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# UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

## Lecture XIV

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# CONTINUED FRACTIONS

## 1. Introduction

The purpose of this paper is twofold. First, to give a small amount of general information on Continued Fractions so that anyone faced with continued fractions will have a starting point for working with them. Secondly, to give various uses of continued fractions to show how they may be applied.

## 2. Definition

A continued fraction may be defined as a sequence of fractions (called partial quotients), arranged in such a manner that the numerator of each succeeding fraction is added to the denominator of the preceding fraction.

The words "continued fraction" will be abbreviated as C.F. hereafter. The C.F. is written

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

or in briefer form

$$a_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots .$$

The $\frac{a_i}{b_i}$ are the partial quotients, where the a's and b's may be functions of any parameters or complex numbers, and $a_o$ may or may not be zero. The C.F. may terminate or may be non-terminating.

3. Approximants and Recursion Formulas.

Denote the C.F. by $\frac{A_n}{B_n}$, $n = 0, 1, 2, \ldots,$ where the value of $n$ tells one at which partial quotient one should evaluate the C.F. That is:

$n = 0$, then $\frac{A_o}{B_o} = a_o$, where one takes $\begin{matrix} A_o = a_o \\ B_o = 1 \end{matrix}$

$n = 1$, then $\frac{A_1}{B_1} = a_o + \frac{a_1}{b_1} = \frac{a_o b_1 + a_1}{b_1}$, where $\begin{matrix} A_1 = a_o b_1 + a_1 \\ B_1 = b_1 \end{matrix}$ .

$n = 2$, then $\frac{A_2}{B_2} = a_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \frac{(a_o b_1 + a_1) b_2 + a_o a_2}{b_1 b_2 + a_2}$

$$\text{where } A_2 = (a_o b_1 + a_1) b_2 + a_o a_2$$

$$B_2 = b_1 b_2 + a_2 .$$

$n = 3$, then $\frac{A_3}{B_3} = a_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3} = \frac{(a_o b_1 b_2 + a_1 b_2 + a_o a_2) b_3 + (a_o b_1 + a_1) a_3}{(b_1 b_2 + a_2) b_3 + b_1 a_3}$

$$\text{where } A_3 = (a_o b_1 b_2 + a_1 b_2 + a_o a_2) b_3 + (a_o b_1 + a_1) a_3$$

$$B_3 = (b_1 b_2 + a_2) b_3 + b_1 a_3 ., \text{ etc.}$$

Now one notices that:

$$A_2 = A_1 b_2 + A_o a_2$$

$$B_2 = B_1 b_2 + B_o a_2 \ ,$$

and that

$$A_3 = A_2 b_3 + A_1 a_3$$

$$B_3 = B_2 b_3 + B_1 a_3 \ .$$

From this one is led to the formulas:

$$A_n = A_{n-1} b_n + A_{n-2} a_n$$

$$B_n = B_{n-1} b_n + B_{n-2} a_n \ .$$

These are fundamental recursion formulas for C.F. The $\frac{A_n}{B_n}$ is called the n'th approximant of the C.F.

## 4. Transformation.

In connection with the approximants one notes that they are unchanged if $a_n$ is replaced by $c_{n-1} c_n a_n$ , and $b_n$ is replaced by $c_n b_n$ , where $n = 1, 2, 3, \ldots$ ; $c_o = 1$ ; and $c_1, c_2, c_3, \ldots \neq 0$ . For a simple verification use $\frac{A_2}{B_2}$ :

$$\frac{A_2}{B_2} = \frac{(c_o a_o c_1 b_1 + c_o c_1 a_1) c_2 b_2 + c_o a_o c_1 c_2 a_2}{c_1 b_1 c_2 b_2 + c_1 c_2 a_2} \ .$$

The $c_1 c_2$ cancel out, and the approximant remains unchanged.

This is of value in putting the C.F. into simpler terms, or in reducing all the a's or all the b's to some particular value.

## 5. Matrix Product

The C.F. can be represented as a matrix product as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$ (using matrix multiplication).

Comparing this with the approximants one has:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix} .$$

Now substituting for $A_{n-1}$, $B_{n-1}$,

$$A_{n-1} = A_{n-2}\, b_{n-1} + A_{n-3}\, b_{n-1}$$

$$B_{n-1} = B_{n-2}\, b_{n-1} + B_{n-3}\, a_{n-1} ,$$

one has:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-2} & A_{n-3} \\ B_{n-2} & B_{n-3} \end{bmatrix} \begin{bmatrix} b_{n-1} & 1 \\ a_{n-1} & 0 \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix}.$$

Continuing this process one gets:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} a_o & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_{n-1} & 1 \\ a_{n-1} & 0 \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix}.$$

As a simple verification find $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$.

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} a_o & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_ob_1 + a_1 & a_o \\ b_1 & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_ob_1b_2 + a_1b_2 + a_oa_2 \\ b_1b_2 + a_2 \end{bmatrix}$$

This compares with

$$\frac{A_2}{B_2} = a_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2} = \frac{a_ob_1b_2 + a_1b_2 + a_oa_2}{b_1b_2 + a_2} .$$

This is an aid to calculating the value of the C.F.

## 6. Determinants

A C.F. also has a determinant form. Write down the values for the various approximants of the A's.

$$\begin{aligned}
a_o &= A_o \\
a_1 + A_ob_1 &= A_1 \\
A_oa_2 + A_1b_2 &= A_2 \\
A_1a_3 + A_2b_3 &= A_3 \\
A_{n-2}a_n + A_{n-1}b_n &= A_n .
\end{aligned}$$

Transposing the A's on the right and taking the coefficients of the A's, one has the determinant:

$$\begin{bmatrix} a_o & -1 & 0 & 0 & 0 & & \cdots\cdots\cdots \\ a_1 & b_1 & -1 & 0 & 0 & & \cdots\cdots\cdots \\ 0 & a_2 & b_2 & -1 & 0 & & \cdots\cdots\cdots \\ 0 & 0 & a_3 & b_3 & -1 & 0 & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 & a_n \quad b_n \end{bmatrix} .$$

This determinant is called a <u>continuant</u>. It corresponds to the C.F.

$$a_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots .$$

<u>Muir's symbol</u> for this is: $K\left( {}_{a_o} \, {}^{a_1}_{b_1} \, {}^{a_2}_{b_2} \, {}^{a_3}_{b_3} \, {}^{\cdots}_{\cdots} \right)$

If one has a C.F. it can be immediately put into the continuant form. To evaluate the C.F. in this form (for example, finding $\frac{A_n}{B_n}$ ) one expands the $(n+1) \times (n+1)$ determinant getting $A_n$, and one expands the cofactor of the term in the first row and first column getting $B_n$. Thus the determinant completely determines the C.F. For example, the upper left hand 3 x 3 minor:

$$\begin{vmatrix} a_o & -1 & 0 \\ a_1 & b_1 & -1 \\ 0 & a_2 & b_2 \end{vmatrix} = a_o b_1 b_2 + a_1 b_2 + a_2 a_o = A_2 .$$

The cofactor of $a_o = \begin{vmatrix} b_1 & -1 \\ a_2 & b_2 \end{vmatrix} = b_1 b_2 + a_2 = B_2$ .

7. Series-convergence

Any series can be expressed as an equivalent C.F. It is known that if the series converges the C.F. also converges. Conversely, if the C.F. converges the series converges, and this relation is important in using C.F. to determine the convergence or divergence of infinite series.

Following are two theorems, stated without proof, which can be used to determine the convergence of a non-terminating C.F., and hence also of a series.

Theorem I: If the sequence of approximants, $\frac{A_o}{B_o}$ , $\frac{A_1}{B_1}$ , ... , $\frac{A_n}{B_n}$ , ... approaches a limit, the C.F. converges.

Theorem II: Using the transformation given previously (Section 4) to make all of the $a_i = 1$ , i.e., $a_o + \frac{1}{b_1'} + \frac{1}{b_2'} + \cdots \frac{1}{b_n'} + \cdots$ , where the $b_i'$ are the transformed $b_i$ , then if the series of $b_i'$ , $\sum b_i'$ , converges the C.F. diverges.

Other theorems may be found in the References, particularly in Perron.

An equivalence relation which will put a series of the form $c_o + c_1x + c_2x^2 + \ldots c_nx^n + \ldots$ into a C.F. is the following:

$$c_o + \frac{c_1x}{1} - \frac{\frac{c_2}{c_1}x}{1 + \frac{c_2}{c_1}x} - \frac{\frac{c_3}{c_2}x}{1 + \frac{c_3}{c_2}x} - \ldots$$ , or reducing this

$$c_o + \frac{c_1x}{1} - \frac{c_2x}{c_1 + c_2x} - \frac{c_1c_3x}{c_2 + c_3x} - \frac{c_2c_4x}{c_3 + c_4x} - \cdots$$

For example, put $e^x$ into a C.F.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x = 1 + \frac{x}{1} - \frac{\frac{1}{2!}x}{1 + \frac{1}{2!}x} - \frac{\frac{2!}{3!}x}{1 + \frac{2!}{3!}x} - \frac{\frac{3!}{4!}x}{1 + \frac{3!}{4!}x} - \cdots$$

$$= 1 + \frac{x}{1} - \cfrac{x}{2 + x - \cfrac{2x}{3 + x - \cfrac{3x}{4 + x - \cfrac{4x}{5 + x - \cdots}}}} .$$

This series simplifies by eliminating the factorials (and the exponents), and can be easily evaluated to any degree of accuracy by the recurrence formulas.

As an example calculate e

$$e = 1 + \cfrac{1}{1 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \cfrac{5}{7 - \cdots}}}}}} .$$

$e = 2.716$ , where the accepted value is $2.718+$ .

## 8. Interpolation Formula

One can devise an interpolation formula, based on reciprocal differences, in the form of a C.F. Given values of $f(x)$ for points $x_o, x_1, x_2, \dots x_n$, where $f(x_i) \neq f(x_j)$.

Define the reciprocal difference of points $x_o, x_1$ as follows:

$$p_1(x_o\, x_1) = \frac{x_o - x_1}{f(x_o) - f(x_1)} ,$$

of points $x_o\, x_1\, x_2$ as:

$$p_2(x_o\, x_1\, x_2) = \frac{x_o - x_2}{p_1(x_o\, x_1) - p_1(x_1\, x_2)} + f(x_1) ,$$

of points $x_o\, x_1\, x_2\, x_3$ as:

$$p_3(x_o\, x_1\, x_2\, x_3) = \frac{x_o - x_3}{p_2(x_o\, x_1\, x_2) - p_2(x_1\, x_2\, x_3)} + p_1(x_1\, x_2) ,$$

of points $x_o\, x_2 \dots x_n$ as:

$$p_n\,(x_o \dots x_n) = \frac{x_o - x_n}{p_{n-1}(x_o \dots x_{n-1}) - p_{n-1}(x_1 \dots x_n)} + p_{n-2}(x_1 \dots x_{n-1}) .$$

In place of $x_o$ put $x$, and solve for $f(x)$, $p_1(x\, x_1)$, etc.

$$f(x) = \frac{x - x_1}{p_1(x\ x_1)} + f(x_1)$$

$$p_1(x\ x_1) = \frac{x - x_2}{p_2(x\ x_1\ x_2) - f(x_1)} + p_1(x_1\ x_2)$$

$$p_2(x\ x_1\ x_2) = \frac{x - x_3}{p_3(x\ x_1\ x_2\ x_3) - p_1(x_1\ x_2)} + p_2(x_1\ x_2\ x_3)$$

$$p_{n-1}(x\ \ldots\ x_{n-1}) = \frac{x - x_n}{p_n(x\ \ldots\ x_n) - p_{n-2}(x_1\ \ldots\ x_{n-1})} + p_{n-1}(x_1\ \ldots\ x_n)\ .$$

Now solving for $f(x)$ and substituting the proper denominators in each case one gets:

$$f(x) = f(x_1) + \cfrac{x - x_1}{p_1(x_1\ x_2) + \cfrac{x - x_2}{p_2(x_1\ x_2\ x_3) - f(x_1) + \cfrac{x - x_3}{p_3(x_1\ x_2\ x_3\ x_4) - p_1(x_1\ x_2) +}}}$$

This is known as <u>Thiele's Interpolation formula</u>.

It is a property of C.F. that if the numerator of any partial quotient vanishes, then this one and all of the following partial quotients do not affect the value of the C.F. Thus when any one of the numerators becomes small enough to neglect, those following can also be neglected.

## 9. Differential Equations.

Given a differential equation of the form,

9.1 $$y = Q_o y' + p_1 y'' ,$$

where $Q_o$ and $p_1$ are continuous and differentiable functions of $x$.

Successively differentiate.

$$y' = Q_o' y' + Q_o y'' + p_1' y'' + p_1 y'''$$

$$y' = \frac{Q_o + p_1'}{1 - Q_o'} y'' + \frac{p_1}{1 - Q_o'} y''' , \text{ where } \frac{Q_o + p_1'}{1 - Q_o'} = Q_1 ,$$

$$\text{and } \frac{p_1}{1 - Q_o'} = p_2 .$$

9.2 $$y' = Q_1 y'' + p_2 y''' , \text{ and similarly;}$$

9.3 $$y'' = Q_2 y''' + p_3 y^{iv}$$

9.4 $$y''' = Q_3 y^{iv} + p_4 y^{v} , \text{ etc.}$$

From 9.1 solve for $$\frac{y}{y'} = Q_o + p_1 / (y'/y'') ;$$

From 9.2 solve for $$\frac{y'}{y''} = Q_1 + p_2 / (y''/ y''') ;$$

From 9.3 solve for $$\frac{y''}{y'''} = Q_2 + p_3 / (y''' / y^{iv}) ; \text{ etc.}$$

In formula $\frac{y}{y'}$ put in the proper denominators for the $p_i$ terms,

then

$$\frac{y}{y'} = Q_o + \cfrac{p_1}{Q_1 + \cfrac{p_2}{Q_2 + \cfrac{p_3}{Q_3 + \cdots}}} .$$

Now consider

$$\frac{y'}{y} = \cfrac{1}{Q_o + \cfrac{p_1}{Q_1 + \cfrac{p_2}{Q_2 + \cfrac{p_3}{Q_3 + \cdots}}}} .$$

If the C.F. terminates, it represents the logarithmic derivative of a solution of the equation, and if it does not terminate then the problem of convergence arises. This was taken up earlier (Section 7), and those theorems may be applied here.

Example: $y = \frac{x}{m} y' + \frac{1}{m} y''$ , where $m$ is an integer; this will give a terminating C.F.

Let one take $m = 4$

9.5 $$y = \frac{x}{4} y' + \frac{1}{4} y'' .$$

Differentiating:

$$y' = \frac{1}{4} y' + \frac{x}{4} y'' + \frac{1}{4} y'''$$

$$\frac{3}{4} y' = \frac{x}{4} y'' + \frac{1}{4} y'''$$

9.6 $$y' = \frac{x}{3} y'' + \frac{1}{3} y'''$$

$$y'' = \frac{1}{3} y'' + \frac{x}{3} y''' + \frac{1}{3} y^{iv} .$$

9.7 $\quad y'' = \frac{x}{2} y''' + \frac{1}{2} y^{iv}$

9.8 $\quad y''' = \frac{x}{1} y^{iv} + \frac{1}{1} y^{v}$

9.9 $\quad 0 = x y^{v} + y^{vi}$ .

Now from

9.5 $$\frac{y}{y'} = \frac{x}{4} + \frac{\frac{1}{4}}{\frac{y'}{y''}} ;$$

9.6 $$\frac{y'}{y''} = \frac{x}{3} + \frac{\frac{1}{3}}{\frac{y''}{y'''}} ;$$

9.7 $$\frac{y''}{y'''} = \frac{x}{2} + \frac{\frac{1}{2}}{\frac{y'''}{y^{iv}}}$$

9.8 $$\frac{y'''}{y^{iv}} = x .$$

Note that $\frac{y^{iv}}{y^{v}}$ is not defined by this process. Then

$$\frac{y}{y'} = \frac{x}{4} + \cfrac{\frac{1}{4}}{\frac{x}{3} + \cfrac{\frac{1}{3}}{\frac{x}{2} + \cfrac{\frac{1}{2}}{x}}}$$

and

$$\frac{y'}{y} = \cfrac{1}{\frac{x}{4} + \cfrac{\frac{1}{4}}{\frac{x}{3} + \cfrac{\frac{1}{3}}{\frac{x}{2} + \cfrac{\frac{1}{2}}{x}}}}$$

and reducing this

$$\frac{y'}{y} = \cfrac{4}{x + \cfrac{3}{x + \cfrac{2}{x + \cfrac{1}{x}}}}$$

If the C.F. is solved for $\frac{A_3}{B_3}$ one gets

$$\frac{A_3}{B_3} = \frac{4x^3 + 12x}{x^4 + 6x^2 + 3} .$$

Thus $y = x^4 + 6x^2 + 3$ .

As a check substitute this in the differential equation:

$$x^4 + 6x^2 + 3 = \frac{x}{4}(4x^3 + 12x) + \frac{1}{4}(12x^2 + 12)$$

$$\underline{4x^4 + 24x^2 + 12} = 4x^4 + 12x^2 + 12x^2 + 12 = \underline{4x^4 + 24x^2 + 12}.$$

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