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Well-posedness and modified scattering for derivative nonlinear Schrödinger equations

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Well-posedness and modified scattering for derivative nonlinear Schrödinger equations

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by

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ABSTRACT OF THE DISSERTATION

Well-posedness and modified scattering for derivative nonlinear Schrödinger equations

by

Donlapark Pornnopparath

Doctor of Philosophy in Mathematics

University of California San Diego, 2018

Professor Ioan Bejenaru, Chair

We consider the initial value problem for various type of nonlinear Schrödinger equations with derivative nonlinearity which cannot be treated by normal perturbative arguments because of the loss in derivative from the nonlinearity.

The first part of the study involves finding the well-posedness in low regularity Sobolev spaces for different types of nonlinearities. The key idea is to capture a part of the solution that resembles the linear Schrödinger dynamic while keeping the remaining part spatial and frequency localized. With this, we can study the interactions between the truncations of the solution at different frequencies and obtain a meaningful perturbative analysis.

In the second part, we study the dynamic of the cubic nonlinear Schrödinger equation in the energy critical Sobolev space by projecting the solution onto different wave packets which are frequency and spatial localized at all time. As a result, we obtain the asymptotic behavior, modified scattering profile and asymptotic completeness of the solution without relying on the integrable structure of the equation.

Chapter 1

Introduction

1.1 Motivations

The Schrödinger equation with polynomial-type nonlinearity,

$$\begin{cases} i\partial_t u + \Delta u = P(u, \overline{u}, \partial_x u, \partial_x \overline{u}) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0, \end{cases} \tag{1.1}$$

where $P : \mathbb{C}^4 \mapsto \mathbb{C}$ is a polynomial of the form

$$P(z) = P(z_1, z_2, z_3, z_4) = \sum_{d \leq |\alpha| \leq l} C_\alpha z^\alpha, \tag{1.2}$$

is a one dimensional model that comes up quite often in nonlinear optics. The nonlinearity often arises from a high-intensity ultrashort light pulse propagating through an optical fiber with high nonlinear coefficients ([43]), for example, semiconductor doped glasses or organic polymers, in which case the refractive index takes a nonlinear form in powers of intensity I of the light pulse: $n = n_0 + n_2 I + n_4 I^2 + \ldots$, where $n_0, n_2, n_4, \ldots$ are refractive index coefficients varying in time. In particular, if we ignore higher intensity orders: $n = n_0 + n_2 I + \ldots + n_{2N} I^N$, the wave equation for the light pulse when the optical fiber has a circular cross section can be written as

$$\Delta_z \mathbf{E} - \frac{1}{c^2}\partial_t^2 \mathbf{P}_L = \frac{1}{c^2}\partial_t^2 \mathbf{P}_{NL}, \tag{1.3}$$

where c is the speed of light, $\mathbf{E}$ is the electric field and $\mathbf{P}_L$ and $\mathbf{P}_{NL}$ are linear and nonlinear parts of the electric polarization written as

$$\mathbf{P}_L = \int_0^\infty (n_0(t'))^2 \mathbf{E}(t - t')\ dt',$$

$$\mathbf{P}_{NL} = c_2|\mathbf{E}|^2 + c_4|\mathbf{E}|^4 + \ldots + c_{2N}|\mathbf{E}|^{2N},$$

where each c_{2i} is a product of n_j's. The equation (1.3) can be solved by the method of separation of variables by writing $\mathbf{E}$ as

$$\mathbf{E} = \hat{\mathbf{e}}\mathbf{R}(\mathbf{r})u(z,t)e^{i\beta z - i\omega t}, \tag{1.4}$$

where $\hat{\mathbf{e}}$ is the direction vector of the polarization and $\mathbf{r}$ is the radius vector in the x-y plane and $u(z,t)$ is the amplitude. We can substitute (1.4) into (1.3) and approximate via Taylor series expansion. When $N = 2$, the resulting equation is

$$i\partial_z u + \partial_{tt} u = ia_0\partial_{ttt} u + a_2|u|^2 u + a_3|u|^4 u + ia_4\partial_t(|u|^2 u) + ia_5\partial_t(|u|^4 u).$$

This equation and its variations had been intensively studied, for examples, in [12], [13], [27], [43], [51] and [52]. If the pulse's width is more than 100 femtosecond, the third-order dispersion term $a_0\partial_{ttt} u$ can be neglected ([1]). Under this assumption, we can generalize the equation to the Nth order refraction index: ([50])

$$i\partial_z u + \partial_{tt} u = \sum_{k=1}^{N} \alpha_k |u|^{2k} u + i \sum_{k=2}^{N} \beta_k \partial_t (|u|^{2k-2} u) + i \sum_{k=2}^{N} \gamma_k \partial_t (|u|^{2k-2}) u. \tag{1.5}$$

Many simplified forms of this equation have been thoroughly investigated. For examples, the Nonlinear Schrödinger equation (NLS):

$$i\partial_z u + \partial_{tt} u = |u|^2 u \tag{1.6}$$

and the Derivative Nonlinear Schrödinger equation (DNLS):

$$i\partial_z u + \partial_{tt} u = i\partial_t (|u|^2 u). \tag{1.7}$$

Equation (1.7) also arises from studies of small-amplitude Alfv́en waves propagating parallel to a magnetic field [40] and large-amplitude magnetohydrodynamic waves in plasmas [44]. There is also recent discovery of rogue waves as solutions for the Darboux transformation of the DNLS (See [56]). More details on the DNLS equation will be explained below.

Before going over the literature, we introduce some notations that we will be using throughout the rest of the thesis.

1.2 Notations

The following notations will be used: The variable x and t always refer to the one-dimensional spatial variable and the time variable, respectively. For $1 \leq p, q \leq \infty$, and $I, J \subseteq \mathbb{R}$, we define the Banach spaces $L_x^p(I)$ and $L_t^q(J)$ by the norms:

$$\|f(x)\|_{L_x^p(I)} := \left(\int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\|g(t)\|_{L_t^q(J)} := \left(\int_J |g(t)|^q \ dx\right)^{\frac{1}{q}}.$$

If there is no confusion, L_x^p will sometimes be shortened to L^p. For any Banach spaces $\mathcal{X}(I)$ and $\mathcal{Y}(J)$ of complex-valued functions on sets I and J and any function f defined on the product space $I \times J$, we define the mixed norm

$$\|f\|_{\mathcal{X}\mathcal{Y}(I\times J)} := \big\| \|f\|_{\mathcal{Y}(J)} \big\|_{\mathcal{X}(I)},$$

and $\|f\|_{\mathcal{X}\mathcal{Y}} = \|f\|_{\mathcal{X}\mathcal{Y}(\mathbb{R}\times\mathbb{R})}$. For $I = [a,b]$, we make a slightly shorter notations $\|f\|_{\mathcal{X}\mathcal{Y}(a,b;J)} := \|f\|_{\mathcal{X}\mathcal{Y}([a,b]\times J)}$ and $\|f\|_{\mathcal{X}\mathcal{Y}(a,b)} := \|f\|_{\mathcal{X}\mathcal{Y}([a,b]\times\mathbb{R})}$. We define the Fourier transform and the inverse Fourier transform of $f(x)$ by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \ dx,$$
$$\check{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi) \ d\xi.$$

Since all of the proofs rely on estimates up to fixed constants, we will make an abuse of notations and drop the constant $\frac{1}{\sqrt{2\pi}}$ from these formulas. For $s \in \mathbb{R}$, we denote by $D^s = (-\Delta)^{s/2}$ the Riesz potential of order $-s$. The Sobolev space H_x^s is defined by the norm

$$\|u\|_{H_x^s} := \|(1-\partial_x^2)^{\frac{s}{2}} u(x)\|_{L_x^2} = \|(1+\xi^2)^{\frac{s}{2}} \widehat{u}(\xi)\|_{L_\xi^2}.$$

We define the Banach space of bounded H^s-valued continuous functions:

$$C_t^0 H_x^s(I\times J) := \left\{ f \in C(I; H_x^s(J)) : \sup_{t\in I} \|f(\cdot,t)\|_{H_x^s(J)} < \infty \right\}.$$

The weighted Sobolev space $H_x^{m,k}$ is defined by

$$\|u\|_{H^{m,k}}^2 = \|(1+|x|^2)^{\frac{k}{2}} (1-\partial_x^2)^{\frac{m}{2}} u\|_{L_x^2}^2.$$

Sometimes we will write these spaces as H^s and $H^{m,k}$ if the variable is well-understood. Let $v \in L_t^\infty L_x^2$. We define the Schrödinger propagator by

$$e^{it\Delta}v(x,t) := \int_{\mathbb{R}} e^{ix\xi - it\xi^2}\hat{v}(\xi)\, d\xi.$$

The notation $a \lesssim b$ and $a \sim b$ means $a \le Cb$ and $ca \le b \le CA$, respectively, for some positive constants c and C, which depend on $P(z)$ but not on the functions involved in these estimates.

We frequently split the frequency space into dyadic intervals, so whenever M and N is mentioned, we assume that $M, N \in 2^{\mathbb{Z}}$. Let $\psi(\xi)$ be a smooth cutoff function supported in $|\xi| \le 4$ and equal 1 on $|\xi| \le 2$. We define $\psi_N = \psi\left(\frac{\xi}{N}\right) - \psi\left(\frac{2\xi}{N}\right)$. Denote by P_N the Littlewood-Paley projection at frequency N, that is

$$\widehat{P_N f}(\xi) = \psi_N(\xi)\hat{f}(\xi)$$

Define $P_{\le N}$ and $P_{>N}$ to be the projections of frequency less than and greater than N:

$$\widehat{P_{\le N} f}(\xi) = \psi_{\le N}\hat{f}(\xi) := \sum_{M \le N} \psi_M(\xi)\hat{f}(\xi),$$

$$\widehat{P_{> N} f}(\xi) = \psi_{> N}\hat{f}(\xi) := \sum_{M > N} \psi_M(\xi)\hat{f}(\xi).$$

We will sometimes shorten the notation as follows: $f_N := P_N f$. For any Banach space $\mathcal{X}$ of functions on $\mathbb{R}$ and $1 \le p \le \infty$, we define the norm $l^p X$ by

$$\|u\|_{l^p\mathcal{X}} := \Big(\sum_{N_i \in 2^{\mathbb{Z}}} \|P_{N_i}u\|_{\mathcal{X}}^p\Big)^{\frac{1}{p}}. \tag{1.8}$$

For $s \ge 0$, we define the homogeneous Sobolev space $\dot{H}^s$ using the Littlewood-Paley projections

$$\|u\|_{\dot{H}^s} := \Big(\sum_{N_i \in 2^{\mathbb{Z}}} N_i^{2s}\|P_{N_i}u\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

For any $A \subseteq \mathbb{R}$, we define an indicator function χ_A by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $\|\chi_{[2^N,2^{N+1}]}(\xi)\widehat{u}(\xi)\|_{L^2_\xi} \sim \|\psi_N(\xi)\widehat{u}(\xi)\|_{L^2_\xi}$, by the duality and orthogonality of $\chi_{[2^N,2^{N+1}]}\widehat{u}$ under the L^2_ξ norm, we can define a norm equivalent to that of H^s in terms of the Littlewood-Paley projections:

$$\|u\|_{H^s} \sim \|P_{\leq 1}u\|_{L^2} + \Big(\sum_{N_i \in 2^{\mathbb{N}}} N_i^{2s}\|P_{N_i}u\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

1.3 Previous results

There are several results regarding the well-posedness of (1.1). In [34], Kenig, Ponce and Vega proved that the equation (1.1) is locally well-posed for a small initial data in $H^{\frac{7}{2}}(\mathbb{R})$. There has been some interest in the special case where $P = i\lambda|u|^k u_x$:

$$\begin{cases} i\partial_t u + \Delta u = i\lambda|u|^k u_x \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0. \end{cases}$$

with $k \in \mathbb{R}$. Hao ([18]) proved that this equation is locally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$ for $k \geq 5$, and Ambrose-Simpson ([2]) proved the result in $H^1(\mathbb{R})$ for $k \geq 2$. Recent studies show that these results can be improved. See Santos ([45]) for the local-wellposedness in $H^{\frac{1}{2}}$ when $k \geq 2$ and Hayashi-Ozawa ([22]) for the local well-posedness in H^2 when $k \geq 1$ and the global well-posedness in H^1 when $k \geq 2$.

Several studies showed that we have better results if P only consists of $\overline{u}$ and $\partial_x\overline{u}$ due to the following heuristic: if u solves the linear Schrödinger equation, then the space-time Fourier transform of $\overline{u}$ is supported away from the parabola $\{(\xi,\tau)|\tau+\xi^2=0\}$, leading to strong dispersive estimates. Grünrock ([15]) showed that for $P=\partial_x(\overline{u}^d)$ or $P=(\partial_x\overline{u})^d$ where $d\geq 3$, the equation (1.1) is locally well-posed for any $s>\frac{1}{2}-\frac{1}{d-1}$ in the former case and $s>\frac{3}{2}-\frac{1}{d-1}$ in the latter. Later, Hirayama ([26]) extended Grünrock's results for $P=\partial_x(\overline{u}^d)$ to the global well-posedness for $s\geq\frac{1}{2}-\frac{1}{d-1}$.

There are also various results for higher dimension analogues of (1.1)

$$\begin{cases} i\partial_t u+\Delta u=P(u,\overline{u},\nabla u,\nabla\overline{u}) \\ u(x,0)=u_0(x), \quad x\in\mathbb{R}^n. \end{cases} \tag{1.9}$$

The most general results in $\mathbb{R}^n$ for $n\geq 2$ is due to Kenig, Ponce and Vega in [34]. For a more specific case, we refer to [4] and [5] where Bejenaru obtained a local well-posedness result for $n=2$ and $P(z)$ is quadratic with low regularity initial data. See also [11] when $n\geq 1$ and $P(u,v)=O(|u|^2+|v|^2)$ or $P(u,v)=O(|u|^3+|v|^3)$. For results in Besov spaces, see [53] for the global well-posedness in $\dot{B}^{s_n}_{1,2}(\mathbb{R}^n)$ where $n\geq 2$ and $s_n=\frac{n}{2}-\frac{1}{d-1}$ which is the critical exponent.

Let us go back to the DNLS equation.

$$\begin{cases} i\partial_t u+\Delta u=i\partial_x(|u|^2u) \\ u(x,0)=u_0\in H^s(\mathbb{R}), s\geq\frac{1}{2}. \end{cases} \tag{1.10}$$

Observe that (1.10) is invariant under the scaling $u(x,t)\mapsto u_\lambda(x,t):=\lambda^{\frac{1}{2}}u(\lambda x,\lambda^2 t)$. Since $\|u_\lambda\|_{\dot{H}^s}=\lambda^s\|u\|_{\dot{H}^s}$, if we follow the scaling heuristic for dispersive equations, the equation

(1.10) is expected to be locally well-posed in any subcritical Sobolev space i.e. any H^s with $s \geq 0$. However, Biagioni and Linares ([6]) have showed that (1.10) is ill-posed for $s < \frac{1}{2}$ in the sense that the solution mapping $u_0 \mapsto u$ fails to be uniformly continuous. This means that our result from Theorem 1.3 when $d = 3$, which is a local well-posedness in $H^{\frac{1}{2}}$, is sharp in this sense.

We mention here a few of many results regarding this equation. The global well-posedness in the energy space $H^1(\mathbb{R})$ was proved by Hayashi and Ozawa in [25]. For data below the energy space, Takaoka has shown in [47] that DNLS is locally well-posed for $s \geq \frac{1}{2}$ using (1.11) with $k = -1$. In [14], Colliander, Keel, Staffilani, Takaoka and Tao used the "I-method" to show the global well-posedness of DNLS for $s > \frac{1}{2}$, assuming the smallness condition $|u_0|_{L^2} < \sqrt{2\pi}$. Later, Miao, Wu and Xu have proved the global well-posedness result for the endpoint case $s = \frac{1}{2}$ using the third generation I-method and same smallness condition in [39]. Lastly, Wu ([54] and [55]) has shown that in the energy-critical case $s = 1$, the smallness threshold is improved to $\|u_0\|_L^2 < 2\sqrt{\pi}$.

We are now shifting focus toward some qualitative aspects of the solutions. Kaup and Newell has shown that the equation in completely integrable, which implies infinitely many conservation laws. Moreover, the inverse scattering method can be applied to obtain soliton solutions which are unstable in a sense that a small perturbation could cause the soliton to disperse (See [32]). Recently, Liu, Perry and Sulem used this method to prove the global well-posedness result in $H^{2,2}(\mathbb{R})$ (see [38]). A study following Wu's above result ([10]) shows an existence of two kinds of solitons: bright solitons with mass $\sqrt{2\pi}$, and lump soliton with mass $2\sqrt{\pi}$. He showed in [54] that there is no blow-up near the $\sqrt{2\pi}$ threshold. On the other hand, the study of Cher, Simpson and Sulem ([10]) has shown some numerical evidence of a blow-up profile that closely resembles the lump soliton.

The main difficulty in studying DNLS is the spatial derivative in nonlinearity. Due to this, all of well-posedness results for DNLS so far involve the *Gauge transformation*:

$$v(x,t) := u(x,t)\exp\left\{ik\int_{-\infty}^{x}|u(y,t)|^2\ dy,\right\} \tag{1.11}$$

where $k \in \mathbb{R}$. In [47], Takaoka used the transformation with $k=-1$ to turn (1.10) into

$$\begin{cases} i\partial_t v + \Delta v = -iv^2\partial_x\overline{v} - \frac{1}{2}|v|^4 v \\ v(x,0) = v_0 \in H^s(\mathbb{R}), s \geq \frac{1}{2}. \end{cases} \tag{1.12}$$

Note that the transformation replaces the term $|u|^2\partial_x u$ with $v^2\partial_v\overline{u}$ which can be treated using the Fourier restriction norm method developed in [7]. In contrast to this type of proofs, we managed to get the local well-posedness of (1.10) (as a part of Theorem 1.3) without using a gauge transformation. The advantage is that the idea can be easily generalized to get a similar result for the equation (1.13)

1.4 Local well-posedness for gDNLS

Our first result is the local well-posedness of (1.1) in Sobolev spaces when the nonlinearity contains an arbitrary number of derivatives.

Theorem 1.1. *In the equation (1.1), let s be any real number such that*

(A) $s \geq \frac{1}{2}$ if $P(x,y,z,w)$ is linear in z and w,

(B) $s \geq \frac{3}{2}$ *otherwise.*

Then there exist a Banach space X^s *and a constant* $C = C(s, d)$ *with the following properties: For any* $u_0 \in H^s(\mathbb{R})$ *such that* $\|u_0\|_{H^s} < C$*, the equation* (1.1) *has a unique solution:*

$$u \in X := \{u \in C_t^0 H_x^s([-1,1] \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ *is Lipschitz continuous from* $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \leq C\}$ *to* X*.*

This shows that, without any restriction to the number of derivatives, we are able to improve Kenig et al.'s result ([34]) from $H^{\frac{7}{2}}$ to $H^{\frac{3}{2}}$. By restricting to only one derivative per term in the nonlinearity, we can improve further to $H^{\frac{1}{2}}$. Moreover, part (A) of Theorem 1.1 extends Hao and Santos's local well-posedness result in $H^{\frac{1}{2}}$ to more general class of nonlinearities. It turns out that the global well-posedness results can be achieved if the nonlinearity is quintic or higher and the endpoint cases are excluded.

Theorem 1.2. *Suppose that* $d \geq 5$ *in* (1.2)*. Let* s *be any number such that*

(A) $s > \frac{1}{2}$ *if each term in* $P(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ *has only one derivative,*

(B) $s > \frac{3}{2}$ *if a term in* $P(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ *has more than one derivative.*

Then the equation (1.1) is globally well-posed in the following sense: There exist a Banach space X^s *and a constant* $C = C(s, d)$ *with the following properties: For any* $u_0 \in H^s(\mathbb{R})$ *such that* $\|u_0\|_{H^s} < C$ *and any time interval* I *containing* 0*, the equation* (1.1) *has a unique solution:*

$$u \in X := \{u \in C_t^0 H_x^s(I \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ is Lipschitz continuous from $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \le C\}$ to X.

1.5 Global well-posedness for a special case of gDNLS in critical Sobolev spaces

Notice that when each term in $P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ has only one derivative, (1.1) is invariance under the scaling $u(x,t) \mapsto u_\lambda(x,t) := \lambda^{\frac{1}{d-1}} u(\lambda x, \lambda^2 t)$. Thus, the critical space is H^{s_0} where $s_0 = \frac{1}{2} - \frac{1}{d-1}$ in the sense that $\|u\|_{H^{s_0}} = \|u_\lambda\|_{H^{s_0}}$. If we follow the heuristic that a dispersive equation is expected to be locally well-posed in any subcritical Sobolev space H^s i.e. $s > s_0$, then the result in part (A) of Theorem 1.2, which requires $s > \frac{1}{2}$, is not optimal in this sense. It turns out that the global well-posedness at critical Sobolev spaces can be achieved if we assume a specific type of the gDNLS equation

$$\begin{cases} i\partial_t u + \Delta u = \partial_x P(u, \overline{u}) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \ge s_0. \end{cases} \tag{1.13}$$

where $P : \mathbb{C}^2 \mapsto \mathbb{C}$ is a polynomial of the form

$$P(z) = P(z_1, z_2) = \sum_{d \le |\alpha| \le l} C_\alpha z^\alpha, \tag{1.14}$$

and $l \ge d \ge 5$.

The following theorem shows that for $d \ge 5$ we have the global well-posedness at the scaling critical Sobolev space.

Theorem 1.3. *Suppose that* $d \geq 5$ *in* (1.14)*. Let* $s_0 = \frac{1}{2} - \frac{1}{d-1}$*. For* $u_0 \in H^s(\mathbb{R})$ *where* $s \geq s_0$*, the equation (1.13) is globally well-posed in the following sense:*

There exist a Banach space X^s *and a constant* $C = C(s,d)$ *with the following properties: For any* $u_0 \in H^s(\mathbb{R})$ *such that* $\|u_0\|_{H^s} < C$ *and any time interval* I *containing* 0*, the equation* (1.13) *has a unique solution:*

$$u \in X := \{u \in C_t^0 H_x^s(I \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \leq 2C\}.$$

Furthermore, the map $u_0 \mapsto u$ *is Lipschitz continuous from* $B_C := \{u_0 \in H^s : \|u_0\|_{H^s} \leq C\}$ *to* X*.*

In the case of $s = s_0$*, the statement above holds true if we replace* H^s *by* $\dot{H}^{s_0}$*.*

This extends Grünrock and Hirayama's results to more general class of nonlinearities. The main ideas behind the proof of Theorem 1.1 and Theorem 1.3 consist of the Duhamel reformulation of the problem, followed by the contraction argument. First, we decompose the nonlinear Duhamel term using (2.28), which was first introduced in [3], to deal with the time integral. Second, we use the local smoothing estimate (2.4) and the maximal function estimate (2.5) to deal with the loss of derivative in nonlinearity. We then combine these tools together as main ingredients for the usual perturbative analysis to obtain the well-posedness results. The proof for Theorem 1.3 in the case $d = 5$ is rather delicate and needs some modulation-frequency argument which is sensitive to the conjugates in the nonlinearity. Therefore, the proof of global well-posedness for $d = 5$, which is motivated by Tao's paper on the quartic generalised KdV equation ([49]), will be treated separately in the last section.

1.6 Global bounds and modified scattering for the DNLS equation

Another problem that comes up quite often in studies of nonlinear dispersive equations is the scattering problem, where one observes the behavior of the solution forward in time and see if it *scatters* to a linear solution. In our case, we consider the dynamic of the solution to the cubic DNLS equation:

$$\begin{aligned} i\partial_t u + \Delta u &= i\partial_x(|u|^2 u) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.15}$$

in the weighted Sobolev space $H^{1,1}(\mathbb{R})$. In particular, we are interested in a global pointwise bound and the scattering profile of the solution. Assume that we can get a pointwise bound in the form

$$\begin{aligned} \|u(x,t)\|_{L^\infty_x} &\lesssim \frac{1}{t^{\frac{1}{2}}}\|u_0\|_{H^{1,1}_x} \\ \|\partial_x u(x,t)\|_{L^\infty_x} &\lesssim \frac{1}{t^{\frac{1}{2}}}\|u_0\|_{H^{1,1}_x}, \end{aligned}$$

then we would expect u to behave like a linear solution for large t since the nonlinearity becomes really small:

$$\|i\partial_x(|u|^2 u)\|_{L^\infty_x} \lesssim \frac{1}{t^{\frac{3}{2}}}\|u_0\|^3_{H^{1,1}_x}.$$

The next step is to find a linear profile of u in the form of

$$u(x,t) \approx CA(x,t)e^{iB(x,t)}$$

for some function A and B. In our work, we use the method pioneered by Ifrim-Tataru in [28], where they solved the scattering problem for the cubic NLS (1.6). The main idea is to test the solution against wave packets $\Psi_v(x,t)$ localized at different frequencies ξ traveling at speed $v = \xi$:

$$\gamma(v,t) = \int u\bar{\Psi}_v \, dx.$$

Since the wave packets are spatial and frequency localized, the PDE equation (1.15) is translated in an ODE of γ in t. We then proceed to solve this ODE and obtain a profile of the solution by approximating u by γ. Note that the method that we just mentioned here is not restrictive to Schrödinger-type equations and has a potential to be a major tool in studying global dynamics of many nonlinear equations. For examples, see the recent works of [redacted]. We summarize the results as follows. Under the smallness condition on the initial data:

$$\|u_0\|_{H^{1,1}} \leq \epsilon \ll 1,$$

we have a global solution satisfying the pointwise bounds

$$\|u\|_{L^\infty} \lesssim \epsilon |t|^{-1/2},$$

$$\|u_x\|_{L^\infty} \lesssim \epsilon |t|^{-1/2}.$$

For large time t, the solution scatters with asymptotic profile

$$u(x,t) = \frac{1}{t^{1/2}} W\left(\frac{x}{t}\right) e^{i|W(x/t)|^2 \frac{x}{t} \log t + i\frac{x^2}{2t}} + err(x,t), \tag{1.16}$$

where W is a function in $H^{1-C\epsilon^2,1}(\mathbb{R})$ for some constant C and err is a small error function which decays in t. Moreover, we prove the asymptotic completeness, which states that for any function W in $H^{1-C\epsilon^2,1}(\mathbb{R})$ and

$$\|W\|_{H^{1+C\epsilon^2,1}(\mathbb{R})} \ll \epsilon \ll 1,$$

there exists an initial data u_0 such that the solution u to the equation (1.15) has the asymptotic profile (1.16). For the full statement of this result, see Theorem 4.1.

1.7 Organization

We organize the materials as follows. In chapter 2, we will mention several linear and smoothing estimates, together with the proofs of the maximal function estimate and bilinear estimate. After that, we introduce the notion of the Fourier restriction spaces $X^{s,b}$ along with several well-known estimates, as we will apply some frequency-modulation analysis in order to prove Theorem 1.3 in the case of $d = 5$. We will also introduce the spaces V^p of functions of bounded p-variation and prove several of their properties which will be used to conclude the results in Chapter 4. In chapter 3, we introduce the solution space X_N and nonlinear space Y_N for functions supported at frequency N, and we will prove the main linear and bilinear estimate for functions in these spaces using a solution decomposition technique from [3]. Then we prove Theorem 1.1 and Theorem 1.3, where the majority of proofs are focused on the multilinear estimates of functions in X_N. In Chapter 4, we prove (1.16) which consists of global boundedness, scattering and asymptotic completeness., using the method of testing against wave packets from [28].

Chapter 2

Elementary Results

2.1 The Linear Schrödinger equation

The corresponding linear equation of (1.1):

$$i\partial_t u + \Delta u = 0,$$

is used as the model for a quantum mechanical particle, e.g. an electron, where $u(x,t)$ is a wave function of the system and the quantity

$$\int_{\mathbb{R}} |u(x,t)|^2 \, dx \tag{2.1}$$

is conserved over time. Normally, we rescale u so that the integral in (2.1) is 1. In this case, for any measurable set $A \in \mathbb{R}$, the integral

$$\int_A |u(x,t)|^2 \, dx$$

gives the probability that the particle is in A at time t. The solutions of (2.1) are waves which at different frequencies travel at different velocities. Hence, localized solutions such as wave packets tend to spread out or *disperse* over time. Such behavior led us to categorize (2.1) as a *dispersive equation*.

2.2 Bernstein type inequality

We begin with the Bernstein inequality for the Littlewood-Paley projections. Note that this is different from the standard result in literatures which is the same estimate but for the space $L_t^q L_x^p$.

Lemma 2.1. *For any pair of* $1 \le p, q \le \infty$*, we have*

$$\|\partial_x P_N f\|_{L_x^p L_t^q} \lesssim N \|P_N f\|_{L_x^p L_t^q} \tag{2.2}$$

Proof. Let $\tilde{P}_N := P_{N/2} + P_N + P_{2N}$ be a Littlewood-Paley projection at a wider frequency interval with corresponding multiplier $\widetilde{\psi}_N$. We can rewrite the term on the left-hand side as

$$\partial_x \tilde{P}_N P_N f = (\partial_x \check{\widetilde{\psi}}_N) * P_N f(x,t).$$

For each x, we have an inequality

$$\|\partial_x P_N f\|_{L_t^q} \le |\partial_x \check{\widetilde{\psi}}_N| * \|P_N f(x,t)\|_{L_t^q}.$$

After taking the L_x^p norm and apply Young's inequality, we have

$$\|\partial_x P_N f\|_{L_x^p L_t^q} \le \|\partial_x \check{\widetilde{\psi}}_N\|_{L_x^1} \|P_N f\|_{L_x^p L_t^q} \lesssim N \|P_N f\|_{L_x^p L_t^q}.$$

□

This lemma helps us quantify derivatives of a function supported in a dyadic frequency interval, which will come in handy in the proofs of multilinear estimates in section 3 - 3.3.

2.3 Stationary phase lemmas

We mention here stationary phase results from harmonic analysis, which will be used in the next subsection. See [46, p.331-334] for their proofs.

Lemma 2.2. *Suppose that ϕ and ψ are smooth functions and ψ is compactly supported in (a, b). If $\phi'(\xi) \neq 0$ for all $\xi \in [a, b]$, then*

$$\left| \int_a^b e^{i\lambda\phi(\xi)} \psi(\xi)\ d\xi \right| \leq \frac{C}{|\lambda|^k}$$

for all $k \geq 0$, where the constant C depends on ϕ, ψ and k.

Lemma 2.3. *Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, ϕ is a real-valued C^2-function in (a, b) and $\phi''(\xi) \gtrsim 1$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(\xi)} \psi(\xi)\ d\xi \right| \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}} \left(|\psi(b)| + \int_a^b |\psi'(\xi)|\ d\xi \right).$$

2.4 Strichartz and local smoothing estimates

In our study, the nonlinear effect of the equation (1.1) with small initial data u_0 plays a major role in the perturbative analysis. As we mentioned in section 1, the main difficulty is a lost of derivative in the nonlinearity. In this regard, we will need the Strichartz estimate for the Schrödinger propagator and the smoothing estimate (2.4) which gives a $\frac{1}{2}$-order derivative gain of the linear solution in a suitable norm. We will also prove a maximal function type estimate (2.5) which will be used for the analysis of the nonlinear term.

Proposition 2.4. *Let $f \in L^2$. Then, we have the following estimates*

$$\|e^{it\Delta} f\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2}, \tag{2.3}$$

where $\dfrac{2}{q} + \dfrac{1}{p} = \dfrac{1}{2}$ *and* $2 \leq p \leq \infty$, *and*

$$\|D^{\frac{1}{2}} e^{it\Delta} f\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^2}. \tag{2.4}$$

Proof. The first inequality is the well-known Strichartz estimate. The proof can be found, for example, in [9] and [48]. The proof of (2.4) can be found in Theorem 4.1 of [33]. □

The following maximal function type estimate tells us that for the linear equation with time-and-frequency localized initial data in $H^s(\mathbb{R})$ where $s \geq \frac{1}{2}$, the solution is well-controlled in $L_x^\gamma L_t^\infty(\mathbb{R} \times I)$, where $I = [-1, 1]$ when $\gamma = 2, 3$ and $I = \mathbb{R}$ when $\gamma \geq 4$.

Proposition 2.5. *Let $u \in L_x^2(\mathbb{R})$.*

1. *If $\gamma = 2$ or 3, assume that* $\operatorname{supp}(|\hat{u}|) \subseteq [N, 4N]$ *where $N \in 2^{\mathbb{N}}$ or* $\operatorname{supp}(|\hat{u}|) \subseteq [0, 1]$, *in which case we consider $N = 1$, then*

$$\|\chi_{[-1,1]}(t) e^{it\Delta} u(x)\|_{L_x^\gamma L_t^\infty} \lesssim N^{\frac{1}{\gamma}} \|u\|_{L_x^2}, \tag{2.5a}$$

2. *If $\gamma \geq 4$, assume that* $\operatorname{supp}(|\hat{u}|) \subseteq [N, 4N]$ *where $N \in 2^{\mathbb{Z}}$, we have*

$$\|e^{it\Delta} u(x)\|_{L_x^\gamma L_t^\infty} \lesssim N^{\frac{\gamma-2}{2\gamma}} \|u\|_{L_x^2}. \tag{2.5b}$$

Remark: We see that the estimate (2.5a) is local in time while (2.5b) is global. By setting $\gamma = d-1$, this leads to the local and global results in Theorem 1.1 and Theorem 1.3.

Proof. We refer to Theorem 2.5 in [33] for a proof of the case $\gamma = 4$. Let $s_0 = s_0(\gamma) = \frac{1}{\gamma}$ for $\gamma = 2, 3$ and $s_0 = \frac{\gamma-2}{2\gamma}$ for $\gamma \geq 5$. We define an operator $T : L_x^2 \to L_x^\gamma L_t^\infty$ by $Tu = \chi_{[-1,1]}(t)e^{it\Delta}u$, yielding $T^*F = \int_{-1}^{1} e^{-it\Delta}F\ dt$. Using the TT^* argument, it follows that (2.5) is equivalent to either of the following estimates for $F \in L_x^2 L_t^1(\mathbb{R}\times\mathbb{R})$ with the same frequency support as u in the cases of $\gamma = 2, 3$.

$$\left\| \int_{-1}^{1} e^{-it\Delta}F(x,t)\ dt \right\|_{L_x^2} \lesssim N^{s_0}\|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1} \tag{2.6}$$

$$\left\| \chi_{[-1,1]}(t)\int_{-1}^{1} e^{i(t-s)\Delta}F(x,s)\ ds \right\|_{L_x^\gamma L_t^\infty} \lesssim N^{2s_0}\|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1}. \tag{2.7}$$

For $\gamma \geq 5$, we have the same estimates but with integrals on $\mathbb{R}$. Thus, it suffices to prove (2.7). First, we assume that $F \in \mathcal{S}(\mathbb{R})$. Since $F = P_{\leq 4N}F$, the inverse Fourier transform of $e^{i(t-s)\xi^2}\widehat{F}$ is defined by

$$\begin{aligned}\mathcal{F}_x^{-1}\left(e^{i(t-s)\xi^2}\widehat{F}(\xi,s)\right) &= c\int_{\mathbb{R}} e^{i(t-s)\xi^2+ix\xi}\widehat{F}(\xi,s)\ d\xi \\ &= \mathcal{F}_x^{-1}\left(e^{-i(t-s)\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) * F(x,s).\end{aligned}$$

Since $-1 \leq t, s \leq 1$ implies $-2 \leq t-s \leq 2$, the term on the right of (2.7) can be replaced by

$$\begin{aligned}&\int_{\mathbb{R}} \mathcal{F}_x^{-1}\left(\chi_{[-2,2]}(t-s)e^{-i(t-s)\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) * F(x,s)\ ds \\ &= \mathcal{F}_x^{-1}\left(\chi_{[-2,2]}(t)e^{-it\xi^2}\psi\left(\frac{\xi}{4N}\right)\right) \star F(x,t) \\ &= c_1 K_1 \star F\end{aligned}$$

where $\star$ denotes the space-time convolution and

$$K_1(x,t) = \int_{\mathbb{R}} e^{-it\xi^2+ix\xi} \chi_{[-2,2]}(t) \psi\left(\frac{\xi}{4N}\right)\ d\xi. \tag{2.8}$$

Similarly, for $\gamma \geq 5$ we have

$$\int_{\mathbb{R}} e^{i(t-s)\Delta} F(x,s)\ ds = c_2 K_2 \star F$$

where

$$K_2(x,t) = \int_{\mathbb{R}} e^{-it\xi^2+ix\xi} \psi\left(\frac{\xi}{4N}\right)\ d\xi. \tag{2.9}$$

To finish the proof, we need the following lemma.

Lemma 2.6. *Let $K_1(x,t)$ and $K_2(x,t)$ be as in (2.8) and (2.9). Then, for $i = 1, 2$*

$$\|K_i\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \lesssim N^{2s_0}. \tag{2.10}$$

We continue the proof of Proposition 2.5. By applying Young's inequality and Lemma 2.6, we obtain

$$\|K_i \star F\|_{L_x^\gamma L_t^\infty} \leq \|K_i\|_{L_x^{\frac{\gamma}{2}} L_t^\infty} \|F\|_{L_x^{\frac{\gamma}{\gamma-1}} L_t^1}$$

as desired. We then finish the proof by the usual density argument. □

Proof of Lemma 2.6. Let $I = [-1,1]$ when $\gamma = 2,3$ and $I = \mathbb{R}$ when $\gamma \geq 4$. We divide $\mathbb{R} \times I$ into three regions

$$\Omega_1 := \{(x,t) \in \mathbb{R} \times I \mid |x| \leq \frac{1}{N}\}$$
$$\Omega_2 := \{(x,t) \in \mathbb{R} \times I \mid |x| \geq 64N|t|\ ,\ |x| > \frac{1}{N}\}$$
$$\Omega_3 := \{(x,t) \in \mathbb{R} \times I \mid |x| < 64N|t|\ ,\ |x| > \frac{1}{N}\},$$

and we will estimate $K_i(x,t)$ in each region. For a fixed $x \in \mathbb{R}$ and $1 \leq i \leq 3$, we define $\Omega_{x,i} := \{t \in I \mid (x,t) \in \Omega_i\}$. We consider the following two cases of values of γ.

Case 1: $\gamma = 2, 3$. Note that in this case we always assume that $N \geq 1$. By a change of variable $\eta = \frac{\xi}{4N}$, we obtain

$$K_1(x,t) = N \int_{\mathbb{R}} \chi_{[-2,2]} e^{-i16tN^2\eta^2 + i4xN\eta} \psi(\eta)\, d\eta$$

A simple estimate on Ω_1 shows that

$$\int_{|x| \leq \frac{1}{N}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \frac{1}{N} \cdot N^{\frac{\gamma}{2}} \Big(\int_{\mathbb{R}} \psi(\eta)\, d\eta \Big)^{\frac{\gamma}{2}} \sim N^{\frac{\gamma-2}{2}} \leq N. \tag{2.11}$$

Next we consider the norm on Ω_2. Note that the integrand in K_1 vanishes if $|\eta| \geq 4$. Factoring out $-i16tN^2\eta^2 + i4xN\eta = -i4xN(\eta - \frac{4tN}{x}\eta^2) := -ixN\phi_1(\eta)$ yields

$$|\phi_1'(\eta)| = |1 - 8\frac{tN}{x}\eta| \geq 1 - 32 \left| \frac{tN}{x} \right| \geq 1 - 32 \cdot \frac{1}{64} = \frac{1}{2},$$

for any $t \in \Omega_{x,2}$. Therefore, ϕ_1 has no critical point in this region. By Lemma 2.2, the integral in K_1 is bounded by $|Nx|^{-k}$ for all $k \geq 0$. In particular, by choosing $k = 2$, we obtain $|K_1(x,t)| \lesssim N(N|x|)^{-2} = N^{-1}|x|^{-2}$. We finish by computing the $L_x^{\frac{\gamma}{2}} L_t^{\infty}$ norm on Ω_2:

$$\int \sup_{t \in \Omega_{x,2}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim N^{(\gamma-1)-\frac{\gamma}{2}} = N^{\frac{\gamma-2}{2}} \leq N. \tag{2.12}$$

Now we consider the norm on Ω_3. Factoring out the exponential term $-i16tN^2\eta^2 + i4xN\eta =$

$-i4tN^2(4\eta^2 - \frac{x\eta}{Nt}) := i4tN^2\phi_2(\eta)$ yields $\phi_2''(\eta) \gtrsim 1$, so we can apply Lemma 2.3 to K_1.

$$\begin{aligned} |K_1(x,t)| &= N \left| \int_{\mathbb{R}} e^{-itN^2\eta^2 + ixN\eta}\psi(\eta)\, d\eta \right| \\ &\lesssim N \cdot \frac{1}{N|t|^{\frac{1}{2}}} \\ &< \frac{64N^{\frac{1}{2}}}{|x|^{\frac{1}{2}}}. \end{aligned} \tag{2.13}$$

Now we compute the $L_x^{\frac{\gamma}{2}}L_t^\infty$ norm of K_1. Observe that the finite time restriction yields $|x| \lesssim N|t| \le 2N$ on Ω_3. Therefore,

$$\int \sup_{t\in\Omega_{x,3}} |K_1(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \int_{|x|<64N|t|} N^{\frac{\gamma}{4}}|x|^{-\frac{\gamma}{4}}\, dx \lesssim N^{\frac{\gamma}{4} - \frac{\gamma-4}{4}} = N. \tag{2.14}$$

Combining (2.11),(2.12) and (2.14), we have that

$$\|K_1\|_{L_x^{\frac{\gamma}{2}}L_t^\infty} \lesssim N^{\frac{2}{\gamma}}.$$

Case 2: $\gamma \ge 5$. Since the estimates in (2.11) and (2.12) do not require any time restriction, we get the same results for K_2.

$$\int_{\Omega_1\cup\Omega_2} |K_2|^{\frac{\gamma}{2}}\, dx \lesssim N^{\frac{\gamma-2}{2}}. \tag{2.15}$$

On Ω_3, we have the same estimate as in (2.13) for K_2. From the fact that $|x| > \frac{1}{N}$ in this region, we have

$$\int \sup_{t\in\Omega_{x,3}} |K_2(x,t)|^{\frac{\gamma}{2}}\, dx \lesssim \int_{|x|>\frac{1}{N}} N^{\frac{\gamma}{4}}|x|^{-\frac{\gamma}{4}}\, dx \lesssim N^{\frac{\gamma}{4} + \frac{\gamma-4}{4}} = N^{\frac{\gamma-2}{2}}. \tag{2.16}$$

Note that we did not use the finite time restriction in this case. Combining (2.15) and (2.16), we have that

$$\|K_2\|_{L_x^{\frac{\gamma}{2}}L_t^\infty} \lesssim N^{\frac{\gamma-2}{\gamma}}.$$

□

To estimate a product of functions as seen in the nonlinearity of DNLS, one usually employs the bilinear estimate which splits the product into estimating individual functions (see [8] where Bourgain proved the estimate in two dimensions).

Theorem 2.7 (Bilinear Strichartz Estimate)**.** *For any $u, v \in L_x^2$, we have*

$$\|P_\lambda(e^{it\Delta}u\overline{e^{it\Delta}v})\|_{L^2_{x,t}} \lesssim \lambda^{-\frac{1}{2}}\|u\|_{L^2}\|v\|_{L^2} \tag{2.17}$$

In addition, if $\hat{u}$ and $\hat{v}$ have disjoint supports and $\alpha = \inf|\mathrm{supp}(\hat{u}) - \mathrm{supp}(\hat{v})| > 0$, then we have

$$\|e^{it\Delta}ue^{it\Delta}v\|_{L^2_{x,t}} \lesssim \alpha^{-\frac{1}{2}}\|u\|_{L^2}\|v\|_{L^2}. \tag{2.18}$$

Proof. We follow the proof in [37, Theorem 2.9]. By duality, this is equivalent to showing that for any $F \in C_c^\infty$,

$$\Big|\int F(\xi-\eta, \xi^2-\eta^2)\psi_{>\lambda}(\xi-\eta)\hat{u}(\xi)\bar{\hat{v}}(\eta)\ d\xi d\eta\Big| \lesssim \lambda^{-\frac{1}{2}}\|F\|_{L^2_{\xi,\tau}}\|\hat{u}\|_{L^2_\xi}\|\hat{v}\|_{L^2_\xi}.$$

For each fixed α and β, let $(\xi_{\alpha\beta}, \eta_{\alpha\beta})$ be a solution to $\alpha = \xi^2 - \eta^2$ and $\beta = \xi - \eta$. We see that the change of variables $(\xi, \eta) \mapsto (\alpha, \beta)$ gives the Jacobian $J = 2(\eta - \xi)$. This together with Cauchy-Schwarz yield

$$\begin{aligned}
&\Big|\int F(\xi-\eta, \xi^2-\eta^2)\psi_{>\lambda}(\xi-\eta)\hat{u}(\xi)\bar{\hat{v}}(\eta)\ d\xi d\eta\Big| \\
&\quad = \Big|\int F(\alpha, \beta)\psi_{>\lambda}(\beta)\hat{u}(\xi_{\alpha\beta})\bar{\hat{v}}(\eta_{\alpha\beta})\frac{1}{J}\ d\alpha d\beta\Big| \\
&\quad \le \|F\|_{L^2_{\xi,\tau}}\Big(\int|\psi_{>\lambda}(\beta)|^2|\hat{u}(\xi_{\alpha\beta})|^2|\hat{v}(\eta_{\alpha\beta})|^2\frac{1}{J^2}\ d\alpha d\beta\Big)^{\frac{1}{2}} \\
&\quad = \|F\|_{L^2_{\xi,\tau}}\left(\int|\psi_{>\lambda}(\xi-\eta)|^2|\hat{u}(\xi)|^2|\hat{v}(\eta)|^2\frac{1}{J}\ d\xi d\eta\right)^{\frac{1}{2}} \\
&\quad \lesssim \lambda^{-\frac{1}{2}}\|F\|_{L^2_{\xi,\tau}}\|\hat{u}\|_{L^2_\xi}\|\hat{v}\|_{L^2_\xi}.
\end{aligned}$$

This concludes the proof of (2.17). The proof for (2.18) is essentially the same, but $\xi - \eta$ is replaced by $\xi + \eta$, $\xi^2 - \eta^2$ is replaced by $\xi^2 + \eta^2$ and there is no $\psi_{>\lambda}$. The conclusion follows from the observation that

$$\frac{1}{|J|} = \frac{1}{2|\eta - \xi|} \gtrsim \frac{1}{\alpha}.$$

□

We will need a variant of this estimate adapted to the X^s space (2.44) for our trilinear estimate (3.3). The details will be explained in the next section.

2.5 The main linear estimate

In this section, we consider a nonlinear Schrödinger equation

$$\begin{aligned} iu_t + \Delta u &= F \\ u(x,0) &= u_0. \end{aligned} \tag{2.19}$$

Let $I = [-1, 1]$ if $d = 3, 4$ and $I = \mathbb{R}$ if $d \geq 5$. A solution $u(x,t) \in \mathbb{R} \times I$ can be represented by the Duhamel formula

$$u(x,t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(s)\, ds. \tag{2.20}$$

In the proof of Theorem 1.1 and Theorem 1.3, the spaces that we use are based on the following norms which take a function u supported at dyadic frequency interval $\sim N$.

$$\begin{aligned} \|u\|_{Y_N} &= \inf\{N^{-\frac{1}{2}}\|u_1\|_{L^1_x L^2_t} + \|u_2\|_{L^1_t L^2_x} \mid u_1 + u_2 = u\} \\ \|u\|_{X_N} &= \|u\|_{L^\infty_t L^2_x} + N^{-s_0}\|u\|_{L^{d-1}_x L^\infty_t} + N^{\frac{1}{2}}\|u\|_{L^\infty_x L^2_t} \\ &\quad + N^{-\frac{1}{2}}\|(i\partial_t + \Delta)u\|_{Y_N}, \end{aligned} \tag{2.21}$$

where $L_t^\infty L_x^2 = L_t^\infty L_x^2(I \times \mathbb{R})$ and $L_x^p L_t^q = L_x^p L_t^q(\mathbb{R} \times I)$. These norms satisfy the following linear estimate, which makes them suitable for the contraction argument.

Theorem 2.8. *Let u be a solution to equation* (2.19)*. Then,*

$$\|P_N u\|_{X_N} \lesssim \|u_0\|_{L_x^2} + \|P_N F\|_{Y_N}. \tag{2.22}$$

This immediately follows from the Duhamel formula and the following three propositions.

Proposition 2.9. *For any $u_0 \in L_x^2(\mathbb{R})$, we have*

$$\|e^{it\Delta} P_N u_0\|_{X_N} \lesssim \|u_0\|_{L_x^2}. \tag{2.23}$$

Proof. This follows from the Strichartz estimate (2.3), the smoothing estimate (2.4) and (2.5a) if $d = 3, 4$ or (2.5b) if $d \geq 5$. □

Proposition 2.10. *For any function $F(x,t)$ such that $P_N F \in L_x^1 L_t^2$, we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} \lesssim \|P_N F\|_{Y_N}. \tag{2.24}$$

Proof. It follows from Minkowski inequality and (2.23) that

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} &\leq \int_{\mathbb{R}} \|e^{i(t-s)\Delta} P_N F(s)\|_{X_N}\, ds \\
&\lesssim \int_{\mathbb{R}} \|P_N F(s)\|_{L_x^2}\, ds \\
&= \|P_N F\|_{L_t^1 L_x^2}.
\end{aligned}$$

Therefore, it suffices to prove that

$$\left\| \int_0^t e^{i(t-s)\Delta} P_N F(s)\, ds \right\|_{X_N} \lesssim N^{-\frac{1}{2}} \|P_N F\|_{L_x^1 L_t^2}. \tag{2.25}$$

Let K_0 be the fundamental solution of Schrödinger equation i.e.

$$K_0(x,t) = \mathcal{F}^{-1}(e^{-it\xi^2}) = \frac{1}{\sqrt{4\pi it}} e^{ix^2/4t}.$$

Thus,

$$\begin{aligned}\int_0^t e^{i(t-s)\Delta} P_N F(x,s)\ ds &= \int_0^t \int_{\mathbb{R}} P_N \left[K_0(x-y,t-s)F(y,s)\right]\ dyds \\ &= \int_{\mathbb{R}} \int_0^t P_N \left[K_0(x-y,t-s)F(y,s)\right]\ dsdy \\ &:= \int_{\mathbb{R}} w_y\ dy,\end{aligned} \tag{2.26}$$

In order to proceed, we will make use of the following lemma.

Lemma 2.11. *For any $N \in 2^{\mathbb{Z}}$, the function w_y defined in* (2.26) *satisfies the following estimate:*

$$\|w_y\|_{X_N} \lesssim N^{-\frac{1}{2}} \|F(y)\|_{L_t^2}. \tag{2.27}$$

Continuing the proof of Proposition 2.10, we see that the estimate (2.25) follows immediately from (2.27). □

Proof of Lemma 2.11. By translation invariance, it suffices to assume that $y = 0$. Denote $F_0(t) := F(0,t)$. To proceed, we use the following decomposition which was first introduced in [3] to deal with Scrödinger maps.

$$w_0(x,t) = -e^{it\Delta}\mathcal{L}v_0(x) - (P_{<N/2^{50}} 1_{x>0}) e^{it\Delta} v_0(x) + h(x,t), \tag{2.28}$$

where $\mathcal{L} : L_x^2(\mathbb{R}) \to L_x^2(\mathbb{R})$ is an operator and

$$\|\mathcal{L}v_0\|_{L_x^2} + \|v_0\|_{L_x^2} + N^{-1}(\|\Delta h\|_{L_{x,t}^2} + \|h_t\|_{L_{x,t}^2}) \lesssim N^{-\frac{1}{2}} \|F_0\|_{L_t^2}. \tag{2.29}$$

To prove the claim, first we rewrite the definition of w_0 as

$$\begin{aligned} w_0(x,t) &= \int_{\mathbb{R}} \chi_{[0,\infty)}(t-s) P_N[K_0(x,t-s)]F_0(s)\ ds \\ &\quad - e^{it\Delta}\int_{-\infty}^{0} P_N[K_0(x,-s)]F_0(s)\ ds \\ &= (\chi_{[0,\infty)}P_N K_0) *_t F_0 - e^{it\Delta}\int_{-\infty}^{0} P_N K_0(x,-s)F_0(s)\ ds, \end{aligned} \tag{2.30}$$

where $*_t$ is the time convolution. The space-time Fourier transform of the first term is equal to

$$\frac{\psi_N(\xi)}{-\tau-\xi^2-i0}\widehat{F}_0(\tau), \tag{2.31}$$

where $\widehat{F_0}$ is the time Fourier transform of F_0. We define

$$\hat{v}_0(\xi) := \psi_N(\xi)\widehat{F}_0(-\xi^2). \tag{2.32}$$

We see that v_0 is supported at frequency $\sim N$. By changing variables we obtain the following estimate,

$$\|v_0\|_{L^2_x} \lesssim N^{-\frac{1}{2}}\|F_0\|_{L^2_t}. \tag{2.33}$$

We apply the spatial Fourier transform to the second term

$$\begin{aligned} \int_{-\infty}^{0} \widehat{P_N K}_0(x,-s)F_0(s)\ ds &= \psi_N(\xi)\int_{-\infty}^{0} e^{is\xi^2}F_0(s)\ ds \\ &= \psi_N(\xi)\mathcal{F}_t(\chi_{(0,\infty]}F_0)(-\xi^2) \\ &:= \widehat{\mathcal{L}v_0}(\xi). \end{aligned} \tag{2.34}$$

We see that $\mathcal{L}v_0$ is supported at frequency $\sim N$. It follows from a change of variables that

$$\|\mathcal{L}v_0\|_{L^2_x} \lesssim N^{-\frac{1}{2}}\|F_0\|_{L^2_t}.$$

Applying the Fourier transform to $e^{it\Delta}v_0$,

$$\mathcal{F}(e^{it\Delta}v_0) = \psi_N(\xi)\widehat{F}_0(-\xi^2)\mathcal{F}_t(e^{-it\xi^2}) = \psi_N(\xi)\widehat{F}_0(-\xi^2)\delta_{\tau+\xi^2}.$$

Assume that $\xi > 0$ and consider the distribution $\delta_{\tau+\xi^2}$. For any $\phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$, by a change of variables

$$\int_0^\infty \phi(\xi,-\xi^2)\ d\xi = \int_{-\infty}^0 \frac{1}{2\sqrt{-\tau}}\phi(\sqrt{-\tau},\tau)\ d\tau.$$

Thus, $1_{\xi>0}\delta_{\tau+\xi^2} = 1_{\tau<0}\frac{1}{2\sqrt{-\tau}}\delta_{\xi-\sqrt{-\tau}}$. Therefore, the following computation holds.

$$\begin{aligned}
&\mathcal{F}\{(P_{<N/2^{50}}1_{x>0})e^{it\Delta}v_0\}(\xi,\tau)\\
&\qquad = (\psi_N(\xi)\widehat{F}_0(-\xi^2)\delta_{\tau+\xi^2}) * \frac{\psi_{<N/2^{50}}(\xi)}{\xi+i0}\\
&\qquad = \left(\frac{\psi_N(\xi)}{2\sqrt{-\tau}}\widehat{F}_0(-\xi^2)\delta_{\xi-\sqrt{-\tau}}\right) * \frac{\psi_{<N/2^{50}}(\xi)}{\xi+i0}\\
&\qquad = \frac{\psi_N(\sqrt{-\tau})\widehat{F}_0(\tau)}{2\sqrt{-\tau}}\frac{\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})}{\xi-\sqrt{-\tau}+i0}\\
&\qquad = \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})\widehat{F}_0(\tau)\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}}\frac{1}{\xi^2+\tau+i0}.
\end{aligned}$$

With this and (2.31), the space-time Fourier transform of the remainder term is given by

$$\begin{aligned}
\hat{h}(\xi,\tau) &= \left(\psi_N(\xi) - \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}}\right)\frac{\widehat{F}_0(\tau)}{-\xi^2-\tau-i0}\\
&:= A(\xi,\tau)\widehat{F}_0(\tau).
\end{aligned} \tag{2.35}$$

The term in the bracket is bounded, supported in $\{0<\xi\sim N\}$ and vanishes when $\xi=\sqrt{-\tau}$, canceling out the singularity. Since the same result holds for $\xi<0$, this implies that

$$\|\Delta h\|_{L^2_{x,t}} + \|\partial_t h\|_{L^2_{x,t}} \sim \|(\xi^2+|\tau|)\hat{h}\|_{L^2_{\xi,\tau}} \lesssim N^{\frac{1}{2}}\|\widehat{F}_0(\tau)\|_{L^2_\tau}. \tag{2.36}$$

The estimate (2.29) then follows from (2.33) and (2.36).

Remark: It is important to note that v_0, Lv_0 and h are supported at frequency $\sim N$, since we will need this fact in any proof that employ the decomposition (2.28).

We are now ready to prove (2.27). By Bernstein's inequality and direct L^2 integration on $A(\xi,\tau)$,

$$
\begin{aligned}
\|h\|_{L_x^{d-1}L_t^\infty} \le \|\mathcal{F}_t h\|_{L_x^{d-1}L_\tau^1} &\lesssim \|\mathcal{F}_t h\|_{L_\tau^1 L_x^{d-1}} \\
&\lesssim N^{\frac{d-3}{2(d-1)}} \|\mathcal{F}_t h\|_{L_\tau^1 L_x^2} \\
&= N^{\frac{d-3}{2(d-1)}} \|\hat{h}\|_{L_\tau^1 L_\xi^2} \\
&\le N^{\frac{d-3}{2(d-1)}} \|A(\xi,\tau)\|_{L^2_{\tau,\xi}} \|\widehat{F}_0(\tau)\|_{L^2_\tau},
\end{aligned}
$$

where $A(\xi,\tau)$ is defined as in (2.35) when $\xi > 0$,. We split the integral in $\|A(\xi,\tau)\|^2_{L^2_{\tau,\xi}}$ as

$$
\begin{aligned}
\|A(\xi,\tau)\|^2_{L^2_{\tau,\xi}} &= \int_{|\xi-\sqrt{-\tau}|<\frac{N}{2^{100}}} |A(\xi,\tau)|^2 \, d\xi d\tau \\
&\quad + \int_{|\xi-\sqrt{-\tau}|\ge\frac{N}{2^{100}}} |A(\xi,\tau)|^2 \, d\xi d\tau \\
&:= A_1 + A_2.
\end{aligned}
$$

Note that $\psi_N(\xi) = \psi_N(\sqrt{-\tau}) + (\xi - \sqrt{-\tau})O(\frac{1}{N})$ as $\xi \to \sqrt{-\tau}$. If $|\xi - \sqrt{-\tau}| < \frac{N}{2^{100}}$, then $\psi_{<N/2^{50}}(\xi - \sqrt{-\tau}) = 1$ and it follows that

$$
\begin{aligned}
&\psi_N(\xi) - \psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi - \sqrt{-\tau})\frac{\xi+\sqrt{-\tau}}{2\sqrt{-\tau}} \\
&\qquad = \frac{\psi_N(\sqrt{-\tau})(\sqrt{-\tau}-\xi)}{2\sqrt{-\tau}} + (\xi-\sqrt{-\tau})O\Big(\frac{1}{N}\Big).
\end{aligned}
$$

Since $A(\xi,\tau)$ is supported in the region $\xi \sim N$, we have that

$$
A_1 \lesssim \int_{\tau\sim -N^2}\int_{\xi\sim N} \frac{1}{-2\tau(\xi+\sqrt{-\tau})^2} + \frac{1}{N^2(\xi+\sqrt{-\tau})^2} \, d\xi d\tau \lesssim \frac{1}{N}.
$$

On the other hand, under the assumptions that, $\xi \sim N$ and $|\xi - \sqrt{-\tau}| \ge \frac{N}{2^{100}}$, we have $|\xi^2 + \tau| = |(\xi + \sqrt{-\tau})(\xi - \sqrt{-\tau})| \gtrsim \frac{N^2}{2^{100}}$. Thus, by a change of variables $(\xi,\tau) \mapsto (\xi,\eta)$

where $\eta := \tau + \xi^2$, we have

$$\begin{aligned} A_2 &\leq \int_{-\infty}^{0} \int_{|\xi-\sqrt{-\tau}|\geq \frac{N}{2^{100}}} \frac{\psi_N(\xi)}{(\xi^2+\tau)^2} + \frac{\psi_N(\sqrt{-\tau})\psi_{<N/2^{50}}(\xi-\sqrt{-\tau})}{-4\tau(\xi+\sqrt{-\tau})^2} \, d\xi d\tau \\ &\lesssim \int_{\xi\sim N} \int_{|\eta|\gtrsim \frac{N^2}{2^{100}}} \frac{1}{\eta^2} \, d\eta d\xi + \int_{\tau\sim -N^2} \int_{\xi\sim N} \frac{1}{-4\tau(\xi+\sqrt{-\tau})^2} \, d\xi d\tau \\ &\lesssim \int_{\xi\sim N} \frac{1}{N^2} \, d\xi + \frac{1}{N} \\ &\lesssim \frac{1}{N}, \end{aligned}$$

and a similar result holds when $\xi < 0$. From this, we can conclude that

$$\|h\|_{L_x^{d-1}L_t^\infty} \lesssim N^{\frac{d-3}{2(d-1)}} \|A(\xi,\tau)\|_{L^2_{\tau,\xi}} \|\widehat{F}_0(\tau)\|_{L^2_\tau} \lesssim N^{\frac{d-3}{2(d-1)}-\frac{1}{2}} \|F(0)\|_{L^2_t}. \tag{2.37}$$

Similarly, we have the following,

$$\|h\|_{L^\infty_{x,t}} \lesssim \|F(0)\|_{L^2_t}. \tag{2.38}$$

In particular, for $d = 3$ and $N \geq 1$, we have that

$$N^{-\frac{1}{2}} \|h\|_{L^2_x L^\infty_t} \leq \|h\|_{L^2_x L^\infty_t} \lesssim N^{-\frac{1}{2}} \|F(0)\|_{L^2_t}. \tag{2.39}$$

Similarly, by Sobolev's embedding,

$$N^{\frac{1}{2}} \|h\|_{L^\infty_x L^2_t} \lesssim N^{\frac{1}{2}} \|h\|_{L^2_t L^\infty_x} \lesssim N^{-1} \|\Delta h\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|F(0)\|_{L^2_t}. \tag{2.40}$$

where we used (2.29) in the last step. Lastly, it follows from (2.37) that

$$\|h\|_{L^\infty_t L^2_x} \leq \|h\|_{L^2_x L^\infty_t} \lesssim N^{-\frac{1}{2}} \|F(0)\|_{L^2_t}. \tag{2.41}$$

Putting together (2.37), (2.40) and (2.41), we are done with estimating h. Similar estimate for the term $1_{x>0} e^{it\Delta} v_0$ follows easily from Strichartz-type estimates (2.3), (2.4) and (2.5).

□

In the proof of Theorem 1.1 in the next section, we will incorporate the low frequency projection $P_{\leq 1}u$ into the spaces X^s and Y^s, which are restricted to the time interval $T = [-1,1]$, in order to obtain the local well-posedness. Therefore, we need an estimate analogous to (2.22) for functions supported at low frequencies, which can be obtained from the two following propositions:

Proposition 2.12. *Let $T = [-1,1]$. For any function $u_0 \in L^2(\mathbb{R})$, we have*

$$\|P_{\leq 1}e^{it\Delta}u_0\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}u_0\|_{L^2_x}. \tag{2.42}$$

Proof. In view of Strichartz's estimate (2.3) with $p = 2$ and $q = \infty$ and (2.5a), it suffices to prove that

$$\|P_{\leq 1}e^{it\Delta}u_0\|_{L^\infty_x L^2_t(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}u_0\|_{L^2_x}.$$

Using the fact that $\widehat{P_{\leq 1}u}_0(\xi,t)$ is compactly supported ξ variable and Plancherel theorem, we have

$$\begin{aligned}\|P_{\leq 1}e^{it\Delta}u_0\|_{L^\infty_x L^2_t(\mathbb{R}\times T)} &\leq \|P_{\leq 1}e^{it\Delta}u_0\|_{L^2_t L^\infty_x(T\times\mathbb{R})} \leq \|\psi(\xi)\hat{u}_0\|_{L^2_t L^1_\xi(T\times\mathbb{R})}\\ &\leq \|\psi(\xi)\hat{u}_0\|_{L^\infty_t L^2_\xi(T\times\mathbb{R})} \qquad = \|P_{\leq 1}u_0\|_{L^2_x}.\end{aligned}$$

□

Proposition 2.13. *Let $T = [-1,1]$. For any function $F(x,t)$ such that $P_{\leq 1}F \in Y_1$, we have*

$$\left\|\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(x,s)\ ds\right\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{Y_1(\mathbb{R}\times T)}. \tag{2.43}$$

Proof. As in the proof of Proposition 2.10, it follows from Minkowski inequality that

$$\left\|\int_0^t e^{i(t-s)\Delta}P_{\leq 1}F(s)\ ds\right\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{L^1_t L^2_x(T\times\mathbb{R})}.$$

Thus, it suffices to prove that

$$\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(s) \; ds \right\|_{X_1(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.$$

Note that for $t \in [0,1]$, we can rewrite

$$\begin{aligned} \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s) \; ds &= \int_{\mathbb{R}} \chi_{[0,1)}(t-s)\chi_{[0,1)}(s) e^{i(t-s)\Delta} P_{\leq 1} F(x,s) \; ds \\ &:= K(x,t) \star \chi_{[0,1)}(t) P_{\leq 1} F(x,t) \end{aligned}$$

where $\star$ is the space-time convolution and

$$K(x,t) = \int_{\mathbb{R}} e^{-it\xi^2 + ix\xi} \chi_{[0,1)}(t) \psi\left(\frac{\xi}{N}\right) \; d\xi,$$

which obeys the estimate (2.10) with $N = 1$. Hence, by Young's inequality

$$\left\| \chi_{[0,1]}(t) \left[K(x,t) \star \chi_{[0,1)}(t) P_{\leq 1} F(x,t) \right] \right\|_{L_x^2 L_t^\infty} \lesssim \|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_x^2 L_t^1}.$$

We use the finite time restriction and apply Bernstein's and Minkowski's inequality.

$$\begin{aligned} \|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_x^2 L_t^1} &\lesssim \|\chi_{[0,1]}(t) P_{\leq 1} F\|_{L_{x,t}^2} \\ &\lesssim \|\chi_{[-1,1]}(t) P_{\leq 1} F\|_{L_t^2 L_x^1} \\ &\leq \|\chi_{[-1,1]}(t) P_{\leq 1} F\|_{L_x^1 L_t^2}. \end{aligned}$$

Since similar proof applies for the time interval $[-1,0]$, we obtain

$$\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s) \; ds \right\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \lesssim \|P_{\leq 1}F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.$$

This estimate has the following two consequences. First, from Minkowski's inequality, we have

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s) \; ds \right\|_{L_t^\infty L_x^2(T\times\mathbb{R})} &\leq \left\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s) \; ds \right\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \\ &\lesssim \|P_{\leq 1}F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}. \end{aligned}$$

Secondly, it follows from Minkowski's inequality, Bernstein's inequality and the finite time restriction that

$$\begin{aligned}\Big\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s)\; ds \Big\|_{L_x^\infty L_t^2(\mathbb{R}\times T)} &\leq \Big\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s)\; ds \Big\|_{L_t^2 L_x^\infty(T\times\mathbb{R})} \\ &\lesssim \Big\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s)\; ds \Big\|_{L_{x,t}^2(\mathbb{R}\times T)} \\ &\lesssim \Big\| \int_0^t e^{i(t-s)\Delta} P_{\leq 1} F(x,s)\; ds \Big\|_{L_x^2 L_t^\infty(\mathbb{R}\times T)} \\ &\lesssim \|P_{\leq 1} F\|_{L_x^1 L_t^2(\mathbb{R}\times T)}.\end{aligned}$$

This concludes the proof of (2.43). □

The essential part of the contraction argument is a multilinear estimate: an estimate of the form $\|\partial_x u_1 \prod_{i=2}^d u_i\|_{Y^s} \lesssim \prod_{i=1}^d \|u_i\|_{X^s}$. One of the main tools that we will use to prove this is the following Bilinear Strichartz estimate for the X^s space.

Theorem 2.14. *Let $N \gg M$ and suppose that u and v are supported at frequency N and M, respectively. Then, we have*

$$\|uv\|_{L_{x,t}^2} \lesssim N^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M}. \tag{2.44}$$

Proof. Let $F_1(x,t) = (i\partial_t + \Delta)u(x,t)$ and $F_2(x,t) = (i\partial_t + \Delta)v(x,t)$. We will prove that

$$\|uv\|_{L_{x,t}^2} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L_x^2} + \|F_1\|_{L_t^1 L_x^2}\right)\left(\|v(0)\|_{L_x^2} + \|F_2\|_{L_t^1 L_x^2}\right) \tag{2.45}$$

$$\|uv\|_{L_{x,t}^2} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L_x^2} + N^{-\frac{1}{2}}\|F_1\|_{L_x^1 L_t^2}\right)\left(\|v(0)\|_{L_x^2} + M^{-\frac{1}{2}}\|F_2\|_{L_x^1 L_t^2}\right) \tag{2.46}$$

$$\|uv\|_{L_{x,t}^2} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L_x^2} + N^{-\frac{1}{2}}\|F_1\|_{L_x^1 L_t^2}\right)\left(\|v(0)\|_{L_x^2} + \|F_2\|_{L_t^1 L_x^2}\right). \tag{2.47}$$

$$\|uv\|_{L_{x,t}^2} \lesssim N^{-\frac{1}{2}} \left(\|u(0)\|_{L_x^2} + \|F_1\|_{L_t^1 L_x^2}\right)\left(\|v(0)\|_{L_x^2} + M^{-\frac{1}{2}}\|F_2\|_{L_x^1 L_t^2}\right). \tag{2.48}$$

To achieve (2.45), we consider the expansion of $u\bar{v}$ after using the Duhamel formula on u and v.

$$u(x,t) = e^{it\Delta}u(0) - i\int_0^t e^{i(t-s)\Delta}F_1(s)\ ds$$
$$v(x,t) = e^{it\Delta}v(0) - i\int_0^t e^{i(t-s)\Delta}F_2(s)\ ds,$$

It follows from the bilinear estimate for free solutions (2.18) that

$$\|e^{it\Delta}u(0)e^{it\Delta}v(0)\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}}\|u(0)\|_{L^2_x}\|v(0)\|_{L^2_x}$$

By the Minkowski inequality, we have that

$$\begin{aligned}\|\int_0^t e^{i(t-s)\Delta}F_1(s)e^{it\Delta}v(0)\ ds\|_{L^2_{x,t}} &\lesssim N^{-\frac{1}{2}}\int_{\mathbb{R}}\|F_1(s)\|_{L^2_x}\|v(0)\|_{L^2_x}\ ds\\ &= N^{-\frac{1}{2}}\|F_1\|_{L^1_tL^2_x}\|v(0)\|_{L^2_x}.\end{aligned}$$

Similarly,

$$\|\int_0^t e^{it\Delta}u(0)e^{i(t-s)\Delta}F_2(s)\ ds\|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}}\|u(0)\|_{L^2_x}\|F_2\|_{L^1_tL^2_x}.$$

With the same proof, we can estimate the last term in the product.

$$\begin{aligned}&\|\int_0^t\int_0^t e^{i(t-s)\Delta}F_1(s)e^{i(t-s)\Delta}F_2(\tilde{s})\ dsd\tilde{s}\|_{L^2_{x,t}}\\ &\lesssim N^{-\frac{1}{2}}\|F_1\|_{L^1_tL^2_x}\|F_2\|_{L^1_tL^2_x},\end{aligned}$$

and (2.45) follows.

To prove (2.46), we recall (2.28) which allows us to decompose u and v as follows

$$u(x,t) = e^{it\Delta}u(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}u_y + (P_{N/2^{50}}1_{x>0})e^{it\Delta}u_y - h_{1,y}(x,t)\ dy \tag{2.49}$$
$$v(x,t) = e^{it\Delta}v(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}v_{y'} + (P_{M/2^{50}}1_{x>0})e^{it\Delta}v_{y'} - h_{2,y'}(x,t)\ dy', \tag{2.50}$$

where $\mathcal{L} : L_x^2 \to L_x^2$ is a bounded operator and $u_y, \mathcal{L}u_y$ and $h_{1,y}$ are defined similarly to (2.35), (2.34) and (2.35), respectively. From the remark following (2.36), we see that these functions are supported at frequency $\sim N$. Similar $v_{y'}, \mathcal{L}v_{y'}, h_{2,y'}$ Moreover, we have

$$\|\mathcal{L}u_y\|_{L_x^2} + \|u_y\|_{L_x^2} + \frac{1}{N}(\|\Delta h_y\|_{L_{x,t}^2} + \|\partial_t h_y\|_{L_{x,t}^2}) \lesssim \frac{1}{N^{\frac{1}{2}}}\|F_1(y,t)\|_{L_t^2}. \tag{2.51}$$

Similar conclusions hold for $v_{y'}, \mathcal{L}v_{y'}$ and $h_{2,y'}$ at frequency $\sim M$ with corresponding nonlinearity $F_2(y',t)$. Consider each term in the product uv. Let $\psi_{N/2^{50}}$ be the function defined by $P_{N/2^{50}}f := \psi_{N/2^{50}} * f$. Observe that for any $G \in L^2$, we have that

$$\begin{aligned}
&\|(P_{N/2^{50}}1_{x>0})e^{it\Delta}u_y G(x)\|_{L_{x,t}^2} \\
&\qquad = \|(\psi_{N/2^{50}} * 1_{x>0})e^{it\Delta}u_y G(x)\|_{L_{x,t}^2} \\
&\qquad \le \int \|1_{x-z>0}e^{it\Delta}u_y(x)G(x)\|_{L_{x,t}^2}|\psi_{N/2^{50}}(z)|\, dz \\
&\qquad \le \int \|e^{it\Delta}u_y(x)G(x)\|_{L_{x,t}^2}|\psi_{N/2^{50}}(z)|\, dz \\
&\qquad \lesssim \|e^{it\Delta}u_y G\|_{L_{x,t}^2}.
\end{aligned} \tag{2.52}$$

With this, we can take care of all the terms involving $P_{N/2^{50}}1_{x>0}$ in the expansion of uv. For any $A, B \in L^2$, we have

$$\|(P_{N/2^{50}}1_{x>0})e^{it\Delta}u_y e^{it\Delta}B\|_{L_{x,t}^2} \lesssim \|e^{it\Delta}u_y e^{it\Delta}B\|_{L_{x,t}^2}$$
$$\|(P_{N/2^{50}}1_{x>0})e^{it\Delta}u_y h_{2,y'}\|_{L_{x,t}^2} \lesssim \|e^{it\Delta}u_y h_{2,y'}\|_{L_{x,t}^2}.$$

Similarly,

$$\|e^{it\Delta}A(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\|_{L_{x,t}^2} \lesssim \|e^{it\Delta}Ae^{it\Delta}v_{y'}\|_{L_{x,t}^2}$$
$$\|h_{1,y}(P_{N/2^{50}}1_{x>0})e^{it\Delta}v_{y'}\|_{L_{x,t}^2} \lesssim \|h_{1,y}e^{it\Delta}v_{y'}\|_{L_{x,t}^2},$$

and lastly,

$$
\begin{aligned}
&\left\| \left[(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} u_y\right] \left[(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} v_{y'}\right] \right\|_{L^2_{x,t}} \\
\lesssim &\left\| e^{it\Delta} u_y \left[(P_{N/2^{50}} 1_{x>0}) e^{it\Delta} v_{y'}\right] \right\|_{L^2_{x,t}} \\
\lesssim &\| e^{it\Delta} u_y e^{it\Delta} v_{y'} \|_{L^2_{x,t}}.
\end{aligned}
$$

Therefore, we only have to worry about the terms of the forms $e^{it\Delta} A e^{it\Delta} B$, $e^{it\Delta} A h_{2,y'}$, $h_{1,y} e^{it\Delta} B$ and $h_{1,y} h_{2,y'}$. Note that any choice of A that is not $u(0)$ is an integral with respect to y. The same holds for B. By the bilinear Strichartz estimate (2.18), one obtains

$$
\| e^{it\Delta} A e^{it\Delta} B \|_{L^2_{x,t}} \lesssim N^{-\frac{1}{2}} \|A\|_{L^2_x} \|B\|_{L^2_x}. \tag{2.53}
$$

We get the desired bound by observing that either we have $\|A\|_{L^2_x} = \|u(0)\|_{L^2_x}$ or $\|A\|_{L^2_x} \lesssim \int_{\mathbb{R}} \|u_y\|_{L^2_x} \, dy \lesssim N^{-\frac{1}{2}} \|F_1\|_{L^1_x L^2_t}$ from (2.51). It remains to estimate the terms that involve $h_{1,y}$ and $h_{2,y}$. By Hölder and Bernstein inequalities, (2.4) and (2.39), We have that

$$
\begin{aligned}
\| e^{it\Delta} A h_{2,y'} \|_{L^2_{x,t}} &\lesssim \| e^{it\Delta} A \|_{L^\infty_x L^2_t} \| h_{2,y'} \|_{L^2_x L^\infty_t} \\
&\lesssim N^{-\frac{1}{2}} M^{-\frac{1}{2}} \|A\|_{L^2_x} \|F_2(y')\|_{L^2_t}.
\end{aligned} \tag{2.54}
$$

By taking $\int_{\mathbb{R}} \cdot \, dy'$ when $A = u(0)$ and $\int_{\mathbb{R}} \int_{\mathbb{R}} \cdot \, dy dy'$ when $A = \mathcal{L} u_y$ or $A = u_y$ on both sides of the inequality and applying (2.29), we get the desired bound. On the other hand, we get the estimate for $\|h_{1,y} e^{it\Delta} B\|_{L^2_{x,t}}$ by observing that from (2.29), we have $\|\Delta h_{1,y}\|_{L^2_{x,t}} \lesssim N^{-\frac{3}{2}} \|F_1\|_{L^2_t}$. Hence,

$$
\begin{aligned}
\| h_{1,y} e^{it\Delta} B \|_{L^2_{x,t}} &\lesssim \| h_{1,y} \|_{L^2_{x,t}} \| e^{it\Delta} B \|_{L^\infty_{x,t}} \\
&\lesssim N^{-\frac{3}{2}} M^{\frac{1}{2}} \|F_1\|_{L^2_x} \|B\|_{L^\infty_t L^2_x} \\
&\leq N^{-\frac{1}{2}} M^{-\frac{1}{2}} \|F_1\|_{L^2_x} \|B\|_{L^\infty_t L^2_x}.
\end{aligned} \tag{2.55}
$$

Lastly, we use (2.39) and (2.40) to estimate the remaining term

$$\begin{aligned} \|h_{1,y}h_{2,y'}\|_{L^2_{x,t}} &\leq \|h_{1,y}\|_{L^\infty_x L^2_t}\|h_{2,y'}\|_{L^2_x L^\infty_t} \\ &\lesssim N^{-1}M^{-\frac{1}{2}}\|F_1(y)\|_{L^2_t}\|F_2(y')\|_{L^2_t}. \end{aligned} \tag{2.56}$$

Taking $\int_{\mathbb{R}}\int_{\mathbb{R}} \cdot \, dydy'$, we obtain (2.46). We are now left to proving (2.47) and (2.48). The proof is a mix of the ideas we used to prove (2.45) and (2.46). For (2.47), we write u using the decomposition (2.49) and v using the Duhamel formula. On the product expansion of $\|uv\|_{L^2_{x,t}}$, we apply the triangle inequality and Minkowski inequality. We then apply the bilinear estimate (2.18) to any term of the form $\|e^{it\Delta}Ae^{it\Delta}B\|_{L^2_{x,t}}$ to get the desired bound. This leaves us with the terms of the form $\|e^{it\Delta}Ah_{2,y'}\|_{L^2_{x,t}}$, on which we can apply (2.54). In the same manner, we can prove (2.48) using the Duhamel formula for u and the decomposition (2.50) for v. We finish the proof by observing that the terms of the form $\|h_{1,y}e^{it\Delta}B\|_{L^2_{x,t}}$ can be bounded using (2.55). □

2.6 $X^{s,b}$ space

This new class of function spaces was introduced by Bourgain ([7]) under an observation that if the we take the space-time Fourier transform of a linear dispersive equation, for example,

$$\mathcal{F}(i\partial_t u + \Delta u)(\xi, \tau) = 0$$

$$(\tau + \xi^2)\mathcal{F}u(\xi, \tau) = 0,$$

then the support of $\mathcal{F}u(\xi,\tau)$ lives in the parabola $\{\tau+\xi^2=0\}$. Since, for a short time, solutions to the Cauchy problem for a nonlinear Schrödinger equation behaves like the linear solution, this suggest that the space-time Fourier transform of these solutions will be supported in a small oneighborhood of this parabola. This observation gives rise to the following norm:

$$\|u\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b\mathcal{F}u(\xi,\tau)\|_{L^2_{\xi,\tau}},$$

which measure the regularity of u by the $\langle\xi\rangle^s$ factor and the "closeness" to the linear solution by the $\langle\tau+\xi^2\rangle^b$ factor. The space that we will be using is a modification of this norm using the Littlewood-Paley projections. For each $N\in 2^{\mathbb{Z}}$, let A_N be a set defined by

$$A_M := \{(\xi,\tau) : M \le |\tau+\xi^2| \le 2M\}. \tag{2.57}$$

Recall that $\tilde{u}(\xi,\tau)$ is the space-time Fourier transform of $u(x,t)$. The $\dot{X}^{0,b,q}$ space is the closure of the test functions under the following norm:

$$\|u\|_{\dot{X}^{0,b,q}} := \Big(\sum_{M\in 2^{\mathbb{Z}}}(N^b\|\tilde{u}\|_{L^2_{\xi,\tau}(A_M)})^q\Big)^{\frac{1}{q}}.$$

Previously, the nonlinear space $\dot{Y}^s$ is based on the space Z_N defined by the following norm on each frequency N.

$$\|u\|_{Z_N} := N^{-\frac{1}{2}}\|u\|_{L^1_xL^2_t}.$$

We modify this by adding the $\dot{X}^{0,-\frac{1}{2},1}$ space.

$$Y_N := Z_N + \dot{X}^{0,-\frac{1}{2},1}.$$

The solution space is defined by

$$\|u\|_{X_N} = \|u(0)\|_{L^2_x} + \|(i\partial_t+\Delta)u\|_{Y_N}$$

$$\|u\|_{\dot{X}^s} = \Big(\sum_{N \in 2^{\mathbb{Z}}} N^{2s} \|P_N u\|_{X_N}^2 \Big)^{\frac{1}{2}}$$
$$\|u\|_{X^s} = \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s}, \tag{2.58}$$

and the nonlinear space is defined by

$$\|u\|_{\dot{Y}^s} = \Big(\sum_{N \in 2^{\mathbb{Z}}} N^{2s} \|P_N u\|_{Y_N}^2 \Big)^{\frac{1}{2}}$$
$$\|u\|_{Y^s} = \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}. \tag{2.59}$$

The following proposition shows that any estimates of free solutions that we proved in Chapter 2 can be extended to functions in X_N using the Schrödinger equation version of Lemma 4.1 from Tao ([49]).

Proposition 2.15 ([49]). *Let S be any space-time Banach space that satisfies the following inequality,*

$$\|g(t)F(x,t)\|_S \ \leq \|g\|_{L_t^\infty} \|F(x,t)\|_S, \tag{2.60}$$

for any $F \in S$ and $g \in L_t^\infty(\mathbb{R})$. Let $T : L^2(\mathbb{R}) \times \ldots \times L^2(\mathbb{R}) \to S$ be a spatial multilinear operator satisfying

$$\|T(e^{it\Delta}u_{1,0}, \ldots, e^{it\Delta}u_{k,0})\|_S \ \lesssim \prod_{i=1}^{k} \|u_{i,0}\|_{L_x^2}$$

for any $u_{1,0}, \ldots, u_{k,0} \in L_x^2(\mathbb{R})$. Then the following estimate

$$\|T(u_1, \ldots, u_k)\|_S \ \lesssim \prod_{i=1}^{k} (\|u_i(0)\|_{L_x^2} + \|(i\partial_t + \Delta)u_i\|_{\dot{X}^{0,-\frac{1}{2},1}}) \tag{2.61}$$

holds true for any $u_1, \ldots, u_k \in \dot{X}^{0,-\frac{1}{2},1}$ provided that u_i is supported at frequency $\sim N_i$ for $1 \leq i \leq k$.

With this proposition, we can obtain several Strichartz-type estimates for X_N that will be useful later on.

Corollary 2.16. *For any $u \in X_N$, we have the following estimates:*

$$\|u\|_{L_t^\infty L_x^2 \cap L_{t,x}^6} \lesssim \|u\|_{X_N} \tag{2.62}$$

$$\|u\|_{L_x^\infty L_t^2} \lesssim N^{-\frac{1}{2}} \|u\|_{X_N} \tag{2.63}$$

$$\|u\|_{L_x^4 L_t^\infty} \lesssim N^{\frac{1}{4}} \|u\|_{X_N} \tag{2.64}$$

Proof. We apply Proposition 2.15 to linear estimates (2.3), (2.4) and (2.5), and bilinear estimates (2.17) and (2.18). □

We also have the bilinear estimate adapted to the space X_N.

Proposition 2.17. *Let N, M and λ be dyadic numbers such that $M \le N$ and $\lambda \lesssim N$. For any functions u and v supported at frequency $\sim N$ and $\sim M$, respectively, we have*

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M} \tag{2.65}$$

In addition, if $\hat{u}$ and $\hat{v}$ have disjoint supports and $\alpha = \inf|\operatorname{supp}(\hat{u}) - \operatorname{supp}(\hat{v})|$, then we have

$$\|uv\|_{L_{t,x}^2} \lesssim \alpha^{-\frac{1}{2}} \|u\|_{X_N} \|v\|_{X_M}. \tag{2.66}$$

Proof. As before, the bilinear estimate for homogeneous solutions (2.17) and (2.18) is the keys to proving these estimates. It suffices to prove (2.65), since (2.66) will follow in a similar manner. Denote $F_1 := (i\partial_t + \Delta)u$ and $F_2 := (i\partial_t + \Delta)v$. Using Proposition 2.15 with $T(u_1, u_2) = u_1 u_2$ to extend the bilinear estimate (2.17), we obtain

$$\|P_{>\lambda}(u\bar{v})\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} (\|u(0)\|_{L_x^2} + \|F_1\|_{\dot{X}^{0,-\frac{1}{2},1}})(\|v(0)\|_{L_x^2} + \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}}).$$

Therefore, it suffices to prove that for any $u \in X_N$ and $v \in X_M$,

$$\|P_{>\lambda}(u\bar{v})\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}}(\|u(0)\|_{L^2_x} + \|F_1\|_{Z_N})(\|v(0)\|_{L^2_x} + \|F_2\|_{Z_M}), \tag{2.67}$$

$$\|P_{>\lambda}(u\bar{v})\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}}(\|u(0)\|_{L^2_x} + \|F_1\|_{Z_N})(\|v(0)\|_{L^2_x} + \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}}), \tag{2.68}$$

$$\|P_{>\lambda}(u\bar{v})\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}}(\|u(0)\|_{L^2_x} + \|F_1\|_{\dot{X}^{0,-\frac{1}{2},1}})(\|v(0)\|_{L^2_x} + \|F_2\|_{Z_N}). \tag{2.69}$$

We use the decomposition from (2.28) for u. However, in this case, the frequency localization at $\frac{N}{2^{50}}$ is replaced by $\frac{\lambda}{2^{50}}$:

$$u(x,t) = e^{it\Delta}u(0) - \int_{\mathbb{R}} e^{it\Delta}\mathcal{L}u_y + (P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_y - h_y(x,t)\; dy,$$

where $\mathcal{L} : L^2_x \to L^2_x$ is a bounded operator and $u_y, \mathcal{L}u_y$ and h_y are defined similarly to (2.32), (2.34) and (2.35), respectively. From the remark following (2.36), we see that these functions are supported at frequency $\sim N$. Moreover, the following estimate still holds even with the frequency replacement.

$$\|\mathcal{L}u_y\|_{L^2_x} + \|u_y\|_{L^2_x} + \frac{1}{N}(\|\Delta h_y\|_{L^2_{x,t}} + \|\partial_t h_y\|_{L^2_{x,t}}) \lesssim \frac{1}{N^{\frac{1}{2}}}\|F_1(y,t)\|_{L^2_t}. \tag{2.70}$$

We consider all the possible terms in $P_{>\lambda}(u\bar{v})$. First, we consider all the terms that involve $P_{\lambda/2^{50}}1_{x>0}$. For any $G \in L^2_x$, we have that

$$\begin{aligned} P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_y G\Big] =& P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\ll\lambda}(e^{it\Delta}u_y G)\Big] \\ &+ P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_y G)\Big] \\ =& P_{>\lambda}\Big[(P_{\lambda/2^{50}}1_{x>0})P_{\gtrsim\lambda}(e^{it\Delta}u_y G)\Big]. \end{aligned}$$

Let $\psi_{N/2^{50}}$ be the function defined by $P_{N/2^{50}}f := \psi_{N/2^{50}} * f$. Consequently,

$$\Big\|P_{>\lambda}\left[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_y G\right]\Big\|_{L^2_{x,t}}$$

$$
\begin{aligned}
&= \left\| P_{>\lambda} \left[(P_{\lambda/2^{50}} 1_{x>0}) P_{\gtrsim\lambda}(e^{it\Delta} u_y G) \right] \right\|_{L^2_{x,t}} \\
&\lesssim \left\| (P_{\lambda/2^{50}} 1_{x>0}) P_{\gtrsim\lambda}(e^{it\Delta} u_y G) \right\|_{L^2_{x,t}} \\
&= \left\| (\psi_{\lambda/2^{50}} * 1_{x>0}) P_{\gtrsim\lambda}(e^{it\Delta} u_y G) \right\|_{L^2_{x,t}} \\
&\leq \int \left\| 1_{x-z>0} P_{\gtrsim\lambda} \left[e^{it\Delta} u_y(x) G(x) \right] \right\|_{L^2_{x,t}} |\psi_{N/2^{50}}(z)| \, dz \\
&\lesssim \| P_{\gtrsim\lambda}(e^{it\Delta} u_y G) \|_{L^2_{x,t}}.
\end{aligned}
$$

In other words, to estimate such terms, we can take out the $P_{\lambda/2^{50}} 1_{x>0}$ factor just like what we did in the proof of Theorem 2.14. Following the same line of proof as for (2.46) but using a different bilinear estimate (2.17) instead of (2.18), we obtain (2.67). To prove (2.68) and (2.69), we will show that for any $v_0 \in L^2_x$ supported at frequency $\sim M$,

$$
\| P_{>\lambda}(u \overline{e^{it\Delta} v_0}) \|_{L^2_{x,t}} \lesssim \lambda^{-\frac{1}{2}} (\| u(0) \|_{L^2_x} + \| F_1 \|_{Z_N}) \| v_0 \|_{L^2_x}, \tag{2.71}
$$

which, in view of Proposition 2.15 with $T(v) = P_{>\lambda}(u\bar{v})$, leads to (2.68). From (2.17) and (2.70), we obtain

$$
\begin{aligned}
\| P_{>\lambda}(e^{it\Delta} u(0) \overline{e^{it\Delta} v_0}) \|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}} \| u(0) \|_{L^2_x} \| v_0 \|_{L^2_x}, \\
\| P_{>\lambda}(e^{it\Delta} \mathcal{L} u_y \overline{e^{it\Delta} v_0}) \|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}} \| \mathcal{L} u_y \|_{L^2_x} \| v_0 \|_{L^2_x}, \\
&\lesssim (\lambda N)^{-\frac{1}{2}} \| F_1(y,t) \|_{L^2_t} \| v_0 \|_{L^2_x} \\
\| P_{\gtrsim\lambda}(e^{it\Delta} u_y \overline{e^{it\Delta} v_0}) \|_{L^2_{x,t}} &\lesssim \lambda^{-\frac{1}{2}} \| u_y \|_{L^2_x} \| v_0 \|_{L^2_x} \\
&\lesssim (\lambda N)^{-\frac{1}{2}} \| F_1(y,t) \|_{L^2_t} \| v_0 \|_{L^2_x}.
\end{aligned}
$$

We use the last inequality to estimate the term in $P_{>\lambda}(u\overline{e^{it\Delta}v_0})$ that involves $P_{\lambda/2^{50}}1_{x>0}$.

$$\left\|P_{>\lambda}\left[(P_{\lambda/2^{50}}1_{x>0})e^{it\Delta}u_y\overline{e^{it\Delta}v_0}\right]\right\|_{L^2_{x,t}} \lesssim \|P_{\gtrsim\lambda}(e^{it\Delta}u_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}}$$
$$\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}.$$

For the remaining term, we use the Hölder inequality, (2.70) and the fact that $\lambda \lesssim N$.

$$\begin{aligned}\|P_{>\lambda}(h_y\overline{e^{it\Delta}v_0})\|_{L^2_{x,t}} &\lesssim \|h_y\|_{L^2_{x,t}}\|e^{it\Delta}v_0\|_{L^\infty_{x,t}} \\ &\lesssim \frac{M^{\frac{1}{2}}}{N^{\frac{3}{2}}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x} \\ &\lesssim (\lambda N)^{-\frac{1}{2}}\|F_1(y,t)\|_{L^2_t}\|v_0\|_{L^2_x}.\end{aligned} \tag{2.72}$$

Recalling that $\|(i\partial_t+\Delta)u\|_{Z_N} = N^{-\frac{1}{2}}\|(i\partial_t+\Delta)u\|_{L^1_xL^2_t}$, these estimates yield (2.71) via the Minkowski inequality. The proof for (2.69) is similar, except at (2.72) where we have the following modification:

$$\begin{aligned}\|P_{>\lambda}(e^{it\Delta}u_0\overline{h}_{y'})\|_{L^2_{x,t}} &\lesssim \|e^{it\Delta}u_0\|_{L^\infty_xL^2_t}\|h_{y'}\|_{L^2_xL^\infty_t} \\ &\lesssim (NM)^{-\frac{1}{2}}\|u_0\|_{L^2_x}\|F_2(y',t)\|_{L^2_t} \\ &\lesssim (\lambda M)^{-\frac{1}{2}}\|u_0\|_{L^2_x}\|F_2(y',t)\|_{L^2_t}.\end{aligned}$$

For the second to last inequality, we used the smoothing estimate (2.4) and (2.37) with $d = 3$. This concludes the proof of (2.65). □

We will also use the following estimate which was taken from Tao ([49]) and modified to be suitable to our spaces.

Proposition 2.18. *Suppose that u is supported at frequency $\sim N$. Then we have*

$$\|u\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|u\|_{X_N}. \tag{2.73}$$

Proof. Consider the Duhamel's formula of u.

$$u(x,t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F_1(s)\ ds - i\int_0^t e^{i(t-s)\Delta}F_2(s)\ ds, \tag{2.74}$$

where $F_1 \in Z_N$ and $F_2 \in \dot{X}^{0,-\frac{1}{2},1}$. For $i = 1, 2$, we split the term

$$\int_0^t e^{i(t-s)\Delta}F_i(s)\ ds = \int_{-\infty}^t e^{i(t-s)\Delta}F_i(s)\ ds - e^{it\Delta}\int_{-\infty}^0 e^{-is\Delta}F_i(s)\ ds.$$

Since the $\dot{X}^{0,\frac{1}{2},\infty}$ seminorm vanishes on any free solution, it suffices to estimate the first term. For F_1, we recall the computation (2.26) from the proof of Lemma 2.11 that the first term is equal to

$$\int w_y\ dy \quad \text{where} \quad \widetilde{w}_y = \frac{\psi_N(\xi)}{-\tau - \xi^2 - i0}\widehat{F}_1(y,\tau).$$

With a direct integration, we see that

$$\begin{aligned}\|\chi_{A_M}\widetilde{w}\|_{L^2_{x,\tau}} &\sim \frac{1}{N^{\frac{1}{2}}}\Big(\int\int_{\xi\sim N}\frac{|\xi|}{(\tau+\xi^2)^2}\chi_{A_M}[\widehat{F}_1(y,\tau)]^2\ d\xi d\tau\Big)^{\frac{1}{2}} \\ &\lesssim \frac{1}{N^{\frac{1}{2}}M^{\frac{1}{2}}}\|F_1(y,t)\|_{L^2_t},\end{aligned}$$

From the definition of $\dot{X}^{0,\frac{1}{2},\infty}$, it follows that

$$\Big\|\int_{-\infty}^t e^{i(t-s)\Delta}F_1(s)\ ds\Big\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|F_1\|_{Z_N}.$$

On the other hand, we consider the space-time Fourier transform

$$\mathcal{F}\int \chi_{(0,\infty)}(t-s)e^{i(t-s)\Delta}F_2(s)\ ds = \frac{\widetilde{F}_{2,M}(\xi,\tau)}{-\tau-\xi^2-i0}.$$

It follows from the Plancherel's theorem that

$$\Big\|\int_{-\infty}^t e^{i(t-s)\Delta}F_2(s)\ ds\Big\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim \|F_2\|_{\dot{X}^{0,-\frac{1}{2},1}},$$

and the conclusion immediately follows. □

2.7 V^p space

Toward the end of Chapter 4, we will employ another space $V^p\mathcal{B}$ of functions of bounded p variation with respect to a Banach space $\mathcal{B}$ defined by the seminorm:

$$\|u\|_{V^p\mathcal{B}} := \sup_{-\infty<t_0<\ldots<t_K<\infty} \left(\sum_{k=1}^{K} \|u(t_k) - u(t_{k-1})\|_{\mathcal{B}}^p \right)^{1/p},$$

and let $V^p_{-,rc}\mathcal{B}$ be the space of right continuous functions f in $V^p\mathcal{B}$ satisfying $f(-\infty) = 0$.

Let $1 \le p < q < \infty$. Since $l^p(\mathbb{N}) \subset l^q(\mathbb{N})$, it follows from the definition that

$$V^p\mathcal{B} \subset V^q\mathcal{B}. \tag{2.75}$$

We refer to [17] for a complete treatment of these spaces as main tools to study well-posedness problems for PDEs. For our purpose, however, we only need the following proposition:

Proposition 2.19 ([35], Corollary 3.3)**.** *Let p and q be indices satisfying*

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}. \quad 4 \le p \le \infty.$$

For any function $u \in V^2 L^2_x$, the following estimate holds:

$$\|e^{it\Delta} u\|_{L^p_t L^q_x} \lesssim \|u\|_{V^2 L^2_x}. \tag{2.76}$$

We mention here the main reason that this space will be employed later on in Section 4.6, namely, the fact that it commutes with the l^2-type spaces.

Proposition 2.20. *For any Banach space $\mathcal{B}$ and any sequence of functions $f_n : \mathbb{R} \to \mathcal{B}$, we have*

$$\Big\| \Big(\sum_{n=1}^{\infty} \|f_n(\cdot)\|_{\mathcal{B}}^2 \Big)^{\frac{1}{2}} \Big\|_{V^2\mathbb{R}} \le \Big(\sum_{n=1}^{\infty} \|f_n\|_{V^2\mathcal{B}}^2 \Big)^{\frac{1}{2}}. \tag{2.77}$$

Proof. We have that

$$\Big\|\Big(\sum_{n=1}^{\infty}\|f_n(\cdot)\|_{\mathcal{B}}^2\Big)^{\frac{1}{2}}\Big\|_{V^2\mathbb{R}} = \sup_{-\infty<t_0<\ldots<t_K<\infty}\left(\sum_{k=1}^{K}\sum_{n=1}^{\infty}\|f_n(t_k)-f_n(t_{k-1})\|_{\mathcal{B}}^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n=1}^{\infty}\left\{\sup_{-\infty<t_0<\ldots<t_K<\infty}\sum_{k=1}^{K}\|f_n(t_k)-f_n(t_{k-1})\|_{\mathcal{B}}^2\right\}\right)^{\frac{1}{2}}$$

$$= \left(\sum_{n=1}^{\infty}\|f_n\|_{V^2\mathcal{B}}^2\right)^{\frac{1}{2}}.$$

□

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Chapter 3

Well-posedness results

3.1 Proof of Theorem 1.1

Let s be the exponent which satisfies the condition in Theorem 1.1. To obtain the local well-posedness, we redefine the spaces X^s and Y^s from (2.21) in a way that the projections on the low frequencies are combined together. Since we assume a finite time restriction, so any spaces mentioned below are defined on the product space $\mathbb{R} \times [-1, 1]$.

$$\begin{aligned}
\|u\|_{Z_N} &= \|u\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty \cap L_{x,t}^6} + N^{-\frac{1}{2}}\|u\|_{L_x^2 L_t^\infty} + N^{\frac{1}{2}}\|u\|_{L_x^\infty L_t^2} \\
\|u\|_{Y_N} &= \inf\{N^{-\frac{1}{2}}\|u_1\|_{L_x^1 L_t^2} + \|u_2\|_{L_t^1 L_x^2} \mid u_1 + u_2 = u\} \\
\|u\|_{X_N} &= \|u\|_{Z_N} + \|(i\partial_t + \Delta)u\|_{Y_N} \\
\|u\|_{X^s} &= \|P_{\leq 1}u\|_{X_1} + \Big(\sum_{N \in 2^{\mathbb{N}}} N^{2s}\|P_N u\|_{X_N}^2\Big)^{\frac{1}{2}} \\
\|u\|_{Y^s} &= \|P_{\leq 1}u\|_{Y_1} + \Big(\sum_{N \in 2^{\mathbb{N}}} N^{2s}\|P_N u\|_{Y_N}^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.1}$$

The previous section prepares us all the estimates we need in order to obtain the linear estimate for the X^s and Y^s spaces; It follows from (2.22), (2.42) and (2.43) that for any $s \geq \frac{1}{2}$,

$$\|u\|_{X^s} \lesssim \|u_0\|_{H^s} + \|F\|_{Y^s}. \tag{3.2}$$

We are now ready to prove the multilinear estimate.

Theorem 3.1. *Let $d \geq 3$. For any $u_1, u_2, \dots, u_d \in X^s$ where $s \geq \frac{1}{2}$, we have the following estimate.*

$$\left\|(\partial_x u_1)\prod_{i=2}^{d} u_i\right\|_{Y^s} \lesssim \prod_{i=1}^{d} \|u_i\|_{X^s}. \tag{3.3}$$

Proof. It suffices to prove that

$$\left\|(\partial_x u_1)\prod_{i=2}^{d} u_i\right\|_{Y^s} \lesssim \|u_1\|_{X^s}\prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}. \tag{3.4}$$

which implies (3.3) since $X^s \subset X^{\frac{1}{2}}$ due to the absence of low frequency projections. In view of (2.42) and (2.43), we can treat $P_{\leq 1}$ as P_1, so it suffices to estimate the summation over high frequencies:

$$\sum_{N,N_1,\dots,N_d} N^s \left\|P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i)\right\|_{Y^s}, \tag{3.5}$$

where $N \geq 1$ and $N_i \geq 1$ for all i in the summation. We can assume that $N_1 \geq N_2 \geq \dots \geq N_d$ and $N \lesssim N_1$. This is because u_1 is the only term in (3.5) that has a derivative, and so any other frequency distribution would lead to a better estimate. We define $c_{N_1,1} = N_1^s \|P_{N_1} u_1\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{\frac{1}{2}} \|P_{N_i} u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. Thus, we see that $\|c_{N_1,1}\|_{l^2(N_1)} = \|u_1\|_{X^s}$ and $\|c_{N_i,i}\|_{l^2(N_i)} = \|u_i\|_{X^{\frac{1}{2}}}$ for $2 \leq i \leq d$. In order to obtain the l^2 summation of $c_{N_i,i}$, we will repeatedly be using the following application of the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{N_j,\dots,N_d} \frac{1}{N_j^a} \prod_{i=j}^{d} c_{N_i,i} \leq \sum_{N_j,\dots,N_d} \prod_{i=j}^{d} \frac{1}{N_i^{\frac{a}{d}}} c_{N_i,i} &\leq \prod_{i=j}^{d} \sum_{N_i \geq 1} \frac{1}{N_i^{\frac{a}{d}}} c_{N_i,i} \\ &\lesssim \prod_{i=j}^{d} \|u_i\|_{X^{\frac{1}{2}}},\end{aligned} \tag{3.6}$$

for any $a > 0$. To prove (3.4), we split the summation over three different kinds of frequency interactions.

$$\begin{aligned}&\sum_{N,N_1,\dots,N_d} N^s \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{Y^s} \\ &= \Big(\sum_{I} + \sum_{II} + \sum_{III} \Big) N^s \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{Y^s}\end{aligned}$$

Each of the summations contains certain ranges of $N, N_1, \dots, N_d$ described by the following cases:

I). $N_1 \gg N_2$ and $N \sim N_1$.

By Hölder inequality, (2.5) with $\gamma = 2$ and (3.6),

$$\begin{aligned}&\sum_{N_1,\dots,N_d} \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^{d} P_{N_i} u_i) \Big\|_{L_x^1 L_t^2} \\ &\qquad \lesssim \sum_{N_i} \| P_{N_1} \partial_x u_1 P_{N_2} u_2 \|_{L_{x,t}^2} \| P_{N_3} u_3 \|_{L_x^2 L_t^\infty} \prod_{i=4}^{d} \| P_{N_i} u_i \|_{L_{x,t}^\infty} \\ &\qquad \lesssim \sum_{N_i} \frac{1}{N_1^{s-\frac{1}{2}} N_2^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i} \\ &\qquad \lesssim \frac{1}{N^{s-\frac{1}{2}}} \sum_{N_i} \frac{1}{N_2^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i} \\ &\qquad \lesssim \frac{1}{N^{s-\frac{1}{2}}} \sum_{N_1 \sim N} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}.\end{aligned}$$

Therefore,

$$\sum_{I} N^{s-\frac{1}{2}} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i) \Big\|_{L_x^1 L_t^2} \lesssim \sum_{N_1 \sim N} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}.$$

Taking the l^2 summation with respect to $N \geq 1$, we obtain (3.4).

II). $N_1 \sim N_2 \gg N_3 \geq \ldots \geq N_d$ and $N \lesssim N_1$.

In this case, we use the bilinear estimate for the product $P_{N_1}\partial_x u_1 P_{N_3}u_3$ and put $P_{N_2}u_2$ in the Strichartz space $L_t^4 L_x^\infty$:

$$\begin{aligned}
&\sum_{N_1,\ldots,N_d} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i) \Big\|_{L_t^1 L_x^2} \\
&\qquad \lesssim \sum_{N_1,\ldots,N_d} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i) \Big\|_{L_t^{\frac{4}{3}} L_x^2} \\
&\qquad \lesssim \sum_{N_i} \|P_{N_1}\partial_x u_1 P_{N_3}u_3\|_{L_{t,x}^2} \|P_{N_2}u_2\|_{L_t^4 L_x^\infty} \prod_{i=4}^{d} \|P_{N_i}u_i\|_{L_{t,x}^\infty} \\
&\qquad \lesssim \sum_{N_i} \frac{1}{N_1^{s-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i} \\
&\qquad \lesssim \Big(\sum_{N_1 \sim N_2} \frac{1}{N_1^s} c_{N_1,1} c_{N_2,2} \Big) \Big(\sum_{N_3,\ldots,N_d} \frac{1}{N_3^{\frac{1}{2}}} \prod_{i=3}^{d} c_{N_i,i} \Big) \\
&\qquad \lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^s} c_{N_1,1} \Big)^{\frac{1}{2}} \prod_{i=2}^{d} \|u_i\|_{\dot{X}^{\frac{1}{2}}},
\end{aligned}$$

where we used (3.6) in the second to last step. Therefore,

$$\sum_{II} N^{2s} \|P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i)\|_{L_t^1 L_x^2}^2 \lesssim \|u_1\|_{X^s}^2 \prod_{i=2}^{d} \|u_i\|_{X^{\frac{1}{2}}}^2.$$

III). $N_1 \sim N_2 \sim N_3 \geq \ldots \geq N_d$ and $N \lesssim N_1$.

We divide the proof into two cases depending on the degree d.

A). $d = 3$.

Even though we cannot use the bilinear estimate in this case, the fact that $N_1 \sim N_2 \sim N_3$ allows us to cancel the derivative loss in $P_{N_1}\partial_x u_1$ by the $\frac{1}{2}$ regularity from $P_{N_2}u_2$ and $P_{N_3}u_3$ via the Hölder inequality:

$$\begin{aligned}
&\sum_{N_1\sim N_2\sim N_3} \Big\| P_N[(P_{N_1}\partial_x u_1)P_{N_2}u_2 P_{N_3}u_3] \Big\|_{L^1_t L^2_x} \\
&\quad\lesssim \sum_{N_1\sim N_2\sim N_3} \Big\| P_N[(P_{N_1}\partial_x u_1)P_{N_2}u_2 P_{N_3}u_3] \Big\|_{L^2_{t,x}} \\
&\quad\lesssim \sum_{N_1\sim N_2\sim N_3} \|P_{N_1}\partial_x u_1\|_{L^6_{t,x}} \|P_{N_2}u_2\|_{L^6_{t,x}} \|P_{N_3}u_3\|_{L^6_{t,x}} \\
&\quad\lesssim \sum_{N_1\sim N_2\sim N_3} \frac{N_1^{1-s}}{N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} c_{N_1,1}c_{N_2,2}c_{N_3,3} \\
&\quad\lesssim \Big(\sum_{N_1\gtrsim N} \frac{1}{N_1^s} c_{N_1,1} \Big)^{\frac{1}{2}} \|u_2\|_{X^{\frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}}},
\end{aligned}$$

where the last step follows from the Cauchy-Schwarz inequality on $c_{N_1,1}c_{N_2,2}c_{N_3,3}$.

B). $d \geq 4$.

We again take advantage of the finite time restriction and put $P_{N_i}u_i$ for $1 \leq i \leq 4$ in suitable Strichartz spaces, namely $L^\infty_t L^2_x$ and $L^4_t L^\infty_x$.

$$\begin{aligned}
&\sum_{N_1,\dots,N_d} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i) \Big\|_{L^1_t L^2_x} \\
&\quad\lesssim \sum_{N_1,\dots,N_d} \Big\| P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{d} P_{N_i}u_i) \Big\|_{L^{\frac{4}{3}}_t L^2_x} \\
&\quad\lesssim \sum_{N_i} \|P_{N_1}\partial_x u_1\|_{L^\infty_t L^2_x} \prod_{i=2}^{4} \|P_{N_i}u_i\|_{L^4_t L^\infty_x} \prod_{i=5}^{d} \|P_{N_i}u_i\|_{L^\infty_{t,x}} \\
&\quad\lesssim \sum_{N_i} \frac{N_1^{1-s}}{N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}} \prod_{i=1}^{d} c_{N_i,i}
\end{aligned}$$

$$\lesssim \Big(\sum_{N_1,N_2,N_3} \frac{1}{N_1^s} c_{N_1,1} c_{N_2,2} c_{N_3,3} \Big) \Big(\sum_{N_4 \dots, N_d} \frac{1}{N_4^{\frac{1}{2}}} \prod_{i=4}^d c_{N_i,i} \Big)$$
$$\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^s} c_{N_1,1} \Big)^{\frac{1}{2}} \prod_{i=2}^d \|u_i\|_{X^{\frac{1}{2}}},$$

In either case, it follows that

$$\sum_{III} N^{2s} \Big\| P_N (P_{N_1} \partial_x u_1 \prod_{i=2}^d P_{N_i} u_i) \Big\|_{L_t^1 L_x^2}^2 \lesssim \|u_1\|_{X^s}^2 \prod_{i=2}^d \|u_i\|_{X^{\frac{1}{2}}}^2 .$$

and this concludes the proof.

□

In view of this theorem, if every term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has only one derivative, then we expect to close the contraction argument in a subspace of $X^{\frac{1}{2}}$. On the other hand, if we replace u_j by $\partial_x u_j$ for some $j \geq 2$, then it follows from (2.2) that $\|\partial_x u_i\|_{X^s} \lesssim \|u_i\|_{X^{s+1}}$ for any $s > 0$, and so (3.4) yields

$$\Big\| (\partial_x u_1)(\partial_x u_j) \prod_{\substack{i=2 \\ i \neq j}}^d u_i \Big\|_{Y^{\frac{3}{2}}} \lesssim \|u_1\|_{X^{\frac{3}{2}}} \|\partial_x u_j\|_{X^{\frac{1}{2}}} \prod_{\substack{i=2 \\ i \neq j}}^d \|u_i\|_{X^{\frac{1}{2}}}$$
$$\lesssim \|u_1\|_{X^{\frac{3}{2}}} \prod_{i=2}^d \|u_i\|_{X^{\frac{3}{2}}},$$

and for any $s \geq \frac{3}{2}$, we have

$$\Big\| (\partial_x u_1)(\partial_x u_j) \prod_{\substack{i=2 \\ i \neq j}}^d u_i \Big\|_{Y^s} \lesssim \|u_1\|_{X^s} \prod_{i=2}^d \|u_i\|_{X^s}.$$

Consequently, in the case that a term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has more than one derivative, we can employ the contraction argument in $X^{\frac{3}{2}}$.

Proof of Theorem 1.1. We define $F(u) := P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$. Let u and v be functions in X^s. We use the main linear estimate (3.2) and simple algebra to obtain

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\partial_x^2} \left[F(u(x,s)) - F(v(x,s)) \right] \, ds \right\|_{X^s} &\le c_1 \left\| F(u) - F(v) \right\|_{Y^s} \\ &\le c_1 c_2 (\|u\|_{X^s}^{d-1} + \|v\|_{X^s}^{d-1}) \|u - v\|_{X^s}, \end{aligned} \tag{3.7}$$

where we used the multilinear estimate (3.3) in the last step.

Let $C := \min\left\{(8c_1c_2)^{-\frac{1}{d-1}}, (4c_2)^{-\frac{1}{d-1}}\right\}$ where c_1 and c_2 are constants in (3.7). Define a Banach space as stated in the theorem:

$$X = \{u \in C_t^0 H_x^s([-1,1] \times \mathbb{R}) \cap X^s : \|u\|_{X^s} \le 2C\}.$$

Let $u_0 \in X$ such that $\|u_0\|_{H^s} \le C$. Then, for $u \in X$, we define an operator

$$Lu := e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} F(u(x,s)) \, ds,$$

Again, by the main linear estimate, we have

$$\begin{aligned} \|Lu\|_{X^s} &\le \|u_0\|_{H^s} + \|F\|_{Y^s} \\ &\le \|u_0\|_{H^s} + c_2 \|u\|_{X^s}^d \\ &\le \frac{3C}{2} < 2C. \end{aligned}$$

Thus, L maps X to X. Moreover, from (3.7),

$$\|Lu - Lv\|_{X^s} \le c_1 c_2 (\|u\|_{X^s}^{d-1} + \|v\|_{X^s}^{d-1}) \|u - v\|_{X^s} \le \frac{1}{4} \|u - v\|_{X^s}.$$

Thus, L is a contraction and the local well-posedness in X immediately follows. $\square$

3.2 Proof of Theorem 1.3 when $d \geq 6$

In the previous sections, we used the time restriction to avoid dealing with low frequencies at $\xi \leq 1$. However, such argument cannot be used to obtain the global well-posedness for the gDNLS with nonlinearity of order $d \geq 5$. Therefore, the function spaces that we use will take these low frequencies into account. Let $s_0(d) = \frac{1}{2} - \frac{1}{d-1} = \frac{d-3}{2(d-1)}$ for $d \geq 5$. The spaces X^s and Y^s in (2.21) are replaced by those defined by the quasi-norms $\dot{X}^s$ and $\dot{Y}^s$ which in turn are defined by the norms X_N and Y_N,

$$\begin{aligned}
\|u\|_{X_N} &= \|u\|_{L_t^\infty L_x^2} + N^{-\frac{1}{4}}\|u\|_{L_x^4 L_t^\infty} + N^{\frac{1}{2}}\|u\|_{L_x^\infty L_t^2} \\
&\quad + N^{-\frac{1}{2}}\|(i\partial_t + \Delta)u\|_{L_x^1 L_t^2} \\
\|u\|_{\dot{X}^s} &= \Big(\sum_{N \in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|_{X_N}^2\Big)^{\frac{1}{2}} \\
\|u\|_{X^s} &= \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s} \\
\|u\|_{Y_N} &= N^{-\frac{1}{2}}\|u\|_{L_x^1 L_t^2} \\
\|u\|_{\dot{Y}^s} &= \Big(\sum_{N \in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|_{Y_N}^2\Big)^{\frac{1}{2}} \\
\|u\|_{Y^s} &= \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.
\end{aligned} \tag{3.8}$$

Thus we have embeddings $X^s \hookrightarrow H^s$ and $X^s \hookrightarrow X^{s_0} \hookrightarrow \dot{X}^{s_0}$ for any $s \geq s_0$. In view of (2.22), we obtain the linear estimate

$$\|u\|_{X^s} \lesssim \|u_0\|_{H^s} + \|F\|_{Y^s}. \tag{3.9}$$

With these choices of spaces, we can establish the multilinear estimate for $d \geq 6$. The proof for the case $d = 5$ is significantly more involved and requires some frequency-modulation analysis, so we will postpone it to the next section.

Theorem 3.2. *Let $d \geq 6$. We have the following estimates.*

1). For any $u_1, u_2, \ldots, u_d \in X^{s_0}$,

$$\Big\|\partial_x \prod_{i=1}^{d} u_i\Big\|_{\dot{Y}^{s_0}} \lesssim \prod_{i=1}^{d} \|u_i\|_{\dot{X}^{s_0}}, \tag{3.10}$$

2). Let $s \geq s_0$. For any $u_1, u_2, \ldots, u_d \in X^s$,

$$\Big\|\partial_x \prod_{i=1}^{d} u_i\Big\|_{Y^s} \lesssim \prod_{i=1}^{d} \|u_i\|_{X^s}. \tag{3.11}$$

Proof. Our goal is to obtain the estimate

$$\sum_N N^{2s+1} \|P_N \prod_{i=1}^{d} u_i\|^2_{L^1_x L^2_t} \lesssim \sum_{j=1}^{d} \|u_j\|^2_{\dot{X}^s} \prod_{i \neq j} \|u_i\|^2_{\dot{X}^{s_0}}, \tag{3.12}$$

which implies (3.10) by choosing $s = s_0$. We get (3.11) by combining two different versions of this estimate with a fixed $s \geq s_0$ and with $s = 0$. We will focus on each term on the left-hand side of (3.11)

$$\begin{aligned} N^{2s-1}\Big\|P_N \partial_x \prod_{i=1}^{d} u_i\Big\|^2_{L^1_x L^2_t} &= N^{2s-1}\Big\|P_N \partial_x \sum_{N_1,\ldots,N_d} \prod_{i=1}^{d} P_{N_i} u_i\Big\|^2_{L^1_x L^2_t} \\ &\lesssim N^{2s+1} \sum_{N_1,\ldots,N_d} \Big\|P_N \prod_{i=1}^{d} P_{N_i} u_i\Big\|^2_{L^1_x L^2_t}, \end{aligned}$$

and study different kinds of frequency interactions. As before, we assume that $N_1 \geq N_2 \geq \ldots \geq N_d$. We define $c_{N_1,1} = N_1^s \|P_{N_i} u_1\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{s_0} \|P_{N_i} u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. We will use the following two estimates for a product of terms with higher and lower frequencies.

1. For $N \lesssim N_1 \sim N_2 \sim \ldots \sim N_{j-1}$ where $j \geq 3$, it follows from the Cauchy-Schwarz inequality that

$$\sum_{N_i} \prod_{i=1}^{j-1} c_{N_i,i} \lesssim \Big(\sum_{N_1 \gtrsim N} c^2_{N_1,1}\Big)^{\frac{1}{2}} \prod_{i=2}^{j-1} \|u_i\|_{\dot{X}^{s_0}}. \tag{3.13}$$

2. For $N_j \geq N_{j+1} \geq \ldots \geq N_d$ and any $\alpha > 0$, Young's inequality and trivial estimate $c_{N_i,i} \leq \|u_i\|_{\dot{X}^{s_0}}$ imply

$$\begin{aligned}\sum_{N_j \geq \ldots \geq N_d} \left(\frac{N_d}{N_j}\right)^{\alpha} \prod_{i=j}^{d} c_{N_i,i} &\leq \prod_{i=j+1}^{d-1} \|u_i\|_{\dot{X}^{s_0}} \sum_{N_j \geq \ldots \geq N_d} \left(\frac{N_d}{N_j}\right)^{\alpha} c_{N_j,j} c_{N_d,d} \\ &\lesssim_{\alpha} \prod_{i=j}^{d} \|u_i\|_{\dot{X}^{s_0}}.\end{aligned} \tag{3.14}$$

These estimates will be used in each case after appropriate uses of Hölder inequality, Bernstein inequality and bilinear estimate (2.44).

By Hölder and Bernstein inequalities,

$$\begin{aligned}&\left\| P_N \prod_{i=1}^{d} u_i \right\|_{L_x^1 L_t^2} \\ &\lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \prod_{i=6}^{d} \|P_{N_i} u_i\|_{L_{x,t}^\infty} \\ &\lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \prod_{i=6}^{d} N_i^{\frac{1}{2}} \|P_{N_i} u_i\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \prod_{i=2}^{5} \frac{1}{N_i^{s_0-\frac{1}{4}}} \prod_{i=6}^{d} N_i^{\frac{1}{2}-s_0} \prod_{i=1}^{d} c_{N_i,i}.\end{aligned}$$

Since $s_0 = \frac{1}{2} - \frac{1}{d-1}$, the sums of the exponents in $\prod_{i=2}^{5} N_i^{s_0-\frac{1}{4}}$ and $\prod_{i=6}^{d} N_i^{\frac{1}{2}-s_0}$ are equal. With the assumption that $N_2 \geq N_3 \geq \ldots \geq N_d$, the right-hand side is bounded by

$$\sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_d}{N_2}\right)^{\frac{1}{4(d-1)}} \prod_{i=1}^{d} c_{N_i,i}. \tag{3.15}$$

To estimate this term, we consider the following two frequency interactions.

1. $N \sim N_1 \gg N_2 \geq \ldots \geq N_d$.

   Using (3.14) on $c_{N_2,2} c_{N_3,3} \ldots c_{N_d,d}$, we can bound (3.15) by

$$\sum_{N_1 \sim N} \frac{1}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}},$$

for each fixed N. We have that

$$\sum_N \Big(\sum_{N_1 \sim N} \frac{N^{2s+1}}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \Big)^2 \prod_{i=2}^{5} \|u_i\|^2_{\dot{X}^{s_0}} \sim \sum_{N_1} c^2_{N_1,1} \prod_{i=2}^{5} \|u_i\|^2_{\dot{X}^{s_0}}$$
$$= \|u_1\|^2_{\dot{X}^s} \prod_{i=2}^{5} \|u_i\|^2_{\dot{X}^{s_0}},$$

which implies (3.12) as desired.

2. $N \lesssim N_1 \sim N_2 \geq \ldots \geq N_d$.

Using (3.13) on $c_{N_1,1} c_{N_2,2}$ and (3.14) on $c_{N_3,3} c_{N_4,4} \ldots c_{N_d,d}$, we can bound (3.15) by

$$\Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c^2_{N_1,1} \Big)^{\frac{1}{2}} \prod_{i=2}^{5} \|u_i\|_{\dot{X}^{s_0}}.$$

Therefore, by switching the order of summations,

$$\sum_N \sum_{N_1 \gtrsim N} \frac{N^{2s+1}}{N_1^{2s+1}} c^2_{N_1,1} \prod_{i=2}^{5} \|u_i\|^2_{\dot{X}^{s_0}} \lesssim \sum_{N_1} c^2_{N_1,1} \prod_{i=2}^{5} \|u_i\|^2_{\dot{X}^{s_0}},$$

which again implies (3.12). This concludes the proof for $d \geq 6$.

□

Using the linear estimate (3.9) and the multilinear estimates (3.10) and (3.11), the proof for Theorem 1.3 follows in the same manner as in Theorem 1.1. Note that we did not use any finite time restriction in any parts of the proof.

3.3 Proof of Theorem 1.3 when $d = 5$

The difficulty in this case arises from the fact that there is no room left to put the lowest frequency term in $L^\infty_{x,t}$. Thus, we will take this case with extra care by adding the

$\dot{X}^{0,b,q}$ spaces mentioned in Section 2.6. We start with proving the multilinear estimate. Note that the position of complex conjugates will be significant in the analysis below.

Theorem 3.3. *For $1 \leq i \leq 5$, let u_i represent u or $\bar{u}$. Then we have the following estimates.*

1). For any $u \in X^{\frac{1}{4}}$,

$$\Big\|\partial_x \prod_{i=1}^{5} u_i\Big\|_{\dot{Y}^{\frac{1}{4}}} \lesssim \|u\|^5_{\dot{X}^{\frac{1}{4}}}, \tag{3.16}$$

2). Let $s \geq \frac{1}{4}$. For any $u \in X^s$,

$$\Big\|\partial_x \prod_{i=1}^{5} u_i\Big\|_{Y^s} \lesssim \|u\|^5_{X^s}. \tag{3.17}$$

Proof. As before, our goal is to obtain the estimate

$$\sum_N N^{2s+2}\Big\|P_N \prod_{i=1}^{5} u_i\Big\|^2_{Y_N} \lesssim \|u\|^2_{\dot{X}^s}\|u\|^8_{\dot{X}^{\frac{1}{4}}}. \tag{3.18}$$

First, we split each term in the left-hand side as the sum of all possible frequency interactions:

$$N^{2s+2}\Big\|P_N\partial_x \prod_{i=1}^{5} u_i\Big\|^2_{Y_N} \lesssim N^{2s+2} \sum_{N_1,\ldots,N_5} \Big\|P_N \prod_{i=1}^{5} P_{N_i} u_i\Big\|^2_{Y_N}.$$

Assume that $N_1 \geq N_2 \geq \ldots \geq N_5$. Define $c_{N_1,1} = N_1^s\|P_{N_1}u\|_{X_{N_1}}$ and $c_{N_i,i} = N_i^{\frac{1}{4}}\|P_{N_i}u\|_{X_{N_i}}$ for $2 \leq i \leq 5$. We make a slight abuse of notation by using $\sum_{N_i}$ for the summation over all possible $N_1, N_2, \ldots, N_5$ when the restrictions on these numbers are clear. We also will be using the Cauchy-Schwarz inequality (3.13) and Young's inequality (3.14).

We split the left-hand side of (3.18) over four different kinds of frequency interactions:

$$\begin{aligned}&\sum_{N,N_1,\ldots,N_5} N^s\Big\|P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{5} P_{N_i}u_i)\Big\|_{Y_N}\\ &= \Big(\sum_I + \sum_{II} + \sum_{III} + \sum_{IV}\Big) N^s\Big\|P_N(P_{N_1}\partial_x u_1 \prod_{i=2}^{5} P_{N_i}u_i)\Big\|_{Y_N}.\end{aligned}$$

Each of the summations contains certain ranges of $N, N_1, \ldots, N_5$ described by the following cases:

I). $N \lesssim N_1 \sim N_2 \sim N_3 \sim N_4 \sim N_5$.

By Hölder and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\Big\|P_N \prod_{i=1}^{5} u_i\Big\|_{L_x^1 L_t^2} &\lesssim \sum_{N_i} \|P_{N_1} u_1\|_{L_x^\infty L_t^2} \prod_{i=2}^{5} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \\ &= \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \prod_{i=1}^{5} c_{N_i,i} \\ &\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4 .\end{aligned}$$

Summing over $N \in 2^{\mathbb{Z}}$, we see that

$$\begin{aligned}\sum_I N^{2s+1} \Big\|P_N \prod_{i=1}^{5} u_i\Big\|_{L_x^1 L_t^2}^2 &\lesssim \sum_{N_1} \sum_{N \lesssim N_1} \left(\frac{N}{N_1}\right)^{2s+1} c_{N_1,1}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8 \\ &\lesssim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8 .\end{aligned}$$

II). $N \sim N_1 \gg N_2 \geq N_3 \geq N_4 \geq N_5$.

By the bilinear estimate (2.65) or (2.66) on $P_{N_1} u_1 P_{N_2} u_2$ (depending on the complex conjugates) and Bernstein inequality on $P_{N_5} u_5$, we have that for each fixed N,

$$\begin{aligned}&\Big\|P_N \prod_{i=1}^{5} u_i\Big\|_{L_x^1 L_t^2} \\ &\lesssim \sum_{N_i} \|P_{N_1} u_1 P_{N_2} u_2\|_{L_{x,t}^2} \prod_{i=3}^{4} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \|P_{N_5} u_5\|_{L_{x,t}^\infty} \\ &\lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \|P_{N_1} u_1\|_{X_{N_1}} \|P_{N_2} u_2\|_{X_{N_2}} \prod_{i=3}^{4} \|P_{N_i} u_i\|_{L_x^4 L_t^\infty} \|P_{N_5} u_5\|_{L_t^\infty L_x^2} \\ &= \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_2}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}.\end{aligned}$$

By Young's inequality (3.14), this term is bounded by

$$\lesssim \sum_{N_1 \sim N} \frac{1}{N_1^{s+\frac{1}{2}}} c_{N_1,1} \|u\|^4_{\dot{X}^{\frac{1}{4}}}.$$

Therefore,

$$\begin{aligned}\sum_{II} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|^2_{L^1_x L^2_t} &\lesssim \sum_N \Big(\sum_{N_1 \sim N} \left(\frac{N}{N_1}\right)^{s+\frac{1}{2}} c_{N_1,1} \Big)^2 \|u\|^8_{\dot{X}^{\frac{1}{4}}} \\ &\lesssim \Big(\sum_{N_1} \sum_{N \sim N_1} c^2_{N_1,1} \Big) \|u\|^8_{\dot{X}^{\frac{1}{4}}} \\ &\sim \|u\|^2_{\dot{X}^s} \|u\|^8_{\dot{X}^{\frac{1}{4}}}.\end{aligned}$$

III). $N \lesssim N_1 \sim N_2 \sim N_{j-1} \gg N_j \geq N_5$ where $j = 3$ or $j = 4$.

This is similar to case II), but instead we use the bilinear estimate on $P_{N_1} u_1 P_{N_j} u_j$.

$$\begin{aligned}&\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L^1_x L^2_t} \\ &\lesssim \sum_{N_i} \|P_{N_1} u_1 P_{N_j} u_j\|_{L^2_{x,t}} \prod_{\substack{2 \leq i \leq 4 \\ i \neq j}} \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_5} u_5\|_{L^\infty_{x,t}} \\ &\lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \|P_{N_1} u_1\|_{X_{N_1}} \|P_{N_j} u_j\|_{X_{N_2}} \prod_{\substack{2 \leq i \leq 4 \\ i \neq j}} \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_5} u_5\|_{L^\infty_t L^2_x} \\ &\lesssim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_j}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality (3.13) on $\prod_{i=1}^{j-1} c_{N_i,i}$ and (3.14) on $\prod_{i=j}^{5} c_{N_i,i}$, we see that

$$\sum_{N_i} \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_j}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i} \lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+1}} c^2_{N_1,1} \Big)^{\frac{1}{2}} \|u\|^8_{\dot{X}^{\frac{1}{4}}}$$

Therefore,

$$\sum_{III} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|^2_{L^1_x L^2_t} \lesssim \Big(\sum_N \sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2s+1} c^2_{N_1,1} \Big) \|u\|^8_{\dot{X}^{\frac{1}{4}}}$$

$$\sim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8.$$

IV). $N \lesssim N_1 \sim N_2 \sim N_3 \sim N_4 \gg N_5$.

In this case, we will take the number of complex conjugates in $u_1u_2u_3u_4$ into consideration. Note that the positions of conjugates does not matter here.

1). $u_1 = u_3 = u$ and $u_2 = u_4 = \bar{u}$. We divide into further subcases by comparing the sizes between N and N_5.

1.1). $N \sim N_5$.

We use Hölder inequality and apply the bilinear estimate (2.66) to $\|P_{N_1}u_1P_{N_5}u_5\|_{L^2_{x,t}}$

$$\begin{aligned}
&\Big\|P_N \prod_{i=1}^5 u_i\Big\|_{L^1_xL^2_t} \\
&\lesssim \sum_{N_i} \|P_{N_1}u_1P_{N_5}u_5\|_{L^2_{x,t}} \prod_{i=2}^3 \|P_{N_i}u_i\|_{L^4_xL^\infty_t} \|P_{N_4}u_4\|_{L^\infty_{x,t}} \\
&\lesssim \sum_{N_i} \frac{N_4^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \|P_{N_1}u_1\|_{X_{N_1}} \|P_{N_5}u_5\|_{X_{N_5}} \prod_{i=2}^3 \|P_{N_i}u_i\|_{L^4_xL^\infty_t} \|P_{N_4}u_4\|_{L^\infty_tL^2_x} \\
&\lesssim \sum_{N_i} \frac{N_4^{\frac{1}{4}}}{N_1^{s+\frac{1}{2}}N_5^{\frac{1}{4}}} \prod_{i=1}^5 c_{N_i,i} \\
&\sim \sum_{N_i} \frac{1}{N_1^{s+\frac{1}{4}}N^{\frac{1}{4}}} \prod_{i=1}^5 c_{N_i,i} \\
&\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+\frac{1}{2}}N^{\frac{1}{2}}} c_{N_1,1}^2\Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4,
\end{aligned}$$

where we used Cauchy-Schwarz, the fact that $N_4 \sim N_1$, $N \sim N_5$ and the trivial inequality $c_{N_5,5} \leq \|u\|_{\dot{X}^{\frac{1}{4}}}$ in the last step. Consequently,

$$\sum_{\substack{IV \\ N\sim N_5}} N^{2s+1}\Big\|P_N \prod_{i=1}^5 u_i\Big\|_{L^1_xL^2_t}^2 \lesssim \Big(\sum_N \sum_{N_1\gtrsim N} \Big(\frac{N}{N_1}\Big)^{2s+\frac{1}{2}} c_{N_1,1}^2\Big)\|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\sim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8.$$

1.2). $N \gg N_5$.

We split $\prod_{i=1}^5 P_{N_i} u_i$ into four terms using low and high frequency projections.

$$P_{N_1} u_1 P_{N_2} u_2 = P_{\ll N}(P_{N_1} u_1 P_{N_2} u_2) + P_{\gtrsim N}(P_{N_1} u_1 P_{N_2} u_2),$$

$$P_{N_3} u_3 P_{N_4} u_4 = P_{\ll N}(P_{N_3} u_3 P_{N_4} u_4) + P_{\gtrsim N}(P_{N_3} u_3 P_{N_4} u_4).$$

Since $N \gg N_5$, so $\prod_{i=1}^4 P_{N_i} u_i$ must be at frequency $\gg N$. Thus, we can assume that each of the resulting terms after the splits contains at least one high frequency projection. Thus, it suffices to estimate $P_{\gtrsim N}(P_{N_1} u_1 P_{N_2} u_2) \prod_{i=3}^5 P_{N_i} u_i$. We use the bilinear estimate (2.65) on $P_{\gtrsim N}(P_{N_1} u_1 P_{N_2} u_2)$,

$$\|P_{\gtrsim N}(P_{N_1} u_1 P_{N_2} u_2)\|_{L^2_{x,t}} \lesssim \frac{1}{N^{\frac{1}{2}}} \|P_{N_1} u\|_{X_{N_1}} \|P_{N_2} u\|_{X_{N_2}}. \tag{3.19}$$

Then, by applying the estimate (3.13) on $c_{N_1,1} c_{N_3,3} c_{N_4,4}$ and (3.14) on $c_{N_2,2} c_{N_5,5}$, we obtain

$$\begin{aligned}
&\Big\| P_N \prod_{i=1}^5 u_i \Big\|_{L^1_x L^2_t} \\
&\lesssim \sum_{N_i} \|P_{\gtrsim N}(P_{N_1} u_1 P_{N_2} u_2)\|_{L^2_{x,t}} \prod_{i=3}^4 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_5} u_5\|_{L^\infty_{x,t}} \\
&\lesssim \sum_{N_i} \frac{N_5^{\frac{1}{2}}}{N^{\frac{1}{2}}} \|P_{N_1} u\|_{X_{N_1}} \|P_{N_2} u\|_{X_{N_2}} \prod_{i=3}^4 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_5} u_5\|_{L^\infty_t L^2_x} \qquad (3.20) \\
&\lesssim \sum_{N_i N} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_2}\right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i} \\
&\sim \sum_{N_i N} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i} \\
&\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N N_1^{2s}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4,
\end{aligned}$$

where we used the Cauchy-Schwarz on $\sum_{N_i} \frac{1}{N^{\frac{1}{2}} N_1^s} c_{N_1,1} c_{N_2,2}$ and the Young's inequality on $\sum_{N_i} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} c_{N_3,3} c_{N_4,4} c_{N_5,5}$. As a result,

$$\sum_{\substack{IV \\ N \gg N_5}} N^{2s+1} \Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2}^2 \lesssim \Big(\sum_N \sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2s} c_{N_1,1}^2 \Big) \|u\|_{\dot{X}^{\frac{1}{4}}}^8$$

$$\sim \|u\|_{\dot{X}^s}^2 \|u\|_{\dot{X}^{\frac{1}{4}}}^8.$$

1.3). $N \ll N_5$.

This is similar to case 1.2), but we split $\prod_{i=1}^5 P_{N_i} u_i$ at N_5 instead of N.

$$P_{N_1} u_1 P_{N_2} u_2 = P_{\ll N_5}(P_{N_1} u_1 P_{N_2} u_2) + P_{\gtrsim N_5}(P_{N_1} u_1 P_{N_2} u_2),$$

$$P_{N_3} u_3 P_{N_4} u_4 = P_{\ll N_5}(P_{N_3} u_3 P_{N_4} u_4) + P_{\gtrsim N_5}(P_{N_3} u_3 P_{N_4} u_4).$$

Since the output is supported at frequency $N \ll N_5$, we can see that $\prod_{i=1}^4 P_{N_i} u_i$ must be supported at frequency $\sim N_5$. Thus, we can assume that each term in the product expansion contains at least one high frequency projection. To estimate the product, we can use (3.19) and (3.20) that we just obtained and replace $N^{-\frac{1}{2}}$ by $N_5^{-\frac{1}{2}}$.

$$\Big\| P_N \prod_{i=1}^{5} u_i \Big\|_{L_x^1 L_t^2} \lesssim \sum_{N_i} \frac{1}{N_5^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\ll \sum_{N_i} \frac{1}{N^{\frac{1}{2}} N_1^s} \left(\frac{N_5}{N_3}\right)^{\frac{1}{4}} \prod_{i=1}^{5} c_{N_i,i}$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N N_1^{2s}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u\|_{\dot{X}^{\frac{1}{4}}}^4,$$

which leads to the same result as in the previous case.

2). $u_1 = u_2 = u_3 = u$, u_4 and u_5 can be either u or $\bar{u}$.

This is the hardest case and requires some frequency-modulation analysis. Suppose that for some $1 \leq j \leq 5$ the space-time Fourier transform of $P_{N_j} u$ is supported in the set

$$\{(\xi, \tau) : |\tau + N_1^2| > \frac{1}{32} N_1^2\}, \tag{3.21a}$$

or that of $P_{N_j} \bar{u}$ (for $4 \leq j \leq 5$) is supported in the set

$$\{(\xi, \tau) : |\tau - N_1^2| > \frac{1}{32} N_1^2\}. \tag{3.21b}$$

Then, (2.73) yields

$$\|P_{N_j} u_j\|_{L^2_{x,t}} \lesssim N_1^{-1} \|P_{N_j} u_j\|_{\dot{X}^{0,\frac{1}{2},\infty}} \lesssim N_1^{-1} \|P_{N_j} u_j\|_{X_{N_j}}.$$

Without loss of generality, assume that $j = 1$. Then by Hölder and Bernstein inequalities,

$$\begin{aligned} \Big\| P_N \prod_{i=1}^5 P_{N_i} u_i \Big\|_{L^1_x L^2_t} &\lesssim \|P_{N_1} u_1\|_{L^2_{x,t}} \prod_{i=2}^3 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \prod_{i=4}^5 \|P_{N_i} u_i\|_{L^\infty_{x,t}} \\ &\lesssim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_4 N_5}{N_1^2} \right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i} \\ &\sim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_3} \right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i}. \end{aligned}$$

On the other hand, if the space-time Fourier transform of $P_{N_5} u_5$ is supported in the set (3.21a) in the case $u_5 = u$ or (3.21b) in the case $u_5 = \bar{u}$, then we have

$$\begin{aligned} \Big\| P_N \prod_{i=1}^5 P_{N_i} u_i \Big\|_{L^1_x L^2_t} &\lesssim \prod_{i=1}^2 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_3} u_3 P_{N_4} u_4 P_{N_5} u_5\|_{L^2_{x,t}} \\ &\lesssim \prod_{i=1}^2 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \prod_{i=3}^4 \|P_{N_i} u_i\|_{L^\infty_t L^4_x} \|P_{N_5} u_5\|_{L^2_t L^\infty_x} \\ &\lesssim N_5^{\frac{1}{2}} \prod_{i=1}^4 \|P_{N_i} u_i\|_{L^4_x L^\infty_t} \|P_{N_5} u_5\|_{L^2_{x,t}} \\ &\lesssim \frac{N_5^{\frac{1}{4}}}{N_1^{s+\frac{3}{4}}} \prod_{i=1}^5 c_{N_i,i} \\ &\sim \frac{1}{N_1^{s+\frac{1}{2}}} \left(\frac{N_5}{N_3} \right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i}. \end{aligned}$$

We then get the desired result by observing that

$$\frac{1}{N_1^{s+\frac{1}{2}}}\left(\frac{N_5}{N_3}\right)^{\frac{1}{4}}\prod_{i=1}^{5}c_{N_i,i}\lesssim\Big(\sum_{N_1\gtrsim N}\frac{1}{N_1^{2s+1}}c_{N_1,1}^2\Big)^{\frac{1}{2}}\|u\|_{\dot X^{\frac{1}{4}}}^4.$$

Thus, we can assume that the space-time Fourier transform of $P_{N_j}u$ is supported in the set

$$\{\xi,\tau:|\tau+N_1^2|\leq\frac{1}{32}N_1^2\},$$

and that of $P_{N_k}\bar{u}$ is supported in

$$\{\xi,\tau:|\tau-N_1^2|\leq\frac{1}{32}N_1^2\}.$$

Here, we introduce Riesz transforms P_+ and P_- defined by

$$\widehat{P_+f}(\xi)=1_{\xi\geq 0}\hat{f},\qquad\widehat{P_-f}(\xi)=1_{\xi<0}\hat{f}.$$

Then, denoting $P_+P_{N_i}:=P_{N_i}^+$ and $P_-P_{N_i}:=P_{N_i}^-$, for $1\leq i\leq 4$, we decompose $P_{N_i}u_i$ into

$$P_{N_i}u_i=P_{N_i}^+u_i+P_{N_i}^-u_i,$$

and consider all the terms that we get from $\prod_{i=1}^5P_{N_i}u_i$. For any term that contains $P_{N_j}^+uP_{N_k}^-u$, $P_{N_j}^+uP_{N_k}^+\bar{u}$ or $P_{N_j}^-uP_{N_k}^-\bar{u}$, where $1\leq j<k\leq 4$, we can apply the bilinear estimates (2.65) and (2.66), then proceed with the Hölder's and Bernstein inequality on $L_x^1L_t^2$ as in the previous cases. For example, if $j=1$ and $k=2$, then we have

$$\begin{aligned}&\Big\|P_N(P_{N_1}^+u_1P_{N_2}^-u_2\prod_{i=3}^{5}P_{N_i}u_i)\Big\|_{L_x^1L_t^2}\\&\lesssim\frac{N_5^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}\prod_{i=1}^{2}\|P_{N_i}u\|_{X_{N_i}}\prod_{i=3}^{4}\|P_{N_i}u_i\|_{L_x^4L_t^\infty}\|P_{N_5}u_5\|_{L_t^\infty L_x^2}\\&\lesssim\frac{1}{N_1^{s+\frac{1}{2}}}\left(\frac{N_5}{N_2}\right)^{\frac{1}{4}}\prod_{i=1}^{5}c_{N_i,i}\\&\sim\frac{1}{N_1^{s+\frac{1}{2}}}\left(\frac{N_5}{N_3}\right)^{\frac{1}{4}}\prod_{i=1}^{5}c_{N_i,i},\end{aligned}$$

Therefore, it suffices to consider the following four terms.

(i) $(\prod_{i=1}^3 P_{N_i}^+ u) P_{N_4}^+ u P_{N_5} u_5$

(ii) $(\prod_{i=1}^3 P_{N_i}^- u) P_{N_4}^- u P_{N_5} u_5$

(iii) $(\prod_{i=1}^3 P_{N_i}^+ u) P_{N_4}^- \bar{u} P_{N_5} u_5$

(iv) $(\prod_{i=1}^3 P_{N_i}^- u) P_{N_4}^+ \bar{u} P_{N_5} u_5$

In either case, simple algebra shows that the space-time Fourier transform of the product is supported at least $\gtrsim N_1^2$ away from the parabola $\tau = -\xi^2$. The worst case is (iii) with $u_5 = u$ where the output's modulation is

$$(3N_1 - N_1 \pm N_5)^2 - 4N_1^2 + N_1^2 \sim N_1^2.$$

Thus, we can put these products in the $\dot{X}^{0,-\frac{1}{2},1}$ space and get a good bound. For example, focusing on (iii), we use Hölder inequality, Bernstein inequality and the boundedness of Riesz transforms.

$$\begin{aligned}
&\Big\| P_N [(\prod_{i=1}^3 P_{N_i}^+ u) P_{N_4}^- \bar{u} P_{N_5} u_5] \Big\|_{\dot{X}^{0,-\frac{1}{2},1}} \\
&\qquad \lesssim \frac{1}{N_1} \Big\| (\prod_{i=1}^3 P_{N_i}^+ u) P_{N_4}^- \bar{u} P_{N_5} u_5 \Big\|_{L^2_{t,x}} \\
&\qquad \lesssim \frac{(N_4 N_5)^{\frac{1}{2}}}{N_1} \prod_{i=1}^3 \| P_{N_i} u \|_{L^6_{t,x}} \prod_{i=4}^5 \| P_{N_i} u \|_{L^\infty_t L^2_x} \\
&\qquad \lesssim \frac{1}{N_1^{s+1}} \left(\frac{N_5}{N_1} \right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i} \\
&\qquad \sim \frac{1}{N_1^{s+1}} \left(\frac{N_5}{N_3} \right)^{\frac{1}{4}} \prod_{i=1}^5 c_{N_i,i} \\
&\qquad \lesssim \Big(\sum_{N_1 \gtrsim N} \frac{1}{N_1^{2s+2}} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \| u \|^4_{\dot{X}^{\frac{1}{4}}}.
\end{aligned}$$

Hence, by summing over N and N_i's, we have

$$\sum_{IV} N^{2s+2}\Big\|P_N[(\prod_{i=1}^{3} P^+_{N_i}u)P^-_{N_4}\bar{u}P_{N_5}u_5]\Big\|^2_{\dot{X}^{0,-\frac{1}{2},1}}$$

$$\lesssim \sum_{N_1}\sum_{N\lesssim N_1}\left(\frac{N}{N_1}\right)^{2s+2} c^2_{N_1,1}\|u\|^8_{\dot{X}^{\frac{1}{4}}}$$

$$\lesssim \|u\|^2_{\dot{X}^s}\|u\|^8_{\dot{X}^{s_0}},$$

as desired.

3). $u_1 = u_2 = u_3 = \bar{u}$, u_4 and u_5 can be either u or $\bar{u}$.

The proof is the same as in the previous case. Note that we get a better result in the sense that the space-time Fourier support of $\prod_{i=1}^5 P_{N_i}u_i$ when $\mathcal{F}_{x,t}u_i$ is supported in (3.22) is $\gtrsim N_1^2$ away from the parabola $\tau = -\xi^2$ without relying on the Riesz transforms. This concludes the proof of the multilinear estimate.

□

3.4 Proof of Theorem 1.2

The proof is similar to what we did in Section 3.2 with the same function spaces:

$$\begin{aligned}
\|u\|_{X_N} &= \|u\|_{L^\infty_t L^2_x} + N^{-\frac{1}{4}}\|u\|_{L^4_x L^\infty_t} + N^{\frac{1}{2}}\|u\|_{L^\infty_x L^2_t} \\
&\quad + N^{-\frac{1}{2}}\|(i\partial_t + \Delta)u\|_{L^1_x L^2_t} \\
\|u\|_{\dot{X}^s} &= \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|^2_{X_N}\Big)^{\frac{1}{2}} \\
\|u\|_{X^s} &= \|u\|_{\dot{X}^0} + \|u\|_{\dot{X}^s} \qquad (3.23) \\
\|u\|_{Y_N} &= N^{-\frac{1}{2}}\|u\|_{L^1_x L^2_t}
\end{aligned}$$

$$\|u\|_{\dot{Y}^s} = \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N u\|_{Y_N}^2\Big)^{\frac{1}{2}}$$

$$\|u\|_{Y^s} = \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.$$

Now we state a multilinear estimate. The proof is shortened as it is similar to that of Theorem 3.2 for the most part.

Theorem 3.4. *Suppose that $d \geq 5$. Let $s, r > \frac{1}{2}$ and $u_i \in X^s$ for $1 \leq i \leq d$. Then we have the following estimate:*

$$\Big\|(\partial_x u_1)\prod_{i=2}^{d} u_i\Big\|_{Y^r} \lesssim \|u_1\|_{X^r}\prod_{i=2}^{d}\|u_i\|_{X^s}, \tag{3.24}$$

Proof. Again, we study the frequency interactions with N being the output frequency and $N_1 \geq N_2 \geq \ldots \geq N_d$ being the input frequencies. For $s > \frac{1}{2}$, we define $c_{N_1,1} = \|P_{N_1}u_1\|_{X_{N_1}}$ and $c_{N_i,i} = \|P_{N_i}u_i\|_{X_{N_i}}$ for $2 \leq i \leq d$. We consider the usual $High \times Low \to High$ and $High \times High \to Low$ interactions:

1. $N \sim N_1 \gg N_2 \geq \ldots \geq N_d$.

   With some abuse of notations, we define $\prod_{i=5}^{d-1} A_i = 1$ if $d = 5$. By Hölder inequality, Young's inequality and the continuous embedding $X^s \hookrightarrow X^{s'} \hookrightarrow \dot{X}^{s'}$ for any $s' > s > \frac{1}{2}$,

$$\begin{aligned}
&\Big\|N^{r-\frac{1}{2}}P_N[(P_{N_1}\partial_x u_1)\prod_{i=2}^{d}P_{N_i}u_i]\Big\|_{L_x^1L_t^2} \\
&\quad\lesssim N^{r-\frac{1}{2}}\sum_{N_i}\|P_{N_1}\partial_x u_1\|_{L_x^\infty L_t^2}\prod_{i=2}^{4}\|P_{N_i}u_i\|_{L_x^4L_t^\infty}\prod_{i=5}^{d-1}\|P_{N_i}u_i\|_{L_{x,t}^\infty}\|P_{N_d}u_d\|_{L_x^4L_t^\infty} \\
&\quad\lesssim \sum_{N_i}\Big(\frac{N}{N_1}\Big)^{r-\frac{1}{2}}\Big(\frac{N_d}{N_2}\Big)^{\frac{1}{4}}c_{N_1,1}(N_2^{\frac{1}{2}}c_{N_2,2})c_{N_d,d}\prod_{i=3}^{4}N_i^{\frac{1}{4}}c_{N_i,i}\prod_{i=5}^{d-1}N_i^{\frac{1}{2}}c_{N_i,i}
\end{aligned}$$

$$\lesssim \sum_{N_1 \sim N} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} c_{N_1,1} \|u_2\|_{\dot{X}^{\frac{1}{2}}} \prod_{i=3}^{4} \|u_i\|_{\dot{X}^{\frac{1}{4}}} \prod_{i=5}^{d-1} \|u_i\|_{\dot{X}^{\frac{1}{2}}} \|u_d\|_{\dot{X}^0}$$

$$\lesssim \sum_{N_1 \sim N} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} c_{N_1,1} \prod_{i=2}^{d} \|u_i\|_{X^s}.$$

Take the l^2 summation and (3.24) follows.

2. $N \lesssim N_1 \sim N_2 \geq \ldots \geq N_d$.

This is similar to the previous case, but we apply Cauchy-Schwarz to $\sum_i c_{N_1,1} c_{N_2,2}$.

$$N^{r-\frac{1}{2}} \Big\| P_N[(P_{N_1}\partial_x u_1) \prod_{i=2}^{d} P_{N_i} u_i] \Big\|_{L_x^1 L_t^2}$$

$$\lesssim \sum_{N_i} \left(\frac{N}{N_1}\right)^{r-\frac{1}{2}} \left(\frac{N_d}{N_3}\right)^{\frac{1}{4}} c_{N_1,1} (N_2^{\frac{1}{4}} c_{N_2,2})(N_3^{\frac{1}{2}} c_{N_3,3})(N_4^{\frac{1}{4}} c_{N_4,4}) c_{N_d,d} \prod_{i=5}^{d-1} (N_i^{\frac{1}{2}} c_{N_i,i})$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2r-1} c_{N_1,1}^2 \Big)^{\frac{1}{2}} \|u_2\|_{\dot{X}^{\frac{1}{4}}} \|u_3\|_{\dot{X}^{\frac{1}{2}}} \|u_4\|_{\dot{X}^{\frac{1}{4}}} \prod_{i=5}^{d-1} \|u_i\|_{\dot{X}^{\frac{1}{2}}} \|u_d\|_{\dot{X}^0}$$

$$\lesssim \Big(\sum_{N_1 \gtrsim N} \left(\frac{N}{N_1}\right)^{2r-1} \|P_{N_1} u_1\|_{X_{N_1}}^2 \Big)^{\frac{1}{2}} \prod_{i=2}^{d} \|u_i\|_{X^s}.$$

Take the l^2 summation to obtain (3.24).

□

The proof of Theorem 1.2 part (A) now follows the same contraction argument as before. To prove part (B) of the theorem, we replace u_j by $\partial_x u_j$ for some $j \geq 2$, and it follows from (2.2) that $\|\partial_x u_i\|_{X^s} \lesssim \|u_i\|_{X^{s+1}}$ for any $s > \frac{1}{2}$. Hence, (3.24) implies that for any $s > \frac{3}{2}$,

$$\Big\| (\partial_x u_1)(\partial_x u_j) \prod_{\substack{i=2 \\ i \neq j}}^{d} u_i \Big\|_{Y^s} \lesssim \|u_1\|_{X^s} \|\partial_x u_j\|_{X^{s-1}} \prod_{\substack{i=2 \\ i \neq j}}^{d} \|u_i\|_{X^{s-1}}$$

$$\lesssim \prod_{i=1}^{d} \|u_i\|_{X^s}.$$

Consequently, in the case that a term in $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ has more than one derivative, we can employ the contraction argument in X^s.

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Chapter 4

Global bounds and modified scattering for DNLS

As promised in Section 1.6, we give the full statement of our result here.

Theorem 4.1. *A) (Global bounds) Assume that*

$$\|u_0\|_{H^{1,1}} \leq \epsilon \ll 1. \tag{4.1}$$

Then the equation (1.15) *with the initial data* u_0 *has a global solution satisfying the pointwise bounds*

$$\|u\|_{L^\infty} \lesssim \epsilon |t|^{-1/2}, \tag{4.2}$$

$$\|u_x\|_{L^\infty} \lesssim \epsilon |t|^{-1/2}. \tag{4.3}$$

B) (Asymptotic profiles) Let u *be a solution to* (1.15)*, then there exists a function*

$W \in H^{1-C\epsilon^2,1}(\mathbb{R})$ such that

$$u(x,t) = \frac{1}{t^{1/2}} W\left(\frac{x}{t}\right) e^{i|W(x/t)|^2 \frac{x}{t}\log t + i\frac{x^2}{2t}} + err_1(x,t), \tag{4.4}$$

$$\hat{u}(x,t) = \frac{1}{t^{1/2}} W(\xi) e^{i|W(\xi)|^2 \frac{x}{t}\log t - i\frac{t\xi^2}{2}} + err_2(x,t), \tag{4.5}$$

$$u_x(x,t) = \frac{ix}{t^{3/2}} W\left(\frac{x}{t}\right) e^{i|W(x/t)|^2 \frac{x}{t}\log t + i\frac{x^2}{2t}} + err_3(x,t), \tag{4.6}$$

where

$$\|err_1\|_{L^\infty_x} \lesssim \epsilon(1+t)^{-\frac{3}{4}+C\epsilon^2}, \qquad \|err_1\|_{L^2_x} \lesssim \epsilon(1+t)^{-\frac{1}{4}+C\epsilon^2}, \tag{4.7}$$

$$\|err_2\|_{L^\infty_x} \lesssim \epsilon(1+t)^{-1+C\epsilon^2}, \qquad \|err_2\|_{L^2_x} \lesssim \epsilon(1+t)^{-1+C\epsilon^2}, \tag{4.8}$$

$$\|err_3\|_{L^\infty_x} \lesssim \epsilon(1+t)^{-\frac{1}{2}+C\epsilon^2}, \qquad \|err_3\|_{L^2_x} \lesssim \epsilon(1+t)^{-1+C\epsilon^2}. \tag{4.9}$$

C) (Asymptotic completeness) Let C be a large constant and W be a function satisfying

$$\|W\|_{H^{1+C\epsilon^2,1}(\mathbb{R})} \ll \epsilon \ll 1. \tag{4.10}$$

Then there exists a function u_0 satisfying (4.1) *such that the equation* (1.15) *with u_0 as the initial data has the solution u with the profile* (4.4),(4.5) *and* (4.6).

Instead of directly tackling (1.15) which involves a derivative in the nonlinearlity, we can decouple it into two cubic NLS equations by defining $u_1 := u\exp(-2i\int_{-\infty}^{x}|u|^2\, dx')$ and $u_2 = \frac{1}{\sqrt{2}}(\partial_x u_1 + i|u_1|^2 u_1)$. We then obtain a system of equations:

$$\begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_{xx} u_1 = -i\sqrt{2}\overline{u}_2 u_1^2 \\ i\partial_t u_2 + \frac{1}{2}\partial_{xx} u_2 = i\sqrt{2}\overline{u}_1 u_2^2 \end{cases} \tag{4.11}$$

This is where we apply the technique from [28], originally employed to study the cubic nonlinear Schrödinger equation:

$$i\partial_t u + \frac{1}{2}\partial_{xx} u = u|u|^2, \qquad u(x,0) = u_0(x).$$

The main idea is to consider the dynamic of the solutions along wave packets $\overline{\Psi}_v$ (to be precisely defined in Section 4.2) traveling with velocity v:

$$\gamma(t,v) := \int u\overline{\Psi}_v \, dx.$$

Since $\overline{\Psi}_v$ is localized around the ray $\Gamma_v := \{x = vt\}$, we can think of $\gamma(t,v)$ as the decay of u along Γ_v. We can then study the dynamic of γ in order to construct a scattering profile for u. To see this technique employed for other equations, see [19], [20], [21], [29], [30], [41] and [42].

We adapt this idea to 4.11 by studying the simultaneous dynamics of u_1 and u_2 along the Γ_v:

$$\gamma_i(t,v) := \int u_i\overline{\Psi}_v \, dx, \qquad i = 1, 2.$$

The ode dynamics for γ_1 and γ_2 will then be used to construct a profile for u_1, u_2 and finally for u.

Note that the space $H^{1,1}$ involves the x operator $f \mapsto xf$ which does not commute with the Schödinger flow, preventing us from applying the usual perturbative argument. To resolve this issue, we introduce a new operator L defined by

$$Lu := (x + it\partial_x)u.$$

Now for the operator P that defines the Schrödinger equation

$$Pu := (i\partial_t + \frac{1}{2}\partial_{xx})u,$$

it is easy to check that $PL = LP$. Moreover, L is a conjugate of x and $t\partial_x$ with respect to the linear flow and multiplication by $e^{\frac{ix^2}{2t}}$, respectively:

$$Le^{it\Delta/2}f = e^{it\Delta/2}xf$$
$$Le^{\frac{ix^2}{2t}}f = te^{\frac{ix^2}{2t}}\partial_x f.$$

We see that the exponential factors on the right-hand side will disappear after taking L^2 norms by the duality. This suggests that the issue can be overcome by analyzing the equations for Lu_1 and Lu_2, as we shall see later in this chapter.

4.1 Local theory

In this section we address the local in time well-posedness of the system (4.11). To fix things, local in time refers to the time interval $[0,1]$. We also reiterate that the data is assumed to be small, that is $\|u_0\|_{H^{1,1}} \ll 1$.

Proposition 4.2. *Let $\mathcal{S} = L_t^4 L_x^\infty \cap L_t^\infty L_x^2$. Assume that $\|u_0\|_{H^1} < \epsilon < 1$. The system* (4.11) *is locally well-posed with the solutions satisfying*

$$\|\partial_x u_1\|_{\mathcal{S}} + \|u_1\|_{\mathcal{S}} + \|u_2\|_{\mathcal{S}} \lesssim \|u_0\|_{H^1}. \tag{4.12}$$

Proof. Note that $|u_1(x,0)| = |u_0(x)|$ and $|u_2(x,0)| \lesssim \frac{1}{\sqrt{2}}(|\partial_x u_0(x)| + |u_0(x)|^3)$. By the standard energy estimate for the cubic NLS, we obtain

$$\begin{aligned}\|u_1\|_{\mathcal{S}} &\lesssim \|u_0\|_{L^2} + \|\overline{u}_2 u_1^2\|_{L_t^1 L_x^2} \\ &\lesssim \|u_0\|_{L^2} + \|u_1\|_{L_t^4 L_x^\infty}\|u_2\|_{L_t^4 L_x^\infty}\|u_1\|_{L_t^\infty L_x^2}.\end{aligned}$$

Similarly,

$$\|u_2\|_{\mathcal{S}} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^\infty L_x^2}.$$

We can combine these two estimates to obtain a linear estimate for $(u_1, u_2) \in \mathcal{S} \times \mathcal{S}$:

$$\|u_1\|_{\mathcal{S}} + \|u_2\|_{\mathcal{S}} \lesssim \|u_0\|_{H^1} + (\|u_1\|_{\mathcal{S}} + \|u_2\|_{\mathcal{S}})^3.$$

For the estimate of $\|\partial_x u_1\|_S$, we use the relation $u_1 = \sqrt{2}u_2 - i|u_1|^2 u_1$ and Bernstein's inequality:

$$\begin{aligned}
\|\partial_x u_1\|_S &\lesssim \|u_2\|_S + \||u_1|^3\|_S \\
&\leq \|u_2\|_S + \|u_1\|_{L_{t,x}^\infty}^2 \|u_1\|_S \\
&\lesssim \|u_2\|_S + \|u_1\|_{L_t^\infty H_x^{1/2}}^2 \|u_1\|_S \\
&\leq \|u_2\|_S + \|u_1\|_{L_t^\infty H_x^1}^2 \|u_1\|_S \\
&\leq \|u_2\|_S + \|\partial_x u_1\|_S^2 \|u_1\|_S.
\end{aligned}$$

The local-wellposedness of (4.11) then follows from the usual contraction argument on the space

$$\{(u_1, u_2) \in S \times S : \|\partial_x u_1\|_{\mathcal{S}} + \|u_1\|_{\mathcal{S}} + \|u_2\|_{\mathcal{S}} \leq C\}$$

for some small constant C. □

By applying the operator L to the equations, we see that Lu_1 and Lu_2 satisfy the following equations:

$$\begin{cases} PLu_1 = -i2\sqrt{2}u_1\overline{u_2}Lu_1 + i\sqrt{2}u_1^2\overline{Lu_2} \\ PLu_2 = i2\sqrt{2}u_2\overline{u_1}Lu_2 - i\sqrt{2}u_2^2\overline{Lu_1} \end{cases} \tag{4.13}$$

With the assumption that $t \leq 1$, the energy estimate yields the following proposition:

Proposition 4.3. *Assume that $t \leq 1$ and $\|u_0\|_{H^{1,1}} < \epsilon < 1$. Then Lu_1 and Lu_2 satisfy the following estimate:*

$$\|Lu_1\|_{L_t^\infty L_x^2} + \|Lu_2\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}}. \tag{4.14}$$

Proof. Using standard energy estimates we obtain the following:

$$\begin{aligned} \|Lu_2\|_{L_t^\infty L_x^2} &\lesssim \|xu_2(0)\|_{L^2} + \|u_2\overline{u_1}Lu_2 + u_2^2\overline{Lu_1}\|_{L_t^1 L_x^2} \\ &\lesssim \|xu_2(0)\|_{L^2} + \|u_1\|_{L_t^4 L_x^\infty}\|u_2\|_{L_t^4 L_x^\infty}\|Lu_2\|_{L_t^\infty L_x^2} \\ &\quad + \|u_2\|_{L_t^4 L_x^\infty}^2\|Lu_1\|_{L_t^\infty L_x^2}. \end{aligned}$$

Combining this with (4.12) and the smallness assumptions gives

$$\|Lu_2\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}} + \|u_2\|_{L_t^4 L_x^\infty}^2\|Lu_1\|_{L_t^\infty L_x^2}$$

In a similar manner,

$$\begin{aligned} \|Lu_1\|_{L_t^\infty L_x^2} &\lesssim \|xu_1(0)\|_{L^2} + \|u_1\|_{L_t^4 L_x^\infty}\|u_2\|_{L_t^4 L_x^\infty}\|Lu_1\|_{L_t^\infty L_x^2} \\ &\quad + \|u_1\|_{L_t^4 L_x^\infty}^2\|Lu_2\|_{L_t^\infty L_x^2} \end{aligned}$$

leads to

$$\|Lu_1\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}} + \|u_1\|_{L_t^4 L_x^\infty}^2\|Lu_2\|_{L_t^\infty L_x^2}.$$

Invoking the smallness of $\|u_1\|_{L_t^4 L_x^\infty}$ and $\|u_2\|_{L_t^4 L_x^\infty}$, allows us to conclude with (4.14).

□

We turn our attention to estimating $\partial_x Lu_1$. By taking the derivative to the first equation in 4.13, we have an equation for $\partial_x(Lu_1)$. To deal with the nonlinear terms for this equation, we will need the following lemma

Lemma 4.4. *Let* $\Gamma_1 := \{(N_1, N_2, N_3) \in (2^{\mathbb{Z}})^3 : N_3 \leq 8 \max\{N_1, N_2\}\}$*. Then, the following estimate is true for all* $u_1, u_2, u_3 \in L^2 \cap L^\infty$*:*

$$\begin{aligned} \|S\|_{L^2} := & \left\| \sum_{(N_1,N_2,N_3)\in\Gamma_1} P_{N_1} u_1 P_{N_2} u_2 P_{N_3} \partial_x u_3 \right\|_{L^2} \\ & \lesssim (\|\partial_x u_1\|_{l^2 L^\infty} \|u_2\|_{l^2 L^\infty} + \|u_1\|_{l^2 L^\infty} \|\partial_x u_2\|_{l^2 L^\infty}) \|u_3\|_{L^2}, \end{aligned} \tag{4.15}$$

$$\|S\|_{L^2} \lesssim \|\partial_x u_1\|_{l^2 L^\infty} \|u_2\|_{L^2} \|u_3\|_{l^2 L^\infty} + \|u_1\|_{l^2 L^\infty} \|\partial_x u_2\|_{L^2} \|u_3\|_{l^2 L^\infty}. \tag{4.16}$$

Proof. We consider several subsets of Γ_1, estimate their contribution and show a bound as above.

i) $N_1 \leq 2^{-10} N_2$ and $N_3 \leq 2^{-10} N_2$: For each fixed $N_2 \in 2^{\mathbb{Z}}$, we have

$$\begin{aligned} \left\| \sum_{\substack{N_3 \leq 8N_2 \\ N_1 \leq N_2}} P_{N_2} u_1 P_{N_2} u_2 P_{N_3} \partial_x u_3 \right\|_{L^2} & \lesssim \|P_{\leq 2^{-10} N_2} u_1\|_{L^\infty} \|P_{N_2} u_2 P_{\leq 8N_2} \partial_x u_3\|_{L^2} \\ & \lesssim \|u_1\|_{L^\infty} \left[\sum_{M \leq 8N_2} \|P_{N_2} u_2\|_{L^\infty} \|P_M \partial_x u_3\|_{L^2} \right] \\ & \lesssim \|u_1\|_{L^\infty} \|P_{N_2} \partial_x u_2\|_{L^\infty} \left[\sum_{M \leq 8N_2} \frac{M}{N_2} \|P_M u_3\|_{L^2} \right] \\ & \lesssim \|u_1\|_{L^\infty} \|P_{N_2} \partial_x u_2\|_{L^\infty} \|P_{\leq N_2} u_3\|_{L^2} \\ & \lesssim \|u_1\|_{L^\infty} \|P_{N_2} \partial_x u_2\|_{L^\infty} \|u_3\|_{L^2}. \end{aligned}$$

Since

$$\sum_{\substack{N_3 \leq 8N_2 \\ N_1 \leq N_2}} P_{N_1} u_1 P_{N_2} u_2 P_{N_3} \partial_x u_3$$

is supported at frequency $\approx N_2$, the estimate 4.15, follows by summing with respect to N_2.

ii) $N_2 \leq 2^{-10} N_1$ and $N_3 \leq 2^{-10} N_2$: This is similar to the above.

iii) $N_1 \approx N_2$: This case is essentially reducible to $N_1 = N_2$, when we estimate as above to obtain

$$\left\| \sum_{N_3 \leq 8N_2} P_{N_2} u_1 P_{N_2} u_2 P_{N_3} \partial_x u_3 \right\|_{L^2} \lesssim \|P_{N_2} u_1\|_{L^\infty} \|P_{N_2} \partial_x u_2\|_{L^\infty} \|u_3\|_{L^2}.$$

The summation with respect to N_2 is performed in a trivial manner:

$$\left\| \sum_{N_2} \sum_{N_3 \leq 8N_2} P_{N_2} u_1 P_{N_2} u_2 P_{N_3} \partial_x u_3 \right\|_{L^2} \lesssim \|u_1\|_{l^2 L^\infty} \|\partial_x u_2\|_{l^2 L^\infty} \|u_3\|_{L^2}.$$

iv) $N_3 \approx N_2$: This case is essentially reducible to $N_3 = N_2$, when we estimate as above to obtain

$$\left\| \sum_{N_1 \leq 2^{-10} N_2} P_{N_2} u_1 P_{N_2} u_2 P_{N_2} \partial_x u_3 \right\|_{L^2} \lesssim \|u_1\|_{L^\infty} \|P_{N_2} \partial_x u_2\|_{L^\infty} \|P_{N_2} u_3\|_{L^2}.$$

The summation with respect to N_2 is performed in a trivial manner:

$$\left\| \sum_{N_2} \sum_{N_1 \leq 2^{-10} N_2} P_{N_1} u_1 P_{N_2} u_2 P_{N_2} \partial_x u_3 \right\|_{L^2} \lesssim \|u_1\|_{L^\infty} \|\partial_x u_2\|_{l^2 L^\infty} \|u_3\|_{L^2}.$$

This finishes the proof of (4.15); the proof of (4.16) is entirely similar.

□

Lemma 4.5. *Assume that for all $t \in \mathbb{R}$, $u_{1,0}(t)$, $u_{2,0}(t)$, $u_{3,0}(t)$ and $F_1(t)$, $F_2(t)$, $F_3(t)$ are functions in $L^2_x(\mathbb{R})$ and $u_i = u_i(t,x)$ satisfies*

$$\widehat{u}_i(s,\xi) = e^{-is\xi^2} \widehat{u}_{i,0}(\xi) + \int_0^s e^{i(\sigma - s)\xi^2} \widehat{F}_i(\sigma, \xi_i) \, d\sigma. \tag{4.17}$$

for $1 \leq i \leq 3$. Then, we have the following estimate

$$\begin{aligned} \left\| \int_0^t u_1 u_2 \overline{\partial_x u_3} \, ds \right\|_{L^2} &\lesssim B + \sup_t \|u_1(t)\|_{L^2_x} \|u_2(t)\|_{L^2_x} \|u_3(t)\|_{L^2_x} \\ &\quad + \sum_{(i,j,k)=1,2,3} \int_0^t \|F_i(s)\|_{L^2_x} \|u_j(s)\|_{L^2_x} \|u_k(s)\|_{L^2_x} \, ds. \end{aligned} \tag{4.18}$$

where

$$B = \min\Big\{ \int_0^t (\|\partial_x u_1(s)\|_{l^2L_x^\infty} \|u_2(s)\|_{l^2L_x^\infty} + \|u_1(s)\|_{l^2L_x^\infty} \|\partial_x u_2(s)\|_{l^2L_x^\infty}) \|u_3(s)\|_{L_x^2}\ ds,$$

$$\int_0^t \|\partial_x u_1(s)\|_{l^2L_x^\infty} \|u_2(s)\|_{L_x^2} \|u_3(s)\|_{l^2L_x^\infty} + \|u_1(s)\|_{l^2L_x^\infty} \|\partial_x u_2(s)\|_{L_x^2} \|u_3(s)\|_{l^2L_x^\infty}\ ds\Big\}.$$

Proof. We begin by writing $u_1u_2\overline{\partial_x u_3} = \sum_{N_1,N_2,N_3} P_1u_1P_2u_2P_3\overline{\partial_x u_3}$. Then, we split the possible combinations of index into two sets:

$$\Gamma_1 := \{(N_1,N_2,N_3) \in (2^{\mathbb{Z}})^3 : N_3 \leq 8\max\{N_1,N_2\}\}$$

$$\Gamma_2 := \{(N_1,N_2,N_3) \in (2^{\mathbb{Z}})^3 : N_3 \geq 8\max\{N_1,N_2\}\}.$$

The first estimate in (4.18) is obtained immediately by using (4.15) on Γ_1.

For Γ_2, we use (4.17). For simplicity, when we write $u_1u_2\overline{\partial_x u_3}$ and work under the assumption that $\widehat{u}_i$ is supported in the set $\left[\frac{N_i}{2}, 2N_i\right]$ with the condition that $(N_1,N_2,N_3) \in \Gamma_2$. Using integration by parts, we have that for $K(\xi,\xi_1,\xi_2,\xi_3) := \xi^2+\xi_3^2-\xi_2^2-\xi_1^2$ and $S(\xi) = \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \xi_1+\xi_2+\xi_3 = \xi\}$, the following holds true

$$\begin{aligned}
&\int_0^t \mathcal{F}\left(u_1u_2\overline{\partial_x u_3}\right)(s,\xi)\ ds \\
&= -\int_0^t \int_{S(\xi)} e^{isK} e^{is\xi_1^2}\widehat{u}_1(\xi_1) e^{is\xi_2^2}\widehat{u}_2(\xi_2)\overline{e^{is\xi_3^2}\xi_3\widehat{u}_3(-\xi_3)}\ d\xi_1 d\xi_2 d\xi_3\ ds \\
&= \left[-\int_{S(\xi)} \frac{1}{iK} e^{isK} e^{is\xi_1^2}\widehat{u_1}(\xi_1) e^{is\xi_2^2}\widehat{u}_2(\xi_2)\overline{e^{is\xi_3^2}\xi_3\widehat{u}_3(-\xi_3)}\ d\xi_1 d\xi_2 d\xi_3\right]_{s=0}^t \\
&\quad + \int_0^t \int_{S(\xi)} \frac{1}{iK} e^{isK}\partial_s\left\{e^{is\xi_1^2}\widehat{u_1}(\xi_1) e^{is\xi_2^2}\widehat{u}_2(\xi_2)\overline{e^{is\xi_3^2}\xi_3\widehat{u}_3(-\xi_3)}\right\}\ d\xi_1 d\xi_2 d\xi_3\ ds.
\end{aligned}$$

Note that, as a consequence of (4.17), we have

$$\partial_s e^{is\xi_i^2}\widehat{u}_i = \partial_s\left\{u_i(0) + \int_0^s e^{i\sigma\xi^2}\widehat{F}_i(\sigma,\xi_i)\ d\sigma\right\} = e^{is\xi^2}\widehat{F}_i(s,\xi_i).$$

We define $v_i(\xi) = e^{is\xi^2}\hat{u}_i(\xi), G_i = e^{is\xi^2}\hat{F}_i, i = 1,2,\ v_3(\xi) = \overline{e^{is\xi^2}\xi\hat{u}_3(-\xi)}$ and $G_3 = \overline{e^{is\xi^2}\xi\hat{F}_3(-\xi)}$. Using that $K(\xi,\xi_1,\xi_2,\xi_3) \sim \xi_3^2 \gtrsim 1$ on Γ_2, we obtain

$$\begin{aligned}\left\|\int_0^t u_1u_2\overline{\partial_x u_3}\ ds\right\|_{L^2(\Gamma_2)} &\lesssim \sup_t \frac{1}{N_3^2}\|v_1 * v_2 * v_3\|_{L^2} \\ &\quad + \frac{1}{N_3^2}\sum_{(i,j,k)=1,2,3}\int_0^t \|\hat{G}_i * \hat{u}_j * \hat{u}_k\|_{L^2}\ ds \\ &\lesssim \sup_t \frac{1}{N_3^2}\|v_1\|_{L^1}\|v_2\|_{L^1}\|v_3\|_{L^2} \\ &\quad + \frac{1}{N_3^2}\sum_{(i,j,k)=1,2,3}\int_0^t \|\hat{F}_i\|\|v_j\|\|u_k\|_{L^2}\ ds.\end{aligned}$$

We use

$$\begin{aligned}\|v_1 * v_2 * v_3\|_{L^2} \lesssim \|v_1\|_{L^1}\|v_2\|_{L^1}\|v_3\|_{L^2} &\lesssim N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2} \\ &\lesssim N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}N_3\|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2}.\end{aligned}$$

We estimate the same way the terms $\|\hat{G}_i * \hat{u}_j * \hat{u}_k\|_{L^2}$ with the rule that the high frequency terms, v_3 or G_3 are placed in L^2, while the low frequency ones in L^1. Putting these estimates together we obtain the bound

$$\lesssim N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}N_3^{-1}\left(\sup_t \|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2} + \sum_{(i,j,k)=1,2,3}\int_0^t \|F_i\|_{L^2}\|u_j\|_{L^2}\|u_k\|_{L^2}\ ds\right).$$

To obtain the actual contribution coming from Γ_2 to the estimate (4.18), we need to sum with respect to N_1, N_2, N_3 subject to the constraint $N_3 \geq 8\max\{N_1, N_2\}$. The summation with respect to N_1, N_2 is easy due to the gains $(N_1N_3^{-1})^{\frac{1}{2}}$ and $(N_2N_3^{-1})^{\frac{1}{2}}$. The one with respect to N_3 uses the almost orthogonality of the output with respect to N_3; precisely $u_1u_2\overline{\partial_x u_3}$ is supported at frequency $\approx N_3$.

□

Now we are ready to obtain an estimate for $\partial_x Lu_1$.

Proposition 4.6. *For $t \leq 1$ we have the estimate*

$$\|\partial_x Lu_1\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}}. \tag{4.19}$$

Proof. By taking the derivative to the first equation of 4.13, we have that $\partial_x Lu_1$ satisfies

$$P\partial_x Lu_1 = \partial_x(-i2\sqrt{2}u_1\overline{u_2}Lu_1 + i\sqrt{2}u_1^2\overline{Lu_2})$$

The Duhamel formula yields

$$\partial_x Lu_1 = e^{it\Delta/2}(xu_x(0)) + \int_0^t e^{i(t-s)\Delta/2}\partial_x(-i2\sqrt{2}u_1\overline{u_2}Lu_1 + i\sqrt{2}u_1^2\overline{Lu_2})\ ds$$

After using the product rule, we can easily estimate some of the terms using the Strichartz inequality and the size of the time interval (being ≤ 1):

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)\Delta/2}(\partial_x u_1)\overline{u_2}Lu_1\ ds\right\|_{L^2} &\lesssim \|(\partial_x u_1)\overline{u_2}Lu_1\|_{L_t^1 L_x^2([0,t]\times\mathbb{R})} \\ &\lesssim \|\partial_x u_1\|_{L_t^4 L_x^\infty}\|u_2\|_{L_t^4 L_x^\infty}\|Lu_1\|_{L_t^\infty L_x^2}.\end{aligned}$$

In a similar manner we obtain:

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)\Delta/2}u_1\overline{u_2}(\partial_x Lu_1)\ ds\right\|_{L^2} &\lesssim \|u_1\|_{L_t^4 L_x^\infty}\|u_2\|_{L_t^4 L_x^\infty}\|\partial_x(Lu_1)\|_{L_t^\infty L_x^2} \\ \left\|\int_0^t e^{i(t-s)\Delta/2}(\partial_x u_1)u_1\overline{Lu_2}\ ds\right\|_{L^2} &\lesssim \|\partial_x u_1\|_{L_t^4 L_x^\infty}\|u_1\|_{L_t^4 L_x^\infty}\|Lu_2\|_{L_t^\infty L_x^2}.\end{aligned}$$

However, since none of the estimates we have proved so far cover any bounds for $\partial_x u_2$ and $\partial_x Lu_2$, more work is needed for the terms $u_1\partial_x(\overline{u_2})Lu_1$ and $u_1^2\partial_x(\overline{Lu_2})$. By applying (4.18) and invoking the equations for u_1, u_2 and Lu_1, we have that

$$
\begin{aligned}
&\left\| \int_0^t e^{i(t-s)\Delta/2} u_1 (\partial_x \overline{u_2}) L u_1 \; ds \right\|_{L^2} \\
&\lesssim \|\partial_x u_1\|_{L_t^4 l^2 L_x^\infty} \|u_2\|_{L_t^4 l^2 L_x^\infty} \|Lu_1\|_{L_t^\infty L_x^2} + \|u_1\|_{L_t^4 l^2 L_x^\infty} \|u_2\|_{L_t^4 l^2 L_x^\infty} \|\partial_x L u_1\|_{L_t^\infty L_x^2} \\
&\quad + \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|Lu_1\|_{L_t^\infty L_x^2} + \|u_1^2 u_2\|_{L_t^1 L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|Lu_1\|_{L_t^\infty L_x^2} \\
&\quad + \|u_1\|_{L_t^\infty L_x^2} \|u_1 u_2^2\|_{L_t^1 L_x^2} \|Lu_1\|_{L_t^\infty L_x^2} \\
&\quad + \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|u_1 \overline{u_2} L u_1 + u_1^2 \overline{L u_2}\|_{L_t^1 L_x^2}.
\end{aligned}
$$

The $L_t^4 l^2 L_x^\infty$ norms can be bounded as follows:

$$
\begin{aligned}
\|\partial_x u_1\|_{L_t^4 l^2 L_x^\infty} = \|\partial_x u_1\|_{L_t^2 l^1 L_x^\infty}^{1/2} \le \|\partial_x u_1\|_{l^1 L_t^2 L_x^\infty}^{1/2} &= \|\partial_x u_1\|_{l^2 L_t^4 L_x^\infty} \\
&\lesssim \|u_0\|_{l^2 H^1} \\
&\sim \|u_0\|_{H^1}.
\end{aligned}
$$

The same applies to $\|u_2\|_{L_t^4 l^2 L_x^\infty}$. Since we already have the bounds for the rest of the linear terms, all that is left are the nonlinear terms which can be easily bounded:

$$
\begin{aligned}
&\|u_1^2 u_2\|_{L_t^1 L_x^2} \lesssim \|u_1\|_{L_t^4 L_x^\infty}^2 \|u_2\|_{L_t^\infty L_x^2} \\
&\|u_1 u_2^2\|_{L_t^1 L_x^2} \lesssim \|u_2\|_{L_t^4 L_x^\infty}^2 \|u_1\|_{L_t^\infty L_x^2} \\
&\|u_1 \overline{u_2} L u_1\|_{L_t^1 L_x^2} \lesssim \|u_1\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|Lu_1\|_{L_t^\infty L_x^2} \\
&\|u_1^2 \overline{L u_2}\|_{L_t^1 L_x^2} \lesssim \|u_1\|_{L_t^4 L_x^\infty}^2 \|Lu_2\|_{L_t^\infty L_x^2}.
\end{aligned}
$$

Combining all these estimates and apply (4.12) and (4.14), all these terms collapse on the right-hand side collapse to a simple estimate.

$$\left\| \int_0^t e^{i(t-s)\Delta/2} u_1 (\partial_x \overline{u_2}) L u_1 \, ds \right\|_{L^2} \lesssim \|u_0\|_{H^{1,1}}^3 + \|u_0\|_{H^{1,1}}^2 \|\partial_x L u_1\|_{L_t^\infty L_x^2}.$$

In a similar manner, we have that

$$\left\| \int_0^t e^{i(t-s)\Delta/2} u_1^2 (\partial_x \overline{L u_2}) \, ds \right\|_{L^2} \lesssim \|u_0\|_{H^{1,1}}^3 + \|u_0\|_{H^{1,1}}^2 \|\partial_x L u_1\|_{L_t^\infty L_x^2}.$$

Therefore we obtain

$$\|\partial_x L u_1\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}} + \|u_0\|_{H^{1,1}}^3 + \|u_0\|_{H^{1,1}}^2 \|\partial_x L u_1\|_{L_t^\infty L_x^2}.$$

and we conclude that

$$\|\partial_x L u_1\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{H^{1,1}}.$$

□

4.2 Testing using wave packets

As mentioned in the introduction, we will use the technique of testing against the wave packet Ψ_v supported on the ray $\xi = v = \frac{x}{t}$. The precise definition is as follows: we let χ be a smooth function with compact support around 0 and $\int \chi(x)\, dx = 1$. We then define

$$\begin{aligned} \phi(t,x) &:= \frac{x^2}{2t}, \\ \Psi_v(t,x) &:= \chi\left(\frac{x - vt}{\sqrt{t}}\right) e^{i\phi(x,t)}, \\ \gamma_i(t,v) &:= \int u_i(t) \overline{\Psi}_v(t,x)\, dx, \qquad i = 1,2. \end{aligned}$$

We also make a convention that when a result holds for γ and u, it means that it holds for γ_i and u_i for $i = 1, 2$.

The following results from Ifrim-Tataru allow us to estimate the solutions of (1.15) with the help of γ_1 and γ_2.

Lemma 4.7 (Lemma 2.2 in [28]). *Assume that $u(t) \in L^2_x \cap L^\infty_x$ and $Lu(t) \in L^2_x$ for all $t \in \mathbb{R}$ and define $\gamma(t, v) := \int u(t)\overline{\Psi}_v(t, x)\, dx$. Then u and γ satisfy the following estimates:*

$$\|\gamma\|_{L^\infty_v} \lesssim t^{1/2}\|u\|_{L^\infty_x}, \qquad \|\gamma\|_{L^2_v} \lesssim \|u\|_{L^2_x}, \qquad \|\partial_v\gamma\|_{L^2_v} \lesssim \|Lu\|_{L^2_x}, \tag{4.20}$$

$$\begin{aligned} \|u(t, vt) - t^{-1/2}e^{i\phi(t,vt)}\gamma(t, v)\|_{L^2_v} &\lesssim t^{-1}\|Lu\|_{L^2_x}, \\ \|u(t, vt) - t^{-1/2}e^{i\phi(t,vt)}\gamma(t, v)\|_{L^\infty_v} &\lesssim t^{-3/4}\|Lu\|_{L^2_x}, \end{aligned} \tag{4.21}$$

We also have the Fourier estimates

$$\begin{aligned} \|\widehat{u}(t, \xi) - e^{-it\xi^2/2}\gamma(t, \xi)\|_{L^2_\xi} &\lesssim t^{-1}\|Lu\|_{L^2_x}, \\ \|\widehat{u}(t, \xi) - e^{-it\xi^2/2}\gamma(t, \xi)\|_{L^\infty_\xi} &\lesssim t^{-1}\|Lu\|_{L^2_x}, \end{aligned} \tag{4.22}$$

The following Lemma tells us that the momentum operator on u corresponds to the position operator on γ.

Lemma 4.8. *Let u and γ be defined as in Lemma 4.7. Then we have the following estimates:*

$$\begin{aligned} \left|u_x(t, vt) - it^{-1/2}e^{i\phi}v\gamma\right| &\lesssim t^{-1/2}(\|u\|_{L^\infty} + t^{-1/4}\|Lu_x\|_{L^2}) \\ \left\|u_x(t, vt) - it^{-1/2}e^{i\phi}v\gamma\right\|_{L^2_v} &\lesssim t^{-1}(\|u\|_{L^2} + \|Lu_x\|_{L^2}). \end{aligned} \tag{4.23}$$

Proof. Note that $\int \chi\, dx = 1$. We start with the triangle inequality.

$$\left|u_x - it^{-1/2}e^{i\phi}v\gamma\right| \le \left|e^{-i\phi}u_x - it^{-1/2}\int \frac{x}{t}u\overline{\Psi}_v\, dx\right| + \left|it^{-1/2}\int \left(\frac{x - vt}{t}\right)u\overline{\Psi}_v\, dx\right|$$

By Hölder inequality, we have

$$\left|it^{-1/2}\int \left(\frac{x - vt}{t}\right)u\overline{\Psi}_v\, dx\right| \lesssim t^{-3/4}\|u\|_{L^2_x}.$$

For the first term, we will use integration by parts. By defining $\tilde{w} := e^{-i\phi}u_x$, we obtain

$$\begin{aligned}\left|e^{-i\phi}u_x - it^{-1/2}\int \frac{x}{t}u\overline{\Psi}_v\ dx\right| &= \left|e^{-i\phi}u_x + t^{-1/2}\int u\chi\left(\frac{x-vt}{\sqrt{t}}\right)\partial_x e^{-i\phi}\ dx\right| \\ &= \left|e^{-i\phi}u_x - t^{-1/2}\int \partial_x\left\{u\chi\left(\frac{x-vt}{\sqrt{t}}\right)\right\}e^{-i\phi}\ dx\right| \\ &\le \left|t^{-1/2}\int[\tilde{w}(t,vt)-\tilde{w}(t,x)]\chi\left(\frac{x-vt}{\sqrt{t}}\right)\ dx\right| \\ &\quad + \left|t^{-1}\int u\chi'\left(\frac{x-vt}{\sqrt{t}}\right)e^{-i\phi}\ dx\right|. \end{aligned} \tag{4.24}$$

Since the second term can be written as $e^{-i\phi(t,vt)}u(t,vt) *_v \chi'(t^{1/2}v)$, we can apply Young's inequality to obtain

$$\left|e^{-i\phi(t,vt)}u(t,vt) *_v \chi'(t^{1/2}v)\right| \lesssim t^{-1/2}\|u\|_{L^\infty_x}$$

and

$$\left\|e^{-i\phi(t,vt)}u(t,vt) *_v \chi'(t^{1/2}v)\right\|_{L^2_v} \lesssim t^{-1}\|u\|_{L^2_x}.$$

For the remaining term, we use Hölder inequality

$$\begin{aligned}\left|e^{-i\phi(t,vt)}u_x(t,vt) - e^{-i\phi}u_x(t,x)\right| &= \left|\int_x^{vt}\partial_y\left[e^{-i\phi(t,y)}u_x(t,y)\right]\ dy\right| \\ &= \left|-\frac{i}{t}\int_x^{vt}e^{-i\phi(t,y)}Lu_x(t,y)\ dy\right| \\ &\le \frac{|x-tv|^{1/2}}{t}\|Lu_x\|_{L^2}\end{aligned} \tag{4.25}$$

Therefore, by defining $z = \frac{x-vt}{\sqrt{t}}$, we have that

$$\begin{aligned}\left|t^{-1/2}\int[\tilde{w}(t,vt)-\tilde{w}(t,x)]\chi\left(\frac{x-vt}{\sqrt{t}}\right)\ dx\right| &\lesssim t^{-3/4}\|Lu_x\|_{L^2_x}\int|z|\chi(z)\ dz \\ &\lesssim t^{-3/4}\|Lu_x\|_{L^2_x}.\end{aligned}$$

It remains to estimate the L_v^2 norm of this term. By (4.25) and Minkowski inequality, we have

$$\begin{aligned}
&\left\|t^{-1/2}\int[\tilde{w}(t,vt)-\tilde{w}(t,x)]\chi\left(\frac{x-vt}{\sqrt{t}}\right)\,dx\right\|_{L_v^2} \\
&\qquad \le \frac{1}{t^{3/2}}\left\|\int\int_x^{vt}|Lu_x(t,y)|\chi\left(\frac{x-vt}{\sqrt{t}}\right)\,dydx\right\|_{L_v^2} \\
&\qquad = \frac{1}{t^{1/2}}\left\|\int\int_0^1|Lu_x(t,vt+t^{1/2}zh)|z\chi(z)\,dhdz\right\|_{L_v^2} \\
&\qquad \le \frac{1}{t}\|Lu_x\|_{L^2}.
\end{aligned}$$

□

These results suggest that it might be better to focus on the dynamic of γ_1, which we can describe in the following lemma.

Lemma 4.9. *γ_1 and γ_2 satisfy the following ODEs.*

$$\dot{\gamma_1}(t,v) = -i(t^{-1}v|\gamma_1|^2+\frac{1}{2}t^{-2}|\gamma_1|^4)\gamma_1+R_1(t,v). \tag{4.26}$$

$$\dot{\gamma_2}(t,v) = i(t^{-1}v|\gamma_1|^2+\frac{1}{2}t^{-2}|\gamma_1|^4)\gamma_2+R_2(t,v), \tag{4.27}$$

where

$$\begin{aligned}
\|R_1\|_{L_v^\infty} &\lesssim t^{-5/4}\|Lu_1\|_{L^2}+\|u_1\|_{L^\infty}^2(t^{-1/4}\|L\partial_x u_1\|_{L^2}+\|u_1\|_{L^\infty}) \\
&\qquad + t^{-1/4}\|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^\infty}\|Lu_1\|_{L^2}+t^{-1/4}\|u_1\|_{L^\infty}^4\|Lu_1\|_{L^2}, \\
\|R_2\|_{L_v^\infty} &\lesssim t^{-5/4}\|Lu_2\|_{L^2}+\|u_1\|_{L^\infty}\|u_2\|_{L^\infty}(t^{-1/4}\|L\partial_x u_1\|_{L^2}+\|u_1\|_{L^\infty}) \\
&\qquad + t^{-1/4}\|\partial_x u_1\|_{L^\infty}\|u_2\|_{L^\infty}\|Lu_1\|_{L^2}+t^{-1/4}\|u_1\|_{L^\infty}^3\|u_2\|_{L^\infty}\|Lu_1\|_{L^2}
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
\|R_1\|_{L^2_v} &\lesssim t^{-3/2}\|Lu_1\|_{L^2} + t^{-1/2}\|u_1\|_{L^\infty}^2(\|L\partial_x u_1\|_{L^2} + \|u_1\|_{L^2}) \\
&\quad + t^{-1/2}\|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^\infty}\|Lu_1\|_{L^2} + t^{-1/2}\|u_1\|_{L^\infty}^4\|Lu_1\|_{L^2}, \\
\|R_2\|_{L^2_v} &\lesssim t^{-3/2}\|Lu_2\|_{L^2} + t^{-1/2}\|u_1\|_{L^\infty}\|u_2\|_{L^\infty}(\|L\partial_x u_1\|_{L^2} + \|u_1\|_{L^2}) \\
&\quad + t^{-1/2}\|\partial_x u_1\|_{L^\infty}\|u_2\|_{L^\infty}\|Lu_1\|_{L^2} + t^{-1/2}\|u_1\|_{L^\infty}^3\|u_2\|_{L^\infty}\|Lu_1\|_{L^2}.
\end{aligned} \tag{4.29}$$

Proof. From the definition of γ_1, we compute the time derivative and then utilize the PDE 4.11 for u_1.

$$\begin{aligned}
\dot{\gamma_1}(t) &= \int \partial_t u_1 \overline{\Psi}_v + u_1 \partial_t \overline{\Psi}_v dx = \int (i\partial_x^2 u_1 - \sqrt{2}\overline{u_2}u_1^2)\overline{\Psi}_v + u_1\partial_t\overline{\Psi}_v \; dx \\
&= \int iu_1\overline{(i\partial_t + \partial_x^2)\Psi_v} \; dx - \sqrt{2}\int \overline{u_2}u_1^2\overline{\Psi}_v \; dx \\
&= \int iu_1\overline{(i\partial_t + \partial_x^2)\Psi_v} \; dx - \int u_1^2 \left(\overline{\partial_x u_1 + \frac{i}{2}|u_1|^2 u_1}\right)\overline{\Psi}_v \; dx.
\end{aligned} \tag{4.30}$$

From direct calculation, we see that for $\tilde{x} = t^{-1/2}(x - vt)$,

$$(i\partial_t + \partial_x^2)\Psi_v = \frac{1}{2t^{1/2}} e^{i\phi}\partial_x\left[\chi'(\tilde{x}) + i\tilde{x}\chi(\tilde{x})\right].$$

We then integrate by parts to obtain

$$\begin{aligned}
\int iu_1\overline{(i\partial_t + \partial_x^2)\Psi_v} \; dx &= \frac{1}{2t^{1/2}}\int i\partial_x\left[\chi'(\tilde{x}) - i\tilde{x}\chi(\tilde{x})\right]u_1 e^{-i\phi} \; dx \\
&= -\frac{1}{2t^{3/2}}\int \left[\chi'(\tilde{x}) - i\tilde{x}\chi(\tilde{x})\right]Lu_1 e^{-i\phi} \; dx.
\end{aligned}$$

Therefore, by Hölder inequality,

$$\left|\int iu_1\overline{(i\partial_t + \partial_x^2)\Psi_v} \; dx\right| \lesssim t^{-5/4}\|Lu_1\|_{L^2}. \tag{4.31}$$

We will now split the term.

$$\int u_1^2\overline{\partial_x u_1 \Psi_v}\ dx = \int u_1^2\overline{\Psi}_v e^{-i\phi}\left(\overline{e^{-i\phi}\partial_x u_1} - \overline{e^{-i\phi(t,vt)}\partial_x u_1(t,vt)}\right)\ dx$$

$$+ \int u_1\overline{\partial_x u_1(t,vt)\Psi}_v e^{i\phi(t,vt)}\left(e^{-i\phi}u_1 - e^{-i\phi(t,vt)}u_1(t,vt)\right)\ dx$$

$$+ \gamma_1\overline{\partial_x u_1(t,vt)}\left(u_1(t,vt) - t^{-1/2}e^{i\phi(t,vt)}\gamma_1\right)$$

$$+ t^{-1/2}e^{i\phi(t,vt)}\gamma_1^2\left(\overline{\partial_x u_1(t,vt)} - it^{-1/2}e^{-i\phi(t,vt)}v\overline{\gamma}_1\right)$$

$$+ it^{-1}v|\gamma_1|^2\gamma_1$$

$$:= R_{31} + R_{32} + R_{33} + R_{34} + it^{-1}v|\gamma_1|^2\gamma_1 + R_3$$

$$:= R_3 + it^{-1}v|\gamma_1|^2\gamma_1.$$

The term R_{31} and can be estimated using (4.25).

$$|R_{31}| \lesssim t^{-1/4}\|u_1\|_{L^\infty}^2\|L\partial_x u_1\|_{L^2}.$$

The same goes for R_{32}, but with u_x replaced by u in (4.25).

$$|R_{32}| \lesssim t^{-1/4}\|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^\infty}\|Lu_1\|_{L^2}.$$

The term R_{33} can be estimated using (4.20) and (4.21)

$$|R_{33}| \lesssim t^{-1/4}\|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^\infty}\|Lu_1\|_{L^2},$$

and R_{34} can be estimated using (4.20) and (4.23).

$$|R_{34}| \lesssim \|u_1\|_{L^\infty}^3 + t^{-1/4}\|u_1\|_{L^\infty}^2\|L\partial_x u_1\|_{L^2}.$$

In conclusion, we have

$$|R_3| \lesssim \|u_1\|_{L^\infty}^2(t^{-1/4}\|L\partial_x u_1\|_{L^2} + \|u_1\|_{L^\infty}) + t^{-1/4}\|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^\infty}\|Lu_1\|_{L^2}.$$

We apply the same idea to the remaining term.

$$\begin{aligned}\frac{i}{2}\int |u_1|^4 u_1 \overline{\Psi}_v \, dx &= \frac{i}{2}\int u_1 \overline{\Psi}_v \left(|u_1|^4 - |u_1(t,vt)|^4\right) \, dx \\ &+ \frac{i}{2}\gamma_1 \left(|u_1(t,vt)|^4 - t^{-2}|\gamma_1|^4\right) \\ &+ \frac{i}{2} t^{-2}|\gamma_1|^4\gamma_1 \\ &:= \frac{i}{2} t^{-2}|\gamma_1|^4\gamma_1 + R_4\end{aligned}$$

where R_4 satisfies the estimate

$$|R_4| \lesssim t^{-1/4}\|u_1\|_{L^\infty}^4 \|Lu_1\|_{L^2}.$$

Consequently, we obtain the L^∞ bound for $R_1 := R_3 + R_4$. The L^2 bound for R_1 can be obtained in a similar manner, using the second estimates in (4.21) and (4.23) instead of the first ones.

The proof for γ_1 is finished after combining $R_1 := R_3 + R_4$. With the same proof focusing on the PDE for u_2, we can obtain similar result for γ_2. □

4.3 Global Well-posedness and global bounds

Assume that the initial data is small in $H^{1,1}$. Then, the DNLS has a unique solution in $L_t^\infty H_x^1$ due to the conservation of mass and energy.

Proposition 4.10. *Assume that u is a solution to* (1.15) *with the initial data satisfying $\|u_o\|_{H^{1,1}} \leq \epsilon < \frac{\sqrt{\pi}}{2\sqrt[4]{28}}$. Then for all $t \in \mathbb{R}$, we have the estimate*

$$\|u(t)\|_{H^1} \lesssim \epsilon. \tag{4.32}$$

Proof. It follows from the conservation of mass that $\|u(t)\|_{L^2_x} \leq \epsilon$. From Gagliardo-Nirenberg inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}} 3\mathrm{Im}|u(t)|^2 u(t)\overline{u_x(t)} + 2|u(t)|^6 \, dx\right| &\leq 14\int_{\mathbb{R}} |u|^6 \, dx + \frac{3}{4}\int_{\mathbb{R}} |u_x|^2 \, dx \\ &\leq \frac{56}{\pi^2}\|u_x\|_{L^2_x}^2 \|u\|_{L^2_x}^4 + \frac{3}{4}\|u_x\|_{L^2_x}^2 \\ &\leq \left(\frac{56\epsilon^4}{\pi^2} + \frac{3}{4}\right)\|u_x\|_{L^2_x}^2 \\ &= \frac{7}{8}\|u_x\|_{L^2_x}^2.\end{aligned}$$

Therefore, $E(u(t)) \sim \|u_x\|_{L^2_x}^2$, and it follows that

$$\|u(t)\|_{H^1_x}^2 \sim M(u(t)) + E(u(t)) = M(u(0)) + E(u(0)) \sim \|u(0)\|_{H^1_x}^2 \leq \epsilon$$

as desired. □

Corollary 4.11. *The equation* (1.15) *with initial data* $u_0 \in H^{1,1}$ *has a unique solution in* H^1.

Proof. This follows from the previous proposition and the H^1 local theory for the DNLS. □

We will now prove pointwise estimates of the solution. First, we will assume two bootstrap assumptions

$$\|u\|_{L^\infty} \leq D\epsilon|t|^{-1/2}, \tag{4.33}$$

$$\|u_x\|_{L^\infty} \leq D\epsilon|t|^{-1/2}. \tag{4.34}$$

where $1 \ll D \ll \epsilon^{-1}$.

This implies that

$$\|u_1\|_{L^\infty} \leq D\epsilon|t|^{-1/2}, \tag{4.35}$$

$$\|\partial_x u_1\|_{L^\infty} \lesssim D\epsilon|t|^{-1/2} + D^3\epsilon^3|t|^{-3/2}, \tag{4.36}$$

$$\|u_2\|_{L^\infty} \lesssim D\epsilon|t|^{-1/2} + D^3\epsilon^3|t|^{-3/2}. \tag{4.37}$$

Under these assumptions, we obtain the following estimates for Lu_i and $L\partial_x u_1$.

Lemma 4.12. *For $t \geq 1$, we have that*

$$\|Lu_i(t)\|_{L^2} \lesssim \epsilon t^{CD^2\epsilon^2} \tag{4.38}$$

for some $C > 1$ independent of D and ϵ.

Proof. Multiply equation ((4.13)) by $\overline{Lu_1}$ and integrate. We have

$$\frac{d}{dt}\|Lu_1(t)\|_{L^2}^2 = \text{Re}\int \left[-i2\sqrt{2}u_1\overline{u_2}|Lu_1|^2 + i\sqrt{2}u_1^2\overline{Lu_1Lu_2}\right]\,dx$$
$$\frac{d}{dt}\|Lu_2(t)\|_{L^2}^2 = \text{Re}\int \left[i2\sqrt{2}u_2\overline{u_1}|Lu_2|^2 - i\sqrt{2}u_2^2\overline{Lu_1Lu_2}\right]\,dx$$

This leads to an inequality

$$\frac{d}{dt}\left[\|Lu_1(t)\|_{L^2}^2 + \|Lu_2(t)\|_{L^2}^2\right]$$
$$\lesssim \left[\|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2 + \|u_1\|_{L^\infty}\|u_2\|_{L^\infty}\right]\left[\|Lu_1(t)\|_{L^2}^2 + \|Lu_2(t)\|_{L^2}^2\right].$$

Note that $\|Lu_i\|_{L^2} \lesssim \|xu_i(0)\|_{L_x^2} \leq \epsilon$ for $i = 1, 2$. By applying Gronwall's inequality, we obtain

$$\begin{aligned}
&\|Lu_1(t)\|_{L^2}^2 + \|Lu_2(t)\|_{L^2}^2 \\
&\qquad \leq \left(\|Lu_1(1)\|_{L^2}^2 + \|Lu_2(1)\|_{L^2}^2\right)e^{\int_1^t \|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2 + \|u_1\|_{L^\infty}\|u_2\|_{L^\infty}\,dt} \\
&\qquad \lesssim \epsilon^2 e^{\int_1^t \|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2 + \|u_1\|_{L^\infty}\|u_2\|_{L^\infty}\,dt} \\
&\qquad \leq \epsilon^2 t^{C_0D^2\epsilon^2}.
\end{aligned} \tag{4.39}$$

For some $C_0 > 2$. $\square$

The last thing we need is an L^2 estimate for $L\partial_x u_1$:

Lemma 4.13. *For $t \geq 1$, we have an estimate*

$$\|L\partial_x u_1\|_{L^2} \lesssim \left(\epsilon + t^{-1}D^2\epsilon^3\right) t^{CD^2\epsilon^2} \lesssim \epsilon t^{CD^2\epsilon^2}. \tag{4.40}$$

where the last inequality has an implicit constant depending on D.

Proof. Recall that we have $\partial_x u_1 = \sqrt{2}u_2 - i|u_1|^2 u_1$. Applying L to both sides, we compute

$$L|u_1|^2 u_1 = it\partial_x(|u_1|^2)u_1 + |u_1|^2 Lu_1.$$

This allows us to estimate

$$\begin{aligned}
\|L\partial_x u_1\|_{L^2} &\lesssim \|Lu_2\|_{L^2} + \|\partial_x u_1\|_{L^\infty}\|u_1\|_{L^2}\|u_1\|_{L^\infty} + \|Lu_1\|_{L^2}\|u_1\|_{L^\infty}^2 \\
&\lesssim \epsilon t^{CD^2\epsilon^2} + D^2\epsilon^3 + D^2\epsilon^3 t^{CD^2\epsilon^2-1} \\
&\leq \left(2\epsilon + t^{-1}D^2\epsilon^3\right) t^{CD^2\epsilon^2}.
\end{aligned}$$

□

We can now close the bootstrap argument and obtain global bounds.

Proposition 4.14. *For any $t \geq 1$, we have that*

$$\|u\|_{L^\infty} \lesssim \epsilon t^{-1/2}, \tag{4.41}$$

$$\|u_x\|_{L^\infty} \lesssim \epsilon t^{-1/2}. \tag{4.42}$$

with the implicit constants not depending on D.

Proof. It suffices to prove in the case $t \geq 0$. We first obtain the global bounds. By defining $w_1 := e^{-i\phi}u_1$, we can use the local well-posedness result to obtain the bound for $t \in (0, 1]$.

$$|u(t)| = |u_1(t)| = |w_1(t)| \lesssim \|\partial_x w_1(t)\|_{L^2}\|w_1(t)\|_{L^2} = t^{-1/2}\|Lu_1(t)\|_{L^2}^{1/2}\|u_1(t)\|_{L^2}^{1/2} \lesssim \epsilon t^{-1/2}. \tag{4.43}$$

To advance from $t = 1$, we will use (4.21), (4.26) and (4.28). First, note that

$$\|u_1(t, vt) - t^{-1/2}e^{i\phi(t,vt)}\gamma_1(t, v)\|_{L^\infty_v} \lesssim t^{-3/4}\|Lu_1\|_{L^2_x} \lesssim \epsilon t^{-\frac{3}{4}+CD^2\epsilon^2}. \tag{4.44}$$

Thus, it suffices to bound γ_1. Since $i|\gamma|^2$ is purely imaginary, we have that

$$|\gamma_1(t)| \leq |\gamma_1(1)| + \int_1^t |R_1(s, v)| \, ds \tag{4.45}$$

From (4.20) and (4.43), we have

$$|\gamma_1(1)| \lesssim t^{1/2}\|u_1(1)\|_{L^\infty} \lesssim \epsilon.$$

From (4.28), (4.33)-(4.37),(4.38) and (4.40), we have

$$|R_1| \lesssim \epsilon(1 + D^2\epsilon^2)t^{-5/4+CD^2\epsilon^2}. \tag{4.46}$$

By integrating this estimate, we obtain

$$\begin{aligned}
\|u(t)\|_{L^\infty} = \|u_1(t)\|_{L^\infty} &\lesssim t^{-1/2}\|\gamma_1(t, v)\|_{L^\infty_v} \\
&\lesssim t^{-1/2}\Big(|\gamma_1(1)| + \int_1^t |R_1(s, v)| \, ds\Big) \\
&\lesssim \epsilon(1 + D^2\epsilon^2)t^{-1/2} \\
&\lesssim \epsilon t^{-1/2},
\end{aligned}$$

from the assumption $1 \ll D \ll \epsilon^{-1}$. We then obtain (4.41) from the bootstrap argument as desired.

We will now prove the bound for u_x. We have that for $t \in (0,1]$

$$\|\partial_x u_1\|_{L^\infty} \lesssim t^{-1/2}\|L(\partial_x u_1)\|_{L^2}^{1/2}\|\partial_x u_1\|_{L^2}^{1/2} \lesssim t^{-1/2}\epsilon. \tag{4.47}$$

Now assume that $t \geq 1$. It follows from the estimate (4.21) that

$$\|u_2(t,vt) - t^{-1/2}e^{i\phi(t,vt)}\gamma_2(t,v)\|_{L_v^\infty} \lesssim t^{-3/4}\|Lu_2\|_{L^2} \lesssim \epsilon t^{-\frac{3}{4}+CD^2\epsilon^2}. \tag{4.48}$$

As before, we have that

$$|\gamma_2(t)| \leq |\gamma_2(1)| + \int_1^t |R_2(s,v)|\, ds \tag{4.49}$$

From (4.20), (4.47) and (4.43), we have

$$|\gamma_2(1)| \lesssim t^{1/2}\|u_2(1)\|_{L^\infty} \lesssim t^{1/2}(\|\partial_x u_1(1)\|_{L^\infty} + \|u_1(1)\|_{L^\infty}^3) \lesssim \epsilon.$$

We then estimate the remainder R_2 in the same way that we did for R_1.

$$|R_2| \lesssim \epsilon(1 + D^2\epsilon^2)t^{-5/4+CD^2\epsilon^2}. \tag{4.50}$$

From this, we see that the bound for u_2 is the same as that of u_1.

$$\|u_2(t)\|_{L^\infty} \lesssim t^{-1/2}\|\gamma_2\|_{L_v^\infty} \lesssim t^{-1/2}\Big(|\gamma_2(1)| + \int_1^t |R_2(s,v)|\, ds\Big) \lesssim \epsilon t^{-1/2}.$$

and we conclude that

$$\begin{aligned}\|u_x(t)\|_{L^\infty} &\lesssim \|u_2(t)\|_{L^\infty} + \|u_1(t)\|_{L^\infty}^3 \\ &\lesssim \|u_2(t)\|_{L^\infty} + \|u_1(t)\|_{L^\infty}^3 \\ &\lesssim \epsilon t^{-1/2}\end{aligned}$$

as desired.

□

Corollary 4.15. *For $t \geq 1$, we have an estimate*

$$\|L\partial_x u_1\|_{L^2} \lesssim \epsilon t^{CD^2\epsilon^2}. \tag{4.51}$$

Proof. The proof is the same as Lemma 4.13, but the implicit constant now does not depend on D because of (4.41) and (4.42). □

4.4 Asymptotic profiles

We will extract profiles from u_1 and u_2 and use them to construct a profile for u. The following estimate will be used to reverse the Gauge transformation from u_1 to u.

Lemma 4.16. *Let $0 \leq s \leq 1$.*

1. *For any $f \in H^s$ and $g \in H^s$,*

$$\left\| f(x) \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{H^s} \lesssim \|f\|_{H^s} + \|f\|_{L^2}\|g\|_{L^2}\|g\|_{H^s}. \tag{4.52}$$

2. *For any $f \in H^{s+1}$ and $g \in H^{s+1}$,*

$$\left\| f(x) \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{H^{s+1}} \lesssim \|f\|_{H^{s+1}} + \|f\|_{L^\infty}\|g\|_{H^s}(\|g\|_{L^\infty} + \|g\|_{L^2}^2). \tag{4.53}$$

Proof. For a proof of (4.52) we refer to [23]. In order to prove (4.53), we will apply the product estimate for Sobolev spaces: for any $\alpha \geq 0$ and $f_1, f_2 \in \dot{H}^\alpha \cap L^\infty$, we have that

$$\|f_1 f_2\|_{H^\alpha} \lesssim \|f_1\|_{L^\infty} \|f_2\|_{H^\alpha} + \|f_1\|_{H^\alpha} \|f_2\|_{L^\infty} \tag{4.54}$$

$$\|f_1 f_2\|_{\dot{H}^\alpha} \lesssim \|f_1\|_{L^\infty} \|f_2\|_{\dot{H}^\alpha} + \|f_1\|_{\dot{H}^\alpha} \|f_2\|_{L^\infty}. \tag{4.55}$$

See, for instance, [31] for proofs of these inequaliies. It follows that

$$\begin{aligned}
&\left\| f \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{H^{s+1}} \\
&\quad \sim \|f\|_{L^2} + \left\| f \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{\dot{H}^{s+1}} \\
&\quad \lesssim \|f\|_{L^2} + \|f\|_{\dot{H}^{s+1}} + \|f\|_{L^\infty} \left\| \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{\dot{H}^{s+1}} \\
&\quad \sim \|f\|_{H^{s+1}} + \|f\|_{L^\infty} \left\| |g|^2 \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{\dot{H}^{s}}.
\end{aligned}$$

Notice that we can estimate the last term using (4.52) and (4.54).

$$\begin{aligned}
\left\| |g(x')|^2 \exp\left(i \int_{-\infty}^{x} |g(x')|^2 \, dx' \right) \right\|_{\dot{H}^{s}} &\lesssim \||g|^2\|_{H^s} + \|g\|_{L^2}^2 \|g\|_{H^s} \\
&\lesssim \|g\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^2}^2 \|g\|_{H^s},
\end{aligned}$$

which gives the desired inequality. □

We begin by recalling the estimates (4.46) in the proof of Proposition 4.14:

$$\|R_1\|_{L_v^\infty} \lesssim \epsilon t^{-5/4 + CD^2\epsilon^2}. \tag{4.56}$$

We can also obtain an L^2 bounds using (4.29), (4.38), (4.51), (4.41) and (4.42):

$$\|R_1\|_{L_v^2} \lesssim \epsilon t^{-3/2 + CD^2\epsilon^2}. \tag{4.57}$$

Moreover, the estimate (4.20) together with the global estimates (4.41) and (4.42) allow us to obtain the following bounds for γ_1 and γ_2:

$$\|\gamma_i\|_{L_v^\infty} \lesssim \epsilon, \qquad \|\gamma_i\|_{L_v^2} \lesssim \epsilon, \tag{4.58}$$

for $i = 1, 2$. We recall the ODE of γ_1 from (4.26):

$$\dot{\gamma_1}(t,v) = -it^{-1}v|\gamma_1|^2\gamma_1 - \frac{1}{2}it^{-2}|\gamma_1|^4\gamma_1 + R_1(t,v). \tag{4.59}$$

Since the last two terms on the right-hand side are integrable on $t \in [1,\infty)$ and are small compared to the first term as $t \to \infty$, we can obtain an approximated solution to this ODE by assuming that these terms vanish. This can be expressed as

$$\gamma_1(t,v) = W_1(v)e^{-iv|W(v)|^2 \log t} + err_0(t,v), \tag{4.60}$$

where $\|err_0\|_{L_v^\infty} \lesssim \epsilon t^{-1/4+CD^2\epsilon^2}$ and $\|err_0\|_{L_v^2} \lesssim \epsilon t^{-1/2+CD^2\epsilon^2}$. Then we can approximate u_1 by γ_1 using (4.21) and (4.38):

$$\begin{aligned} u_1(t,x) &= \frac{1}{t^{1/2}}W_1\left(\frac{x}{t}\right)e^{-i|W_1(x/t)|^2\frac{x}{t}\log t + i\frac{x^2}{2t}} + err_p(t,x) \\ &:= \tilde{u}_1(t,x) + err_p(t,x). \end{aligned} \tag{4.61}$$

where $\|err_p\|_{L_x^\infty} \lesssim \epsilon t^{-3/4+CD^2\epsilon^2}$ and $\|err_p\|_{L_x^2} \lesssim \epsilon t^{-1+CD^2\epsilon^2}$. By setting $t = 1$, we have that

$$\|W_1(x)\|_{L_x^\infty} \lesssim \|u_1(1,x)\|_{L_x^\infty} \lesssim \epsilon, \qquad \|W_1(x)\|_{L_x^2} \lesssim \|u_1(1,x)\|_{L_x^2} \lesssim \epsilon. \tag{4.62}$$

Since $u(t,x) = u_1(t,x)\exp(2i\int_\infty^x |u_1(t,x')|^2\, dx')$, we can give an expression for u:

$$u(t,x) = \tilde{u}_1(t,x)\exp\left(2i\int_{-\infty}^x |\tilde{u}_1|^2 + 2\mathrm{Re}[\tilde{u}_1\overline{err}_p(t)] + |err_p(t)|^2\, dx'\right) + err_1'(t,x) \tag{4.63}$$

Note that (4.62) implies

$$\|\tilde{u}_1 err_p\|_{L^1_x} \leq \frac{1}{t^{1/2}} \|W_1(x/t)\|_{L^2_x} \|err_p\|_{L^2_x} \lesssim \epsilon^2 t^{-1+CD^2\epsilon^2}.$$

Thus we can take the integrand in (4.63) as a small perturbation of $|\tilde{u}_1|^2$. Therefore, as $t \to \infty$, we can write

$$u(t,x) = \tilde{u}_1(t,x) \exp\left(2i \int_{-\infty}^{x} |\tilde{u}_1|^2 \, dx'\right) + err_1(t,x)$$

where $\|err_1\|_{L^\infty_x} \lesssim \epsilon t^{-3/4+CD^2\epsilon^2}$ and $\|err_1\|_{L^2_x} \lesssim \epsilon t^{-1+CD^2\epsilon^2}$.

By defining $W(x) := W_1(x) \exp\left(2i \int_{-\infty}^{x} |W_1(x')|^2 \, dx'\right)$, we obtain

$$u(t,x) = \frac{1}{t^{1/2}} W\left(\frac{x}{t}\right) e^{-i|W(x/t)|^2 \frac{x}{t} \log t + i\frac{x^2}{2t}} + err_1(t,x), \tag{4.64}$$

as desired. The same technique can also be applied to prove (4.6). First, we apply the estimate (4.23) on u_1 to obtain

$$\partial_x u_1(t,x) = \frac{ix}{t^{3/2}} W_1\left(\frac{x}{t}\right) e^{-i|W_1(x/t)|^2 \frac{x}{t} \log t + i\frac{x^2}{2t}} + err_3'(t,x), \tag{4.65}$$

where $\|err_3'\|_{L^\infty_x} \lesssim \epsilon t^{-3/4+CD^2\epsilon^2}$ and $\|err_3'\|_{L^2_x} \lesssim \epsilon t^{-1+CD^2\epsilon^2}$. The result then follows from

$$\begin{aligned}\partial_x u(t,x) &= (\partial_x u_1(t,x) + 2iu_1(t,x)|u_1(t,x)|^2) \exp\left(2i \int_{-\infty}^{x} |u_1(x')|^2 \, dx'\right) \\ &= \frac{ix}{t^{3/2}} W\left(\frac{x}{t}\right) e^{-i|W(x/t)|^2 \frac{x}{t} \log t + i\frac{x^2}{2t}} + err_3(t,x).\end{aligned}$$

To find the regularity of W, we go back to (4.65) and use $u_2 = \frac{1}{\sqrt{2}}(\partial_x u_1 + i|u_1|^2 u_1)$ to obtain

$$u_2(t,vt) = \frac{iv}{\sqrt{2}t^{1/2}} W_1(v) \, e^{-iv|W_1(v)|^2 \log t + i\phi(t,vt)} + O_{L^2_v}(\epsilon t^{-3/2+CD^2\epsilon^2}). \tag{4.66}$$

From (4.21) and (4.38), we have that

$$u_2(t, vt) = \frac{1}{t^{1/2}} e^{i\phi(t,vt)} \gamma_2(t, v) + O_{L^2_v}(\epsilon t^{-1+CD^2\epsilon^2}).$$

It follows that

$$\gamma_2(t, v) = \frac{iv}{\sqrt{2}} W_1(v) e^{-iv|W_1(v)|^2 \log t} + O_{L^2_v}(\epsilon t^{-1/2+CD^2\epsilon^2}). \tag{4.67}$$

We then multiply both sides by $e^{iv|W_1(v)|^2 \log t}$ and observe that the exponent contains a product of $|W_1(v)| = |\gamma_1(t, v)| + O_{L^2_v}(\epsilon t^{-1/2+CD^2\epsilon^2})$ and $v|W_1(v)| = |\gamma_2(t, v)| + O_{L^2_v}(\epsilon t^{-1/2+CD^2\epsilon^2})$. Therefore, (4.67) gives us an L^2_v approximation for $vW_1(v)$ for all large t.

$$\begin{aligned}
&\|vW_1(v) + i\sqrt{2}\gamma_2(t, v) e^{i|\gamma_1(t,v)||\gamma_2(t,v)| \log t}\|_{L^2_v} \\
&\lesssim \|\gamma_2\|_{L^\infty_v}(\|\gamma_1\|_{L^2_v}\|\gamma_2\|_{L^\infty_v} + \|\gamma_1\|_{L^\infty_v} + \|\gamma_2\|_{L^\infty_v})\epsilon t^{-1/2+CD^2\epsilon^2} \log t \\
&\lesssim \epsilon t^{-1/2+CD^2\epsilon^2} \log t.
\end{aligned}$$

From (4.20), we have

$$\left\|\partial_v\left[\gamma_2(t, v) e^{i|\gamma_1(t,v)||\gamma_2(t,v)| \log t}\right]\right\|_{L^2_v} \lesssim \epsilon t^{CD^2\epsilon^2} \log t.$$

Therefore, we obtain the asymptotic for $vW_1(v)$:

$$vW_1(v) = O_{H^1_v}(\epsilon t^{CD^2\epsilon^2} \log t) + O_{L^2_v}(\epsilon t^{-1/2+CD^2\epsilon^2} \log t). \tag{4.68}$$

We can make the same analysis on (4.60) alone to obtain the asymptotic for $W_1(v)$:

$$W_1(v) = O_{H^1_v}(\epsilon t^{CD^2\epsilon^2} \log t) + O_{L^2_v}(\epsilon t^{-1/2+CD^2\epsilon^2} \log t). \tag{4.69}$$

One can multiply both sides of (4.68) and (4.69) by $\exp\left(2i \int_{-\infty}^{x} |W_1(x')|^2 \, dx'\right)$, apply (4.52) and (4.62) to obtain the same asymptotics for $vW(v)$. The regularity of W can then be achieved by interpolation:

$$\|W\|_{H_v^{1-C_0\epsilon^2,1}} \lesssim \epsilon,$$

for large enough C_0.

It remains to prove (4.5), a profile for $\hat{u}$. We can approximate $\hat{u}_1$ by γ_1 using (4.22) and (4.38):

$$\hat{u}_1(t,\xi) = W_1(\xi)e^{-i\xi|W_1(\xi)|^2 \log t - it\xi^2/2} + err_f(t,\xi),$$

where $\|err_f\|_{L^\infty_\xi} \lesssim \epsilon t^{-1/4+CD^2\epsilon^2}$ and $\|err_f\|_{L^2_\xi} \lesssim \epsilon t^{-1/2+CD^2\epsilon^2}$. To obtain a similar profile for $\hat{u}$, we need the factorization technique for the Schrödinger propagator $\mathcal{U}(t) := e^{it\Delta/2}$ from [24] and [16]. By defining $M_t = e^{\frac{ix^2}{2t}}$ and $\mathcal{D}$, a dilation operator, by

$$(\mathcal{D}_t\phi)(x) = \frac{1}{(it)^{1/2}}\phi\Big(\frac{x}{t}\Big),$$

we have that for any $f \in L^2_x$ and any $t \geq 1$,

$$\begin{aligned}
[\mathcal{U}(t)f](t,x) &= [\mathcal{F}^{-1}e^{-it\xi^2/2}\mathcal{F}_x f](t,x) \\
&= \mathcal{F}^{-1}\mathcal{F}\left[\frac{1}{(2\pi it)^{1/2}}e^{\frac{ix^2}{2t}} * f(x)\right] \\
&= \frac{1}{(2\pi it)^{1/2}}\int_{\mathbb{R}} e^{\frac{i(x-\xi)^2}{2t}} f(\xi)\, d\xi \\
&= e^{\frac{ix^2}{2t}}\frac{1}{(it)^{1/2}}\left[\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-\frac{ix\xi}{t}} e^{\frac{i\xi^2}{2t}} f(\xi)\, d\xi\right] \\
&= [M_t\mathcal{D}_t\mathcal{F}M_t f](t,x).
\end{aligned}$$

Note that f in the first line depends on x while the one in the last line depends on ξ. From this, we have $[\mathcal{U}(-t)f](t,x) = [M_{-t}\mathcal{F}^{-1}\mathcal{D}_{\frac{1}{t}}M_{-t}f](t,x)$. By applying this to our function u, it follows that

$$\begin{aligned}
\mathcal{F}\mathcal{U}(-t)u &= \mathcal{F}\mathcal{U}(-t)\left[u_1(t,x)\exp\left(2i\int_{-\infty}^{x}|u_1(x')|^2\ dx'\right)\right]\\
&= \mathcal{F}M_{-t}\mathcal{F}^{-1}\mathcal{D}_{\frac{1}{t}}M_{-t}\left[u_1(t,\xi)\exp\left(2i\int_{-\infty}^{\xi}|u_1(y)|^2\ dy\right)\right]\\
&= \mathcal{F}M_{-t}\mathcal{F}^{-1}\left[\left(\mathcal{D}_{\frac{1}{t}}M_{-t}u_1\right)\exp\left(2i\int_{-\infty}^{t\xi}|u_1(y)|^2\ dy\right)\right]\\
&= \mathcal{F}M_{-t}\mathcal{F}^{-1}\left[\left(\mathcal{D}_{\frac{1}{t}}M_{-t}u_1\right)\exp\left(2i\int_{-\infty}^{\xi}t|M_{-t}u_1(t\xi')|^2\ d\xi'\right)\right]\\
&= \mathcal{F}M_{-t}\mathcal{F}^{-1}\left[\left(\mathcal{D}_{\frac{1}{t}}M_{-t}u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|D_{\frac{1}{t}}M_{-t}u_1|^2\ d\xi'\right)\right]\\
&= \mathcal{F}M_{-t}\mathcal{F}^{-1}\left[\left(\mathcal{F}M_t\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|D_{\frac{1}{t}}M_{-t}u_1|^2\ d\xi'\right)\right]\\
&= \left(\mathcal{F}\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|D_{\frac{1}{t}}M_{-t}u_1|^2\ d\xi'\right)+G(t,\xi),
\end{aligned}$$

where

$$\begin{aligned}
G(t,\xi) &= \mathcal{F}(M_{-t}-1)\mathcal{F}^{-1}\left[\left(\mathcal{F}M_t\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|D_{\frac{1}{t}}M_{-t}u_1|^2\ d\xi'\right)\right]\\
&\quad+\left(\mathcal{F}(M_t-1)\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|D_{\frac{1}{t}}M_{-t}u_1|^2\ d\xi'\right)\\
&= \mathcal{F}(M_{-t}-1)\mathcal{F}^{-1}\left[\left(\mathcal{F}M_t\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|\mathcal{F}M_{-t}\mathcal{U}(-t)u_1|^2\ d\xi'\right)\right]\\
&\quad+\left(\mathcal{F}(M_t-1)\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|\mathcal{F}M_{-t}\mathcal{U}(-t)u_1|^2\ d\xi'\right)\\
&:= G_1+G_2.
\end{aligned}$$

Note that for any small $0<\gamma<\frac{1}{2}$, we have $|M_{-t}-1|\lesssim_\gamma \left|\frac{x^2}{t}\right|^{\frac{\gamma}{2}}$. Therefore, for any $\frac{1}{2}<\alpha\le 1-\gamma$, it follows from the Sobolev embedding, (4.52) and duality that

$$\begin{aligned}
\|G_1\|_{L^\infty_\xi} &\lesssim \|G_1\|_{H^\alpha}\\
&\lesssim \frac{1}{t^{\frac{\gamma}{2}}}\left\|\left(\mathcal{F}M_t\mathcal{U}(-t)u_1\right)\exp\left(2i\int_{-\infty}^{\xi}|\mathcal{F}M_{-t}\mathcal{U}(-t)u_1|^2\ d\xi'\right)\right\|_{H^{\alpha+\gamma}}
\end{aligned}$$

$$\lesssim \frac{1}{t^{\frac{\gamma}{2}}} \left(\|\mathcal{F} M_t \mathcal{U}(-t) u_1\|_{H^1} + \|\mathcal{F} M_t \mathcal{U}(-t) u_1\|_{L^2}^2 \|\mathcal{F} M_t \mathcal{U}(-t) u_1\|_{H^1} \right)$$
$$= \frac{1}{t^{\frac{\gamma}{2}}} \left(\|\mathcal{U}(-t) u_1\|_{H^{0,1}} + \|u_1\|_{L^2}^2 \|\mathcal{U}(-t) u_1\|_{H^{0,1}} \right).$$

and similarly

$$\begin{aligned}
\|G_2\|_{L^\infty_\xi} &= \|\mathcal{F}(M_t - 1)\mathcal{U}(-t) u_1\|_{L^\infty_\xi} \\
&\lesssim \|\mathcal{F}(M_t - 1)\mathcal{U}(-t) u_1\|_{H^\alpha} \\
&\lesssim \frac{1}{t^{\frac{\gamma}{2}}} \|x^{\alpha+\gamma} \mathcal{U}(-t) u_1\|_{L^2} \\
&\leq \frac{1}{t^{\frac{\gamma}{2}}} \|\mathcal{U}(-t) u_1\|_{H^{0,1}}.
\end{aligned}$$

Since $L = \mathcal{U}(t) x \mathcal{U}(-t)$, we can use (4.38) and set $\gamma = \frac{1}{2} - 2CD^2\epsilon^2$ to find a global bound for G.

$$\begin{aligned}
\|G\|_{L^\infty_\xi} &\lesssim \frac{1}{t^{\frac{\gamma}{2}}} \left[\|u_1\|_{L^2} + \|Lu_1\|_{L^2} + \|u_1\|_{L^2}^2 (\|u_1\|_{L^2} + \|Lu_1\|_{L^2}) \right] \\
&\lesssim \epsilon t^{-\frac{\gamma}{2} + CD^2\epsilon^2} \\
&\lesssim \epsilon t^{-1/4 + 2CD^2\epsilon^2}.
\end{aligned}$$

We can estimate G in L^2_ξ using the same proof. Since there is no need for the Sobolev embedding, we can take $\alpha = 0$ and $\gamma = 1 - 2CD^2\epsilon^2$, which yield

$$\|G\|_{L^2_\xi} \lesssim t^{-1/2 + 2CD^2\epsilon^2}.$$

Continuing the computation of $\mathcal{F}\mathcal{U}(-t)u$, we observe that the asymptotic (4.61) of u_1 implies $|D_{\frac{1}{t}} M_{-t} u_1(t,\xi)| \leq |W(\xi)| + t^{\frac{1}{2}} |err_p(t, tx)|$. Therefore,

$$\mathcal{F}\mathcal{U}(-t)u = \left(\mathcal{F}\mathcal{U}(-t) u_1 \right) \exp\left(2i \int_{-\infty}^{\xi} |D_{\frac{1}{t}} M_{-t} u_1|^2 \, d\xi' \right) + \epsilon t^{-1/4 + 2CD^2\epsilon^2}$$

$$= \Big(\mathcal{F}\mathcal{U}(-t)u_1\Big) \exp\left(2i\int_{-\infty}^{\xi} |W_1(\xi')|^2 \, d\xi'\right) + err_2'(t,\xi)$$

$$= W_1(\xi)e^{-i\xi|W_1(\xi)|^2 \log t} \exp\left(2i\int_{-\infty}^{\xi} |W_1(\xi')|^2 \, d\xi'\right) + err_2''(t,\xi)$$

$$= W(\xi)e^{-i\xi|W(\xi)|^2 \log t} + err_2''(t,\xi).$$

The error term can be approximated (ignoring the lower order terms in t) as follows:

$$\begin{aligned}
\|err_2''\|_{L^\infty_\xi} &\le \|err_f\|_{L^\infty_\xi} + \|err_2'\|_{L^\infty_\xi} + \epsilon t^{-1/4+2CD^2\epsilon^2} \\
&\lesssim \|err_f\|_{L^\infty_\xi} + t^{1/2}\|\mathcal{F}\mathcal{U}(-t)u_1\|_{L^\infty_\xi}\|W_1 err_p(t,tx)\|_{L^1_\xi} + \epsilon t^{-1/4+2CD^2\epsilon^2} \\
&\lesssim \|err_f\|_{L^\infty_\xi} + \|W_1\|_{L^\infty_\xi}\|W_1\|_{L^2_\xi}\|err_p(t)\|_{L^2_\xi} + \epsilon t^{-1/4+2CD^2\epsilon^2} \\
&\lesssim \epsilon t^{-1/4+2CD^2\epsilon^2} \\
\|err_2''\|_{L^2_\xi} &\le \|err_f\|_{L^2_\xi} + \|err_2'\|_{L^2_\xi} + \epsilon t^{-1/2+2CD^2\epsilon^2} \\
&\lesssim \|err_f\|_{L^\infty_\xi} + t^{1/2}\|u_1\|_{L^2_\xi}\|W_1 err_p(t,tx)\|_{L^1_\xi} + \epsilon t^{-1/2+2CD^2\epsilon^2} \\
&\lesssim \epsilon t^{-1/2+2CD^2\epsilon^2}.
\end{aligned}$$

By defining $err_2(t,\xi) := e^{-it\xi^2} err_2''(t,\xi)$, we obtain the wave profile for $\hat{u}$ as stated in (4.5).

4.5 The asymptotic completeness

We will now prove *(C)* in Theorem 4.1, which is the asymptotic completeness of equation (1.15). Let $W \in H^{1+C\epsilon^2,1}(\mathbb{R})$. Roughly speaking, we want to show the existence of a small initial data $u_0 \in H^{1,1}$ whose profile in (4.4) is associated with W. Let W_1 be the gauge transformation of W:

$$W_1 = W \exp\left(2i \int_{-\infty}^{x} |W(x')|^2 \, dx'\right).$$

An application of (4.53) and (4.62) shows that

$$\|W_1\|_{H^{1+C\epsilon^2,1}} = \left\| W \exp\left(2i \int_{-\infty}^{x} |W(x')|^2 \, dx'\right) \right\|_{H^{1+C\epsilon^2,1}} \lesssim \|W\|_{H^{1+C\epsilon^2,1}}.$$

Therefore, it suffices to find initial data $u_{1,0} \in H^{1,1}$ and $u_{2,0} \in H^{0,1}$ so that the system of equations (4.11) has a solution whose profiles in (4.61) and (4.66) are associated with W_1, because we can then obtain u_0 from $u_0 = u_1 \exp(\int_{-\infty}^{x} |u_1|^2 \, dx')$. To simplify the proof below, we will make an abuse of notations and replace W_1 by W. Here, we assume a stronger bound on W:

$$\|W\|_{H^{1+2\delta,1}(\mathbb{R})} \le M \ll 1 \quad \text{where} \quad M, \delta > 0 \;\text{ and }\; M^2 \ll \delta. \tag{4.70}$$

Let v_1 and v_2 be the asymptotic profiles of u_1 and u_2 defined by

$$\begin{aligned} v_1(t,x) &:= \frac{1}{t^{1/2}} W\left(\frac{x}{t}\right) e^{i|W(x/t)|^2(x/t)\log t + ix^2/(2t)}, \\ v_2(t,x) &:= \frac{ix}{\sqrt{2}t^{3/2}} W\left(\frac{x}{t}\right) e^{i|W(x/t)|^2(x/t)\log t + ix^2/(2t)}. \end{aligned}$$

In the argument below, we will take a look at L^∞ and L^2 behavior of $(i\partial_t + \frac{1}{2}\partial_x^2)v_i$ and $L(i\partial_t + \frac{1}{2}\partial_x^2)v_i$ for $i = 1, 2$, which requires the spatial regularity of W up to the third order. For this reason, we create a new profile that resemble W and has the desired regularity using the frequency cutoff.

$$\mathcal{W}(t,v) := P_{\le t^{1/2}} W(v)$$

where $\mathcal{F}(P_{\le t^{1/2}}W)(t,\xi) := \psi_{\le 1}(\xi/t^{1/2})\widehat{W}(t,\xi)$. Then the approximate profiles are

$$w_1(t,x) := \frac{1}{\sqrt{t}}\mathcal{W}\left(t,x/t\right)e^{i|\mathcal{W}(t,x/t)|^2(x/t)\log t + ix^2/(2t)},$$
$$w_2(t,x) := \frac{ix}{\sqrt{2}t^{3/2}}\mathcal{W}\left(t,x/t\right)e^{i|\mathcal{W}(t,x/t)|^2(x/t)\log t + ix^2/(2t)}.$$

By Bernstein's inequality, we have

$$\|\mathcal{W}(t,v) - W(v)\|_{L^2_v} \lesssim Mt^{-1/2-\delta}, \quad \|\mathcal{W}(t,v) - W(v)\|_{L^\infty_v} \lesssim Mt^{-1/4-\delta}.$$

We see that w_1 is a good approximation of v_1.

$$|v_1 - w_1| \lesssim \frac{1}{\sqrt{t}}|W(x/t) - \mathcal{W}(t,x/t)| + \frac{\log t}{\sqrt{t}}\left|\frac{x}{t}\mathcal{W}(t,x/t)(|W(x/t)|^2 - |\mathcal{W}(x/t)|^2)\right|$$

Some factors in the second term can be bounded by Sobolev embedding.

$$|W(v)| + (|v|+1)|\mathcal{W}(v)| \lesssim \|W\|_{H^{1+2\delta,1}(\mathbb{R})} \le M.$$

Therefore, we can see that w_1 is a good approximation of v_1.

$$\|v_1 - w_1\|_{L^2_x} \lesssim Mt^{-1/2-\delta}(1 + M^2\log t),$$
$$\|v_1 - w_1\|_{L^\infty} \lesssim Mt^{-3/4-\delta}(1 + M^2\log t).$$

To get an approximation for v_2, we note that w_2 contains the term $\frac{x}{t}\mathcal{W}(t,x/t)$. Applying the Fourier transform,

$$\mathcal{F}(vP_{\le t^{1/2}}W) \approx \partial_\xi[\psi(\xi/t^{1/2}\widehat{W}(\xi)] = \partial_\xi\psi(\xi/t^{1/2})\widehat{W}(\xi) + \psi(\xi/t^{1/2})\partial_\xi\widehat{W}(\xi),$$

it follows from the duality and Young's inequality that

$$\begin{aligned} \|vP_{\le t^{1/2}}W\|_{L^2_v} &\lesssim \|P_{\le t^{1/2}}vW\|_{L^2_v} + t^{-1/2}\|W\|_{L^2_v} \\ \|vP_{\le t^{1/2}}W\|_{L^\infty} &\lesssim \|P_{\le t^{1/2}}vW\|_{L^\infty} + t^{-1/2}\|W\|_{L^\infty}. \end{aligned} \tag{4.71}$$

It follows that for v_2 and w_2,

$$\|v_2 - w_2\|_{L_x^2} \lesssim Mt^{-1/2-\delta}(1 + M^2 \log t),$$

$$\|v_2 - w_2\|_{L^\infty} \lesssim Mt^{-3/4-\delta}(1 + M^2 \log t).$$

Thus we know that w_1 and w_2 are good approximations of v_1 and v_2. Let u_1 and u_2 be any solution to (4.11). By defining $U_i := u_i - w_i$ for $i = 1, 2$, we see that if the profile of u_i is associated with W_1, then $U_i(\infty) := \lim_{t\to\infty} U(t, x) = 0$ for all $x \in \mathbb{R}$ and vice versa. Therefore, using the information that u_1 and u_2 solve (4.11), we can reformulate the problem as a PDE for U_i with zero initial data at $t = \infty$. To do this, we let n, m to be a permutation of the indices $1, 2$ and define the error function f_n by

$$f_n := (i\partial_t + \frac{1}{2}\partial_x^2)w_n - i(-1)^n\sqrt{2}\,\overline{w_m}w_n^2, \tag{4.72}$$

and then consider the equations for U_n:

$$(i\partial_t + \frac{1}{2}\partial_x^2)U_n = -i(-1)^n\sqrt{2}(\overline{U_m + w_m})(U_n + w_n)^2 + i(-1)^n\sqrt{2}\,\overline{w_m}w_n^2 - f_n.$$

This can be rewritten as

$$(i\partial_t + \frac{1}{2}\partial_x^2)U_n = N_n(U, w) - f_n, \qquad U_n(\infty) = 0, \tag{4.73}$$

where $U := (U_1, U_2)$ and $w := (w_1, w_2)$ and

$$N_n(U, w) = i(-1)^m\sqrt{2}(U_n^2\overline{U}_m + U_n^2\overline{w}_m + 2U_n\overline{U}_m w_n + 2U_n w_n\overline{w}_m + \overline{U}_m w_n^2).$$

The equation can be rewritten as

$$U_n(t, x) = \Phi(t)N_n(U, w) - \Phi(t)f_n$$

where $\Phi(t)f := i\int_t^\infty e^{\frac{i(t-s)}{2}\Delta} f(s)\, ds$. The Strichartz-type estimate for this problem can be stated as follows:

$$\|\Phi(t)f\|_{L_t^\infty L_x^2(T,\infty)} + \|\Phi(t)f\|_{L_t^4 L_x^\infty(T,\infty)} \lesssim \|f\|_{L_t^1 L_x^2(T,\infty)}. \tag{4.74}$$

We will solve for U_n using the contraction argument. The solution space X is defined by

$$\|U_n\|_X := \sup_{T\geq 1} \frac{T^{1/2+\delta}}{(1+M^2\log T)^2} \left(\|U_n\|_{L_t^\infty L_x^2(T,2T)} + \|U_n\|_{L_t^4 L_x^\infty(T,2T)}\right), \tag{4.75}$$

and the space for LU_n is defined by

$$\|LU_n\|_{\tilde{X}} := \sup_{T\geq 1} \frac{T^{\delta}}{(1+M^2\log T)^3} \left(\|LU_n\|_{L_t^\infty L_x^2(T,2T)} + \|LU_n\|_{L_t^4 L_x^\infty(T,2T)}\right). \tag{4.76}$$

To make a contraction argument, we need the following estimate for the inhomogeneous terms, whose proof is postponed until the next section.

Lemma 4.17. *For $n = 1, 2$ the function f_n defined in* (4.72) *satisfies the following estimate:*

$$\|\Phi f_n\|_X + \|\Phi L f_n\|_{\tilde{X}} \lesssim M. \tag{4.77}$$

In view of (4.77), it suffices to prove that the map $U \mapsto (N_1(U,w), N_2(U,w))$ is a contraction for a small ball in $X \times X$. This can be done by proving the following estimate

$$\sum_{n=1}^{2} \|N_n(U,w) - N_n(\tilde{U},w)\|_{L_t^1 L_x^2(T,\infty)} \lesssim \sum_{n=1}^{2} \|U_n - \tilde{U}_n\|_X (M + \|U_n\|_X^2 + \|\tilde{U}_n\|_X^2). \tag{4.78}$$

Let C be the implicit constant in (4.78). Then, by choosing $M \ll C^{-1}$, the map is a contraction on the ball $\{(U_1,U_2) \in X \times X : \|U_1\|_X + \|U_2\|_X\} \leq CM$. Consequently, we obtain solutions U_1 and U_2 satisfying

$$\|U_1\|_X + \|U_2\|_X \lesssim M \tag{4.79}$$

The estimate (4.78) can be simplified by taking $\tilde{U} = 0$, resulting in

$$\sum_{n=1}^{2} \|N_n(U,w)\|_{L_t^1 L_x^2(T,\infty)} \lesssim \sum_{n=1}^{2} M\|U_n\|_X + \|U_n\|_X^3, \tag{4.80}$$

and the proof of (4.78) will follow in the same manner. To begin the proof, we divide $[T,\infty)$ into dyadic intervals on each of which we will estimate. In the following computations, we let $k, l \in \{1,2\}$

$$\|U_k w_k w_l\|_{L_t^1 L_x^2(T,2T)} \lesssim T\|w_k\|_{L^\infty_{x,t}(T,2T)}\|w_l\|_{L^\infty_{x,t}(T,2T)}\|U_k\|_{L_t^\infty L_x^2(T,2T)}$$

$$\|U_k U_l w_m\|_{L_t^1 L_x^2(T,2T)} \lesssim T^{3/4}\|w_m\|_{L^\infty_{x,t}(T,2T)}\|U_k\|_{L_t^\infty L_x^2(T,2T)}\|U_l\|_{L_t^4 L_x^\infty(T,2T)},$$

$$\|U_n^2 U_m\|_{L_t^1 L_x^2(T,2T)} \lesssim T^{1/2}\|U_m\|_{L_t^\infty L_x^2(T,2T)}\|U_n\|^2_{L_t^4 L_x^\infty(T,2T)}.$$

From the definitions of w_1 and w_2, we have

$$\begin{aligned}
\|U_k w_k w_l\|_{L_t^1 L_x^2(T,2T)} &\lesssim M^2 T^{-1/2-\delta}(1+M^2\log T)^2\|U_k\|_X \\
\|U_k U_l w_k\|_{L_t^1 L_x^2(T,2T)} &\lesssim M T^{-3/4-2\delta}(1+M^2\log T)^4\|U_k\|_X\|U_l\|_X \\
\|U_n^2 U_m\|_{L_t^1 L_x^2(T,2T)} &\lesssim T^{-1-3\delta}(1+M^2\log T)^6\|U_m\|_X\|U_n\|_X^2.
\end{aligned} \tag{4.81}$$

Those account for all terms in $N_1(U,w)$ and $N_2(U,w)$. After taking the summations in T, we obtain (4.80) as desired.

We will now find the L_x^2 bounds for LU_1 and LU_2. Note that both functions satisfy the system of equations in V_1 and V_2:

$$(i\partial_t + \frac{1}{2}\partial_x^2)V_n = LN_n(U,w) - Lf_n, \qquad LU_n(\infty) = 0. \tag{4.82}$$

By the uniqueness of the solution obtained from the contraction argument, we are guaranteed that (LU_1, LU_2) is the only solution to (4.82). We rewrite these equations as

$$\begin{aligned}
LU_n(t,x) &= \Phi(t)LN_n(U,w) - \Phi(t)Lf_n \\
&= P(LU_n) + Q(LU_m) + g_n - Lf_n,
\end{aligned}$$

where

$$P(LU_n) = i(-1)^m 2\sqrt{2}(LU_nU_n\overline{U}_m + LU_nU_n\overline{w}_m + LU_n\overline{U}_mw_n + LU_nw_n\overline{w}_m),$$

$$Q(LU_m) = i(-1)^n\sqrt{2}(U_n^2\overline{LU_m} - 2U_n\overline{LU_m}w_n - \overline{LU_m}w_n^2),$$

$$g_n = i(-1)^m\sqrt{2}(2\overline{U}_mw_nLw_n - U_n^2\overline{Lw_m}$$

$$+ 2U_n\overline{U}_mLw_n + 2U_nLw_n\overline{w}_m - 2U_nw_n\overline{Lw_m}).$$

We estimate as in (4.81), using (4.79) for U_1 and U_2. Notice that the worst terms, namely the last terms of $P(LU_n)$ and $Q(LU_m)$, give the lowest order of decay in T.

$$\|P(LU_n)\|_{L_t^1L_x^2(T,2T)} \lesssim M^2T^{-\delta}(1 + M^2\log T)^3\|LU_n\|_{\tilde{X}},$$

$$\|Q(LU_m)\|_{L_t^1L_x^2(T,2T)} \lesssim M^2T^{-\delta}(1 + M^2\log T)^3\|LU_m\|_{\tilde{X}}.$$

After taking the summation over dyadic $T \geq 1$, we have

$$\sum_{n=1}^{2}\|P(LU_n)\|_{L_t^1L_x^2(T,\infty)} \lesssim \delta^{-1}M^2T^{-\delta}(1 + M^2\log T)^3\sum_{n=1}^{2}\|LU_n\|_{\tilde{X}} \tag{4.83}$$

Let C be the implicit constant. In order to get a contraction map, we can pick any positive $M \ll \left(\frac{\delta}{C}\right)^{\frac{1}{2}}$. To estimate g_n, we need the bounds on Lw_n which we get from the direct calculations.

$$\|Lw_1\|_{L_x^2} \lesssim \|\partial_v\mathcal{W}\|_{L_v^2} + \log t\|\mathcal{W}\|_{L_v^\infty}^2\|v\partial_v\mathcal{W}\|_{L_v^2}$$

$$\lesssim M(1 + M^2\log t),$$

$$\|Lw_2\|_{L_x^2} \lesssim t^{-1}\|\mathcal{W}\|_{L_v^2} + \|v\partial_v\mathcal{W}\|_{L_v^2} + \log t\|v\mathcal{W}\|_{L_v^\infty}^2\|\partial_v\mathcal{W}\|_{L_v^2}$$

$$\lesssim M(1 + M^2\log t).$$

Consequently,

$$\|g_n\|_{L_t^1L_x^2(T,2T)} \lesssim M^3T^{-1/4-2\delta}(1 + M^2\log T)^3.$$

Taking the summation over dyadic $T \geq 1$, we have

$$\|g_n\|_{L_t^1 L_x^2(T,\infty)} \lesssim M^3 T^{-1/4-2\delta}(1 + M^2 \log T)^3.$$

We then take $T = 1$. From the Strichartz estimate (4.74) and (4.83), we conclude that

$$\sum_{n=1}^{2} \|LU_n\|_{L_t^\infty(1,\infty;L_x^2)} \lesssim M$$

as desired.

4.6 Proof of Lemma 4.17

In view of (4.74), it suffices to estimate $\|f_n\|_{L_t^1 L_x^2(T,\infty)}$ and $\|Lf_n\|_{L_t^1 L_x^2(T,\infty)}$. We begin with computing f_n. We denote a function $W_{\leq t^{1/2}}(t, v)$ by

$$\widehat{W}_{\leq t^{1/2}}(t, \xi) = \psi_{<1}\left(\frac{\xi}{t^{1/2}}\right) \widehat{W}(t, \xi).$$

Observe that

$$\frac{d}{dt}\psi_{<1}\left(\frac{\xi}{t^{1/2}}\right) = \frac{\xi}{t^{3/2}}\psi'_{<1}\left(\frac{\xi}{t^{1/2}}\right).$$

where the hidden constants in the approximation only depend on our choice of ψ_1. We see that

$$\partial_t \mathcal{W}(t, v) = \int_{\mathbb{R}} \frac{\xi}{t^{3/2}}\psi'_{<1}\left(\frac{\xi}{t^{1/2}}\right) \widehat{W}(\xi) e^{iv\xi}\, d\xi := \frac{1}{t} W_{t^{1/2}}(v).$$

Consequently,

$$
\begin{aligned}
f_1 = \frac{1}{t^{1/2}} e^{i\frac{x^2}{2t} + i\frac{x}{t}\log t|\mathcal{W}|^2} \Bigg\{ & \frac{1}{t}\left[W_{t^{1/2}} + 2i\frac{x}{t}\log t \mathcal{W}\Re(W_{t^{1/2}}\overline{\mathcal{W}})\right] \\
& + \frac{1}{2t^2}\left[\mathcal{W}'' + 2i\frac{x}{t}\log t\mathcal{W}\Re(\mathcal{W}''\overline{\mathcal{W}}) + 2i\log t\mathcal{W}'|\mathcal{W}|^2\right] \\
& + \frac{1}{t^2}\left[i\frac{x}{t}\log t\mathcal{W}'\Re(\mathcal{W}'\overline{\mathcal{W}}) + i\frac{x}{t}\log t\mathcal{W}|\mathcal{W}'|^2 - 2\mathcal{W}\left(\frac{x}{t}\log t\Re(\mathcal{W}'\overline{\mathcal{W}})\right)^2\right. \\
& \left. - (\log t)^2\mathcal{W}|\mathcal{W}|^4 - \frac{2x}{t}(\log t)^2\mathcal{W}|\mathcal{W}|^2\Re(\mathcal{W}'\overline{\mathcal{W}})\right]\Bigg\} \\
f_2 = \frac{i}{\sqrt{2}t^{1/2}} e^{i\frac{x^2}{2t} + i\frac{x}{t}\log t|\mathcal{W}|^2} \Bigg\{ & \frac{x}{t^2}\left[W_{t^{1/2}} + 2i\frac{x}{t}\log t \mathcal{W}\Re(W_{t^{1/2}}\overline{\mathcal{W}})\right] \\
& + \frac{x}{2t^3}\left[\mathcal{W}'' + 2i\frac{x}{t}\log t\mathcal{W}\Re(\mathcal{W}''\overline{\mathcal{W}}) + 2i\log t\mathcal{W}'|\mathcal{W}|^2\right] \\
& + \frac{x}{t^3}\left[i\frac{x}{t}\log t\mathcal{W}'\Re(\mathcal{W}'\overline{\mathcal{W}}) + i\frac{x}{t}\log t\mathcal{W}|\mathcal{W}'|^2 - 2\mathcal{W}\left(\frac{x}{t}\log t\Re(\mathcal{W}'\overline{\mathcal{W}})\right)^2\right. \\
& \left. - (\log t)^2\mathcal{W}|\mathcal{W}|^4 - \frac{2x}{t}(\log t)^2\mathcal{W}|\mathcal{W}|^2\Re(\mathcal{W}'\overline{\mathcal{W}})\right] \\
& + \frac{1}{t^2}\left[\mathcal{W}' + 2i\frac{x}{t}\log t\mathcal{W}\Re(\mathcal{W}'\overline{\mathcal{W}}) + i\log t\mathcal{W}|\mathcal{W}|^2\right]\Bigg\}.
\end{aligned}
\tag{4.84}
$$

From the definition (4.70) and (4.71), Bernstein's inequality yields

$$
\begin{aligned}
& \|\langle v\rangle \mathcal{W}\|_{L^\infty} \lesssim M, \quad \|\langle v\rangle \mathcal{W}'\|_{L^\infty} \lesssim Mt^{1/4-\delta}, \\
& \|\langle v\rangle \mathcal{W}'\|_{L^2_v} \lesssim M, \quad \|\langle v\rangle \mathcal{W}''\|_{L^2_v} \lesssim Mt^{1/2-\delta}, \quad \|\langle v\rangle \mathcal{W}'''\|_{L^2_v} \lesssim Mt^{1-\delta}, \\
& \|\langle v\rangle W_{t^{1/2}}\|_{L^2_v} \lesssim Mt^{-1/2-\delta}, \quad \|\langle v\rangle W'_{t^{1/2}}\|_{L^2_v} \lesssim Mt^{-\delta}.
\end{aligned}
\tag{4.85}
$$

Therefore, we obtain the estimate

$$
\|f_n\|_{L^2_x} \lesssim Mt^{-3/2-\delta}(1 + M^2\log t)^2. \tag{4.86}
$$

By integration in time and the Strichartz estimate (4.74), we obtain the bound for Φf_n in (4.77). To estimate Lf_n, we use $L(e^{i\frac{x^2}{2t}}g(x/t)) = ie^{i\frac{x^2}{2t}}\partial_v g(x/t)$ to obtain

$$
\|Lf_n\|_{L^2_x} \lesssim Mt^{-1-\delta}(1 + M^2\log t)^3. \tag{4.87}
$$

However, an integration in time gives an extra δ^{-1} factor.

$$\|Lf_n\|_{L^1_tL^2_x(T,2T)} \lesssim \delta^{-1}MT^{-\delta}(1+M^2\log T)^3.$$

Applying (4.74) directly gives us an extra δ^{-1} factor which does not imply (4.17) since $\delta \gg M^2$, so more careful analysis on Lf_n is required. Notice that the problem arises from the terms in Lf_n that give time decay $t^{-1-\delta}$ in L^2_x; For example, we estimate the first term in Lf_2, ignoring terms with higher decay in t:

$$\begin{aligned}\left\|L\left[\frac{x}{t^{5/2}}e^{i\frac{x^2}{2t}+i\frac{x}{t}\log t|\mathcal{W}|^2}W_{t^{1/2}}\left(\frac{x}{t}\right)\right]\right\|_{L^2_x} &\lesssim \frac{1}{t^{3/2}}\left\||\frac{x}{t}W_{t^{1/2}}\left(\frac{x}{t}\right)|+|W'_{t^{1/2}}\left(\frac{x}{t}\right)|\right\|_{L^2_x}\\ &= \frac{1}{t}\||vW_{t^{1/2}}(v)|+|W'_{t^{1/2}}(v)|\|_{L^2_v}\\ &\lesssim Mt^{-1-\delta},\end{aligned}$$

where we used (4.85) for the last inequality. In fact, the terms that give $t^{-1-\delta}$ decay are precisely those that contain $\mathcal{W}'''$ or $W'_{t^{1/2}}$. Since the rest of the terms do not contribute a factor of δ^{-1} to $\|Lf_n\|_{L^2_x}$ via (4.74), it suffices to estimate these terms. In order to do so, we let

$$h_n := iL\frac{1}{t^{3/2}}e^{i\frac{x^2}{2t}}Z_n(t,x/t) = \frac{1}{t^{1/2}}e^{i\frac{x^2}{2t}}\partial_x(Z_n(t,x/t)) = \frac{1}{t^{3/2}}e^{i\frac{x^2}{2t}}\partial_v Z_n(t,v)$$

where $v=\frac{x}{t}$ and

$$\begin{aligned}Z_1 &:= e^{iv\log t|\mathcal{W}|^2}\left(W_{t^{1/2}}+2iv\log t\mathcal{W}\mathrm{Re}(W_{t^{1/2}}\overline{\mathcal{W}})+\frac{\mathcal{W}''}{2t}+i\frac{v}{t}\log t\mathcal{W}\mathrm{Re}(\mathcal{W}''\overline{\mathcal{W}})\right)\\ Z_2 &:= \frac{i}{\sqrt{2}}vZ_1.\end{aligned}$$

The proof is finished after we estimate Φh_n in $\tilde{X}$. First, using (4.85) and their weighted version, we have

$$\|\partial_v^j Z_n\|_{L^2_v} \lesssim t^{-1+\frac{j}{2}}(1+M^2\log t)^{j+1}\|\langle v\rangle W''_{\le t^{1/2}}\|_{L^2_v}, \qquad j=0,1,2. \tag{4.88}$$

Since

$$\|\langle v\rangle W''_{\leq t^{1/2}}\|_{L^2_v} \lesssim t^{\frac{1}{2}-\delta}\|W\|_{H_v^{1+2\delta,1}} \leq t^{\frac{1}{2}-\delta}M, \tag{4.89}$$

inequality (4.88) with $j=1$ tells us that $\partial_v Z_n$ (and hence h_n) decays in L^2_v sense in the region of frequencies greater than $t^{1/2}$.

With that in mind, we compute

$$\begin{aligned}
\Phi h_n(t) &= ie^{it\Delta/2}\mathcal{F}_\xi^{-1}\left\{\int_t^\infty \int \frac{1}{s^{3/2}} e^{-ix\xi} e^{is\xi^2/2} e^{i\frac{x^2}{2s}} \partial_v Z_n(s, x/s)\ dxds\right\} \\
&= ie^{it\Delta/2}\mathcal{F}_\xi^{-1}\left\{\int_t^\infty \int \frac{1}{s^{1/2}} e^{is(\xi-v)^2/2} \partial_v Z_n(s,v)\ dvds\right\} \\
&= ie^{it\Delta/2}\mathcal{F}_\xi^{-1}\left\{\int_t^\infty \frac{1}{s^{1/2}} e^{is\xi^2/2} * \partial_v Z_n(s,\xi) ds\right\} \\
&= ie^{it\Delta/2}\int_t^\infty s^{-1} e^{i\frac{x^2}{2s}} \widehat{\partial_v Z_n}(s,-x) ds \\
&= e^{it\Delta/2}\int_t^\infty s^{-1} x e^{i\frac{x^2}{2s}} \widehat{Z_n}(s,-x) ds.
\end{aligned} \tag{4.90}$$

To estimate the right hand side under the norm of $L^1_t L^2_x(T,\infty)$, we claim that the following estimate holds:

$$\|t^{-1} x e^{i\frac{x^2}{2t}} \widehat{Z_n}(t,x)\|_{l^2_x L^1_t L^2_x(T,\infty)} \lesssim MT^{-\delta}(1+M^2\log T)^3. \tag{4.91}$$

where the l^2_x norm is the l^2 sum taken with respect to the dyadic intervals in x. Notice that there is no more δ^{-1} factor.

Assuming that (4.91) is true, we then define $z_n(t) := e^{-it\Delta/2}\Phi h_n(t)$. In other words,

$$z_n(t) = \int_t^\infty s^{-1} x e^{i\frac{x^2}{2s}} \widehat{Z_n}(s,-x) ds$$

From (4.88) with $j=1$, we see that $z_n : [T,\infty) \to L^2_x(\mathbb{R})$ is continuous and the estimate (4.91) implies that $z_n \in l^2_x \dot{W}^{1,1}_t L^2_x(T,\infty)$. Even though the l^2_x norm and the $\dot{W}^{1,1}_t$ norm cannot be interchanged, we can embed $\dot{W}_t^{1,1}$ in a larger space which allows us to do so. First, we introduce the space V^p of functions of bounded p variation with respect to a Banach space $\mathcal{B}$ defined by the seminorm:

$$\|z\|_{V^p\mathcal{B}(T,\infty)} := \sup_{T=t_0<\ldots<t_K<\infty} \left(\sum_{k=1}^{K} \|z(t_k) - z(t_{k-1})\|_{\mathcal{B}}^p \right)^{1/p}.$$

Notice that V^1 is the space of functions of bounded variation and $\dot{W}_t^{1,1}L_x^2(T,\infty)$ is the space of absolutely continuous functions from $[T,\infty)$ to $L_x^2(\mathbb{R})$, so we have an embedding $\dot{W}_t^{1,1}L_x^2(T,\infty) \subset V^1 L_x^2(T,\infty)$. Therefore, in view of (2.75) and (2.77), we have the following chain of inclusions:

$$l_x^2 \dot{W}_t^{1,1} L_x^2(T,\infty) \subset l_x^2 V^2 L_x^2(T,\infty) \subset V^2 l_x^2 L_x^2(T,\infty) = V^2 L_x^2(T,\infty).$$

Since z_n is continuous and $z_n(\infty) = 0$, so it satisfies the hypothesis in Proposition 2.19 after a time reflection $t \mapsto -t$. Therefore, by (2.76),

$$\|e^{it\Delta/2} z_n(t)\|_{L_t^\infty L_x^2(T,\infty)} + \|e^{it\Delta/2} z_n(t)\|_{L_t^4 L_x^\infty(T,\infty)} \lesssim \|z_n\|_{V^2 L_x^2(T,\infty)}.$$

It follows that

$$\begin{aligned} \|\Phi h_n(t)\|_{L_t^\infty L_x^2(T,\infty)} + \|\Phi h_n(t)\|_{L_t^4 L_x^\infty(T,\infty)} &\lesssim \|z_n\|_{V^2 L_x^2(T,\infty)} \\ &\lesssim M T^{-\delta}(1 + M^2 \log T)^3. \end{aligned}$$

This leads to the desired estimate $\|\Phi L f_n\|_{\tilde{X}} \lesssim M$.

It remains to prove (4.91). We divide this into two cases:

Case 1: $|x| < T^{1/2}$. From (4.88) with $j = 0$ and (4.89), we obtain

$$\|t^{-1} x \widehat{Z_n}(t,x)\|_{l_x^2 L_t^1 L_x^2(T,\infty)} \lesssim T^{1/2} \|t^{-1} \widehat{Z_n}(t,x)\|_{l^\infty L_t^1 L_x^2(T,\infty)}$$

$$\lesssim T^{1/2}\|t^{-2}(1+M^2\log t)\langle v\rangle W''_{\le t^{1/2}}\|_{l^\infty L^1_t L^2_v(T,\infty)}$$

$$\lesssim T^{-1/2}(1+M^2\log T)\|\langle v\rangle W''_{\le t^{1/2}}\|_{L^2_v}$$

$$\lesssim MT^{-\delta}(1+M^2\log T).$$

Case 2: $|x|\ge T^{1/2}$. We consider each term in the l^2 sum where $|x|\approx R\ge T^{1/2}$. In the following integral with respect to t, we apply the estimate (4.88) with $j=0$ for $R<t^{1/2}$ and $j=2$ for $R\ge t^{1/2}$ (note that $\|\partial_v^2\widehat{Z_n}\|\approx R^2\|\widehat{Z_n}\|$ in this region),

$$\begin{aligned}
&\|t^{-1}x\widehat{Z_n}(t,x)\|_{L^1_tL^2_x(T,\infty;|x|\approx R)}\\
&\quad\lesssim \int_T^{R^2}\frac{1}{Rt}(1+M^2\log t)^3\|\langle v\rangle W''_{\le t^{1/2}}\|_{L^2_v(|v|\approx R)}\,dt\\
&\qquad+\int_{R^2}^\infty \frac{R}{t^2}(1+M^2\log t)\|\langle v\rangle W''_{\le t^{1/2}}\|_{L^2_v(|v|\approx R)}\,dt\\
&\quad\le R\left(\int_T^{R^2}\frac{1}{R^2t^2}(1+M^2\log t)^6\|\langle v\rangle W''_{\le t^{1/2}}\|^2_{L^2_v(|v|\approx R)}\,dt\right)^{1/2}\\
&\qquad+\frac{1}{R}\left(\int_{R^2}^\infty \frac{R^2}{t^2}(1+M^2\log t)^2\|\langle v\rangle W''_{\le t^{1/2}}\|^2_{L^2_v(|v|\approx R)}\,dt\right)^{1/2}\\
&\quad\lesssim\left(\int_T^\infty\frac{1}{t^2}(1+M^2\log t)^6\|\langle v\rangle W''_{\le t^{1/2}}\|^2_{L^2_v(|v|\approx R)}\,dt\right)^{1/2}.
\end{aligned}$$

Taking the l^2 sum with respect to $R\in 2^{\mathbb{Z}}$, we have

$$\|t^{-1}x\widehat{Z_n}(t,x)\|^2_{l^2_xL^1_tL^2_x(T,\infty)}\lesssim\int_T^\infty\frac{1}{t^2}(1+M^2\log t)^6\|\langle v\rangle W''_{\le t^{1/2}}\|^2_{L^2_v}\,dt. \tag{4.92}$$

Note that by the duality,

$$\begin{aligned}
\|\langle v\rangle W''_{\le t^{1/2}}\|_{L^2_v}&\lesssim\|P_{\le t^{1/2}}vW''\|_{L^2_v}+t^{-1/2}\|W''_{\le 4t^{1/2}}\|_{L^2_v}\\
&\lesssim t^{-\delta}\|P_{\le t^{1/2}}\langle D\rangle v\langle D\rangle^{1+2\delta}W\|_{L^2_v}+\|P_{\le t^{1/2}}W\|_{H^{1+2\delta}_v}\\
&\quad+t^{-\delta}\|W_{\le 4t^{1/2}}\|_{H^{1+2\delta}_v}.
\end{aligned}$$

It suffices to use the first term to estimate the integral in (4.92), as the other terms will give lower orders of t in the following argument. Since $M^2 \ll \delta$, we have that

$$\begin{aligned}&\int_T^\infty t^{-2-2\delta}(1+M^2\log t)^6\|P_{\le t^{1/2}}\langle D\rangle v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\ dt\\&\lesssim T^{-2\delta}(1+M^2\log T)^6\int_T^\infty t^{-2}\|P_{\le t^{1/2}}\langle D\rangle v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\ dt.\end{aligned} \tag{4.93}$$

Now we split the integral over dyadic intervals

$$\begin{aligned}&\int_T^\infty t^{-2}\|P_{\le t^{1/2}}\langle D\rangle v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\ dt\\&=\sum_{2^N>\frac{T^{1/2}}{2}}\int_{2^{2N}}^{2^{2N+2}} t^{-2}\left\{\sum_{\frac{T^{1/2}}{2}<2^k<2t^{1/2}}\|P_{2^k}\langle D\rangle v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\right\}\ dt\\&\lesssim\sum_{2^k>\frac{T^{1/2}}{2}}\sum_{N\ge k}\int_{2^{2N}}^{2^{2N+2}}\frac{2^{2k}}{2^{4N}}\|P_{2^k}v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\ dt\\&\le\sum_{2^k>\frac{T^{1/2}}{2}}\sum_{N\ge k}\frac{2^{2k}}{2^{2N}}\|P_{2^k}v\langle D\rangle^{1+2\delta}W\|_{L_v^2}^2\\&\lesssim\|W\|_{H_v^{1+2\delta,1}}^2\\&\le M.\end{aligned}$$

Combined with (4.93), we get (4.91) as desired.

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Chapter 4, in full, is currently being prepared for submission for publication of the material. Pornnopparath, Donlapark. The dissertation author was the primary investigator and author of this material.

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