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ABSTRACT

Gorman Engel curves are extended to incomplete systems. The roles of Slutsky symmetry and homogeneity/adding up are isolated in the rank and functional form restrictions for Gorman systems. Symmetry determines the rank condition. The maximum rank is three for incomplete and complete systems. Homogeneity/adding up determines the functional form restrictions in complete systems. There is no restriction on functional form in an incomplete system. Every full rank and minimal deficit reduced rank Gorman system has a representation as a polynomial in a single function of income. This generates a complete taxonomy of indirect preferences for Gorman systems. Using this taxonomy, we develop models of incomplete Gorman systems that nest rank and functional form and satisfy global regularity conditions. All results are completely derived with elementary and straightforward methods that should be of wide interest.

KEY WORDS: Aggregation, functional form, Gorman Engel curves, incomplete demand systems, rank, weak integrability

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1. Introduction

Incomplete information is the rule not the exception. We are almost always interested in a small subset of the list of possible items purchased and used by consumers. This has three impacts on demand models. The budget constraint is a strict inequality. The demands are not 0° homogeneous in the prices of the goods being modeled and income. And if only *some* of the goods are modeled, then there is no compelling reason for the demands for the goods we do not model to have the same structure as those that we do. They may or may not have the same structure. We simply have no way to know. The upshot is incomplete systems are more applicable and in several ways more interesting than complete systems. Beyond this, is there any other reason to extend the known results on Gorman Engel curves for complete systems to incomplete systems? We offer two simple compelling examples to motivate our interest in this question.

First, consider a consumer receiving utility over goods, $\boldsymbol{q}$, and leisure, ℓ, with an average cost of leisure (or, equivalently, the average wage rate) a function of labor supplied, $w(\ell)$, and who is endowed with a unit of total time, $\ell \in [0,1]$. Now take the total "expenditure" on goods consumption and leisure as given for the moment. Then the consumer's choice problem can be stated as

$$v(\boldsymbol{p}, m) \equiv \sup\left\{u(\boldsymbol{q}, \ell) : \boldsymbol{p}'\boldsymbol{q} + w(\ell)\ell = m,\ \boldsymbol{q} \geq \boldsymbol{0},\ \ell \in [0,1]\right\}. \tag{1}$$

Assuming an interior solution, it is easy to show that goods demands, $\boldsymbol{q} = \boldsymbol{h}(\boldsymbol{p}, m)$, satisfy Roy's identity,

$$\boldsymbol{h}(\boldsymbol{p},m) = -\frac{\partial v(\boldsymbol{p},m)/\partial \boldsymbol{p}}{\partial v(\boldsymbol{p},m)/\partial m} \tag{2}$$

but are not 0° homogeneous in ($\boldsymbol{p}$,m) and do not satisfy adding up.

Let $\tilde{w}(\ell) = w(\ell) + w'(\ell)\ell$ be the marginal cost of leisure, let $\tilde{\boldsymbol{p}} = [\boldsymbol{p}^{\mathsf{T}}\ \tilde{w}]^{\mathsf{T}}$, and express the inverse Hessian of the Lagrangean for (1) as

$$\boldsymbol{H} = \begin{bmatrix} u_{\boldsymbol{qq}^{\mathsf{T}}} & u_{\boldsymbol{q}\ell} \\ u_{\ell\boldsymbol{q}^{\mathsf{T}}} & u_{\ell\ell} - v_m(2w' + w''\ell) \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{d} \\ \boldsymbol{d}^{\mathsf{T}} & e \end{bmatrix}, \tag{3}$$

with $\boldsymbol{C} = u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} + \left[u_{\ell\ell} - v_m(2w' + w''\ell) - u_{\boldsymbol{q}\ell}^{\mathsf{T}} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} u_{\boldsymbol{q}\ell}\right]^{-1} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} u_{\boldsymbol{q}\ell} u_{\boldsymbol{q}\ell}^{\mathsf{T}} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1}$,

$$\boldsymbol{d} = -\left[u_{\ell\ell} - v_m(2w' + w''\ell) - u_{\boldsymbol{q}\ell}^{\mathsf{T}} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} u_{\boldsymbol{q}\ell}\right]^{-1} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} u_{\boldsymbol{q}\ell},$$

and $e = \left[u_{\ell\ell} - v_m(2w' + w''\ell) - u_{\boldsymbol{q}\ell}^{\mathsf{T}} u_{\boldsymbol{qq}^{\mathsf{T}}}^{-1} u_{\boldsymbol{q}\ell}\right]^{-1}$,

with each term evaluated at $(\boldsymbol{h}(\boldsymbol{p},m), \ell(\boldsymbol{p},m))$ and subscripts denoting partial derivatives. Then the n×n matrix of Slutsky substitution terms,

$$\boldsymbol{S} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{p}^{\mathsf{T}}} + \frac{\partial \boldsymbol{h}}{\partial m}\boldsymbol{h}^{\mathsf{T}} = v_m\left[\boldsymbol{C} - (\tilde{\boldsymbol{p}}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{p}})^{-1}[\boldsymbol{C}\,\boldsymbol{d}]\tilde{\boldsymbol{p}}\tilde{\boldsymbol{p}}^{\mathsf{T}}[\boldsymbol{C}\,\boldsymbol{d}]^{\mathsf{T}}\right], \tag{4}$$

is symmetric, negative semidefinite. Hence, the set of goods demands, $\boldsymbol{q} = \boldsymbol{h}(\boldsymbol{p},m)$, has all of the properties of an incomplete demand system. Moreover, even if we assume that $u(\boldsymbol{q},\ell)$ has the same structure with respect to $\boldsymbol{q}$ and ℓ, any joint dependence between the average return to labor and labor supply implies that the functional form of the demand for leisure will differ from that of the demands for consumption goods.

The second example extends the first to a dynamic lifecycle consumption model with bequests and an uncertain consumer lifespan. To keep the example as simple as possible, let $v(\boldsymbol{p},m)$ be as above and interpret it as the optimal periodic utility flow given market prices and a total expenditure level on goods and leisure. Given a lifespan of *t* periods into the future, let the cumulative lifetime utility of the consumer be $\int_0^t e^{-\rho\tau} v(\boldsymbol{p},m(\tau))d\tau + e^{-\delta t} b(W(t))$, where ρ is the discount rate for periodic utility, δ is the (possibly different) discount rate for the benefits of leaving a bequest to the consumer's heirs, *W*(*t*) is the value of remaining wealth left at the end of the lifespan, and $b(W(t))$ is the utility generated from leaving the bequest. To simplify matters further, assume that the only source of uncertainty is the lifespan of the consumer.

Let $f(t)$ be the probability density function for death at *t*, so that $1-F(t)$ is the probability of a lifespan no more than *t* periods into the future. Define the hazard rate of death at *t* as $\eta(t) = f(t)/[1-F(t)] = -d\ln[1-F(t)]/dt$, equivalently, $1-F(t) = e^{-\int_0^t \eta(\tau)d\tau}$. Then the expected utility flow over all possible lifetimes can be written as

$$E(U) = \int_0^\infty \left[\int_0^t e^{-\rho\tau} v(\boldsymbol{p},m(\tau))d\tau + e^{-\delta t} b(W(t)) \right] f(t)dt$$

$$= \left[\int_0^t e^{-\rho\tau} v(\boldsymbol{p},m(\tau))d\tau \right] F(t) \Big|_0^\infty + \int_0^\infty \left[e^{-\delta t} b(W(t)) f(t) - e^{-\rho t} v(\boldsymbol{p},m(t)) F(t) \right] dt$$

$$= \int_0^\infty \left[e^{-\int_0^t \eta(\tau)d\tau - \rho t} v(\boldsymbol{p},m(t)) + e^{-\delta t} f(t) b(W(t)) \right] dt, \qquad (5)$$

invoking the natural assumptions $F(0)=0$ and $\lim_{t\to\infty} F(t)=1$.

Let N_0 be the initial value of nonlabor assets, let r be the certain and constant market discount rate, define full initial wealth of the consumer by $W_0 = N_0 + \int_0^{\infty} e^{-rt} w dt$, and to simplify the story as much as possible, assume that w is independent of the hours worked each period. Then the wealth transition equation is

$$\dot{W} = rW - m,\ W_0 \text{ fixed}. \tag{6}$$

If the consumer maximizes $E(U)$ subject to (6), then the optimal solution path for m can be characterized by (6) and the differential equation for consumption expenditures,

$$\dot{m} = \frac{\left[(\eta + \rho - r)v_m - \eta e^{(\rho-\delta)t} b'(W)\right]}{v_{mm}}. \tag{7}$$

If consumers are not more impatient than the market, $\eta + \rho \le r$, utility is concave in consumption and leisure expenditure, $v_{mm} < 0$, and wealth is no greater than the level that maximizes the utility generated from bequests, $W \le \arg\max\{B(x)\}$, then consumption and wealth accumulate early in the planning horizon, $\dot{m} > 0$ and $\dot{W} > 0$, and savings is the residual claimant on the unobservable budget constraint, $\dot{W} \equiv s = rW - m > 0$.

As in the previous example, we have an incomplete demand system for goods consumption. Since bequests are realized only *ex post*, it is virtually impossible to model this aspect of the consumption decision. The issues raised in these examples extend to kinks and other nonlinearities in the earnings function due to such issues as overtime regulations or income taxes, as well as uncertain future prices, incomes, and rates of return, multiple assets held by consuming households, and time-varying discount rates. In both cases, and indeed almost all cases we can conceive of, an incomplete system clearly seems more appropriate than a complete one.

The question of interest here, then, is do any of the known results on aggregation for complete systems of Gorman Engel curves extend to incomplete demand systems, and if they do, in what form? The purpose of this paper is to present a comprehensive extension of Gorman Engel curves to incomplete systems. In doing so, we isolate the role of symmetry from that of homogeneity and adding up in determining the rank and form of Gorman systems. Symmetry determines the rank restriction. The maximum rank is three for incomplete as well as complete systems. Homogeneity and adding up determine the functional form restriction in complete systems. There is no functional form restriction for an incomplete system.

Each full rank or minimal deficit reduced rank Gorman system can be represented as a polynomial in a single function of income. This produces a complete taxonomy of closed form expressions for the indirect preferences in Gorman systems. We develop two classes of incomplete Gorman systems to nest rank and functional form that satisfy global regularity, can be estimated with aggregate data, and can be used for inferences on the consumption and welfare of identifiable consumer groups.

Many of our results are constructed from the existing literature by combining and synthesizing very different approaches. In particular, we make extensive use of methods in Lie (1880; translated with commentary in Hermann 1975), Gorman (1981), van Daal and Merkies (1989), Lewbel (1990), and Russell and Farris (1993, 1998). We only use basic calculus and the most elementary of methods from the theory of differential equations to generate our results in the simplest, most direct, and understandable approach we could find, attempting to make the results accessible and of interest to as wide an audience as possible.

Our plan for the rest of the paper is as follows. In the next section, we set up the model used throughout the paper and briefly review existing results for complete systems within the context of this framework. The third section develops and discusses our extension of these results to incomplete systems, concentrating on the full rank and minimally deficit reduced rank cases. We characterize the indirect preference functions in the full rank cases, and present and discuss a class of preferences that gives rise to a minimal deficit reduced rank model with any number of income terms. The fourth section develops two classes of incomplete demand systems that nest rank and functional form. The final section summarizes these results and discusses some of our applications. To reduce the mathematical and notational burden, almost all of the proofs and derivations are contained in the Appendix.

2. A Review of Complete Gorman Systems

One of Terence Gorman's legacies is his contribution to consistent aggregation in demand (Gorman 1953, 1961, 1981). He first derived necessary and sufficient conditions for the existence of a representative consumer (Gorman 1953), and then obtained the indirect preference functions for this case (Gorman 1961), known as the *Gorman polar form*. Muellbauer (1975, 1976) extended the quasi-homothetic case of Gorman to a single nonlinear function of income, obtaining the *price independent generalized linear* (PIGL) and *price independent generalized logarithmic* (PIGLOG) demand systems. Gorman (1981) then extended these results, deriving the class of functional forms for all complete demand systems that can be expressed in terms of a finite number of additive income terms each multiplied by a vector of price functions.

The *rank* of a system of Gorman Engel curves is the number of linearly independent columns in the matrix of price functions that multiply the income variables. Every complete system of *Gorman Engel curves* satisfies two conditions. The rank of the system is at most three. If the rank of the system is three, when the demands are written with budget shares on the left, then the income functions that are not constant (one must be constant) are real powers of income, integer powers of log–income, or sine and cosine functions of log–income. This is the foundation of an important literature on aggregation for complete systems (Deaton and Muellbauer 1980; Lewbel 1987, 1988, 1989, 1990; Russell 1983, 1996; Russell and Farris 1993, 1998; and van Daal and Merkies 1989).

To review, extend, and relate the known results on complete Gorman systems to incomplete demand systems in a coherent framework, we need a few definitions, some notation, and a convention. Let $\boldsymbol{p} \in \mathbb{R}_{++}^{n_q}$ be the vector of market prices for the goods of interest, $\boldsymbol{q} \in \mathbb{R}_{+}^{n_q}$, let $\tilde{\boldsymbol{p}} \in \mathbb{R}_{++}^{n_{\tilde{q}}}$ be the vector of market prices for all other goods, $\tilde{\boldsymbol{q}} \in \mathbb{R}_{+}^{n_{\tilde{q}}}$, let $m \in \mathbb{R}_{++}$ be total consumption expenditure (hereafter, *income*, for brevity), let $z = \tilde{\boldsymbol{p}}^{\mathsf{T}}\tilde{\boldsymbol{q}} = m - \boldsymbol{p}^{\mathsf{T}}\boldsymbol{q} > 0$ be the total expenditure on all other goods, and let $\boldsymbol{s} \in \mathbb{R}^{r}$ be a vector of demand shifters. The *nominal expenditure function* is defined by

$$E(\boldsymbol{P},\tilde{\boldsymbol{P}},\boldsymbol{s},u) \equiv \min\left\{\boldsymbol{P}^{\mathsf{T}}\boldsymbol{q} + \tilde{\boldsymbol{P}}^{\mathsf{T}}\tilde{\boldsymbol{q}} : u(\boldsymbol{q},\tilde{\boldsymbol{q}},\boldsymbol{s}) \geq u\right\} \tag{8}$$

We assume $E : \mathbb{R}_{++}^{n_q} \times \mathbb{R}_{++}^{n_{\tilde{q}}} \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}_{++}$ is analytic and has neoclassical properties. In particular, it is 1° homogeneous in all prices $(\boldsymbol{P},\tilde{\boldsymbol{P}})$.

Now let $\pi : \mathbb{R}_{+}^{n_{\tilde{q}}} \to \mathbb{R}_{+}$ be any known, positively linearly homogeneous function of the prices of other goods, $\tilde{\boldsymbol{P}} \in \mathbb{R}_{+}^{n_{\tilde{q}}}$, and without any loss in generality, write the *deflated expenditure function* as[1]

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u) \equiv E\left(\boldsymbol{P}\big/\pi(\tilde{\boldsymbol{P}}),\tilde{\boldsymbol{P}}\big/\pi(\tilde{\boldsymbol{P}}),\boldsymbol{s},u\right)\Big/\pi(\tilde{\boldsymbol{P}}). \tag{9}$$

If $n \geq n_q + 1$, then $e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)$ is increasing in ($\boldsymbol{p}$,u), concave in $\boldsymbol{p}$, not 1° homogeneous in $\boldsymbol{p}$, and $\boldsymbol{p}^{\mathsf{T}} e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u) < e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)$ (LaFrance and Hanemann 1989).[2] But if $n_q = n$, then $\tilde{\boldsymbol{P}}$ has no elements and we will adopt the convention that $\pi(\tilde{\boldsymbol{P}}) \equiv 1$, $e(\boldsymbol{p},\boldsymbol{s},u)$ is 1° homogeneous in $\boldsymbol{p}$ and satisfies the adding up condition (the system is complete). We assume an interior solution for $\boldsymbol{q}$ so there are no binding non-negativity constraints. Thus,

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[1]Because $\partial e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\big/\partial\boldsymbol{p} \equiv \partial E(\boldsymbol{P},\tilde{\boldsymbol{P}},\boldsymbol{s},u)\big/\partial\boldsymbol{P}$, deflating by $\pi(\tilde{\boldsymbol{P}})$ does not alter the functional relationship between income and $\boldsymbol{P}$ at this level of generality. However, it plays a significant role when we restrict the demand equations to be a member of the class of Gorman Engel curves (see the next footnote).

[2]Lewbel (1989) derived a rank four system when income is deflated by an index of all prices. Russell and Farris (1998) showed that Lewbel's rank four case is somewhat singular. The income functions cannot be linearly independent and the log derivative of the price index deflating income must be the negative of one of the price functions. In the present model setup, income is deflated by some function of *all other prices*. The second condition identified by Russell and Farris (1998) therefore cannot be met for an incomplete system since $\pi(\tilde{\boldsymbol{P}})$ does not depend on $\boldsymbol{P}$. As a result, Lewbel's (1989) rank four case is impossible in this framework. The advantage of this restriction is it permits us to identify the connections among incomplete deflated income systems, complete nominal income systems, and the theory of Lie transformation groups on the real line.

symmetry is the main mathematical property of interest, although we also consider curvature later. Within this framework, a system of Gorman Engel curves might be written as

$$\frac{\partial e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{p}} = \sum_{k=1}^{K} \beta_k(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s}) H_k\left(e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right). \tag{10}$$

Now let $\boldsymbol{x} = [g_1(p_1)\cdots g_{n_q}(p_{n_q})]^\top \equiv \boldsymbol{g}(\boldsymbol{p})$ be an n_q–vector of diffeomorphisms of $\boldsymbol{p}$, so that each $g_i \in \mathcal{C}^\infty$ is a strictly increasing function with a smooth inverse $p_i(x_i)$, and let $y = f(m)$ be a diffeomorphism of deflated income with inverse $\phi(y)$. Then, rather than (10), we might write the demand system in terms of the variables $\boldsymbol{x}$ and y as[3]

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = \sum_{i=1}^{K} \alpha_i(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) h_i\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right). \tag{11}$$

However, since $y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)$, we also can write (10) as

$$\begin{aligned}\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{x}} &= f'\left(e(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)\frac{\partial \boldsymbol{p}(\boldsymbol{x})^\top}{\partial \boldsymbol{x}}\frac{\partial e(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{p}} \\ &= \sum_{k=1}^{K}\frac{\partial \boldsymbol{p}(\boldsymbol{x})^\top}{\partial \boldsymbol{x}}\beta_k(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},\boldsymbol{s})\frac{H_k\left(\phi\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)\right)}{\phi'\left(\phi\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)\right)} \\ &\equiv \sum_{k=1}^{K}\alpha_k(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right).\end{aligned} \tag{12}$$

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[3] Gorman (1981) *chose* the coordinate system by taking logarithms of prices and income. He invoked adding up early in his argument to deduce that a constant must be one of the income functions. In contrast, Russell (1996) and Russell and Farris (1993, 1998) applied the *coordinate free* methods of exterior differential calculus and the theory of Lie transformation groups, using adding up only at the end of their argument to reproduce Gorman's restriction on functional form. It can be shown that in partial differential equation systems with the present structure, a change in coordinates can be made so that a constant is always one of the functions of *y* (see Hermann 1975:147-150 and the Appendix).

These steps are reversible: functional separability of e from $\boldsymbol{p}$ is equivalent to functional separability of y from $\boldsymbol{x}$ and Gorman systems are independent of coordinates. Our first result is Lemma 1, showing us that symmetry also is independent of the coordinate system, while curvature is not.

Lemma 1. Let $e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{z}, u)$ *be the deflated expenditure function, let* $y = f(e)$, $f \in \mathbb{C}^2$, $f' > 0$, *with inverse* $m = \phi(y)$, *let* $x_i = g_i(p_i)$, $g_i \in \mathbb{C}^2$, $g_i' > 0$, *for each* $i = 1, \ldots, n_q$, *and write the deflated expenditure function as*

$$e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{z}, u) = \phi\left[y(\boldsymbol{g}(\boldsymbol{p}), \boldsymbol{z}, u)\right].$$

Then (a) $\dfrac{\partial^2 e}{\partial \boldsymbol{p} \partial \boldsymbol{p}^\top}$ *is symmetric if and only if* $\dfrac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\top}$ *is symmetric; and (b) if* $\phi'' \leq 0$, $g_i'' \leq 0 \ \forall \ i$, *and y is concave in* $\boldsymbol{x}$, *then e is concave in* $\boldsymbol{p}$.

Although very simple and completely intuitive, part (a) of this lemma proves to be extremely useful in the developments below. The reason is that this property lets us freely switch from one representation of y or $\boldsymbol{x}$ to another whenever this proves to be convenient and informative without a need to reconsider the implications for integrability.

We also will require a pair of conditions on the number of goods in the demand system, the number of income functions, and the functional relationships between the price and income functions that will guarantee that the demand system has a unique representation and is well-identified. The first of these is that the $\{h_k(y)\}_{k=1}^K$ are linearly independent with respect to the constants in K–dimensional space. That is, there can be no $\boldsymbol{c} \in \mathbb{R}^K$ satisfying $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{c}^\top \boldsymbol{h}(y^1) = 0 \ \ \forall \ y^1 \in \mathcal{N}(y) \subset \mathbb{R}$, where $\mathcal{N}(y)$ is an open neighborhood of any arbitrary point in the interior of the domain for y. The reason that we need this condition can be understood best by supposing that it was not satisfied. Then $\forall\ \boldsymbol{d} \in \mathbb{R}^K$ we can add $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\boldsymbol{d}\boldsymbol{c}^{\mathsf{T}}\boldsymbol{h}(y) \equiv \boldsymbol{0}_{n_q}$ to the system of demand equations without changing it at all,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) h_k(y) + \sum_{\ell=1}^{K} d_\ell \alpha_\ell(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \sum_{k=1}^{K} c_k h_k(y)$$

$$= \sum_{k=1}^{K} \left[\alpha_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + c_k \sum_{\ell=1}^{K} d_\ell \alpha_\ell(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \right] h_k(y) \tag{13}$$

$$= \boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \left[\boldsymbol{I} + \boldsymbol{d}\boldsymbol{c}^{\mathsf{T}} \right] \boldsymbol{h}(y) .$$

The matrix $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})$ certainly could not be identified in such a case and the demand model would make little sense.

Similarly, we require the columns of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})$ to be linearly independent with respect to the *K*–dimensional constants. For this, there can be no $\boldsymbol{c} \in \mathbb{R}^K$ satisfying $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{x}^1, \tilde{\boldsymbol{p}}, \boldsymbol{s})\boldsymbol{c} = \boldsymbol{0}\ \ \forall\ \boldsymbol{x}^1 \in \mathcal{N}(\boldsymbol{x})$, where here $\mathcal{N}(\boldsymbol{x})$ is an open neighborhood of any point in the interior of the domain of $\boldsymbol{x}$. For, if this did not hold, then $\forall\ \boldsymbol{d} \in \mathbb{R}^K$, if we add $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\boldsymbol{c}\boldsymbol{d}^{\mathsf{T}}\boldsymbol{h}(y) \equiv \boldsymbol{0}_{n_q}$ to the system we do not change it,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \left[\alpha_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + d_k \sum_{\ell=1}^{K} c_\ell \alpha_\ell(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \right] h_k(y)$$

$$= \boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \left[\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^{\mathsf{T}} \right] \boldsymbol{h}(y) . \tag{14}$$

Again, the matrix $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})$ cannot be identified and the demand system makes no sense.

We therefore assume throughout that the dimensions of $\boldsymbol{A}$ and $\boldsymbol{h}$ have been reduced as necessary to guarantee a unique representation (Gorman 1981: 358-59; Russell and Farris 1998: 201-202).

The rest of this section summarizes and synthesizes what is known about complete systems that have Gorman's functionally separable structure as given in (10) above. Gorman (1981) proved that all complete demand systems in this class must have a rank of $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{s})$ that is at most three. If the rank of $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{s})$ is three, then the system must take one of the following three possible forms:

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x},\boldsymbol{s})m + \sum_{k=1}^{K} \alpha_k(\boldsymbol{x},\boldsymbol{s})m[\ln(m)]^k \text{ ;} \tag{15}$$

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x},\boldsymbol{s})m + \sum_{\kappa\in S} \beta_\tau(\boldsymbol{x},\boldsymbol{s})m^{1-\kappa} + \sum_{\kappa\in S} \gamma_\tau(\boldsymbol{x},\boldsymbol{s})m^{1+\kappa} \text{ ,} \tag{16}$$

for S a set of nonzero constants; or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x},\boldsymbol{s})m + \sum_{\tau\in T} \beta_\tau(\boldsymbol{x},\boldsymbol{s})m\sin\left(\tau\ln(m)\right) + \sum_{\tau\in T} \gamma_\tau(\boldsymbol{x},\boldsymbol{s})m\cos\left(\tau\ln(m)\right) \text{,} \tag{17}$$

for T a set of positive constants.[4] This includes PIGLOG models and extensions that are polynomials in $\ln(m)$, simple polynomials in income, and PIGL models and extensions with power functions of the form m^κ, in addition to the trigonometric form (17).

A Gorman Engel curve system has *full rank* (Lewbel 1990) if the rank of $\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$ equals the number of columns and therefore also the number of income func-

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[4] Another important implication of Gorman's (1981) constructive proof, which Lewbel (1990) pointed out and exploited to great advantage in deriving the solutions for the full rank three complete system cases, is that if the rank of $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{s})$ is three and there are exactly three income terms, then $K = 2$ in (15), S has one element, κ, appearing once with a negative sign and once with a positive sign in the exponents in (16), and T has one element, τ, appearing in one sine and one cosine term in (17).

tions, $h_k(y)$. Full rank one complete systems are homothetic,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x}, \boldsymbol{s})m\,, \tag{18}$$

due to adding up. In budget share form, all full rank one complete systems are zero–order polynomials in income.

Muellbauer (1975, 1976) showed that full rank two complete systems are either PIGL or PIGLOG; either

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x}, \boldsymbol{s})m + \alpha_1(\boldsymbol{x}, \boldsymbol{s})m^{1-\kappa}\,. \tag{19}$$

for some $\kappa \neq 0$, or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x}, \boldsymbol{s})m + \alpha_1(\boldsymbol{x}, \boldsymbol{s})m\ln(m)\,. \tag{20}$$

Note that a Bernoulli first-order differential equation system,

$$\frac{\partial e(\boldsymbol{x}, \boldsymbol{s}, u)^{\kappa}}{\partial \boldsymbol{x}} = \kappa e(\boldsymbol{x}, \boldsymbol{s}, u)^{\kappa-1}\left(\frac{\partial e(\boldsymbol{x}, \boldsymbol{s}, u)}{\partial \boldsymbol{x}}\right) = \beta_0(\boldsymbol{x}, \boldsymbol{s}) + \beta_1(\boldsymbol{x}, \boldsymbol{s})\left[e(\boldsymbol{x}, \boldsymbol{s}, u)^{\kappa}/\kappa\right], \tag{21}$$

has the PIGL form,

$$\frac{\partial e(\boldsymbol{x}, \boldsymbol{s}, u)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x}, \boldsymbol{s})e(\boldsymbol{x}, \boldsymbol{s}, u) + \alpha_1(\boldsymbol{x}, \boldsymbol{s})e(\boldsymbol{x}, \boldsymbol{s}, u)^{1-\kappa}\,, \tag{22}$$

while a logarithmic first-order differential equation system,

$$\frac{\partial \ln\left[e(\boldsymbol{x}, \boldsymbol{s}, u)\right]}{\partial \boldsymbol{x}} = \frac{\partial e(\boldsymbol{x}, \boldsymbol{s}, u)/\partial \boldsymbol{x}}{e(\boldsymbol{x}, \boldsymbol{s}, u)} = \alpha_0(\boldsymbol{x}, \boldsymbol{s}) + \alpha_1(\boldsymbol{x}, \boldsymbol{s})\ln\left[e(\boldsymbol{x}, \boldsymbol{s}, u)\right], \tag{23}$$

has the PIGLOG form

$$\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u) + \alpha_1(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u)\ln\left[e(\boldsymbol{x},\boldsymbol{s},u)\right]. \tag{24}$$

Hence, all full rank two complete systems are first-order polynomials in a single function of income.

Finally, combining and synthesizing some of the results in Gorman (1981), van Daal and Merkies (1989), and Lewbel (1990), every full rank three Gorman complete system can be represented as a quadratic polynomial in one of three possible functions of income. A quadratic Bernoulli first–order differential equation system,

$$\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)^{\kappa}}{\partial \boldsymbol{x}} = \kappa e(\boldsymbol{x},\boldsymbol{s},u)^{\kappa-1}\left(\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}}\right)$$

$$= \beta_0(\boldsymbol{x},\boldsymbol{s}) + \beta_1(\boldsymbol{x},\boldsymbol{s})\left[e(\boldsymbol{x},\boldsymbol{s},u)^{\kappa}/\kappa\right] + \beta_2(\boldsymbol{x},\boldsymbol{s})\left[e(\boldsymbol{x},\boldsymbol{s},u)^{\kappa}/\kappa\right]^2, \tag{25}$$

has the generalized PIGL form,

$$\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = \alpha_0(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u) + \alpha_1(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u)^{1-\kappa} + \alpha_2(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u)^{1+\kappa}. \tag{26}$$

This includes the quadratic expenditure system of Howe, Pollak and Wales (1979) and van Daal and Merkies (1989) as the special case where $\kappa = 1$. Second, the quadratic logarithmic first–order differential equation system,

$$\frac{\partial \ln\left[e(\boldsymbol{x},\boldsymbol{s},u)\right]}{\partial \boldsymbol{x}} = \frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)/\partial \boldsymbol{x}}{e(\boldsymbol{x},\boldsymbol{s},u)}$$

$$= \alpha_0(\boldsymbol{x},\boldsymbol{s}) + \alpha_1(\boldsymbol{x},\boldsymbol{s})\ln\left[e(\boldsymbol{x},\boldsymbol{s},u)\right] + \alpha_2(\boldsymbol{x},\boldsymbol{s})\left\{\ln\left[e(\boldsymbol{x},\boldsymbol{s},u)\right]\right\}^2, \tag{27}$$

has the generalized PIGLOG form,

$$\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = e(\boldsymbol{x},\boldsymbol{s},u)\left\{\alpha_0(\boldsymbol{x},\boldsymbol{s}) + \alpha_1(\boldsymbol{x},\boldsymbol{s})\ln\left[e(\boldsymbol{x},\boldsymbol{s},u)\right] + \alpha_2(\boldsymbol{x},\boldsymbol{s})\left[\ln\left(e(\boldsymbol{x},\boldsymbol{s},u)\right)\right]^2\right\}. \quad (28)$$

Finally, the quadratic complex exponential first–order differential equation system,[5]

$$\frac{\partial}{\partial \boldsymbol{x}}\left[(\iota\tau)^{-1} e(\boldsymbol{x},\boldsymbol{s},u)^{\iota\tau}\right] = e(\boldsymbol{x},\boldsymbol{s},u)^{\iota\tau-1}\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}}$$

$$= ½\left[\alpha_1(\boldsymbol{x},\boldsymbol{s}) + \iota\alpha_2(\boldsymbol{x},\boldsymbol{s})\right] + \alpha_0(\boldsymbol{x},\boldsymbol{s})e(\boldsymbol{x},\boldsymbol{s},u)^{\iota\tau} + ½\left[\alpha_1(\boldsymbol{x},\boldsymbol{s}) - \iota\alpha_2(\boldsymbol{x},\boldsymbol{s})\right]e(\boldsymbol{x},\boldsymbol{s},u)^{2\iota\tau}, \quad (29)$$

has the trigonometric form[6]

$$\frac{\partial e(\boldsymbol{x},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = e(\boldsymbol{x},\boldsymbol{s},u)\left\{\alpha_0(\boldsymbol{x},\boldsymbol{s}) + ½\iota\alpha_1(\boldsymbol{x},\boldsymbol{s})\left(m^{-\iota\tau} - m^{\iota\tau}\right) + ½\alpha_2(\boldsymbol{x},\boldsymbol{s})\left(m^{-\iota\tau} + m^{\iota\tau}\right)\right\}$$

$$= e(\boldsymbol{x},\boldsymbol{s},u)\left\{\alpha_0(\boldsymbol{x},\boldsymbol{s}) + \alpha_1(\boldsymbol{x},\boldsymbol{s})\sin\left[\tau\ln\left(e(\boldsymbol{x},\boldsymbol{s},u)\right)\right] + \alpha_2(\boldsymbol{x},\boldsymbol{s})\cos\left[\tau\ln\left(e(\boldsymbol{x},\boldsymbol{s},u)\right)\right]\right\}. \quad (30)$$

We make use of de Moivre's theorem to obtain (30) from (29),

$$e^{\pm\iota\tau y} = 1 \pm \tfrac{1}{1!}(\iota\tau y) \pm \tfrac{1}{2!}(\iota\tau y)^2 \pm \tfrac{1}{3!}(\iota\tau y)^3 + \cdots$$

$$= \left[1 - \tfrac{1}{2!}(\tau y)^2 + \tfrac{1}{4!}(\tau y)^6 + \cdots\right] \pm \iota\left[\tfrac{1}{1!}(\tau y) - \tfrac{1}{3!}(\tau y)^3 + \tfrac{1}{5!}(\tau y)^5 + \cdots\right]$$

$$= \cos(\tau y) \pm \iota\sin(\tau y), \quad (31)$$

with $y = \ln[e(\boldsymbol{x},\boldsymbol{s},u)]$ and the price functions in (29) chosen to give real solutions in (30).

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[5]Including $(\iota\tau)^{-1}$ on the left-hand-side is innocuous, since $1/\iota = -\iota$. The right-hand-side can be multiplied by $\iota\tau$ with this absorbed in the complex conjugate price vectors without changing the nature of the result.

[6]Expressions (25)–(30) are not derived explicitly and do not appear in the literature on complete Gorman systems. We deduced them from a careful reading and synthesis of arguments in Gorman (1981), van Daal and Merkies (1989), and Lewbel (1990). It also is clear from the results of Russell and Farris (1993) that quadratic forms like these must exist in all full rank three cases.

Russell (1983, 1996) and Russell and Farris (1993, 1998) establish the connection between Gorman systems and Lie transformation groups on the real line. Russell (1983) argues that Gorman's theorem follows from Sophus Lie's results on the maximal rank of transformation groups (Lie 1880, translated with comments in Hermann 1975). Jerison (1993) presents a counterexample based on a polynomial demand system with more than three income terms that is not a Lie transformation group. But Russell and Farris (1993) prove Russell's claim in full rank systems by showing that a full rank Gorman system is a special case of the quadratic system

$$\boldsymbol{q} = \frac{\alpha_0 + \alpha_1 f(m) + \alpha_2 f(m)^2}{f'(m)}, \tag{32}$$

for some smooth, monotone function $y = f(m)$. They also show that it is the adding up condition that restricts the functional form to those cases identified in Gorman (1981). Russell and Farris (1998) extend their earlier results to show that the example presented by Jerison (1993) is the *only* one possible in a generic sense (also, see Russell 1996).[7] That is, if there are $K \geq 3$ linearly independent income functions and a maximal number of the *Lie brackets*, $h_k(y)h_\ell'(y) - h_k'(y)h_\ell(y),\ k < \ell$, are contained in the space spanned by the functions $\{h_k(y)\}_{k=1}^K$, then a representation of *y* exists supporting the polynomial form

$$\boldsymbol{q} = \frac{\alpha_1 + \alpha_2 f(m) + \alpha_3 f(m)^2 + \cdots + \alpha_K f(m)^{K-1}}{f'(m)}. \tag{33}$$

[7] The sobriquet *generic* indicates our conjecture that Theorem 4 of Russell and Farris (1998) gives a precise meaning to the intriguing last paragraph and footnote in Gorman (1981).

Of course, the functional form restrictions found by Gorman (1981) continue apply for complete systems in this case as well as the full rank cases.

In all of the work reviewed here, only Russell and Farris (1993) even mention an incomplete system. They argue (page 319) that (32) completely characterizes the class of full rank incomplete systems for *any* smooth and monotone function of income. But this is not an entirely correct statement. We show in the Appendix that homogeneity restricts the functional form in the full rank one and two cases. *All* demand equations, whether or not they are a complete or a Gorman system, must be 0° homogeneous in *all* prices and income. Many smooth, monotone functions will not become 0° homogeneous after multiplication by a function of prices, $e^{\lambda m} \;\forall\; \lambda > 0$ is a simple example. Income has to be deflated by some function of the prices of other goods, as in equation (9), to obtain the requisite 0° homogeneity property for an incomplete system. This detail notwithstanding, in the next section we extend Gorman Engel curves to incomplete demand systems, starting from equation (11) and making use of expressions almost identical to (32) and (33).

3. Incomplete Gorman Systems

For the rest of the paper, we assume $n \geq n_q + 1$ so the demand system is incomplete, all prices and income are deflated by $\pi(\tilde{\boldsymbol{P}})$ as in equation (9), and the demands have the functionally separable structure of equation (11). We begin with the main characterization theorem for this class of systems. Proposition 1 converts some of the results presented by Russell and Farris (1993, page 317; 1998, Theorem 1, page 189; and 1998, Theorem 4, page 193) to the current framework, and restates them somewhat differently. Since linear and functional dependencies lead to well-known difficulties in empirical applications, we focus only the on full rank and minimal deficit reduced rank cases.

Proposition 1. If the system of demand equations has Gorman's (1981) functionally separable structure,

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \alpha_k(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right),\ K<\infty,$$

is weakly integrable, and $rank\left[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\right]=K$*, then* $K \le 3$*, and a representation for* $y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)$ *exists such tat*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \begin{cases} \alpha_1, & K=1 \\ \alpha_1+\alpha_2 y, & K=2 \\ \alpha_1+\alpha_2 y+\alpha_3 y^2, & K=3 \end{cases}.$$

If $K \ge 3$*, and a maximal number of Lie brackets,* $h_k h'_\ell - h'_k h_\ell\ \forall\ k<\ell$*, are locally contained in the space spanned by* $\{h_1 \cdots h_K\}$*, then* $rank\left[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\right]=3$*, and there is a representation for y such that*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1+\alpha_2 y+\cdots+\alpha_K y^{K-1}.$$

In their work on this topic, Russell and Farris (1993, 1998) use the methods of differential topology, exterior differential calculus, and the theory of Lie algebras on the real line. These subjects of high level mathematics can be very difficult to master and apply successfully. In the Appendix, we apply purely elementary methods to derive and explain all of the results stated in this paper. One of our goals is to make the structure of Gorman systems more accessible to and much better understood by a wider range of readers. Several of the results stated here are new. Except where otherwise noted, to the best of our knowledge all of the derivations presented in the Appendix are original.

We briefly outline the steps involved in proving Proposition 1 here. First, this is a purely local result, so we only need to show the stated properties for a small neighborhood of an arbitrary point in the interior of the domain of the demands. Second, we can exploit monotonicity of the expenditure function to conclude that at least one income function satisfies $h_k'(y) \neq 0$ at least locally. Without loss in generality, let this be $h_1(y)$ and define the new variable $\gamma(y) \equiv \int^y ds/h_1(s)$. Then $\gamma'(y) \equiv 1/h_1(y)$ by the fundamental theorem of calculus, which lets us rewrite the system (11) in the rescaled form

$$\frac{\partial\gamma(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u))}{\partial\boldsymbol{x}} = \gamma'(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u))\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial\boldsymbol{x}}$$

$$= \frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)/\partial\boldsymbol{x}}{h_1'\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)} = \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) + \sum_{i=2}^{K}\alpha_i(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\frac{h_i\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)}{h_1\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)}$$

$$\equiv \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) + \sum_{i=2}^{K}\alpha_i(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\tilde{h}_i\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right). \tag{34}$$

Third, we can now apply Lemma 1, which lets us redefine the transformation y without changing the structure of the partial differential equations or integrability. In particular, the composite function $\gamma(f(m))$ is the new definition that follows directly from (34) and we can solve the integrability problem for the simpler representation,

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \boldsymbol{x}} = \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) + \sum_{i=2}^{K} \alpha_i(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s}) h_i\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right). \tag{35}$$

The fourth step is to invoke the symmetry conditions. After some algebra, this can be written as a system of ½$n_q(n_q-1)$ equations in ½$K(K-1)$ *Lie bracket*s, $h'_k h_\ell - h_k h'_\ell$,

$$\sum_{k=2}^{K}\sum_{\ell=1}^{k-1}\left(\alpha_{ik}\alpha_{j\ell} - \alpha_{jk}\alpha_{i\ell}\right)\left(h'_k h_\ell - h_k h'_\ell\right) = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{jk}}{\partial x_i} - \frac{\partial \alpha_{ik}}{\partial x_j}\right) h_k, \ \forall\ 1 \le j < i = 2, \cdots, n_q. \tag{36}$$

This system is much simpler in matrix form, $\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$, with $\boldsymbol{B}$ ½$n_q(n_q-1)\times$½$K(K-1)$, $\boldsymbol{C}$ ½$n_q(n_q-1)\times K$, $\boldsymbol{h}$ $K\times 1$, and $\tilde{\boldsymbol{h}}$ ½$K(K-1)\times 1$. Thus, for this to be a well-posed system we must have at least as many equations as unknowns, $n_q \ge K$. We assume throughout that this is the case. Premultiplying both sides by $\boldsymbol{B}^\top$ generates an equivalent square system, $\boldsymbol{B}^\top\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^\top\boldsymbol{C}\boldsymbol{h}$. This reveals the crux of the rank condition. $\boldsymbol{B}^\top\boldsymbol{B}$ inherits its rank from $\boldsymbol{A}$, which is K, but has dimension ½$K(K-1)\times$½$K(K-1)$. Existence of a unique solution for $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ therefore implies that $K \le 3$.

Now, when $\boldsymbol{B}^\top\boldsymbol{B}$ has full rank, the least squares formula gives $\tilde{\boldsymbol{h}}$ uniquely in terms of $\boldsymbol{h}$ as $\tilde{\boldsymbol{h}} = (\boldsymbol{B}^\top\boldsymbol{B})^{-1}\boldsymbol{B}^\top\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}$. Note that $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y, but not on $(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$, while $\boldsymbol{D}$ depends only on $(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$, but not on y. This implies that the elements of $\boldsymbol{D}$ are constant. Linear independence of the income functions and existence of a unique solution imply that not all of the elements in any row of $\boldsymbol{D}$ can vanish. Also note that $\boldsymbol{D}$ has dimension ½$K(K-1)\times K$. When $K = 1$, $\boldsymbol{D}$ has zero rows and there are no Lie brackets. When $K = 2$, $\boldsymbol{D}$ has one row and two columns. When $K = 3$, $\boldsymbol{D}$ has three rows and three columns. If $K > 3$, $\boldsymbol{D}$ has more rows than columns and there are more Lie brackets than income functions. Least squares cannot be applied to find $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ since $\boldsymbol{B}^{\top}\boldsymbol{B}$ must then be singular. The main question in this case is whether there are any redundant equations in the under–identified system $\boldsymbol{B}^{\top}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\top}\boldsymbol{C}\boldsymbol{h}$, and if so how many. We will return to this issue below.

The fifth step is to identify the representations in each full rank case. This is accomplished by combining (35) with the solutions for the Lie brackets $\tilde{\boldsymbol{h}}$ to obtain a complete system of linear, first-order, ordinary differential equations with constant coefficients subject to a set of linear side conditions. Rank one follows from (35) with $K = 1$. Rank two is considerably more involved. Starting with the single Lie bracket equation, $h_1(y)h_2'(y) - h_1'(y)h_2(y) = d_{12}^1 h_1(y) + d_{12}^2 h_2(y)$, we note that, without loss in generality, we can let $d_{12}^1 \neq 0$. We then make a change variables to $\tilde{h}_1(y) = d_{12}^1 h_1(y) + d_{12}^2 h_2(y)$, so that $\tilde{h}_1'(y) = d_{12}^1 h_1'(y) + d_{12}^2 h_2'(y)$, and to $\tilde{h}_2(y) = h_2(y)/d_{12}^1$, so that $\tilde{h}'(y)_2 = h_2'(y)/d_{12}^1$. Note that this simply redefines the price vectors with the linear transformations $\tilde{\alpha}_1 = \alpha_1/d_{12}^1$ and $\tilde{\alpha}_2 = -(d_{12}^2/d_{12}^1)\alpha_1 + d_{12}^1\alpha_2$, which does not affect the rank of $\boldsymbol{A}$ or the question of integrability in any fundamental way by Lemma 1. Some straightforward algebra implies $\tilde{h}_1(y)\tilde{h}_2'(y) - \tilde{h}_1'(y)\tilde{h}_2(y) = \tilde{h}_1(y)$ and integration gives $\tilde{h}_2(y) = h_1(y)\int^y ds/\tilde{h}_1(s)$. Dropping the tildes and applying the above definition of $\gamma(y)$ then produces the representation in the proposition for $K = 2$.

Rank 3 is even more complicated and we leave many details to the Appendix.

However, the main ideas are essentially the same as before. We make a of change variables with the composite function $\gamma(f(m))$ to imply that a representation for y exists that includes the constant function as one of the income terms. We then obtain a system of three equations in three unknowns for the three unique and nontrivial Lie brackets in the form

$$h_2'(y) = d_{12}^1 + d_{12}^2 h_2(y) + d_{12}^3 h_3(y)\,,$$

$$h_3'(y) = d_{13}^1 + d_{13}^2 h_2(y) + d_{13}^3 h_3(y)\,, \tag{37}$$

$$h_2(y)h_3'(y) - h_2'(y)h_3(y) = d_{23}^1 + d_{23}^2 h_2(y) + d_{23}^3 h_3(y)\,.$$

The first two constitute a complete system of linear, ordinary, first-order differential equations with constant coefficients. These are straightforward to solve for and ***D***. The third equation is a constraint on the set of matrices ***D*** that are compatible with integrability. Solving the first two equations and then checking the third for consistency, the only possibility is repeated vanishing roots. The a complete solution then takes the form

$$h_k(y) = a_k + b_k y + c_k y^2,\ k = 2,3\,, \tag{38}$$

for constants $\{a_k, b_k, c_k\}_{k=2}^3$. Redefining the price functions by $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$, $\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$, and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$ gives the representation for $K = 3$.

This part of the proposition states that a definition for $y = f(m)$ can always be found such that every full rank weakly integrable Gorman system is at most a quadratic form. This property holds whether or not the system is complete.

The proof for the reduced rank case when $K \geq 3$ is completed in two parts. The representation part follows almost exactly the same steps as Russell and Farris (1998). Beginning with a full rank three system, if we add a fourth income function, then two out of the three new Lie brackets are contained in the vector space spanned by the functions $\{1\ y\ y^2\ h_4(y)\}$ if and only if $h_4(y) = y^3$. The arguments are much the same as before, though fewer and far simpler. In particular, we add a single linear, first-order, ordinary differential equation with constant coefficients in the form

$$h_4'(y) = d_{14}^1 + d_{14}^2 y + d_{14}^3 y^2 + d_{14}^4 h_4(y)\,, \tag{39}$$

and without loss in generality, a single side condition due to integrability in the form

$$yh_4'(y) - h_4(y) = d_{24}^1 + d_{24}^2 y + d_{24}^3 y^2 + d_{24}^4 h_4(y)\,. \tag{40}$$

Both of these are compatible if and only if $d_{14}^4 = 0$, which then implies, again without any loss in generality (by constructing linear combinations of the α_i similar to the ones defined above), that $h_4(y) = y^3$. A simple induction completes this part of the proof.

Closely following Russell and Farris (1998, Theorem 4), this part of the proposition states that polynomials produce a maximal number of redundancies and a minimal number of defects for any Gorman system. The important thing to note is that this result is based on a weak necessary condition. It only requires that two of the new Lie brackets out of $K-2$ candidates, ignoring previous Lie brackets that are outside of any space with a lower dimension, are in the vector space spanned by the income functions. Any polynomial Lie bracket of the form $h_k h_\ell' - h_k' h_\ell = (\ell - k) y^{k+\ell-3}$, $k < \ell$, will be contained in the vector space spanned by the basis $\{1\, y\, y^2 \cdots y^{K-1}\}$ if and only if $k+\ell \le K+2$. In all cases, the difference between the number of spanned and redundant brackets is the number of income terms. The relationships among the number of income terms, Lie brackets, Lie brackets in the vector space spanned by $\{1\, y\, y^2 \cdots y^{K-1}\}$, *defects* (Lie brackets outside of this space), and *redundancies* (Lie brackets inside the space that repeat a power of *y*) are shown in table 1 for all values of $K < \infty$.

Table 1. Number of Income Functions, Lie Brackets, Defects, and Redundancies.

K	**Lie Brackets**	**Spanned Brackets**	**Defects**	**Redundancies**
1	0	–	–	–
2	1	1	0	0
3	3	3	0	0
4	6	5	1	1
5	10	8	2	3
6	15	11	4	5
7	21	15	6	8
8	28	19	9	11
9	36	24	12	15
10	45	29	16	19
⋮	⋮	⋮	⋮	⋮
K even	$\frac{1}{2}K(K-1)$	$\frac{1}{4}K(K+2)-1$	$\frac{1}{4}K(K-4)+1$	$\frac{1}{4}K(K-2)-1$
K odd	$\frac{1}{2}K(K-1)$	$\frac{1}{4}(K+1)^2-1$	$\frac{1}{4}(K+1)(K-5)+2$	$\frac{1}{4}(K-1)^2-1$

Last we show that any polynomial system with $K \geq 3$ has $rank[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})] \leq 3$. The argument is constructive and relies only on continuity of the symmetry conditions for powers of y from K+1 to 2K–1, demonstrating that if the system has a polynomial representation of the form $\partial y / \partial \boldsymbol{x} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k y^{k-1}$, then the matrix of price vectors must satisfy $\boldsymbol{\alpha}_k \equiv \varphi_k \boldsymbol{\alpha}_K$, $\forall\ k \geq 3$, for some $\varphi_k : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$, $k = 3, \ldots, K$.

Given Proposition 1 characterizing all full rank Gorman systems, we can identify the closed form solutions for the indirect preferences in the full rank cases. For this purpose, it is sufficient to recover the transformed, deflated expenditure function as (complete details are contained in the Appendix),

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \begin{cases} \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u), & K = 1 \\ \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u), & K = 2 \\ \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) - \dfrac{\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\left[\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) - \beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\right]}, & K = 3,\ \lambda \leq 0 \\ \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) - \dfrac{\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\tan\left(\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) - \beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\right)}, & K = 3,\ \lambda > 0 \end{cases} \tag{41}$$

where $\beta_1, \beta_2, \beta_3 : \mathbb{R}^{n_q} \times \mathbb{R}_+^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$, $\theta : \mathbb{R}_+^{n_{\tilde{q}}} \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}$, and λ is the constant term in the integral $\varphi(z) = \int^z ds / (1 + \lambda s^2)$ taken from Lewbel (1987, 1990) and van Daal and Merkies (1989). The cases $\lambda \leq 0$ and $\lambda > 0$ represent real and complex roots, respectively, in the system of Ricatti partial differential equations,

$$\frac{\partial z}{\partial \boldsymbol{x}} = (1+\lambda z^2)\frac{\partial \gamma_3}{\partial \boldsymbol{x}}. \tag{42}$$

Here $z = -\gamma_2/(y-\gamma_1)$ and $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R}^{n_q} \times \mathbb{R}_+^{n_{\bar{q}}} \times \mathbb{R}^r \to \mathbb{R}$ are price functions equivalent to the corresponding price functions in van Daal and Merkies (1989). When the roots are real, the β_i and γ_i satisfy $\beta_1 = \gamma_1 + \kappa\gamma_3$, $\beta_2 = 2\kappa\gamma_3$, $\beta_3 = e^{2\gamma_2}$, with κ defined by $-\lambda = \kappa^2$, while in the complex root case, the β_i and γ_i are related by $\beta_1 = \gamma_1$, $\beta_2 = \kappa\pi\gamma_2$, and $\beta_3 = \kappa\gamma_3$, with κ now defined by $\lambda = -(\iota\kappa)^2$, $\iota = \sqrt{-1}$.

In differential topology, the space of all real projective transformation groups is commonly associated with the *special linear group two*, $\mathfrak{sl}(2)$, which is generally defined by the set of all 2×2 real matrices,

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

that have unit determinants, $\alpha\delta - \beta\gamma = 1$. The associated inverses,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix},$$

are members of $\mathfrak{sl}(2)$, as well as the identiy map $\boldsymbol{I}_2$.

Any real projective transformation group can be written in the form

$$y = \frac{\alpha\theta + \beta}{\gamma\theta + \delta} \Leftrightarrow \theta = \frac{\delta y - \beta}{-\gamma y + \alpha}, \ \forall\ \alpha\delta - \beta\gamma = 0. \tag{43}$$

The set of all 2×2 matrix inverses in $\mathfrak{sl}(2)$ are one-to-one and onto the inverse functions for this group, and $\boldsymbol{I}_2$ defines the identity map in both spaces. Simple algebra gives

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left(\alpha\frac{\partial\beta}{\partial \boldsymbol{x}} - \beta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right) + \left[\left(\beta\frac{\partial\gamma}{\partial \boldsymbol{x}} - \gamma\frac{\partial\beta}{\partial \boldsymbol{x}}\right) - \left(\alpha\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right)\right] y + \left(\gamma\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\gamma}{\partial \boldsymbol{x}}\right) y^2 . \quad (44)$$

This representation defines a large class of indirect utility functions to generate Gorman systems in the form

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, m) = \upsilon\left\{\frac{\delta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}) f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}) f(m) + \alpha(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s})}, \tilde{\boldsymbol{p}}, \boldsymbol{s}\right\},\ \alpha\delta - \beta\gamma \equiv 1 . \quad (45)$$

Equivalently, the deflated and transformed expenditure function for this class of preferences is

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \frac{\alpha(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})},\ \alpha\delta - \beta\gamma \equiv 1 . \quad (46)$$

From this we see immediately the connection between the class of full rank three Gorman systems and the projective transformation group with real parameters. Note that $\gamma \neq 0$ is required for a full rank three system, so that we can rescale the price functions to obtain

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \frac{\tilde{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \tilde{\beta}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \tilde{\delta}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})} \quad (47)$$

with $\tilde{\alpha} = \alpha/\gamma$, $\tilde{\beta} = \beta/\gamma$, and $\tilde{\delta} = \delta/\gamma$. It is straightforward to convert (47) to the form in equation (41) above. In the Appendix, we show that the case of complex roots in the full rank three case generates a member of the complex projective transformation group with price functions in the deflated, transformed expenditure function that are complex-valued. Thus, the primary piece of the puzzle that was missed by Howe, Pollak and Wales (1979) in their derivation of a solution to integrability of the quadratic expenditure system is the class of expenditure functions with complex-valued price functions.

The results discussed above do not preclude higher order polynomials, only more than three income terms with a matrix of *linearly independent* price functions. We demonstrate this with an example extending Jerison (1993). Let the indirect utility function be

$$v(\boldsymbol{x}, \tilde{\boldsymbol{p}}, y, \boldsymbol{s}) = \upsilon\left[\left(\frac{\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) - y}\right)^{\eta} - \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}); \tilde{\boldsymbol{p}}, \boldsymbol{s}\right], \tag{48}$$

where we assume $\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) > y$ for monotonicity and let η be any real number in the interval $[1,\infty)$. Applying Roy's identity, we generate an incomplete demand system as

$$\boldsymbol{q} = \frac{1}{f'(m)} \frac{\partial \boldsymbol{x}^{\mathsf{T}}}{\partial \boldsymbol{p}} \left[\frac{\partial \gamma}{\partial \boldsymbol{x}} - \frac{\partial \beta}{\partial \boldsymbol{x}}\left(\frac{\gamma - y}{\beta}\right) + \left(\frac{\beta}{\eta}\right)\frac{\partial \delta}{\partial \boldsymbol{x}}\left(\frac{\gamma - y}{\beta}\right)^{\eta+1}\right]. \tag{49}$$

Under certain restrictions on η, this takes the form of proposition 1 and illustrates the full nature of its implications. First note that there are three linearly independent functions of y on the right-hand-side of (49). When $\eta = 1$, we have a quadratic in y. But the parameter η can be any integer in $[1,\infty)$ and preferences will remain well-behaved with appropriate choices for the functions β, γ, δ. If η is an integer greater than one, expanding the last term in square brackets with the binomial formula implies that all powers of y from 0 to $\eta+1$ appear on the right. The model cannot be reduced to a quadratic for any $\eta > 1$. The first two terms in square brackets involve the powers 0 and 1 in y and the sub-matrix of price vectors on the powers of y from 2 through $\eta+1$ has rank equal to one.

However, η also can assume any real non–integer value in $[1,\infty)$ and preferences will remain well–behaved with appropriate choices of the functions $\{\beta,\gamma,\delta\}$. In such a case, the last term in square brackets on the right-hand-side of (49) is analytic with a convergent Taylor series expansion over the set of positive values for $\gamma - y$. The vectors of price functions for all powers of y greater than one are all proportional and the matrix of price functions, even with an infinite number of columns, has rank at most three. Thus, if there is a finite number of terms with functional separability between the prices of interest and income, then we must have a polynomial in y. But a very large set of well–defined demand models exists beyond quadratic polynomials, and each element can be represented as an irreducible polynomial of higher order than a quadratic, and may even have an infinite number of income terms.

4. Nesting Rank and Functional Form

In the two decades since its introduction by Deaton and Muellbauer, the AIDS has been widely used in demand analysis. The vast majority of empirical applications follows Deaton and Muellbauer's suggestion and replaces the translog price index that deflates income with Stone's index, which generates the LA-AIDS. Although Deaton and Muellbauer (1980: 317-320) cautioned against and avoided the practice, most empirical applications of the LA-AIDS include tests for and the imposition of an approximate version of Slutsky symmetry by restricting the log-price coefficient matrix to be symmetric. Examples include Anderson and Blundell (1983), Buse (1998), Moschini (1995), Moschini and Meilke (1989), and Pashardes (1993). In this section, we present conditions for integrability of the LA-AIDS and a simple method for nesting the homothetic solution within homothetic PIGL systems. We then extend this to rank two non-homothetic PIGL and rank three QPIGL forms. Finally, we present a similar nesting procedure for a generalized quadratic utility function

If it is integrable, the LA-AIDS can be written as

$$\boldsymbol{w} = \frac{\partial \ln e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)}{\partial \ln \boldsymbol{p}} = \alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s}) + \boldsymbol{B} \ln \boldsymbol{p} + \gamma \left[\ln e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) - (\ln \boldsymbol{p})^{\mathsf{T}} \frac{\partial \ln e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)}{\partial \ln \boldsymbol{p}} \right]. \quad (50)$$

Let $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) \equiv \ln[e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)]$, $\boldsymbol{x} \equiv \ln(\boldsymbol{p})$, with $\boldsymbol{p}(\boldsymbol{x}) \equiv [e^{x_1} \cdots e^{x_{n_q}}]^{\mathsf{T}}$, and rewrite (50) in the form

$$(\boldsymbol{I} + \gamma \boldsymbol{x}^{\mathsf{T}}) \frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \boldsymbol{B}\boldsymbol{x} + \gamma y . \quad (51)$$

Then we have the following from LaFrance (2004).

Proposition 2. If the LA-AIDS model is weakly integrable on the open set $\mathcal{N} \subset \mathbb{R}^{n_q}$ *defined by* $1 + \gamma^{\mathsf{T}} \boldsymbol{x} \neq 0 \ \forall \ \boldsymbol{x} \in \mathcal{N}$*, then either (a)* $\gamma \neq \boldsymbol{0}$ *and* $\boldsymbol{B} = \beta_0 \gamma \gamma^{\mathsf{T}}$ *for some* $\beta_0 \in \mathbb{R}$*, with logarithmic expenditure function*

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s})^{\mathsf{T}} \boldsymbol{x} + \beta_0 \left[(1 + \gamma^{\mathsf{T}} \boldsymbol{x}) \ln(1 + \gamma^{\mathsf{T}} \boldsymbol{x}) - \frac{\gamma^{\mathsf{T}} \boldsymbol{x}}{(1 + \gamma^{\mathsf{T}} \boldsymbol{x})} \right] + (1 + \gamma^{\mathsf{T}} \boldsymbol{x}) \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)$$

or (b) $\gamma = \boldsymbol{0}$ *and* $\boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}}$*, with logarithmic expenditure function,*

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s})^{\mathsf{T}} \boldsymbol{x} + \tfrac{1}{2} \boldsymbol{x}^{\mathsf{T}} \boldsymbol{B} \boldsymbol{x} + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) .$$

Case (b) is a homothetic demand model with the same structure as the homothetic Linear Incomplete Demand System (LIDS) in LaFrance (1985). This provides a mechanism to nest the homothetic LA-AIDS and LIDS using Box-Cox transformations. If we define $m(\kappa) \equiv (m^{\kappa} - 1)/\kappa$, $p_i(\lambda) \equiv (p_i^{\lambda} - 1)/\lambda$, and $\boldsymbol{p}(\lambda) \equiv [p_1(\lambda) \cdots p_{n_q}(\lambda)]^{\mathsf{T}}$, then we can write a class of weakly integrable, homothetic PIGL-IDS models in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}[\alpha + \boldsymbol{B}\boldsymbol{p}(\lambda)], \tag{52}$$

where $\boldsymbol{P}^{\lambda} \equiv \textbf{diag}[p_i^{\lambda}]$. The deflated expenditure function for (52) is

$$\mathrm{e}(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) \equiv \left\{1 + \kappa\left[\alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s})^{\mathsf{T}} \boldsymbol{p}(\lambda) + \tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)\right]\right\}^{1/\kappa}, \tag{53}$$

The demands in (52) are homothetic, with income elasticities of $1-\kappa \ \forall \ \kappa \in \mathbb{R}$.

We next extend this mechanism to nest the rank and functional form to a rank 2, non-homothetic, integrable AIDS-IDS model,

$$\boldsymbol{w} = \alpha + \boldsymbol{B}\ln(\boldsymbol{p}) + \gamma\left[\ln(m) - \alpha_0 - \alpha^{\mathsf{T}}\ln(\boldsymbol{p}) - \tfrac{1}{2}\ln(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\ln(\boldsymbol{p})\right]. \tag{54}$$

To do so, we require a third proposition. The next result shows that (54) is a special case of a complete class of incomplete demand systems that can be characterized in the following way. We first return to the case where $y \equiv f(m)$ and $x_i \equiv g_i(p_i) \ \forall \ i = 1, \ldots, n_q$ are arbitrary diffeomorphisms. We then suppose that the demands for $\boldsymbol{q}$ can be written in terms of a linear function of y and linear and quadratic functions of $\boldsymbol{x}$, with no interaction terms between $\boldsymbol{x}$ and y,

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)}{\partial x_i} = \alpha_i + \beta^{\mathsf{T}}{}_i\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\Delta_i\boldsymbol{x} + \gamma_i y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u), \ \forall \ i = 1, \ldots, n_q, \tag{55}$$

and, without any loss in generality, we can assume that $\gamma_1 \neq 0$ and that each $n_q \times n_q$ matrix, Δ_i, is symmetric $\forall \ i$. Then we have the following.

Proposition 3. The system of partial differential equations,

$$\frac{\partial y}{\partial x_i} = \alpha_i + \beta_i^{\mathsf{T}} \boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}} \Delta_i \boldsymbol{x} + \gamma_i y,\ i = 1,\ldots,n_q,$$

where $\gamma_1 \neq 0$ *and each* $n_q \times n_q$ *matrix,* Δ_i *is symmetric* $\forall$ *i, is weakly integrable if and only if it can be written in the form*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{p} + \gamma\left[y - \alpha_0 - \tilde{\alpha}^{\mathsf{T}} \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}} \tilde{\boldsymbol{B}} \boldsymbol{x} \right]$$

where α_0 *is a scalar,* $\tilde{\alpha} = \alpha - \alpha_0\gamma$ *is an* $n_q \times 1$ *vector,* $\tilde{\boldsymbol{B}}$ *is a symmetric* $n_q \times n_q$ *matrix satisfying* $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \gamma\alpha^{\mathsf{T}}$, *where* $\boldsymbol{B} = \left[\beta_1 \cdots \beta_{n_q}\right]$, *and* $\Delta_i = -\gamma_i \tilde{\boldsymbol{B}}\ \forall\ i$.

When $f(\cdot)$ and the $g_i(\cdot)$ are logarithmic transformations, we have an AIDS-IDS. This is the only functional form in this class that can be a complete system, elucidating an important difference between complete and incomplete demand systems. In this case adding up determines the functional form in a complete system, while incomplete systems with the same structure admit any functional form.

With the above Box-Cox definitions for $m(\kappa)$ and $\boldsymbol{p}(\lambda)$, we can write an integrable non-homothetic PIGL-IDS that is linear in the Box-Cox expenditure term and linear and quadratic in the Box-Cox price terms as

$$\boldsymbol{w} = m^{-\kappa} \boldsymbol{P}^{\lambda} \left\{ \alpha + \boldsymbol{B}\boldsymbol{p}(\lambda) + \gamma\left[m(\kappa) - \alpha_0 - \alpha' \boldsymbol{p}(\lambda) - \tfrac{1}{2}\boldsymbol{p}(\lambda)' \boldsymbol{B} \boldsymbol{p}(\lambda) \right] \right\}. \tag{56}$$

For all (κ, λ) pairs, this model allows us to estimate the income aggregation function through the Box-Cox parameter κ. If $\kappa = \lambda = 0$ we obtain the AIDS-IDS, if $\kappa = \lambda = 1$ we obtain the linear-quadratic IDS (LQ-IDS) of LaFrance (1990), and for all (κ, λ) pairs we obtain an integrable PIGL-IDS.[8] The deflated expenditure function for (56) is

$$\mathrm{e}(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u) \equiv \left\{1+\kappa\left[\alpha_0(\tilde{\boldsymbol{p}},\boldsymbol{s})+\alpha(\tilde{\boldsymbol{p}},\boldsymbol{s})^{\mathsf{T}}\boldsymbol{p}(\lambda)+½\boldsymbol{p}(\lambda)^{\mathsf{T}}\boldsymbol{B}\boldsymbol{p}(\lambda)+\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)e^{\gamma^{\mathsf{T}}\boldsymbol{p}(\lambda)}\right]\right\}^{1/\kappa}. \quad (57)$$

Next we extend this to demand models that include linear and quadratic terms in the Box-Cox transformation of deflated income (QES-IDS). From proposition 1, we can do this with little loss in generality by extending the rank two PIGL-IDS expenditure function to a rank three version by applying the QES transformation developed by Howe, Pollak, and Wales (1979). One convenient (and carefully selected) choice in this class is the deflated expenditure function,

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u) \equiv \left\{1+\kappa\left[\alpha_0+\alpha^{\mathsf{T}}\boldsymbol{p}(\lambda)+½\boldsymbol{p}(\lambda)^{\mathsf{T}}\boldsymbol{B}\boldsymbol{p}(\lambda)-\frac{e^{\gamma^{\mathsf{T}}\boldsymbol{p}(\lambda)}}{\left(\delta^{\mathsf{T}}\boldsymbol{p}(\lambda)e^{\gamma^{\mathsf{T}}\boldsymbol{p}(\lambda)}+\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right)}\right]\right\}^{1/\kappa}. \quad (58)$$

An application of Roy's identity gives the QPIGL-IDS extension of the AIDS-IDS in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\boldsymbol{P}^{\lambda}\left\{\alpha+\boldsymbol{B}\boldsymbol{p}(\lambda)+\gamma\left[m(\kappa)-\alpha_0-\alpha'\boldsymbol{p}(\lambda)-\tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\right]\right.$$

$$\left.+\left[\boldsymbol{I}+\gamma'\boldsymbol{p}(\lambda)\right]\delta\left[m(\kappa)-\alpha_0-\alpha'\boldsymbol{p}(\lambda)-\tfrac{1}{2}\boldsymbol{p}(\lambda)'\boldsymbol{B}\boldsymbol{p}(\lambda)\right]^2\right\}. \quad (59)$$

So long as α and $\boldsymbol{B}$ do not vanish simultaneously, which is necessary for the model to be able to be able to attain rank three, it follows that: (a) $\gamma \neq \mathbf{0}, \delta \neq \mathbf{0}$ is necessary and suffi-

[8] See Agnew (1998) for a comprehensive development and application of this full rank two PIGL-IDS.

cient for a full rank three QPIGL-IDS; (b) $\gamma \neq \mathbf{0}$, $\delta = \mathbf{0}$ is necessary and sufficient for a full rank two, non-homothetic PIGL-IDS; (c) $\gamma = \mathbf{0}$, $\delta \neq \mathbf{0}$ is necessary and sufficient for a rank two QPIGL-IDS that excludes the linear term in the deflated and transformed superlative income variable; and (d) $\gamma = \delta = \mathbf{0}$ is necessary and sufficient for a rank one homothetic PIGL-IDS. Thus, we obtain a rich class of models that permits nesting, testing and estimating the rank and functional form of the income terms in incomplete demand systems with a generalized AIDS structure.

We apply the same methods developed above to produce a full rank three generalized quadratic-type direct utility function and a generalized translog-type indirect utility function (Christensen, Jorgenson, and Lau, 1975). First define the functions

$$\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = \boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} + 2\gamma'\boldsymbol{x} + 1\,, \tag{60}$$

$$\eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = \alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s}) - \alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s})^{\mathsf{T}}\boldsymbol{x}\,, \tag{61}$$

where $\alpha(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ is a vector of functions of other prices and demographics, $\alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s})$ is a scalar function of other prices and demographics, $\boldsymbol{B}$ is an $n_q \times n_q$ matrix of parameters, and γ is a vector of parameters. The starting point for this application of Proposition 1 is the class of indirect utility functions defined by

$$v(\boldsymbol{x}, y, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = \upsilon\left\{-\frac{\sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}}{[y - \eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})]} - \frac{\delta^{\mathsf{T}}\boldsymbol{x}}{\sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}}, \tilde{\boldsymbol{p}}, \boldsymbol{s}\right\}, \tag{62}$$

which is equivalent to the transformed and deflated expenditure function in the form

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) - \left(\frac{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\delta^{\mathsf{T}} \boldsymbol{x} + \sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)}} \right) \tag{63}$$

Roy's identity gives a rank three QPIGL-IDS in the form

$$\boldsymbol{q} = \frac{1}{f'(m)} \frac{\partial \boldsymbol{x}^{\mathsf{T}}}{\partial \boldsymbol{p}} \left\{ \alpha + \left[1 - \delta^{\mathsf{T}} \boldsymbol{x} \left(\frac{y - \eta}{\varphi} \right) \right] \left(\frac{y - \eta}{\varphi} \right) (\boldsymbol{B}\boldsymbol{x} + \gamma) + \frac{(y - \eta)^2}{\varphi} \delta \right\}. \tag{64}$$

The Box-Cox transformations, $y = (m^{\kappa} - 1)/\kappa$ and $x_i = (p_i^{\lambda} - 1)/\lambda \ \forall \ i = 1, \cdots, n_q$, imply that $\kappa = \lambda = 0$ gives a rank three extension of a generalized translog-type indirect utility model, $\kappa = \lambda = 1$ gives a rank three extension of a generalized quadratic-type direct utility model, and all values of κ and λ give a rank three QPIGL-IDS. Rank two is obtained with $\delta = \boldsymbol{0}$. If $\eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) \equiv 0$ and $\delta = \boldsymbol{0}$, we have a rank one homothetic model. We again are able to nest both rank and functional form in a single unifying framework.

An extremely useful view of this incomplete demand system arises from noting that the demands for ***q*** satisfy the partial differential equations,

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \left[1 - \delta^{\mathsf{T}} \boldsymbol{x} \left(\frac{y - \eta}{\varphi} \right) \right] \left(\frac{y - \eta}{\varphi} \right) (\boldsymbol{B}\boldsymbol{x} + \gamma) + \frac{(y - \eta)^2}{\varphi} \delta. \tag{65}$$

By Lemmas 2 and 3 in the Appendix, model permits us to determine necessary and sufficient conditions for symmetry and sufficient conditions for concavity of y in $\boldsymbol{x}$, hence of e in $\boldsymbol{p}$, entirely from (65) for this class of models. Calculating the second-order partial derivatives and careful (and tedious) grouping, canceling, and algebraic manipulations give

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}} = \left[1-\delta^{\mathsf{T}}\boldsymbol{x}\left(\frac{y-\eta}{\varphi}\right)\right]\left(\frac{y-\eta}{\varphi}\right)\left[\boldsymbol{B}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\gamma)(\boldsymbol{Bx}+\gamma)^{\mathsf{T}}\right]$$

$$+\,2\frac{(y-\eta)^3}{\varphi^2}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\gamma)\boldsymbol{x}^{\mathsf{T}}\right]\delta\delta^{\mathsf{T}}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)\boldsymbol{x}(\boldsymbol{Bx}+\gamma)^{\mathsf{T}}\right], \qquad (66)$$

so that symmetry of $\boldsymbol{B}$ is necessary and sufficient for symmetry of $\partial^2 y/\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}$, and therefore also for symmetry of $\partial^2 e/\partial \boldsymbol{p}\partial \boldsymbol{p}^{\mathsf{T}}$ by Lemma 1 above.

Recall we use the transformation $\tilde{u} = -u^{-1}$ (again taken from Howe, Pollak, and Wales 1978) of the Gorman polar form of the quasi-indirect utility function when $\delta = \mathbf{0}$. That is, we take the negative reciprocal of $(y-\eta)/\sqrt{\varphi}$, which is the generalized quadratic quasi-indirect utility function.[9] In this case $y = \eta$ is the bliss point and monotonicity requires $y-\eta < 0$, while $\varphi > 0$ is required for the radical to be well-defined in Gorman's choice for normalizing the utility index. When $\|\delta\| < \varepsilon$ for small enough $\varepsilon > 0$, we have $1-\delta^{\mathsf{T}}\boldsymbol{x}(y-\eta)/\varphi > 0$. This inequality must be satisfied, at least in a neighborhood of each point in the interior of the domain of the demand system, if preferences are well-behaved. This is equivalent to the condition that if we add $-\delta^{\mathsf{T}}\boldsymbol{x}/\sqrt{\varphi}$ to $\tilde{u} = -u^{-1}$, then we do not change the sign of the (cardinal) utility index. Indeed, this condition is required for the Howe, Pollak and Wales (1979) transformation from u to $-u^{-1}$ to remain well-defined and it can be shown that preferences become ill-behaved when it is violated.

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[9] We use *generalized quadratic* to refer to the fact that indirect preferences are defined in terms of deflated and transformed prices and income, $\boldsymbol{x}$ and y, respectively, rather than directly in terms of $\boldsymbol{p}$ and m.

In any case, we would expect the second-order income effects to be small relative to the first-order income effects on the demands. In other words $\delta^{\mathsf{T}}\boldsymbol{x}$ should be small relative to φ. The upshot is that, so long as $1-\delta^{\mathsf{T}}\boldsymbol{x}(y-\eta)/\varphi>0$, $y-\eta<0$ and $\varphi>0$, the second line of (66) will be a symmetric, negative semidefinite, rank one matrix. Lemmas 2 and 3 in the Appendix show that $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$ is necessary and $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}+\gamma\gamma^{\mathsf{T}}$ is sufficient, for $\partial^2 y\big/\partial\boldsymbol{x}\partial\boldsymbol{x}^{\mathsf{T}}$ to then be symmetric, negative semidefinite. Under the conditions in Lemma 1 above, this in turn is necessary and sufficient for the weak integrability of this incomplete demand system throughout the open set

$$\Im\equiv\left\{(\boldsymbol{p},\tilde{\boldsymbol{p}},m,\boldsymbol{s})\in\mathbb{R}^{n_q}_{++}\times\mathbb{R}^{n_{\tilde{q}}}_{++}\times\mathbb{R}_{++}\times\mathbb{R}^{r}:\varphi>0,y-\eta<0,1-\delta^{\mathsf{T}}\boldsymbol{x}(y-\eta)/\varphi>0\right\}. \quad (67)$$

These curvature restrictions apply only to the parameters of the model and are straightforward to implement. We recently have experienced success applying them to U.S. food consumption (Beatty and LaFrance 2001; LaFrance and Beatty 2003).

5. Conclusions

In this paper, we extend the literature on aggregation to incomplete demand systems. In stark contrast to complete demand systems, there is no restriction on the class of functional forms for the income variables. On the other hand, the maximal rank of an incomplete Gorman system is three. This follows purely from Slutsky symmetry.

We also use Box-Cox transformations of the prices of the goods of interest and a separate Box-Cox transformation on income to generate two large classes of nested functional forms. One makes it possible to test for the rank and functional form of generalized AIDS models. The other permits the same analysis to be applied to a generalized translog-type indirect utility function and a generalized quadratic-type direct utility function.

We have found both frameworks for nesting incomplete demand systems to be empirically tractable as well as substantial improvements over the traditional rank two alternatives (Beatty and LaFrance, 2000; LaFrance, Beatty, Pope and Agnew, 2000, 2002; and LaFrance and Beatty, 2003). In both classes of nested functional forms, rank three appears to be essential for most, though not all, of the data sets we have applied this framework to. In addition, the point estimates for the Box-Cox parameters on prices and income tend to fall much closer to unity than to zero. Lemma 1 suggests that this may be a generic circumstance because the logarithmic transformation of income to define y makes satisfying curvature difficult. However, both restrictions ($\kappa = \lambda = 1$ or 0, respectively) are rejected at all reasonable levels of significance in every data set we have used to empirically investigate this question. This suggests that the generalizations developed here to nest the rank and functional form of Gorman systems should become useful tools in applied demand analysis.

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MATHEMATICAL APPENDIX

A.1 SYMMETRY AND CURVATURE

Lemma 1. Let $e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{z},u)$ *be the deflated expenditure function, let* $y = f(e)$, $f \in \mathbb{C}^2$, $f' > 0$, *with inverse* $m = \phi(y)$, *let* $x_i = g_i(p_i)$, $g_i \in \mathbb{C}^2$, $g_i' > 0$, *for each* $i = 1,\ldots,n_q$, *and write the deflated expenditure function as*

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{z},u) = \phi\left[y(\boldsymbol{g}(\boldsymbol{p}),\boldsymbol{z},u)\right].$$

Then (a) $\dfrac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}^{\mathsf{T}}}$ *is symmetric if and only if* $\dfrac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}}$ *is symmetric; and (b) if* $\phi'' \le 0$, $g_i'' \le 0\ \forall\ i$, *and y is concave in* ***x***, *then e is concave in* ***p***.

Proof: We have

$$\frac{\partial e}{\partial \boldsymbol{p}} = \phi'(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial y}{\partial \boldsymbol{x}}, \tag{A.1}$$

so that

$$\frac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}^{\mathsf{T}}} = \phi''(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial y}{\partial \boldsymbol{x}}\frac{\partial y}{\partial \boldsymbol{x}'}\,\mathbf{diag}\left[g_i'\right] + \phi'(y)\,\mathbf{diag}\left[g_i''\frac{\partial y}{\partial x_i}\right]$$

$$+\ \phi'(y)\,\mathbf{diag}\left[g_i'\right]\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}}\,\mathbf{diag}\left[g_i'\right]. \tag{A.2}$$

The first two terms on the right are automatically symmetric, so that symmetry of the left-hand-side is equivalent to symmetry of the Hessian matrix on the far right-hand-side. The first two matrices on the right are negative semidefinite when $\phi'' \le 0$ and $g_i'' \le 0\ \forall\ i$, so that if $\dfrac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}}$ is negative semidefinite, then $\dfrac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}^{\mathsf{T}}}$ is as well. ■

A.2 DIFFERENTIAL EQUATIONS FOR PIGL AND PIGLOG FUNCTIONAL FORMS

Consider the quasi-linear ordinary differential equation

$$\frac{y'(x)}{y(x)} = \frac{d\ln\left(y(x)\right)}{dx} = \alpha(x) + \beta(x) f\left(y(x)\right). \tag{A.3}$$

This differential equation lies at the heart of the functional form question originally posed by Muellbauer (1975, 1976). In particular, the simplest form of this question is, "What is the class of functions $f\left(y\right)$ that can satisfy (A.3) and the 0° homogeneity condition,

$$\alpha'(x)x + \beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0\,?\text{"} \tag{A.4}$$

It turns out that there are only two possibilities: a special case of Bernoulli's equation,

$$\frac{y'}{y} = \alpha_0 + \beta_0\left(\frac{y}{x}\right)^{\kappa},\ \kappa \neq 0\,; \tag{A.5}$$

or a special case of the logarithmic transformation,

$$\frac{y'}{y} = \alpha_0 + \beta_0 \ln\left(\frac{y}{x}\right). \tag{A.6}$$

The reason for this can be obtained by analyzing the implications of (A.4) directly. First, consider the case where $\alpha'(x)x = 0$, so that $\alpha(x) = \alpha_0$, a constant. Then (A.4) reduces to

$$\beta'(x)xf(y) + \beta(x)f'(y)y \equiv 0, \tag{A.7}$$

or equivalently,

$$\frac{d\ln(f)}{d\ln(y)} = \frac{f'(y)}{f(y)}y = -\frac{\beta'(x)}{\beta(x)}x = -\frac{d\ln(\beta)}{d\ln(x)} = \kappa, \tag{A.8}$$

where κ is a constant because the left-hand-side is independent of x, while the right-hand-side is independent of y. Without any loss in generality, the solutions are $f(y) = y^{\kappa}$ and $\beta(x) = \beta_0 x^{-\kappa}$.

Now suppose that $\alpha'(x)x \neq 0$, so that

$$\beta'(x)xf(y) + \beta(x)f'(y)y = -\alpha'(x)x. \tag{A.9}$$

Since the right-hand-side is again independent of y, at least one of the terms on the left also must be independent of y. If $f'(y) = 0$, so that $f(y) = f_0$ is constant, we obtain the degenerate case where the functions of y on the right-hand-side of (A.3) are not linearly independent. Hence, it must be that $\beta'(x)x = 0$, i.e., $\beta(x) = \beta$, a constant, and

$$f'(y)y = \frac{df(y)}{d\ln(y)} = -\frac{\alpha'(x)x}{\beta} = \lambda\,, \tag{A.10}$$

where λ is a constant again because the left-hand-side is independent of x and the right-hand-side is independent of y. Solving the left side gives

$$f(y) = \lambda \ln(y) + \gamma\,, \tag{A.11}$$

while the right-hand-side can be rewritten as

$$\frac{d\alpha(x)}{d\ln(x)} = -\lambda\beta\,, \tag{A.12}$$

which has solution

$$\alpha(x) = \alpha - \lambda\beta\ln(x)\,. \tag{A.13}$$

Combining (A.11) and (A.13), we obtain (A.6), with $\alpha_0 = \alpha + \beta\gamma$ and $\beta_0 = \beta\lambda$.

The implication is that, for ranks one and two demand models in this class, the admissible forms of $f(y)$ are completely determined by homogeneity.

When we consider *incomplete demand systems*, we do not have homogeneity in the prices of interest or adding up to restrict the functional form. For Bernoulli's differential equation,

$$y' = \alpha(x)y + \beta(x)y^{1-\kappa},\ \kappa \neq 0\,, \tag{A.14}$$

if we note that $\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y'$, we can rewrite this as the linear ordinary differential equation in $f(y) = y^{\kappa}/\kappa$,

$$\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y' = \left(\kappa\alpha(x)\right)\left(y^{\kappa}/\kappa\right) + \beta(x)\,, \tag{A.15}$$

with complete solution

$$y(x) = \left[\kappa e^{\int^x \kappa\alpha(s)ds}\left(\int^x e^{-\int^s \kappa\alpha(t)dt}\beta(s)ds + c\right)\right]^{1/\kappa}. \tag{A.16}$$

Similarly, the logarithmic first-order linear differential equation is

$$\frac{d\ln(y)}{dx} = \frac{y'}{y} = \alpha(x) + \beta(x)\ln\left(y\right), \tag{A.17}$$

with complete solution

$$y(x) = \exp\left\{ e^{\int^x \beta(s)ds} \left(\int^x e^{-\int^s \beta(t)dt} \alpha(s)ds + c \right) \right\}. \tag{A.18}$$

The generic nature of both of these differential equations is that they can be written as simple linear first-order ordinary differential equations,

$$\frac{df\,(y(x))}{dx} = \alpha(x) + \beta(x) f\,(y(x)). \tag{A.19}$$

When y is deflated income and the demands do not absorb all of the budget, homogeneity and adding up do not impose any restriction on the class of functions $f\,(y)$ that can solve this differential equation, and the complete class of solutions is

$$y(x) = f^{-1}\left[e^{\int^x \kappa\alpha(s)ds} \left(\int^x e^{-\int^s \kappa\alpha(t)dt} \beta(s)ds + c \right) \right]. \tag{A.20}$$

A.3 Proofs of the Propositions

Proposition 1. If the system of demand equations has Gorman's (1981) functionally separable structure,

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) h_k\left(y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) \right),\ K < \infty,$$

is weakly integrable, and $rank\left[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\right] = K$*, then* $K \le 3$ *and there is a representation for* $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, \boldsymbol{s}, u)\right)$ *such that*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \begin{cases} \alpha_1 & K = 1 \\ \alpha_1 + \alpha_2 y & K = 2\,. \\ \alpha_1 + \alpha_2 y + \alpha_3 y^2 & K = 3 \end{cases}$$

If $K \ge 3$ *and a maximal number of Lie Brackets,* $h_k h'_\ell - h'_k h_\ell\ \forall\ k < \ell$*, are locally contained in the space spanned by* $\{h_1 \cdots h_K\}$*, then* $rank\left[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\right] = 3$ *and y has a representation such that*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 y + \cdots + \boldsymbol{\alpha}_K y^{K-1}.$$

Proof: (Also, see Russell and Farris (1993), Theorem 2, and Russell and Farris (1998), Theorem 4 for alternative proofs using exterior differential calculus.) By Young's theorem, the second-order cross partial derivatives of y with respect to $\boldsymbol{x}$ must be symmetric for integrability,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \sum_{k=1}^{K} \left(\frac{\partial \alpha_{ik}}{\partial x_j} h_k + \alpha_{ik} h_k' \sum_{\ell=1}^{K} \alpha_{j\ell} h_\ell \right)$$

$$= \sum_{k=1}^{K} \left(\frac{\partial \alpha_{jk}}{\partial x_i} h_k + \alpha_{jk} h_k' \sum_{\ell=1}^{K} \alpha_{i\ell} h_\ell \right) = \frac{\partial^2 y}{\partial x_j \partial x_i} \quad \forall \; i \neq j. \tag{A.21}$$

These can be re-expressed in terms of $\frac{1}{2} n_q(n_q-1)$ vanishing differences,

$$0 = \sum_{k=1}^{K} \left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i} \right) h_k + \sum_{k=1}^{K} \sum_{\ell=1}^{K} \alpha_{ik} \alpha_{j\ell} \left(h_k' h_\ell - h_k h_\ell' \right) \; \forall \; 1 \leq j < i = 2, \cdots, n_q. \tag{A.22}$$

In the double sum on the right-hand-side if (A.22), when $k = \ell$, the term $\alpha_{ik}\alpha_{jk}$ is multiplied by the Lie Bracket, $h_k' h_k - h_k h_k' = 0$. On the other hand, when $k \neq \ell$, the Lie Bracket $h_k' h_\ell - h_k h_\ell'$ appears twice, once multiplied by $\alpha_{ik}\alpha_{j\ell}$ and once multiplied by $-\alpha_{i\ell}\alpha_{jk}$. Therefore, rewrite (A.22) as

$$0 = \sum_{k=1}^{K} \left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i} \right) h_k + \sum_{k=2}^{K} \sum_{\ell=1}^{k-1} \left(\alpha_{ik} \alpha_{j\ell} - \alpha_{jk} \alpha_{i\ell} \right) \left(h_k' h_\ell - h_k h_\ell' \right), \; 1 \leq j < i = 2, \cdots, n_q, \tag{A.23}$$

a linear system of $\frac{1}{2} n_q(n_q-1)$ equations in the $\frac{1}{2} K(K-1)$ Lie Brackets $h_k' h_\ell - h_k h_\ell'$.

The first step in the proof of the proposition is to restate (A.23) in matrix form. Define

$$\boldsymbol{B} = \begin{bmatrix} \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \cdots & \alpha_{2k}\alpha_{1\ell} - \alpha_{1k}\alpha_{2\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{i2}\alpha_{j1} - \alpha_{j2}\alpha_{i1} & \cdots & \alpha_{ik}\alpha_{j\ell} - \alpha_{ik}\alpha_{j\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n_q 2}\alpha_{n_q-1,1} - \alpha_{n_q-1,2}\alpha_{n_q 1} & \cdots & \alpha_{n_q k}\alpha_{n_q-1,\ell} - \alpha_{n_q-1,k}\alpha_{n_q \ell} & \cdots & \alpha_{n_q K}\alpha_{n_q-1,K-1} - \alpha_{n_q-1,K}\alpha_{n_q K-1} \end{bmatrix},$$

$$\boldsymbol{C} = -\begin{bmatrix} \frac{\partial \alpha_{11}}{\partial x_2} - \frac{\partial \alpha_{21}}{\partial x_1} & \cdots & \frac{\partial \alpha_{1K}}{\partial x_2} - \frac{\partial \alpha_{2K}}{\partial x_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{i1}}{\partial x_j} - \frac{\partial \alpha_{j1}}{\partial x_i} & \cdots & \frac{\partial \alpha_{iK}}{\partial x_j} - \frac{\partial \alpha_{jK}}{\partial x_i} \\ \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{n_q 1}}{\partial x_{n_q-1}} - \frac{\partial \alpha_{n_q-1,1}}{\partial x_{n_q}} & \cdots & \frac{\partial \alpha_{n_q K}}{\partial x_{n_q-1}} - \frac{\partial \alpha_{n_q-1K}}{\partial x_{n_q}} \end{bmatrix},$$

$$\boldsymbol{h} = \begin{bmatrix} h_1 & \cdots & h_K \end{bmatrix}^{\mathsf{T}},$$

and $$\tilde{\boldsymbol{h}} = \begin{bmatrix} h_2' h_1 - h_2 h_1' & \cdots & h_k' h_\ell - h_k h_\ell' & \cdots & h_K' h_{K-1} - h_K h_{K-1}' \end{bmatrix}^{\mathsf{T}}.$$

$\boldsymbol{B}$ is $\tfrac{1}{2} n_q(n_q-1) \times \tfrac{1}{2} K(K-1)$, $\boldsymbol{C}$ is $\tfrac{1}{2} n_q(n_q-1) \times K$, $\boldsymbol{h}$ is $K \times 1$, and $\tilde{\boldsymbol{h}}$ is $\tfrac{1}{2} K(K-1) \times 1$. This gives the symmetry conditions in matrix form as

$$\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}. \tag{A.24}$$

For this to be a well-posed system of equations we must have at least as many equations as unknowns, which is equivalent to $n_q \geq K$. Assume this is so. Premultiply both sides of (A.24) by $\boldsymbol{B}^{\mathsf{T}}$ to get the square system, $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h}$. The rank result of Lie (1880) when $\boldsymbol{B}$ has full column rank is that $\tfrac{1}{2} K(K-1) \leq K$, equivalently, $K \leq 3$ (Hermann 1975: 143-146). The reason is a direct result of linear algebra. The rank of $\boldsymbol{B}$ is inherited from the rank of $\boldsymbol{A}$, and when $n_q = K$ the determinant of $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is a multiple of the determinant of $\boldsymbol{A}$ (Hermann 1975: 141).[1] Since $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is of order $\tfrac{1}{2} K(K-1) \times \tfrac{1}{2} K(K-1)$ and has rank no greater than K (the rank of $\boldsymbol{A}$), it follows that $K \leq 3$, completing the proof of the first part of the proposition for the full rank case.

The next step is to obtain the representation result for the full rank case. Assume that $\boldsymbol{B}$ has full column rank. The least squares formula for $\tilde{\boldsymbol{h}}$ as a function of $\boldsymbol{h}$ is

$$\tilde{\boldsymbol{h}} = (\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}. \tag{A.25}$$

The vectors $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y and not on $(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})$, while the matrix $\boldsymbol{D}$ depends

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[1] It can be shown (see the Appendix by Robert Bryant in Russell and Farris 1998) that the problem always can be reduced through changes in the coordinates for $\boldsymbol{x}$ in such a way that $n_q = K$.

only on $(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})$ and not on y. It follows that the elements of $\boldsymbol{D}$ are *absolute constants*; a fundamental property that we require below. Since $\boldsymbol{B}$ is of order $\tfrac{1}{2}n_q(n_q - 1) \times \tfrac{1}{2}K(K-1)$ and $\boldsymbol{C}$ is of order $\tfrac{1}{2}n_q(n_q - 1) \times K$, it follows that $\boldsymbol{D}$ is of order $\tfrac{1}{2}K(K-1) \times K$. That is, when $K = 1$, $\boldsymbol{D}$ has zero rows (there are no Lie Bracket*s*), when $K = 2$, $\boldsymbol{D}$ has one row and two columns), and when $K = 3$, $\boldsymbol{D}$ has three rows and three columns. If $K > 3$, then $\boldsymbol{D}$ would have more rows than columns (i.e., more Lie Bracket*s* than income functions), and the full rank condition cannot be satisfied. We address each full rank case in turn.

Rank 1:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 h_1(y). \tag{A.26}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^\top} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} h_1(y) + h_1'(y) \alpha_1 \alpha_1^\top \tag{A.27}$$

is symmetric. Hence, $\partial \alpha_1 / \partial \boldsymbol{x}^\top$ must be symmetric, which is necessary and sufficient for the existence of a function, $\beta : \mathbb{R}_{++}^{n_q} \times \mathbb{R}_{++}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$, such that $\partial \beta / \partial \boldsymbol{x} = \alpha_1$. Rewrite the demands as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}} h_1(y), \tag{A.28}$$

and separate the variables (recall that $h_1(y) \neq 0$ is required for $\partial y / \partial \boldsymbol{x} \gg \boldsymbol{0}$) to obtain

$$\gamma(y) \equiv \int^y h_1(s)^{-1} ds = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u). \tag{A.29}$$

From this we have

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y) \frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)} \frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1. \tag{A.30}$$

Therefore, a representation for y exists (by composing γ and f), such that $\partial y / \partial \boldsymbol{x} = \alpha_1$ and $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)$, with $\alpha_1 = \partial \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) / \partial \boldsymbol{x}$.

Rank 2:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 h_1(y) + \alpha_2 h_2(y). \tag{A.31}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^\top} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} h_1 + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^\top} h_2 + \left(\alpha_1 h_1' + \alpha_2 h_2'\right)\left(\alpha_1 h_1 + \alpha_2 h_2\right)^\top \tag{A.32}$$

is symmetric. Expanding gives

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^\top} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} h_1 + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^\top} h_2 + \alpha_1\alpha_1^\top h_1' h_1 + \alpha_1\alpha_2^\top h_1' h_2 + \alpha_2\alpha_1^\top h_2' h_1 + \alpha_2\alpha_2^\top h_2' h_2 , \tag{A.33}$$

and the terms $\alpha_1\alpha_1^\top h_1' h_1$ and $\alpha_2\alpha_2^\top h_2' h_2$ are automatically symmetric. Since α_1 and α_2 are linearly independent, $\alpha_2 \neq c\alpha_1$ for any $c \in \mathbb{R}$. Otherwise, the rank of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) = [\alpha_1\, \alpha_2]$ is only 1, not 2. Hence, $\alpha_1\alpha_2^\top$ is not symmetric. Since h_1 and h_2 are functionally independent (equivalently, are locally linearly independent), $h_1' h_2 \neq h_1 h_2'$. Otherwise, $h_2 = ch_1$ for some constant $c \in \mathbb{R}$; a contradiction. Hence, we can premultiply the reduced symmetry conditions by $\alpha_1^\top$ and postmultiply by α_2 to obtain

$$\begin{aligned}
&\alpha_1^\top \left(\frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} h_1 + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^\top} h_2 + \alpha_1\alpha_2^\top h_1' h_2 + \alpha_2\alpha_1^\top h_2' h_1 \right) \alpha_2 = \\
&\alpha_1^\top \frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} \alpha_2 h_1 + \alpha_1^\top \frac{\partial \alpha_2}{\partial \boldsymbol{x}^\top} \alpha_2 h_2 + \alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 h_1' h_2 + (\alpha_1^\top\alpha_2)^2 h_2' h_1 = \\
&\alpha_1^\top \frac{\partial \alpha_1^\top}{\partial \boldsymbol{x}} \alpha_2 h_1 + \alpha_1^\top \frac{\partial \alpha_2^\top}{\partial \boldsymbol{x}} \alpha_2 h_2 + (\alpha_1^\top\alpha_2)^2 h_1' h_2 + \alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 h_2' h_1 = \\
&\alpha_1^\top \left(\frac{\partial \alpha_1^\top}{\partial \boldsymbol{x}} h_1 + \frac{\partial \alpha_2^\top}{\partial \boldsymbol{x}} h_2 + \alpha_2\alpha_1^\top h_1' h_2 + \alpha_1\alpha_2^\top h_2' h_1 \right) \alpha_2 .
\end{aligned} \tag{A.34}$$

Group common terms in the $\alpha_i^\top \alpha_j$ and rearrange to write

$$\left[\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 - (\alpha_1^\top\alpha_2)^2\right](h_1 h_2' - h_1' h_2) = \alpha_1^\top \left(\frac{\partial \alpha_1}{\partial \boldsymbol{x}^\top} - \frac{\partial \alpha_1^\top}{\partial \boldsymbol{x}} \right) \alpha_2 h_1 + \alpha_1^\top \left(\frac{\partial \alpha_2}{\partial \boldsymbol{x}^\top} - \frac{\partial \alpha_2^\top}{\partial \boldsymbol{x}} \right) \alpha_2 h_2 . \tag{A.35}$$

Solving for the Lie Bracket, $h_1 h_2' - h_1' h_2$, we have

$$h_1 h_2' - h_1' h_2 = \left[\frac{\alpha_1^\top \left(\partial \alpha_1 / \partial \boldsymbol{x}^\top - \partial \alpha_1^\top / \partial \boldsymbol{x}\right) \alpha_2}{\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 - (\alpha_1^\top\alpha_2)^2} \right] h_1 + \left[\frac{\alpha_1^\top \left(\partial \alpha_2 / \partial \boldsymbol{x}^\top - \partial \alpha_2^\top / \partial \boldsymbol{x}\right) \alpha_2}{\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 - (\alpha_1^\top\alpha_2)^2} \right] h_2$$

$$\equiv c_1 h_1 + c_1 h_2 , \tag{A.36}$$

with c_1 and c_2 absolute constants, both of which cannot vanish. Without loss in generality, let $h_1 \neq 0$ (both h_i cannot vanish simultaneously and neither can vanish over an open set). Dividing both sides of (A.36) by h_1 and solving for h_2' gives

$$h_2'(y) = c_1 - \frac{h_1'(y)}{h_1(y)} + \frac{c_2}{h_1(y)} h_2(y) . \tag{A.37}$$

Let $c_1 \neq 0$ (reverse the roles of h_1 and h_2, if necessary) and make a change of variables to $\tilde{h}_1 = c_1 h_1 + c_1 h_2$, with $\tilde{h}_1' = c_1 h_1' + c_1 h_2'$, and to $\tilde{h}_2 = h_2 / c_1$, with $\tilde{h}_2' = h_2' / c_1$. Then

$$\tilde{h}_1 \tilde{h}_2' - \tilde{h}_1' \tilde{h}_2 = (c_1 h_1 + c_2 h_2)(h_2'/c_1) - (c_1 h_1' + c_2 h_2')(h_2/c_1) = h_1 h_2' - h_1' h_2 . \tag{A.38}$$

We now have

$$\tilde{h}_1 \tilde{h}_2' - \tilde{h}_1' \tilde{h}_2 = h_1 h_2' - h_1' h_2 = c_1 h_1 + c_2 h_2 = \tilde{h}_1 . \tag{A.39}$$

In other words (abusing notation by dropping the ~'s), we form particular linear combinations of the h_i such that

$$h_1 h_2' - h_1' h_2 = h_1 \neq 0 , \tag{A.40}$$

equivalently,

$$h_2' - \frac{h_1'}{h_1} h_2 = 1 . \tag{A.41}$$

Since

$$\frac{d}{dy}\left(\frac{h_2}{h_1}\right) = \frac{h_2'}{h_1} - \frac{h_1'}{h_1^2} h_2 = \frac{1}{h_1} , \tag{A.42}$$

direct integration gives

$$\int \frac{d}{dy}\left(\frac{h_2(y)}{h_1(y)}\right) dy = \int \frac{dy}{h_1(y)} , \tag{A.43}$$

equivalently,

$$h_2(y) = h_1(y)\int^{y} \frac{ds}{h_1(s)}. \tag{A.44}$$

Define $\gamma(y) = \int^{y} h_1(s)^{-1} ds$ and rewrite (A.31) as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left[\alpha_1 + \alpha_2\gamma(y)\right]h_1(y). \tag{A.45}$$

Since $\gamma'(y) = \frac{d}{dy}\int^{y} h_1(s)^{-1} ds = h_1(y)^{-1}$, this is equivalent to

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y)\frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)}\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \alpha_2\gamma(y). \tag{A.46}$$

We thus can change the definition of $y = f(m)$ to incorporate $\gamma(y)$ through composition, and any full rank 2 system has a representation for *y* such that $\partial y/\partial \boldsymbol{x} = \alpha_1 + \alpha_2 y$.

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 h_1(y) + \alpha_2 h_2(y) + \alpha_3 h_3(y) \tag{A.47}$$

The derivations in this case are considerably more involved. We make use of several previous results and techniques from the theory of differential equations to simplify and reduce the calculations. Let $h_1(y) \neq 0$, define $\gamma(y) = \int^{y} h_1(s)^{-1} ds$, and rewrite (A.47) as

$$\begin{aligned}\frac{\partial \gamma}{\partial \boldsymbol{x}} &= \gamma'(y)\frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)}\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \alpha_2\left(\frac{h_2(y)}{h_1(y)}\right) + \alpha_3\left(\frac{h_3(y)}{h_1(y)}\right) \\ &\equiv \alpha_1 + \alpha_2\tilde{h}_2(y) + \alpha_3\tilde{h}_3(y) \end{aligned} \tag{A.48}$$

By Lemma 1 symmetry is coordinate free. Therefore, consider the representation (again dropping the ~'s and redefining *y* if necessary)

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \alpha_1 + \alpha_2 h_2(y) + \alpha_3 h_3(y). \tag{A.49}$$

The least squares conversion of the symmetry conditions gives

$$h_2'(y) = c_{12}^1 + c_{12}^2 h_2(y) + c_{12}^3 h_3(y),$$

$$h_3'(y) = c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y), \tag{A.50}$$

$$h_2(y)h_3'(y) - h_2'(y)h_3(y) = c_{23}^1 + c_{23}^2 h_2(y) + c_{23}^3 h_3(y),$$

where the c_{ij}^k are constants and cannot all be zero in any given equation. The first two equations form a complete system of linear, ordinary differential equations with constant coefficients. These would be straightforward to solve if the system were not constrained by the third equation (the Lie Bracket for $h_2(y)$ and $h_3(y)$).

Our plan of attack is to calculate the complete solution to the two-equation system of differential equations and then check for consistency with the third equation. This second step restricts the set of values that the c_{ij}^k can assume in an integrable system. Differentiate the first equation with respect to *y* and substitute out $h_3'(y)$ and $h_3(y)$,

$$h_2''(y) = c_{12}^2 h_2'(y) + c_{12}^3 h_3'(y)$$

$$= c_{12}^2 h_2'(y) + c_{12}^3 \left[c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y) \right]$$

$$= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{12}^3 c_{13}^3 h_3(y) \tag{A.51}$$

$$= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{13}^3 \left[h_2'(y) - c_{12}^1 + c_{12}^2 h_2(y) \right]$$

$$= \left(c_{12}^2 + c_{13}^3\right) h_2'(y) + \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) h_2(y) + \left(c_{13}^1 c_{12}^3 - c_{12}^1 c_{13}^3\right).$$

The homogeneous part of this second-order differential equation is

$$h_2''(y) - \left(c_{12}^2 + c_{13}^3\right) h_2'(y) - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) h_2(y) = 0. \tag{A.52}$$

Trying $h_2(y) = e^{\lambda y}$ produces the characteristic equation

$$\lambda^2 - \left(c_{12}^2 + c_{13}^3\right)\lambda - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) = 0, \tag{A.53}$$

with characteristic roots

$$\lambda = ½\left[c_{12}^2 + c_{13}^3 \pm \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)} \right]. \tag{A.54}$$

If $c_{12}^2 = c_{12}^3 = c_{13}^3 = 0$, then $\lambda = 0$ is the only root, and the complete solution has the form

$$h_2(y) = a_2 + b_2 y + c_2 y^2\,,$$

$$h_3(y) = a_3 + b_3 y + c_3 y^2\,, \tag{A.55}$$

where the $\{a_k, b_k, c_k\}_{k=2}^{3}$ are constants. Define $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$, $\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$, and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$. Then we have

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 y + \tilde{\alpha}_3 y^2\,. \tag{A.56}$$

The last step in this part of the proof is to show that this is the only possibility in the full rank 3 case. If any of $c_{12}^2 \neq 0$, $c_{12}^3 \neq 0$, or $c_{13}^3 \neq 0$, then we need to consider distinct and repeated roots separately. With distinct roots, the complete solution to the two ordinary differential equations takes the general form

$$h_2(y) = a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}\,,$$

$$h_2(y) = a_3 + b_3 e^{\lambda_1 y} + c_3 e^{\lambda_2 y}\,, \tag{A.57}$$

where the $\{a_k, b_k, c_k\}_{k=2}^{3}$ again are constants. As before, define $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$, $\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$ and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$, and rewrite (A.49) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 e^{\lambda_1 y} + \tilde{\alpha}_3 e^{\lambda_2 y}\,. \tag{A.58}$$

The equation for the Lie Bracket $h_2 h_3' - h_2' h_3$ now takes the form

$$(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)y} = c_{23}^1 + c_{23}^2 e^{\lambda_1 y} + c_{23}^3 e^{\lambda_2 y}\,, \tag{A.59}$$

where $\lambda_2 - \lambda_1 = \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)}$ and $\lambda_2 + \lambda_1 = c_{12}^2 + c_{13}^3$; a contradiction for all $(\lambda_1, \lambda_2) \neq (0,0)$, for either real or complex roots. Therefore, the roots must be real and equal, $\lambda = ½\left(c_{12}^2 + c_{13}^3\right)$. Once again form the above linear combinations of the α_k's, let $h_2(y) = e^{\lambda y}$ and $h_3(y) = y e^{\lambda y}$, and rewrite (A.49) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 e^{\lambda y} + \tilde{\alpha}_3 y e^{\lambda y}\,. \tag{A.60}$$

In this case, the equation for the Lie Bracket, $h_2 h_3' - h_2' h_3$, takes the form

$$e^{2\lambda y} = c_{23}^1 + c_{23}^2 e^{\lambda y} + c_{23}^3 y e^{\lambda y}, \tag{A.61}$$

a contradiction for all $\lambda \neq 0$. Hence, only a repeated vanishing root is possible and a representation for y exists in any full rank 3 system such that

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \alpha_1 + \alpha_2 y + \alpha_3 y^2 . \tag{A.62}$$

This completes the proof of the full rank representation part of the proposition.

The next step in the proof of the proposition is to show that polynomials constitute the class of *minimal deficit* demand systems when $K > 3$. This is accomplished by an inductive argument, and we proceed with the induction beginning with $K = 4$. When $K = 4$ there are a total of six Lie Brackets, but the dimension of the vector space spanned by the basis $\{h_1\, h_2 \cdots h_4\}$ is only four. We know from the theory of Lie algebras on the real line that at least one of the Lie Brackets must lie outside of this space. We have shown that by redefining y and modifying the α_k's to accommodate the change in y, $\{1\, y\, y^2\}$ is the largest Lie algebra on the real line. The structure of this vector space is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2 h_3' - h_2' h_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}. \tag{A.63}$$

If we add a fourth income function to this system, the above derivations apply with minor modifications. Therefore, add $h_4(y)$ to $\{1\, y\, y^2\}$. The Lie Bracket conditions are

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2 h_3' - h_2' h_3 \\ h_4' \\ h_2 h_4' - h_2' h_4 \\ h_3 h_4' - h_3' h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \\ h_4' \\ y h_4' - h_4 \\ y^2 h_4' - 2y h_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c_{14}^1 & c_{14}^2 & c_{14}^3 & c_{14}^4 \\ c_{24}^1 & c_{24}^1 & c_{24}^1 & c_{24}^1 \\ c_{34}^1 & c_{34}^1 & c_{34}^1 & c_{34}^1 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ h_4 \end{bmatrix}. \tag{A.64}$$

At most two of the new Lie Bracket equations can be consistent. Without loss in generality, consider the fourth and fifth,

$$h_4'(y) = c_{14}^1 + c_{14}^2 y + c_{14}^3 y^2 + c_{14}^4 h_4(y),$$

$$y h_4'(y) - h_4(y) = c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 h_4(y). \tag{A.65}$$

These are two linear, first-order ordinary differential equations in $h_4(y)$. The most direct route is to solve the first and check the second for consistency. If $c_{14}^4 = 0$, then integrating the first equation gives

$$h_4(y) = c + c_{14}^1 y + \tfrac{1}{2} c_{14}^2 y^2 + \tfrac{1}{3} c_{14}^3 y^3, \tag{A.66}$$

where *c* is a constant of integration. Applying similar modifications to the α_k's as before, we have $h_k(y) = y^{k-1}$, $k = 1,2,3,4$. The second equation becomes

$$y h_4'(y) - h_4(y) = 3y^3 - y^3 = 2y^3 = 0 \cdot 1 + 0 \cdot y + 0 \cdot y^2 + 2y^3, \tag{A.67}$$

which is contained in the vector space spanned by $\{1\; y\; y^2\; y^3\}$. Of course, the Lie Bracket, $h_3 h_4' - h_3' h_4 = 3y^4 - 2y^4 = y^4$ falls outside of this vector space, as it must.

If $c_{14}^4 \neq 0$, integrating by parts twice gives the complete solution as

$$h_4(y) = -\left[\frac{c_{14}^1}{c_{14}^4} + \frac{c_{14}^2}{\left(c_{14}^4\right)^2} + \frac{2c_{14}^3}{\left(c_{14}^4\right)^3}\right] - \left[\frac{c_{14}^2}{c_{14}^4} + \frac{c_{14}^3}{\left(c_{14}^4\right)^2}\right] y - \frac{c_{14}^3}{c_{14}^4} y^2 + c e^{c_{14}^4 y}, \tag{A.68}$$

where *c* is again a constant of integration. Once more using the above device to adjust the α_k's, we have $h_4(y) = e^{c_{14}^4 y}$, and the second equation becomes

$$y h_4'(y) - h_4(y) = \left(c_{14}^4 y - 1\right) e^{c_{14}^4 y} = c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 e^{c_{14}^4 y}, \tag{A.69}$$

a contradiction. Hence, the structure with four income functions and a maximum number of Lie Brackets spanned by the income functions $\{1\; y\; y^2\; y^3\}$ is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_4' \\ h_2 h_3' - h_2' h_3 \\ h_2 h_4' - h_2' h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ 3y^2 \\ y^2 \\ 2y^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix}. \tag{A.70}$$

Only one (the minimal possible number) Lie Bracket, $h_3 h_4' - h_3' h_4$, out of the total of six, falls outside of this vector space.

The induction is completed by identical steps to show that if a basis with *K* functions is

$\{1\, y\, y^2 \cdots y^{K-1}\}$ and we add a K+1[st] function, $h_{K+1}(y)$, then the maximal increase in the number of spanned Lie Brackets occurs when $h_{K+1}(y) = y^K$.

The final step in the proof of this proposition is to show that for the polynomial class of Gorman Engel curve systems, $rank[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})] \leq 3$. We proceed constructively by showing that if the system of demands has the polynomial representation,[2]

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=0}^{K} \boldsymbol{\alpha}_k y^k , \tag{A.71}$$

and is weakly integrable, then there exist $\varphi_k : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R},\ k = 2, \ldots, K$ such that

$$\boldsymbol{\alpha}_k \equiv \varphi_k \boldsymbol{\alpha}_K \ \ \forall\ k \geq 2 . \tag{A.72}$$

Integrability is equivalent to symmetry of the matrix

$$\sum_{k=0}^{K} \frac{\partial \boldsymbol{\alpha}_i}{\partial \boldsymbol{x}^\top} y^k + \sum_{k=1}^{K} \sum_{\ell=0}^{K} k \boldsymbol{\alpha}_k \boldsymbol{\alpha}_\ell^\top y^{k+\ell-1} . \tag{A.73}$$

By continuity, symmetry requires that each like power of y has a symmetric coefficient matrix, and all of the matrices for powers K+1 through 2K–2 involve nontrivial symmetry conditions without involving any of the $\partial \boldsymbol{\alpha}_k / \partial \boldsymbol{x}^\top$ terms. The matrix on y^{2K-1} only involves $\boldsymbol{\alpha}_K \boldsymbol{\alpha}_K^\top$ and is symmetric. Combine terms in like powers of y and apply a backward recursion beginning with the matrix on y^{2K-2}, so that

$$(K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_K^\top + K\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-1}^\top \tag{A.74}$$

is symmetric if and only if $\boldsymbol{\alpha}_{K-1} \equiv \varphi_{K-1}\boldsymbol{\alpha}_K$ for some $\varphi_{K-1} : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$. Similarly,

$$(K-2)\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_K^\top + (K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-1}^\top + K\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-2}^\top \tag{A.75}$$

is symmetric if and only if $\boldsymbol{\alpha}_{K-2} \equiv \varphi_{K-2}\boldsymbol{\alpha}_K$ for some $\varphi_{K-2} : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$. Applying the recursive argument, consider the matrix on y^{2K-4},

[2] Switching indexes from {1,…,K} to {0,…,K} greatly simplifies the algebra and notation in this part of the proof without affecting the structure of the underlying problem in any way.

$$(K-3)\alpha_{K-3}\alpha_K^{\top} + (K-2)\alpha_{K-2}\alpha_{K-1}^{\top} + (K-1)\alpha_{K-1}\alpha_{K-2}^{\top} + K\alpha_K\alpha_{K-3}^{\top}. \qquad \text{(A.76)}$$

The middle two terms are symmetric, because $\alpha_{K-2}\alpha_{K-1}^{\top} = \varphi_{K-2}\varphi_{K-1}\alpha_K\alpha_K^{\top} = \alpha_{K-1}\alpha_{K-2}^{\top}$. The matrix $\left(\alpha_{K-3}\alpha_K^{\top} + \alpha_K\alpha_{K-3}^{\top}\right)$ is automatically symmetric. Therefore, the matrix on y^{2K-4} is symmetric if and only if $\alpha_K\alpha_{K-3}^{\top}$ is symmetric, which occurs if and only if $\alpha_{K-3} \equiv \varphi_{K-3}\alpha_K$ for some $\varphi_{K-3} : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$. This completes the argument when $3 \le K \le 5$.

If $K > 5$, for each j satisfying $4 \le j \le K-1$, group like terms, substitute $\alpha_{K-i} \equiv \varphi_{K-i}\alpha_K$ for each $i < j$, and appeal to symmetry of the matrix $\alpha_{K+1-j}\alpha_K^{\top} + \alpha_K\alpha_{K+1-j}^{\top}$. Then symmetry sequentially requires that each matrix of the following form is symmetric:

$$(j-1)\alpha_K\alpha_{K+1-j}^{\top} + \sum_{i=1}^{j-2}(K-i)\varphi_{K-i}\varphi_{K+1+i-j}\alpha_K\alpha_K^{\top}. \qquad \text{(A.77)}$$

This holds if and only if $\alpha_{K+1-j} \equiv \varphi_{K+1-j}\alpha_K$ for $\varphi_{K+1-j} : \mathbb{R}^{n_q} \times \mathbb{R}^{n_{\tilde{q}}} \times \mathbb{R}^r \to \mathbb{R}$. When $j = 4$ we have the result for α_{K-3}; when $j = K-1$ we have it for α_2; and $\forall\ K > 2$, we have $\alpha_k \equiv \varphi_k\alpha_K\ \forall\ k = 2,\ldots,K$ so that $rank[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})] \le 3$. ■

Proposition 2. If the LA-AIDS model,

$$\frac{\partial \ln e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \ln \boldsymbol{p}} = \alpha(\tilde{\boldsymbol{p}},\boldsymbol{s}) + \boldsymbol{B}\ln\boldsymbol{p} + \gamma\left[\ln e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u) - (\ln\boldsymbol{p})^{\top}\frac{\partial \ln e(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{\partial \ln \boldsymbol{p}}\right]$$

is weakly integrable on the open set $\mathcal{N} \subset \mathbb{R}^{n_q}$ defined by $1 + \gamma'\boldsymbol{x} \ne 0\ \forall\ \boldsymbol{x} \in \mathcal{N}$, then either (a) $\gamma \ne \boldsymbol{0}$ and $\boldsymbol{B} = \beta_0\gamma\gamma'$ for some $\beta_0 \in \mathbb{R}$, with logarithmic expenditure function

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u) = \alpha(\tilde{\boldsymbol{p}},\boldsymbol{s})^{\top}\boldsymbol{x} + \beta_0\left[(1+\gamma^{\top}\boldsymbol{x})\ln(1+\gamma^{\top}\boldsymbol{x}) - \frac{\gamma^{\top}\boldsymbol{x}}{(1+\gamma^{\top}\boldsymbol{x})}\right]$$

$$+(1+\gamma^{\top}\boldsymbol{x})\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u),$$

or (b) $\gamma = \boldsymbol{0}$ and $\boldsymbol{B} = \boldsymbol{B}'$, with logarithmic expenditure function,

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u) = \alpha(\tilde{\boldsymbol{p}},\boldsymbol{s})^{\top}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\top}\boldsymbol{B}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u).$$

Proof: See LaFrance (2004). ■

Proposition 3. The system of partial differential equations,

$$\frac{\partial y}{\partial x_i} = \alpha_i + \beta_i^{\mathsf{T}} \boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}} \Delta_i \boldsymbol{x} + \gamma_i y,\ i = 1,\ldots,n_q,$$

where $\gamma_1 \neq 0$ *and each* $n_q \times n_q$ *matrix,* Δ_i *is symmetric* $\forall$ *i, is weakly integrable if and only if it can be written in the form*

$$\frac{\partial y}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{p} + \gamma\left[y - \alpha_0 - \tilde{\alpha}^{\mathsf{T}} \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}} \tilde{\boldsymbol{B}} \boldsymbol{x} \right]$$

where α_0 *is a scalar,* $\tilde{\alpha} = \alpha - \alpha_0\gamma$ *is an* $n_q \times 1$ *vector,* $\tilde{\boldsymbol{B}}$ *is a symmetric* $n_q \times n_q$ *matrix satisfying* $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \gamma\alpha^{\mathsf{T}}$, *where* $\boldsymbol{B} = \left[\beta_1 \cdots \beta_{n_q}\right]$, *and* $\Delta_i = -\gamma_i \tilde{\boldsymbol{B}}\ \forall\ i$.

Proof: Symmetry of the Slutsky substitution terms is equivalent to symmetry of the $n_q \times n_q$ matrix with typical element

$$s_{ij} = \beta_{ij} + \sum_{k=1}^{n_q} \delta_{ijk} x_k + \gamma_i \left[\alpha_j + \sum_{k=1}^{n_q} \beta_{jk} x_k + \tfrac{1}{2} \sum_{k=1}^{n_q} \sum_{\ell=1}^{n_q} \delta_{jk\ell} x_k x_\ell + \gamma_j y \right]. \qquad \text{(A.78)}$$

To show necessity, we will derive the implications of symmetry, $s_{ij} = s_{ji}\ \forall\ i, j$. These implications can be conveniently grouped into three sets:

(a) $$\beta_{ij} + \gamma_i \alpha_j = \beta_{ji} + \gamma_j \alpha_i;$$

(b) $$\sum_{k=1}^{n_q} (\delta_{ijk} + \gamma_i \beta_{jk}) x_k = \sum_{k=1}^{n_q} (\delta_{jik} + \gamma_j \beta_{ik}) x_k \text{; and}$$

(c) $$\gamma_i \sum_{k=1}^{n_q} \sum_{l=1}^{n_q} \delta_{jkl} x_k x_l = \gamma_j \sum_{k=1}^{n_q} \sum_{l=1}^{n_q} \delta_{ikl} x_k x_l .$$

From (a), it follows that $\alpha \equiv \hat{\alpha} - \gamma\hat{\alpha}_0$, where $\hat{\alpha}_i = (\beta_{i1} - \beta_{1i}) / \gamma_1$ and $\hat{\alpha}_0 = -\alpha_1 / \gamma_1$. Substituting the right-hand-side for each α_i back into (a) implies $B \equiv [\beta_{ij} + \gamma_i \hat{\alpha}_j]$ is symmetric, equivalently $\left[\beta_{ij}\right] = \boldsymbol{B} - \gamma\alpha^{\mathsf{T}}$ for some symmetric matrix $\boldsymbol{B}$. Now turning to (b), we will use a specialized result of LaFrance and Hanemann (1989, *Theorem 2*, p. 266) for these kinds of problems to obtain $\delta_{ijk} + \gamma_i\beta_{jk} = \delta_{jik} + \gamma_j\beta_{ik}\ \forall\ i, j, k$. We will return to this in a moment. First, however, we need to apply the same result of LaFrance and Hanemann to (c) to get $\gamma_i\delta_{jkl} = \gamma_j\delta_{ikl}\ \forall\ i, j, k, l$. This, in turn, implies that for each *i*, the *n*×*n* matrix $[\delta_{ikl}] = \gamma_i \boldsymbol{C}$ where $\boldsymbol{C}$ is a symmetric matrix with typical element $c_{kl} = \delta_{1kl}/\gamma_1$. Combining this with (b) gives $\gamma_i(c_{jk} + b_{jk}) = \gamma_j(c_{ik} + b_{ik})\ \forall\ i, j, k$. Exploiting $\gamma_1 \neq 0$ and the symmetry of both $\boldsymbol{B}$ and $\boldsymbol{C}$ then give $(b_{ij} + c_{ij}) = (b_{11} + c_{11})\gamma_i\gamma_j/\gamma_1^2$, so that $\boldsymbol{B}$ and $\boldsymbol{C}$ are related by $\boldsymbol{C} = -(\boldsymbol{B} + \varepsilon\gamma\gamma')$, where $\varepsilon = -(b_{11} + c_{11})/\gamma_1^2$. Combining all of these implications, the transformed demands can be written in matrix notation as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \hat{\alpha} - \hat{\alpha}_0\gamma + (\boldsymbol{B} - \gamma\hat{\alpha}^{\mathsf{T}})\boldsymbol{x} - \tfrac{1}{2}\gamma\boldsymbol{x}^{\mathsf{T}}(\boldsymbol{B} + \varepsilon\gamma\gamma^{\mathsf{T}})\boldsymbol{x} + \gamma y$$

$$= \hat{\alpha} + \boldsymbol{B}\boldsymbol{x} + \gamma\left[y - \hat{\alpha}_0 - \hat{\alpha}^{\mathsf{T}}\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}(\boldsymbol{B} + \varepsilon\gamma\gamma^{\mathsf{T}})\boldsymbol{x}\right]. \qquad \text{(A.79)}$$

Adding and subtracting $\varepsilon\gamma$ and $\varepsilon\gamma\gamma'\boldsymbol{x}$ has no affect on the transformed demands. Therefore, let $\tilde{\alpha} = \hat{\alpha} + \varepsilon\gamma$, $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \varepsilon\gamma\gamma'$, and $\alpha_0 = \hat{\alpha}_0 + \varepsilon$, and rewrite the partial differential equations in the equivalent form

$$\frac{\partial y}{\partial \boldsymbol{x}} = \tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{x} + \gamma\left(y - \alpha_0 - \tilde{\alpha}'\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{B}}\boldsymbol{x}\right). \qquad \text{(A.80)}$$

Finally, the integrating factor $e^{-\gamma^{\mathsf{T}}\boldsymbol{x}}$ makes the partial differential equations exact,

$$\frac{\partial}{\partial \boldsymbol{x}}\left[y\, e^{-\gamma^{\mathsf{T}}\boldsymbol{x}}\right] = \left[\tilde{\alpha} + \tilde{\boldsymbol{B}}\boldsymbol{x} - \gamma\left(\alpha_0 + \tilde{\alpha}^{\mathsf{T}}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{B}}\boldsymbol{x}\right)\right]e^{-\gamma^{\mathsf{T}}\boldsymbol{x}}$$

$$= \frac{\partial}{\partial \boldsymbol{x}}\left[\left(\alpha_0 + \tilde{\alpha}^{\mathsf{T}}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{B}}\boldsymbol{x}\right)e^{-\gamma^{\mathsf{T}}\boldsymbol{x}}\right]. \qquad \text{(A.81)}$$

The complete class of solutions to this system of partial differential equations is

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \alpha_0(\tilde{\boldsymbol{p}}, \boldsymbol{s}) + \tilde{\alpha}(\tilde{\boldsymbol{p}}, \boldsymbol{s})^{\mathsf{T}}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{B}}\boldsymbol{x} + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)\, e^{\gamma^{\mathsf{T}}\boldsymbol{x}}. \qquad \text{(A.82)}$$

Sufficiency is shown by applying Hotelling's/Shephard's lemma. ■

A.4 Characterizing Indirect Preferences

In this section, we characterize the class of indirect preferences for each of the full rank cases and present and discuss an example of indirect preferences that gives rise to a rank three demand model with more than three income terms.

Rank 1:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}}. \tag{A.83}$$

Simply integrating gives

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u). \tag{A.84}$$

This is the *translation group* representation of indirect preferences for the rank one case. Solving for the deflated expenditure function gives

$$e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \varphi\big(\beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}) + \theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)\big), \tag{A.85}$$

where $\varphi(\cdot)$ is the inverse of $f(\cdot)$. Equivalently, the indirect utility function has the form

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, m) = \psi\big(f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}), \tilde{\boldsymbol{p}}, \boldsymbol{s}\big), \tag{A.86}$$

where ψ is the inverse of θ with respect to u. Since $\boldsymbol{q} = \boldsymbol{diag}[g_i'] \times \big(\partial\beta/\partial\boldsymbol{x}\big)\big/ f'$, the demands for $\boldsymbol{q}$ in the rank 1 case are homothetic with income elasticity $-mf''(m)/f'(m)$. If $f(m) = m^{\kappa}$, the common income elasticity $1-\kappa$ is constant, but only equals one in the limiting case $f(m) = \ln(m)$. More general transformations do not result in a constant income elasticity, although it must be independent of all prices and demographic variables in this class of demand systems.

Rank 2:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \alpha_2 y. \tag{A.87}$$

Symmetry in this case implies

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\mathsf{T}}} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^{\mathsf{T}}} y + \alpha_2 \alpha_1^{\mathsf{T}} + \alpha_2 \alpha_2^{\mathsf{T}} y = \frac{\partial \alpha_1^{\mathsf{T}}}{\partial \boldsymbol{x}} + \frac{\partial \alpha_2^{\mathsf{T}}}{\partial \boldsymbol{x}} y + \alpha_1 \alpha_2^{\mathsf{T}} + \alpha_2 \alpha_2^{\mathsf{T}} y. \tag{A.88}$$

Eliminating the symmetric matrix $\alpha_2 \alpha_2^{\mathsf{T}} y$ from both sides and equating the matrices that multiply like powers of y implies

$$\frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} + \alpha_2 \alpha_1^{\mathsf{T}} = \frac{\partial \alpha_1^{\mathsf{T}}}{\partial \boldsymbol{x}} + \alpha_1 \alpha_2^{\mathsf{T}}, \tag{A.89}$$

and that $\partial\alpha_2 \big/ \partial \boldsymbol{x}^{\mathsf{T}}$ is symmetric. The latter property implies the existence of a function

$\beta:\mathbb{R}_{++}^{n_q}\times\mathbb{R}_{++}^{n_{\tilde{q}}}\times\mathbb{R}^r\to\mathbb{R}$ such that $\partial\beta/\partial\boldsymbol{x}=\alpha_2$. It follows that $\partial\alpha_1/\partial\boldsymbol{x}^{\mathsf{T}}+\left(\partial\beta/\partial\boldsymbol{x}\right)\alpha_1^{\mathsf{T}}$ is symmetric. Equivalently, we can rewrite (A.87) in the form

$$\frac{\partial y}{\partial\boldsymbol{x}}=\alpha_1+\frac{\partial\beta}{\partial\boldsymbol{x}}y\,, \tag{A.90}$$

with $\dfrac{\partial\alpha_1}{\partial\boldsymbol{x}^{\mathsf{T}}}-\alpha_1\dfrac{\partial\beta}{\partial\boldsymbol{x}^{\mathsf{T}}}=\dfrac{\partial\alpha_1^{\mathsf{T}}}{\partial\boldsymbol{x}}-\dfrac{\partial\beta}{\partial\boldsymbol{x}}\alpha_1^{\mathsf{T}}$, symmetric. We can apply the integrating factor $e^{-\beta}$ by noting that

$$\frac{\partial}{\partial\boldsymbol{x}}\left(ye^{-\beta}\right)=\left(\frac{\partial y}{\partial\boldsymbol{x}}-\frac{\partial\beta}{\partial\boldsymbol{x}}y\right)e^{-\beta}, \tag{A.91}$$

and

$$\frac{\partial}{\partial\boldsymbol{x}^{\mathsf{T}}}\left(\alpha_1e^{-\beta}\right)=\left(\frac{\partial\alpha_1}{\partial\boldsymbol{x}^{\mathsf{T}}}-\alpha_1\frac{\partial\beta}{\partial\boldsymbol{x}^{\mathsf{T}}}\right)e^{-\beta} \tag{A.92}$$

is symmetric. This implies the existence of a function $\gamma:\mathbb{R}_{++}^{n_q}\times\mathbb{R}_{++}^{n_{\tilde{q}}}\times\mathbb{R}^r\to\mathbb{R}$ such that $\partial\gamma/\partial\boldsymbol{x}=\alpha_1e^{-\beta}$, and integrating gives the transformed deflated expenditure function as

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)=e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)\,. \tag{A.93}$$

Let $e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}\equiv\delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$, abuse notation and relabel $e^{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$ as $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})$, and rewrite (A.93) in the form

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)=\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+\delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)\,. \tag{A.94}$$

This quasi-linear form is the *translation and scaling group* representation of indirect preferences in the full rank two case. From this we can write the deflated expenditure function as

$$e(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)=\varphi\big(\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+\delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)\big), \tag{A.95}$$

and the indirect utility function as

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},m)=\psi\left(\frac{f(m)-\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}{\delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})},\tilde{\boldsymbol{p}},\boldsymbol{s}\right). \tag{A.96}$$

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \alpha_1 + \alpha_2 y + \alpha_3 y^2 . \tag{A.97}$$

We present two equivalent closed form expressions for the solution to this case. One establishes the connection to the *projective transformation group*. The other applies when (A.97) has a pair of purely complex roots and provides a direct solution for the trigonometric form of indirect preferences.

First, we note that the methods of van Daal and Merkies (1989) for solving integrability of the complete quadratic expenditure system apply without change to our problem. The only difference is that the homogeneity properties they identified do not apply here. Thus, there is no need to reproduce their steps. They show that (A.97) is integrable if and only if there exist functions, $\beta_1, \beta_2, \beta_3 : \mathbb{R}^{n_q}_{++} \times \mathbb{R}^{n_{\tilde{q}}}_{++} \times \mathbb{R}^r \to \mathbb{R}$, and $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta_1}{\partial \boldsymbol{x}} + \gamma_2(\beta_2)\beta_3 \frac{\partial \beta_2}{\partial \boldsymbol{x}} + \frac{\partial \beta_3}{\partial \boldsymbol{x}} \frac{(y - \beta_1)}{\beta_3} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \frac{(y - \beta_1)^2}{\beta_3} . \tag{A.98}$$

This can be rewritten in the form

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{y - \beta_1}{\beta_3}\right) = \frac{1}{\beta_3}\left(\frac{\partial y}{\partial \boldsymbol{x}} - \frac{\partial \beta_1}{\partial \boldsymbol{x}}\right) - \frac{(y - \beta_1)}{\beta_3^2}\frac{\partial \beta_3}{\partial \boldsymbol{x}} = \left[\gamma_2(\beta_2) + \frac{(y - \beta_1)^2}{\beta_3^2}\right]\frac{\partial \beta_2}{\partial \boldsymbol{x}} . \tag{A.99}$$

We can simplify this even further by making two simple changes of variables. First, let $w = (y - \beta_1)/\beta_3$, so that

$$\frac{\partial w}{\partial \boldsymbol{x}} = \left[\gamma_2(\beta_2) + w^2\right]\frac{\partial \beta_2}{\partial \boldsymbol{x}} \tag{A.100}$$

Second, let $z = -1/w$, so that

$$\frac{\partial z}{\partial \boldsymbol{x}} = \left[1 + \gamma_2(\beta_2) z^2\right]\frac{\partial \beta_2}{\partial \boldsymbol{x}} . \tag{A.101}$$

Now, if $\gamma(\beta_2) \equiv \lambda$ is constant, then we can separate the variables so that

$$\frac{dz}{1 + \lambda z^2} = \frac{\partial \beta_2}{\partial x_i} \ \forall\ i = 1, \cdots, n_q . \tag{A.102}$$

This is an *exact* system of partial differential equations and the solution is found by direct integration,

$$\phi\left(\frac{-\beta_3}{y-\beta_1}\right) \equiv \int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = \beta_2 + \theta\,, \tag{A.103}$$

where $\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u)$ is the "constant of integration." This is readily recognized as the solution obtained by van Daal and Merkies (1989) and applied by Lewbel (1990) to full rank three QPIGL and QPIGLOG complete systems.

But we can go considerably further. One reason for doing this is to obtain closed form expressions for the indirect preferences. A second reason is to show the connection between this representation and the *projective transformation group* representation of indirect preferences that is standard in the theory of Lie groups. A third reason is that in one case we obtain the trigonometric form of indirect preferences that is implied by Gorman (1981) and is presented without derivation in Lewbel (1990), but heretofore has not been obtained explicitly from the structure of a set of demand equations with complex roots.

Suppose that $\lambda > 0$, define $\lambda = -(\iota\kappa)^2 = \kappa^2$, with $\iota = \sqrt{-1}$, let $\tau = \iota\kappa$ be a purely complex constant, write $1 + \lambda z^2 = (1+\tau z)(1-\tau z)$, and apply the method of partial fractions to obtain

$$\frac{1}{1-(\tau z)^2} = \frac{½}{(1-\tau z)} + \frac{½}{(1+\tau z)}\,. \tag{A.104}$$

This implies that

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1-\tau z)} + ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\tau z)}$$

$$= ½\ln\left[\frac{1+\tau\left(\frac{-\beta_3}{y-\beta_1}\right)}{1-\tau\left(\frac{-\beta_3}{y-\beta_1}\right)}\right] = \beta_2 + \theta\,. \tag{A.105}$$

Exponentiating and abusing notation by relabeling $e^{2\beta_2}$ as β_2 and $e^{2\theta}$ as θ gives,

$$\frac{y-\beta_1-\tau\beta_3}{y-\beta_1+\tau\beta_3} = \beta_2\theta\,. \tag{A.106}$$

Solving for the deflated and transformed expenditure function then gives

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)=\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+\tau\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\left(\frac{1+\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)}{1-\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)}\right). \qquad \text{(A.107)}$$

This is an element of the *complex projective transformation group* in θ. Alternatively, solving for the quasi-indirect utility function,

$$\theta=\upsilon(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},y)=\frac{y-\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})-\tau\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}{\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\left[y-\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+\tau\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})\right]}. \qquad \text{(A.108)}$$

In this case, (A.108) is an element of the complex projective transformation group in *y*.

In the case where $\lambda>0$, we also can derive an alternative, but equivalent, expression for the indirect preferences by using a third change of variables to $s=\kappa z$, where $\lambda=\kappa^2>0$, so that

$$\int^{-\beta_3/(y-\beta_1)}\frac{dz}{\left[1+(\kappa z)^2\right]}=\int^{-\kappa\beta_3/(y-\beta_1)}\frac{ds}{\kappa(1+s^2)}=\frac{1}{\kappa\pi}\tan^{-1}\left(\frac{-\kappa\beta_3}{y-\beta_1}\right)=\beta_2+\theta\,. \text{(A.109)}$$

The indirect utility function therefore can be written as

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},m)=\psi\left\{\frac{1}{\kappa\pi}\tan^{-1}\left(\frac{-\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})}{f(m)-\beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})}\right)-\beta_2(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s}),\tilde{\boldsymbol{p}},\boldsymbol{s}\right\} \text{(A.110)}$$

Alternatively, the deflated and transformed expenditure function can be written as

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s},u)=\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})-\frac{\cot\left\{\kappa\pi\left[\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})+\theta(\tilde{\boldsymbol{p}},\boldsymbol{s},u)\right]\right\}}{\kappa\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}. \qquad \text{(A.111)}$$

Clearly (A.109)–(A.111) have the associated trigonometric form for indirect preferences that we wished to find and identify.

Now assume $\lambda<0$, define $-\lambda=\kappa^2$, and write $1+\lambda z^2=(1+\kappa z)(1-\kappa z)$, so that partial fractions imply

$$\frac{1}{1+\lambda z^2}=\frac{1}{1-(\kappa z)^2}=\frac{½}{(1-\kappa z)}+\frac{½}{(1+\kappa z)}. \qquad \text{(A.112)}$$

Integrating as before now gives

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1-\kappa z)} + ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\kappa z)}$$

$$= ½\ln\left[\frac{1+\kappa\left(\dfrac{-\beta_3}{y-\beta_1}\right)}{1-\kappa\left(\dfrac{-\beta_3}{y-\beta_1}\right)}\right] = \beta_2 + \theta\,. \tag{A.113}$$

Notice, in particular, that the only difference between (A.105) and (A.113) is the purely complex root τ and the purely real root κ, respectively. Therefore, proceeding precisely as before, we obtain (A.107) and (A.108) with κ simply replacing τ everywhere. Hence, the class of indirect utility functions in both cases is

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},\boldsymbol{s},m) = \psi\left\{\frac{f(m)-\beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})-\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})}{\beta_2(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})\left[f(m)-\beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})+\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},\boldsymbol{s})\right]},\tilde{\boldsymbol{p}},\boldsymbol{s}\right\}, \tag{A.114}$$

where ψ is once again the inverse of θ with respect to y, and where κ is either purely real or purely complex. The part of v that is associated with $(\boldsymbol{p},m)$ is an element of the (either real or complex) projective transformation group over $y = f(m)$.

To finalize our characterization and exposition, without loss in generality, we can abuse notation further by relabeling $\beta_1 + \kappa\beta_3$ as β_1 and $(-\beta_1 + \kappa\beta_3)\beta_2$ as β_3, for κ either real or complex. Then we have the three–parameter relationship

$$\theta = \frac{y-\beta_1}{\beta_2 y + \beta_3} \Leftrightarrow y = \frac{\beta_1 + \beta_3\theta}{1-\beta_2\theta}. \tag{A.115}$$

Thus, the closed form solutions that can be found in all full rank three cases are members of the projective transformation group. The quasi-indirect utility function, θ, is the inverse group transformation of the (deflated and transformed) expenditure function, y. No additional flexibility in income or prices is obtained with a complex κ even though the trigonometric form in (A.111) is an interesting case. Therefore, for the rest of this section, we assume that κ is real.

When κ is real, the space of all projective transformation groups is referred to in differential topology as *special linear group two* and is denoted by $\mathfrak{sl}(2)$. It is standard practice in Lie group theory to identify the space $\mathfrak{sl}(2)$ with the set of 2×2 real matrices $\boldsymbol{A} = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$ with unit determinant, $\alpha\delta - \beta\gamma = 1$. Indeed, we have $\boldsymbol{A}^{-1} = \begin{bmatrix}\delta & -\beta\\ -\gamma & \alpha\end{bmatrix}$ as a member of this group, and if we write

$$y = \frac{\alpha\theta + \beta}{\gamma\theta + \delta} \Leftrightarrow \theta = \frac{\delta y - \beta}{-\gamma y + \alpha}, \tag{A.116}$$

we can see immediately that 2×2 matrix inverses in this set are one-to-one and onto with the inverse functions of the projective transformation group, while $\boldsymbol{I}_2$ is the identity map in both spaces. Simple algebra then shows that

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left(\alpha\frac{\partial\beta}{\partial \boldsymbol{x}} - \beta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right) + \left[\left(\beta\frac{\partial\gamma}{\partial \boldsymbol{x}} - \gamma\frac{\partial\beta}{\partial \boldsymbol{x}}\right) - \left(\alpha\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right)\right] y + \left(\gamma\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\gamma}{\partial \boldsymbol{x}}\right) y^2 . \tag{A.117}$$

The usefulness of this representation is that integrablity is represented clearly, concisely, and simply in the form of a subset of four out of a total of six possible Lie brackets between the $\{\alpha,\beta,\gamma,\delta\}$ functions with respect to $\boldsymbol{x}$. A (very large) set of full rank three indirect utility functions generating members of Gorman's class of functionally separable demand systems is therefore given by

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, m) = \upsilon\left\{\frac{\delta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}) f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s}) f(m) + \alpha(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}, \boldsymbol{s})}, \tilde{\boldsymbol{p}}, \boldsymbol{s}\right\}, \ \alpha\delta - \beta\gamma \equiv 1. \tag{A.118}$$

Equivalently, we can represent this class of preference systems in terms of the deflated and transformed expenditure function,

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \frac{\alpha(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}, \ \alpha\delta - \beta\gamma \equiv 1 . \tag{A.119}$$

Finally, it is worth noting that $\gamma \neq 0$ is required for a full rank three system, and we can define this class of preferences in terms of Lie's (1880) rank three transformation group,

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s}, u) = \frac{\tilde{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \tilde{\beta}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})}{\theta(\tilde{\boldsymbol{p}}, \boldsymbol{s}, u) + \tilde{\delta}(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{s})} \tag{A.120}$$

where $\tilde{\alpha} = \alpha/\gamma$, $\tilde{\beta} = \beta/\gamma$, and $\tilde{\delta} = \delta/\gamma$. We have established equivalence of full rank three demand systems of Gorman (1981), the rank three transformation group of Lie (1880), and the projective transformation group $\mathfrak{sl}(2)$ using elementary methods. ■

A.5 A CLASS OF GORMAN SYSTEMS WITH MORE THAN THREE INCOME TERMS

The results above do not preclude higher order polynomials, only more than three income terms with a matrix of *linearly independent* price functions. We illustrate this with an example motivated by Jerison (1993). Let the indirect utility function be given by

$$v(\boldsymbol{x},\tilde{\boldsymbol{p}},y,\boldsymbol{s})=\upsilon\left[\left(\frac{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})}{\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})-y}\right)^{\eta}-\delta(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s});\tilde{\boldsymbol{p}},\boldsymbol{s}\right], \tag{A.121}$$

where we assume $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}},\boldsymbol{s})>y$ for monotonicity and let η be any real number in the interval [1,∞). Applying Roy's identity, we generate an incomplete demand system as

$$\boldsymbol{q}=\frac{\mathbf{diag}\left[g_i'(p_i)\right]}{f'(m)}\left[\frac{\partial\gamma}{\partial\boldsymbol{x}}-\frac{\partial\beta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)+\left(\frac{\beta}{\eta}\right)\frac{\partial\delta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)^{\eta+1}\right]. \tag{A.122}$$

Under certain conditions on η, this has the form of proposition one and illustrates the full nature of its implications. First note that there are three linearly independent functions of y on the right-hand-side of (A.122). When $\eta = 1$, we have a quadratic in y. But the parameter η can be any integer in [1,∞) and preferences will remain well-behaved with appropriate choices for the functions β, γ, δ. If η is an integer greater than one, expanding the last term in square brackets with the binomial formula implies that all powers of y from 0 to η+1 appear on the right. The model cannot be reduced to a quadratic for any $\eta > 1$. The first two terms in square brackets involve the powers 0 and 1 in y. The matrix of price functions on the powers of y from 2 through η+1 has rank at most equal to one.

However, η also can assume any real non–integer value in [1,∞) and preferences will remain well–behaved with appropriate choices of the functions {β,γ,δ}. In such cases, the last term in square brackets on the right-hand-side of (A.122) is analytic with a convergent Taylor series expansion over the set of positive values for $\gamma - y$. The vectors of price functions for all powers of y greater than one are all proportional and the matrix of price functions, even with an infinite number of columns, has rank at most three. Thus, if there is a finite number of terms with functional separability between the prices of interest and income, then we must have a polynomial in y. But a very large set of well–defined demand models exists beyond quadratic polynomials, and each element can be represented as an irreducible polynomial of higher order than a quadratic, and may even have an infinite number of income terms. ■

A.6 SEMIDEFINITE MATRICES

*Lemma 2. Let the n×n real-valued matrix **A** be symmetric and positive semidefinite. Then* $\forall\ \boldsymbol{x}\in\mathbb{R}^n:\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}>0$, *the matrix* $\boldsymbol{A}-(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{A}$ *is symmetric and positive semidefinite, with **x** contained in its null space.*

Proof: Since $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x} > 0$ by hypothesis, $\forall\ \boldsymbol{z} \in \mathbb{R}^n$, $\boldsymbol{z}^{\mathsf{T}}\left[\boldsymbol{A} - (\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\right]\boldsymbol{z} \geq 0$ if and only if $\boldsymbol{z}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x}) \geq (\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z})^2$. Let the matrix $\boldsymbol{Q}$ satisfy $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}$ and define $\boldsymbol{v} = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{z}$ and $\boldsymbol{w} = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{x}$. Then $\boldsymbol{z}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x}) \geq (\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z})^2$ if and only if $\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}(\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w}) \geq (\boldsymbol{v}^{\mathsf{T}}\boldsymbol{w})^2$. The latter is an n–dimensional statement of the Cauchy-Schwarz inequality, and this inequality continues to apply when some of the elements of $\boldsymbol{v}$ and or $\boldsymbol{w}$ vanish, which can occur if $\boldsymbol{A}$ has less than full rank. Finally, inspection verifies that

$$\left[\boldsymbol{A} - (\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\right]\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0},$$

so that $\boldsymbol{x}$ is contained in the null space of the matrix $\left[\boldsymbol{A} - (\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{A}\right]$. ■

Lemma 3: A necessary condition for the symmetric matrix $\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix}$ *to be positive semidefinite is* $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$*, where* $\boldsymbol{L}$ *is (upper) triangular, while a sufficient condition is* $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} + \gamma\gamma^{\mathsf{T}}$.

Proof: For necessity, note that if the complete matrix is positive semidefinite, then the upper left $n_q \times n_q$ submatix $\boldsymbol{B}$ must be as well. This implies the existence of a (possibly reduced rank) Choleski factorization $\boldsymbol{B}$ in the form $\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$. For sufficiency, note that

$$\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} + \gamma\gamma^{\mathsf{T}} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \gamma \\ \boldsymbol{0}^{\mathsf{T}} & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{L}^{\mathsf{T}} & \boldsymbol{0} \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix}.$$ ■