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by

Shannon Wang

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August 2024

Applications of counting and classifying states by symmetry in AdS/CFT

Dedicated to my parents, David Wang and Tina Su.

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Publications

1. Donald Marolf, **Shannon Wang**, Zhencheng Wang, Probing phase transitions of holographic entanglement entropy with fixed area states, Published in: JHEP 12 (2020) 084
2. David Berenstein, **Shannon Wang**, BPS coherent states and localization, Published in: JHEP 08 (2022), 164
3. Adolfo Holguin, **Shannon Wang**, Giant gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal $N = 4$ SYM, Published in: JHEP 10 (2022), 078
4. Adolfo Holguin, **Shannon Wang**, Zi-Yue Wang, Multi-matrix correlators and localization, Published in: JHEP 04 (2024), 030

Abstract

Applications of counting and classifying states by symmetry in AdS/CFT

by

Shannon Wang

In the last few years, there has been a surge of interest in studying the AdS/CFT duality through the generating functions for half and quarter BPS correlators in $N = 4$ SYM theory. We arrive at these functions through representation theory; while the required calculations usually involve combinatorics and are computationally tedious, an exact formula– and in the case of quarter BPS correlators, an exact formula for the gauge group $SU(2)$ and a prescription for arriving at a more generic formula for $N > 2$– can be found through the application of the Harish-Chandra-Itzykson-Zuber integral formula, which makes use of the localization theorem. This thesis focuses on developing the mathematical tools needed to probe AdS giant gravitons and their interpretations as BPS coherent states.

Contents

Chapter 1

Introduction

Over the last two decades, there has been a wealth of works exploring giant gravitons, particles that mediate gravity and appear in the framework of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]; in particular, research that interprets BPS coherent states as giant gravitons [19, 20, 21, 22, 23, 24, 25] has gained interest. These objects are represented as matrix integrals, which are laborious to compute; while there are combinatorial formulas that may shed light on their behavior, they are still unwieldy and often intractable. More recently, there has been a surge in effort to simplify these calculations using localization [26, 27, 28, 21, 22, 25], which is shorthand for solving representation theory problems with geometry [29, 30, 31]. This dissertation is devoted to the efforts to study the generating functions of BPS coherent states through the application of the Harish Chandra integral formula [32, 31], which yields a closed form solution through localization methods, to the characters of irreducible unitary representations of different gauge groups.

In this introduction, we will briefly review the basics of the AdS/CFT correspondence. We will start with an introduction to the concepts of conformal field theory before moving on to a speedy review of the holographic duality; finally, we will discuss the operator-state correspondence, which forms the bedrock of this dissertation. This sets the stage for introducing the mathematics that we will explore over the next three chapters and motivates the efforts to build mathematical tools for counting and classifying states by symmetry.

1.1 Conformal Field Theory

We begin with the basics of conformal field theories, which are simply quantum field theories with additional symmetries under conformal transformations. A conformal field theory is a quantum field theory with scaling symmetry and usually involves an extended symmetry group called the conformal group. The local operators in a CFT may be decomposed into irreps of this group; we denote these operators as $\mathcal{O}_i$ and label them with their weight Δ_i, or their scaling dimension. The scaling symmetry comes from conformal transformations, which occur through a change of coordinates $x^\mu \to \tilde{x}^\mu(x)$ that leaves the infinitesimal line elements invariant up to a local scale factor $\Omega(x)$ [33]:

$$\eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \Omega(x)^2 \eta_{\rho\sigma} dx^\rho d\xi^\sigma \tag{1.1}$$

Translations and rotations are examples of conformal transformations; invariant translations give rise to conserved currents. We consider the stress-energy tensor, which is the matrix of such conserved currents and arises from an infinitesimal change to the metric. Then for an arbitrary set of local operators $\{\mathcal{O}_j(x_j)\}$, where x_j denotes the position of the jth insertion, our stress-energy tensor operator can be defined as [33]:

$$\delta\left(\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_N(x_N)\rangle_g\right) = \int \sqrt{-g}(x)(x)\delta g_{\mu\nu}(x)\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_N(x_N)\rangle_g \tag{1.2}$$

Computing the variations to the metric arising from infinitesimal Weyl rescalings shows that the stress tensor must be traceless; this condition has led to the identification of many conformal field theories.

We now focus on the infinitesimal generators that yield the conformal isometries of flat space. They are given by [34]:

$$
\begin{aligned}
P_\mu &= \partial_\mu \\
L_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu
\end{aligned}
\tag{1.3}
$$

We may show that the plane with the origin removed is equivalent to a cylinder with a base that is a $d-1$ dimensional sphere and a height of $\tau = \log(r)$ [34], where r is as defined in the original metric of the plane (missing its origin) [34]:

$$
ds^2 = r^2 \left(\frac{dr^2}{r^2} + d\Omega^2 \right) \tag{1.4}
$$

Clearly the cylinder enjoys translation invariance under τ; the generator for τ translations is then [34]:

$$
D = \partial_\tau = x^\mu \partial_\mu \tag{1.5}
$$

Finally, if we perform a space inversion such that $\tau \to -\tau$ and define a new radial coordinate such that $\tilde{r} = 1/r$, we may write the generator for translations as K. We will see a similar set of generators appear in AdS physics later.

If we apply these generators to a correlator of two operators, $\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\rangle$, we arrive at the two point function [35]:

$$
\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\rangle = C_{\Delta_i,\Delta_j} \frac{\delta_{\Delta_i,\Delta_j}}{|r|^{\Delta_i+\Delta_j}}, \tag{1.6}
$$

where the matrix C_{ij} is the Zamolodchikov metric. We are mainly interested in the two-point functions for BPS operators in $\mathcal{N}=4$ supersymmetric Yang-Mills, which have chiral adjoint scalar fields X, Y, Z with respect to a $\mathcal{N}=1$ decomposition. They can be computed as [36]:

$$\langle \chi_R\left(\bar{Z}\right)(x)\chi_{R'}\left(Z\right)(0)\rangle = \frac{1}{C^n|x|^{2n}}\delta_{R,R'}f_R, \tag{1.7}$$

where R, R' are representations of $U(N)$, C^n is the normalization constant, χ_R, $\chi_{R'}$ are characters of the representations, and $f_R = \prod_{i,j\in\text{boxes}}(N+i-j)$ is the product of all the labels of the boxes of the Young diagram for R.

1.2 Anti-de Sitter Space

We may understand anti-de Sitter space, AdS_{d+1}, as a hyperboloid embedded in $\mathbb{R}^{2,d}$, whose coordinates are $y^{-1}, y^0, \cdots y^d$; y^{-1} and y^0 are timelike and the rest are spacelike. Then our constraint equation is [37]:

$$\left(y^{-1}\right)^2 + \left(y^0\right)^2 - \sum\left(y^i\right)^2 = 1 \tag{1.8}$$

Our induced metric is as follows [37]:

$$ds^2_{flat} = -\left(y^{-1}\right)^2 - \left(y^0\right)^2 + d\vec{y}\cdot d\vec{y} \tag{1.9}$$

In global coordinates, we have [37]:

$$ds^2 = -\cosh(\rho)^2dt^2 + d\rho^2 + \sinh(\rho)^2 d\Omega^2_{d-1} \tag{1.10}$$

The two coordinate systems are related by the following mapping [34]:

$$\begin{aligned} y^{-1} + iy^0 &= \cosh(\rho)\exp(it) \\ \vec{y} &= \sinh(\rho)\hat{n} = \vec{\xi}, \end{aligned} \tag{1.11}$$

where $\hat{n}$ is a unit vector of the embedding of Ω_{d-1} into $\mathbb{R}^d$ in a unit sphere. We may observe from the metric in global coordinates a time translation invarianace as well as rotational symmetries of a sphere; the set of equations above then gives the rotation group and a generator that commutes with the group. We can write down the following generators [34]:

$$\begin{aligned} R_{ij} &\equiv x^i\partial_j - x^j\partial_i \\ H &\equiv x^{-1}\partial_0 - x^0\partial_{-1} \\ \vec{K} &\equiv \xi\vec{\partial_y} - 2\vec{y}\partial_{\bar{\xi}} \\ \vec{P} &\equiv \bar{\xi}\vec{\partial_y} - 2\vec{y}\partial_{\xi} \end{aligned} \tag{1.12}$$

Once we have established our generators, we may start reproducing the representation theory we previously saw in the conformal field theory. We may do so by asserting that the time translation we observe from the AdS metric maps to the time translation as previously seen on the conformal field cylinder; the symmetry generators match. Then the quantum theories are equivalent and the energy in CFT should be equivalent to the energy in AdS [34]. If we attempt to solve the wave equation in AdS, we find that the primary state, or the solution, is an eigenstate of $i\partial_t$ with an eigenvalue of Δ is mapped to the definition of energy in the one-particle sector, which is to say that a representation of a conformal group with conformal dimension Δ can be mapped to a set of single particle states in AdS. This establishes a correspondence between local operators and single particle states, leading to what we call the operator state correspondence [34].

1.3 The Operator State Correspondence

We start with an operator insertion at the origin, which is a singularity that we can proceed to cut out by cutting a circle of radius r_0 around the origin. As described previously in our introduction to conformal field theory, we choose spherical coordinates and rescale the AdS metric so that the volume is constant; then our metric becomes [34]:

$$d\tilde{s}^2 \equiv r^2 \left[\frac{dr^2}{r^2} + d\Omega_{D-1}^2 \right] = e^{2\tau} \left[d\tau^2 + d\omega_{D-1}^2 \right] \tag{1.13}$$

The right hand side of the metric describes the product of a sphere and a line, which yields an infinite cylinder with a sphere as its base; in two dimensions, the sphere is a circle and thus the base is flat. Thus we arrive at a relationship between a punctured sphere and the cylinder $S^{D-1} \times \mathbb{R}$. We can relate cutting the path integral along a fixed r to cutting the path integral along a fixed τ on the cylinder. If we interpret τ as time, we see that we relate infinite τ in the past to $r = 0$; likewise, $\tau = 0$ relates to a point at an infinite distance. When we insert an operator at the origin, we are really moving it to the infinite past in τ. Our inserted operator changes quantum fields at the origin; this can be represented by a quantum state of the Lorentzian cylinder corresponding to this operator. This is the essence of the operator-state correspondence.

1.4 1/2 BPS States and Giant Gravitons

We start with the basis of the supersymmetry generators Q_A. The BPS bound is [38]:

$$\{Q_A, Q_B^\dagger\} = E\delta_{AB} - K_{AB}, \tag{1.14}$$

where E is the mass/energy operator and K_{AB} is a matrix representing the charges.

When some of the eigenvalues of K_{AB} equal E, the corresponding components on the right hand side of the equation vanish, and we have an annihilated state, which we call a BPS state. When half of the generators annihilate the state, the state is a half-BPS state. BPS primary operators are built from the traces of products of the scalar Z field.

Open strings end with Dirichlet boundary conditions on D-branes, which in turn emit closed strings. A pointlike graviton, then, is a vibrational mode of a closed string, and is thought to be an elementary particle carrying the gravitational field. We turn our attention to the giant graviton, which carries the same quantum numbers as a pointlike graviton, but rather than being emitted by a brane, it blows up into a spherical brane of increasing size as its angular momentum increases [39]. The Pfaffian can be considered half of a maximal giant graviton, which is identified as $\det(\Lambda)$, since the maximal giant graviton wraps around the non-trivial cycle twice.

The giant graviton in $AdS_5 \times S^5/\mathbb{Z}_2$, which is dual to $\mathcal{N} = 4$ SYM, is a half BPS D3 brane wrapped around $S^3 \in S^5$ and rotating transversely in S^5. The dual-giant graviton is a half BPS D3 brane wrapped around $S^3 \in AdS_5$ and rotating in a maximal circle around S^5 [2, 5]. Counting half BPS states, then, is equivalent to counting multiple graviton configurations.

1.5 Outline and Summary

In Chapter 2, we lay out the foundation of this dissertation. We start by introducing coherent states averaged over a gauge group action– the unitary group in this chapter– to study the correlators of half BPS states in $N = 4$ SYM theory. The resulting overlaps are a generating function of correlators; if we rewrite them in terms of the Harish-Chandra-Itzykzon-Zuber (HCIZ) integral, we find that this formula allows us to compute the normalization of the two point functions with the characters obtained in [36]. We repeat these calculations and seek to extend these generalizations for A_{n-1} quivers following from solvable integrals over unitary groups. All of these computations make use of localization methods.

We will show that when we promote the parameters of this generative function to collective coordinates, the coherent states' effective action is controlled by a dominant saddle in the regime where the states describe single AdS giant gravitons, allowing us to probe their physics. We then discuss methods to add open strings to these gravitons and show that the resulting calculations rely on correlators of the matrix components of the unitaries in the ensemble given by the HCIZ integral. Fnally, we will discuss how Grassmann integrals give rise to sphere giants, how we may achieve dominant saddles, how these saddles allow for a $1/N$ expansion that arises for open strings, and how to add these open strings. We seek to generalize this coherent state formulation to 1/4 and 1/8 BPS states by framing them as more general integrals over unitary groups.

In Chapter 3, we seek to extend the analysis performed in Chapter 2 to $Sp(2N)$, $SO(2N+1)$, and $SO(2N)$ gauge theories. We repeat the calculations and arrive at generating functions for half BPS correlators in $N = 4$ SYM theories with gauge groups $Sp(2N)$, $SO(2N + 1)$, and $SO(2N)$ by computing the norms of the corresponding BPS coherent states, which are built from operators involving Harish-Chandra integrals. These operators may be interpreted as localized giant gravitons in the bulk of anti-de-Sitter space.

We then choose our basis for the sector of states with no cross-caps to be ordinary Schur functions; we may do this, because we are operating in theories that may be constructed as orientifold projections of a $SU(2N)$ theory. This allows us to observe the relations between the symmetric functions appearing in the expansion of our BPS coherent states and symplectic Schur functions, thus allowing us to perform our calculations using unitary characters, despite working with symplectic and orthogonal gauge groups.

Finally, we make note of some connections to Schubert calculus and Gromov-Witten invariants and discuss how the Harish-Chandra integral may be extended to such problems.

In Chapter 4, we return to the topic in Chapter 2 and seek to extend our initial computation over the unitary gauge group to quarter BPS states. We attempt to generalize the HCIZ integral by adding multiple commuting matrices; this allows us to probe the generating functions of 1/4-BPS states in $N = 4$ super Yang-Mills at finite N. The generalized result allows us to compute the overlaps of two or more generating functions; we are interested in such overlaps because they describe two-point correlators in the free-field limit.

We start with a discussion of the four-matrix HCIZ integral in the $U(2)$ context. We find a non-compact formula– through laborious means– for $N = 3$ and attempt to lay out a prescription for solving a multi-matrix HCIZ integral for $N > 2$. We find that much of the complexity arises from the connections to the restricted Schur polynomial operator basis. Since we show that we may readily generalize our results to an arbitrary number of matrices, we note that should we find a way to solve this integral– which is equivalent to solving a Laplacian of commuting matrices– we may develop new tools to study more generic BPS operators.

1.6 Permissions and Attributions

1. The content of chapter 1 is partially based on the results of collaborations with David Berenstein, Adolfo Holguin, and Zi-Yue Wang, and has previously appeared in the Journal of High Energy Physics [21, 22, 25]. It is reproduced here with the permission of the International School of Advanced Studies: `https://jhep.sissa.it/jhep/help/JHEP/CR_OA.pdf`.

2. The content of chapter 2 is the result of a collaboration with David Berenstein, and

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Chapter 2

BPS Coherent States and Localization

2.1 Introduction

There is a classic combinatorial result for two point functions of gauge invariant half BPS operators in $\mathcal{N}=4$ SYM [36]. Let X, Y, Z be the chiral adjoint scalar field of $\mathcal{N}=4$ SYM with respect to an $\mathcal{N}=1$ decomposition. Let R, R' be two different representations of $U(N)$ characterized by Young diagrams with n boxes and let χ_R be the character of $U(N)$ in the corresponding representation. Then the following is true:

$$\langle \chi_R(\bar{Z})(x)\chi_{R'}(Z)(0)\rangle = \frac{1}{C^n|x|^{2n}}\delta_{R,R'}f_R, \tag{2.1}$$

where C is a normalization constant that depends on conventions. The quantity f_R is a product over the labels of the boxes of the Young diagram associated to R, and is defined as:

$$f_R = \prod_{i,j\in \text{boxes}} (N+i-j), \tag{2.2}$$

where i moves to the right along the rows, j moves vertically downward along the columns, and both indices start at $(i,j)=(1,1)$ in the leftmost upper corner.

Various arguments suggest that this result is not renormalized [40, 41] (see also [42] and references therein). These combinatorial calculations have been extended to other (free field) quiver setups in the works [43, 44, 45] (the results are written succinctly in [46] in terms of generalized free fermions).

Part of the importance of the characters, apart from their orthogonality, is that they can also be interpreted geometrically in terms of D-branes, particularly, giant gravitons [47]. Giant gravitons expanding along the sphere directions (sphere giants for us) arise from column representations [3], and giant gravitons expanding in AdS [48, 49] (AdS giants for us) arise from large row representations [36]. These have served as a starting point to compute the anomalous dimensions of D-branes and the open strings ending on them. There is a combinatorial formalism developed in the works [6, 50, 51, 52] to add open strings. The main issue with these approaches is that they are computationally very difficult to master; the required combinatoric calculus is laborious. We can ask if there is another way to arrive at these results that might lessen the burden of computations and provide additional intuition to the dynamics of these setups.

When one works in less supersymmetric situations, such as with 1/4 or 1/8 BPS states, there are generalized orthogonal bases at zero coupling called restricted Schur bases (see [53, 54] and references therein). However, as soon as one turns on the coupling constant of $\mathcal{N} = 4$ SYM, one expects that the dynamics (at least semiclassically) reduce to some type of model of commuting matrices [14]. Such commuting matrix models are an ad-hoc uncontrolled approximation of the dynamics; they can be used to mimic the droplet picture of half BPS states in terms of free fermions in two dimensions [12] and extend the picture to more complicated setups with less supersymmetry where there is eigenvalue repulsion, but no fermions. The two dimensional droplet picture can also be seen directly in supergravity solutions [55]. Is there a systematic way to do calculations with these more general states that has less supersymmetry and embodies the spirit of commuting matrices, but is actually a complete field theory calculation that can be done ab initio?

In this chapter, we will see that the answers to both of these questions is yes– we can lessen the burden of computations for half BPS states (with strings attached) and find an exact *commuting matrix* model that captures 1/4 and 1/8 BPS states. The technique we introduce will reproduce all the results in equation (2.1) from a generating function. Similarly, we will discover a generating function that captures all 1/4 and bosonic 1/8 BPS states that survive at one loop. In this second problem, the basis we find is implicit, rather than the explicit character basis described above.

Let us write the basic idea. The first step is to realize that when studying local operator insertions in the $\mathcal{N} = 4$ conformal field theory, one can equally well describe the states on the cylinder $S^3 \times \mathbb{R}$ for a real quantum system, rather than the Euclidean formulation. That is, one uses radial quantization to turn the problem into quantum mechanics. Following [12], one replaces the matrix scalar operator $Z(0) \leftrightarrow a_z^\dagger$ with the raising operator of the s-wave of the field Z on the sphere including the matrix indices. Here, these indices are implicit. The free field correlators of Z appearing in (2.1) can equally well be described by overlaps of states in the Fock space of states of the $a_Z^\dagger$ that are gauge invariant. At this stage, we have only applied the operator state correspondence in the conformal field theory.

The next step is to think of the dynamics of the oscillators semiclassically by introducing coherent states. We start with the following object:

$$|\Lambda\rangle \sim \exp(\mathrm{tr}(\Lambda a^\dagger))\,|0\rangle\,, \tag{2.3}$$

where Λ is an $N \times N$ matrix of parameters. The trace indicates that we have a general linear combination of all possible raising operators.

Coherent states have the property that they are overcomplete. They have minimal uncertainty; they behave classically, but they are also eigenstates of the lowering operators. A formalism that can deal with these states is in principle able to deal with all the information of the Hilbert space, because of overcompleteness. The obvious problem with the object introduced in (2.3) is that it is not gauge invariant. We solve this issue by projecting the answer onto gauge invariant states, which we achieve by introducing an averaging over the gauge group. That is, we correct our naïve coherent state by the following:

$$|\Lambda\rangle = \frac{1}{\int dU} \int dU \exp(\mathrm{tr}(U\Lambda U^{-1} a^\dagger)) |0\rangle . \tag{2.4}$$

Because we projected an overcomplete basis to the set of gauge invariant states, we have an overcomplete basis of the gauge invariant states. One can check that the state defined this way is still a coherent state as far as gauge invariant combinations of lowering operators are concerned:

$$\mathrm{tr}(a_Z^n) |\Lambda\rangle = \mathrm{tr}(\Lambda^n) |\Lambda\rangle \tag{2.5}$$

These expressions only depend on the eigenvalues of Λ. At this stage, we can think of Λ as a diagonal matrix without loss of generality.

The matrix integrals that appear are well known. If $a^\dagger$ were a c-number matrix rather than a set of operators, then these would be the integrals of Harish-Chandra-Itzykson-Zuber [32, 56] (we will call this intgeral the HCIZ integral in this chapter). Such integrals can be computed in a variety of ways. We refer the reader to the review paper [57] (and also [58]) for a list of methods and references. We will liberally make use of the collected results in that paper. An important observation is that the HCIZ integral can be computed by localization [29]. The overlaps

$$I(\Lambda', \Lambda) = \langle\Lambda'|\Lambda\rangle \tag{2.6}$$

can be computed exactly with the HCIZ integral. Upon writing equation (2.4) in a character expansion, we can recover all of the overlaps in (2.1) by comparing it to the character expansion of the overlap integral itself (2.6).

We are repackaging a lot of non-trivial combinatorial information in the manipulations of the coherent state object itself. The fact that the final result is an exact sum over saddles makes it possible to understand approximations to calculations that are not apparent in the combinatorial expressions that were performed to arrive at (2.1). This idea extends to insertions of open strings, which we will describe in this formalism as well. The idea is to understand which saddle dominates and in what regimes. Once we have the coherent states, we can promote the Λ eigenvalue parameters to collective coordinates and find a coherent state effective action for the parameters Λ that describe the dynamics we are interested in. We extend this idea to A_{n-1} quiver theories and to states that preserve less supersymmetry. The new idea is that for 1/4 and 1/8 BPS states, we need to introduce more than one matrix $\Lambda_{X,Y,Z}$. When we insist on the 1-loop anomalous dimension of these states vanishing, we find that the three matrices must commute and be able to be diagonalized simultaneously. We thus find a generalization of the HCIZ integral that satisfies some of the conditions for evaluation by localization and embodies the commuting matrix model reduction to eigenvalues. The point is that the matrices that commute are not the original fields. They are the collective coordinate parameters of the states in question.

The idea of localization in $\mathcal{N} = 4$ SYM is important for many other observables. In particular, Wilson loop correlators reduce to matrix model computations [59, 60]. These are exact results, which arise from a localization argument [26]; all the important computations are done with free fields. For the results leading to equation (2.1), this also holds true: the computation arises from free fields. Thus in this chapter, we are seeing a new application of the Harish-Chandra-Itzykson-Zuber integral; the fact that it can be described by localization arguments becomes important as we try to find approximations to the physics by looking at the dominant saddle. A general review of localization methods can be found in [28].

The chapter is organized as follows. In section 2.2 we start with a model of a single harmonic oscillator and a gauged pair of harmonic oscillators to establish the method we will use later. The goal is to show that the denominators in a generating function of coherent states encode the information of the norms of states that are defined algebraically from the vacuum. Then in section 2.3, we introduce the main types of objects we study in this chapter: coherent states in matrix models averaged over a gauge orbit. We show how to compute overlaps of these states in terms of the Harish-Chandra-Itzykson-Zuber integrals and study various generalizations of these ideas to simply laced quivers. We show that this method reproduces many results that are known in the literature. We also discuss the fact that in the integral representation, one gets exact sums over saddles. This becomes important later on when we discuss approximations of the dominant saddle and other extensions of these ideas.

In section 2.4 we study generating functions made by determinants rather than coherent states. These are related to sphere giant gravitons in $AdS_5 \times S^5$. The point is to show that these objects admit an integral representation with a dominant saddle. The idea is to introduce fermions so that the determinant arises from Grassman integrals. Overlaps can be computed with the help of the Hubbard-Stratonovich trick and the fermions can be eliminated completely in terms of a pair of complex auxiliary variables. The integral over these variables reproduces many results. We show how these fermionic variables allow us to introduce open strings attached to the giant gravitons and demonstrate how this formalism simplifies other approaches in the literature. In section 2.5 we show how the fact that there are dominant saddles in the integrals allow one to not only promote the parameters of the gauge invariant coherent states to collective coordinates, but also calculate the effective action for them. In the HCIZ formula, these parameters are associated to multiple AdS giant gravitons. We explain how open strings are added to these configurations as well. We then turn to the problem of studying multiple sphere giant gravitons and argue that the correct multi-giant generalization involves products of determinants. This uses additional information involving character formulas and the Cauchy identity. We also explain how one has a Hilbert space of strings attached to multiple giants and explain the origin of the Gauss' law constraint. In section 2.6 we extend the idea of averaged coherent states to 1/4 and 1/8 bosonic BPS states. These require matrix parameters that commute with each other as is expected from the moduli space of vacua of these theories. We show that the saddles for half BPS states survive and focus on the dominant saddle for a single large eigenvalue and explain some of the differences that appear in the collective coordinate representation of these states. Finally, we close with a brief discussion of our results and present a possible extension of the ideas we discuss here in 2.7.

2.2 Warmup: the harmonic oscillator and the gauged double harmonic oscillator

Let us start with the simplest problem of a single harmonic oscillator. This example is intended to showcase the method we will use later in more complex settings. The idea is to consider a harmonic oscillator in the Hamiltonian formalism, described by a Weyl algebra constructed with a raising and a lowering operator, $[a, a^\dagger] = 1$. The ground state is described by $a|0\rangle = 0$. Consider now the following generating function of states:

$$F[z] = |z\rangle = \exp(za^\dagger)|0\rangle . \tag{2.7}$$

Because $(a^\dagger)^k |0\rangle$ is a complete basis of states, in principle $F(Z)$ contains (all of the) information about the full Hilbert space of states.

For the time being, the variable z is a formal parameter. If we call the non-normalized state $|n\rangle = (a^\dagger)^n |0\rangle$, we can ask how to compute its norm from $F[z]$ and indeed, the overlaps $\langle m|n\rangle$ for all m, n. Once we decide that $F[z]$ is well defined, we can think of it like a state $|z\rangle$ where z is an actual complex variable and not just a formal parameter. The idea is to compute the overlap:

$$\langle \xi|z\rangle = \bar{F}[\bar{\xi}] * F[z] = \langle 0| \exp(\bar{\xi} a) \exp(z a^\dagger) |0\rangle \,, \tag{2.8}$$

where $\bar{F}$ is the adjoint of the generating function and $\bar{\xi}$ is another formal parameter. We should notice that in the bra-ket notation, $(|\xi\rangle)^\dagger = \langle \xi|$ and includes an implicit complex conjugation. We make this explicit in $\bar{F}$ and implicit in $\langle \xi|$. Hopefully, this will not lead to confusion.

There are two ways to do the calculation. First, we can expand the double series to obtain:

$$\langle \xi|z\rangle = \sum_{m,n=0}^{\infty} \frac{\bar{\xi}^m z^n}{m!n!} \langle m|n\rangle \,. \tag{2.9}$$

The other way to do the calculation is to contract the $a^\dagger$ and the a (using the Baker-Campbell-Hausdorff formula) to obtain:

$$\langle \xi|z\rangle = \exp(\bar{\xi} z) = \sum_{n=0}^{\infty} \frac{\bar{\xi}^n z^n}{n!}. \tag{2.10}$$

Comparing the two formulas, we find that the coefficient of $\bar{\xi}^m z^n$ for $n \neq m$ vanishes, which is to say that the states $|m\rangle$ and $|n\rangle$ are orthogonal to each other if $n \neq m$. We also find, comparing the coefficients of $\bar{\xi}^n z^n$, that:

$$\frac{\langle n|n\rangle}{n!n!} = \frac{1}{n!}, \tag{2.11}$$

so that $\langle n|n\rangle = n!$. This can be proved immediately using the raising/lowering operator algebra. The point is that $n!$ is the denominator of the terms of $\bar{\xi}^n z^n$ in the exponential function.

Now, because the exponential function has an infinite radius of convergence, the overlaps are well defined for any value of the complex variables $\bar{\xi}, z$. In particular, the norm:

$$\langle z|z\rangle = \exp(\bar{z}z) > 0, \tag{2.12}$$

is positive definite if $\bar{z}$ is the complex conjugate of z and defines an L^2 normalizable state in the Hilbert space.

Coherent states also satisfy $a\,|z\rangle = z\,|z\rangle \simeq \partial_{a^\dagger} F[z]$, so it is easy to evaluate matrix elements of $(a^\dagger)^k a^m$ from the generating function, giving us:

$$\bar{F}[\bar{\xi}] * (a^\dagger)^k a^m * F[z] = z^m \bar{\xi}^k \langle \bar{\xi}|z\rangle\,, \tag{2.13}$$

which lets us identify operationally $a^\dagger \sim \partial_z$ and $a \sim \partial_{\bar{\xi}}$ as far as normal ordered computations go. The point is that the generating function is not only a generating function of states, but can also be used to compute matrix elements by comparing the double expansion (2.9) with the evaluation formula similar to (2.10).

We now go to our second example, where we have two oscillator algebras with raising operators $a^\dagger, b^\dagger$, and consider the symmetry generator $\hat{Q} = a^\dagger a - b^\dagger b$. We want to build a generating function as above, using gauge invariant states where $Q = 0$. A naïve guess is to do the following:

$$\exp(zb^\dagger a^\dagger)\left|0\right\rangle . \tag{2.14}$$

This turns out not to be optimal: the Baker-Campbell-Hausdorff trick doesn't yield a simple answer. Another option is to use a simple coherent state:

$$\left|\alpha,\beta\right\rangle = \exp(\alpha a^\dagger + \beta b^\dagger)\left|0\right\rangle = \sum \frac{(\alpha^m \beta^n)}{m!n!}\left|m\right\rangle \otimes \left|n\right\rangle , \tag{2.15}$$

but we notice that the generating function also contains non gauge invariant states. We need to project them onto the $n = m$ subset. Because we start with full coherent states, we have all of the information of the Hilbert space, including the states that are not gauge invariant. If we perform the correct projection, we should retain *all* the information that is gauge invariant in the generating function.

This can be done if we notice that the formal parameters α, β can be made to transform under a $U(1)$ symmetry that tracks the charges of $a^\dagger$ and $b^\dagger$. That is, we take $\alpha \to \exp(i\theta)\alpha$ and $\beta \to \exp(-i\theta)\beta$. Then we find:

$$\left|\alpha,\beta,\theta\right\rangle = \exp(\alpha e^{i\theta} a^\dagger + \beta e^{-i\theta} b^\dagger)\left|0\right\rangle = \sum \frac{(\alpha^m \beta^n)}{m!n!}\exp(i(m-n)\theta)\left|m\right\rangle \otimes \left|n\right\rangle , \tag{2.16}$$

and if we seek to only obtain the states with $n = m$, we can average over θ. That is, we consider a generating function of the form:

$$F[\alpha,\beta] = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(\alpha e^{i\theta} a^\dagger + \beta e^{-i\theta} b^\dagger)\left|0\right\rangle , \tag{2.17}$$

where we average over the group action on the operators. This formal functional is given by:

$$F[\alpha,\beta] = \frac{1}{2\pi}\int d\theta \exp(i\hat{Q}\theta)\exp(\alpha a^\dagger + \beta b^\dagger)\left|0\right\rangle , \tag{2.18}$$

where $\hat{Q}$ is the charge operator defined previously.

This is almost a coherent state, except for the group projection. It is straightforward to compute the overlap:

$$\bar{F}[\bar{\alpha}, \bar{\beta}] * F[\alpha, \beta] = \frac{1}{(2\pi)^2} \iint_0^{2\pi} d\tilde{\theta} d\theta \exp\left[\bar{\alpha}\alpha \exp(i(\theta - \tilde{\theta})) + \bar{\beta}\beta \exp(i(\tilde{\theta} - \theta))\right] \quad (2.19)$$

We can now shift variables to $\theta' = \theta - \tilde{\theta}, \tilde{\theta}$, so that one group integral becomes trivial, leaving the other to be explicitly evaluated. We find that:

$$\bar{F}[\bar{\alpha}, \bar{\beta}] * F[\alpha, \beta] = \sum_{n=0}^{\infty} \frac{(\bar{\alpha}\alpha\bar{\beta}\beta)^n}{n!n!} = I_0\left(2\sqrt{\bar{\alpha}\alpha\bar{\beta}\beta}\right) \quad (2.20)$$

which only depends on the gauge invariant combination of parameters $\alpha\beta$ and $\bar{\alpha}\bar{\beta}$. It can also be written explicitly in terms of a Bessel function. At this stage we can set $\alpha = \beta$ if we want to, as they do not have an independent meaning any longer. We also find through comparing coefficients that:

$$(\langle n| \otimes \langle n|) (|m\rangle \otimes |m\rangle) = (n!)^2 \delta_{n,m} \quad (2.21)$$

where again, the norm of the state is the denominator in the (integrated) generating function.

We can check that this is an eigenstate of the gauge invariant composite ab operator, finding that

$$abF[\alpha, \beta] = \alpha\beta F[\alpha, \beta]. \quad (2.22)$$

It is this property that makes these states more convenient: they act as coherent states for the composite gauge invariant operators built from lowering operators. This property can be readily used to compute matrix elements. In this example, the algebra is fairly straightforward, so the calculations can be done without the generating functions.

We can do one more variation on this calculation. The idea is to use the charge $Q = a^\dagger a - b^\dagger b - k$ where k is an integer. In this case, the state $|0\rangle \times |0\rangle$ has charge $-k$ and is not gauge invariant. The gauge invariant states are $|k+n\rangle \otimes |n\rangle$. In the double sum of the coherent state:

$$|\alpha, \beta, \theta\rangle = \exp(\alpha e^{i\theta} a^\dagger + \beta e^{-i\theta} b^\dagger)\,|0\rangle = \sum \frac{(\alpha^m \beta^n)}{m!n!} \exp(i(m-n)\theta)\,|m\rangle \otimes |n\rangle\,, \tag{2.23}$$

we need to project onto states where $m - n = k$. This is straightforward. We use the Fourier transform coefficients of the generating function:

$$\bar{F}[\alpha, \beta]_k = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(-ik\theta) \exp(\alpha e^{i\theta} a^\dagger + \beta e^{-i\theta} b^\dagger)\,|0\rangle \tag{2.24}$$

so that the overlap integral is

$$\begin{aligned} F[\bar{\alpha}, \bar{\beta}]_k * F[\alpha, \beta]_k &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(-ik\theta) \exp(\bar{\alpha}\alpha e^{i\theta} + \bar{\beta}\beta e^{-i\theta}) && (2.25) \\ &= \sum_{n=0}^{\infty} \frac{(\bar{\alpha}\alpha)^{n+k} (\bar{\beta}\beta)^n}{(n+k)!n!} && (2.26) \\ &= (\bar{\alpha}\alpha)^k \left(\sqrt{\bar{\alpha}\alpha\bar{\beta}\beta}\right)^{-k/2} I_k\left(2\sqrt{\bar{\alpha}\alpha\bar{\beta}\beta}\right) && (2.27) \end{aligned}$$

Again, the norm of the fixed charge states is the denominator $(n+k)!n!$ in the sum, and the generating function can be explicitly written in terms of Bessel functions. It is a convergent power series for all α, β in the complex plane. The norm is well defined if α and $\bar{\alpha}$ transform oppositely, which they do if they are complex conjugates of each other. These are also coherent states in the sense of being an eigenvalue of the gauge invariant composite ab operator with eigenvalue $\alpha\beta$. Notice that in this case, α and β appear slightly differently in the overlap. We can think of this as an anomaly. We can also take states given by $\alpha^{-k}F[\alpha,\beta]_k$, which are still coherent states; in that case, the final answer only depends on the product $\alpha\beta$, so we can take them to be equal to each other if we want to.

2.3 Half BPS coherent states in $\mathcal{N}=4$ SYM and some generalizations

We now turn to the problem of finding coherent states for the half BPS states $\mathcal{N}=4$ SYM that are gauge invariant. These states are special in that they are created by a single matrix of raising operator $(a_Z^\dagger)^i_j$. Under the operator state correspondence, the matrix valued operator inserted at the origin is equivalent to the raising operators $Z(0)\leftrightarrow a_Z^\dagger$, where $a^\dagger$ is the raising operator for the s-wave of the field Z on S^3, when studying $\mathcal{N}=4$ SYM on the cylinder [12]. We first consider a naïve coherent state:

$$F[\Lambda]=\exp(\mathrm{tr}(\Lambda\cdot a_Z^\dagger))\,|0\rangle \tag{2.28}$$

with a matrix valued Λ set of parameters. This is a coherent state for the gauge invariant traces $\mathrm{tr}(a^k)$, so that

$$\mathrm{tr}(a^k)F[\Lambda]=\mathrm{tr}(\Lambda^k)F[\Lambda]. \tag{2.29}$$

Since these traces generate all the gauge invariant states from the vacuum, we notice that the only information that we get from Λ is contained in the traces of powers of Λ. This is equivalent to knowing only the eigenvalues of Λ. In that sense, most of the parameters are redundant. We take Λ to be diagonal in what follows. The next problem we have to deal with is that this is not a gauge invariant state. We now introduce the $U(N)$ group action on these states and average over the group. This will look as follows:

$$F[\Lambda] = \frac{1}{Vol(U(N))} \int dU \exp\left(\mathrm{Tr}\left(U\Lambda U^{-1} a_Z^\dagger\right)\right) |0\rangle\,, \tag{2.30}$$

where dU is the Haar measure. The volume of $Vol(U(N)) = \int dU$ and we will call it Vol for short. For fixed U, the integrand will have the same coherent state properties with respect to $\Lambda, \tilde{\Lambda}$ if $\Lambda \to \tilde{\Lambda} = U\Lambda U^{-1}$ (they are related by conjugation), so that $\tilde{\Lambda}$ and Λ have the same eigenvalues and traces. We can think either of the matrix Λ transforming with U at fixed eigenvalues, or the matrix operator $a_Z^\dagger$ transforming with U. In the first case, we can think of the combination $\mathrm{tr}(\Lambda \cdot a_Z^\dagger)$ as being gauge invariant if both Λ and $a^\dagger$ transform opposite to each other. Diagonalizing Λ is a gauge choice and we are summing over the gauge orbit. In the second case, we may think of this as transforming $a^\dagger$ and projecting onto the gauge invariant states at fixed Λ. Either way, we should think of this integral as generating all of the possible half-BPS states.

Right now, we define Λ as an external matrix such that when we act on F, the lowering operators act as $a_Z \sim U \mathrm{diag}\,(\lambda_1, ...\lambda_N)\, U^{-1} = U\Lambda U^{-1}$ at fixed U. The U disappear inside traces. We now wish to find the inner product of $\bar{F}[\bar{\Lambda}']$ and $F[\Lambda]$. Using Eqn. 2.30 and the Baker-Campbell-Hausdorff formula, we arrive at:

$$\begin{aligned}\bar{F}[\bar{\Lambda}] * F[\Lambda] &= \frac{1}{Vol^2} \int dU^* dU \, \langle 0| \exp\left(\mathrm{Tr}\left(U^* \bar{\Lambda}' U^{*-1} a_Z\right)\right) \exp\left(\mathrm{Tr}\left(U\Lambda U^{-1} a_Z^\dagger\right)\right) |0\rangle \\ &= \frac{1}{Vol^2} \int dU^* dU \exp\left(\mathrm{Tr}\left(U\Lambda U^{-1} U^* \bar{\Lambda}' U^{*-1}\right)\right)\end{aligned} \tag{2.31}$$

Here, there is an implicit convention for transposes in $\bar{F}$ for the contraction of the raising/lowering operators that lets us concatenate the matrices in the order shown. Any other way of doing the contraction will give a similar answer with U^* either transposed or inverted in the formulas. They are all equivalent under a change of variables in the Haar measure. Notice that the expression above depends only on the combination $U^{-1}U^*$ and its inverse. We can therefore call a new group variable $\tilde{U} = U^{-1}U^*$ and still keep U. Since the Haar measure is group invariant, at fixed U, we have $d\tilde{U} = dU^*$, which allows us to write $dUdU^* = dUd\tilde{U}$. The integral over U can then be done – it cancels one factor of the volume. The end result is that the integral simplifies into:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \frac{1}{Vol} \int d\tilde{U} \exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right) \tag{2.32}$$

This integral is of the same type as the original definition of the coherent state that gave rise to (2.30), but it is now also a complex analytic function of $\Lambda, \bar{\Lambda}$, instead of a formal state in the Hilbert space. This is a well known integral: the Harish-Chandra-Itzykzon-Zuber integral (HCIZ) whose value can be computed via localization [29]. This can not be directly done in the original generating function of states because the operator matrix $(a^\dagger)^i_j$ cannot be diagonalized.

The integral localizes to solutions of:

$$\mathrm{Tr}(\tilde{U}^{-1}[\delta U, \Lambda]\tilde{U}\bar{\Lambda}') = 0 \tag{2.33}$$

This is equivalent to:

$$[\Lambda, \tilde{U}\bar{\Lambda}'\tilde{U}^{-1}] = 0 \tag{2.34}$$

so that Λ and $\tilde{U}\Lambda'\tilde{U}^{-1}$ are diagonalized simultaneously. This means that the labels of diagonal components λ_i and λ'_i differ by a permutation σ. We can take the matrices to be diagonal and described by $\lambda_i, \lambda'_{\sigma(i)}$, which is to say that U is a permutation matrix. The correct space for matrices U is $U(N)/U(1)^N$, where the $U(1)^N$ can be taken as the matrices that commute with Λ automatically. The saddle value of the integrand is

$\vec{\lambda} \cdot \vec{\lambda}'_\sigma = \sum \lambda_i \lambda'_{\sigma(i)}$.

We know from [58] that we can expand our integrand from Eq. 2.30 through a character expansion, giving a formula of the type:

$$F[\Lambda] = \sum_R \frac{1}{f_R} \chi_R(\Lambda) \chi_R(a_Z^\dagger) \left|0\right\rangle , \tag{2.35}$$

where we have a denominator f_R that we will compute later. The denominator is found in equation (34) of [58], or the denominator in equation (2.11b) of [57] if we divide by the prefactor of the equation. We will recompute the answer by using the exact evaluation of the integral.

We can do the same with the explicit HCIZ integral:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \sum_R \frac{1}{f_R} \chi_R(\bar{\Lambda}) \chi_R(\Lambda) \tag{2.36}$$

Comparing coefficients of the characters of the matrices Λ to the double sum

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \sum_{R,R'} \frac{1}{f_R f_{R'}} \chi_{R'}(\bar{\Lambda}) \chi_R(\Lambda) \left\langle 0 \right| \chi_{R'}(a) \chi_R(a^\dagger) \left| 0 \right\rangle , \tag{2.37}$$

we arrive at $\langle 0|\chi_{R'}(a)\chi_R(a^\dagger)|0\rangle = 0$ if $R \neq R'$, and we also find:

$$\frac{\langle 0|\chi_R(a)\chi_R(a^\dagger)|0\rangle}{f_R^2} = \frac{1}{f_R} \tag{2.38}$$

That is, the characters are orthogonal to each other, and the norm of each of the characters is the denominator f_R. This should be contrasted with the explicit combinatorial derivation in [36]. The reader can check that the answer quoted above and the result of the combinatorial formula match each other.

Now, we will compute f_R directly. This is something that can be done directly from the evaluation of the HCIZ integral. The first step is to understand that the representations appearing in the equation are labeled by Young diagrams for $U(N)$. Each diagram is characterized by the length of the rows, which appear in descending order $j_1 \geq j_2 \geq \dots j_N$.

We need the explicit Weyl character formula:

$$\chi_{j_i}(\Lambda) = \frac{\det\left(\lambda_k^{j_i+N-i}\right)}{\Delta(\Lambda)}, \tag{2.39}$$

which is written as a ratio of determinants, where $\Delta(\Lambda)$ is the Vandermonde determinant of Λ (we can also obtain it from the numerator by setting $j_i = 0$ for all i).

The second item we need is the explicit value of the HCIZ integral:

$$I(\Lambda, \bar{\Lambda}) = \frac{1}{Vol}\int d\tilde{U} \exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right) = \Omega\frac{\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right)}{\Delta(\Lambda)\Delta(\bar{\Lambda}')}, \tag{2.40}$$

where Ω is a normalization constant. The determinant in the numerator is a sum over permutations, it is an explicit sum over all the $N!$ possible saddles that are solutions of equations (2.34).

We can find f_R by first multiplying the result by the product of the Vandermonde determinants. This way, we obtain:

$$I(\Lambda, \bar{\Lambda})\Delta(\Lambda)\Delta(\bar{\Lambda}') = \Omega\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right) = \sum_{\vec{æ}}\frac{1}{f_{\vec{æ}}}\det\left(\lambda_k^{j_i+N-i}\right)\det\left(\bar{\lambda}_k'^{j_i+N-i}\right), \tag{2.41}$$

where we are now labeling the representations R by the vector of values $\vec{æ}$ determining the Young diagram. We will consider on the right hand side the monomials of the type $\lambda_1^{r_1}\dots\lambda_N^{r_N}$ with $r_1 > r_2 > \dots > r_N$. These monomials are in one to one correspondence with the characters. This corresponds to the unique term in the numerator of the determinant that is the product of the diagonal entries:

$$\det\left(\lambda_k^{j_i+N-i}\right) \to \prod_i \lambda_i^{j_i+N-i} + \dots \tag{2.42}$$

Now, we expand the exponentials in the determinant $\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right)$ of the evaluated HCIZ integral and use the multilinearity of the determinant to arrive at:

$$\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right) = \sum_{[n]} \frac{1}{[n]!}\det((\lambda_i\bar{\lambda}'_j)^{n_i}) = \sum_{[n]} \frac{1}{[n]!}\det(\bar{\lambda}'^{n_i}_j)\prod_i \lambda_i^{n_i} \tag{2.43}$$

where $[n]$ is the multi-index $n_1, \dots, n_N$, while $[n]!$ is the product $\prod_j n_j!$. Restricting to the monomials with the correct descending order forces us to take $n_1 > n_2 \dots$ in the sum. We see that we get an explicit sum over characters if we set $n_i = j_i + N - i$, which also have this descending value property. We therefore find that the denominators can be readily computed:

$$f_{\vec{æ}} = [n]!\Omega^{-1} = \Omega^{-1}\prod_i (j_i + N - i)!, \tag{2.44}$$

Setting $f_{\vec{0}} = 1$ by using $\langle 0|0\rangle = 1$, we find $\Omega = \prod_{i=1}^N (N-i)!$.

This is the same answer that was obtained by direct combinatorial methods. Similar localization formulas exist for other groups [29]. What is less straightforward is the corresponding character expansion. This is explained cursorily in [57]. The goal would be to reproduce the combinatorial formulas in [61, 62] and check if the bases agree. We will not pursue this calculation in this chapter. Instead, we will look at other integrals for simply laced quiver theories (A_{n-1} quivers) to show how the method works in those cases as well.

Let us start with a gauge theory of $U(N_1)\times U(N_2)$ matrices (we start with $N_1 = N_2$) and consider a pair of bifundamental fields $a_{12}^\dagger, a_{21}^\dagger$ in the $(\bar{N}_1, N_2)$ and the $(N_2, \bar{N}_1)$

representations. We want to build the same type of coherent states as above. We start with:

$$F[\Lambda] \sim \exp\left(\mathrm{Tr}(\Lambda_{21}\cdot a^\dagger_{12} + \Lambda_{12}\cdot a^\dagger_{21})\right)|0\rangle \tag{2.45}$$

The idea is that Λ_{ij} is in the dual space of $a^\dagger_{ji}$, so we reverse the order of the i, j labels. In this sense, the lowering operators are also reversed $a_{ji} = (a^\dagger_{ij})^\dagger$. Just like before, we need to average over the group $U(N_1)\times U(N_2)$. This is done by the following procedure:

$$F[\Lambda] = \frac{1}{\prod_i Vol_i}\int\prod_{i=1}^{2} dU_i \exp\left(\mathrm{Tr}(\Lambda_{21}U_1 a^\dagger_{12}U_2^{-1} + \Lambda_{12}U_2 a^\dagger_{21}U_1^{-1}\right)|0\rangle \tag{2.46}$$

where all the contractions are matrix multiplications.

With the usual use of the Baker-Campbell-Hausdorff formula, we arrive at a formula where we end up replacing $a^\dagger_{12}$ by the $\bar{\Lambda}'_{12}$ matrix. We obtain:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \frac{1}{\prod_i Vol_i}\int\prod_{i=1}^{2} dU_i \exp\left(\mathrm{Tr}(\Lambda_{21}U_1\bar{\Lambda}'_{12}U_2^{-1} + \Lambda_{12}U_2\bar{\Lambda}'_{21}U_1^{-1})\right) \tag{2.47}$$

These are well known generalizations of the HCIZ integral, first solved in [63]. We now assume, without loss of generality, that the matrices $\Lambda, \bar{\Lambda}'$ are diagonal. The eigenvalues are:

$$\Lambda_{12} \equiv \mathrm{diag}(\lambda^1_{(12)}\dots\lambda^{\min(N_1,N_2)}_{(12)}) \tag{2.48}$$

$$\bar{\Lambda}_{12} \equiv \mathrm{diag}(\bar{\lambda}^1_{(12)}\dots\bar{\lambda}^{\min(N_1,N_2)}_{(12)}) \tag{2.49}$$

Here, the (ij) label of the matrix is in the lower component, and the upper components label the eigenvalues. If $N_1 \neq N_2$, the matrix has diagonal entries to the extent that it is permitted, and the off-diagonal elements vanish. Let $A^\alpha = \prod_{ij}\lambda^\alpha_{(ij)}$ be the diagonal product of the Λ matrices. Similarly $\bar{A}^\beta = \prod_{ij}\lambda^\beta_{(ij)}$. The integral can also be done with localization methods. First, when $N_1 = N_2$, we need to be careful over what we mean by localization. The original localization formula for the HCIZ integral is evaluated on a compact complex manifold $U(N)/U(1)^N$. It is important that we do the same here. However, we are doing integrals over the full $U(N_i)$ and not just $U(N_i)/U(1)^{N_i}$. We need to separate the $U(1)^{N_i}$ explicitly. These are tori, so we have variables $\exp(i\theta_i^\alpha)$ for each group. We choose these to be multiplying U on the left. We define $U_1 = \mathrm{diag}(i\theta_1^\alpha) * \tilde{U}_1$ where $U_1^* \in U(N_1)/U(1)^{N_1}$, which is a proper complex space on which localization can be had.

When we do so, we get a similar equation to (2.34); this time, however, we need to restrict ourselves to δU that are strictly off-diagonal (they need to be orthogonal to the tori that we selected). The result is the same: Λ and a conjugate of $\bar{\Lambda}$ are mutually diagonal for all Λ. This gives rise to a common permutation of the λ variables. The value of the "action" at the saddle is:

$$S = \sum_\alpha \lambda_{ij}^\alpha \bar{\lambda}_{ji}^{\sigma(\alpha)} \exp(i\theta_i^\alpha - i\theta_j^\alpha), \tag{2.50}$$

which still depends explicitly on the angles θ_i. The localization integral is over the off-diagonal pieces. We still need to do an integral over the phases θ_i^α. We may simplify the integral by absorbing the phases into the λ_i. The denominator is computed using the method of images, under the assumption that S is real. The result is:

$$\sqrt{\det(\delta_U^2 S)} = \Delta(A)\Delta(\bar{A}), \tag{2.51}$$

a product of Vandermonde determinants. The localization formula (for each saddle) we need is then given by the following integral:

$$I_{sad} = \frac{1}{\Delta(A)\Delta(\bar{A})} \int \prod d\theta \exp\left[\sum_{\alpha,i} \lambda_{ij}^{\alpha} \bar{\lambda}_{ji}^{\sigma(\alpha)} \exp(i\theta_i^{\alpha} - i\theta_j^{\alpha})\right] \tag{2.52}$$

The integral is done by expanding the exponential as a series and using the binomial expansion. We find:

$$\frac{1}{\Delta(A)\Delta(\bar{A})} \sum_{[n]} \frac{1}{[n]!^2} A^{[n]} \bar{A}_{\sigma}^{[n]} \tag{2.53}$$

This can be understood as N copies of the computation in (2.19), so we get a product of Bessel functions. Each saddle also has a sign, $(-1)^{\sigma}$. We find that up to a normalization constant Ω, we have:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \frac{\Omega}{\Delta(A)\Delta(\bar{A})} \det\left(J_0\left[2\sqrt{(A^{\alpha}\bar{A}^{\beta})}\right]\right) \tag{2.54}$$

Because of the determinant structure, it admits a character expansion. This is another way to arrive at the answer by a direct computation. Again, it is the denominator of the characters that count. We replace $[n]! \to ([n]!)^2$ in all the formulas in (2.43).

$$f_{\vec{æ}} = [n]!^2 \Omega^{-1} = \Omega^{-1} \prod_k (j_k + N - k)!^2, \tag{2.55}$$

including the normalization factor Ω, which is determined by $f_{\vec{0}} = 1$.

Once we go to more general A_{n-1} quivers, with all groups of the same rank, the integrals that need to be done are of the type (2.52), but with the sum over i containing more terms, as many as there are in the A_{n-1} quivers. The most general formulas obtained this way contain different ranks for the different groups:

$$f_{\vec{æ}} = \Omega^{-1} \prod_{i,k} (j_k + N_i - k)!, \tag{2.56}$$

again normalized to $f_{\vec{0}} = 0$ (see [57, 58]). The Bessel function gets replaced by a generalized hypergeometric series, given by:

$$\Phi(A\bar{A}) = \sum_m \frac{1}{\prod_i (m + N_i - N_0)!} (A^\alpha \bar{A}^{\sigma(\alpha)})^m \tag{2.57}$$

where $N_0 = \min(N_i)$ and the determinant that generalizes equation (2.54) is a determinant of a $N_0 \times N_0$ matrix, so that the overlap reads:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \frac{\Omega}{\Delta(A)\Delta(\bar{A})} \det\left(\Phi\left[(A^\alpha \bar{A}^\beta)\right]\right) \tag{2.58}$$

When the ranks of the groups are not the same, the localization integral actually does not work. Let us explain this in the simplest setup where we use a $U(N) \times U(1)$ quiver. The integral we need to do is:

$$I = \frac{1}{2\pi}\frac{1}{Vol} \int dU d\phi \exp\left(\bar{a} \cdot U \cdot \ a \exp(i\phi) + \exp(-i\phi) b U^{-1} \bar{b}\right), \tag{2.59}$$

where a is a column vector, and b is a row vector, given explicitly by:

$$\vec{a} = \begin{pmatrix} \tilde{a} \\ 0 \\ \vdots \end{pmatrix}, \quad \vec{b} = (\tilde{b}, 0 \ldots 0), \tag{2.60}$$

where the product $\vec{b} \cdot \vec{a} = \tilde{a}\tilde{b}$ is gauge invariant. Notice that the vectors $\vec{a}$ and $\vec{b}$ are invariant under a common $U(N-1)$ group. Thus, when we do the integral, we should do the integral over the quotient space:

$$S^{2N-1} \sim U(N)/U(N-1), \tag{2.61}$$

which is a round sphere of dimension $2N-1$. This is not a complex manifold, but the quotient $\mathbb{C}P^N = S^{2N-1}/U(1)$ is such a space. This complex geometry would be the one where localization would take place. Instead of that, let us choose the metric of the sphere as follows:

$$ds^2 = \cos^2(\theta)d\phi_1^2 + d\theta^2 + \sin^2(\theta)d\Omega^2_{2N-3} \tag{2.62}$$

The action has a pair of rotated vectors dotted into another such vector. This inner product has a $\cos(\theta)\exp(i\phi_1)$ factor in it. We find that:

$$I \sim \int \sin(\theta)^{2N-3}\cos(\theta)d\theta d\phi_{12}\exp\left(\bar{\tilde{a}}\tilde{a}\cos(\theta)\exp(i\phi_{12}) + \tilde{b}\cos(\theta)\bar{\tilde{b}}\exp(-i\phi_{12})\right) \tag{2.63}$$

where the angle $\theta \in [0, \pi/2]$ and ϕ_{12} is a relative angle. We can get the same type of answers if we use Euler angle parameterizations in $U(2)$ and $U(3)$ (see [64, 65] to get Euler angles for $SU(3)$ and $SU(N)$).

We can do the integral explicitly in two different orders. We can expand the exponential series and pick the Fourier terms that have vanishing momentum. We obtain:

$$I \sim \sum_n \int \sin(\theta)^{2N-3}\cos(\theta)d\theta(\bar{\tilde{a}}\tilde{a}\tilde{b}\bar{\tilde{b}})^n \frac{1}{(2n)!}\binom{2n}{n}\cos(\theta)^{2n} \tag{2.64}$$

The individual integrals can be written in terms of Γ functions. Up to a normalization factor, we get:

$$I \sim \sum_n (\bar{\tilde{a}}\tilde{a}\tilde{b}\bar{\tilde{b}})^n \frac{1}{n!n!}\frac{\Gamma[N-1]\Gamma[n+1]}{\Gamma[n+N]} = \sum_n (\bar{\tilde{a}}\tilde{a}\tilde{b}\bar{\tilde{b}})^n \frac{1}{n!}\frac{\Gamma[N-1]}{(n+N-1)!} \tag{2.65}$$

The denominator takes the form we expect: $f_n = (n+1-N_0)!(n+N-N_0)!$, where $N_0 = 1$. Everything is fixed if we require that the leading term of the series is equal to one. The second way to do the integral is to introduce a new variable $x = \cos(\theta)$ and let

$y = \bar{\tilde{a}}\tilde{a}\exp(i\phi_{12}) + \tilde{b}\bar{\tilde{b}}\exp(-i\phi_{12})$. Then the integral takes the form:

$$I \propto \int d\phi \int_0^1 dx(1-x^2)^{N-2} x \exp(xy) \tag{2.66}$$

The first few of the answers, for $N = 2, 3$ are:

$$I_2 = \int d\phi \left(\frac{1}{y^2} + \frac{e^y(y-1)}{y^2} \right) \tag{2.67}$$

$$I_3 = \int d\phi y^{-4}(-6 + y^2 + 2e^y(3 - 3y + y^2)) \tag{2.68}$$

and this suggests that there are two saddles: one at $\theta = 0$ and the other at $\theta = \pi/2$. The first saddle has action y, and the other has action 0. The measure factor from the saddle should be the maximum inverse power of y in the expression. But we notice that there are curious factors of y in the numerator. This is because the two endpoints of the θ integral correspond to manifolds of different dimensions. For $\theta = 0$ in (2.62) we get a circle parametrized by ϕ_1, whereas for $\theta = \pi/2$, we get a sphere of dimension $2N-3$. One of the two "critical points" in $S^{2N-3}/U(1)$, if we can call them like that, is not isolated and the other one is. In that sense, a naive notion of localization fails. The Duistermaat-Heckman theorem requires isolated critical points. In the case of $N = 2$, both of the critical points lead to circles shrinking to zero size, but these circles are not the same circles. The theorem of localization only pertains to the fixed points under the same $U(1)$ action. In spite of this, because we can do the integral in the other order (where we integrate the angle variables ϕ first), we get an expression that is a quotient of determinants that admits a character expansion. That is enough to show the orthogonality of character wave functions and compute the norm in terms of a denominator that fits the description above. Since the expression looks like sums over saddles with denominators, we can abuse the language of localization if need be.

For conformal field theories in four dimensions, we usually find ourselves in cases where all the N_i are equal to each other, so the process of localization is valid.

2.4 Determinants and strings attached to them

Consider the following gauge invariant object:

$$G[\lambda] = \det(\lambda - a^\dagger) \left|0\right\rangle , \tag{2.69}$$

where λ is a c-number formal variable. This expression can be expanded in characters (more precisely, subdeterminants) as follows:

$$G[\lambda] = \sum_{n=0}^{N} (-\lambda^n) sdet_{N-n}(a^\dagger) \tag{2.70}$$

We can consider the overlap:

$$\bar{G}[\bar{\lambda}] * W[\lambda] \tag{2.71}$$

Our goal right now is to find an expression of this overlap that can be computed with saddle point methods, as a saddle of a specific integral. The idea is to write the determinant as a fermionic integral (we follow the setup [66, 67, 68, 69], see also [70]):

$$\det(\lambda - a^\dagger) = \iint d\bar{\xi} d\xi \exp(\bar{\xi}(\lambda - a^\dagger)\xi), \tag{2.72}$$

where the fermions $\xi, \bar{\xi}$ are column and row vectors of size N. The determinant is the result of a fermion integral over an auxiliary set of fermions that can be taken to transform under $U(N)$ as a fundamental or antifundamental. Notice that the term in the exponential is again linear in $a^\dagger$. What this means is that we may apply the Baker-

Campbell-Hausdorff trick again in the overlap computation. The overlap we need is of the form:

$$\bar{G}[\bar{\lambda}] * G[\lambda] = \iint d\bar{\chi} d\chi d\bar{\xi} d\xi \exp(\bar{\xi}\lambda\xi + \bar{\chi}\bar{\lambda}\chi - \bar{\chi}\xi\bar{\xi}\chi), \tag{2.73}$$

where the minus sign of the quartic term comes from a fermion sign.

As is standard, we use the Hubbard-Stratonovich trick by inserting a complex boson Gaussian integral to find:

$$\bar{G}[\bar{\lambda}] * G[\lambda] = \iint d\bar{\chi} d\chi d\bar{\xi} d\xi d\bar{\phi} d\phi \exp(\bar{\xi}\lambda\xi + \bar{\chi}\bar{\lambda}\chi - \bar{\phi}\phi + i\phi\bar{\xi}\chi + i\bar{\phi}\bar{\chi}\xi) \tag{2.74}$$

The fermion integral is now diagonal in the $U(N)$ indices, so we get that:

$$\begin{aligned} \bar{G}[\bar{\lambda}] * G[\lambda] &= \iint d\bar{\phi} d\phi \exp(-\bar{\phi}\phi) \det \begin{pmatrix} \lambda & i\phi \\ i\bar{\phi} & \bar{\lambda} \end{pmatrix}^N & (2.75) \\ &= \iint d\bar{\phi} d\phi (\bar{\lambda}\lambda + \bar{\phi}\phi)^N \exp(-\bar{\phi}\phi) & (2.76) \end{aligned}$$

We can now do the integral by expanding the polynomial in $\bar{\lambda}\lambda$ using the binomial theorem, finding that the integral over $\bar{\phi}\phi$ can be expressed in terms of of Γ functions:

$$\bar{G}[\bar{\lambda}] * G[\lambda] = \Omega \sum_{k=0}^{N} \binom{N}{k} (\bar{\lambda}\lambda)^k \Gamma[N-k+1] = \Omega N! \sum_{k=0}^{N} \frac{(\bar{\lambda}\lambda)^k}{k!} \tag{2.77}$$

up to a normalization constant of Ω, which has been implicit in the measure of the integrals. We need to normalize the answer so that the term with $(\bar{\lambda}\lambda)^N$ (the vacuum overlap) has a coefficient equal to one. This means that $\Omega = 1$. We find in a straightforward manner that the subdeterminants are orthogonal and that their norm is $|sdet_{N-K}|^2 = N!/k!$; we can compare this expression to the results in [3, 71].

The results presented here are very direct. We notice that because we have an integral expression, we can evaluate it using a saddle point approximation by varying over $r = \bar{\phi}\phi$. The saddle is the minimum of:

$$-\bar{\phi}\phi + N\log(\bar{\lambda}\lambda + \bar{\phi}\phi) \tag{2.78}$$

Equivalently, using $r \equiv \bar{\phi}\phi$:

$$-1 + \frac{N}{r + \bar{\lambda}\lambda} = 0, \tag{2.79}$$

we find that $r = N - \bar{\lambda}\lambda$. For the saddle point method to be a good approximation, we need the saddle to be close to the positive real axis, which is the line over which we integrate r. When $\bar{\lambda}$ and λ are complex conjugates of each other, this requires that:

$$\bar{\lambda}\lambda < N \tag{2.80}$$

In this setup, $\bar{\phi}\phi$ is of order N. Necessarily, so is λ, which we think of as a parameter.

We find:

$$\bar{G}[\bar{\lambda}] * G[\lambda] \sim \Omega' \exp(-r) \simeq N! \exp(\bar{\lambda}\lambda) \tag{2.81}$$

If we compare this expression to (2.77), we see that the exact answer is a truncated exponential, and that the saddle point approximation gives the exponential function. This fact was first seen in [71], but not as a saddle point with respect to the integral representation. The saddle makes it clear that we have a $1/N$ expansion, because of the specific N dependence of the logarithm. This is induced when we integrate out the fermions.

2.4.1 Adding open strings

The idea of the saddle point calculation is that in the end, $\bar{\lambda}$ and λ become complex variables. The generating function $G[\lambda]$ is to be considered as a state in the Hilbert space of states. We can build other states around this state. Consider a collection of words W_j made of raising operators different from $a^\dagger$ (like the ones that appear in spin chains of $\mathcal{N} = 4$ SYM). We can consider more general states that are of the form:

$$G[\lambda, W] = \int d\bar{\xi} d\xi \exp(\bar{\xi}(\lambda - a^\dagger)\xi) \prod_j (\bar{\xi} W_j \xi) |0\rangle . \tag{2.82}$$

These are open words W_j with fermion flavors attached at the boundaries. The boundary fermions on the words make these states gauge invariant as well: the $\bar{\xi}, \xi$ transform under $U(N)$. In this formalism, the introduction of the fermion variables suggests that there are emergent degrees of freedom in the determinant. When we write the determinant as an integral over the fermion variables, we “integrate out" these degrees of freedom when we do the integral. This has been studied in [70, 72].

Keeping the fermions in more places than just the determinant allows us to affix strings to the defect. Hence, the determinants behave like D-branes; indeed, they are supposed to be sphere giant gravitons. Without the explicit fermions, one would get a formalism similar to [71], which is more cumbersome. Attaching strings combinatorially for single determinant branes was pioneered in [8, 4, 9]. This formalism with fermions does the same work more economically and moreover has a well defined saddle, which allows one to make approximations useful for computations.

Let us now show, mimicking [71], that any such W should not begin or end in the letter $a^\dagger$, because we would be overcounting. Consider $W = W' a^\dagger$. The idea is to write $a^\dagger = \lambda + (a^\dagger - \lambda)$. The term with just λ is a c-number, so it can be taken out and written in terms of shorter words, in this case W'. There is a second term, which can be written as a fermion derivative as follows:

$$G[\lambda, W'(a^\dagger - \lambda)] = -\int d\bar{\xi} d\xi \bar{\xi} W' \partial_{\bar{\xi}} \exp(\bar{\xi}(\lambda - a^\dagger)\xi) |0\rangle \tag{2.83}$$

Now we integrate the fermion derivative by parts and find that:

$$G[\lambda, W'(a^\dagger - \lambda)] = -\int d\bar{\xi} d\xi \exp(\bar{\xi}(\lambda - a^\dagger)\xi) \mathrm{tr}(W') |0\rangle , \tag{2.84}$$

which is usually interpreted as the object G with a closed string $\mathrm{tr}(W')$. These manipulations are straightforward, whereas the original combinatorial calculation was more challenging. The original combinatorial setup with the λ acting as collective coordinates allows one to understand the boundary conditions for the closed spin chain in more detail [16, 15, 17] and is useful at higher loop orders [73, 74, 75], but it becomes prohibitive to understand how the various diagram contribute at various orders in $1/N$. The saddle approximation and the introduction of fermions help facilitate the latter goal. The fermion variables make it easy to generalize further beyond one string and should prove helpful to understanding how strings split and join more generally once non-planar interactions are added. These types of words with fermions can be generated by interactions; the Hamiltonian lowering operators can bring down powers of the ξ fermions. The integration by parts can also act on other insertions of $\bar{\xi}$, producing the splitting and joining of words. Computing overlaps of states with strings will involve fermion correlators (this is how computations are done in [69], for example). These are easy to compute at the saddle point obtained at (2.34), and involve the 2×2 inverse of the quadratic form appearing in the fermion integral. Namely, we have a Feynman rule for a fermion propagator that eliminates the fermion insertions in the strings. The fermion propagator is:

$$\pi = \begin{pmatrix} \lambda & i\phi \\ i\bar{\phi} & \bar{\lambda} \end{pmatrix}^{-1} \tag{2.85}$$

Exploring these issues in detail is beyond the scope of the present chapter.

2.5 Collective coordinates

So far, we have defined coherent states labeled by either eigenvalue parameters $\Lambda \sim (\lambda^1, \dots \lambda^N)$, or by a generating function made of a single parameter λ in the case of determinants. In all of these cases we have found that the overlaps of these states with different parameters can be well described by saddle points of an integral. When the integral is done by localization, the expansion in terms of a sum over a finite number of saddles is exact. When the states have the same parameters (that is, when we are computing the norm of a state), there is usually a single saddle that dominates, as we will describe. In such cases, the physics can become semiclassical. The parameters λ^i that describe the individual state can be promoted to collective coordinates, which will allow us to describe the dynamics of the state in terms of a simplified dynamics of the λ parameters as functions of time. We will demonstrate this process in this section.

So far, we have not discussed the Hamiltonian of the system. We have described raising and lowering operators in a harmonic oscillator context, but at no point did we make it explicit that these are solutions of the dynamics of a quantum system with a Hamiltonian. We want to understand how to do this directly from the λ parameters. Rather than solve the oscillator dynamics of $a^\dagger$ and port it over to the λ, we want to have an effective action for the λ itself that reproduces it. The reason for doing this is that eventually the dynamics of BPS states get corrected when we add other oscillators. In that sense, we get an effective action of collective coordinates and additional excitations, which interact with one another. These interactions lead to corrections of motion in λ, but the fact that the saddles are in some sense strong saddles means that these can still be treated semiclassically and the states will have big overlaps with the coherent states described so far.

The effective action of the collective coordinates on their own is usually written as a first order formulation as follows:

$$S = \int dt \, \langle \lambda | \, i\partial_t \, | \lambda \rangle - \langle \lambda | \hat{H} | \lambda \rangle \, , \tag{2.86}$$

where the first term is a Berry phase. The states $|\lambda\rangle$ are required to be normalized. Applying the variational principle to the action produces an approximation to the Schrödinger equation, restricted to the states of the prescribed form. Our idea is to use the saddle point expressions directly to compute the effective action S. This is very similar to what was done in [71] for a single sphere giant graviton.

The main idea behind computing $\hat{H} \, |\lambda\rangle$ is that the energy in the generating series is equal to the number of raising operators in the expansion of the exponential. This is identical to counting powers of Λ. In that sense, we take the un-normalized $|\vec{\lambda}\rangle$, which is strictly holomorphic, and find that:

$$\hat{H} \, |\lambda\rangle = \sum \lambda^i \partial_{\lambda_i} \, |\lambda\rangle \tag{2.87}$$

After this evaluation, we can rescale the state by multiplying by a c-number (the square root of the norm of the state). If we define the normalization constant as:

$$N(\lambda) = \langle \lambda | \lambda \rangle = \bar{F}[\bar{\Lambda}] * F[\Lambda], \tag{2.88}$$

where $\bar{\Lambda}$ is the complex conjugate matrix to the Λ diagonal matrix, we find after a straightforward computation that:

$$\langle\lambda|\,\hat{H}\,|\lambda\rangle = \sum \lambda^i \partial_{\lambda^i} \log(N(\lambda)) = \sum_i \lambda^i p_i, \tag{2.89}$$

where p_i are a new set of variables, given by derivatives of $K = \log(N(\lambda))$.

Similarly, the Berry phase term is given by:

$$\begin{aligned}
\lim_{\tilde{\lambda}\to\lambda} \frac{\langle\tilde{\lambda}|}{\sqrt{N(\tilde{\lambda})}} i\partial_t \frac{|\lambda\rangle}{\sqrt{N(\lambda)}} &= \lim_{\tilde{\lambda}\to\lambda} \langle\tilde{\lambda}|\lambda\rangle \frac{1}{\sqrt{N(\tilde{\lambda})}} i\partial_t \frac{1}{\sqrt{N(\lambda)}} + \frac{1}{\sqrt{N(\tilde{\lambda})N(\lambda)}} \langle\tilde{\lambda}|i\partial_t|\lambda\rangle \\
&= i\sum_i \dot{\lambda}^i \partial_i \log(N(\lambda)) + \text{total time derivative} \\
&\simeq \sum i p_i \dot{\lambda}^i,
\end{aligned} \tag{2.90}$$

where we drop the term that is a total derivative of $\log(N)$, since total derivatives do not contribute to the action. The action we get is surprisingly simple:

$$S = \int dt \sum i p_i \dot{\lambda}^i - p_i \lambda_i. \tag{2.91}$$

In this equation, we should think of p_i as the canonical conjugates of λ^i. Solving the equations of motion immediately gives $\lambda_i = \lambda_i(0)\exp(-it)$, which is the correct classical behavior for the coherent state fields. The λ^i are well defined for the HCIZ integral, but when we consider some of the generalizations, we realize that we should use the A^i variables rather than the λ directly. The A^i variables are products of Λ for a quiver; only the A eigenvalues enter N. We want the p_i variables to be independent variables, so we rewrite the action in terms of the A variables:

$$S = \int dt \sum i p_i \dot{A}^i - n p_i A^i, \tag{2.92}$$

where n is the number of nodes in the quiver diagram.

A more pressing question is evaluating p_i. We turn to (2.41), which allows us to compute p_i. We want to express the result in the saddle point approximation and check where it is valid. Consider a single large eigenvalue parameter of λ^1 (and make the others as small as needed). We want to know which saddles contribute. It is clear that the exponential in the saddle satisfies the following inequality:

$$|\exp(\bar{\lambda}^{\sigma(1)}\lambda^1 + \dots)| < |\exp(\bar{\lambda}^1\lambda^1 + \bar{\lambda}^2\lambda^2 + \dots)|, \tag{2.93}$$

which can be proved by the Cauchy-Schwearz inequality. We see that we require $\pi(1) = 1$. If we add more large eigenvalues, it becomes obvious that the dominant saddle is the one of the identity permutations by the same method. Keeping in mind that the other λ are small, our expression for the overlap is approximated by:

$$\log(N(\lambda^1, \bar{\lambda}^1)) \sim \lambda^1\bar{\lambda}^1 - (N-1)\log(\bar{\lambda}^1\lambda^1), \tag{2.94}$$

where we only take into account the λ^1 dependence and the dominant saddle. We find this way that:

$$p_1\lambda^1 = \bar{\lambda}^1\lambda^1 - (N-1) \tag{2.95}$$

We expect the energy of the configuration to be positive. This gives a lower bound on the collective coordinate λ, as follows:

$$\bar{\lambda}^1\lambda^1 > N, \tag{2.96}$$

where we have taken the large N approximation. This is the complementary regime to the one found in (2.80). The analysis in [76] looked directly at the expansions of truncated exponentials to do this. Here, we see that the saddle point encodes this information systematically. More importantly, if we say that $\lambda \sim \sqrt{N}$ in a scaling sense, then the energy stored in the state is of order N. This is usually associated with the energy of a D-brane. Indeed, these eigenvalues must be collective coordinates for AdS giant gravitons.

We also notice that $\bar{\lambda}\lambda$ is the Kahler potential of a flat manifold. This is recovered easily in the present formulation. Let us define:

$$K = \log(N(\bar{\Lambda}, \Lambda)) \tag{2.97}$$

The wave functions we constructed begin as holomorphic functions of the λ^i. Therefore, they already represent a complex structure of the parameters. The action we wrote induces a symplectic structure on the λ, p coordinates; we note that p is essentially $\bar{\lambda}$. We find that the symplectic form (expressed in terms of the $\bar{z}, z$ coordinates) is given by:

$$\begin{aligned} \omega = d(\sum p_i dz^i) &= d\sum \partial_i K dz^i = d(\partial K) = (\partial + \bar{\partial})\partial K \\ &= (\bar{\partial}_j \partial_i K) d\bar{z}^j \wedge dz^i = \sum_i d\bar{z}^i \wedge dz^i \end{aligned} \tag{2.98}$$

We find from the dominant saddle point computation that if we include all eigenvalues, the logarithmic correction to K does not affect the metric. The metric is flat. This matches what we expect. The eigenvalues λ parametrize the coherent states we have described. More importantly, in the semi-classical limit, the coherent states encode a particular field theory configuration, where the field we are quantizing (Z in this case) has a classical vacuum expectation value $Z \sim \Lambda$. Notice that because the HCIZ integral is exactly a sum over saddles, all corrections to K that could further arise are the contribution of other saddles: they are to be considered as a non-perturbative effect.

A similar result holds for the determinant calculation, where we would again find the flat Käler metric, as the overlap is a simple exponential. The one difference is how to compute the energy. When we count powers of $(a^\dagger)^k$ in the determinant, they are paired with powers of λ^{N-k}. In this case, the Hamiltonian is therefore $N - \lambda\partial_\lambda$. Putting everything together, we find that the energy in this case is:

$$H_{det} = N - \bar{\lambda}\lambda = \bar{\phi}\phi. \tag{2.99}$$

For the energy to be greater than zero, we need $\bar{\lambda}\lambda < N$, which coincides with the condition that the Hubbard-Stratonovich field ϕ has a good saddle. Again, if we look at the scaling of the energy, it scales like N. This again must be interpreted as a D-brane. These are the sphere giant gravitons.

Let us now turn to the A_{n-1} quivers. We need to understand the asymptotic expansion of the generalized hypergeometric function (see also [57]) for the dominant saddle:

$$\Phi_n(A\bar{A}) = \sum_m \frac{1}{(m!)^n}(A\bar{A})^m, \tag{2.100}$$

which replaces the exponential; in the denominator, we have $(A\bar{A})^N$. Notice that since this only depends on $A = \prod_{i=1}^n(\lambda^1_{(i,i+1)})$, we can take all λ variables to be identical (we should do this anyhow as they are not gauge invariant on their own). We can do the same with $\bar{\lambda}$. From our integral representation of Φ, we find that:

$$\Phi_n(A\bar{A}) = \frac{1}{(2\pi)^n}\iint \prod d\theta_i \exp(\lambda\bar{\lambda}\exp(i(\theta_i - \theta_{i+1}))) \tag{2.101}$$

Now, $\bar{\lambda}\lambda$ should be large, so we can saddle the integral over the angles. The critical point for the maximum is when all the angles are equal to one other. At this point, the saddle is for real values of θ. If the λ are not equal to one other, then the saddles move in the complex plane, and we will need to worry about both the real and imaginary parts of the variables θ.

Asymptotically, we find that:

$$\Phi_n(A\bar{A}) \sim \exp(n\lambda\bar{\lambda}) = \exp(n(A\bar{A})^{1/n}) \tag{2.102}$$

We obtain two results from this. Our first result is:

$$p \sim \bar{A}(A\bar{A})^{1/n-1} - N/A, \tag{2.103}$$

where we have included the measure term. In this case, p is not the complex conjugate of A and a holomorphic correction. The energy function is $\sum \lambda_{i,i+1}\partial_{i,i+1}$. This measures the degree with respect to λ. Since A is composed of a product of n lambdas, we need to multiply the degree of A by n to get the Hamiltonian. We find that the effective action is

$$S = i\int p\dot{A} - (n)pA = \int dt \left(ip\dot{A} - n[(\bar{A}A)^{1/n} - (N-1)]\right) \tag{2.104}$$

as expected. The Kähler potential gives rise to a flat geometry, but the complex structure is that of $\mathbb{C}/\mathbb{Z}_n$, as one would expect from the quiver theory. We get a similar constraint from the positive energy of a single large eigenvalue, namely that $\bar{\lambda}\lambda > N-1$. The Kähler form we get for the one eigenvalue is:

$$\omega = \frac{1}{n}(\bar{A}A)^{1/n-1} d\bar{A} \wedge dA \tag{2.105}$$

Notice that these coherent states also match the generalized coherent states introduced in [46] that were constructed combinatorially. That the energy is essentially the Kähler potential is a semiclassical result expected for BPS states [14, 77]

Consider now a $U(1)\times U(N+1)$ theory in the example where we showed that naive localization does not work. We still get a function similar to ϕ, but now we have:

$$\Phi=\sum_{m=0}^{\infty}\frac{1}{m!(m+N)!}(A\bar{A})^m=(A\bar{A})^{-N/2}I_N(2\sqrt{A\bar{A}}) \tag{2.106}$$

and because one of the terms is a $U(1)$, there is no denominator. The hypergeometric function, when written in terms of Bessel functions, looks as if it does have a denominator, with $N/2$ other eigenvalues. It is as if gauging $U(N)$ only counts for half as many eigenvalues. We want to understand the corresponding expressions at medium to moderate large N.

There are two regimes we want to consider: large A and small A. For large A, we need to use the asymptotic expansion of the Bessel function. This time, we choose to start from a different integral representation of $I_v(z)$ and use Watson's lemma to compute the asymptotic expansion of $I_v(z)$. We begin with:

$$I_v(z)=\frac{(2z)^{-1/2}e^z}{\sqrt{\pi}\Gamma(v+\frac{1}{2})}\int_0^1 e^{-2zt}t^{v-\frac{1}{2}}(1-t)^{v-\frac{1}{2}}dt, \tag{2.107}$$

where we've stipulated that $\mathcal{R}e(v)>-\frac{1}{2}$. We use Watson's lemma, which holds that an integral $F_\lambda(z)$, defined such that:

$$F_\lambda(z)=\int_0^\infty t^{\lambda-1}f(t)e^{-zt}dt, \tag{2.108}$$

where $\mathcal{R}e(\lambda)>0$ and the function $f(t)$ has a Taylor series expansion :

$$f(t)\approx\sum_{n=0}^{\infty}a_nt^n, \tag{2.109}$$

as $t \to 0_+$, can be approximated as:

$$F_\lambda(z) \approx \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda)}{z^{n+\lambda}} \tag{2.110}$$

in the large argument limit for z. For our case, in the large argument limit, our asymptotic expansion of $I_v(z)$ becomes:

$$I_v(z) = \frac{e^z}{\sqrt{2\pi z}\,\Gamma\left(v+\frac{1}{2}\right)} \sum_{k=0}^{\infty} (-1)^k \binom{v-\frac{1}{2}}{k} \frac{\Gamma\left(k+v+\frac{1}{2}\right)}{z^k} \tag{2.111}$$

Putting it all together, we get that:

$$\log(\phi) \sim 2(\bar{A}A)^{1/2} - (N/2+1/2)\log(A\bar{A}) - \frac{4N^2-1}{2\sqrt{\bar{A}A}}, \tag{2.112}$$

so for very large A, we once again arrive at the Kähler potential of $\mathbb{C}/\mathbb{Z}_2$ when we are allowed to ignore the $1/\sqrt{A\bar{A}}$ term. The expression above is valid when the second term is smaller than the first in the series (2.111), or when $(4N^2-1)/(2\sqrt{\bar{A}A})^{-1} < 1$. So $A\bar{A} \sim N^2$ has a N^2 scaling. This is equivalent to $\sqrt{A} \sim \sqrt{N}$, which gives the estimate in terms of the naive eigenvalues λ, rather than the composite object A.

Notice, however, that there is a large correction at large N. For small A, we use the series directly to find that:

$$\Phi \sim \sum_n \frac{1}{n!N^n}(A\bar{A})^n \sim \exp(A\bar{A}/N), \tag{2.113}$$

where we have used a large N approximation for the denominator. This way, we have:

$$\log(\Phi) \sim (A\bar{A})/N, \tag{2.114}$$

so the metric becomes that of the flat complex plane, rather than a cone. This applies so long as $A\bar{A}$ is not too large, so that the approximation of the denominators is valid near the terms of the series that contribute the most. This means that the tip of the cone is flattened (rounded) over a rather large range. We can think of this geometric calculation as quantum effects deforming the singularity away in a manner analogous to [78, 79], where the extra "fractional" branes at the tip of a cone lead to a measurable deformation of the geometry.

This effect can be thought of as a toy model of geometric transitions. The crossover from the small A to the large A happens when the formulas are roughly comparable to each other; that is, when $(\bar{A}A)^{1/2} \sim A\bar{A}/N$, or equivalently, where $(A\bar{A})^{1/2} \sim N$, which is the same estimate we obtained from the asymptotic series.

2.5.1 More open strings

In our discussion so far, we have been able to introduce open strings for the sphere giant graviton D-brane (the determinant). We now want to do this for the coherent state excitation parameterized by a few large eigenvalues λ^i. We notice that in the space of matrices associated with Λ, we already have a diagonal matrix; to each eigenvalue, we can associate an eigenvector $|v\rangle_i$. When we multiply the matrix by $U\Lambda U^{-1}$, this vector is rotated to $U|v\rangle_i$. We can stretch a sting between eigenvalues ij if we add words W as follows:

$$|\Lambda, W\rangle \sim \int dU \exp\left(\mathrm{tr}(U\Lambda U^{-1}a^\dagger)\right) \langle v|_j U^{-1} W U |v\rangle_i |0\rangle\,, \tag{2.115}$$

We may add more words in a similar manner. Notice that we are still doing integrals over $U(N)$ and that now the vectors v transform with the phases of $U(1)^N$ that do not belong to the coset space $U(N)/U(1)^N$. The total phase must cancel, so we find that for each $|v\rangle_i$, there must be another $\bar{v}_i$ somewhere else, which is to say that we get the Gauss' law constraint for $U(1)^N$. This is as expected. The Coulomb branch in Z breaks the gauge group to $U(1)^N$, tied to diagonalizing the expectation values of Z. When we consider saddles in the integral, the saddles in U are very pronounced if for a single eigenvalue $|\lambda\bar{\lambda}|$ that is large of order N. This means that we can ignore contributions from the W to the saddles. Instead, we evaluate U in the corresponding permutation matrix and keep the integration over the phases explicit. This procedure can lead to an effective action of D-branes with strings. Let us understand this for a pair of large eigenvalues λ_{12} and the word $a_X^\dagger$. We need to add the contribution $\langle v|_1 U^{-1} a_X^\dagger U |v\rangle_2$. Now we act with the 1 loop effective Hamiltonian in $\mathcal{N}=4$ SYM, which is written as a contribution coming strictly from F-terms as in [7] (here we use a slightly modified version of [80] with raising and lowering operators, similar to the notation in [13]):

$$g_{YM}^2 \mathrm{tr}\left([a_X^\dagger, a_Z^\dagger][a_Z, a_X]\right) \tag{2.116}$$

When acting on the simple word, the extra lowering operator a_Z brings down a copy of $U\Lambda U^{-1}$, either to the left or to the right. For more general states, the reader may refer to the works [81, 82] (see [83] for a review of the integrability program). These two pieces come as:

$$g_{YM}^2 \langle v|_1 U^{-1} U\Lambda U^{-1}[a_X^\dagger, a_Z^\dagger] U |v\rangle_2 - \langle v|_1 U^{-1}[a_X^\dagger, a_Z^\dagger] U\Lambda U^{-1} |v\rangle_2 \tag{2.117}$$

Cancelling the U and noting that $|v\rangle_{12}$ are eigenstates of Λ, we obtain the answer:

$$g_{YM}^2(\lambda_1 - \lambda_2) \langle v|_1 U^{-1}[a_X^\dagger, a_Z^\dagger] U |v\rangle_2 \tag{2.118}$$

We now perform the same trick when computing with the dual vector, so that $a_Z^\dagger$ brings down powers of $\bar{\Lambda}$. Again, we obtain an integral that involves U^{*-1}, and we pick the identity saddle, so that we get:

$$H_{1-loop} \sim g_{YM}^2 |(\lambda_1 - \lambda_2)|^2 \tag{2.119}$$

With the N scaling of λ, we get a finite contribution in the t'Hooft limit. This is exactly what is expected from other approaches that are based on coherent states without the U integral formalism [73, 76, 84, 85]. This should be contrasted with the difficult combinatoric computations that lead to "open spring theory" [86], which contains the same physics. More generally, such correlators of matrix elements of U have been studied in [87, 88, 89], where some exact expressions can be found. It should be interesting to develop that further.

Developing this idea further is beyond the scope of the present chapter. The eventual goal of such a program would be to simplify the types of analysis found in the works [90, 91]. It is likely that this basis is close to the so called Gauss graph basis described in [92, 93].

There is a second problem we would like to discuss: how to include more than one sphere giant graviton in this discussion. The idea of how to do it correctly should be motivated by what we have seen already, namely, that there is a good character expansion formalism. For sphere giants, we should have a generalization of the type:

$$G[M] \sim \sum_R \frac{1}{s_R} \chi_R(a_z^\dagger) \chi_{R^T}(M^{-1}), \tag{2.120}$$

where the characters of the generalized eigenvalues M involve the dual partition R^T. We are also including some denominator expressions s_R in case they are needed. We should also have inverse powers of M; as in our example of a single sphere giant, counting $a^\dagger$ runs opposite to counting powers of λ. Alternatively, we can go ahead and introduce new variables $u = 1/\lambda$; then, counting powers of u counts the powers of $a^\dagger$. Curiously, inversion of coordinates also seems to play an important role for the spin chain in the $SL(2)$ sector for open strings [85]. Based also on the introduction of fermions to deal with a single determinant, we would expect that such a generalization involves a fermion integral with more fermions. The proportionality constant in front of G should be λ^N for a single eigenvalue (that is, when M is of rank 1). This suggests using $(\det M)^N$ as a normalization factor more generally, which is suggestive of an integral over fermionic rectangular matrices of size $\mathrm{rank}(M) \times N$.

In the paper [71], multiple giants are introduced as products of determinants. We will justify this idea further. There is an expansion in characters from an algebraic identity (see for example [58]):

$$\sum_R \chi_R(t)\chi_R(a_Z^\dagger) = \frac{1}{\det\left(I \otimes I - t \otimes a_Z^\dagger\right)} \tag{2.121}$$

where t is an arbitrary $M \times M$ matrix that is tied to the “coordinates" of M AdS giant gravitons. A straightforward evaluation of the norm of the left shows that the state is not normalizable unless $t = 0$. This is because the operators $a_Z^\dagger$ are unbounded, so the Taylor expansion around $t = 0$ is not convergent. In this sense, this is a formal expansion.

Because the formula is an inverse determinant, it can be written as a bosonic integral. In the study of Wilson loops, it is noted that if one bosonizes a determinant Wilson loop, one passes from single column Young tableaux to row Young tableaux [94], thus switching between two types of “dual" D-brane representation: D5 and D3 branes. Here we see how the same idea can be written; the idea is then to fermionize the determinant above written as a bosonic integral to find:

$$\det\left(I \otimes I - t \otimes a_z^\dagger\right) = \int d\bar{\chi} d\chi \exp\left(\mathrm{tr}(\bar{\chi}\chi - \bar{\chi} a_Z^\dagger \chi t)\right) \tag{2.122}$$

In this case, to each eigenvalue t_s, we associate a flavor of fermions. For each such t_s eigenvalue, we can isolate fermions on the left and and on the right, and thus get the open strings attached to different giants. This would reproduce the ideas of [71], but would use the fermion language. To generalize the calculation, when one considers the Hubbard-Stratonovich trick, one should introduce an $M \times M$ collective field. One would like to have a saddle of this field that aligns in the "identity" permutation. The t variables here are exactly like the inverse of the λ parameters. Notice also that in matrix quantum models of rectangular matrices with fermions, one usually ends up pairing representations with dual young diagrams of the two groups under which the matrices transform [10, 53, 95].

We can also check that if we use the second Cauchy identity for Schur functions, we arrive at:

$$\sum_R \chi_R(t)\chi_{R^T}(a_Z^\dagger) = \det\left(1 + t \otimes a_Z^\dagger\right) \tag{2.123}$$

so that up to a sign, we get the correct generating series that we wanted in terms of characters.

Now, adding open strings becomes very simple. In the basis where t is diagonal, we can have fermions ψ^i and $\bar{\psi}_j$. Sandwiching words between these fermions allows one to form general states with open strings. The Gauss' law constraint becomes trivial: there is a $U(1)^M$ charge under which the fermion integral is invariant. This symmetry sends $\bar{\psi}_j \to \exp(-i\theta_j)\bar{\theta}_j$ and $\psi^j \to \exp(i\theta_j)\psi^j$, one for each parameter t_j. We find that there needs to be as many ψ^j as $\bar{\psi}_j$, that is, the same number of positively charged and negatively charged particles with respect to the D-brane $U(1)$ charge. When two of the t coincide, there is an enhanced $U(2)$ symmetry of the fermionic integral. This should be the generating series counterpart of how to attach strings to sphere giants combinatorially by adding boxes to Young diagrams [6, 50].

There should also be a more general theory of coherent states that has both sphere giants and AdS giants appearing more democratically, as one expects from the strict infinite N limit, where they can be constructed directly by making note of the symmetry between Young diagrams and their transposes [96]. In the strict $N \to \infty$ limit this can be made very precise. For example, the overlaps of multiple giants [19] are easily seen to be given by formulae that can be expanded in terms of Cauchy's character formulae. The parameters of those coherent states play the role of eigenvalue collective coordinates. A general setup for finite N that is democratic between these probably involves both fermion and boson integrals. It is likely that there is a supermatrix model that does this.

2.6 Coherent states for $1/4$ and $1/8$ BPS states

We are now ready to tackle coherent states for 1/4 and 1/8 BPS bosonic states. The idea, following our previous development, is to start with averaged coherent states:

$$F[\Lambda_Z, \Lambda_X, \Lambda_Y] = \frac{1}{Vol} \int dU \exp\left(\mathrm{tr}(U\Lambda_X U^{-1} a_X^\dagger + U\Lambda_Y U^{-1} a_Y^\dagger + U\Lambda_Z U^{-1} a_Z^\dagger)\right) |0\rangle \tag{2.124}$$

We now want to concentrate on the states that are 1/4 and 1/8 BPS at one loop order. The effective Hamiltonian is given by:

$$H = \mathrm{tr}\left([a_X^\dagger, a_Y^\dagger][a_Y, a_X]\right) + \text{cyclic}, \tag{2.125}$$

and since H is a sum of squares, we get $H \geq 0$ as an operator. When we let H act on F, we see that we get a result that is identically equal to zero when the Λ matrices commute. We thus insist that the parameters $\Lambda_X, \Lambda_Y, \lambda_Z$ are commuting matrices, as they should be. The coherent states are a semiclassical approximation to expectation values of fields. The moduli space of vacua occurs when the classical expectation values of the fields (after gauge fixing) commute. Namely, we need the expectation values to commute: $[X,Y]=0\dots$.

Now we perform our usual manipulations contracting $a_X^\dagger, a_X$ etc, to find the overlap:

$$\begin{aligned}\bar{F}[\bar{\Lambda}_Z,\bar{\Lambda}_X,\bar{\Lambda}_Y]*F[\Lambda_Z,\Lambda_X,\Lambda_Y] &= \\ \frac{1}{Vol}\int dU\exp\left(\mathrm{tr}(U\Lambda_X U^{-1}\bar{\Lambda}_X+U\Lambda_Y U^{-1}\bar{\Lambda}_Y+U\Lambda_Z U^{-1}\bar{\Lambda}_Z)\right)|0\rangle\end{aligned} \tag{2.126}$$

This answer has a manifest $U(3)$ symmetry of rotations where the matrices $\Lambda_X,\Lambda_Y,\Lambda_Z$ transform as a 3 of $U(3)$, and the conjugate $\bar{\Lambda}$ transform as the $\bar{3}$.

Here are a couple of observations. First, the action in the integral is also evaluated on a $U(N)/U(1)^N$ geometry; thus, the phases of U acting on the left disappear. Some of the critical points of U are the same as those in the HCIZ integral: permutation matrices. There may be additional ones. The conditions for critical points are:

$$[\bar{\Lambda}_X,U\Lambda_X U^{-1}]+[\bar{\Lambda}_Y,U\Lambda_Y U^{-1}]+[\bar{\Lambda}_Z,U\Lambda_Z U^{-1}]=0 \tag{2.127}$$

It is clear that when U is a permutation matrix, they are critical points.

If $\bar{\Lambda}$ and Λ are real, then the term in the exponential is real. In that case, Morse theory for the compact manifold over which we are doing the integral suggests that for small enough perturbations in Λ_X,Λ_Y slightly away from zero, the set of isolated critical points does not change (these depend continuously on the action when thought of as a Morse function on the manifold we are interested in). That means that we can evaluate a 1-loop approximation around the same saddles and get an approximation for the overlap. These should be dominant. If the integral above is localizable (which we have not proved), then the sum over all saddles (including complex saddles) is exact. Considering that one can localize on 1/8 BPS Wilson loops in [27], the idea that an integral like the one above or a variation of it is amenable to exact localization is very plausible. If additional saddles are needed, they will be complex.

We will do the naive saddle sum now, over the saddles we know. We find that for each saddle, we have a permutation matrix $U \sim P$, and the saddle gives:

$$S_\pi = \sum_i \lambda_X^i \bar{\lambda}_X^{\pi(i)} + \lambda_Y^i \bar{\lambda}_Y^{\pi(i)} + \lambda_Z^i \bar{\lambda}_Z^{\pi(i)}. \tag{2.128}$$

The square root of the measure at the saddle is evaluated readily to a product:

$$\mu_\pi = \prod_{i<j} (\vec{\lambda}_i - \vec{\lambda}_j) \cdot (\vec{\bar{\lambda}}_{\pi(i)} - \vec{\bar{\lambda}}_{\pi(j)}) \tag{2.129}$$

Such measures appear in the computation of the volume of the gauge orbit in [14] (they can also be extended to other orbifolds or more general setups [18, 77, 97]). Unlike in the case of the HCIZ integral, this measure does not factorize holomorphically. Moreover, different saddles have different denominators. This suggests that some poles will not cancel to give rise to a polynomial in the λ variables. If the poles are not cancelled, then there are two possibilities: either localization does not work, or there exist additional complex saddles in the complexified U variables that need to be taken into account.

The measure does reduce to the product of Vandermonde determinants when we get rid of the X, Y variables $\lambda_X = \lambda_y = 0$. It is also invariant under the $SU(3)$ rotations in X, Y, Z, so the answer is consistent from the $SU(3)$ group theory considerations. Notice that μ^2 does change sign with the permutations in the same way the Vandermonde does, so the accompanying sign in the saddle point evaluation at $\lambda_x = \lambda_y = 0$ should be kept. We find an approximation given by

$$\bar{F}[\bar{\Lambda}_Z, \bar{\Lambda}_X, \bar{\Lambda}_Y] * F[\Lambda_Z, \Lambda_X, \Lambda_Y] = \sum_\pi (-1)^\pi \frac{\exp(S_\pi)}{\mu_\pi} \tag{2.130}$$

We can also try to go to our collective coordinate formulation. In that case, $\bar{\Lambda}$ is the adjoint of Λ, and the trivial permutation dominates (which one can show by the Cauchy-Schwarz inequality). In that limit, we find that:

$$\bar{F}[\bar{\Lambda}_Z, \bar{\Lambda}_X, \bar{\Lambda}_Y] * F[\Lambda_Z, \Lambda_X, \Lambda_Y] \sim \frac{\exp(\mathrm{tr}(\bar{\Lambda}\Lambda))}{\prod_{i<j} |\vec{\lambda}_i - \vec{\lambda}_j|^2}, \tag{2.131}$$

and that the energy would evaluate to:

$$\mathrm{tr}(\bar{\Lambda}\Lambda) - \lambda\partial_\lambda \ln(\prod_{i<j} |\vec{\lambda}_i - \vec{\lambda}_j|^2) = \mathrm{tr}(\bar{\Lambda}\Lambda) - N(N-1), \tag{2.132}$$

where we are using the fact that the measure is a homogeneous function (see [14]). For a single large eigenvalue to be well behaved, we require that the energy in that eigenvalue (evaluated with $N-1$ eigenvalues set to zero) is positive compared to the result when $N \to N-1$. This gives a $SU(3)$ covariant version of the eigenvalue being larger than N. That is:

$$\vec{\bar{\lambda}} \cdot \vec{\lambda} > N - 1 \tag{2.133}$$

We know that the equation of motion of λ_i is $i\dot{\lambda}_i = \lambda_i$, and that the Hamiltonian is essentially $\sum \bar{\lambda}_i \lambda_i$ up to a constant. This suggests that the canonical conjugate to λ_i is $\bar{\lambda}_i$. Indeed, an action based on that prescription alone would give the correct equations of motion. Moreover, the Kähler potential would be that of a flat geometry. Notice, however, that we have a quantum correction from the measure.

This is important. One way of interpreting this correction has to do with the counting of states. When we write the generating function for a single large eigenvalue, we get:

$$\bar{F} * F \sim \exp(\bar{\lambda}\lambda)/(\bar{\lambda}\lambda)^2 \to \sum_{m=0}^{\infty} N! \frac{(\bar{\lambda}\lambda)^m}{(m+N)!}, \tag{2.134}$$

where we only keep the regular part of the answer, as the full generating series has no singularities at the origin. In this notation, we are suppressing the $SU(3)$ labels, and we only keep the polynomial part of the answer. We see that there is one state per monomial $\lambda_X^{n_1}\lambda_Y^{n_2}\lambda_Z^{n_3}$ from expanding each of the terms of the sum; the expansion gives a sum of squares. The number of states at energy k is the number of states of the k-th completely symmetric representation of $SU(3)$. This has dimension $(1+k)(2+k)/2$. Semiclassically, this should be the volume of phase space between energy k and $k+1$; our phase space is the complex manifold we are discussing. We should now substitute $k = E \sim \bar{\lambda}\lambda - N = r^2 - N$, where r^2 is the norm. We find that the volume at fixed r scales like $(r^2-N)^2 r dr/2$, which is different from that of a flat geometry at the origin. At large r, the correction does not matter and the metric becomes scaling, but at finite r, we get the wrong counting of states. The representation theory has also been studied directly in [98], using different methods.

If we have a few branes with large eigenvalues, the measure gets a correction from the product of two measures that is singular when the large eigenvalues coincide. This indicates the enhanced symmetry of the integral– when the eigenvalues coincide, the critical point leaves $SU(2)$ invariant, and the fixed point is not isolated. This is associated with the enhanced gauge symmetry of coinciding branes.

The main point we are making is that the idea of saddles dominating is still accurate and the type of constructions that are used for attaching strings to these setups still hold, including energies like equation (2.119), properly covariantized to be $U(3)$ invariant. The BPS states here are only implicit. They are expressed as polynomials of the eigenvalues λ multiplying functions of oscillators that we have not computed explicitly, and written as integrals over the group. They are also invariant under combined permutations of the eigenvalues. This should be contrasted with other approaches to this problem [99, 100].

Another option is to keep Λ_Z finite and expand the exponential in the other variables

$\Lambda_{X,Y}$ as a power series. Results to each order would then be given in terms of correlators of U, U^{-1} matrix elements in the “HCIZ ensemble." Such correlators have been studied in [87, 88, 89] and they do correspond to elaborate sums over the saddles. Such formal expansions would add open strings to the eigenvalues of the Λ_Z configuration and would tie in with the open string formalism briefly mentioned in this chapter.

2.7 Discussion

In this chapter, we have discussed a new application of the Harish-Chandra-Itzykson-Zuber integrals and their generalizations to study correlators of BPS states in $\mathcal{N} = 4$ SYM. The main idea was to introduce coherent states and average them over a group orbit to obtain gauge invariant states. We were able to reproduce various results that were obtained originally from combinatorial arguments. We then promoted the parameters in the generating function to collective coordinates of states. We exploited the fact that the integrals in question are written as sums of saddles to find that there is a dominant saddle. This allows one to make various approximations to find the effective action of these collective coordinates directly. We showed how a similar structure could be found with determinants that also lead to dominant saddles and explained why this is the correct generalization in terms of the type of algebraic structure that arises.

We also showed how to introduce open strings in all of these these setups. We found that computations done this way require knowledge of correlators of matrix elements of unitary matrices in the HCIZ ensemble. This method provides a complementary approach to study anomalous dimensions of open strings that we are currently investigating. Again, the fact that there is a dominant saddle for the setup allows simplifications to be made in computations. We demonstrated that these open strings need to satisfy a Gauss' law constraint that becomes evident in the formulation, including the non-abelian enhancement that occurs when D-branes coincide.

We extended these ideas to more general group integrals for 1/4 and 1/8 BPS states. We believe that because 1/8 BPS WIlson loops are computable using localization, one can find a formalism where the correlators of these states are computed this way, rather than the combinatorial approaches found in current literature.

It would be interesting if similar ideas can be used to study the ABJM theory [101] correlators. In that case, Wilson loops can be computed with supermatrix models [102], derived using localization methods [103]. The associated spin chain is integrable [104].

We have not yet applied our ideas to the study of higher point functions along the lines of [105] which may also be computed using localization methods, nor to the fact that the HCIZ integrals are also tau functions of integrable systems (see [58]). All of these avenues suggest a rich setup of possible applications of the ideas presented in this chapter to the computation of protected quantities in various setups.

Chapter 3

Giant Gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal $N = 4$ SYM

3.1 Introduction

Recently, there has been a renewed interest in determinant operators in large N holographic gauge theories and their string dual description as giant gravitons [66, 106, 70, 72, 107]; the dimension of these operators is order N, which makes them ideal to probe sub-AdS physics. A natural basis for gauge invariant operators is the Schur functions, which are characters of the unitary and symmetric groups. Combinatorial methods for computing correlation functions in free $\mathcal{N} = 4$ SYM were developed in [36, 108]. More recent works have emphasized the utility of an effective action approach obtained by recasting the determinant operators as fermionic integrals and integrating out the super Yang-Mills fields. In this description, the non-perturbative physics of the problem can be obtained from a saddle point approximation for an effective action in terms of a set of collective fields [66, 70].

A similar prescription for AdS giant gravitons was proposed in [21], where it was realized that the norms of BPS states are encoded in the expansion of the Harish-Chandra-Itzykson-Zuber (HCIZ) integral, which appears in the evaluation of the norms of a certain class of gauge invariant coherent states:

$$\mathcal{O}_\Lambda(0) = \int_{SU(N)} dU \exp\left(\mathrm{Tr}\left[\Lambda U a_Z^\dagger U^\dagger\right]\right). \tag{3.1}$$

This sheds light on why the group characters evaluated on the Yang-Mills fields may serve as an orthogonal basis, even though they are only orthogonal with respect to the Haar measure, and gives a different interpretation of the norms of BPS states as the coefficients in the expansion of the HCIZ integral. This technique has the advantage of repackaging the combinatorics of the Schur functions into integrals over the unitary group.

The Harish-Chandra integrals have natural generalizations to the B, C, D series, $Sp(2N)$ and $SO(M)$. For a choice of simple Lie group G, the HCIZ integral has an exact formula in terms of a sum over the saddle points:

$$\mathcal{H}(x, y) = \int e^{\langle \mathrm{Ad}_g(x), y\rangle} dg = c_{\mathfrak{g}} \sum_{w\in W} \frac{\epsilon(w) e^{\langle w(x), y\rangle}}{\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)}. \tag{3.2}$$

Each saddle point of the integral corresponds to a Weyl reflection, and the denominators are given by the discriminant of the Lie algebra. These integrals have received less attention than the unitary HCIZ integral, which serves as a single plaquette model in lattice gauge theory.

The bulk of the work on probing finite N physics is limited to field theories with $U(N)$ and $SU(N)$ gauge groups (see [109, 110, 111, 112]), but more recently, there has been some interest in extending these studies to field theories with $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$ gauge groups [113, 114, 115]. There is good reason for this surge of interest: maximally supersymmetric Yang-Mills theory with symplectic and orthogonal groups are dual to type IIB strings on $AdS_5 \times \mathbb{RP}^5$ [2]. Depending on the choice of the orientifold projection, the gauge group of the theory is either $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$; S-duality relates the spectrum of the $Sp(2N)$ and the $SO(2N+1)$ theories, while the $SO(2N)$ theories are self-dual. The exact matching of the spectrum for the symplectic and orthogonal theories is poorly understood, due to the combinatorial difficulty associated with constructing states of these theories.

In this chapter, we study BPS coherent states of $\mathcal{N} = 4$ SYM for special orthogonal and symplectic groups. The norms of such states are given precisely by a Harish-Chandra integral over the corresponding group. By explicitly expanding the integral, we find that these coherent states serve as generating functions for gauge invariant states in the gauge theory, and the corresponding coefficients in the expansion give their norms. In principle, this gives a way of constructing an orthogonal basis of states for these theories from group theoretic data for the corresponding gauge group. We argue that these generating functions are only able to capture information about the "unitary" part of the gauge symmetry, which is to say that operators we find in the expansion match in form to operators in the unitary theory. In section 3.2, we review the construction of gauge invariant coherent states for the $SU(N)$ theory. In section 3.3, we generalize this to the symplectic case and argue that the odd special orthogonal case is related to the symplectic case by a rank-level duality that exchanges a Young diagram with its conjugate diagram. We repeat the calculations for the even orthogonal case. In section 3.4, we discuss other attempts at finding an orthogonal basis for $Sp(2N)$, $SO(2N+1)$, or $SO(2N)$ and how our results can be interpreted in a relevant context. Finally, we conclude with a discussion of a few open questions and future directions of work.

3.2 Review of the $U(N)$ case

We begin with a brief review of BPS coherent states in $U(N)$. The same analysis may be applied to any free gauge theory with an adjoint scalar field Z. We know from [21] that given a naïve coherent state $F[\Lambda]$ of the form:

$$\exp(\mathrm{Tr}(\Lambda \cdot a_Z^\dagger))\,|0\rangle\,, \tag{3.3}$$

where Λ is taken to be a diagonal matrix-valued set of parameters and $a_z^\dagger$ is the raising operator for the s-wave of the field Z on S^3 in [11], we may introduce an auxiliary $U(N)$ group action and average over the group, which allows us to rewrite a gauge invariant coherent state as:

$$F[\Lambda] = \frac{1}{Vol(U(N))}\int dU\,\exp(\mathrm{Tr}(U\Lambda U^{-1}a_Z^\dagger))\,|0\rangle\,, \tag{3.4}$$

where dU is the Haar measure. Our normalization factor $Vol(U(N)) = \int dU$; we can set it equal to one for the sake of brevity. We may compute the overlap of $F[\Lambda]$ as defined in Eq. (3.4) with its adjoint $\bar{F}[\bar{\Lambda}]$ by evaluating the HCIZ integral:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \int d\tilde{U}\,\exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right). \tag{3.5}$$

We see that we have sidestepped most of the Wick contractions of the matrix operators $\left(a^\dagger\right)^i_j$, which would make $F[\Lambda]$ difficult to compute in the form it takes in Eq. (3.4). $F[\Lambda]$ can be evaluated through a character expansion, as described in [116]:

$$F[\Lambda] = \sum_R \frac{1}{f_R}\chi_R(\Lambda)\chi_R(a_Z^\dagger)\,|0\rangle \tag{3.6}$$

We may also rewrite Eq. (3.5) through a character expansion:

$$\bar{F}[\bar{\Lambda}] * F[\Lambda] = \sum_R \frac{1}{f_R} \chi_R \left(\bar{\Lambda}\right) \chi_R \left(\Lambda\right) \tag{3.7}$$

We can then compare the coefficients of the characters from the equation above to what we would obtain from multiplying Eq. (3.6) by its adjoint and find:

$$\langle 0| \chi_R(a)\chi_R(a^\dagger) |0\rangle = f_R \tag{3.8}$$

It becomes obvious that we must compute f_R to evaluate the overlap of $\chi_R(a)$ and $\chi_R(a^\dagger)$. The thing to keep in mind is that the representations R in the coherent state $F[\Lambda]$ correspond to Young diagrams for $U(N)$, which are characterized by the indices $j_1 \geq j_2 \geq \dots j_N$, where each index j_i iterates over row i. Because these are characters of the unitary group, they may be rewritten with the Weyl character formula:

$$\chi_{j_i}(\Lambda) = \frac{\det\left(\lambda_k^{j_i+N-i}\right)}{\Delta(\Lambda)}, \tag{3.9}$$

where λ_k are the eigenvalues of Λ and $\Delta(\Lambda)$ is the Vandermonde determinant of Λ. Then we may rewrite the HCIZ integral as a product of these expanded characters:

$$I(\Lambda, \bar{\Lambda}) = \int d\tilde{U} \exp\left(\mathrm{Tr}\left(\tilde{U}^{-1}\Lambda\tilde{U}\bar{\Lambda}'\right)\right) = \Omega\frac{\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right)}{\Delta(\Lambda)\Delta(\bar{\Lambda}')} \tag{3.10}$$

where Ω is a normalization constant. We rewrite the numerator to reintroduce f_R:

$$\Omega \det\left(\exp(\lambda_i\bar{\lambda}'_j)\right) = \sum_{\vec{j}} \frac{1}{f_{\vec{j}}} \det\left(\lambda_k^{j_i+N-i}\right) \det\left(\bar{\lambda}_k'^{j_i+N-i}\right) \tag{3.11}$$

We have relabeled R with the indices $\vec{j}$, and have rewritten the equation above accordingly. The expressions inside the determinants are monomials and correspond to the term $\prod_i \lambda_i^{j_i+N-i} + \dots$ in $\det\left(\lambda_k^{j_i+N-i}\right)$. Thus we may expand the exponential in Eq.

(3.11) as:

$$\det\left(\exp(\lambda_i\bar{\lambda}'_j)\right) = \sum_{[n]} \frac{1}{[n]!}\det((\lambda_i\bar{\lambda}'_j)^{n_i}) = \sum_{[n]} \frac{1}{[n]!}\det(\bar{\lambda}'^{n_i}_j)\prod_i \lambda_i^{n_i} + \dots, \tag{3.12}$$

where we have made use of the multilinearity of the determinant. The factor $[n]$ encapsulates $n_1, \dots, n_N$; then $[n]! = \prod_j n_j!$. We see that we are limited to $n_1 > n_2 \dots$ when we restrict ourselves to the monomials with the correct descending order; when we set $n_i = j_i + N - i$, we arrive at an explicit sum over the characters. Thus our denominator $f_{\vec{j}}$ may be computed as:

$$f_{\vec{j}} = \Omega^{-1}\prod_i (j_i + N - i)!, \tag{3.13}$$

We may set $f_{\vec{0}} = 1$, as $\langle 0|0\rangle = 1$. Then we arrive at:

$$\Omega = \prod_{i=1}^{N}(N - i)! \tag{3.14}$$

From this we can easily read off the norms of the states $\chi_R(a^\dagger)$:

$$\langle \chi_{R'}(a)\chi_R(a^\dagger)\rangle = \delta_{R,R'}\frac{\prod_i (j_i + N - i)!}{\prod_{i=1}^{N}(N - i)!}, \tag{3.15}$$

which agrees with the well-known result of [36].

3.3 Symplectic and orthogonal cases

Before repeating the analysis for the other simple lie groups, we should comment on the interpretation of the $Sp(2N)$ and $SO(N)$ theories as orientifold projections of a unitary theory. To do this, we first consider a simple toy model correponding to a single harmonic oscillator. As it turns out, this simple model captures a lot of the qualitative behaviour of the answer for symplectic and orthogonal groups.

3.3.1 A toy model for the orientifold projection

As a warm-up, we consider a single quantum harmonic oscillator:

$$[a, a^\dagger] = 1. \tag{3.16}$$

A natural basis of states for this system is the eigenstates of the occupation number operator $\hat{n}\,|n\rangle = n\,|n\rangle$. One thing that we may do with this system is to define a parity operator $\Omega = (-1)^{\hat{n}}$ and further divide the set of states into those that are mutual eigenvectors of $\hat{n}$ and Ω. This gives an orthogonal decomposition of the Hilbert space of the harmonic oscillator into sectors of positive and negative parity $\mathcal{H} \cong \mathcal{H}_+ \bigoplus \mathcal{H}_-$, and divides all the states into even and odd states under the orientation reversal transformation

$$\begin{aligned} P &: x \to -x \\ P &: p \to -p, \end{aligned} \tag{3.17}$$

where x and p are the position and momentum operators. Because the raising operators are monomials in x and p, the odd parity states are created with odd numbers of raising operators and vice versa. The operators $\frac{1}{2}(1 \pm \Omega)$ respectively serve as orthogonal projection operators into $\mathcal{H}_+$ and $\mathcal{H}_-$.

What we would like to do is build coherent states in each of these two sectors of the theory. For instance, we can project a coherent state into the sector of positive parity by applying the operator $\frac{1}{2}(1 + \Omega)$:

$$\frac{1}{2}\left(1+\Omega\right)|\alpha\rangle = \frac{1}{2}\left(1+e^{\pi i\hat{n}}\right)e^{\alpha a^\dagger}|0\rangle = \frac{1}{2}\left(e^{\alpha a^\dagger}+e^{-\alpha a^\dagger}\right)|0\rangle = \cosh\left(\alpha a^\dagger\right)|0\rangle\,. \quad (3.18)$$

We call this state $|\alpha, +\rangle$. One nice property of this state is that it is an eigenstate of a^2 with eigenvalue α^2. In this sense, we can call this a coherent state for the positive chirality sector of the model. By a similar computation, the overlap between any two of these coherent states is given by:

$$\langle\beta^*, +|\alpha, +\rangle = \cosh\left(\alpha\beta\right). \quad (3.19)$$

The case for negative parity requires more care, and will be the case that is relevant to the analysis of the $Sp(2N)$ and $SO(2N+1)$ theories. If we project a coherent state into the sector of negative chirality, we obtain the state:

$$\frac{1}{2}\left(1-\Omega\right)|\alpha\rangle = \sinh\left(\alpha a^\dagger\right)|0\rangle\,. \quad (3.20)$$

The issue is that this state is not a coherent state in the usual sense; when we act on the state with a lowering operator, the state won't return to the original state since the minimum ocupation number that appears in the series is $|1\rangle$. Rather, this state is also an eigenvector of a^2 with eigenvalue α^2. Since the original vacuum state is annihilated by the projector $\frac{1}{2}\left(1-\Omega\right)$, the true vacuum in this sector is the state occupation number one $|1\rangle$. By a relabeling of the states for the odd sector, the coherent state can be written as

$$|\alpha, -\rangle = -i\,\text{sinc}\left(i\alpha a^\dagger\right)|\tilde{0}\rangle\,, \quad (3.21)$$

where $\text{sinc}(x) = \frac{\sin x}{x}$, and the new vacuum is $|\tilde{0}\rangle = |1\rangle$. A simple computation yields the norm of this coherent state:

$$\langle \beta^*, - | \alpha, - \rangle = \frac{\sinh \alpha\beta}{\alpha\beta}. \tag{3.22}$$

3.3.2 The symplectic HCIZ integral

We now seek to expand our definition for a well-defined BPS operator averaged over the unitary group to the symplectic group:

$$F_{Sp(2N)}[\Lambda] = \frac{1}{\mathrm{Vol}(Sp(2N))} \int_{Sp(2N)} dg \exp(\mathrm{Tr}(g\Lambda g^{-1} a_Z^\dagger)) \left|0\right\rangle, \tag{3.23}$$

where dg is the Haar measure for the symplectic group and $\mathrm{Vol}(Sp(2N)) = \int_{Sp(2N)} dg$ is a normalization factor, which we can always rescale to one. The group elements of $Sp(2N)$ can be represented by $2N \times 2N$ matrices that are both unitary and symplectic:

$$\begin{aligned} g^\dagger g &= \mathbf{1}_{2N} \\ g^T \Omega g &= \Omega, \end{aligned} \tag{3.24}$$

where Ω is a choice of anti-symmetric symplectic matrix:

$$\Omega = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}. \tag{3.25}$$

The symplectic condition (3.24) translates into the orientifold projection of the Chan-Paton indices for the open strings ending on a stack of $2N$ $D3$ branes [2]. This forces the raising and lowering operators of the $Sp(2N)$ theory to satisfy the orientifold projection condition:

$$\Omega\, a_Z^\dagger\, \Omega = (a_Z^\dagger)^T = -a_Z^\dagger, \tag{3.26}$$

where the transpose is taken on the group indices, which we omit for clarity. This means that any operator made from traces of odd numbers of fields will automatically vanish. We choose to normalize the commutation relations for the raising and lowering operators by a factor of $\frac{1}{2}$, which will make the computation of the norm of the coherent state more transparent:

$$[(a_Z)^i_j, (a^\dagger_Z)^l_k] = \frac{1}{2}\left(\delta^l_j\delta^i_k - \Omega_{jk}\Omega^{lj}\right). \tag{3.27}$$

As with the unitary case, we wish to compute the overlap between two coherent states. This is done by applying the Campbell-Hausdorff formula; since the raising and lowering operators have different relations from the unitary case, we must check that commuting the exponentials really simplifies the norm into the form where it can be evaluated by a Harish-Chandra integral. After some algebra, we see that in the symplectic case, the exponentials can be commuted as follows:

$$\begin{aligned}\left[\mathrm{Tr}\left(g a_z g^\dagger \Lambda\right), \mathrm{Tr}\left(h a^\dagger_z h^\dagger \bar{\Lambda}'\right)\right] &= \frac{1}{2}\mathrm{Tr}\left(gh\Lambda(gh)^\dagger\bar{\Lambda}'\right) + \frac{1}{2}\mathrm{Tr}\left(g\Lambda g^\dagger \Omega h^T \bar{\Lambda}'^T (h^T)^{-1}\Omega\right) \\ &= \mathrm{Tr}\left(gh\Lambda(gh)^\dagger\bar{\Lambda}'\right).\end{aligned} \tag{3.28}$$

The second term in (3.28) is equivalent to the first term after using the group relations (3.24). This means that once again, we can compute the operator's overlap with its adjoint with the symplectic Harish-Chandra integral:

$$\bar{F}_{Sp(2N)}[\bar{\Lambda}] * F_{Sp(2N)}[\Lambda] = \int d\tilde{g} \exp\left(\mathrm{Tr}\left(\tilde{g}^{-1}\Lambda\tilde{g}\bar{\Lambda}'\right)\right) = \mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}'), \tag{3.29}$$

where $\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}')$ is given in [31]:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{2N-1}(2p+1)!\right)\frac{\det\left[\sinh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{2N}}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}^{(2)}\right)\prod_{i=1}^{2N}\lambda_i\bar{\lambda}'_i}. \tag{3.30}$$

The denominator in this formula is computed using the Weyl denominator formula for the corresponding discriminant, as demonstrated in [117, 118]:

$$\boldsymbol{\Delta}_{\mathfrak{sp}(2N)}(\lambda) = \prod_{j}^{N} \lambda_j \prod_{1\leq j<k\leq N} \left(\lambda_j^2 - \lambda_k^2\right) = \det(\Lambda)\, \Delta\left(\Lambda^2\right) \tag{3.31}$$

Thus we may rewrite Eq. (3.30) as:

$$\boldsymbol{\Delta}_{\mathfrak{sp}(2N)}(\lambda)\boldsymbol{\Delta}_{\mathfrak{sp}(2N)}(\bar{\lambda}')\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{N-1}(2p+1)!\right) \det\left[\sinh(2\Lambda_j\bar{\Lambda}'_k)\right]. \tag{3.32}$$

The numerator can be simplified by using the identity that $\sinh(2\Lambda_j\bar{\Lambda}'_k)$ is a modified Bessel function of the first kind of order $\nu = \frac{1}{2}$, and expanding the determinant. We know that:

$$\sinh\left(2\Lambda\bar{\Lambda}'\right) = \sqrt{\pi\Lambda\bar{\Lambda}'}\, I_{\frac{1}{2}}\left(2\Lambda\bar{\Lambda}'\right) = \sum_{m=0}^{\infty} \frac{2^{m+1}}{m!\,(2m+1)!!} \left(\Lambda\bar{\Lambda}'\right)^{2m+1} \tag{3.33}$$

Then we can use the Cauchy-Binet formula to expand the determinant:

$$\det\left[\sinh\left(2\Lambda_i\bar{\Lambda}'_j\right)\right] = \sum_{m_i}\prod_{i}^{N} \frac{2^{m_i+1}}{m_i!\,(2m_i+1)!!} \det\left[\Lambda_j^{2m_i+1}\right] \det\left[\bar{\Lambda}'^{2m_i+1}_j\right] \tag{3.34}$$

Thus Eq. (3.30) becomes:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \sum_{m_i}\prod_{i}^{N} \frac{2^{m_i+1}\,(2i-1)!}{m_i!\,(2m_i+1)!!} \frac{\det\left[\Lambda_j^{2m_i}\right]\det\left[\bar{\Lambda}'^{2m_i}_j\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j)} \tag{3.35}$$

Once again, if we set $m_i = \mu_i + N - i$, we may rewrite Eq. (3.23) as an explicit sum over the Schur polynomials:

$$\mathcal{H}_{Sp(2N)}(\Lambda, \bar{\Lambda}') = \sum_{\mu} \frac{1}{f_\mu} \chi_\mu(\Lambda^2) \chi_\mu(\bar{\Lambda}'^2), \tag{3.36}$$

where the coefficient in the expansion is given by

$$f_\mu = \prod_{i}^{N} \frac{(\mu_i + N - i)!\,(2\mu_i + 2N - 2i + 1)!!}{2^{\mu_i + N - i + 1}\,(2i - 1)!}, \tag{3.37}$$

and the sum is taken over all integer partitions μ.

This form of the expansion is natural from the point of view of the orientifold projection, since we projected out all the states with an odd number of raising operators acting on the vacuum state. Similarly, the operator that creates the coherent state must have a formal expansion of a similar form:

$$\mathcal{O}_\Lambda = \int_{Sp(2N)} dg \exp\left(\mathrm{Tr}\left(g \Lambda g^{-1} a_Z^\dagger\right)\right) = \sum_{\mu} \frac{1}{f_\mu} \chi_\mu(\Lambda^2) \chi_\mu(\left(a_Z^\dagger\right)^2) \tag{3.38}$$

This indicates that just as in the unitary case, the norms of states are given by the inverse of the coefficients that appear in the expansion of the Harish-Chandra integral.

3.3.3 Special orthogonal groups

Odd special orthogonal group

It is known that the Harish-Chandra integral for the odd orthogonal group is the same as that for the symplectic group. This can be thought of as a result of the S-duality of $\mathcal{N} = 4$ super Yang-Mills theory; S-duality exchanges the $Sp(2N)$ and $SO(2N+1)$, while $SO(2N)$ is S-duality invariant [2]. This means that the spectrum of the $Sp(2N)$ and the $SO(2N+1)$ theories are related by a change of basis. We will argue that this change of basis is simply the transpose operation on the Young diagram μ associated to a given representation.

One reason to suspect that this is the case comes from the Schur-Weyl duality for odd orthogonal and symplectic groups. It is well-known that the centralizer algebra associated to the k-fold tensor product of fundamental representations of $SU(N)$ is the group algebra of the symmetric group $\mathbb{C}S_k$. This means that the k-fold tensor product of fundamental representations of $SU(N)$ decomposes into tensor products of irreducible representations of S_k and $SU(N)$:

$$V_{SU(N)}^{\otimes k} \cong \bigoplus_{\lambda} \pi^{\lambda} \otimes U_{\lambda}. \tag{3.39}$$

This is more complicated for the symplectic and orthogonal groups, since the corresponding centralizer algebra is no longer a group algebra, but rather the algebra associated to the Brauer monoid. One way to understand this is that the symplectic and orthogonal lie algebras have additional invariant tensors compared to the unitary case. For tensor products of fundamental representations of unitary groups, the only invariant tensors allowed are the identity and permutation operators:

$$\begin{aligned} \mathbb{I}(V_a \otimes V_b) &\to V_a \otimes V_b \\ \mathbb{P}(V_a \otimes V_b) &\to V_b \otimes V_a. \end{aligned} \tag{3.40}$$

Clearly these operations are invertible and generate the symmetric group S_k. For orthogonal groups, there is an additional invariant tensor, called the trace operation:

$$\mathbb{K}(V_a \otimes V_b) \to \mathbb{C}. \tag{3.41}$$

These tensors are well known in the integrable spin chain literature, and are the same kind of tensors that appear in the $SO(6)$ integrable spin chain [83]. Unlike the identity and permutation operators, the trace operation is not invertible, and together with the identity, it generates the Temperley-Lieb algebra $TL_k(2N)$ [119, 120]; the linear span of these three operations generates the Brauer algebra $B_k(2N)$. The importance of Brauer centralizer algebras has been emphasized in [121, 122], where they were used to diagonalize two-point functions in the space of gauge theory operators and their adjoints. These operators correspond to bound states of non-holomorphic giants. Brauer centralizer algebras have also been used to construct coherent states in [20].

Returning to the tensor decomposition of the k-fold tensor product of fundamentals of $SO(2N+1)$, the corresponding decompostition is [120]:

$$V_{SO(2N+1)}^{\otimes k} \cong \bigoplus_{k=0}^{\lfloor f/2 \rfloor} \bigoplus_{\lambda \vdash f-2k} D_\lambda \otimes V_\lambda, \tag{3.42}$$

with D_λ and V_λ respectively denoting the irreducible representations of the Brauer algebra and $SO(2N+1)$. The analogous statement for the symplectic group $Sp(2N)$ exchanges N with $-N$ and V_λ with W_{λ^T}, where W_{λ^T} is the irreducible representation of $Sp(2N)$ associated to the diagram conjugate to λ:

$$V_{Sp(2N)}^{\otimes k} \cong \bigoplus_{k=0}^{\lfloor f/2 \rfloor} \bigoplus_{\lambda \vdash f-2k} D_\lambda \otimes W_{\lambda^T}. \tag{3.43}$$

Since the Harish-Chandra integral involves group averages of powers of traces of the form $\mathrm{Tr}\,(g\Lambda g^{-1}\Lambda')$, it is natural to expect that every term in expansion for the odd orthogonal groups should match to a term with the corresponding transposed Young diagram in the expansion for the symplectic integral. This might appear surprising, since the number of boxes that can appear in a column is bounded from above by N, while the number of boxes in a row can be arbitrary. One way of understanding this apperent mismatch is that the fundamental degrees of freedom in one description might be mapped to a bound state by S-duality. In reality, representations with arbitrary numbers of boxes in a column are possible, but will not be irreducible.

Special even orthogonal group

Extending our definition for a well-defined BPS operator to the even special orthogonal group requires a little more work. We modify the definition of $F[\Lambda]$ to reflect averaging over the even special orthogonal group:

$$F_{SO(2N)}[\Lambda] = \int dO \exp(\mathrm{Tr}(O\Lambda O^{-1} a_Z^\dagger)) |0\rangle . \tag{3.44}$$

As before, the overlap of $F[\Lambda]$ and its adjoint is the corresponding Harish-Chandra integral:

$$\bar{F}_{SO(2N)}[\bar{\Lambda}] * F_{SO(2N)}[\Lambda] = \int d\tilde{O} \exp\left(\mathrm{Tr}\left(\tilde{O}^{-1}\Lambda\tilde{O}\bar{\Lambda}'\right)\right) = \mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}'), \tag{3.45}$$

where $\mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}')$ is given by [31]:

$$\mathcal{H}_{SO(2N)}(\Lambda, \bar{\Lambda}') = \left(\prod_{p=1}^{N-1}(2p)!\right) \frac{\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^N + \det\left[\sinh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^N}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}'^{(2)}\right)}. \tag{3.46}$$

We note that Eq. (3.44) is invariant under an additional symmetry:

$$O \to \tilde{I}O, \tag{3.47}$$

where $\tilde{I}$ is a diagonal matrix with determinant equal to ± 1. To get rid of this redundancy, we could integrate over the entire orthogonal group $O(N)$. For $SU(N)$, $Sp(2N)$ and $SO(2N+1)$, this process does not change the value of the integral. This is similar to what happens in the Kazakov-Migdal model in [123], where the additional abelian part of the gauge field decouples from the collective field effective action. We also note that even though the whole integral is invariant under the parity transformation

$$\begin{aligned} \tilde{P} &: \Lambda \to -\Lambda \\ \tilde{P} &: \Lambda' \to -\Lambda', \end{aligned} \tag{3.48}$$

the overlap is not invariant under the individual reflections of each of the eigenvalue matrices. This is because the second term is odd under transformation by individual reflections of the matrices Λ and Λ'. Since each state must be individually invariant under this reflection, we choose to use the Harish-Chandra integral for $O(2N)$:

$$\mathcal{H}_{O(2N)} = \left(\prod_{p=1}^{N-1}(2p)!\right) \frac{\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{N}}{\Delta\left(\Lambda^{(2)}\right)\Delta\left(\bar{\Lambda}'^{(2)}\right)}. \tag{3.49}$$

This is precisely the matrix analogue of the norm of the coherent state for the positive parity states of a harmonic oscillator. The main difference between each of the orientifold projections is that the vacuum of each theory is charged differently under parity; the symplectic case formally begins at occupation number one of the parent theory, while the even orthogonal case begins at occupation number zero.

We can now repeat the analysis of the previous sections with $\det\left[\cosh\left(2\Lambda_j\bar{\Lambda}'_k\right)\right]_{j,k=1}^{N}$. We know that:

$$\cosh\left(2\Lambda\bar{\Lambda}'\right) = \sqrt{\pi\Lambda\bar{\Lambda}'}I_{-\frac{1}{2}}\left(2\Lambda\bar{\Lambda}'0\right) = \sum_{m=0}^{\infty}\frac{2^m}{m!\,(2m-1)!!}\left(\Lambda\bar{\Lambda}'\right)^{2m} \tag{3.50}$$

Applying the Cauchy-Binet formula yields:

$$\det\left[\cosh\left(2\Lambda_i\bar{\Lambda}'_j\right)\right] = \sum_{m_i}\prod_{i}^{N}\frac{2^{m_i}}{m_i!\,(2m_i-1)!!}\det\left[\Lambda_j^{2m_i}\right]\det\left[\bar{\Lambda}_j'^{2m_i}\right] \tag{3.51}$$

Then the Harish-Chandra integral for $O(2N)$ becomes:

$$\mathcal{H}_{O(2N)}(\Lambda,\Lambda') = \sum_{m_i}\prod_{i}^{N}\frac{2^{m_i}(2i-2)!}{m_i!\,(2m_i-1)!!}\frac{\det\left[\Lambda_j^{2m_i}\right]\det\left[\bar{\Lambda}'^{2m_i}_j\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j)} \tag{3.52}$$

By setting $m_i = \mu_i + N - i$, the expression once again becomes a sum over Schur polynomials:

$$\mathcal{H}_{O(2N)}(\Lambda,\Lambda') = \sum_{\mu}\frac{1}{h_\mu}\chi_\mu(\Lambda^2)\chi_\mu((\Lambda')^2), \tag{3.53}$$

where the coefficient is now given by:

$$h_\mu = \frac{(\mu_i+N-i)!\,(2\mu_i+2N-2i-1)!!}{2^{\mu_i+N-i}(2i-2)!}. \tag{3.54}$$

Once again, we can expand the operator itself as a formal sum:

$$\int_{O(N)} dO\exp\left(O\Lambda O^T a_Z^\dagger\right) = \sum_{\mu}\frac{1}{h_\mu}\chi_\mu(\Lambda^2)\chi_\mu((a_Z^\dagger)^2), \tag{3.55}$$

which implies that the norm of the states are given by h_μ.

We chose to get rid of the redundancy by integrating over $O(2N)$ rather than $SO(2N)$; in doing so, we have chosen a specific partition function. The drawback to choosing $O(2N)$ as our gauge group is that we eliminate the Pfaffian operator, which is defined as:

$$Pf(\Lambda)^2 = \det(\Lambda), \tag{3.56}$$

where Λ is a $2n \times 2n$ skew-symmetric matrix. If we make another choice and integrate over $SO(2N)$ instead, our Harish-Chandra integral becomes:

$$\mathcal{H}_{SO(2N)}(\Lambda, \Lambda') = \sum_{m_i} \prod_i^N \frac{2^{m_i}(2i-2)!}{m_i!\,(2m_i-1)!!} \frac{\det\left[\Lambda^{2m_i}\right]\det\left[\bar{\Lambda}'^{2m_i}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j)} + \sum_{n_i} \prod_i^N \frac{2^{n_i+1}\,(2i)!}{n_i!\,(2n_i+1)!!} \frac{\det\left[\Lambda^{2n_i+1}\right]\det\left[\bar{\Lambda}'^{2n_i+1}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)(\bar{\lambda}'^2_i-\bar{\lambda}'^2_j)} \tag{3.57}$$

We see that the Pfaffian of $SO(2N)$, which changes sign under a single reflection, makes an appearance in the term we previously discarded. If we write $\Lambda = X_j + iX_k$, where X_j and X_k are two of the six scalar fields X_i in the adjoint representation of $SO(2N)$ $\mathcal{N} = 4$ SYM, then $Pf(\Lambda)$ corresponds to a single BPS $D3$ brane wrapped around the non-trivial three-cycle of $\mathcal{RP}^5$ [2, 5]. It can be considered half of a maximal giant graviton, which is identified as $\det(\Lambda)$, since the maximal giant graviton wraps around the non-trivial cycle twice.

3.4 A change of basis

One approach to diagonalizing two-point functions is to build an orthogonal basis for $Sp(2N)$ and $SO(2N)$ using local operators, as done in [114, 115], which built on the restricted Schur polynomials introduced in [110]. This is achieved by introducing a tensor T in $V^{\otimes 2n}$ that has $2n$ indices and taking the sum over Wick contractions as a sum over permutations in $V^{\otimes 2n}$, or over $S_n[S_2]$. T is then decomposed into irreducible components that don't mix under S_{2n} when computing the two-point function. Operators are then built using projectors that commute with all of the permutations; it can be shown that these operators diagonalize the two-point function. Because these operators should be invariant in $SO(2N)$, their indices should contract in pairs. Each index corresponds to a box in the Young diagram R for a tensor in representation R. The Young diagrams that correspond to non-zero, gauge-invariant operators have an even number of boxes in each column and row, which is to say that $2n$ is divisible by 4, and that a square Young diagram composed of four boxes may be used a building block for the Young diagram R. Then the number of gauge invariant operators that can be built from n fields is the number of partitions of $n/2$. We now reproduce the formula for computing two-point functions in the operator basis defined in [114, 115] for both $Sp(2N)$ and $SO(2N)$:

$$\langle \mathcal{O}_R(Z)\bar{\mathcal{O}}_S(Z)\rangle = \delta_{RS}2^n\left(\frac{d_{R/4}}{d_R}\right)^2 \prod_{i\in\text{even boxes in }R} c_i, \tag{3.58}$$

where R, S are Young diagrams with $2n$ boxes; $R/4$ is a Young diagram with $n/2$ boxes that corresponds to the Young diagram R; $d_{R/4}, d_R$ are respectively the dimensions of the representations; and c_i is the factor $N + a - b$ assigned to each box, where a is the column index and b is the row index.

It is difficult to match our results exactly to that of [114, 115], given the difference in bases. Nevertheless, we may still observe a few similarities. A natural expectation is that the HCIZ integral for a particular group has an expansion in terms of the irreducible characters of the corresponding group. An argument for this would be as follows: first we consider the exponential of the trace $\mathrm{Tr}\left(g\Lambda g^{-1}a_Z^\dagger\right)$. We can then expand this exponential and exchange the order of the sum or integration to evaluate the Harish-Chandra integral as in the unitary case:

$$\mathcal{O}_\Lambda = \sum_{m=0}^{\infty}\frac{1}{m!}\int dg\mathrm{Tr}\left(g\Lambda g^{-1}a_Z^\dagger\right)^m. \tag{3.59}$$

We may try to express each term as a character of the corresponding group by taking traces of Eq. (3.42) or Eq. (3.43). Formally, this gives an expansion for the Harish-Chandra integral as a sum of infinitesimal characters evaluated on the Lie algebra. For

$U(N)$, this is not a problem, because the formulas for Schur polynomials make sense when evaluated on the Lie algebra. This does not seem to be the case for $Sp(2N)$ and $SO(N)$. Even then, one may try to make sense of this formal expansion in order to get a formula for the coefficients. If one extrapolates the answer for the unitary case, the expectation would be that the coefficients are ratios of dimensions of irreducible representatiations of the group and the corresponding centralizer algebra. This turns out to be partially true, since the coefficient associated to single row representations in Eq. (3.36) seems to agree precisely with the ratio of

$$c_\mu = \frac{2^m d^\mu}{m! D_\mu}, \tag{3.60}$$

where μ is a partition of m, d^μ is the dimension of the irreducible representation of the symmetric group S_m, and D_μ is the dimension of the corresponding symplectic group representation. This is clearly different from Eq. (3.58), but we note that the number of partitions of $n/2$ is the dimension of $S_{n/2}$, which is equivalent to $d_{R/4}$. Thus we have preserved the characteristic of the coefficient as a function of the ratio of the dimension of the irrep of the corresponding symmetric group to the dimension of the gauge group representation.

We now make the observation that since we perform the character expansion using Schur polynomials, which present as ratios of determinants, our basis is directly linked to free fermions; after all, the Schur functions correspond to free fermion wave functions [36, 11]. We return to the results of [115], where it is shown that the character of the local operator can be written in terms of a Schur polynomial of the matrix of the operator's eigenvalues. Thus the character of the operator has the interpretation of the Slater determinant of $N/2$ single particle wave functions, or $N/2$ fermions moving in an external harmonic oscillator potential. So we may conclude that our basis describes the same dynamics as the operator basis constructed in [114, 115].

3.5 Discussion

In this chapter, we extended the method of computing the norms of half BPS coherent states through localization [21] to theories with the gauge groups $Sp(2N)$, $SO(2N + 1)$, and $SO(2N)$. We did this by constructing coherent states averaged over a group orbit from each group and computing the norm of these states through the symplectic and special orthogonal Harish-Chandra integrals. The integration over the group may be viewed as a sort of path integral over the emergent world-volume gauge symmetry of a stack of N giant gravitons inside $AdS_5 \times \mathbb{RP}^5$; the norm of the state gives the effective action of this theory. Curiously enough, these types of integrals first appeared in models of induced QCD. By expanding the Harish-Chandra integrals, we found that each integral admits an expression as a sum of unitary characters. This matches what one would expect of an orientifold projection of a $U(2N)$ gauge theory; all the states that are spanned by the coherent states are "doubled" versions of those in the original theory. In particular, the coherent states considered here do not span the complete spectrum of the free $Sp(2N)$ and $SO(2N)$ theories. This is because the Harish-Chandra integral is only able to capture information from tensor contractions of the invariant tensors of the unitary group (meaning all products of traces). It is likely that some of the data corresponding to worldsheets with cross-caps is missing.

As in the unitary case, the coefficient associated with the characters in this series expansion computes the overlap of the corresponding Schur polynomials of the operators $(a)^i_j$ and $(a^\dagger)^i_j$. Our method should be contrasted to other constructions of basis of operators for the $Sp(2N)$ and $SO(2N)$ theories [114, 115], since our construction uses group theoretic objects more closely associated to each group. We conclude with some comments and an outline of open questions.

Connection to symplectic and orthogonal characters

A natural question is to ask is why Schur polynomials appear in the expansion for the symplectic Harish-Chandra integral, as opposed to sympletic Schur polynomials. If one tries to evaluate the symplectic Schur polynomials on a Cartan element of the Lie algebra in the most naïve way,

$$sp_\lambda(X) = \frac{\det\left[x_j^{\lambda_i - i+1} - \bar{x}_1^{\lambda_i - i+1}\right]}{\det\left[x_j^{N-j+1} - \bar{x}_1^{N-j+1}\right]}, \tag{3.61}$$

by replacing $\bar{x}_i$ with $-x_i$ instead of $1/x_i$, one obtains a suggestive formula:

$$sp_\lambda(X) sp_\lambda(Y) \sim \frac{\det\left[x_a^{\lambda_b+N-b+1}\right]}{\Delta(x_i^2)\prod_{c=1}^N x_c} \frac{\det\left[y_a^{\lambda_b+N-b+1}\right]}{\Delta(y_i^2)\prod_{c=1}^N y_c}. \tag{3.62}$$

This can be recognized as the terms in the expansion for the function:

$$\frac{\det\left(\sinh(x_i y_j)\right)}{\boldsymbol{\Delta}_{\mathfrak{sp}(\mathfrak{n})}(x)\boldsymbol{\Delta}_{\mathfrak{sp}(\mathfrak{n})}(y)} \sim \mathcal{H}_{Sp(2N)}(x,y). \tag{3.63}$$

The main difficulty with making this a precise equality comes from the fact that the denominator and numerator of Eq. (3.61) have zeros that need to cancel between each other, leaving an ambiguity for the normalization of the symplectic characters. Another issue is that different choices of representations appear to lead to the same polynomial. This is expected, since irreducible representations can appear with multiplicities in the decompostion of tensor products. However, by adding information from the centralizer algebra, one should be able to differentiate between irreducible representations. This additional data is precisely the $1/N$ corrections coming from cross-caps. This idea seems to suggest that there might be a refined version of the Harish-Chandra integral that takes into account the contributions from cross-cap states that are missing in the original integral. This would give an explicit connection between the representation theory of the Weyl group of $Sp(2N)$ [117], and the Brauer algebra [120].

Another connection between the symplectic Harish-Chandra integral and the symplectic characters comes from their generalizations to continuous Schur polynomials. A similar $\sinh[\lambda_j x_i]$ term makes an appearance in the continuous symplectic Schur function, which is defined in [124] as:

$$sp_{\Lambda}^{\text{cont}}(X) = \frac{\det\left[\sinh\left(\lambda_j x_i\right)\right]}{\prod_{1\leq i<j\leq N}\left(\lambda_i^2-\lambda_j^2\right)\prod_{i=1}^{N}\left(\lambda_i\right)}. \tag{3.64}$$

Notice that up to a factor of the discriminant of x_i, the continuous Schur functions agree with the symplectic Harish-Chandra integral. The continuous Schur function may then be written in the form of Eq. (3.36), where the determinant is folded into the coefficient f_μ. The presence of the Harish-Chandra integral implies that localization occurs in this calculation. An important point is that the continuous Schur functions are defined by a diferent integral formula in [124]:

$$sp_{\Lambda}^{cont}(X) = \int_{GT_{2N}(X)} \prod_{k=1}^{N} e^{\lambda_k(2|z_{2k-1}|-|z_{2k-2}|-|z_{2k}|)} dz_{i,j}, \tag{3.65}$$

where $GT_{2N}(\Lambda)$ is the set of all continuous Gelfand-Tsetlin patterns of shape Λ. Roughly, speaking, an integer point μ in this space can be associated to a Young diagram μ. The fact that this integral evaluates to what appears to be a sum of over integer points (Young diagrams) seems to suggest that there is some sort of localization in this space. Curiously enough, this integral is somewhat remniscient of a momentum space amplitude. For instance, taking λ to be big with λx held fixed, the value of the integral divided by the appropriate discriminant remains fixed, but the integration region shrinks to points where

$$2|z_{2k-1}| - |z_{2k-2}| - |z_{2k}| = 0. \tag{3.66}$$

Each of these points should correspond to a particular symplectic Schur polynomial . This is somewhat suggestive of some sort of worldsheet localization for a tensionless string [125, 126], where the integral over the worldsheet moduli space is expected to localize to a certain set of integer points.

Connection to quantum Schubert calculus

It is well known that the Schur symmetric functions are related to Schubert classes, which form an integer basis for the cohomology ring of the Grassmannian. The product of two Schubert classes may be expanded as a linear combination of Schubert classes summed over the given partitions ν [127]:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu. \tag{3.67}$$

The different Schubert classes are represented by $\sigma_\mu, \sigma_\nu, \sigma_\lambda$ and $c^\nu_{\lambda\mu}$ represent the Littlewood-Richardson coefficients. A well-known result is that every Schubert class can be associated with a Schur polynomial; the connection is seen by noting that the cohomology product mirrors the way the product of two Schur functions can be expanded as a linear combination of ordinary Schur functions [128]:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu. \tag{3.68}$$

This is directly related to the three point function of coherent states for the $U(N)$ theory. This is because after applying the Campbell-Hausdorff formula, one obtains an integral over a complex Grassmanian.

It should be noted that the skew Schur polynomial can be expanded in the same basis in a similar way [128]:

$$s_{\lambda/\mu} = \sum_{T \in SSYT(\lambda/\mu)} \mathbf{x}^{\mathbf{T}} = \sum_{\nu} \mathbf{c}^{\lambda}_{\mu\nu} \mathbf{s}_{\nu}, \tag{3.69}$$

where $SSYT$ refers to the skew Schur Young tableaux. We note that the even symplectic Schur functions we touch upon in the previous subsection can be written as a sum of skew Schur functions summed over Frobenius coordinates.

We are interested in computing the Gromov-Witten invariants, which appear in the *quantum* product of two Schubert classes, which is defined on the small quantum cohomology ring of the Grassmannian, $QH^*(Gr_{kn})$. $QH^*(Gr_{kn})$ is defined as the tensor product of the cohomology ring of the Grassmannian and the polynomial ring $\mathbb{Z}[q]$, where q is a variable of degree n. The quantum product of two Schubert classes, then, is defined in [127] as:

$$\sigma_\lambda * \sigma_\mu = \sum_{d,\nu} q^d C^{\nu,d}_{\lambda\mu} \sigma_\nu, \tag{3.70}$$

where d is a non-negative integer such that $|\nu| = |\lambda| + |\mu| - dn$, and $C^{\nu,d}_{\lambda\mu}$ are the Gromov-Witten invariants. Toric Schur functions are defined in [127] to correspond to cylindric diagrams of shape $\lambda[r]/\mu[s]$, which are defined as finite subsets of $\mathcal{C}_{nk} = \mathbb{Z}^2/(-k, n-k)\mathbb{Z}$. We label these toric Schur functions with the shape $\lambda/d/\mu$, where $d = r - s$. Since skew Schur functions are toric Schur functions when $d = 0$, it should not come as a surprise that toric Schur functions may be expanded in the Schur basis just as the former are in [127]:

$$s_{\lambda/d/\mu} = \sum_{\nu} C^{\lambda,d}_{\mu\nu} s_\nu \tag{3.71}$$

The main difference is that the Gromov-Witten invariants have replaced the classical Littlewood-Richardson coefficients. This replacement should correspond to replacing a centralizer algebra by a Hecke algebra. Thus once again, it should not come as a surprise that the Gromov-Witten invariants can be given as an alternating sum of classical Littlewood-Richardson coefficients, as demonstrated in [129]. The problem with this approach is that there are too many Littlewood-Richardson coefficients to keep track of, which makes the computation unwieldy. It is known that Gromov-Witten invariants in other contexts may be computed through localization [130, 131]. It would be interesting if the Harish-Chandra integral can be extended to the toric Schur functions to obtain a combinatorial formula for the Gromov-Witten invariants. This may also shed light on the problem of computing the normalization of three-point functions, where the Gromov-Witten invariants appear as coefficients.

Chapter 4

Multi-matrix correlators and localization

4.1 Introduction

Large operators in large N gauge theories are an important subject of study with relevance to nuclear physics, theories of quantum gravity and strings. Although there has been enormous success in computing the spectrum of anomalous dimensions of light operators in models such as maximally supersymmetric Yang-Mills theory in the planar limit, very little is known about how to tackle generic operators whose dimensions can scale with a power of N. This is an interesting problem for holography [1] and for understanding the structure of conformal field theories more generally [132]. One of the difficulties one faces when trying to address these types of problems is that the intuitions from the planar limit are often unjustified for large operators; one must sum over both planar and non-planar diagrams and it is not a priori clear which diagrams dominate in the large N limit. A promising approach is to replace single and multi-trace operators with a different basis that is better behaved at finite N [36], and then perform a systematic expansion around protected states in the large N limit. In the case of maximally supersymmetric Yang-Mills theory, this has been implemented at finite N [50, 133, 15, 134]. Even though the expressions found through these techniques at finite N are quite explicit, it is usually difficult to take the large N limit of such quantities.

More recently, there have been works showing that certain generating functions can be used to perform computations in the free-field theory limit [66, 21, 106, 22, 135]. This technique has been succesfully implemented in the computation of three-point correlators involving large operators made out of a single matrix field [69, 136, 24], as in the half-BPS sector of $\mathcal{N}=4$ SYM [12, 55], where the dual gravitational description is explicitly realized from the gauge theory. An explicit mapping between BPS states made out of more than one matrix and asymptotically $AdS_5 \times S^5$ geometries is still lacking, though a compelling description in terms of *bubbling geometries* seems to exist [14, 137, 138]. The study of generating functions for multi-matrix correlators was outlined in [106, 139] for certain classes of operators, and more generally in [21]. Our goal is to elucidate some of the details regarding the generating functions of $\frac{1}{4}$ and $\frac{1}{8}$- BPS operators in $\mathcal{N}=4$ SYM. We do this by proposing a fixed-point formula for the overlap of generic coherent state generating functions; this gives us an integral formula that generalizes the Harish-Chandra-Itzykson-Zuber (HCIZ) formula to multiple pairs of matrices. Integrals of this type appear naturally in the study of multi-matrix models of commuting random matrices.

This chapter is structured as follows. In section 4.2, we review the generating function techniques, focusing on the case of $\frac{1}{4}$ BPS operators in $\mathcal{N}=4$ SYM. We argue that the form of these operators is protected, so we can restrict to eigenstates of the one-loop dilatation operator. We then evaluate the norm of the generating function for the $U(2)$ theory by explicit integration to motivate our fixed point formula for general N. Finally, we give a prescription for extending the HCIZ formula to the multiple-matrix model using the heat kernel method as outlined in [140]; we will discuss our results for $U(2)$ and $U(3)$ and the insights we may glean from them to extrapolate a general formula for $U(N)$. In section 4.3, we connect our results to the construction of restricted Schur polynomials and outline how to generalize to operators associated with Young diagrams with arbitrary number of rows or columns. We will briefly discuss our attempts to arrive at a general formula via the character expansion method. Finally, we conclude with some future directions.

4.2 Multi-matrix Generating Functions

We are interested in studying operators in gauge theories that are made out of more than one matrix-valued scalar field. In particular, we will work with $\frac{1}{4}$-BPS operators in $U(N)$ $\mathcal{N} = 4$ SYM on the cylinder $\mathbb{R} \times S^3$. At weak coupling, these operators can be built out of symmetrized products of two of the three complex scalar fields of the theory X, Y. Generalizing to more than two matrices is straightforward. This class of operators transforms in the $[p, q, p]$ representations of the $SU(4)_R$ symmetry, and the operators are generically of multi-trace form. We will concentrate on scalar primary states at an equal time slice for simplicity. Unlike $\frac{1}{2}$-BPS operators, which can be built explicitly in the free theory, $\frac{1}{4}$-BPS operators of the interacting theory are different from those of the free theory. The lifting of states due to non zero gauge coupling can be treated pertubatively and the loop corrections to dilatation operators annihilate operators that are made out of symmetric products of X and Y. This problem was studied in detail for small operators in [141], but for generic large operators, explicit constructions in terms multi-traces are cumbersome. An alternative expansion in terms of characters was introduced in [50], which the authors call the *restricted Schur polynomial basis*. This basis is convenient for dealing with the mixing between the different trace structures since it diagonalizes the matrix of two point functions for all values of N.

4.2.1 Generating $\frac{1}{4}$ BPS States

Yet another way of generating $\frac{1}{4}$-BPS states can be found by studying operators of the form:

$$|\Lambda_X, \Lambda_Y\rangle = \frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \; \exp\left(\mathrm{Tr}\left[UXU^\dagger\Lambda_X + UYU^\dagger\Lambda_Y\right]\right)|0\rangle\,. \tag{4.1}$$

If we insist that the coherent state parameters Λ_X and Λ_Y commute, $|\Lambda_X, \Lambda_Y\rangle$ is annihilated by the one-loop dilatation operator; it was shown in [135] that this persists to two-loop order. In [142], it was conjectured that the space of BPS states in $\mathcal{N}=4$ SYM is given by the kernel of the one-loop dilatation operator at all values of the coupling; we will take this as a working assumption and work with the set of states annihilated by the Beisert one-loop dilatation operator:

$$\hat{D}_2^{SU(2)} = g^2 \mathrm{Tr}\left[[X, Y][\partial_X, \partial_Y]\right]. \tag{4.2}$$

Because the states (4.1) are coherent states of $\bar{X}, \bar{Y}$ [21], they form an overcomplete basis of states for any value of N. This has many computational advantages, mostly due to the fact that taking the large N limit is very straightforward, but translating back into a complete orthogonal basis of operators can be complicated. This may be solved by computing the norm of the coherent states. By exploiting the Campbell-Hausdorff formula, we arrive at an integral of the form:

$$\langle\bar{\Lambda}_X, \bar{\Lambda}_Y|\,|\Lambda_X, \Lambda_Y\rangle = \frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \; \exp\left(\mathrm{Tr}\left[U\bar{\Lambda}_X U^\dagger\Lambda_X + U\bar{\Lambda}_Y U^\dagger\Lambda_Y\right]\right). \tag{4.3}$$

Since we can in principle expand (4.1) in terms of an orthonormal basis, we may use this overlap to determine the coefficients relating the multi-trace basis of operators to an orthogonal basis by expanding in a series and matching the coefficients as done in [21]. The precise tool relating the multi-trace basis operators and the character expansion in this case is the Weingarten calculus [143]; an example illustrating this technique can be found in [144]. The main obstacle we face is evaluating the integral (4.3) for generic coherent state parameters. To our knowledge, these types of integrals have not been studied before, and a closed form expression for them is needed. Our main goal will be to evaluate this class of integrals for any value of N. Although we only explicitly study the case of $U(N)$ integrals, the methods should apply generally and should generalize to $SO(N)$ and $Sp(N)$ groups as well as to quivers. These types of integrals are also a natural object to study in the context of matrix models, since they arise in the study of multi-matrix models of commuting matrices.

4.2.2 The Four-Matrix Model in $SU(2)$

Before proceeding to the case of general N, we will study the following integral

$$I_2 = \int_{SU(2)} dU \; e^{\mathrm{Tr}\left[UAU^\dagger \bar{A} + UBU^\dagger \bar{B}\right]} \tag{4.4}$$

for commuting matrices $A, B, \bar{A}, \bar{B}$. We will first approximate I_2 by a saddle point approximation; the critical points of the function in the exponential are given by the solutions to the equations

$$[A, U^\dagger \bar{A} U] + [B, U^\dagger \bar{B} U] = 0. \tag{4.5}$$

For generic enough matrices, this is only satisfied if each of the two terms vanishes individually

$$[A, U^\dagger \bar{A} U] = [B, U^\dagger \bar{B} U] = 0. \tag{4.6}$$

The only problematic cases occur when a subset of the eigenvalues of B is a permutation of a subset of eigenvalues of $-A$. From here on, we assume that the eigenvalues are generic enough that this does not happen. This means that, generically, the saddle points are labelled by permutation matrices U_π. We are then left with a Gaussian integral around each of the saddle points, which can be evaluated easily; this results in a "one-loop determinant" factor given by:

$$D_2(a, \bar{a}, b, \bar{b}) = (a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_2)\left(\bar{b}_1 - \bar{b}_2\right) \tag{4.7}$$

This gives an approximate value for the integral (up to a convention dependent normalization factor):

$$I_2 \simeq \frac{e^{a_1\bar{a}_1 + a_2\bar{a}_2 + b_1\bar{b}_1 + b_2\bar{b}_2} - e^{a_1\bar{a}_2 + a_2\bar{a}_1 + b_1\bar{b}_2 + b_2\bar{b}_1}}{(a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_2)\left(\bar{b}_1 - \bar{b}_2\right)}. \tag{4.8}$$

At first sight, it is not clear that this approximation is reliable, since there is no large parameter in the exponential. To gain more intuition, we evaluate I_2 through an explicit computation.

First, we must parameterize our unitary matrix U; then, we need to compute the

Haar measure. We start with the following matrices:

$$\begin{aligned} A &= \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \ B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \\ \bar{A} &= \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{pmatrix} \end{aligned} \tag{4.9}$$

We then seek to parametrize our unitary matrix. We know that any arbitrary $SU(2)$ matrix must meet the following conditions:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in \mathbb{C}^{2\times 2} \ \Big| \ |a|^2 + |b|^2 = 1 \right\} \tag{4.10}$$

For ease of computation, we choose to parameterize U with Euler angles:

$$U = \begin{pmatrix} e^{-i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} & -e^{i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} \\ e^{-i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} & e^{i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{4.11}$$

We seek to rewrite the Haar measure dU in terms of $J(\theta,\gamma,\alpha)d\theta d\gamma d\alpha$, where $J(\theta,\gamma,\alpha)$ is the Jacobian. We may do so by computing the inverse of the unitary matrix and multiplying it by its partial derivatives with respect to the Euler angles. We start by finding the inverse of U:

$$U^{-1} = \begin{pmatrix} e^{i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} & e^{i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} \\ -e^{-i\frac{\gamma-\alpha}{2}} \sin\frac{\theta}{2} & e^{-i\frac{\gamma+\alpha}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{4.12}$$

Then we calculate the partial derivatives with respect to γ, α, and θ and multiply by the inverse. We obtain:

$$\begin{aligned}
U^{-1}\frac{\partial U}{\partial \gamma} &= \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix} \\
U^{-1}\frac{\partial U}{\partial \alpha} &= \begin{pmatrix} -\frac{i}{2}\cos\theta & \frac{i}{2}e^{i\gamma}\sin\theta \\ \frac{i}{2}e^{-i\gamma}\sin\theta & \frac{i}{2}\cos\theta \end{pmatrix} \\
U^{-1}\frac{\partial U}{\partial \theta} &= \begin{pmatrix} 0 & -\frac{1}{2}e^{i\gamma} \\ \frac{1}{2}e^{-i\gamma} & 0 \end{pmatrix}
\end{aligned} \tag{4.13}$$

We calculate the Jacobian matrix using the following basis $\epsilon_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\epsilon_2 = \begin{pmatrix} 0 & ie^{i\gamma} \\ ie^{-i\gamma} & 0 \end{pmatrix}$, and $\epsilon_3 = \begin{pmatrix} 0 & -e^{i\gamma} \\ e^{-i\gamma} & 0 \end{pmatrix}$:

$$\mathcal{J} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\cos\theta & 0 \\ 0 & \frac{1}{2}\sin\theta & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{4.14}$$

The Jacobian $J(\theta, \gamma, \alpha)$ we seek is the determinant of $\mathcal{J}$:

$$\det(J) = \frac{1}{8}|\sin\theta| \tag{4.15}$$

We see that it is only dependent on θ. Our integral becomes:

$$\begin{aligned}
I_2 &= \frac{1}{8}\int_0^{\pi} d\theta \int_0^{4\pi} \frac{d\gamma}{4\pi} \int_0^{2\pi} \frac{d\alpha}{2\pi} |\sin\theta| e^{\mathrm{Tr}\left[\bar{A}UAU^\dagger + \bar{B}UBU^\dagger\right]} \\
&= \frac{1}{8}\int_0^{\pi} d\theta |\sin\theta| e^{\frac{1}{2}\left((a_1+a_2)(\bar{a}_1+\bar{a}_2)+(b_1+b_2)\left(\bar{b}_1+\bar{b}_2\right)+\left((a_1-a_2)(\bar{a}_1-\bar{a}_2)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)\cos\theta\right)}
\end{aligned} \tag{4.16}$$

Our critical points are $\theta = 0$ and $\theta = \pi$, so we can remove the absolute value bars. Then we evaluate our integral:

$$\begin{aligned} I_2 &= \frac{1}{8}\int_0^{\pi} d\theta \sin\theta e^{\frac{1}{2}\left((a_1+a_2)(\bar{a}_1+\bar{a}_2)+(b_1+b_2)\left(\bar{b}_1+\bar{b}_2\right)+\left((a_1-a_2)(\bar{a}_1-\bar{a}_2)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)\cos\theta\right)} \\ &= \frac{e^{a_1\bar{a}_1+a_2\bar{a}_2+b_1\bar{b}_1+b_2\bar{b}_2} - e^{\bar{a}_1 a_2+a_1\bar{a}_2+\bar{b}_1 b_2+b_1\bar{b}_2}}{4\left((a_1-a_2)(\bar{a}_1-\bar{a}_2)+(b_1-b_2)\left(\bar{b}_1-\bar{b}_2\right)\right)} \end{aligned} \tag{4.17}$$

This is precisely the same result that the saddle point approximation yields. From the intermediate steps, it is clear that there are never any terms that mix the eigenvalues of A and B; if we set either $A = 0$ or $B = 0$, we immediately recover the HCIZ formula for $U(2)$.

4.2.3 Proof of Localization for $U(2)$

One important issue to understand is why the integral I_2 has an exact saddle point approximation, while the naive saddle point approximation for I_N fails to be exact for $N > 2$. The idea is to try to follow the proof [140] for the HCIZ integral and see exactly how the analysis differs for multi-matrix models. One key observation is that the integral $I_N(A, B, \bar{A}, \bar{B})$ is an eigenfunction of a holomorphic Laplacian:

$$-\left[\frac{\partial}{\partial A_{ij}}\frac{\partial}{\partial A_{ji}} + \frac{\partial}{\partial B_{ij}}\frac{\partial}{\partial B_{ji}}\right] I_N(A,B,\bar{A},\bar{B}) = \mathrm{Tr}[\bar{A}^2+\bar{B}^2] I_N(A,B,\bar{A},\bar{B}). \tag{4.18}$$

Before continuing, it is worthwile to explain what we mean by holomorphic in this context, and why this is important. One way in which the integral I_N appears is as the Jacobian factor for a Gaussian matrix integral over a pair of commuting normal matrices

$$\begin{aligned} \mathcal{Z} &= \int_{[A,\bar{A}]=0} [dAd\bar{A}] \int_{[B,\bar{B}]=0} [dBd\bar{B}]\ \exp\left\{\mathrm{Tr}[A\bar{A}] + \mathrm{Tr}[B\bar{B}]\right\} \delta\big([A,B]\big) \\ &= \int d\mu(a,\bar{a},b,\bar{b})\ I_N(A,B,\bar{A},\bar{B}) \end{aligned} \tag{4.19}$$

As it stands, this expression is formal unless we specify a contour of integration for the eigenvalues of $A, \bar{A}$ and $B, \bar{B}$. A choice of contour corresponds to a choice of polarization in the space of eigenvalues; this makes the eigenvalues of A and $\bar{A}$ canonically conjugate. This is quite natural from the interpretation of the integral as the norm of a coherent state in matrix quantum mechanics, where the collective coordinates a_i and b_i are holomorphic phase space coordinates. The barred coordinates are then conjugate momentum variables. Thus, the correct Laplacian operator has to be constructed from the metric of the space of commuting normal matrices. This is exactly the quantization discussed in [12].

We rescale the matrices by a constant factor t; the resulting equation implies that the integral is related to a holomorphic heat kernel on the space of commuting matrices:

$$K_t(a, \bar{a}, b, \bar{b}) = t^{-N} \int dU \exp\left\{ -\frac{1}{t} \mathrm{Tr}[UAU^\dagger \bar{A} + UBU^\dagger \bar{B}] \right\} \tag{4.20}$$

As we take t to zero, the integral will be very well approximated by the saddle point approximation. We see that the integral approaches a delta function; we can use the kernel itself to propagate this initial condition to a finite t. This would imply that the integral comes from a sum over the real saddle points of the integral. If the kernel is a plane wave, then the integral localizes, which is to say that the steepest descent contour gives exactly a Gaussian integral centered around each saddle point. This occurs if the heat equation for the kernel corresponds to a Schrödinger equation for an integrable system, since we can in principle change the variables into a set of action-angle variables, where the wavefunction is a plane wave. Whenever the kernel cannot be written this way, true localization fails, and instead, the integral is given by a sum over thimbles, with the kernel giving a parametrization of the integration contour.

Now we review some coordinate transformations for the Laplacian in the space of normal matrices. Given that the squared distance of two $n \times n$ normal matrices A and A' is $d(A, A') = \mathrm{Tr}|A - A'|^2$ and invariant under unitary transformations $A \to UAU^\dagger$, the metric is:

$$ds^2 = \sum_{ij} |dA_{ij}|^2. \tag{4.21}$$

We know that the Laplace-Beltrami operator is:

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_i \left(g^{ij} \sqrt{g} \partial_j \right) \tag{4.22}$$

We now perform a coordinate transformation $A = U\Omega a U^\dagger$ to rewrite the matrix in terms of its n eigenvalues a_i and $n(n-1)$ angular variables θ_α. Setting $dH = iU^\dagger dU$, we may rewrite our invariant distance as:

$$\begin{aligned} ds^2 &= g_{i\bar{j}} da_i d\bar{a}_j + g_{\alpha\beta} d\theta^\alpha d\theta^\beta \\ &= \sum_i |da_i|^2 + \sum_{i,j} |a_i - a_j|^2 dH_{ij} dH_{ji}, \end{aligned} \tag{4.23}$$

where we have defined:

$$g_{\alpha\beta} = 2 \sum_{i<j} |a_i - a_j|^2 \mathrm{Re}(\partial_\alpha H_{ij})(\partial_\beta H^*_{ij}) = (VDV^\dagger)_{\alpha\beta}, \tag{4.24}$$

where $V_{\alpha,ij} = \partial_\alpha H_{ij}$ and D is a diagonal matrix with elements $|a_i - a_j|^2$. The square root of the metric tensor's determinant is then:

$$\sqrt{g} = |\Delta|^4 |\det V|, \tag{4.25}$$

where Δ is the Vandermonde determinant of the eigenvalues a_i. Because the new metric tensor is block diagonal with an eigenvalue sector and a unitary sector, its inverse is block diagonal as well with an eigenvalue sector and a unitary sector, and thus the Laplacian may be separated into two operators, one for each sector. In its entirety, the Laplacian is:

$$\nabla_A^2 = \frac{1}{|\Delta|^4}\sum_i \frac{\partial}{\partial a_i}|\Delta|^4\frac{\partial}{\partial a_i} + \frac{1}{|\Delta|^2}\frac{1}{\det|V|}\sum_\alpha \frac{\partial}{\partial \theta_\alpha}(V^{-1})^*_{\alpha,ij}|\det V|\sum_\beta (V^{-1})_{ij,\beta}\frac{\partial}{\partial \theta_\beta} \tag{4.26}$$

We now consider the space of two $N \times N$ commuting normal matrices A and B. After diagonalization the metric for this space becomes:

$$ds^2 = |da_i|^2 + |db_i|^2 + \sum_{i,j}\left(|a_i - a_j|^2 + |b_i - b_j|^2\right) dH_{ij}dH_{ji}, \tag{4.27}$$

Then the square root of the metric tensor's determinant becomes:

$$\sqrt{g} = \prod_{i<j}\left(|a_i - a_j|^2 + |b_i - b_j|^2\right)^2 |\det V| = \mu^2|\det V|, \tag{4.28}$$

and we rewrite $\prod_{i<j}(|a_i - a_j|^2 + |b_i - b_j|^2)$ as μ.

We may rewrite the holomorphic Laplacian as [12]:

$$\begin{aligned}\nabla_{A,B}^2 = &\frac{1}{\mu^2}\left[\sum_k \frac{\partial}{\partial a_k}\mu^2\frac{\partial}{\partial a_k} + \sum_k \frac{\partial}{\partial b_k}\mu^2\frac{\partial}{\partial b_k}\right] \\ &+ \sum_{i<j}\frac{1}{\mu^2}\frac{1}{\det|V|}\sum_\alpha \frac{\partial}{\partial \theta_\alpha}(V^{-1})^*_{\alpha,ij}|\det V|\sum_\beta (V^{-1})_{ij,\beta}\frac{\partial}{\partial \theta_\beta}\end{aligned} \tag{4.29}$$

Notice that all of the eigenvalue dependance is on the first two terms and because the integral averages over the angular variables of A and B it is annihilated by the last term, so we will omit it from now on. So now our problem is reduced to finding eigenfunctions for this operator. As is common in matrix quantum mechanics, one can often reabsorb the measure factor μ into the definition of the eigenfuntion, so we will express $I_N(A, B, \bar{A}, \bar{B})$

in terms of an auxiliary function $\Psi_N(A, B, \bar{A}, \bar{B})$:

$$I_N(A, B, \bar{A}, \bar{B}) = \mu \, \Psi_N(A, B, \bar{A}, \bar{B}). \tag{4.30}$$

After this rescaling, the Laplacian operator becomes a sum of two terms, one being the flat space Laplacian and the other an effective potential:

$$\nabla^2_{A,B} \, I_N(A, B, \bar{A}, \bar{B}) = \frac{1}{\mu} \left[\nabla_a^2 + \nabla_b^2\right] \Psi_N - \frac{1}{\mu^2} \left(\left[\nabla_a^2 + \nabla_b^2\right] \mu\right) \Psi_N = \frac{\lambda \, \Psi_N}{\mu}. \tag{4.31}$$

So far our discussion applies to general rank of matrices. Focusing on $N = 2$, we can easily check that the potential term vanishes. This is because μ is linear in a_i and b_i. In this case, the problem reduces to finding eigenfunctions for the Laplace operator in flat space:

$$\left[\nabla_a^2 + \nabla_b^2\right] \Psi_2 = \lambda \Psi_2. \tag{4.32}$$

The solutions to this equation are plane waves:

$$\Psi_2 \sim \prod_i e^{a_i \bar{a}_i + b_i \bar{b}_i}. \tag{4.33}$$

This ansatz does not respect the symmetry properties of the integral under simultaneous permutations of a_i and b_i, so the correct solution is a symmetrized sum of plane waves:

$$\Psi_2 = \mathcal{C}_2 \, \frac{1}{2!} \sum_{\pi \in S_2} (-1)^\pi \prod_i e^{a_i \bar{a}_{\pi(i)} + b_i \bar{b}_{\pi(i)}}. \tag{4.34}$$

After dividing by the measure factor, we reproduce the expected answer. At this point, it becomes clear that the heat kernel proof works for the $SU(2)$ integral, since the kernel is Gaussian and the saddle point approximation as $t \to 0$ can be propagated forward to obtain the integral for finite t. Intuitively, we should be able to localize the integral much like the single matrix case, because the wavefunction Ψ_2 is an eigenfuction of an integrable (free) Hamiltonian. In the case where $B = \bar{B} = 0$, the measure factor μ reduces to a Vandermonde determinant, which is also annihilated by the flat space Laplacian; thus we see that the usual HCIZ integral is associated with a wavefunction of a free fermion or boson. But this is not the case for $B \neq 0$ and $N > 2$, since the potential term does not vanish. Note that this does not necessarily mean that the integral is not localizable. An example that comes to mind are the integrals of the Harish-Chandra type for the symplectic groups, which are associated with the wavefunctions of integrable Calogero models. While these integrals are known to localize by the heat kernel methods [30], in this case, it was noted that the naive localization argument nevertheless still fails [145], and that one must include additional instanton solutions to the WKB approximation.

Returning to our guess for Ψ_N, it becomes clear that a closed form for $I_N(A, B, \bar{A}, \bar{B})$ must include μ as its denominator. This is because μ is the natural integration measure for the eigenvalues a_i, b_i. The fact that the eigenvalue is $\mathrm{Tr}[\bar{A}^2 + \bar{B}^2]$ also suggests that the denominator should have an exponential factor:

$$\mu\, I_N(A, B, \bar{A}, \bar{B}) = \Psi_N \sim \sum_{\pi \in S_N} c_\pi \prod_i e^{a_i \bar{a}_{\pi(i)} + b_i \bar{b}_{\pi(i)}}\ \chi(a_i, b_i, \bar{a}_{\pi(i)}, \bar{b}_{\pi(i)}). \tag{4.35}$$

By a symmetry argument it is also plausible that the numerator is also given by a determinant. We seek the missing factor in the numerator; we know that the Bethe ansatz is unlikely to provide a solution to our differential equation, because our effective potential appears to contain three-body interactions. Finding such a formula would amount to finding eigenfunctions of $\nabla_{A,B}$ along the lines of [12, 146], but in our case we are only interested in the ground state wavefunction in the effective potential. While a complete analysis is beyond the scope of this chapter, we may still lay out a prescription for finding an analytical solution to our modified integral. We previously argued that such a solution must have μ as its denominator; following [12], the problem can be simplified by rewriting the equation for I_N and removing the effective potential in the Laplacian at the cost of adding first order derivative terms. Thus we may rewrite (4.35) as:

$$\Psi_N = \sum_{\pi \in S_N} c_\pi \prod_i e^{a_i \bar{a}_{\pi(i)} + b_i \bar{b}_{\pi(i)}} \ \xi(a_i, b_i, \bar{a}_{\pi(i)}, \bar{b}_{\pi(i)})\mu = \sum_{\pi \in S_N} f_\pi \xi_\pi \mu, \tag{4.36}$$

where we have set $f_\pi = c_\pi \prod_i e^{a_i \bar{a}_{\pi(i)} + b_i \bar{b}_{\pi(i)}}$. We see then that we have:

$$\begin{aligned}
\lambda \Psi_N &= \left[\nabla_a^2 + \nabla_b^2\right] \Psi_N - \frac{1}{\mu}\left(\left[\nabla_a^2 + \nabla_b^2\right]\mu\right)\Psi_N \\
&= \nabla_a \cdot \nabla_a \left(\sum_{\pi \in S_N} f_\pi \xi_\pi \mu\right) + \nabla_b \cdot \nabla_b \left(\sum_{\pi \in S_N} f_\pi \xi_\pi \mu\right) \\
&- \frac{1}{\mu}\left(\left[\nabla_a^2 + \nabla_b^2\right]\mu\right)\left(\sum_{\pi \in S_N} f_\pi \xi_\pi \mu\right) \\
&= \nabla_a \cdot \left(\mu \sum_{\pi \in S_N} \nabla_a \left(f_\pi \xi_\pi\right) + \nabla_a(\mu) \sum_{\pi \in S_N} \left(f_\pi \xi_\pi\right)\right) \\
&+ \nabla_b \cdot \left(\mu \sum_{\pi \in S_N} \nabla_b \left(f_\pi \xi_\pi\right) + \nabla_b(\mu) \sum_{\pi \in S_N} \left(f_\pi \xi_\pi\right)\right) \\
&- \left(\left[\nabla_a^2 + \nabla_b^2\right]\mu\right)\left(\sum_{\pi \in S_N} f_\pi \xi_\pi\right)
\end{aligned} \tag{4.37}$$

Expanding, we arrive at:

$$
\begin{aligned}
\lambda\Psi_N &= 2\nabla_a(\mu)\cdot\left(\sum_{\pi\in S_N}\nabla_a(f_\pi\xi_\pi)\right)+\mu\sum_{\pi\in S_N}\nabla_a^2(f_\pi\xi_\pi)+\nabla_a^2(\mu)\sum_{\pi\in S_N}(f_\pi\xi_\pi)\\
&+2\nabla_b(\mu)\cdot\left(\sum_{\pi\in S_N}\nabla_b(f_\pi\xi_\pi)\right)+\mu\sum_{\pi\in S_N}\nabla_b^2(f_\pi\xi_\pi)+\nabla_b^2(\mu)\sum_{\pi\in S_N}(f_\pi\xi_\pi)\\
&-\left(\left[\nabla_a^2+\nabla_b^2\right]\mu\right)\left(\sum_{\pi\in S_N}f_\pi\xi_\pi\right)\\
&=2\nabla_a(\mu)\cdot\left(\sum_{\pi\in S_N}\nabla_a(f_\pi\xi_\pi)\right)+\mu\sum_{\pi\in S_N}\nabla_a^2(f_\pi\xi_\pi)+2\nabla_b(\mu)\cdot\left(\sum_{\pi\in S_N}\nabla_b(f_\pi\xi_\pi)\right)\\
&+\mu\sum_{\pi\in S_N}\nabla_b^2(f_\pi\xi_\pi)
\end{aligned}
\tag{4.38}
$$

We compute the terms containing the second derivatives and find:

$$
\begin{aligned}
\lambda\Psi_N &= 2\sum_{\pi\in S_N}\left(f_\pi\nabla_a\mu\cdot\nabla_a\xi_\pi+\xi_\pi\nabla_a\mu\cdot\nabla_a f_\pi+f_\pi\nabla_b\mu\cdot\nabla_b\xi_\pi+\xi_\pi\nabla_b\mu\cdot\nabla_b f_\pi\right)\\
&+\mu\sum_{\pi\in S_N}\left(f_\pi\nabla_a^2\xi_\pi+f_\pi\nabla_b^2\xi_\pi+\xi_\pi\nabla_a^2 f_\pi+\xi_\pi\nabla_b^2 f_\pi+2\nabla_a\xi_\pi\cdot\nabla_a f_\pi+2\nabla_b\xi_\pi\cdot\nabla_b f_\pi\right)
\end{aligned}
\tag{4.39}
$$

We know that $\mu\sum_{\pi\in S_N}\xi_\pi\nabla_a^2 f_\pi+\xi_\pi\nabla_b^2 f_\pi=\lambda\Psi_N$. Thus we can simplify our differential equation:

$$
\begin{aligned}
0 &= 2\sum_{\pi\in S_N}\left(f_\pi\nabla_a\mu\cdot\nabla_a\xi_\pi+\xi_\pi\nabla_a\mu\cdot\nabla_a f_\pi+f_\pi\nabla_b\mu\cdot\nabla_b\xi_\pi+\xi_\pi\nabla_b\mu\cdot\nabla_b f_\pi\right)\\
&+\mu\sum_{\pi\in S_N}\left(f_\pi\nabla_a^2\xi_\pi+f_\pi\nabla_b^2\xi_\pi+2\nabla_a\xi_\pi\cdot\nabla_a f_\pi+2\nabla_b\xi_\pi\cdot\nabla_b f_\pi\right)
\end{aligned}
\tag{4.40}
$$

We see then that we have a multivariable PDE; while there are numerical methods to approximate an analytical solution, they are all computationally laborious and ill-suited to solving PDEs containing more than two independent variables. Nevertheless, given the nature of the HCIZ integral, we note that there is another method to compute an analytical expression for the missing factor that, while tedious, is still tractable. That is to evaluate the integral explicitly and use this to determine the missing factors. We solve the integral for $U(3)$ in Appendix A.1. We will reproduce the result directly below:

$$I_3 = \sum_{q=0}^{\infty} \sum_{\substack{f+2p+4k=q \\ f,p,k\geq 0}} \sum_{m,n=0}^{k} \sum_{\substack{j+l=n \\ 0\leq j\leq m \\ 0\leq l\leq k-m}} \sum_{g=0}^{p+k} \sum_{\substack{r+s+t+h=g \\ 0\leq r\leq p \\ 0\leq s\leq j \\ 0\leq t\leq m-j \\ 0\leq h\leq k-m}} 16\pi e^{w_1} \times \frac{1}{(k+1)(k+2)(p+4k-2m-2n+2)(q-2m-2g+2)} \times \frac{(-1)^{-l-h} w_2^f w_3^r w_4^{p-r} w_5^s w_6^{j-s} w_7^t w_8^{m-j-t} w_9^{2(k-m)}}{2^{2(k-m)+1} f!(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!} \tag{4.41}$$

The expressions for w_i are listed in (A.29); we see immediately that the a_i and $\bar{a}_j$ eigenvalues do not mix with the b_i and $\bar{b}_j$ eigenvalues. We make note of the $SU(2)$ symmetry between a_i and b_i, and $\bar{a}_j$ and $\bar{b}_j$; we note that if we remove the b_i and $\bar{b}_j$, we recover the expression for the two-matrix Harish-Chandra integral evaluated over the $U(3)$ group. Given that we may remove b_i and $\bar{b}_j$, replace a_i and $\bar{a}_j$ with $u_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ and $\bar{u}_j = \begin{pmatrix} \bar{a}_j \\ \bar{b}_j \end{pmatrix}$, and still recover the same expressions for w_i, we could naively expect to recover an expression similar to the original Harish-Chandra integral [140]:

$$I(u, \bar{u}) = \Omega \frac{\det\left(\exp(u_i \bar{u}_j)\right)}{\Delta(u)\Delta(\bar{u})} \tag{4.42}$$

if we use u_i and $\bar{u}_j$ in lieu of a_i, $\bar{a}_j$, b_i, and $\bar{b}_j$; set Ω as the normalization constant; and specify that when multiplying the Vandermondes in the denominator, one must take the dot product of $(u_i - u_j)$ and $(\bar{u}_i - \bar{u}_j)$. But upon this substitution, we find that we only recover:

$$I(u,\bar{u}) = \Omega \frac{\det\left(\exp(u_i \bar{u}_j)\right)}{\mu} \tag{4.43}$$

We are missing the factors of $\chi(a_i, b_i, \bar{a}_j, \bar{b}_j)$ in the numerator. We can solve for the missing factors by examining the case of $U(3)$; given that the effective potential is the same general form for $N > 2$, we may extrapolate the missing factor for a general $U(N)$ formula from the $U(3)$ results.

In the case of $U(3)$, we can expand the original HCIZ integral:

$$I(\Lambda, \bar{\Lambda}) = \int dU \exp\left(\mathrm{Tr}\left(U^{-1}\Lambda U \bar{\Lambda}\right)\right) = \Omega \frac{\det\left(\exp(\lambda_i \bar{\lambda}_j)\right)}{\Delta(\Lambda)\Delta(\bar{\Lambda})} \tag{4.44}$$

until we arrive at a series that takes the form of (A.39) with modified w_i to reflect the omission of the b_i and $\bar{b}_j$. We examine the effects of replacing a_i and a_j with u_i and u_j at each step; we then compare the results to an expansion of (4.42). The additional terms that replacing a_i and a_j with u_i and u_j yields must sum up to the missing factors. We briefly sketch out the start of such an expansion. We note that for $SU(3)$, we have:

$$\begin{aligned}
\det(\exp(a_i \bar{a}_j + b_i \bar{b}_j)) &= -e^{a_3\bar{a}_1 + a_2\bar{a}_2 + a_1\bar{a}_3 + b_3\bar{b}_1 + b_2\bar{b}_2 + b_1\bar{b}_3} + e^{a_2\bar{a}_1 + a_3\bar{a}_2 + a_1\bar{a}_3 + b_2\bar{b}_1 + b_3\bar{b}_2 + b_1\bar{b}_3} \\
&+ e^{a_3\bar{a}_1 + a_1\bar{a}_2 + a_2\bar{a}_3 + b_3\bar{b}_1 + b_1\bar{b}_2 + b_2\bar{b}_3} - e^{a_1\bar{a}_1 + a_3\bar{a}_2 + a_2\bar{a}_3 + b_1\bar{b}_1 + b_3\bar{b}_2 + b_2\bar{b}_3} \\
&- e^{a_2\bar{a}_1 + a_1\bar{a}_2 + a_3\bar{a}_3 + b_2\bar{b}_1 + b_1\bar{b}_2 + b_3\bar{b}_3} + e^{a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 + b_1\bar{b}_1 + b_2\bar{b}_2 + b_3\bar{b}_3} \\
&= -e^{s_1} + e^{s_2} + e^{s_3} - e^{s_4} - e^{s_5} + e^{s_6}
\end{aligned} \tag{4.45}$$

We have set:

$$\begin{aligned}
s_1 &= a_3\bar{a}_1 + a_2\bar{a}_2 + a_1\bar{a}_3 + b_3\bar{b}_1 + b_2\bar{b}_2 + b_1\bar{b}_3 \\
s_2 &= a_2\bar{a}_1 + a_3\bar{a}_2 + a_1\bar{a}_3 + b_2\bar{b}_1 + b_3\bar{b}_2 + b_1\bar{b}_3 \\
s_3 &= a_3\bar{a}_1 + a_1\bar{a}_2 + a_2\bar{a}_3 + b_3\bar{b}_1 + b_1\bar{b}_2 + b_2\bar{b}_3 \\
s_4 &= a_1\bar{a}_1 + a_3\bar{a}_2 + a_2\bar{a}_3 + b_1\bar{b}_1 + b_3\bar{b}_2 + b_2\bar{b}_3 \\
s_5 &= a_2\bar{a}_1 + a_1\bar{a}_2 + a_3\bar{a}_3 + b_2\bar{b}_1 + b_1\bar{b}_2 + b_3\bar{b}_3 \\
s_6 &= a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 + b_1\bar{b}_1 + b_2\bar{b}_2 + b_3\bar{b}_3
\end{aligned} \tag{4.46}$$

If we expand each term as a Taylor series, we may rewrite our determinant as:

$$\begin{aligned}
\det(\exp(a_i\bar{a}_j + b_i\bar{b}_j)) = &\sum_m \frac{1}{m!}(s_2 - s_1)\sum_{n=0}^{m-1} s_2^n s_1^{m-n-1} \\
&- \sum_m \frac{1}{m!}(s_4 - s_3)\sum_{n=0}^{m-1} s_4^n s_3^{m-n-1} \\
&+ \sum_m \frac{1}{m!}(s_6 - s_5)\sum_{n=0}^{m-1} s_6^n s_5^{m-n-1}
\end{aligned} \tag{4.47}$$

We note that:

$$\begin{aligned}
s_2 - s_1 &= (a_2 - a_3)(\bar{a}_1 - \bar{a}_2) + (b_2 - b_3)(\bar{b}_1 - \bar{b}_2) \\
s_4 - s_3 &= (a_1 - a_3)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_3)(\bar{b}_1 - \bar{b}_2) \\
s_6 - s_5 &= (a_1 - a_2)(\bar{a}_1 - \bar{a}_2) + (b_1 - b_2)(\bar{b}_1 - \bar{b}_2)
\end{aligned} \tag{4.48}$$

We see immediately that if we remove the b_i and b_j, the factors listed above cancel out factors in the Vandermonde determinants of the original HCIZ integral; however, once we add in b_i and b_j, some of the factors in (4.48) no longer cancel factors in μ. This suggests that there are missing saddle points and that the missing factors should add the terms needed to restore the overall factor of μ in the numerator. We note that our results should generalize to an arbitrary number of matrices; we would simply modify μ to account for the additional matrices and add the relevant derivative terms to (4.40), as well as modify u_i and $\bar{u}_j$ to account for the new eigenvalues.

We leave off here and save this computation for future works.

4.3 Connection with Restricted Schur Polynomials and Collective Coordinates

A natural question to ask is: what sort of basis of operators do the coherent states (4.1) actually generate? This is quite non-trivial, since there are in principle many different ways of orthogonalizing the two point function of $\frac{1}{4}$-BPS operators at finite N. As a concrete example, we can take the simplest multi-matrix coherent state we obtain from choosing the coherent state parameters to be rank one projectors for arbitrary N. These states will describe semi-classical configurations of single quarter BPS giant gravitons. Similar generating functions were introduced in [106, 23]; we will clarify the relationship between them and the coherent states studied here. This should help in generalizing to higher rank cases corresponding to bound states of AdS giants. The idea is to consider the following state:

$$|\lambda_x, \lambda_y\rangle = \int_{\mathbb{CP}^{N-1}} d\varphi^\dagger d\varphi \; e^{\lambda_x \varphi^\dagger X \varphi + \lambda_y \varphi^\dagger Y \varphi} |0\rangle . \tag{4.49}$$

To evaluate this, we need a formula for the moments of $\varphi_i^\dagger \varphi_j$ with respect to the flat Haar measure on $\mathbb{CP}^{N-1}$. The measure can be rewritten as follows:

$$\int_{\mathbb{CP}^{N-1}} d\varphi^\dagger d\varphi = \int_{i\mathbb{R}} ds \; \int d\bar{\phi} d\phi \; e^{-s\left(\bar{\phi}\phi - 1\right)} = \int_{\mathcal{C}} e^s \; \int \; d\bar{\phi} d\phi \; e^{-s\bar{\phi}\phi}. \tag{4.50}$$

In other words, we can trade the integral over projective space for a regular Gaussian integral at the cost of introducing an additional contour integral over an auxiliary parameter s. Then the moments have a simple expression in terms of the projection operators $P_{(k)}$ [106]:

$$\int_{\mathbb{CP}^{N-1}} d\varphi^\dagger \prod_{l=1}^{k} \left(\varphi^\dagger\right)^{i_l} \varphi_{j_l} = \oint \frac{ds\, e^s}{s^{N+k}}\, k!\, \left(P_{(k)}\right)^I_J = \frac{k!}{(N+k-1)!} \left(P_{(k)}\right)^I_J . \tag{4.51}$$

Borrowing the results of [106], we may rewrite the coherent state as a sum of the so-called restricted Schur polynomial operators $\chi_{(k_1+k_2),(k_1)\,(k_2)}(X,Y)$:

$$|\lambda_x, \lambda_y\rangle = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_x^{k_1} \lambda_2^{k_2}}{(N+k_1+k_2-1)!} \chi_{(k_1+k_2),(k_1)\,(k_2)}(X,Y)\, |0\rangle\, . \tag{4.52}$$

Now we would like to understand the analogue of this formula in the general case.

First, we need to recall the definition of the restricted Schur polynomials:

$$\chi_{R,(r,s)\,\alpha\beta}(X,Y) = \mathrm{Tr}[P_{R,(r,s)\,\alpha\beta}\, X^n \otimes Y^m]. \tag{4.53}$$

Here, R is a Young diagram associated with an irreducible representation of S_{n+m}; the labels (r,s) correspond to an irreducible representation of $S_n \times S_m$ contained in R. The object $P_{R,(r,s)\,\alpha\beta}$ can be understood as follows: starting with $S_m \times S_n \subset S_{m+n}$, we can find representations $r \times s$ sitting within R. Generically, the representation $r \times s$ can appear more than once inside of R, so one needs to keep track of how one embeds $r \times s$ into R. If the multiplicity of (r,s) is $n_{(r,s)}$ and its dimension $d_{(r,s)}$, then a generic element of S_{n+m} will be block diagonalized into $\left(n_{(r,s)} d_{(r,s)}\right) \times \left(n_{(r,s)} d_{(r,s)}\right)$ blocks. The matrix indices α, β keep track of this information, where α and β range from 1 to $n_{(r,s)}$. The $P_{R,(r,s)\,\alpha\beta}$ are then intertwining operators between each of these blocks. More formally, we can label each of the embeddings of $r \times s$ by an index γ and consider the space $R_\gamma \subset R$. The restricted Schur polynomial is then given by:

$$\chi_{R,R_\gamma}(X,Y) = \frac{1}{m!n!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\gamma}[\Gamma_R(\sigma)] \mathrm{Tr}[\sigma X^n \otimes Y^m], \tag{4.54}$$

where $\Gamma_R(\sigma)$ is the matrix representing σ [50]. The most complicated part of the restricted Schur polynomials is the evaluation of $\mathrm{Tr}_{R_\gamma}[\Gamma_R(\sigma)]$, which involves building R_γ explicitly.

By expanding the exponential and evaluating the unitary integrals, we obtain:

$$\begin{aligned} &\frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \ \exp\left(UXU^\dagger \Lambda_X + UYU^\dagger \Lambda_Y\right) = \\ &\sum_{n,m} \frac{1}{m!n!} \sum_{\sigma,\tau \in S_{n+m}} \mathrm{Tr}[\sigma \Lambda_X^n \otimes \Lambda_Y^m] \mathrm{Tr}[\tau X^n \otimes Y^m]\, \boldsymbol{Wg}(\sigma\tau^{-1}, N), \end{aligned} \tag{4.55}$$

where $\boldsymbol{Wg}(\sigma, N)$ is the Weingarten function. Explicit combinatorial formulas for Weingarten functions are well known from the work of Collins (see [143] for an elementary introduction); before delving into specific details, we should contrast this with the situation where one of the $\Lambda_{X,Y}$ is zero. In this case, the resulting sum can be recast as a diagonal sum of the products of unitary characters; right now, we have a complicated sum of traces. For a moment, let us consider the situation for a single matrix. The resulting sum is:

$$\begin{aligned} &\frac{1}{\mathrm{Vol}\,[U(N)]} \int dU \ \exp\left(UXU^\dagger \Lambda_X\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma,\tau \in S_n} \mathrm{Tr}[\sigma \Lambda_X^n] \mathrm{Tr}[\tau^{-1} X^n]\, \boldsymbol{Wg}(\sigma\tau^{-1}, N) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma,\tau \in S_n} \mathrm{Tr}[\sigma \Lambda_X^n] \mathrm{Tr}[\tau^{-1} X^n] \sum_{\lambda \vdash n} \frac{1}{n!\, f_\lambda} \chi^\lambda(\tau^{-1}\sigma) \chi^\lambda(1) \\ &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{1}{f_\lambda} s_\lambda(X) s_\lambda(\Lambda_X). \end{aligned} \tag{4.56}$$

The last line is obtained from the character expansion of the integral, which was computed in [21]. Then for two matrices, we have:

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}\left[U(N)\right]}\int dU\;\exp\left(UXU^{\dagger}\Lambda_X+UYU^{\dagger}\Lambda_Y\right)\\
&=\sum_{n,m}\frac{1}{m!n!(n+m)!}\sum_{\lambda\vdash n+m}\frac{1}{f^{\lambda}}\sum_{\sigma,\tau\in S_{n+m}}\chi^{\lambda}(\sigma)\chi^{\lambda}(\tau)\mathrm{Tr}[\sigma\Lambda_X^{n}\otimes\Lambda_Y^{m}]\mathrm{Tr}[\tau X^{n}\otimes Y^{m}].
\end{aligned} \tag{4.57}$$

Clearly this has a similar structure to the definition of the restricted Schur polynomials (4.54), but the restricted characters have been replaced with ordinary symmetric group characters instead. We can always formally rewrite each of the terms in the series as a sum over restricted characters by decomposing the trace over R into a sum of traces over each of the R_γ :

$$\chi^{R}(\sigma)=\mathrm{Tr}_R[\Gamma_R(\sigma)]=\sum_{\alpha}\sum_{(r,s)_{\alpha\alpha}\subset R}\mathrm{Tr}_{(r,s)_{\alpha\alpha}}[\Gamma_R(\sigma)]. \tag{4.58}$$

This allows us to rewrite the integral as a sum of restricted Schur polynomial operators. However, this sum does not capture every restricted character; the problem comes from elements $\sigma\in S_{n+m}$ that are not elements of $S_n\times S_m$. Generically, the representation R of S_n will be expressed as a sum of the irreducible representations R_α of $S_n\times S_m$ with multiplicities n_α. This means that σ is not fully block diagonal on the space $n_\alpha R_\alpha$, and indeed, restricted traces on each of these blocks lead to different orthogonal states in the free theory. This is expected, since the coherent state only generates operators that have vanishing one-loop anomalous dimension, and the dilatation operator acts non-trivially on generic restricted Schur polynomial operators. This means that the operators obtained from diagonal traces over $n_\alpha R_\alpha$ have vanishing one-loop anomalous dimension, while off-diagonal operators should be associated with excited states with open strings. We expect that operators that are approximate eigenstates of the dilatation operator in the large N limit take the form of open string modifications of these coherent states, and are likely more closely related to the Gauss graph operators as in [92].

An interesting direction to take would be to construct a generating function for all restricted Schur polynomials. Clearly, something as simple as (4.1) cannot work. This can be traced back to the fact that the sum over S_{n+m} has many redundancies owing to the fact that we can conjugate by an element of $S_n \times S_m$ while leaving the traces fixed. This is the statement that we can permute the n X's and m Y's among themselves while simultaneously permuting the $\Lambda_{X,Y}$'s. As explained in [147], there is an equivalence relation between elements of S_{n+m} in such a way that

$$\sigma \sim \tau \Leftrightarrow \mathrm{Tr}[\sigma A^n \otimes B^m] = \mathrm{Tr}[\tau A^n \otimes B^m], \tag{4.59}$$

which happens exactly when σ can be conjugated into τ by an element of $S_n \times S_m$. In other words, the construction of restricted Schur polynomials is equivalent to constructing generalized class functions on restricted conjugacy classes, which means that the coherent state generating function (4.1) cannot differentiate between different restricted Schur polynomials by itself for the simple reason that the Weingarten function is a class function. If we want to replace the characters in (4.1) with restricted characters, we must either change the domain of integration or integrate against an appropriate measure factor that is sensitive to this information. This is equivalent to finding an analytic formula for restricted characters, which may be recast as a Schrödinger problem over the space of commuting matrices [14, 146]. The point is that the norm of the quarter-BPS coherent state is related to the heat kernel over the space of commuting matrices, or equivalently to the Green's function of the Schrödinger equation. In practice, the coherent states still form an overcomplete basis of operators that can be used for computations, even if we do not currently know how to project into a particular primary state; the leading contribution at large N will come from the saddle point approximation.

In Appendix A.2, we checked a few low order terms by explicit calculation and found that the quarter-BPS coherent state generating function is given by a sum of product of restricted Schur polynomials much like the half-BPS coherent state. As we explained, this should hold for all the terms in the series, but checking higher order terms is difficult. This can be taken as further evidence that the coherent states span all possible BPS states and makes manifest many of the ideas in [14], since free field theory correlators can be encoded in integrals of polynomial functions of the collective coordinates. In the saddle point approximation, the integration measure is given by

$$\int d\vec{\Lambda} \cdot d\vec{\bar{\Lambda}} \, \langle \bar{\Lambda}_X, \bar{\Lambda}_Y | \, | \Lambda_X, \Lambda_Y \rangle \simeq \int \prod_i d\lambda_i^x d\bar{\lambda}_i^x d\lambda_i^y d\bar{\lambda}_i^y \prod_{i<j} \left(\vec{\lambda}_i - \vec{\lambda}_j \right) \left(\vec{\bar{\lambda}}_i - \vec{\bar{\lambda}}_j \right) \, e^{\vec{\lambda}_i \cdot \vec{\bar{\lambda}}_i}. \tag{4.60}$$

After an appropriate choice of contour for which λ and $\bar{\lambda}$ are canonically conjugate variables, this reproduces the strong coupling ansatz in [14]. It should also be clear that this measure describes the ground state wavefunction. Although our analysis is strictly on the weak coupling regime, we note that the dilatation operator in the $SU(2)$ sector can only act by permutations; this sector is closed, so it is very plausible that the quarter-BPS states are not renormalized. Even at weak coupling, the vacuum structure becomes quite non-trivial, modifying the Coulomb branch analysis for small collective coordinates, since the eigenvalues behave as strongly coupled bosons at low energies. This modifies the topology of the moduli space of vacua near the center of mass of the eigenvalues, even for half-BPS configurations. Since this sector preserves as much supersymmetry as $\mathcal{N} = 2$ SYM, it is quite plausible that the g_{YM} corrections are under control for sufficiently small modifications of BPS operators. At strong coupling, the expectation is that such states describe rotating strings propagating on bubbling geometries; the energy of these strings will follow the dispersion relation of a centrally extended BPS state

$$\Delta - J = \sqrt{Q^2 + |M|^2}, \tag{4.61}$$

with the central charge M being related to the length of the string in the bubbling geometry times the string tension. This is purely a kinematic effect; all of the dynamics should be encoded in the central charge M. Near the core of the geometry, the naive Coulomb branch analysis certainly breaks down, but S-duality considerations suggest that the string tension is not renormalized [148], so the corrections to curvature on the moduli space should come from finite N effects. This should be captured by non-planar corrections involving the exchange of gravitons between the background and a probe string. Such geometries can be engineered by integrating against the wavefunctions that break the $SO(6)_R$ symmetry of the vacuum. It is not hard to come up with such wavefunctions (for instance a non-symmetric Gaussian perturbation), and there is a schematic mapping between the eigenvalue distribution at large N and bubbling geometry [137]. The relation (4.60) should be corrected with additional $1/N$ effects, since the naive saddle point approximation fails to give the exact overlap, but these effects should only be relevant when we try to probe eigenvalues are placed in non-generic configurations. These should be thought of as microstate configurations for coarse-grained eigenvalue droplet configurations associated to superstar geometries. For $\frac{1}{8}$ BPS states, the analysis is more subtle; we have to take into account the effects from fermions since we would be working in a $SU(2|3)$ subsector. One should be able to ignore the effect of the fermions for large enough semiclassical operators. This makes this class of coherent states ideal for studying near-BPS limits around large operators without having to deal with the mixing of multi-trace structures.

4.4 Discussion

In this chapter, we studied multi-matrix coherent states for bosonic matrices that generate $\frac{1}{4}$ and $\frac{1}{8}$ BPS states in $\mathcal{N}=4$ SYM. We showed that the norm of these coherent states admits a fixed point formula generalizing the Harish-Chandra-Itzykson-Zuber formula for gauge group $U(2)$, and provided evidence of an expansion in terms of restricted Schur polynomials for $U(N)$. This gives in principle a way of generating expressions for BPS states for any value N in $\mathcal{N}=4$ SYM. One technical obstacle we face is that our construction does not give an alternative construction of the so-called restricted Schur polynomial operators [50]. This is related to the expectation that there is a hidden symmetry under which different operators are charged. One idea is that determining the Casimir charges should be enough to differentiate between different operators, but this problem is quite non-trivial even in the $\frac{1}{2}$ BPS sector [149]. It is also unclear how to implement this idea efficiently at large N since the number of Casimirs needed to distinguish between different operators grows with the complexity of the operators. Despite this obstacle, our results are important for computing correlators of $\frac{1}{4}$ and $\frac{1}{8}$ BPS operators dual to bound states of giant gravitons [150] and generic bubbling geometries [137]. Understanding the precise map between the overcomplete 'eigenvalue basis' of coherent states and specific orthogonal bases of operators remains an important problem. We conclude with a few more immediate directions for future work.

$\frac{1}{16}$ BPS States and Black Hole Microstate Operators

One of the more interesting generalizations would be to the case of $\frac{1}{16}$ BPS operators. By now, there is ample evidence that there exists a class of $\frac{1}{16}$ BPS operators describing the microstates of supersymmetric black holes in $AdS_5 \times S^5$ [151, 152, 153, 154]. Recently, there have been some studies of these types of states for small values of N [155, 156];

see [157] for a more general discussion. Our results imply that one should be cautious in extrapolating results about operators for the $U(2)$ theory, since multi-matrix states appear to be qualitatively different for other values of N. We expect that most of the interesting qualities of such operators are missing from the $U(2)$ and $SU(2)$ theory. It would be nice to develop more systematic techniques to build these types of operators. In principle, there are no obstructions to generalizing our techniques to this setup, with the working assumption that finding states with vanishing one-loop anomalous dimension is enough [142]. The idea would essentially be to build a superfield coherent state [158]:

$$\int dU \;\exp\left\{\int d^3\theta \int dz \mathrm{Tr}\left[U\Psi U^\dagger \Phi\right]\right\}|0\rangle\,, \tag{4.62}$$

where $\Psi(z,\theta)$ is the $\mathbb{C}^{2|3}$ superfield discussed in [158, 153], and Φ is an auxiliary superfield of coherent state parameters. The combined effect of the exponentiation and integration over the unitary matrices is to generate all possible gauge invariant tensor contractions. One should expect that the operators generated by this generating function are generalizations of the $SU(2|3)$ restricted Schur polynomials constructed in [53]. Generically, the terms in the expansion of (4.62) will not be of multi-graviton form, so they are natural candidates for microstates of supersymmetric black holes. In practice, the main disadvantage of an expression like (4.62) is that it might not be practically useful, in the sense that the expansion necesarily involves an infinite number of matrix fields associated with covariant derivatives acting on the fields. One way of avoiding this difficulty is to use generating functions such as the ones studied in [135]. Alternatively, one can view the auxiliary superfield Φ as a full-fledged dynamical collective coordinate. One would then hope that integrating out the SYM fields leads to an effective matrix quantum mechanics describing (near)-BPS black hole microstates, with the lightcone coordinate z acting as a time variable.

Three Point Correlators, Bubbling Geometries, and Twisted Holography

Although eventually we would like to study black holes, it is important to build intuition from simpler examples. One class of such examples is the BPS bubbling geometries [137] generalizing LLM geometries [55]. Although the droplet description of such states in supergravity is compelling, a precise mapping between the weak coupling BPS states is not fully developed[1]. The coherent states (4.55) have a more natural connection to such geometries[24]. A worthwhile exercise would be to study correlators of single trace chiral primaries in the background of heavy coherent states corresponding to both giant gravitons or bubbling geometries; see [159] for some finite N results. The holographic renomalization techniques of [160] are also applicable in these cases, but it would be interesting to develop more efficient computational techniques in supergravity along the lines of [161]. A good toy model for this would be to study these types of questions in Twisted Holography [162]. In that set-up, the eigenvalue droplets should be related to bubbling on a 2 dimensional complex base with holomorphic coordinates X, Y, with the vacuum configuration being a droplet with the topology of S^3; this is the deformed conifold description of $SL(2,\mathbb{C})$. More generic eigenvalue configurations should lead to other non-compact Calabi-Yau threefolds such as in [163, 164]. The expectation is that the geometry on both sides of the duality is encoded by a spectral curve [70] on which stacks of branes are supported, which should appear as the spectral curve of the corresponding matrix model. Understanding this could help in clarifying the dictionary between collective coordinates and bubbling geometries in the AdS_5 case.

[1]For instance, it is unclear whether the solutions found in [137] exhaust the set of all $\frac{1}{4}$ and $\frac{1}{8}$ BPS states.

Bound States of Giants and Branes at Angles

One reasonable goal would be to understand coherent states associated with pairs of eigenvalues, which are built from simpler integrals over the Grassmanian $Gr(2, N)$. The main difficulty of such a character expansion already appears in this simpler case:

$$|\lambda_x, \lambda_y\rangle = \int_{Gr(2,N)} dV^\dagger dV \; e^{\mathrm{Tr}[VXV^\dagger\lambda_x + VYV^\dagger\lambda_y]}, \tag{4.63}$$

where the λ_α are not 2×2 diagonal matrices. The norm of this state has a rather explicit integral form over 2×2 matrices over a compact domain; thus an explicit evaluation should be feasible. We expect that this state has a non-trivial expansion in terms of restricted Schur polynomials [106]. Since the domain of integration is simpler than that of the general case, the integral might lead itself to a saddle point analysis. One might be able to explicitly compute Lefshetz thimbles for this case and determine whether localization fails or not, or whether complex saddle points are needed. It would be interesting to construct multi-matrix analogues of the generating functions found in [139] for determinant operators. This would help in constructing the precise operators dual to intersecting giants [165], and particularly in understanding their integrable boundary states [166].

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Appendix A

Appendix Title

A.1 The Four-Matrix Model in $U(3)$

We now consider the following integral:

$$I = \int dU(3) \exp(\mathrm{Tr}(\bar{A}UAU^\dagger) + \mathrm{Tr}(\bar{B}UBU^\dagger)), \tag{A.1}$$

for $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\bar{B} = \begin{pmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{pmatrix}$.

We know that we can parameterize our $U(3)$ matrix as:

$$U = e^{i\lambda_3\alpha}e^{i\lambda_2\beta}e^{i\lambda_3\sigma}e^{i\lambda_5\theta}e^{i\lambda_3 a}e^{i\lambda_2 b}e^{i\lambda_3 c}e^{i\lambda_8\phi}, \tag{A.2}$$

where λ_i denotes the ith generators of $U(3)$. We list the relevant $SU(3)$ generators below

[64]:

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{A.3}$$

Because $U(3) = SU(3) \times U(1)$, we multiply U by an additional phase $e^{i\psi}$. This yields the parameterization of $U(3)$ that we will use to compute the argument of the exponential in Eq. (A.1). Simplifying and expanding, we arrive at:

$$\begin{aligned} \mathrm{Tr}\left(\bar{A}UAU^\dagger + \bar{B}UBU^\dagger\right) &= (a_3\bar{a_3} + b_3\bar{b_3})\cos^2\theta + (a_2\bar{a_2} + b_2\bar{b_2})\cos^2\beta\cos^2 b \\ &+ (a_1\bar{a_1} + b_1\bar{b_1})\cos^2\beta\cos^2\theta\cos^2 b + (\bar{a_1}a_2 + \bar{b_1}b_2)\cos^2 b\sin^2\beta \\ &+ (a_1\bar{a_2} + b_1\bar{b_2})\cos^2\theta\cos^2 b\sin^2\beta + (\bar{a_1}a_3 + \bar{b_1}b_3)\cos^2\beta\sin^2\theta \\ &+ (a_1\bar{a_3} + b_1\bar{b_3})\cos^2 b\sin^2\theta + (\bar{a_2}a_3 + \bar{b_2}b_3)\sin^2\beta\sin^2\theta \\ &+ (a_1\bar{a_2} + b_1\bar{b_2})\cos^2\beta\sin^2 b + (\bar{a_1}a_2 + \bar{b_1}b_2)\cos^2\beta\cos^2\theta\sin^2 b \\ &+ (a_1\bar{a_1} + b_1\bar{b_1})\sin^2\beta\sin^2 b + (a_2\bar{a_2} + b_2\bar{b_2})\cos^2\theta\sin^2\beta\sin^2 b \\ &+ (a_2\bar{a_3} + b_2\bar{b_3})\sin^2\theta\sin^2 b \\ &- \frac{1}{4}(a_1\bar{a_1} - \bar{a_1}a_2 - a_1\bar{a_2} + a_2\bar{a_2} + b_1\bar{b_1} - \bar{b_1}b_2 - b_1\bar{b_2} + b_2\bar{b_2}) \\ &\times e^{-2i(\sigma+a)}\cos\theta\sin 2\beta\sin 2b \\ &- \frac{1}{4}(a_1\bar{a_1} - \bar{a_1}a_2 - a_1\bar{a_2} + a_2\bar{a_2} + b_1\bar{b_1} - \bar{b_1}b_2 - b_1\bar{b_2} + b_2\bar{b_2}) \\ &\times e^{2i(\sigma+a)}\cos\theta\sin 2\beta\sin 2b \end{aligned} \tag{A.4}$$

We plug this into the integral. We may use the method outlined in [64] to compute the Haar measure; we arrive at:

$$dU = \sin 2\beta\sin 2\theta\sin 2b\sin^2\theta d\alpha d\beta d\sigma d\theta dadbdcd\phi d\psi \tag{A.5}$$

We cite the angle limits from [64]:

$$\begin{aligned} &0 \le \alpha, \sigma, a, c, \psi < \pi \\ &0 \le \beta, b, \theta < \frac{\pi}{2} \\ &0 \le \phi < 2\pi \end{aligned} \tag{A.6}$$

We seek to integrate over σ and a first. We observe that:

$$-e^{2i(\sigma+a)} - e^{-2i(\sigma+a)} = -2\cos(\sigma + a) \tag{A.7}$$

Returning to our integral, we note that rather than integrate over σ, we may perform a change of variables and integrate over $\sigma + a$ and change our integration limits to $[0, 2\pi]$. We would integrate over a twice, but the integral over a is trivial; as long as we note that the integral over the angular variable a is normalized to 1, we may proceed with integrating. Then the relevant integral is:

$$\begin{aligned} I_\sigma &= \int_0^{2\pi} e^{\left(-\frac{1}{2}(a_1\bar{a_1} - \bar{a_1}a_2 - a_1\bar{a_2} + a_2\bar{a_2} + b_1\bar{b_1} - \bar{b_1}b_2 - b_1\bar{b_2} + b_2\bar{b_2})\sin 2\beta \sin 2b \cos\theta \sin(\sigma+a)\right)} d\sigma \\ &= 2\pi I_0\left(\frac{1}{2}(a_1\bar{a_1} - \bar{a_1}a_2 - a_1\bar{a_2} + a_2\bar{a_2} + b_1\bar{b_1} - \bar{b_1}b_2 - b_1\bar{b_2} + b_2\bar{b_2})\sin 2\beta \sin 2b \cos\theta\right) \end{aligned} \tag{A.8}$$

where I_0 is the modified Bessel function of the first kind of order zero. There are no factors of a left in the integrand; as long as $\int_0^\pi da$ is normalized by a factor of π, we may consider the previous integral to have simultaneously integrated over both σ and a.

We now seek to integrate over θ. First, we collect the relevant terms and group the coefficients for simplicity's sake. We have:

$$
\begin{aligned}
v_1 &= \sin 2\beta \sin 2b \\
v_2 &= (a_2\bar{a}_2 + b_2\bar{b}_2)\cos^2\beta\cos^2 b + (\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2 b \sin^2\beta \\
&\quad + (a_1\bar{a}_2 + b_1\bar{b}_2)\cos^2\beta\sin^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1)\sin^2\beta\sin^2 b \\
v_3 &= (a_3\bar{a}_3 + b_3\bar{b}_3) + (a_1\bar{a}_1 + b_1\bar{b}_1)\cos^2\beta\cos^2 b + (a_1\bar{a}_2 + b_1\bar{b}_2)\cos^2 b\sin^2\beta \\
&\quad + (\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2\beta\sin^2 b + (a_2\bar{a}_2 + b_2\bar{b}_2)\sin^2\beta\sin^2 b \\
v_4 &= (\bar{a}_1 a_3 + \bar{b}_1 b_3)\cos^2\beta + (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b \\
&\quad + (\bar{a}_2 a_3 + \bar{b}_2 b_3)\sin^2\beta + (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \\
v_5 &= \frac{1}{2}(a_1\bar{a}_1 - \bar{a}_1 a_2 - a_1\bar{a}_2 + a_2\bar{a}_2 + b_1\bar{b}_1 - \bar{b}_1 b_2 - b_1\bar{b}_2 + b_2\bar{b}_2)\sin 2\beta\sin 2b
\end{aligned}
\tag{A.9}
$$

Our integral over θ thus becomes:

$$
I_\theta = 2\pi \int_0^{\frac{\pi}{2}} v_1 e^{v_2 + v_3\cos^2\theta + v_4\sin^2\theta} I_0(v_5\cos\theta)\sin 2\theta\sin^2\theta d\theta, \tag{A.10}
$$

where I_θ is the integral over θ in I from eq. (A.1) and v_i denote the grouped coefficients. We now observe that we may rewrite $\sin 2\theta$ as $2\sin\theta\cos\theta$. Then our integral becomes:

$$
I_\theta = -4\pi \int_0^{\frac{\pi}{2}} v_1 e^{v_2 + v_4 + (v_3 - v_4)\cos^2\theta} I_0(v_5\cos\theta)\cos\theta(1 - \cos^2\theta) d\cos\theta, \tag{A.11}
$$

Setting $x = \cos\theta$, our integral becomes:

$$
\begin{aligned}
I_\theta &= 4\pi \int_0^1 v_1 e^{v_2 + v_4 + (v_3 - v_4)x^2} I_0(v_5 x) x(1 - x^2) dx \\
&= 4\pi v_1 e^{v_2 + v_4} \int_0^1 e^{(v_3 - v_4)x^2} I_0(v_5 x) x(1 - x^2) dx
\end{aligned}
\tag{A.12}
$$

We find the Taylor expansion of $e^{(v_3-v_4)x^2}$:

$$e^{(v_3-v_4)x^2} = \sum_{k=0}^{\infty} \frac{(v_3-v_4)^k x^{2k}}{k!} \tag{A.13}$$

Now, we expand the Bessel function in series. We find that:

$$I_0(v_5 x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+1)} \left(\frac{v_5 x}{2}\right)^{2m} \tag{A.14}$$

Putting everything together, we arrive at:

$$\begin{aligned} I_\theta &= 4\pi v_1 e^{v_2+v_4} \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(v_3-v_4)^k x^{2k}}{k!}\right) \left(\sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+1)} \left(\frac{v_5 x}{2}\right)^{2k}\right) x(1-x^2)dx \\ &= 4\pi v_1 e^{v_2+v_4} \\ &\times \int_0^1 \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(\frac{(v_3-v_4)^m}{m!}\right) \left(\frac{v_5^{2(k-m)}}{2^{2(k-m)}(k-m)!\Gamma(k-m+1)}\right) x^{2k+1}(1-x^2)dx \\ &= 4\pi v_1 e^{v_2+v_4} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{1}{2(k+1)(k+2)} \left(\frac{(v_3-v_4)^m}{m!}\right) \left(\frac{v_5^{2(k-m)}}{2^{2(k-m)}(k-m)!\Gamma(k-m+1)}\right) \end{aligned} \tag{A.15}$$

We now seek to integrate over β. Before we start, we first examine the combinations

$v_2 + v_4$ and $v_3 - v_4$:

$$\begin{aligned} v_2 + v_4 &= (a_2\bar{a}_2 + b_2\bar{b}_2)\cos^2\beta\cos^2 b + (\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2 b\sin^2\beta \\ &+ (a_1\bar{a}_2 + b_1\bar{b}_2)\cos^2\beta\sin^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1)\sin^2\beta\sin^2 b \\ &+ (\bar{a}_1 a_3 + \bar{b}_1 b_3)\cos^2\beta + (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b \\ &+ (\bar{a}_2 a_3 + \bar{b}_2 b_3)\sin^2\beta + (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \\ &= \big((a_2\bar{a}_2 + b_2\bar{b}_2)\cos^2 b + (a_1\bar{a}_2 + b_1\bar{b}_2)\sin^2 b + \bar{a}_1 a_3 + \bar{b}_1 b_3\big)\cos^2\beta \\ &+ \big((\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1)\sin^2 b + \bar{a}_2 a_3 + \bar{b}_2 b_3\big)\sin^2\beta \\ &+ (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b + (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \end{aligned} \tag{A.16}$$

We note that $v_1 = \sin 2\beta \sin 2b$, which means we can repeat the process of rewriting $\sin 2\beta$ as $2\sin\beta\cos\beta$, but absorbing $\cos\beta$ behind the derivative instead. Then we rewrite the expression above as:

$$\begin{aligned} v_2 + v_4 &= \big((a_2\bar{a}_2 + b_2\bar{b}_2)\cos^2 b + (a_1\bar{a}_2 + b_1\bar{b}_2)\sin^2 b + \bar{a}_1 a_3 + \bar{b}_1 b_3\big)\big(1 - \sin^2\beta\big) \\ &+ \big((\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1)\sin^2 b + \bar{a}_2 a_3 + \bar{b}_2 b_3\big)\sin^2\beta \\ &+ (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b + (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \\ &= \bar{a}_1 a_3 + \bar{b}_1 b_3 + \big(a_1\bar{a}_3 + b_1\bar{b}_3 + a_2\bar{a}_2 + b_2\bar{b}_2\big)\cos^2 b \\ &+ \big(a_2\bar{a}_3 + b_2\bar{b}_3 + a_1\bar{a}_2 + b_1\bar{b}_2\big)\sin^2 b \\ &+ \big(\bar{a}_2 a_3 + \bar{b}_2 b_3\big)\sin^2\beta \\ &+ \big((\bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_2 - b_2\bar{b}_2)\cos^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_2 - b_1\bar{b}_2)\sin^2 b\big)\sin^2\beta \end{aligned} \tag{A.17}$$

We then turn to $v_3 - v_4$:

$$\begin{aligned}
v_3 - v_4 &= (a_3\bar{a}_3 + b_3\bar{b}_3) + (a_1\bar{a}_1 + b_1\bar{b}_1)\cos^2\beta\cos^2 b + (a_1\bar{a}_2 + b_1\bar{b}_2)\cos^2 b\sin^2\beta \\
&+ (\bar{a}_1 a_2 + \bar{b}_1 b_2)\cos^2\beta\sin^2 b + (a_2\bar{a}_2 + b_2\bar{b}_2)\sin^2\beta\sin^2 b \\
&- (\bar{a}_1 a_3 + \bar{b}_1 b_3)\cos^2\beta - (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b \\
&- (\bar{a}_2 a_3 + \bar{b}_2 b_3)\sin^2\beta - (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \\
&= (a_3\bar{a}_3 + b_3\bar{b}_3) - (a_1\bar{a}_3 + b_1\bar{b}_3)\cos^2 b - (a_2\bar{a}_3 + b_2\bar{b}_3)\sin^2 b \\
&+ \big((a_1\bar{a}_1 + b_1\bar{b}_1)\cos^2 b + (\bar{a}_1 a_2 + \bar{b}_1 b_2)\sin^2 b - (\bar{a}_1 a_3 + \bar{b}_1 b_3)\big)\left(1 - \sin^2\beta\right) \\
&+ \big((a_1\bar{a}_2 + b_1\bar{b}_2)\cos^2 b + (a_2\bar{a}_2 + b_2\bar{b}_2)\sin^2 b - (\bar{a}_2 a_3 + \bar{b}_2 b_3)\big)\sin^2\beta \\
&= a_3\bar{a}_3 + b_3\bar{b}_3 - \bar{a}_1 a_3 - \bar{b}_1 b_3 + (a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_3 - b_1\bar{b}_3)\cos^2 b \\
&+ (\bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_3 - b_2\bar{b}_3)\sin^2 b \\
&+ \big((a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_1 - b_1\bar{b}_1)\cos^2 b + (a_2\bar{a}_2 + b_2\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2)\sin^2 b\big)\sin^2\beta \\
&+ \left(\bar{a}_1 a_3 + \bar{b}_1 b_3 - \bar{a}_2 a_3 - \bar{b}_2 b_3\right)\sin^2\beta
\end{aligned} \tag{A.18}$$

We once again regroup and relabel our coefficients for ease of computation:

$$\begin{aligned}
u_1 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 + (a_1\bar{a}_3 + b_1\bar{b}_3 + a_2\bar{a}_2 + b_2\bar{b}_2)\cos^2 b + (a_2\bar{a}_3 + b_2\bar{b}_3 + a_1\bar{a}_2 + b_1\bar{b}_2)\sin^2 b \\
u_2 &= \bar{a}_2 a_3 + \bar{b}_2 b_3 + (\bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_2 - b_2\bar{b}_2)\cos^2 b + (a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_2 - b_1\bar{b}_2)\sin^2 b \\
u_3 &= a_3\bar{a}_3 + b_3\bar{b}_3 - \bar{a}_1 a_3 - \bar{b}_1 b_3 + (a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_3 - b_1\bar{b}_3)\cos^2 b \\
&+ (\bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_3 - b_2\bar{b}_3)\sin^2 b \\
u_4 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 - \bar{a}_2 a_3 - \bar{b}_2 b_3 + (a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_1 - b_1\bar{b}_1)\cos^2 b \\
&+ (a_2\bar{a}_2 + b_2\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2)\sin^2 b \\
u_5 &= (a_1\bar{a}_1 - \bar{a}_1 a_2 - a_1\bar{a}_2 + a_2\bar{a}_2 + b_1\bar{b}_1 - \bar{b}_1 b_2 - b_1\bar{b}_2 + b_2\bar{b}_2)\sin 2b
\end{aligned} \tag{A.19}$$

We set $y = \sin\beta$. Then our integral over β becomes:

$$\begin{aligned} I_\beta = 8\pi \sin 2b \int_0^1 & e^{u_1+u_2 y} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{1}{2(k+1)(k+2)} \left(\frac{\left(u_3+u_4y^2\right)^m}{m!}\right) \\ & \times \left(\frac{\left(u_5 y\sqrt{1-y^2}\right)^{2(k-m)}}{2^{2(k-m)}(k-m)!\Gamma(k-m+1)}\right) y dy \end{aligned} \tag{A.20}$$

We now examine $\left(u_3+u_4y^2\right)^m$. We know that we can use the binomial expansion to express it as:

$$\left(u_3+u_4y^2\right)^m = \sum_{j=0}^{m} \binom{m}{j} u_3^j \left(u_4y^2\right)^{m-j} \tag{A.21}$$

Then we have:

$$\frac{\left(u_3+u_4y^2\right)^m}{m!} = \sum_{j=0}^{m} \frac{1}{(m-j)!j!} u_3^j \left(u_4y^2\right)^{m-j} \tag{A.22}$$

We then examine $\left(y\sqrt{1-y^2}\right)^{2(k-m)}$. First, we note that we can rewrite this expression as $\left(y^2-y^4\right)^{k-m}$. Then, using the binomial series, we find:

$$\begin{aligned} \left(y^2-y^4\right)^{k-m} &= \sum_{l=0}^{k-m} \binom{k-m}{l} (-1)^{k-m-l} y^{2l} y^{4(k-m-l)} \\ &= \sum_{l=0}^{k-m} \binom{k-m}{l} (-1)^{k-m-l} y^{4k-4m-2l} \end{aligned} \tag{A.23}$$

Then we have:

$$\begin{aligned} \frac{\left(u_5 y\sqrt{1-y^2}\right)^{2(k-m)}}{2^{2(k-m)}(k-m)!\Gamma(k-m+1)} &= \frac{u_5^{2(k-m)} \sum_{l=0}^{k-m} \binom{k-m}{l} (-1)^{k-m-l} y^{4k-4m-2l}}{2^{2(k-m)}(k-m)!\Gamma(k-m+1)} \\ &= \sum_{l=0}^{k-m} \frac{(-1)^{k-m-l} u_5^{2(k-m)} y^{4k-4m-2l}}{2^{2(k-m)} l!(k-m-l)!\Gamma(k-m+1)} \end{aligned} \tag{A.24}$$

We now expand:

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{2(k+1)(k+2)}\left(\sum_{j=0}^{m}\frac{u_3^j u_4^{m-j}}{(m-j)!j!}y^{2(m-j)}\right)\\
&\times\left(\sum_{l=0}^{k-m}\frac{(-1)^{k-m-l}u_5^{2(k-m)}y^{4k-4m-2l}}{2^{2(k-m)}l!(k-m-l)!\Gamma(k-m+1)}\right)\\
&=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{y^{4k-2m}}{2(k+1)(k+2)}\left(\sum_{j=0}^{m}\frac{u_3^j u_4^{m-j}}{(m-j)!j!}y^{-2j}\right)\\
&\times\left(\sum_{l=0}^{k-m}\frac{(-1)^{k-m-l}u_5^{2(k-m)}y^{-2l}}{2^{2(k-m)}l!(k-m-l)!\Gamma(k-m+1)}\right)\\
&=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\le j\le m\\0\le l\le k-m}}\frac{1}{2(k+1)(k+2)}\frac{u_3^j u_4^{m-j}}{(m-j)!j!}\frac{(-1)^{k-m-l}u_5^{2(k-m)}y^{4k-2m-2n}}{2^{2(k-m)}(l)!(k-m-l)!\Gamma(k-m+1)}
\end{aligned}
\tag{A.25}
$$

Finally, we note that we can expand $e^{u_2 y}$ using the Taylor series:

$$
e^{u_2 y}=\sum_{p=0}^{\infty}\frac{u_2^p y^p}{p!}
\tag{A.26}
$$

Then I_β evaluates to:

$$
\begin{aligned}
&8\pi\sin 2be^{u_1}\int_0^1\sum_{p=0}^{\infty}\frac{u_2^p y^p}{p!}\\
&\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\le j\le m\\0\le l\le k-m}}\frac{1}{2(k+1)(k+2)}\frac{u_3^j u_4^{m-j}}{(m-j)!j!}\frac{(-1)^{k-m-l}u_5^{2(k-m)}y^{4k-2m-2n+1}}{2^{2(k-m)}(l)!(k-m-l)!\Gamma(k-m+1)}dy\\
&=8\pi\sin 2be^{u_1}\sum_{q=0}^{\infty}\sum_{\substack{p+4k=q\\p,k\ge 0}}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\le j\le m\\0\le l\le k-m}}\frac{u_2^p}{(k+1)(k+2)p!(q-2m-2n+2)}\frac{u_3^j u_4^{m-j}}{(m-j)!j!}\\
&\times\frac{(-1)^{k-m-l}u_5^{2(k-m)}}{2^{2(k-m)+1}(l)!(k-m-l)!\Gamma(k-m+1)}
\end{aligned}
\tag{A.27}
$$

Finally, we integrate over b. As before, we take $\sin 2b$ and rewrite it as $2\sin b\cos b$. Then we absorb $\cos b$ behind the derivative and set $z = \sin b$. We now integrate over z from 0 to 1. We rewrite u_i to reflect this change:

$$\begin{aligned}
u_1 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 + a_1\bar{a}_3 + b_1\bar{b}_3 + a_2\bar{a}_2 + b_2\bar{b}_2 \\
&\quad + (a_2\bar{a}_3 + b_2\bar{b}_3 + a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_3 - b_1\bar{b}_3 - a_2\bar{a}_2 - b_2\bar{b}_2)\sin^2 b \\
u_2 &= \bar{a}_2 a_3 + \bar{b}_2 b_3 + \bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_2 - b_2\bar{b}_2 \\
&\quad + (a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_2 - b_1\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2 + a_2\bar{a}_2 + b_2\bar{b}_2)\sin^2 b \\
u_3 &= a_3\bar{a}_3 + b_3\bar{b}_3 - \bar{a}_1 a_3 - \bar{b}_1 b_3 + a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_3 - b_1\bar{b}_3 \\
&\quad + (\bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_3 - b_2\bar{b}_3 - a_1\bar{a}_1 - b_1\bar{b}_1 + a_1\bar{a}_3 + b_1\bar{b}_3)\sin^2 b \\
u_4 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 - \bar{a}_2 a_3 - \bar{b}_2 b_3 + a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_1 - b_1\bar{b}_1 \\
&\quad + (a_2\bar{a}_2 + b_2\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2 - a_1\bar{a}_2 - b_1\bar{b}_2 + a_1\bar{a}_1 + b_1\bar{b}_1)\sin^2 b \\
u_5 &= (a_1\bar{a}_1 - \bar{a}_1 a_2 - a_1\bar{a}_2 + a_2\bar{a}_2 + b_1\bar{b}_1 - \bar{b}_1 b_2 - b_1\bar{b}_2 + b_2\bar{b}_2)\sin 2b
\end{aligned} \tag{A.28}$$

We set:

$$\begin{aligned}
w_1 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 + a_1\bar{a}_3 + b_1\bar{b}_3 + a_2\bar{a}_2 + b_2\bar{b}_2 \\
w_2 &= a_2\bar{a}_3 + b_2\bar{b}_3 + a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_3 - b_1\bar{b}_3 - a_2\bar{a}_2 - b_2\bar{b}_2 \\
w_3 &= \bar{a}_2 a_3 + \bar{b}_2 b_3 + \bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_2 - b_2\bar{b}_2 \\
w_4 &= a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_2 - b_1\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2 + a_2\bar{a}_2 + b_2\bar{b}_2 \\
w_5 &= a_3\bar{a}_3 + b_3\bar{b}_3 - \bar{a}_1 a_3 - \bar{b}_1 b_3 + a_1\bar{a}_1 + b_1\bar{b}_1 - a_1\bar{a}_3 - b_1\bar{b}_3 \\
w_6 &= \bar{a}_1 a_2 + \bar{b}_1 b_2 - a_2\bar{a}_3 - b_2\bar{b}_3 - a_1\bar{a}_1 - b_1\bar{b}_1 + a_1\bar{a}_3 + b_1\bar{b}_3 \\
w_7 &= \bar{a}_1 a_3 + \bar{b}_1 b_3 - \bar{a}_2 a_3 - \bar{b}_2 b_3 + a_1\bar{a}_2 + b_1\bar{b}_2 - a_1\bar{a}_1 - b_1\bar{b}_1 \\
w_8 &= a_2\bar{a}_2 + b_2\bar{b}_2 - \bar{a}_1 a_2 - \bar{b}_1 b_2 - a_1\bar{a}_2 - b_1\bar{b}_2 + a_1\bar{a}_1 + b_1\bar{b}_1 \\
w_9 &= 2(a_1\bar{a}_1 - \bar{a}_1 a_2 - a_1\bar{a}_2 + a_2\bar{a}_2 + b_1\bar{b}_1 - \bar{b}_1 b_2 - b_1\bar{b}_2 + b_2\bar{b}_2)
\end{aligned} \tag{A.29}$$

Then our integral becomes:

$$I_b = 16\pi e^{w_1} \int_0^1 e^{w_2 z^2} \times \sum_{q=0}^{\infty} \sum_{\substack{p+4k=q \\ p,k\geq 0}} \sum_{m=0}^{k} \sum_{n=0}^{k} \sum_{\substack{j+l=n \\ 0\leq j\leq m \\ 0\leq l\leq k-m}} \frac{(w_3+w_4z^2)^p (w_5+w_6z^2)^j (w_7+w_8z^2)^{m-j}}{(k+1)(k+2)(q-2m-2n+2)p!j!(m-j)!} \times \frac{(-1)^{k-m-l}(w_9 z\sqrt{1-z^2})^{2(k-m)}}{2^{2(k-m)+1} l!(k-m-l)!\Gamma(k-m+1)} z dz \tag{A.30}$$

As before, we note that:

$$\frac{(w_3+w_4z^2)^p}{p!} = \sum_{r=0}^{p} \frac{1}{(p-r)!r!} w_3^r \left(w_4 z^2\right)^{p-r} \tag{A.31}$$

and

$$\frac{(w_5+w_6z^2)^j}{j!} = \sum_{s=0}^{j} \frac{1}{(j-s)!s!} w_5^s \left(w_6 z^2\right)^{j-s} \tag{A.32}$$

and

$$\frac{(w_7+w_8z^2)^{m-j}}{(m-j)!} = \sum_{t=0}^{m-j} \frac{1}{(m-j-t)!t!} w_7^t \left(w_8 z^2\right)^{m-j-t} \tag{A.33}$$

We find that:

$$\begin{aligned}\frac{(w_3+w_4z^2)^p(w_5+w_6z^2)^j(w_7+w_8z^2)^{m-j}}{p!j!(m-j)!} &= \left(\sum_{r=0}^{p}\frac{w_3^r w_4^{p-r}}{(p-r)!r!}z^{2p-2r}\right)\\ &\times\left(\sum_{s=0}^{j}\frac{w_5^s w_6^{j-s}}{(j-s)!s!}z^{2j-2s}\right)\\ &\times\left(\sum_{t=0}^{m-j}\frac{w_7^t w_8^{m-j-t}}{(m-j-t)!t!}z^{2m-2j-2t}\right)\\ &= \sum_{g=0}^{p+m}\sum_{\substack{r+s+t=g\\0\le r\le p\\0\le s\le j\\0\le t\le m-j}}\frac{w_3^r w_4^{p-r}w_5^s w_6^{j-s}}{(p-r)!r!(j-s)!s!}\\ &\times\frac{w_7^t w_8^{m-j-t}z^{2(p+m-g)}}{(m-j-t)!t!}\end{aligned} \tag{A.34}$$

We also know that $\left(z\sqrt{1-z^2}\right)^{2(k-m)}$ can be written as:

$$\left(z\sqrt{1-z^2}\right)^{2(k-m)} = \sum_{h=0}^{k-m}\binom{k-m}{h}(-1)^{k-m-h}z^{4k-4m-2h} \tag{A.35}$$

Then we have:

$$\begin{aligned}\frac{(-1)^{k-m-l}(w_9z\sqrt{1-z^2})^{2(k-m)}}{2^{2(k-m)+1}l!(k-m-l)!\Gamma(k-m+1)} &= \frac{(-1)^{k-m-l}w_9^{2(k-m)}}{2^{2(k-m)+1}l!(k-m-l)!\Gamma(k-m+1)}\\ &\times\sum_{h=0}^{k-m}\binom{k-m}{h}(-1)^{k-m-h}z^{4k-4m-2h}\\ &= \sum_{h=0}^{k-m}\frac{(-1)^{-l-h}w_9^{2(k-m)}z^{4k-4m-2h}}{2^{2(k-m)+1}l!h!(k-m-l)!(k-m-h)!}\end{aligned} \tag{A.36}$$

We compute:

$$\sum_{q=0}^{\infty}\sum_{\substack{p+4k=q\\p,k\geq 0}}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\leq j\leq m\\0\leq l\leq k-m}}\frac{1}{(k+1)(k+2)(q-2m-2n+2)}$$
$$\times\left(\sum_{g=0}^{p+m}\sum_{\substack{r+s+t=g\\0\leq r\leq p\\0\leq s\leq j\\0\leq t\leq m-j}}\frac{w_3^r w_4^{p-r}w_5^s w_6^{j-s}w_7^t w_8^{m-j-t}}{(p-r)!r!(j-s)!s!(m-j-t)!t!}z^{2(p+m-g)}\right)$$
$$\times\left(\sum_{h=0}^{k-m}\frac{(-1)^{-l-h}w_9^{2(k-m)}z^{4k-4m-2h}}{2^{2(k-m)+1}l!h!(k-m-l)!(k-m-h)!}\right)$$

and arrive at:

$$\sum_{q=0}^{\infty}\sum_{\substack{p+4k=q\\p,k\geq 0}}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\leq j\leq m\\0\leq l\leq k-m}}\frac{1}{(k+1)(k+2)(q-2m-2n+2)}$$
$$\times\sum_{g=0}^{p+k}\sum_{\substack{r+s+t+h=g\\0\leq r\leq p\\0\leq s\leq j\\0\leq t\leq m-j\\0\leq h\leq k-m}}\frac{(-1)^{-l-h}w_3^r w_4^{p-r}w_5^s w_6^{j-s}w_7^t w_8^{m-j-t}w_9^{2(k-m)}}{2^{2(k-m)+1}(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!}$$
$$\times z^{2(2k+p-m-g)}$$

Once again, we expand $e^{w_2 z}$ using the Taylor series:

$$e^{w_2 z}=\sum_{f=0}^{\infty}\frac{w_2^f z^f}{f!} \tag{A.37}$$

Then our integral becomes:

$$16\pi e^{w_1}\int_0^1\left(\sum_{f=0}^{\infty}\frac{w_2^f z^f}{f!}\right)\sum_{q=0}^{\infty}\sum_{\substack{p+4k=q\\p,k\geq 0}}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{\substack{j+l=n\\0\leq j\leq m\\0\leq l\leq k-m}}\frac{1}{(k+1)(k+2)(q-2m-2n+2)}$$
$$\times\sum_{g=0}^{p+k}\sum_{\substack{r+s+t+h=g\\0\leq r\leq p\\0\leq s\leq j\\0\leq t\leq m-j\\0\leq h\leq k-m}}\frac{(-1)^{-l-h}w_3^r w_4^{p-r}w_5^s w_6^{j-s}w_7^t w_8^{m-j-t}w_9^{2(k-m)}}{2^{2(k-m)+1}(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!}$$
$$\times z^{2(2k+p-m-g)}zdz$$

This is a hideous series, and we would be forgiven for thinking that we should define a new index that matches $2(2k+p-m-g)$. But q does the job, if more subtly, and so we will retain q and rewrite $2(2k+p-m-g)$ as $2(q-2k-m-g)$. Then we can integrate over z and find:

$$16\pi e^{w_1}\int_0^1 \sum_{d=0}^{\infty} \sum_{\substack{2q-4k+f=d\\ p+4k=q\\ q,k,p,f\geq 0}} \sum_{m=0}^{k}\sum_{n=0}^{k} \sum_{\substack{j+l=n\\ 0\leq j\leq m\\ 0\leq l\leq k-m}} \frac{1}{(k+1)(k+2)(q-2m-2n+2)}$$
$$\times \sum_{g=0}^{p+k} \sum_{\substack{r+s+t+h=g\\ 0\leq r\leq p\\ 0\leq s\leq j\\ 0\leq t\leq m-j\\ 0\leq h\leq k-m}} \frac{(-1)^{-l-h} w_2^f w_3^r w_4^{p-r} w_5^s w_6^{j-s} w_7^t w_8^{m-j-t} w_9^{2(k-m)}}{2^{2(k-m)+1} f!(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!}$$
$$\times z^{d-2m-2g+1}dz$$
$$= 16\pi e^{w_1} \sum_{q=0}^{\infty} \sum_{\substack{2p+4k+f=q\\ k,p,f\geq 0}} \sum_{m,n=0}^{k} \sum_{\substack{j+l=n\\ 0\leq j\leq m\\ 0\leq l\leq k-m}} \sum_{g=0}^{p+k} \sum_{\substack{r+s+t+h=g\\ 0\leq r\leq p\\ 0\leq s\leq j\\ 0\leq t\leq m-j\\ 0\leq h\leq k-m}} \frac{1}{(k+1)(k+2)(p+4k-2m-2n+2)}$$
$$\times \frac{(-1)^{-l-h} w_2^f w_3^r w_4^{p-r} w_5^s w_6^{j-s} w_7^t w_8^{m-j-t}}{2^{2(k-m)+1}(q-2m-2g+2) f!(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!}$$
$$\times \frac{w_9^{2(k-m)}}{(k-m-l)!(k-m-h)!} \tag{A.38}$$

For clarity, we rewrite our integral as:

$$I = \sum_{q=0}^{\infty} \sum_{\substack{f+2p+4k=q\\ f,p,k\geq 0}} \sum_{m,n=0}^{k} \sum_{\substack{j+l=n\\ 0\leq j\leq m\\ 0\leq l\leq k-m}} \sum_{g=0}^{p+k} \sum_{\substack{r+s+t+h=g\\ 0\leq r\leq p\\ 0\leq s\leq j\\ 0\leq t\leq m-j\\ 0\leq h\leq k-m}} \frac{16\pi e^{w_1}}{(k+1)(k+2)}$$
$$\times \frac{1}{(p+4k-2m-2n+2)(q-2m-2g+2)}$$
$$\times \frac{(-1)^{-l-h} w_2^f w_3^r w_4^{p-r} w_5^s w_6^{j-s} w_7^t w_8^{m-j-t} w_9^{2(k-m)}}{2^{2(k-m)+1} f!(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!} \tag{A.39}$$

We now seek to simplify this answer. First, starting from $q = 0$, we list the first few possible combinations for q, f, p, and k; the results can be found in table A.1.

We see that if we hold all the other indices constant and expand the sum over q and

q	f	p	k	q	f	p	k	q	f	p	k	q	f	p	k
0	0	0	0	4	4	0	0	6	6	0	0	8	2	1	1
1	1	0	0	5	1	0	1	7	1	1	1	8	2	3	0
2	0	1	0	5	1	2	0	7	1	3	0	8	4	0	1
2	2	0	0	5	3	1	0	7	3	2	0	8	4	2	0
3	1	1	0	5	5	0	0	7	5	1	0	8	6	1	0
3	3	0	0	6	0	1	1	7	7	0	0	8	8	0	0
4	0	0	1	6	0	3	0	8	0	0	2	9	1	0	2
4	0	2	0	6	2	2	0	8	0	2	1	9	1	2	1
4	2	1	0	6	4	1	0	8	0	4	0	9	1	4	0

Table A.1: We list the first few combinations of q, f, p, and k that satisfy the constraints on the indices.

f, then we may extract a factor of $\frac{1}{q-2m-2g+2}\frac{w_2^f}{f!}$ from the terms associated with the set of fixed indices. Since $q-f=2p+4k$, we may rewrite this factor as $\frac{1}{f+2p+2k-2m-2g+2}\frac{w_2^f}{f!}$. Summing from zero to infinity, we find that:

$$\sum_{f=0}^{\infty}\frac{1}{f+2p+4k-2m-2g+2}\frac{\omega_2^f}{f!}=(-w_2)^{-2p-4k+2m+2g-2}\Gamma(2p+4k-2m-2g+2,0,-w_2) \tag{A.40}$$

Then we have:

$$\begin{aligned} I=&\sum_{p,k\geq 0}^{\infty}\sum_{m,n=0}^{k}\sum_{\substack{j+l=n\\0\leq j\leq m\\0\leq l\leq k-m}}\sum_{g=0}^{p+k}\sum_{\substack{r+s+t+h=g\\0\leq r\leq p\\0\leq s\leq j\\0\leq t\leq m-j\\0\leq h\leq k-m}}\frac{\pi e^{w_1}\Gamma(2p+4k-2m-2g+2,0,-w_2)}{(k+1)(k+2)(p+4k-2m-2n+2)}\\ &\times\frac{1}{(-w_2)^{2p+4k-2m-2g+2}}\\ &\times\frac{(-1)^{-l-h}w_3^r w_4^{p-r}w_5^s w_6^{j-s}w_7^t w_8^{m-j-t}w_9^{2(k-m)}}{2^{2(k-m)-3}(p-r)!r!(j-s)!s!(m-j-t)!t!l!h!(k-m-l)!(k-m-h)!}\end{aligned} \tag{A.41}$$

A.2 Expansion in Terms of Restricted Characters

We seek to compute our four-matrix Harish-Chandra integral through a character expansion. We start with:

$$I_{R,R'} = \int \delta U \, \chi_R(UXU^\dagger \bar{X})\chi_{R'}(UYU^\dagger \bar{Y}) \tag{A.42}$$

If we have $A \in \mathrm{Hom}(V, V)$, $A' \in \mathrm{Hom}(V', V')$, then:

$$\mathrm{Tr}_V A \; \mathrm{Tr}_{V'} A' = \mathrm{Tr}_{V \otimes V'} A \otimes A' \tag{A.43}$$

We arrive at:

$$I_{R,R'} = \int \delta U \, \chi_{R \otimes R'} \left(U_R \otimes U_{R'} \, X_R \otimes Y_{R'} \, U_R^\dagger \otimes U_{R'}^\dagger \, \bar{X}_R \otimes \bar{X}_{R'} \right) \tag{A.44}$$

Now we may decompose our product representation into irreducible representations:

$$I_{R,R'} = \sum_{S \in R \otimes R'} \int \delta U \, \chi_S \left(U_S Z_S U_S^\dagger \bar{Z}_S \right) \tag{A.45}$$

where

$$Z_S \in X_R \otimes Y_{R'} \tag{A.46}$$

For example, if we set $R = \square$, $R' = \square$, then we have:

$$\boxed{x} \otimes \boxed{y} = \boxed{x}\!\boxed{y} \oplus \begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array} \tag{A.47}$$

$$\boxed{x}\boxed{y} = \frac{1}{2}\left([x][y] + [xy]\right), \quad \begin{array}{c}\boxed{x}\\ \boxed{y}\end{array} = \frac{1}{2}\left([x][y] - [xy]\right) \tag{A.48}$$

From here on out, we use symbols interchangeably to represent both the representation S and the character $\chi_S(Z_S)$ associated with it. We write the trace of the fundamental representation matrices as $[x] \equiv \sum_{i=1}^{N} x_i$. We see then that S could be the symmetric/anti-symmmetric combination of X and Y in its fundamental representation. If we seek to evaluate our integral with the Young diagrams we listed earlier, we arrive at:

$$I_{R,R} = \frac{1}{D_S}\chi_S(X \otimes Y)\chi_S(\bar{X} \otimes \bar{Y}) + \frac{1}{D_{S'}}\chi_{S'}(X \otimes Y)\chi_{S'}(\bar{X} \otimes \bar{Y}), \quad S = \boxed{x}\boxed{y}, \quad S' = \begin{array}{c}\boxed{x}\\ \boxed{y}\end{array} \tag{A.49}$$

where D_S is the dimension of the R representation of the $GL(|R|)$ group.

We now compute our second example, where we hae set $R = \square\square$, $R' = \square$. Then we have:

$$\boxed{x}\boxed{x} \otimes \boxed{y} = \boxed{x}\boxed{x}\boxed{y} \oplus \begin{array}{l}\boxed{x}\boxed{x}\\ \boxed{y}\end{array} \tag{A.50}$$

$$\boxed{x}\boxed{x}\boxed{y} = \frac{1}{6}\left([x]^2[y] + [x^2][y] + 2[x][xy] + 2[x^2y]\right), \tag{A.51}$$

$$\begin{array}{l}\boxed{x}\boxed{x}\\ \boxed{y}\end{array} = \frac{1}{3}\left([x]^2[y] + [x^2][y] - [x][xy] - [x^2y]\right) \tag{A.52}$$

[1]

$$I_{R,R'} = \frac{1}{D_S}\chi_S\left(X \otimes Y\right)\chi_S\left(\bar{X} \otimes \bar{Y}\right) + \frac{1}{D_{S'}}\chi_{S'}\left(X \otimes Y\right)\chi_{S'}\left(\bar{X} \otimes \bar{Y}\right) \tag{A.54}$$

$$S = \begin{array}{|c|c|c|}\hline x & x & y \\ \hline\end{array}, \quad S' = \begin{array}{|c|c}\hline x & \multicolumn{1}{c|}{x} \\ \hline y & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \tag{A.55}$$

Similarly, if $R = \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$, $R' = \begin{array}{|c|}\hline \, \\ \hline\end{array}$, then

$$\begin{array}{|c|}\hline x \\ \hline x \\ \hline\end{array} \otimes \begin{array}{|c|}\hline y \\ \hline\end{array} = \begin{array}{|c|c}\hline x & \multicolumn{1}{c|}{y} \\ \hline x & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline x \\ \hline x \\ \hline y \\ \hline\end{array} \tag{A.56}$$

$$\begin{array}{|c|c}\hline x & \multicolumn{1}{c|}{y} \\ \hline x & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} = \frac{1}{3}\left([x]^2[y] - [x^2][y] + [x][xy] - [x^2y]\right) \quad \begin{array}{|c|}\hline x \\ \hline x \\ \hline y \\ \hline\end{array} = \frac{1}{6}\left([x]^2[y] - [x^2][y] - 2[x][xy] + 2[x^2y]\right)$$

[2]

These diagrams describe what are called the *restricted Schur polynomials* in literature.

[1]We must be careful about how we project our tensors onto the Young tableaux. We have:

$$T^{ab} = T^{ba}, \quad T^{ab|c} = T_S^{abc} + T_{S'}^{ab,c} \quad T_S^{abc} = \frac{1}{3}(T^{ab|c} + T^{bc|a} + T^{ca|b}), \quad T_{S'}^{ab,c} = \frac{1}{3}(2T^{ab|c} - T^{bc|a} - T^{ca|b})$$

where T_S is totally symmetric as expected. However, projecting $T_{S'}$ requires more delicate handling. The tensor satisfies

$$T_{S'}^{ab,c} + T_{S'}^{bc,a} + T_{S'}^{ca,b} = 0 \tag{A.53}$$

[2]Similarly,

$$T^{ab} = -T^{ba}, \quad T^{ab|c} = T_S^{abc} + T_{S'}^{ab,c} \tag{A.57}$$

$$T_S^{abc} = \frac{1}{3}(T^{ab|c} + T^{bc|a} + T^{ca|b}), \quad T_{S'}^{ab,c} = \frac{1}{3}(2T^{ab|c} - T^{bc|a} - T^{ca|b}) \tag{A.58}$$

We have dealt with each term seperately; we now combine our results for different representations to compute the initial four-matrix Harish-Chandra integral. We find:

$$I = \sum_{R,R'} d_R d_{R'} \int \delta U\, \chi_R(UXU^\dagger \bar{X})\chi_{R'}(UYU^\dagger \bar{Y}) \tag{A.59}$$

$$= \sum_{R,R'} d_R d_{R'} \sum_{S\in R\otimes R'} \int \delta U\, \mathrm{Tr}_S \left[U_S Z_S U_S^\dagger \bar{Z}_S\right] \tag{A.60}$$

$$= \sum_{R,R'} d_R d_{R'} \sum_{S\in R\otimes R'} \frac{1}{D_S} \mathrm{Tr}_S(Z_S)\mathrm{Tr}_S(\bar{Z}_S), \quad Z = X_R \otimes Y_{R'} \tag{A.61}$$

where d_R is the dimension of R representation of $S_{|R|}$ group, divided by $|R|!$.

Equivalently, the integral can also be written as:

$$I = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \int \delta U\, \left(\mathrm{Tr}[UXU^\dagger \bar{X}]\right)^m \left(\mathrm{Tr}[UYU^\dagger \bar{Y}]\right)^n \tag{A.62}$$

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \sum_{S\in V^{\otimes m}\otimes V^{\otimes n}} \frac{1}{D_S} \mathrm{Tr}_S(Z_S)\mathrm{Tr}_S(\bar{Z}_S) \tag{A.63}$$

The first few terms are:

$$I = 1 + \begin{ytableau} x \end{ytableau}^2 + \begin{ytableau} y \end{ytableau}^2 + \begin{ytableau} x & y \end{ytableau}^2 + \begin{ytableau} x \\ y \end{ytableau}^2 + \frac{1}{2}\begin{ytableau} x & x \end{ytableau}^2 + \frac{1}{2}\begin{ytableau} x \\ x \end{ytableau}^2 + \frac{1}{2}\begin{ytableau} y & y \end{ytableau}^2 + \frac{1}{2}\begin{ytableau} y \\ y \end{ytableau}^2 \tag{A.64}$$

$$+ \frac{1}{3!}\left(\begin{ytableau} x & x & x \end{ytableau}^2 + 2\begin{ytableau} x & x \\ x \end{ytableau}^2 + \begin{ytableau} x \\ x \\ x \end{ytableau}^2\right) + \frac{1}{2!}\left(\begin{ytableau} x & x & y \end{ytableau}^2 + \begin{ytableau} x & x \\ y \end{ytableau}^2 + \begin{ytableau} x & y \\ x \end{ytableau}^2 + \begin{ytableau} x \\ x \\ y \end{ytableau}^2\right) \tag{A.65}$$

$$+ \frac{1}{2!}\left(\begin{ytableau} x & y & y \end{ytableau}^2 + \begin{ytableau} y & y \\ x \end{ytableau}^2 + \begin{ytableau} x & y \\ y \end{ytableau}^2 + \begin{ytableau} y \\ y \\ x \end{ytableau}^2\right) + \frac{1}{3!}\left(\begin{ytableau} y & y & y \end{ytableau}^2 + 2\begin{ytableau} y & y \\ y \end{ytableau}^2 + \begin{ytableau} y \\ y \\ y \end{ytableau}^2\right) + \ldots \tag{A.66}$$

where the square of a Young diagram represents the product of the character of $X \otimes Y$ and the character of the same representation of $\bar{X} \otimes \bar{Y}$, divided by the dimension D_R of this representation, e.g.

$$\boxed{x}\boxed{y}^2 \equiv \frac{1}{D_S}\chi_S(Z_S)\chi_S(\bar{Z}_S) \tag{A.67}$$

$$= \frac{2}{(N+1)N}\frac{[X][Y]+[XY]}{2}\frac{[\bar{X}][\bar{Y}]+[\bar{X}\bar{Y}]}{2}, \tag{A.68}$$

$$S = \boxed{x}\boxed{y} \tag{A.69}$$

The expansion above matches (4.17), the $N = 2$ integral formula, precisely.

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