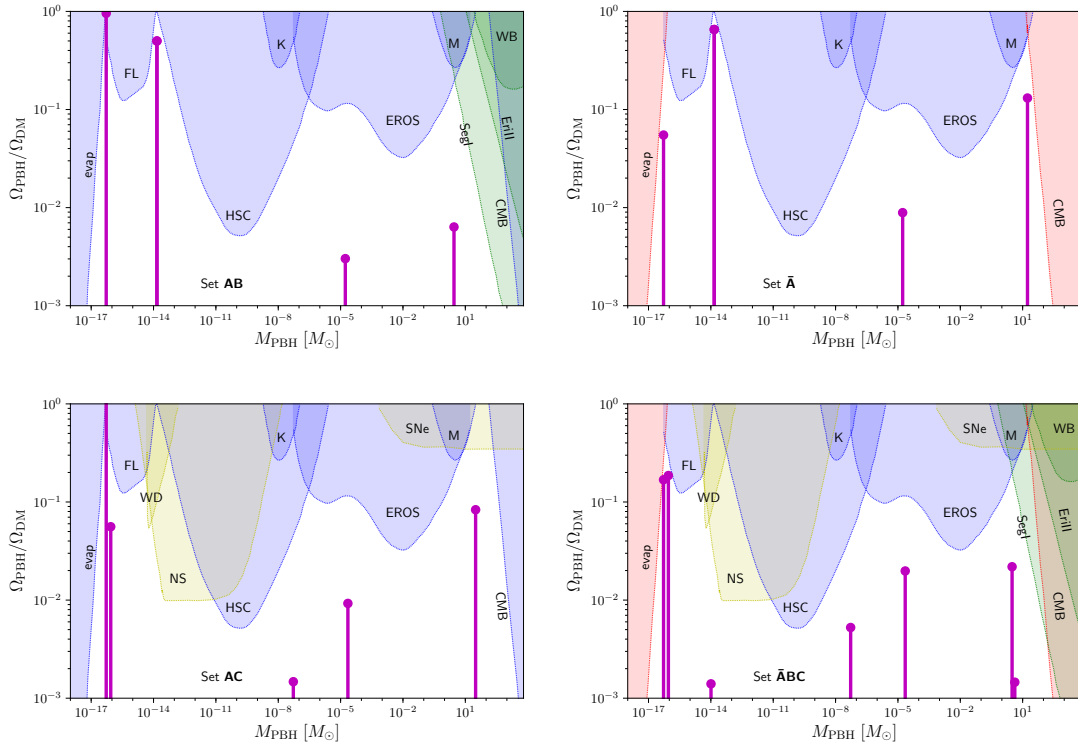
capture [26] and a recently claimed constraint from SNe lensing in the LIGO window [27]. The constraints from evaporation and from Planck in **A** have been estimated differently in the literature, with important consequences for our analysis. Set **A** itself contains relatively non-restrictive estimates of these constraints. We incorporate more stringent versions (see section 4) of these constraints in a set  $\bar{\mathbf{A}}$ , which is otherwise identical to **A**.

We determine optimal mass functions for sets **A**,  $\bar{\mathbf{A}}$ , and all of their combinations with sets **B** and **C**. The results are summarized in table 1 and illustrated in fig. 1. We do not include cosmological constraints on the total matter density, so these values of  $f_{\text{max,all}}$  may exceed 1. In particular, note that all combinations containing **A** have  $f_{\text{max,all}} > 1$ , while all combinations containing  $\bar{\mathbf{A}}$  and **C** have  $f_{\text{max,all}} < 1$ . The set  $\bar{\mathbf{A}}$  on its own has marginal status if only monochromatic mass functions are considered, but clearly  $f_{\text{max,all}} > 1$  in this case. With the constraints we consider in this work,  $f_{\text{PBH}} = 1$  is always allowed when using the less stringent set **A**, regardless of additional constraints.

## 4 Discussion

With the maximization procedure introduced in section 3.3, it is simple to determine the maximum PBH density consistent with constraints. We stress that this is a bound that applies for mass functions of all forms. Thus, given a set of observational constraints, we can determine a model-independent bound on the density of PBH.

Further, the functional form of the optimal mass function elucidates the dependence of constraints on the variance of the mass function. In the single-constraint case, we showed that an extended mass function never outperforms the optimal monochromatic mass function. Indeed, in this case, increasing the variance of a narrow mass function will only relax constraints if  $f_{\text{max}}$  is concave-up in the neighborhood of interest. When multiple constraints are considered, the relationship between the variance of the mass function and the allowed density is less obvious. Our semi-analytical optimum mass functions all exhibit some non-zero spread, and they definitively allow higher PBH densities than any zero-variance (i.e., monochromatic) mass function. However, extending a monochromatic mass function only slightly, without overlapping additional points of the semi-analytical optimum mass function, is not useful for relaxing constraints. In this respect, our findings are consistent with those of [16, 17].



**Figure 1.** The semi-analytical optimum mass function for four sets of constraints. Constraint functions for monochromatic mass functions are shown in blue (**A**), red ( **$\bar{\text{A}}$** ), green (**B**), and yellow (**C**). Vertical lines denote the locations of Dirac deltas in the semi-analytical optimum mass function, with height indicating the weight given to each one.

Our results quantify the risks of using monochromatic mass functions to assess the overall status of the PBH dark matter paradigm. So long as one window in the constraint functions dominates over the others, the difference between  $f_{\text{max,all}}$  and  $f_{\text{mono}}$  is generally very small. Set **A** is a clear example of such a case, and the correction is of order 0.1%. On the other hand, if PBH are constrained to a similar extent in two distinct windows, the correction can be large. The most dramatic example is provided by set  **$\bar{\text{A}}$** , for which  $f_{\text{max,all}}$  is larger than  $f_{\text{mono}}$  by 52%. We conclude that, at worst, the bound on the total PBH density is related to the monochromatic version by a factor of  $O(1)$ .

The optimal mass functions themselves (fig. 1) do not correspond to any well-motivated production scenario that we are aware of, and we certainly do not claim that the maximal density can be attained by producing PBH monochromatically at a discrete collection of masses spanning 15 orders of magnitude. Instead, the panels of fig. 1 should be interpreted as tool to relate monochromatic constraint functions to their impact on the allowed total density of PBH. In particular, an immediate and non-trivial conclusion that can be drawn from the figures is that the addition of any new constraint which does not overlap the peaks of the optimal mass function will not reduce  $f_{\text{max,all}}$ .

The most substantial differences in  $f_{\text{max,all}}$  arise from differences between **A** and  **$\bar{\text{A}}$** . Set  **$\bar{\text{A}}$**  contains more stringent forms of constraints from CMB anisotropy and PBH evaporation. The CMB constraint is strongly dependent on modeling poorly-understood accretion pro-

cesses. Both versions of the constraint used in this work are drawn from [13]: the version in set **A** is obtained by considering only collisional ionization of the accreted gas, while the version in set  $\bar{\mathbf{A}}$  is obtained by including photoionization as well. The evaporation constraint is sensitive to uncertainties in the spectrum of extragalactic background radiation. We adopt the extreme cases considered by [16], with the relaxed form contained in set **A** and the more stringent form in set  $\bar{\mathbf{A}}$ . The values of  $f_{\text{max,all}}$  in table 1 demonstrate that the present observational status of PBH dark matter is strongly dependent on the constraints adopted. However, to rule out  $f_{\text{PBH}} = 1$ , it is necessary to both take the more stringent constraints  $\bar{\mathbf{A}}$  in place of **A**, and to include at least one of the constraints from set **C**: supernova microlensing [27], neutron star capture [26], and white dwarf explosions [25].

The supernova microlensing constraint is the most recent of those we consider, and its robustness is the subject of ongoing discussion in the literature [see e.g. 8]. We note that this constraint is dominant in the LIGO window only when dynamical constraints from set **B** are neglected, so the addition of this constraint alone to set **AB** or  $\bar{\mathbf{A}}\mathbf{B}$  will have a small impact on  $f_{\text{max,all}}$ . The constraint from neutron star capture is also subject to astrophysical uncertainties, since it is dependent on the dark matter density in the cores of galactic clusters [26]. We consider the relatively restrictive constraint obtained by taking  $\rho_{\text{DM}} = 10^4 \text{ GeVcm}^{-3}$ . The strength of the constraint scales linearly with  $\rho_{\text{DM}}$ , and more conservative estimates take  $\rho_{\text{DM}}$  smaller by an order of magnitude or more. However, this constraint is most effective in a window shared with constraints from white dwarf explosions, so even if one of the two is subject to substantial uncertainties, the effect of set **C** on  $f_{\text{max,all}}$  remains large.

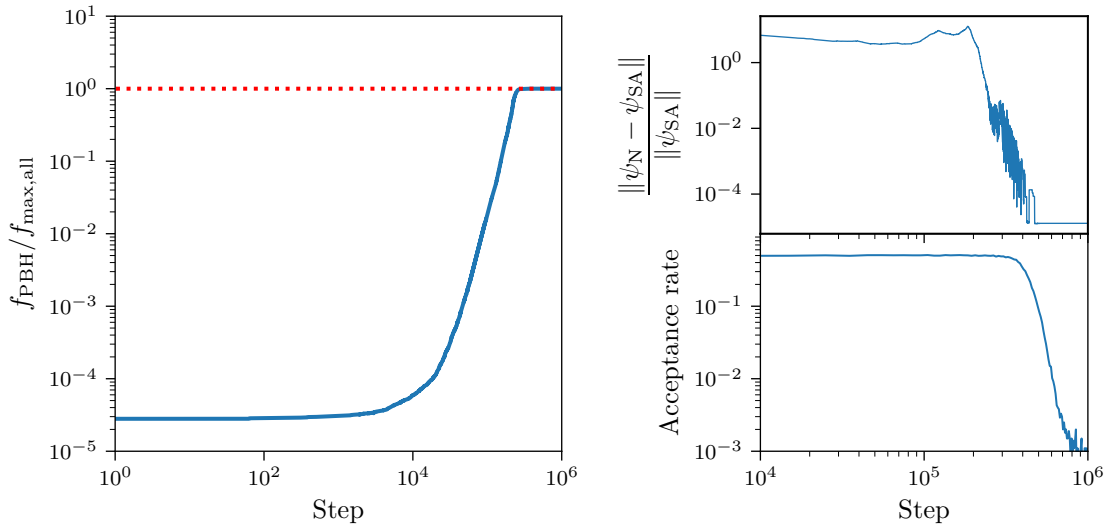
## 5 Conclusions

We have found the form of the mass function which maximizes the PBH density subject to observational constraints, and we have used this to calculate an upper bound on the fraction of dark matter in PBH. Depending on the constraints adopted, we find  $f_{\text{max,all}}$  as large as 27.25 (set **A**) or as small as 0.405 (set  $\bar{\mathbf{A}}\mathbf{BC}$ ). The scenario in which all dark matter is composed of PBH is ruled out by stringent limits from evaporation and Planck if combined with the constraints from white dwarf explosions, neutron star capture and SNe lensing (set **C**). However, if relaxed constraints from evaporation and Planck are adopted, PBH dark matter is not ruled out by the addition of any other constraints we consider in this work.

Our method provides a fast and robust technique to determine the total allowed density of PBH given a set of constraints ( $f_{\text{max,all}}$ ), independent of the form of the PBH mass function. The optimal mass function itself allows an easy test of the impact of additional constraints on  $f_{\text{max,all}}$ . While the optimal mass function is not exactly monochromatic, it is very nearly so for realistic constraints. The optimal mass function corresponding to each set of constraints we consider is approximately monochromatic, with additional components scaling the total allowed fraction by no more than an  $O(1)$  factor. Our results explain the findings of [16, 17] that extended mass functions are generally more strongly constrained than monochromatic mass functions, and confirm that the monochromatic maximum density  $f_{\text{mono}}$  is a good approximation of the allowed density across all mass functions.

## A Comparison to numerical optimization

Given a set of constraints, it is also possible to use numerical methods to find a mass function which maximizes the PBH density. There are significant caveats to such an approach. Most



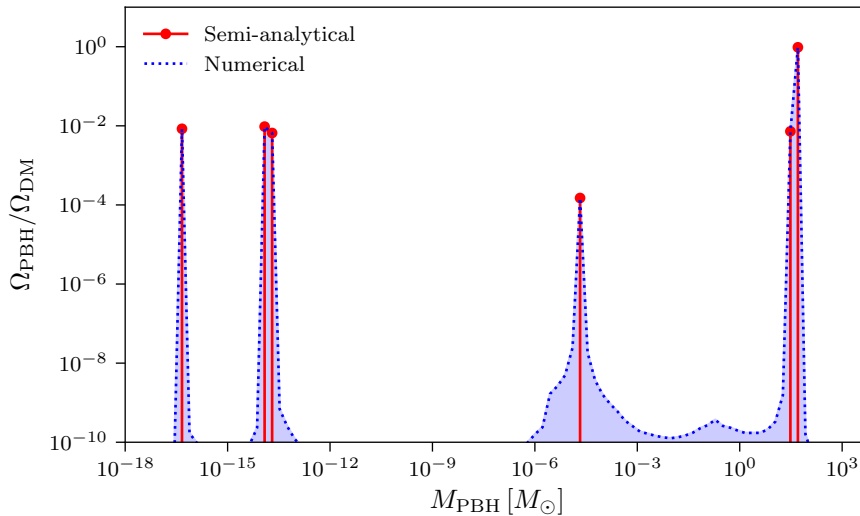
**Figure 2.** Left:  $f_{\text{PBH}}$  attained in each step during numerical maximization, shown as a fraction of the semi-analytical  $f_{\text{max,all}}$ . The dashed red line indicates  $f_{\text{PBH}} = f_{\text{max,all}}$ . Right top:  $L^2$  norm of the difference between  $\psi_N$  (numerical) and  $\psi_{\text{SA}}$  (semi-analytical) mass functions for each step, shown as a fraction of  $\|\psi_{\text{SA}}\|$ . In computing the norm,  $\psi_{\text{SA}}$  is treated as a step function on the mass bins. Right bottom: acceptance rate in bins of  $10^4$  steps.

importantly, a maximization algorithm may converge to a local optimum rather than a global optimum. Additionally, computational costs may render numerical approaches impractical unless the functions involved are discretized sparsely. Even so, numerical optimization can be used to validate our analytical results: if the same set of masses is used for discretization, then the numerical result should never reach a greater normalized mass (cf. eq. (3.12)) than that of our corresponding semi-analytical result. Numerical methods can also be used to check that our semi-analytical optimum is a stationary point of the normalized mass functional.

We implement these validation steps using a simple Monte Carlo algorithm, as follows: we begin with an initial mass function of the form  $\psi_0(M) \propto M^{-1}$ , which assigns equal PBH density to each log-spaced mass bin. We then perturb the value of  $\psi_0$  in a random bin  $k$  by a value selected from a Gaussian distribution with mean 0 and variance  $\sigma^2\psi_0(M_k)^2$ , where  $\sigma$  is a parameter of the maximization. We denote the resulting mass function by  $\psi_1(M)$ . If  $\psi_1(M_k) \geq 0$  and  $\mathcal{M}[\psi_1] > \mathcal{M}[\psi_0]$ , we accept the step, replace  $\psi_0$  by  $\psi_1$ , and repeat. For simplicity, we do not accept any steps which reduce the normalized mass. This is not necessary in order to test whether our semi-analytical optimum mass function is a stationary point. We also reject steps which increase the normalized mass by less than  $10^{-10}$  to avoid exceeding the numerical precision of the semi-analytical result.

In order to make the problem numerically tractable, we use only  $10^2$  log-spaced mass bins. This discretization is different from the one used in table 1, and it does not capture sharp features of the constraints. Consequently, in order to compare the numerical results with semi-analytical results, we regenerate the semi-analytical mass function with the same discretization. Note that this affects both the form of the optimal mass function and the calculated  $f_{\text{max,all}}$ .

We implement the numerical optimization with  $\sigma = 10^{-2}$ . In what follows, we denote the numerical mass function by  $\psi_N$ , and the semi-analytical optimum by  $\psi_{\text{SA}}$ . The left-hand



**Figure 3.** Blue: numerically-optimized mass function  $\psi_N$  after  $10^6$  steps. Red: semi-analytical optimum  $\psi_{SA}$ . Each curve shows the integral of the mass function in each bin, i.e., the total contribution of that bin to  $f_{PBH}$ .

side of fig. 2 shows  $f_{PBH}$  for the numerical mass function at each step as a fraction of the semi-analytical  $f_{\max,all}$ . The numerical  $f_{PBH}$  converges to  $f_{\max,all}$  and immediately stabilizes, and in particular, in no step does  $f_{PBH}$  exceed  $f_{\max,all}$ .

In principle,  $\psi_N$  need not converge to  $\psi_{SA}$  even given that  $f_{PBH}$  converges to  $f_{\max,all}$ , since the mass function with maximal density is not necessarily unique. However, in the top-right panel of fig. 2, we show that  $\psi_N$  tends to  $\psi_{SA}$  in the  $L^2$  norm. To compute this distance consistently, we treat the Dirac deltas of  $\psi_{SA}$  as constant functions in their respective bins. As an additional test of convergence, we compute the acceptance rate, i.e., the fraction of steps which are accepted, during each window of  $10^4$  iterations. The acceptance rate vanishes as  $\psi_N$  approaches  $\psi_{SA}$ , which further demonstrates that  $\psi_{SA}$  is a stationary point of the normalized mass.

The numerical and semi-analytical mass functions are shown in fig. 3. In order to compare Dirac deltas with the smooth mass function  $\psi_N$ , the figure shows the integral of the mass function in each bin rather than  $\psi_N$  and  $\psi_{SA}$  themselves. It is clear that in this case, the numerical algorithm converges to the semi-analytical optimum. We have established via analytical arguments that this is not simply a local optimum, but indeed the global maximum of the normalized mass.

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