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High Energy Problems, Low Energy Solutions

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High Energy Problems, Low Energy Solutions

A dissertation submitted in partial satisfaction of the
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by

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2019

The dissertation of Shauna Michelle Kravec is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California San Diego

2019

DEDICATION

To Toni Michelle Kravec, who I wish could be here more than ever.

EPIGRAPH

The explorer who will not come back or send back his ships to tell his tale is not an explorer, only an adventurer; and his sons are born in exile.

—Ursula K. Le Guin, The Dispossessed

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SM Kravec, Sridip Pal, "Non-Relativistic Conformal Field Theories in the Large Charge Sector ", *Journal of High Energy Physics*, 2019.

SM Kravec, John McGreevy, Brian Swingle, "All-Fermion Electrodynamics and Fermion Number Anomaly Inflow ", *Physical Review D*, 2015.

SM Kravec, John McGreevy "A Gauge Theory Generalization of the Fermion-Doubling Theorem ", *Physical Review Letters*, 2013.

AJ Breindel, RE Stuart, WJ Bock, DN Stelter, SM Kravec, EM Conwell, "Hole Wave Functions and Transport with Deazaadenines Replacing Adenines in DNA ", *Journal of Physical Chemistry B*, 2013.

SM Kravec, CD Kinz-Thompson, EM Conwell, "Localization of a Hole on an Adenine-Thymine Radical Cation in B-Form DNA in Water ", *Journal of Physical Chemistry B*, 2011.

ABSTRACT OF THE DISSERTATION

High Energy Problems, Low Energy Solutions

by

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This dissertation covers topics in the intersection of high energy and condensed matter physics. It is motivated by the question, ‘Given information about physics at the highest energy scale, how does that constrain the theory at low energies?’ This is a difficult question as complexity can be ‘emergent’, leading to a rich and unpredictable variety of possibilities.

In the first half of the thesis we discuss ‘symmetry protected topological phases’; states of matter whose low-energy physics is described by an ‘invertible’ topological field theory. Such theories encode ‘anomalies’ and imply exotic surface states when defined on a manifold with boundary. We study a model in five dimensions whose anomalous boundary is electromagnetism, but where the elementary electric and magnetically charged particles are fermions.

In the second half we discuss ‘non-relativistic conformal field theories’ at finite charge density. One possibility for the low-energy physics of such systems is that of a superfluid ground-state, realized experimentally in systems of ultra cold fermi gases. Additionally, such theories have a ‘state-operator correspondence’ which relates their operator spectrum to states in a harmonic trap. This enables us to use the field theory of the superfluid to calculate properties of the operator spectrum systemically in the limit of large charge.

Chapter 1

Introduction

My research lies at the intersection between condensed matter and high energy theory. Central to both of these fields is the question: What can we say about the behavior of quantum field theories at low-energy? Broadly interpreted, this is the problem of characterizing different phases of matter. In the generic case, where the field theory is strongly interacting and perturbation theory is not reliable, this is a very difficult question and one where experiment has proved to be a valuable guide. For example, a gas of electrons in two dimensions can be metallic despite strong magnetic field, and a gas of quarks and gluons confines to neutral hadrons at low temperatures.

What are some of the behaviors of quantum field theories at low energy? One possible outcome is a mass gap for all excitations. At energies below the gap, this may appear to be a boring possibility. However, not all gapped phases are equal. One non-trivial possibility is that the low-energy physics is described by a topological quantum field theory. These topological phases are distinct from more conventional gapped systems, and host a wide variety of exotic phenomena like the fractionalization of charge and statistics, anomalous boundary states, and topologically protected degeneracy. Experimental realizations include the quantum hall states, as well as topological insulators and superconductors.

The simplest possible topological phases are known as symmetry protected topological phases or SPTs. These systems dont host fractionalized particles, and appear trivial from a bulk point of view. However if they are defined on a space with a boundary, they give rise to exotic physics at that edge. If considered in isolation, the theories that describe the edges of SPTs have anomalies. In order to truly be symmetric, and sometimes even well defined, they require the extra dimensional bulk. The topological field theories that describe SPTs exactly encode the physics of the anomaly.

In the first half of this thesis (Chapters 2 and 3), a question we (John McGreevy, Brian Swingle, and I) answered was: Are there quantum field theories in D=3+1 dimensions which are only realized as the boundary of some D=4+1 topological phase? From a high energy point of view, this is equivalent to discovering new anomalies. A particularly subtle example is that of Maxwell theory, where both the electron and the monopole are fermions. This all-fermion electrodynamics is actually impossible to realize without either a charge-neutral fermion or an entire extra dimension. We argued this on the basis of electromagnetic duality, and gave a construction of the five-dimensional topological field theory which captures the anomaly.

As the edge states of many topological phases demonstrate, another possible outcome is the existence of gapless excitations at low-energies. An important case is systems tuned to a critical point. At these special places in parameter space, there is an emergent scale symmetry. The low-energy physics is then often described by a conformal field theory. The importance of conformal field theories is well appreciated in both condensed matter, for their use in predicting critical exponents and other universal properties, and in high energy physics, as highly symmetric quantum field theories and for their role in the AdS/CFT correspondence.

While we know a lot about conformal field theories on the basis of symmetry alone, it is generically difficult to calculate many quantities of interest. This is again due to the presence of strong interactions. This has led people to develop many interesting and powerful techniques, such as the conformal bootstrap. Another approach, developed recently, applies to CFTs with additional global symmetry. States of the CFT with finite charge densities are amenable to an effective field theory description, controlled in the limit of large density. This is essentially a condensed matter of CFTs. On the sphere such states are related to charged operators. For example, the ground state energy determines the lowest lying operator dimension. This allows one to use the EFT to calculate the operator spectrum and correlation functions.

As described in the second half of this thesis (Chapters 4 and 5), we (Sridip Pal and I) extended this approach to the case where scale invariance exists, but time and space are distinguished. These 'non-relativistic conformal field theories' (NRCFTs) appear as effective descriptions of several condensed matter systems, such as ultracold fermi gases at unitarity. They have a state-operator correspondence, which relates operators to states in a harmonic potential. By assuming a superfluid ground state in this potential, something which has been experimentally demonstrated for unitary fermions, we were able to construct the relevant EFT to arbitrary order and in any dimension. Using this EFT, we were able to compute the dimensions of charged scalar operators, the spectrum of low-lying excited states, as well as correlation functions of charged operators. We also computed the dimensions of operators with large spin and charge. These are especially noteworthy, as they correspond to vortex configurations of the superfluid. Together, these results constrain the large charge sector of any NRCFT with a superfluid ground state.

Chapter 2

A gauge theory generalization of the fermion-doubling theorem

2.1 Abstract

It is possible to characterize certain states of matter by properties of their edge states. This implies a notion of 'surface-only models': models which can only be regularized at the edge of a higher-dimensional system. After incorporating the fermion-doubling results of Nielsen and Ninomiya into this framework, we employ this idea to identify new obstructions to symmetry-preserving regulators of quantum field theory. We focus on an example which forbids regulated models of Maxwell theory with manifest electromagnetic duality symmetry.

2.2 Introduction

This paper is about obstructions to symmetry-preserving regulators of quantum field theories (QFTs), in 3+1 spacetime dimensions. The most famous example of such an obstruction is the theorem of Nielsen and Ninomiya [2, 3, 4, 5]. The basic statement of this theorem forbids regularization of free fermions preserving chiral symmetry[1]. We will approach the study of such obstructions by thinking about certain states of matter, in one higher dimension, with an energy gap (*i.e.* the energy of the first excited state is strictly larger than the energy of the groundstate, even in thermodynamic limit). More precisely, we will study the low-energy effective field theories of such states; below the energy gap, these are topological field theories (TFTs) in $4+1$ dimensions. Such states will be difficult to realize in the laboratory. We will use them to demonstrate an obstruction to any regularization of Maxwell theory which preserves manifest electric-magnetic duality.

To convey the idea of how the study of such extra-dimensional models can be useful for understanding the practical question of symmetry-preserving regulators of 3+1-dimensional QFTs, we must digress on the subject of realizations of symmetries in QFT and in condensed matter. A basic question in condensed matter physics is to enumerate the possible gapped phases of matter. Two gapped phases are equivalent if they are adiabatically connected (varying the parameters in the Hamiltonian whose ground state they are to get from one to the other, without closing the energy gap). An important possible distinguishing feature of different phases is how the symmetries of the Hamiltonian are realized. This leads to Landau's criterion which characterizes states by what symmetries of the Hamiltonian are broken by the groundstate. Considering this to be understood, it is interesting to refine the question to "How do we distinguish gapped phases that do not break any symmetries?"

A sophisticated answer to this question, vigorously advocated by Wen [6, 7], is *topological order*. A phase with topological order can be characterized by three related properties[2]:

1. *Fractionalization of symmetries*, that is, emergent excitations which carry statistics or quantum numbers which are fractions of those of the constituents.

2. *Topology-dependent groundstate degeneracy*; this is a consequence of property 1, since the

[1] In odd spacetime dimensions, chiral symmetry is replaced by parity symmetry in this statement.

[2] These are sufficient conditions; a complete characterization is possible in two dimensions in terms of adiabatic modular transformations [7] the generalization to higher dimensions seems not to be known yet.

groundstates must represent the algebra of flux insertion operators associated with adiabatic braiding of quasiparticle-antiquasiparticle pairs.

3. *Long-range entanglement*, which may be quantified [8, 9] by the topological entanglement entropy γ, a universal piece of the von Neumann entropy of the reduced density of region A of surface area $\ell(A)$ in the groundstate: $S(A) = \ell(A)\Lambda - \gamma$ (where Λ is nonuniversal). For abelian states, γ is the logarithm of the number of torus groundstates, and vanishes for states with short-ranged entangled states [10].

Topological order is interesting and difficult. Recently, a simpler question has been fruitfully addressed: "What are possible (gapped) phases that don't break symmetries and don't have topological order?" (For a nice review, see the second part of [11].) In this paper we will use the spatial-topology-independence of the groundstate degeneracy as our criterion for short-range entanglement (SRE). The E_8 state in 2+1 dimensions [12, 13, 14, 15] is a known exception to this characterization.

A way to characterize such nearly-trivial states is to study them on a space with boundary. A gapped state of matter in $d+1$ dimensions with short-range entanglement can be (at least partially) characterized (within some symmetry class of Hamiltonians) by (properties of) its edge states (*i.e.* what happens at an interface with the vacuum, or with another SRE state). The idea is simple: if we cannot adiabatically deform the Hamiltonian in time from one state to another, we must also not be able to deform the Hamiltonian *in space* from one state to another, without something interesting happening in between. The SRE assumption is playing an important role here: we are assuming that the bulk state has short-ranged correlations, so that changes we might make at the surface cannot have effects deep in the bulk.

A useful refinement of this definition incorporates symmetries of the Hamiltonian: An *symmetry-protected topological (SPT) state*, protected by a symmetry group G, is a SRE, gapped state, which is not adiabatically connected to a product state by local Hamiltonians preserving G. Prominent examples include free fermion topological insulators in 3+1d, protected by $U(1)$

and $\mathbb{Z}_2^{\mathcal{T}}$, which have an odd number of Dirac cones on the surface. Free fermion topological insulators have been classified [16, 17]. Interactions affect the connectivity of the phase diagram in both directions: there are states which are adiabatically connected only through interacting Hamiltonians [18], and there are states which only exist with interactions, including all SPT states of bosons [19, 20, 21, 22].

A simplifying property is that the set of SPT states (protected by a given symmetry group G) forms a group.

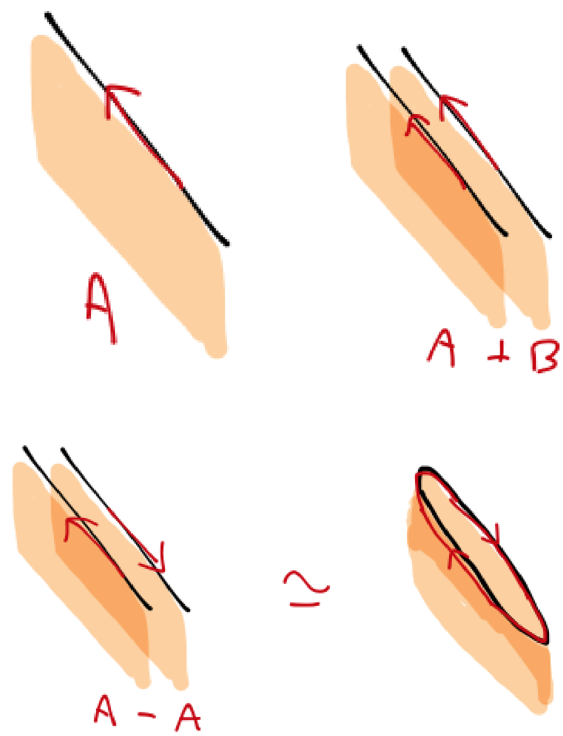

Figure 2.1: Visual depiction of the group operation on SPT states.

The addition law is tensor product of Hilbert spaces, and addition of all interaction terms allowed by G. The inverse operation is simply addition of the mirror image. We emphasize that with topological order, destroying the edge states does not modify the nontrivial physics (*e.g.* fractional charges) in the bulk: states with topological order do not form a group. A conjecture for the identity of this group (in d space dimensions, for given G) is the group cohomology group $H^{d+1}(G,U(1))$ [23]. Exceptions to this conjecture have been found [20, 21, 24, 25] and are understood by [13]. For our present purposes, we do not need to know how to classify these states. The existence of this group struture has the following implication [20, 21, 24, 25, 26], which we may pursue by examples.

Suppose that the edge theory (*e.g.* with vacuum) of a $(D+1)$-dimensional SPT_G state were realized *otherwise* – that is, intrinsically in D dimensions, with a local Hamiltonian respecting G. Then we could paint the conjugate local theory on the surface of a $(D+1)$-dimensional space hosting the SPT state without changing anything about the bulk. By allowed (G-preserving, local) deformations of the surface Hamiltonian, we could then completely destroy the edge states. But this contradicts the claim that we could characterize the $(D+1)$-dimensional SPT state by its edge theory. We conclude that the edge theories of SPT_G states cannot be regularized intrinsically in D dims, while preserving G. We will refer to them as "surface-only models". More generally, we can consider the interface between *pairs* of SRE states; the edge theory for interface between any pair of SPT states is also a surface-only theory.

The Nielsen-Ninomiya result is implied by this logic. Consider free massive (relativistic, for convenience) fermions in 4+1 dimensions: $S = \int d^{4+1}x\bar{\Psi}\left(\partial\!\!\!/ + m\right)\Psi$. The two signs of m label distinct phases. One proof of this arises by coupling to an external gauge field via the fermion number current $\Delta S = \int d^5xA^\mu\bar{\Psi}\gamma_\mu\Psi$. The effective action involves a quantized topological term:

$$\log\int[D\Psi]e^{\mathbf{i}S_{4+1}[\Psi,A]} = \frac{m}{|m|}\int\frac{\mathrm{d}^5x}{24\pi^2}\varepsilon_{abcde}A_aF_{bc}F_{de}.$$

The domain wall between the two phases hosts an exponentially localized chiral fermion field [27, 28], a fact which has been useful for lattice simulations [28, 29].

A more famous $D = 3+1$ analog begins with a massive fermion in 3+1 dimensions:

$$S = \int d^{3+1}x\bar{\Psi}\left(\partial\!\!\!/ + m + \mathbf{i}\hat{m}\gamma^5\right)\Psi$$

If we demand time-reversal ($\mathcal{T}$) invariance, $\hat{m} = 0$, and $\pm m$ label distinct SPT states protected by $\mathbb{Z}_2^{\mathcal{T}}$. Coupling to an external gauge field

$$\log \int [D\Psi] e^{\mathbf{i} S_{3+1}[\Psi, A]} = \frac{m}{|m|} \int \frac{\mathrm{d}^4 x}{32\pi^2} \varepsilon^{abcd} F_{ab} F_{cd}.$$

produces a quantized magnetoelectric effect with $\theta = 0, \pi$ [30]. The domain wall hosts a single Dirac cone in 2+1d. (We must emphasize that in both the above examples the protecting symmetry which distinguishes these states from the trivial insulator includes charge conservation.)

This suggests the following strategy for generalizing these results away from free fermions. Study a simple (unitary) gapped or topological field theory in 4+1 dimensions without topological order, with symmetry G. Consider the model on the disk with some boundary conditions. The resulting edge theory is a "surface-only theory with respect to G" – it cannot be regulated by a local $3+1$-dim'l model while preserving G.

What does it mean to be a surface-only state? Such a model is perfectly consistent and unitary; it can be realized at the edge of some gapped bulk theory. However, it cannot be regularized in a local way consistent with the symmetries in absence of the bulk.

It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations. Why 'probably'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV. An example of a SPT-forbidden symmetry emerging in the IR occurs in critical Heisenberg spin chains, where the spin symmetry is enhanced to an $\mathrm{O}(4)$ which rotates the spin order into the valence bond solid order.

It also does not rule out symmetric UV completions that include gravity, or decoupling limits of gravity/string theory. (UV completions of gravity have their own complications!) String theory strongly suggests the existence of Lorentz-invariant states of gravity with chiral fermions and lots of supersymmetry (the $E_8 \times E_8$ heterotic string, chiral matter on D-brane intersections, self-dual tensor fields...) some of which can be decoupled from gravity.

2.3 A simple family of topological field theories in 4+1 dimensions

Here we will study simple field theories in 4+1 dimensions, whose path integral is gaussian. We follow an analog of the K-matrix approach used in [14] to study 2+1 dimensional SPT states.

Specifically, consider $2N_B$ 2-form potentials B^I_{MN}, in $4+1$ dimensions, with action

$$S_{\rm CS}[B] = \frac{K_{IJ}}{2\pi}\int_{\mathbb{R}\times\Sigma_4} B^I \wedge {\rm d}B^J \ . \tag{2.1}$$

$M,N = 0..4$ is a 4+1 dimensional spacetime index, $I,J = 1..2N_B$, and Σ is a smooth 4-manifold (sometimes with boundary).

In $4\ell+1$ dims, K is a *skew*-symmetric integer $2N_B \times 2N_B$ matrix. This basic difference from 2+1 dimensional CS theory (where the K-matrix is symmetric) arises from the fact that the wedge product of two-forms is symmetric, and hence $B\wedge {\rm d}B = \frac{1}{2}{\rm d}(B\wedge B)$, and thus the symmetric part of K produces a total derivative. The action is independent of a choice of metric on $\mathbb{R}\times\Sigma_{2p}$. Just as one may add an irrelevant Maxwell term to CS gauge theory, these models may be considered as the $g\to\infty$ limit of the non-topological models with ('topologically massive' [31]) propagating two-forms with action

$$S = \int \frac{\tau_{IJ}}{M}{\rm d}B^I \wedge \star {\rm d}B^J + S_{\rm CS}[B] \ ; \tag{2.2}$$

the mass scale M determines the energy gap.

These models have a long history, including [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. The canonical quantization of closely-related models was discussed in [36, 37]. (The difference from the models we study comes from issues related to the compactness of the gauge group, as in (3.4) below. This includes the quantization of the level and the resulting conclusion about the finiteness of the dimension of the Hilbert space.) The linking number of (homologically-trivial) pairs of surfaces was constructed as an observable of this theory in [38, 37].

Moore [43] refers to these models as both trivial and difficult. They are trivial in that the path integral is gaussian, and difficult in the sense that there are lots of subtleties in defining what's being integrated over. We will not highlight all these subtleties, but we will endeavor to not be wrong[3]. Moore and collaborators [43] have studied such generalizations of CS theory and their edge states with great care. In particular, the paper [41] constructs the partition function for the theory we study below and relates it to a piece of the $\mathcal{N} = 4$ super Yang-Mills partition function.

This relation arises because these models are realized as a 'topological sector' of type IIB strings on $AdS_5 \times X_5$ [39]; the simplest case with two 2-forms arises for $X_5 = S^5$, where the forms may be identified as $B^1 \equiv B_{NSNS}, B^2 \equiv C_{RR}$ which couple to fundamental strings and D1-branes respectively; the CS term (2.1) arises from

$$S_{IIB} \ni \frac{1}{2\pi}\int_{AdS_5\times S^5} F^{(5)}_{RR} \wedge B \wedge \mathrm{d}C = \frac{N}{2\pi}\int_{\mathbf{R}\times\Sigma} B \wedge \mathrm{d}C$$

and allows for the ending of N F-strings holographically dual to the baryon vertex of $\mathcal{N} = 4$ SYM [45, 46]. S-duality of IIB string theory acts by interchanging the two-forms. This symmetry of (2.1) will play a crucial role below.

We focus on the Abelian case. The action S is invariant under the gauge redundancies

$$B^I \simeq B^I + \mathrm{d}\lambda^I \tag{2.3}$$

[3]This paper [44] gives a readable description of the dangers here. We can go a long way toward avoiding running afoul of the subtleties which are resolved by formulating the model in terms of differential cohomology if we only consider manifolds whose homology has no torsion. Even with this restriction we can say something interesting. More subtle distinctions between 4+1d phases that only arise on manifolds with torsion in their homology will have to wait for future work.

where λ^I are one-forms. We will impose large gauge equivalences:

$$B^I \simeq B^I + n^\alpha \omega_\alpha, \quad [\omega^\alpha] \in H^2(\Sigma, \mathbb{Z}), \;\; n^\alpha \in \mathbb{Z}^{b^2(\Sigma)} \tag{2.4}$$

where $b^2 \equiv \dim H^2(\Sigma, \mathbb{Z})$ is the 2nd Betti number of Σ. In the case of 2+1 CS gauge theory of $\mathbb{R} \times \Sigma_2$, the analogous identification arises naturally from large gauge transformations [34, 35]

$$A \equiv A + \mathbf{i} g^{-1} \mathrm{d}g, \;\; \text{with} \quad g(x) = e^{\mathbf{i} \sum_{\alpha=1}^{b^1(\Sigma_2)} \int_{x_0}^{x} n^\alpha \omega_\alpha}$$

where ω_α form a basis of $H^1(\Sigma_2, \mathbb{Z})$, and x_0 is a base point. In the 4+1 dimensional case, we don't know what a group-valued one-form is, but retain the natural identification (3.4). (A mathematical formalism which produces this identification in a 'natural' way is described in [44].) This identification was not imposed in the otherwise-identical models studied in [36].

This class of models has been used [47, 48, 49, 40, 41, 50, 51, 43] to 'holographically' *define* the partition function of the edge theory. (These papers focus mainly on the case of bulk spacetime dimension $D = 4\ell + 3$: 1+1d chiral CFTs, conformal blocks of the 5+1d (2,0) theory.) In this paper, we are using this same relation to a different end.

Finally, we note that the simplest model (2.1) with one pair of two-forms is equivalent to a $\mathbb{Z}_k$ *2-form* gauge theory [49, 52]. The case we will be most interested in, with $k = 1$, can therefore be regarded as a kind of '$\mathbb{Z}_1$ gauge theory', which nevertheless has a something to teach us.

2.4 Bulk physics

2.4.1 When is this an EFT for an SPT state?

For this subsection we suppose that $\partial\Sigma$ is empty. We wish to ascertain the size of the Hilbert space, and its dependence on the topology of Σ. Thinking of this as the EFT for some 4+1 dimensional gapped state of matter, at energies below the gap, this number is the groundstate degeneracy (up to $e^{-(\text{system size})\cdot\text{gap}}$ finite-size effects). If this degeneracy is dependent on the topology of Σ, then this state has topological order in the bulk [6]. A closely related calculation appears in section 6 of [44], in the case of a theory of p-forms in $2p+1$ dimensions, for the case of odd p.

Many aspects of the problem are analogous to the case of 2+1 dimensional CS 1-form gauge theory. The kinetic term in (2.2) is analogous to the Maxwell term, and g has units of energy. With $g < \infty$ in (2.2), there are excited states of energy $E \propto g$, analogous to higher Landau levels. As in 2+1d, this is an example of an inclusion of metric dependence which does not change the nature of the phase. We will focus on the limit $g \to \infty$.

And as in the 2+1d case, the identification (3.4) means that the gauge-inequivalent operators are labelled by cohomology classes, here $[\omega_\alpha] \in H^2(\Sigma, \mathbb{Z})$:

$$\mathcal{F}_\omega(m) \equiv e^{2\pi \mathbf{i} m_I \int_\omega B^I}$$

where the identification on B implies $m^I \in \mathbb{Z}$. These operators satisfy a Heisenberg algebra:

$$\mathcal{F}_{\omega_\alpha}(m)\mathcal{F}_{\omega_\beta}(m') = \mathcal{F}_{\omega_\beta}(m')\mathcal{F}_{\omega_\alpha}(m)e^{2\pi \mathbf{i} m_I^\alpha m_J'^\beta (K^{-1})^{IJ} I_{\alpha\beta}}.$$

This kind of algebra is familiar from the quantum Hall effect, and from 2+1 dimensional CS theory. Its only irreducible representation is the Hilbert space of a particle on a 'fuzzy torus.' of dimension $2N_B b^2(\Sigma)$. The number of states depends on the intersection form on the second homology of Σ:

$$\int_\Sigma \omega_\alpha \wedge \omega_\beta = I_{\alpha\beta}.$$

This is a $b^2(\Sigma) \times b^2(\Sigma)$ *symmetric* matrix, which has various properties guaranteed by the theory of 4-manifold topology [53]. For a smooth, compact, oriented, simply-connected 4-manifold, it is unimodular (*i.e.* has $|\det I| = 1$). It is even if Σ is a spin manifold (*i.e.* it admits spinor fields, *i.e.* its first Steiffel-Whitney class vanishes). Its signature is not definite.

Consider the simplest nontrivial case where $b^2(\Sigma) = 1$, which case I is a 1×1 matrix. The simplest example is $\Sigma = \mathbb{P}^2$ where $I = 1$ [53].

We may arrive at the same conclusion by expanding the action in a basis of cohomology representatives:

$$B^I = \sum_{\alpha=1}^{b^2(\Sigma_4,\mathbb{Z})} \omega_\alpha b^{I\alpha}(t), \quad \text{span}\{\omega_\alpha\} = H^2(\Sigma_4,\mathbb{Z}),$$

so

$$S = \frac{K_{IJ}}{2\pi} \int dt \int_{\Sigma_4} \omega_\alpha \wedge \omega_\beta b^{I\alpha} \dot{b}^{J\beta} = \frac{K_{IJ}}{2\pi} \int dt\, I_{\alpha\beta} b^{I\alpha} \dot{b}^{J\beta}$$

which describes a particle in $b^2(\Sigma)$ dimensions with a magnetic field in each pair of dimensions of strength k.

First we further assume that there is just one pair of B-fields, $N_B = 1$ and (WLOG by a $GL(N_B,\mathbb{Z})$ rotation) we take $K = k\mathbf{i}\sigma^y$. Then the hilbert space is that of a particle on a periodic 1d lattice with k sites:

$$\mathcal{F}_1|\Omega\rangle = |\Omega\rangle, \; |w\rangle = \mathcal{F}_2^w|\Omega\rangle, \;\; w = 1...k.$$

Here $\mathcal{F}_I \equiv \mathcal{F}_I(1)$. (Note that $\mathcal{F}_I^k = 1\!\!1$.)

Now keep this simplest Σ but take more pairs of B-fields. We can skew-diagonalize the K-matrix, so that we find N_B copies of the previous discussion, with various k, which are the skew-eigenvalues of K:

$$K = \begin{pmatrix} 0 & k_1 & 0 & 0 & \dots \\ -k_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & k_2 & \dots \\ 0 & 0 & -k_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

So the dimension of the Hilbert space of the bulk in this case is

$$\dim \mathcal{H}(b_2(\Sigma) = 1, I = 1) = \prod_{a=1}^{N_B} k_a = \text{Pf}\, K,$$

the pfaffian.

If $I > 1$, it multiples k_a in the commutation relation. This gives

$$\dim \mathcal{H}(b_2(\Sigma) = 1, I) = \prod_{a=1}^{N_B} k_a = \text{Pf}\ (IK)\,.$$

Now consider a more general 4-manifold where $b_2 > 1$. Then there are $b_2 N_B$ pairs of canonically-conjugate variables. By $GL(2b_2N_B, \mathbb{Z})$ rotations, we can choose a basis to diagonalize $I \otimes \mathcal{K}$. Since I is symmetric, this matrix is still skew-symmetric. If we call its skew-eigenvalues λ_A, and the number of groundstates is

$$\boxed{\dim \mathcal{H} = \prod_{A=1}^{b_2(\Sigma)N_B} \lambda_A = \text{Pf}\ (I \otimes \mathcal{K})}\,.$$

The conclusion from this discussion is that if $I \otimes K$ has any skew-eigenvalues which are larger than one, the system has bulk topological order: a topology-dependent groundstate degeneracy.

2.4.2 This is a model of bosons

We observe that if Σ is a spin manifold, then we may take the entries in K to be half-integer, while preserving the consistency of the theory (in particular, the fact that the number of groundstates is an integer). A choice of spin structure was not required for the calculation we did here, but would be for other observables. Such models would provide low-energy effective theories for fermionic models, analogous to odd k CS gauge theories in two dimensions, as is familiar from the integer quantum Hall effect and was studied in great detail by [50].

The preceding statements provide low-energy evidence that the models (2.1) for integer K can arise as the low-energy EFT for models of bosons. High-energy evidence for this claim comes from [54], which constructs a local bosonic lattice model which produces this EFT at low energies.

2.5 Surface states

2.5.1 2+1d case with boundary

Here we briefly review some relevant properties of 2+1d CS gauge theory on a spacetime of the form $\mathbb{R} \times \Sigma$ with a spatial boundary, $\partial\Sigma$ [34, 35, 49, 50, 6]. We will further assume for convenience that $H^p(\Sigma, \partial\Sigma) = \delta^{0,p}$ – the topology (specifically, the relative cohomology) of the bulk is trivial.

The gauge group consists only of gauge transformations which go to the identity at the boundary. Transformations which act nontrivially do not leave the CS action invariant; if we wished to mod out by them we would need to add degrees of freedom to cancel this variation, and would arrive at the same description. The CS equation of motion for A_0 is a constraint which imposes $0 = F$ which means that

$$A = g^{-1}\mathrm{d}g \tag{2.5}$$

which we can choose to vanish in the bulk. In the abelian case we can take $A = \mathrm{d}\phi$, with $\phi \simeq \phi + 2\pi$ compact. We arrive at a theory of compact free bosons.

The Hilbert space of edge states is specified universally by the bulk – the CS action evaluated on (2.5) gives $S_0 = \int_{\mathbf{R}\times\partial\Sigma} \frac{k^{IJ}}{4\pi}\partial_t\phi^I\partial_x\phi^J$ which determines the canonical equal-time commutators

$$[\phi^I(x), \phi^J(x')] = 2\pi \mathbf{i} k^{-1}_{IJ}\theta(x - x').$$

However, the Hamiltonian governing these bosons comes entirely from the choice of boundary conditions, since the bulk action is linear in time derivatives. The non-universal velocity of the chiral edge modes is not encoded in the Chern-Simons action; this is natural from the quantum Hall point of view, where this velocity is determined by the slope of the potential confining the Hall droplet. The non-universal velocity term arises if we choose the 'boosted' boundary condition $A_0 + vA_x|_{\partial\Sigma} = 0$, where x is the coordinate along the boundary. (Note that this is a boundary condition, not a gauge choice – v affects the physics.) This gives $S[\phi] = \frac{k}{4\pi}\int(\partial_t + v\partial_x)\phi\partial_x\phi$, which shows that the edge boson is chiral; the sign of v is correlated with the sign of k in order that the energy be bounded below. Alternatively, and more covariantly, we may use units with $v = 1$ and impose $A - \star A|_{\partial\Sigma} = 0$ [43]. We may neglect the possibility of a matrix of velocities if we include a coupling to disorder [55, 56], which we should unless we are interested in SPT states protected by translation invariance.

2.5.2 4+1d abelian two-form gauge theory with boundary

As in the previous discussion, we consider the bulk spacetime to be $\mathbb{R} \times \Sigma_4$ with nontrivial boundary $\partial\Sigma_4$, and trivial topology in the bulk. The gauge transformations must act trivially at the boundary.

We will focus on the simplest case where $K = k\mathbf{i}\sigma^2$. Consider

$$\begin{aligned} S[B,C] &= \frac{k}{2\pi}\int_{\mathbf{R}\times\Sigma_4} (B\wedge \mathrm{d}C - C\wedge \mathrm{d}B) \\ &+ \int_{\mathbf{R}\times\partial\Sigma_4} \left(\frac{1}{4g^2} C\wedge\star_4 C + \frac{k}{4\pi} B\wedge C\right) \end{aligned}$$

A convenient boundary condition (imposed by free variation of (2.6) with the boundary terms indicated) is:

$$\left(\frac{k}{2\pi}B - \frac{1}{2g^2}\star_4 C\right)|_{\partial\Sigma_4} = 0 \ .$$

The path integral over B produces a delta function forcing C to be flat [49]:

$$\int [DB] e^{\mathbf{i}S} = \delta[\mathrm{d}C] \quad \Longrightarrow \quad C = \mathrm{d}a \ .$$

$$S[C = \mathrm{d}a] = \boxed{\frac{1}{4g^2}\int_{\mathbf{R}\times\partial\Sigma_4} \mathrm{d}a\wedge\star_4 \mathrm{d}a}$$

This is ordinary Maxwell theory. We know how to regularize this theory with a local lattice boson model [4] We infer from the logic described above that we are forced to break some symmetry of (2.1) in order to realize its edge states in a local way. What symmetry must we break in writing such a local model?

We answer this burning question in the next section, after which we comment on more general boundary conditions. The general abelian case produces a collection of copies of Maxwell theory. This can be seen by block-diagonalizing K by a $\mathrm{GL}(2N_B,\mathbb{Z})$ rotation.

[4]For example,

$$\begin{aligned} \mathbf{H} &= -\sum_{\text{vertices}, v\in\Delta_0} \left(\sum_{\ell\in s(v)} n_\ell - q_v\right)^2 \\ &- \sum_{p\in\Delta_2}\prod_{\ell\in\partial(p)} e^{\mathbf{i}b_\ell} + h.c. - \Gamma\sum_{\ell\in\Delta_1} n_\ell^2. \end{aligned} \tag{2.6}$$

$\Delta_p \equiv \{p\text{-simplices}\}$. $s(v) \equiv \{$edges incident on v (oriented ingoing)$\}$ and $[b_p, n_p] = \mathbf{i}$ is a number-phase representation; $b_p \equiv b_p + 2\pi, n_p \in \mathbb{Z}$.

2.6 Symmetries

CS gauge theories in 2+1d have bulk ‘topological’ conserved currents. When these theories arise from quantum Hall states, some linear combination of these currents can be interpreted as electron number:

$$0 = \partial^\mu J_\mu \quad \Longrightarrow \quad J_\mu \equiv \frac{1}{2\pi}\varepsilon_{\mu\nu\rho}\partial_\nu A_\rho \ .$$

J is conserved when A is single-valued. In 4+1d, the analog is pairs of string currents

$$J^I_{\mu\nu} \equiv \frac{1}{2\pi}\varepsilon_{\mu\nu\rho\sigma\lambda}\partial_\rho B^I_{\sigma\lambda} \ .$$

We can use these symmetries to demonstrate that different K label nontrivial, different states.

In $D = 2+1$ CS theory, we can couple the particle currents $J^I \equiv \star \mathrm{d}A^I$ to *external* 1-form potentials, $\mathcal{A}_I$:

$$\log \int [DA^I] e^{\mathbf{i}\int \frac{k}{4\pi} A\mathrm{d}A + \mathbf{i}\int J_I \mathcal{A}^I} = \int_{2+1} \left(4\pi k^{-1}\right)_{IJ} \mathcal{A}^I \mathrm{d}\mathcal{A}^J$$

This term in the effective action demonstrates a quantized Hall response; in the absence of topological order ($\det k = 1$), $\sigma_{xy}\frac{h}{e^2}$ is an integer. This is one way to distinguish the integer quantum Hall state from a trivial insulator. In the 2+1d case, the thermal Hall response $\propto c_L - c_R$ makes this distinction even in the absence of charge conservation.

The analogous stratagem in $D = 4+1$ is to couple the string currents $J^I = \star \mathrm{d}B^I$ to *external* 2-form potentials, $\mathcal{B}_I$:

$$\log \int [DB^I] e^{\mathbf{i}\int \frac{K}{4\pi} B\mathrm{d}B + \mathbf{i}\int J_I \mathcal{B}^I} = \int \left(4\pi K^{-1}\right)_{IJ} \mathcal{B}^I \mathrm{d}\mathcal{B}^J \ .$$

This term exhibits a quantized ‘string Hall’ response to external 3-form field strengths, which again is an integer in the absence of topological order, $\mathrm{Pfaff} K = 1$). This response distinguishes this state from the completely trivial state of bosons in 4+1 dimensions.

- Translation invariance is a red herring. In fact, breaking translation invariance with static disorder helps to produce a uniform speed of light in the edge theory. Indeed, the lattice model [54] should have the same edge states.

- Stringy symmetries: $J^B_{\ell 0}|_{bdy} = E_\ell, J^C_{\ell 0}|_{bdy} = -B_\ell$. $E_\ell \equiv \partial_t a_\ell - \partial_\ell a_t$ $B_\ell \equiv \varepsilon_{\ell ij}(\partial_i a_j - \partial_j a_i)$ are ordinary E&M fields

$$J^C_{y0} = \varepsilon_{ijk}\partial_i C_{jk} = \varepsilon_{ijk}\partial_i\partial_j a_k = \vec{\nabla}\cdot\vec{B}$$

$$J^B_{y0} = \varepsilon_{ijk}\partial_i B_{jk} = \varepsilon_{ijk}\partial_i \varepsilon_{jkl} E_\ell = \vec{\nabla}\cdot\vec{E}.$$

  This is ordinary charge, of course it has to be conserved.

- $\mathcal{C}$: $(B,C) \to -(B,C)$ is $(\vec{E},\vec{B}) \to -(\vec{E},\vec{B})$. This is preserved in pure U(1) lattice gauge theory.

- $\mathcal{TP}$: $t \to -t, x^M \to -x^M, \mathbf{i} \to -\mathbf{i}$, $B \to -B, C \to C$ as two-forms. Acts in the usual way on the EM field as $(E,B) \to (E,-B)$.

- **Electric-magnetic Duality:** $(B,C) \to (C,-B)$ The global symmetry which interchanges B and C is a manifest symmetry of the bulk theory. It acts on the boundary gauge field as electromagnetic duality $(\vec{E},\vec{B}) \to (\vec{B},-\vec{E})$.

- Just as in string theory and gauge theory [57], the $\mathbb{Z}_2$ EM transformation just described is a subgroup of a classical SL$(2,\mathbb{R})$ symmetry acting on $(B,C)^T$ as a doublet $\begin{pmatrix} B \\ C \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$ with $ad - bc = 1$. The continuous parts of this group, where $a,b,c,d \notin \mathbb{Z}$

do not preserve the integrality condition (3.4), and are not symmetries of the quantum theory. The $\mathrm{SL}(2,\mathbb{Z})$ subgroup is generated by

$$SS = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

SS is the EM duality transformation above, and $\mathbf{T}$ adds a boundary term $\int_{\mathbf{R}\times\partial\Sigma} \frac{k}{4\pi} C \wedge C$; this shifts the θ angle of the surface gauge theory by 2π.

This EM duality symmetry is effectively unbreakable in the 4+1d CS theory. Even with arbitrary boundary conditions, breaking Lorentz invariance, the scaling freedom in the relationship between B, C and $a, \tilde{a}$, and in the duality map itself always allow a symmetry operation exchanging a and $\tilde{a}$. The general boundary condition is of the form

$$0 = (B + c_1 C + c_2 \star B + c_3 \star C)\,|_{\partial\Sigma} \; .$$

Of the three real coefficients c_i, one may be absorbed in the speed of light, a second may be absorbed in the relationship between C and the Maxwell potential a, and the third may be absorbed in the duality map, $(B, C) \to (\lambda C, \lambda^{-1} B)$ [5].

The only way to break the manifest EM duality symmetry of the bulk is to add charged matter, in the form of strings which end at the boundary. Any matter we add has a definite charge vector (q_e, q_m). Even if the matter comes in dual pairs, condensing matter with a definite charge vector will gap out the boundary photon (and the matter with the dual charge vector).

Charged matter in the surface Maxwell theory arises from strings which are minimally coupled to the bulk 2-forms and terminate at the boundary. The precise spectrum of matter is information additional to the low-energy bulk action we have described; it is specified in a lattice

[5] This transformation also acts on the large gauge identification map (3.4), and on the surface acts as the usual (fake) rescaling of the charge lattice by $(eq_e, gq_m) \to (\lambda e q_e, \lambda^{-1} g q_m)$ preserving Dirac quantization $ge = 2\pi$.

UV completion [54].

We conclude that it is not possible to regularize Maxwell theory in a way which retains manifest electromagnetic duality symmetry. This is consistent with the existence of $\mathrm{U}(1)$ lattice gauge theory, and its UV completions in terms of lattice models with local Hilbert spaces [58, 12], in which EM duality is not manifest on the lattice. Of course, we still do expect this symmetry to emerge in the IR limit of these models.

2.7 Discussion

2.7.1 7d CS theory and the (2,0) superconformal theory

There are many generalizations of the kind of nearly-trivial bulk theory we have studied above. Consider the following 6+1d Chern-Simons theory

$$S_7[C^{(3)}] = \frac{k}{4\pi}\int_{\mathbf{R}\times\Sigma_6} C^{(3)} \wedge \mathrm{d}C^{(3)}$$

For $k=1$, there is no topological order, and the model is completely trivial in the bulk. To study the edge states, we examine solutions of the bulk equations of motion $C^{(3)} = dc^{(2)}$, with the boundary condition: $C_{0ij} = v(\star_6 C)_{ij}$. The resulting action is

$$S_7[C^{(3)} = dc^{(2)}] = \frac{k}{4\pi}\int_{\mathbf{R}\times\partial\Sigma_6} \varepsilon\partial c^{(2)} \cdot \left(\partial_t c^{(2)} + v\varepsilon\partial c^{(2)}\right).$$

c is a self-dual 2-form potential in 5+1d. In this case (as in the 2+1d CS theory and the integer QHE), no symmetry is required to protect the edge states. We conclude that this model cannot be regularized intrinsically in 5+1 dimensions.

This model describes the 'topological sector' of the A_0 *(2,0) superconformal theory* in 6d [39, 40, 51] – the worldvolume theory of an M5-brane in M-theory. The conjecture that these exotic QFTs can be consistently decoupled from gravity underlies much recent progress in string theory (*e.g.* [59] and references thereto). In particular, by compactification, its existence makes various deep 4d QFT dualities manifest. This model reduces to the simple model without fermions or scalars when the partition function preserves enough supersymmetry [40]. We conclude that the $(2,0)$ theory cannot be regularized preserving a sufficient amount of supersymmetry to allow this. It is possible that further inquiry in this direction can provide guidance for better definitions of string theory.

2.7.2 Would-be gauge anomalies and surface-only models

Many surface-only obstructions, including the Nielsen-Ninomiya examples, are directly related to *anomalies* in the protecting symmetry: They would be gauge anomalies if we tried to gauge the protecting symmetry. This property has been used to identify nontrivial SPT states in [60].

We described an obstruction to regularizing a self-dual 2-form theory in $D = 5+1$; this, too, can be understood in terms of a known anomaly. Just as for chiral CFTs (the chiral boson theory is a theory of a self-dual 1-form field strength), a *gravitational anomaly* was relevant here. In 1+1 dimensions, the chiral central charge $c_L - c_R$, which determines the magnitude of the thermal Hall effect, measures an anomaly that obstructs coupling the theory to gravity. A similar gravitational anomaly arises for the 6d self-dual tensor theory [61].

Are all of surface-only obstructions to realizing a symmetric regulator merely anomalies we would find should we try to gauge the symmetry? Thinking of anomalies as ‘symmetries broken by *any* regulator’ the answer would seem to be ‘yes’ [26]. It is true that the NN theorem can be interpreted as the statement that no chirally-symmetric regulator can produce the correct chiral anomaly in a background gauge field coupled to the vector current.

It is not clear, however, whether our present notion of ‘anomaly’ is general enough to make this the correct answer. Here is some evidence that there can be obstructions more general than obstructions to gauging symmetries:

1. A known example [60] is SPT states protected by time-reversal $\mathbb{Z}_2^{\mathcal{T}}$. It is not clear what it would mean to gauge an antiunitary symmetry (all numbers are real?), so it is not clear that obstructions to regularization arising from demanding $\mathbb{Z}_2^{\mathcal{T}}$ symmetry can be thought of as would-be discrete gauge anomalies.

2. From our discussion above, we can conclude that it is impossible to gauge electromagnetic duality in Maxwell theory. Previous literature suggesting that it is impossible to gauge EM duality includes [62, 63, 64]. (In other models in other dimensions, it is possible to gauge analogs of EM duality [65].) We do not know how to describe this phenomenon in terms of a known $\mathbb{Z}_2$ anomaly. We note, however, that a model which resulted from identifying configurations related by the S transformation could not be CPT invariant, since CPT in $4k$ spacetime dimensions relates helicity ± 1 states.

   We observe (following [49]) that the model we get on the surface is exactly in the form given by [57]. We are forced to conclude that the manifestly EM-duality-invariant (but not manifestly Lorentz-invariant) model described in [57] cannot be regularized. (Note that Lorentz invariance is not the protecting symmetry: breaking Lorentz-invariance in the bulk of the 5d CS theory has no effect on the surface states.)

3. Finally, it would be very interesting to find obstructions to supersymmetry-preserving regulators. Gauging supersymmetry leads to supergravity. Consistency of quantum supergravity is much harder to verify than the absence of quantum violations of the current.

2.8 Appendix: A simple but non-unitary TFT

Here we consider analogous gaussian models where the fields are grassman-valued p-forms in 2p+1 dimensions:

$$S = \frac{K_{IJ}}{4\pi} \int \gamma^I \wedge d\gamma^J.$$

Our motivation for thinking about this was that this reverses the behavior of the K matrices as a function of the parity of p, relative to the case of bosonic forms. That is: in 2+1 dimensions (odd p), the K-matrix is antisymmetric, while in 4+1 dimensions (even p) the K-matrix is symmetric.

There is an analog in 1+1 dimensions which is called the *bc* CFT, an example of which arises in string theory. There b, c are grassmann fields with action $\int b\bar{\partial}c$. there one can consider the spin of (b, c) to be $(\lambda, 1-\lambda)$ respectively; it is a CFT for any λ with (chiral) central charge $c_\lambda = -3(2\lambda - 1)^2 + 1$. $\lambda = \frac{1}{2}$ is free complex fermions, $c_{\frac{1}{2}} = 1$. The spin doesn't make any difference in terms of counting of degrees of freedom in 1+1d, it just affects the form of the stress tensor. If we take the spins to be $0, 1$ ($\lambda = 0$), so that we have a one-form and a zero-form we get $c = -2$. It is a non-unitary theory (the central charge is positive in a unitary CFT).

Such a system violates the usual connection between spin and statistics, which is general for unitary relativistic quantum field theories (indeed we will find that it is not unitary). If we add a Maxwell-like term $\frac{1}{m}(\partial\gamma)^2$ then surely this action will propagate ghosts. But in the topological limit $m \to 0$, nothing propagates, and we might hope it is unitary, but has no relativistic UV completion. As we will see by finding the space of states on a closed manifold, it is not.

If we study the quantization of this model on a closed manifold Σ_{2p} we can expand

$$\gamma^I = \sum_{\alpha=1}^{b^p(\Sigma_{2p}, \mathbb{Z})} \omega_\alpha \theta^{\alpha I}$$

and find

$$S = \int \mathrm{d}t \, \frac{K_{IJ}}{4\pi} \, I_{\alpha\beta} \theta^{\alpha I} \dot{\theta}^{\beta J} \, ,$$

with $I_{\alpha\beta} = \int_{\Sigma_{2p}} \omega_\alpha \wedge \omega_\beta$ the intersection form on p-cycles. Quantizing this system leads to the canonical anticommutation relations

$$\{\theta^{\alpha I}, \theta^{\beta J}\} = 4\pi\hbar \left(K^{-1}\right)^{IJ} \left(I^{-1}\right)^{\alpha\beta} .$$

In any dimension $2p$ there are manifolds with indefinite signature of the intersection form on the middle homology. If the RHS is of indefinite signature, the Hilbert space representing this algebra has states of negative norm.

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Chapter 3

All-fermion electrodynamics and fermion number anomaly inflow

3.1 Abstract

We demonstrate that $3+1$-dimensional quantum electrodynamics with fermionic charges, fermionic monopoles, and fermionic dyons arises at the edge of a $4+1$-dimensional gapped state with short-range entanglement. This state cannot be adiabatically connected to a product state, even in the absence of any symmetry. This provides independent evidence for the obstruction found by [66] to a $3+1$-dimensional -distance completion of all-fermion electrodynamics. The nontriviality of the bulk is demonstrated by a novel fermion number anomaly.

3.2 Introduction

Topological Phases

What are phases of matter? What properties distinguish phases from each other? These questions are of central importance to the study of condensed matter physics.

Recent work has emphasized the importance of symmetry, topology, and entanglement in distinguishing gapped[1] phases beyond the physics of spontaneous symmetry breaking. These are often referred to as "topological phases".

Some such phases are described, at low energies, by topological quantum field theories in some spacetime dimension D.[2] Examples include the Chern-Simons theory in $D = 2+1$ and its role in describing quantum hall fluids [67] as well as BF-theory in describing topological phases in $D = 3+1$.[68]

It is often interesting to consider these theories defined on a manifold with spatial boundary. For example the Chern-Simons theories generically host a rational conformal field theory on the $D = 1+1$ boundary. We will refer this as the "edge physics" of a given theory.

Among these theories we can distinguish two classes: theories which are "invertible" and those which are not.[69]

From a physics point of view, invertibility means that within the Hilbert space of the theory one can identify an "inverse ground state" $|(g.s)^{-1}\rangle$ such that if you took two copies of the Hilbert space and consider $|\psi\rangle = |g.s\rangle|(g.s)^{-1}\rangle$ one can find a quasi-local unitary operator U which transforms $|\psi\rangle$ to a product state.[70] Moreover this procedure should be possible on any manifold on which the field theory is defined, independent of its topology. This rules out the possibility of ground state degeneracy which depends on topology.[3]

[1] i.e. the energy of the first excited state is strictly larger than the energy of the groundstate, even in thermodynamic limit

[2] Following Sachdev's convention, we'll use $D = d+1$ to denote the number of spacetime dimensions.

[3] As one may guess a distinguishing feature of the "non-invertible" theories is a topology dependent groundstate degeneracy. These phases are referred to as "topologically ordered" or "long-range entangled". For a review, see *e.g.* [7]

This state $|(g.s)^{-1}\rangle$ itself occurs as the ground state of an invertible topological phase sharing the same global symmetries. This leads to a natural group structure for these theories where the group addition law is tensor product of Hilbert spaces, and addition of all terms to the Hamiltonian allowed by some global symmetry G.[4]

Identifying these groups is known as ”classifying” invertible topological phases and has been the subject of much theoretical activity. [23, 17, 70, 71]

Such phases are often referred to as ”short-range entangled”, ”symmetry protected topological phases”, or ”SPTs” though the role of global symmetry is somewhat opaque from the discussion thus far. Indeed, non-trivial invertible phases which do not require a global symmetry G to be protected[5] are rare and interesting.

Much more common are examples where the topological phase requires a non-trivial global symmetry G to be respected in order to distinguish these phases from trivial gapped systems. Examples include the free-fermion topological insulators protected by time-reversal, the Haldane chain in one dimension protected by $SO(3)$ spin rotation symmetry or time reversal, as well as the host of models described in ref [23].

Up to now the only known examples that don't require symmetry are (copies of) the fermionic chiral $(p+ip)$ superfluid states[6], Kitaev's E_8 state of bosons [13, 14, 15] (both in $2+1$ dimensions), and Kitaev's majorana chain in $1+1$ dimensions [72] (provided we assume fermion number is unbreakable).

In this paper, we construct another example of a short-range entangled topological phase not protected by any symmetry. It is made from bosons in $D = 4+1$ dimensions and its edge hosts a version of electrodynamics where all charged objects are fermions.

[4] Sometimes one further restricts terms permissible. For example in the study free-fermion theories one only allows terms which are quadratic in the fermionic operators. These groups may not be isomorphic to the group found when allowing all generic interactions. In this context it can be referred to as the ”collapse” of the free-fermion classification by interactions. See ref. [18]

[5] i.e. the Hamiltonian cannot be continuously/adiabatically deformed to a trivial theory by the action of a local unitary without closing the gap. This is what is meant to distinguish different gapped phases.

[6] in which the $\mathbb{Z}_2$ symmetry is ungauged and the vortices are not dynamical objects

Edge Physics

Following the aforementioned examples, it is believed that such invertible topological phases are characterized by their edge states.[7] This must be the case as the physics in the ”bulk” dimension appears trivial.[8]

This implies that the physics which may arise at the edge of a D-dimensional SPT (and any low-energy effective field theory description thereof) must have features which may not arise intrinsically, in the absence of the extra-dimensional bulk.

That is, there must not be a local lattice model (or other regulator) in strictly $d-1$ spatial dimensions which regulates the edge theory and preserves all of its symmetries. For example, there is no way to regulate a chiral fermion in one dimension, by virtue of its gravitational anomaly, and there is no way to regulate free chiral fermions in three dimensions due to the chiral anomaly.[9]

This realization [1, 74] implies that the study of SPT states may be used to identify obstructions to symmetric regulators of quantum field theory (QFT). In simple examples, such an obstruction can be identified with an ’t Hooft anomaly coefficient [75], a well-known obstruction to gauging a global symmetry of a field theory. When realized at the edge, the bulk theory cancels the anomaly by anomaly inflow [27]. However, there are examples, particularly for discrete symmetries, where there is no previously-known anomaly.[10]

Examples of such obstructions which go beyond familiar global anomalies include many interesting states in $2+1$ dimensions, such as the algebraic vortex liquid [1], time-reversal-invariant $\mathbb{Z}_2$ gauge theory where all quasiparticles are fermions (the “all-fermion toric code”) [1, 25], other topologically ordered states in $2+1$ dimensions [80, 24, 81, 82, 83, 84], and a simple three dimensional example [74].

[7] The subject is reviewed in [11, 73].

[8] It lacks fractionally charged excitations, always has a unique ground state, satisfies the area-law of entanglement entropy without any interesting corrections, etc.

[9] As a reminder chiral in $D = 1+1$ simply means left or right moving. In odd spacetime dimension chiral symmetry is replaced by discrete parity symmetry.

[10] Formal attempts to interpret SPT obstructions in these terms include [76, 77, 78, 71, 79].

This paper may be regarded as a sequel to [74], which identified an obstruction to a regulator for 'pure' $\mathsf{U}(1)$ gauge theory which manifestly preserves electromagnetic duality.[11] While this is a gaussian model, such a no go result is interesting given attempts to construct such manifestly duality-symmetric realizations [57]. Further, it shows the impossibility of *gauging* electromagnetic duality, a conclusion which was argued from a very different point of view in [62, 63, 64].

Here we point out that a stronger obstruction may be found by adding 'matter' to the bulk model studied in [74]. The model we find at the surface is $3+1$-dimensional electrodynamics where all of the minimally-charged (electrically and/or magnetically) particles are *fermions*. This system has been discussed recently in [66], which demonstrated that it does not admit an interface with vacuum – it is not 'edgeable'.

To be precise, ref. [66] showed that all-fermion electrodynamics cannot be realized in $3+1$-dimensions if the microscopic regulator consists entirely of bosonic degrees of freedom. If we add to the microscopic physics gauge-invariant fermion degrees of freedom, then we can bind the gauge invariant fermion to the minimally charged fermionic objects to produce minimally charged bosonic objects. Bosonic electrodynamics of course can be regulated in strict $3+1$-dimensions, by $\mathsf{U}(1)$ lattice gauge theory [86], or (more locally) by a $\mathsf{U}(1)$ toric code [87, 88].

We note that the classification of [71] includes a nontrivial state in $4+1$ dimensions without symmetry. Ref. [89] attributes fermionic excitations to its surface states. We anticipate that the independent construction in this paper can be interpreted as a physics-based realization of the machinery in that work.

Why is the bulk nontrivial?

That the edge of the $4+1$-dimensional system realizes all-fermion electrodynamics, combined with an argument that all-fermion electrodynamics cannot be regulated in $3+1$ dimensions,

[11] The edge theory of this model was studied further in [85].

implies that the bulk is a non-trivial $4+1$-dimensional state of matter. Ref. [66] (Appendix D) has given one such argument for the absence of a $3+1$d regulator of all-fermion electrodynamics. Hence the bulk is a nontrivial state of matter; any representative groundstate of which is not adiabatically deformable to a product state. We provide two independent demonstrations of bulk non-triviality, one from the point of view of the edge (in §3.3.3), and one that uses directly the bulk (in §3.6).

Since no symmetry is required to define the bulk state, it is a topological phase of matter which is protected from all weak Hamiltonian perturbations. However, it is still short-range entangled [70, 90]: two copies of the bulk state can be deformed into a product state, so it is its own 'inverse state'.

In the context of microscopic *bosonic* phases, the only other known example of a short-range entangled state which is distinct from the trivial phase in the absence of any symmetry is the E_8 state in $D=2+1$ dimensions.[12]

As stated above, the distinguishing feature of the E_8 state is its chiral edge modes at an interface with the vacuum. A sharp and universal characterization of these chiral edge modes is the thermal Hall response: heat will be transported uni-directionally without dissipation along the boundary of the sample. In the language of anomalies, the nontriviality of this example is demonstrated by the chiral central charge $c_- \equiv c_L - c_R$ of the edge states. c_- represents a gravitational anomaly of the edge CFT, and this is a construction of gravitational anomaly inflow.

In the $D=4+1$ dimensional example studied here, the analogous signature of the nontriviality of the state seems to be fermion number anomaly inflow, as we show in §3.6.

We demonstrate that this effect also occurs in the $D=3+1$ boson SPT protected by time reversal symmetry studied in [68, 91, 1, 25, 24].[13] This phase is an example of an SPT which lies outside the group cohomology classification of ref. [23] and we refer to it as such.

[12] as well as multiple copies of this state, which comprise an integer classification

[13] A related phenomenon was described for edge states of 3+1d SPTs whose protecting group contains $\mathsf{U}(1)$ in [24]. In that case, the anomaly occurs upon gauging the $\mathsf{U}(1)$.

A possible surface termination of this SPT consists of an all-fermion toric code, a model which has no $D = 2+1$ realization with time reversal symmetry. Our claim implies that the preservation of time-reversal in the all-fermion toric code comes at the cost of the conservation of fermion number!

We emphasize that the main conclusion of this paper pertains to models made from *bosons* in $D = 4+1$ dimensions. As we show, the addition of microscopic gauge-invariant fermions to the system removes any obstruction to realizing the edge physics in strict $D = 3+1$ dimensions. Such a gauge-invariant local fermion cannot arise at the edge of a bosonic system. From the point of view of a lattice field theorist attempting to regularize the given low-energy field theory, having to add an extra species of massive fermion at the cutoff may not seem like a huge price. However, we regard the demonstration that such a step is *required* as fascinating and requiring a systematic understanding.

The paper is structured as follows. First (§3.3), we review the physics of two-form Chern-Simons ("*BdC*") theory in $4+1$ dimensions and show that it admits an edge which supports all-fermion electrodynamics. The group of electromagnetic duality transformations, which can be realized as an exact symmetry of the bulk *BdC* theory, plays an important role in the analysis. Second (§3.4), by considering the path integral of all-fermion electrodynamics on $\mathbb{CP}^2$, we show that all-fermion electrodynamics cannot have a bosonic regulator. This constitutes a proof of bulk non-triviality via edge non-regularizability. Third (§3.5), we show how to construct the bulk non-trivial state from layers of ordinary (e.g., with bosonic charges) electrodynamics by condensing dyon strings. Finally, we show how to interpret the obstruction in terms of a fermion number anomaly of the all-fermion electrodynamics (§3.6) and show that similar physics is realized in the non-trivial time-reversal ($\mathcal{T}$) protected bosonic SPT in $3+1$ dimensions (§3.7).

3.3 The *BdC* model coupled to matter

3.3.1 *BdC* summary

We begin by describing the action of the *BdC* theory and reviewing its basic properties [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Consider 2-forms B^I_{MN} ($I = 1..N_B$ labels the form, MN are the spacetime indices) in $4+1$ dimensions, with the topological action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbf{R}\times\Sigma} B^I \wedge \mathrm{d}B^J \tag{3.1}$$

where Σ denotes the space of interest.

To process this action, we need a little exterior algebra: a p-form α_p and a q-form β_q satisfy $\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$ and $\mathrm{d}(\alpha_p \wedge \beta_q) = \mathrm{d}\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge \mathrm{d}\beta_q$. Hence we have $B^I \wedge B^J = B^J \wedge B^I$ and

$$B^I \wedge \mathrm{d}B^J = \mathrm{d}(B^I \wedge B^J) - \mathrm{d}B^I \wedge B^J \ , \tag{3.2}$$

so up to a total derivative the action is anti-symmetric in IJ. Thus K is an anti-symmetric $2N_B \times 2N_B$ matrix. Shortly we show that in order for (3.1) to govern the low-energy effective field theory of a short-range entangled bulk state, K must also be an integer matrix with $\det(K) = 1$. Also, since $B \wedge \mathrm{d}B = \frac{1}{2}\mathrm{d}(B \wedge B)$ is a total derivative, we must have an even number of such two-forms.

We note that we view the topological field theory action (3.1) as the extreme low-energy effective field theory for a gapped state of matter. In particular we can consider the addition of generic irrelevant bulk terms like the bulk Maxwell term

$$\frac{\tau_{IJ}}{g^2} \int_{\mathbf{R}\times\Sigma} dB^I \wedge \star dB^J \tag{3.3}$$

(where $\star$ is the Hodge duality operation).

So long as the action of (3.1) is not a total derivative, i.e. if the number of two-forms is even, then the term (3.3) does not produce a gapless excitation in the $D = 4+1$ bulk.

This is analogous to the situation in $D = 2+1$ for Chern-Simons theory. The models described may be considered as the $g \to \infty$ limit of the non-topological models with ("topologically massive" [31]) propagating two-forms with action given by the sum of (3.1) and (3.3).

The local gauge transformations $B^I \simeq B^I + d\lambda^I$ are redundancies of the model. An important further ingredient of the definition of the model [49, 41, 43] is the 'large gauge' identifications:

$$B^I \simeq B^I + n^\alpha \omega_\alpha, \quad [\omega^\alpha] \in H^2(\Sigma, \mathbb{Z}), \ \ n^\alpha \in \mathbb{Z}^{b^2(\Sigma)} , \tag{3.4}$$

where the betti number $b^2(\Sigma) \equiv \dim H^2(\Sigma, \mathbb{Z})$ is the dimension of the second integer cohomology of Σ. This requires the entries of K to be integers[14].

The equations of motion following from (3.1) are, $\forall I$,

$$K_{IJ} dB^J = 0. \tag{3.5}$$

When K has full rank, these equations are solved by flat two-form fields, which are identified by local gauge equivalences, and there are therefore no local degrees of freedom. As a result, the gauge-inequivalent operators (analogs of Wilson loop operators) are labelled by cohomology classes

$$\mathcal{F}_\omega(m) \equiv e^{2\pi \mathbf{i} m_I \int_\omega B^I} \tag{3.6}$$

with $[\omega] \in H^2(\Sigma, \mathbb{Z})$. The identification (3.4) on B implies $m^I \in \mathbb{Z}$.

Using equal-time canonical commutators for B^I, the flux operators (3.6) satisfy a Heisenberg algebra:

$$\mathcal{F}_{\omega_\alpha}(m) \mathcal{F}_{\omega_\beta}(m') = \mathcal{F}_{\omega_\beta}(m') \mathcal{F}_{\omega_\alpha}(m) e^{2\pi \mathbf{i} m_I^\alpha m_J'^\beta (K^{-1})^{IJ} I_{\alpha\beta}}. \tag{3.7}$$

[14] In this paper we will only discuss this model on manifolds without torsion homology. For the machinery required to lift this restriction, see [44].

Here

$$I_{\alpha\beta} \equiv \int_{\Sigma} \omega_\alpha \wedge \omega_\beta$$

is the intersection form on $H^2(\Sigma, \mathbb{Z})$, which is a $b^2(\Sigma) \times b^2(\Sigma)$ symmetric matrix.

In analogy with Chern-Simons theory, the algebra (3.7) is realized on the space of groundstates. The ground state degeneracy is given by the dimensions of irreducible representations of (3.7).

Consider the minimal case (relevant later on) where $\Sigma = \mathbb{CP}^2$, for which $b^2(\mathbb{CP}^2) = 1$ and $I = \mathbb{1}$. The smallest representation of the algebra (3.7) is then $|\mathrm{Pf}(K)|$-dimensional. Because we wish to study invertible systems, which have a unique ground state on all manifolds, we require $\det K = \mathrm{Pf}^2(K) = 1$. [74]

The BdC theory is a special case of (3.1) where we take $N_B = 2$ and let $B^1 = B$, $B^2 = C$, and $K = ki\sigma^y$; we must set $k = 1$ for this state to be short-range entangled.[15]

We now review its physics on a space with boundary [74, 85]. In the presence of a boundary, the solutions of the equations of motion produce physical excitations: a one-form field a localized at the boundary. This mode is physical because gauge transformations which are nontrivial at the boundary do not preserve (3.1). Boundary terms (whose coefficients are non-universal) produce the Maxwell action for a. In particular, the boundary condition arising from variation of an action with the leading irrelevant operators (*i.e.* the bulk Maxwell terms (3.3)) is:

$$\left(\frac{k}{2\pi} B - \frac{1}{2g^2} \star_4 C\right)|_{\partial\Sigma_4} = 0 \,.$$

Upon a convenient rescaling, the identification of boundary degrees of freedom is:

$$B = da, \quad C = \star da \,. \tag{3.8}$$

An important symmetry of the topological action (3.1) is the group $\mathsf{SL}(2N_B, \mathbb{Z})$ of field

[15] When $k > 1$ the system has topological ground state degeneracy depending on $b^2(\Sigma)$, namely $k^{b^2(\Sigma)}$ groundstates

redefinitions that preserve the identifications (3.4). We emphasize that this symmetry is not necessary for the 4+1 bulk to be distinct from a trivial phase; indeed, this symmetry may be broken by UV physics, but it turns out to be very convenient to analyze certain topological features of the physics assuming this symmetry holds. In the case of the BdC theory, the group is $\mathsf{SL}(2,\mathbb{Z})$ and it is closely related to the group of duality transformations on the boundary electrodynamics.

The action of $\mathsf{SL}(2,\mathbb{Z})$ on B,C is in the fundamental representation

$$\begin{pmatrix} B \\ C \end{pmatrix} \to \mathsf{M} \begin{pmatrix} B \\ C \end{pmatrix}$$

with $\mathsf{M} \in \mathsf{SL}(2,\mathbb{Z})$. The '$\mathsf{T}$' transformation $\mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a symmetry because $B \wedge dB$ is a total derivative; by (3.8), this transformation shifts the theta angle of the surface gauge theory by 2π. The 'S' transformation $\mathsf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a symmetry because of (3.2), and acts as electromagnetic duality on the boundary gauge field. These two transformations generate $\mathsf{SL}(2,\mathbb{Z})$. Notice that on B,C, the $\mathbb{Z}_2$ center of the duality group acts nontrivially (this is charge conjugation at the edge).

3.3.2 Coupling to strings (matter)

Just as a one-form gauge field A couples minimally to the worldline of a charge, $\int_{\text{worldline}} A$, a two-form gauge field B couples minimally to the worldsheet of a string, $\int_{\text{worldsheet}} B$. Adding matter to Chern-Simons theory is usually [6] described in terms of a statistics vector, l_I, so that the quasiparticle (here, 'quasistring') current is the two-form $l_I \star dB^I$. If B^I are normalized as in (3.1), the l_I must be integers, so that $e^{\mathbf{i}\oint_\Sigma l_I B^I}$ is periodic under shifts of the periods of B^I over all topologically nontrivial 2-cycles Σ.

Gauge invariance under $B^I \simeq B^I + d\lambda^I$ requires that strings not end in the bulk of the sample. However, strings can end at an interface with vacuuum. Then because of the identification (3.8), the ends of the strings are electric and magnetic charges under the boundary gauge field a. Indeed, given a string which terminates at a boundary, the coupling $\int_{\text{worldsheet}} B$ reduces to the coupling $\int_{\text{worldline}} a$ by Stokes' theorem.

We discuss in detail below the statistics of the surface particles arising at the ends of the bulk string matter. As a preliminary, note that the modular group $\mathsf{SL}(2,\mathbb{Z})$ acts on the string matter as well. This action is necessary to preserve the coupling between string worldsheets and two-form fields.

3.3.3 Edge physics

We now consider an edge of the $D = 4+1$ dimensional BdC bulk which supports $\mathsf{U}(1)$ electrodynamics in $D = 3+1$ dimensions [74]. As anticipated in the introduction, the crucial question is: what are the statistics of the basic charged particles on the edge?

Because the edge electrodynamics is a stable phase of matter and because the statistics of the charged particles is topological data, these statistics must be stable to the breaking of all symmetries in the problem. Hence to determine the statistics we may assume extra symmetry and be confident that we have the correct statistics even if we later break the symmetry (for example by allowing the electron and monopole to have different masses) to realize the generic situation.

Thus suppose that we preserve the manifest $\mathsf{SL}(2,\mathbb{Z})$ duality symmetry of the BdC theory. Duality symmetry implies that the charge e and the monopole m have the same statistics, since they are related by the symmetry. For $\mathsf{G} = \mathsf{U}(1)$, the full duality group is $\mathsf{SL}(2,\mathbb{Z})$, and it acts on the charge vector by $\begin{pmatrix} q_e \\ q_m \end{pmatrix} \to \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathsf{SL}(2,\mathbb{Z})} \begin{pmatrix} q_e \\ q_m \end{pmatrix}$. In particular, the transformation $(\mathsf{T}^{\mathsf{t}}\mathsf{S})^{-1}$ takes the charge to the $(1,1)$ dyon $\varepsilon \equiv em$. The boundstate with these quantum numbers must therefore have the same statistics as the charge and the monopole. Since these are particles in

$3+1$ dimensions, they may be either all bosons or all fermions.

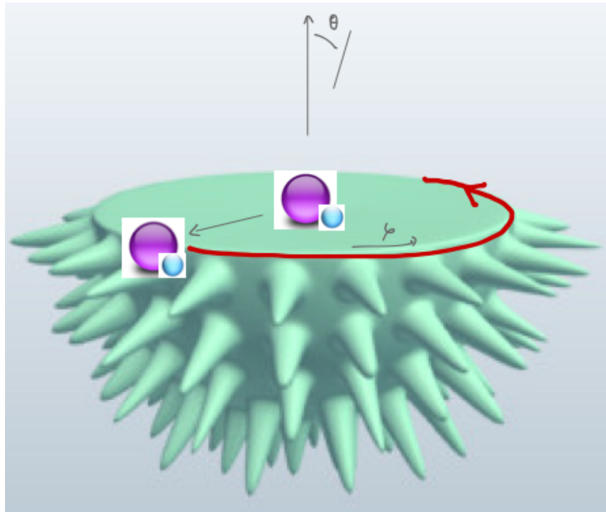

Figure 3.1: A depiction of the calculation of dyon statistics. The spikes represent the flux produced by the dyon at the center.

Naively both possibilities are allowed, but in fact, if e and m have the same statistics, then ε must be a fermion. This phenomenon is sometimes called 'spin from isospin' [92, 93] (when the electrodynamics is UV completed by $\mathsf{SU}(2)$ gauge theory with an adjoint higgs field). Note that we must assume there are no gauge-invariant fermions around, otherwise we could bind such a fermion to the dyon without changing its charges and turn it into a boson.

To see this efficiently, consider two identical dyons well-separated in space compared to any cutoff scales. Since they are identical particles, moving one of them adiabatically in an arc of angle π around the other results in the same state (up to an innocuous center-of-mass translation).

The Berry phase acquired in doing so is

$$\varphi = e \int_0^\pi d\varphi \underbrace{\mathcal{A}_\varphi(\theta = \frac{\pi}{2}, \varphi)}_{\text{Dirac monopole field}} \overset{\text{Dirac}}{=} \pi g e \ .$$

If g and e have the minimal charges, saturating the Dirac quantization condition, then

$$\psi(x_1, x_2) = e^{\mathbf{i}\varphi}\psi(x_2, x_1) = -\psi(x_2, x_1)$$

and these particles are fermions. The extra $\frac{\hbar}{2}$ unit of angular momentum comes from the electromagnetic field. Note that any exchange phase coming from the constituent e and m particles cancels because we assumed they were both bosons or both fermions.

Thus we reach the remarkable conclusion that the model with a duality-symmetric spectrum of all bosons is not even self-consistent! On the other hand, an all-fermion spectrum is self-consistent: because of the additional $\frac{\hbar}{2}$ unit of angular momentum in the electromagnetic fields, the dyon boundstate of two fermions is still a fermion [94].

To prove that the bulk is non-trivial we argue by contradiction and suppose that all-fermion electrodynamics can be realized in strict $D = 3+1$ dimensions with microscopic bosons only. Then we could place a field theory realization on $\mathbb{CP}^2$ since the theory is bosonic and requires no spin structure for its definition. However, something bad happens, which we describe next, in §3.4.

Hence there must be no UV completion in the same dimension with only microscopic bosons. Since the BdC theory provides a UV completion of all-fermion electrodynamics with only bosons at its edge, it follows that the bulk *BdC* phase is necessarily distinct from the trivial phase. Alternatively, the results of [66] also imply that all-fermion electrodynamics cannot be realized in strict $D = 3+1$ dimensions without gauge-invariant fermions, so again we conclude that the bulk *BdC* phase is distinct from the trivial phase.

3.4 The Bad Thing that Happens on $\mathbb{CP}^2$

To show the impossibility of a bosonic regulator of all-fermion QED, we show that there is no consistent way to define the partition function on $\mathbb{CP}^2$. To make the argument we suppose:

Postulate 1: A $\mathsf{U}(1)$ gauge theory with gapped matter (and hence the value of the $\mathsf{U}(1)$ gauge theory path integral on a closed manifold M, modulo non-universal garbage) is specified by the theta angle and the coupling and by the spectrum of charges.

But what theta angle and coupling you ask? What data about the spectrum? More specifically, we suppose:

Improved Postulate 1: The value of the gauge theory path integral on a closed manifold M, modulo non-universal garbage, depends only on the bare coupling $\tau = \theta + \frac{4\pi\mathbf{i}}{g^2}$ (and $\bar{\tau}$), and on a choice of statistics for the excitations with minimal electric and magnetic charges, e, m. We include the dependence on the particle masses and various other couplings in the category of 'non-universal garbage'.

A crucial point here is that the effective theta angle (at energies below the gap to charged excitations) may receive contributions from integrating out the matter, as is familiar from the study of topological insulators (*e.g.* [95, 96]).

A useful perspective then, is that all such gauge theories may be realized by starting with a theory of some bosonic or fermionic matter with a $\mathsf{U}(1)$ global symmetry, possibly in a non-trivial SPT state, and gauging that $\mathsf{U}(1)$ symmetry. This is equivalent to coupling "pure" $\mathsf{U}(1)$ gauge theory to bosonic or fermionic matter in various $\mathsf{U}(1)$ protected SPTs. A possibility which we must also discuss is a case with *no* charged matter, studied with related intent in [97, 74].

Let us consider the action of duality on the gauge theory partition function. We are free to relabel the gauge fields using the electric-magnetic duality group $\tau \to \frac{a\tau+b}{c\tau+d}$, $(a,b,c,d \in \mathbb{Z}, ad-bc=1)$ but we must keep track of the particle statistics as well. We will be most interested in the T transformation which takes $\theta \to \theta + 2\pi$. Recall [94] that shifting the theta angle produces a spectral flow on the charge lattice: monopoles acquire electric charge proportional to $\frac{\theta}{2\pi}$.

Therefore (in the absence of other data, an absence for which we argue below) the choice of statistics of the charged matter gives an invariant meaning to the duality frame. Denote the statistics labels on the gauge theory as follows: BBF if e is a boson, m is a boson and (therefore) em is a fermion, BFB if e is a boson, m is a fermion, em is a boson, etc. Note that by the spin-from-isospin argument, this labeling is redundant (the statistics of em is determined by those of e and m), but it will help emphasize the important distinction between the all-fermion case and the other cases. If we allow neutral fermions, then we have both bosons and fermions in each charge sector, and the labeling scheme breaks down; we assume no neutral fermions. If there are no charged particles, then any duality transformation in $\mathsf{SL}(2,\mathbb{Z})$ is a redundancy: a relabeling of fields.

For example, the Witten effect [94] implies that for any four-manifold M,

$$\mathsf{T} : Z_M(\tau, BBF) = Z_M(\tau+1, BFB).$$

On the other hand, consider the case where $M = \mathbb{CP}^2$; this example is interesting because it has a two-cycle h with unit self-intersection. This means that a line bundle with $\mathbf{c}_1 = h$ has

$$\frac{1}{4\pi^2}\int_{\mathbb{CP}^2} F \wedge F = 1.$$

Therefore the partition sum is

$$Z_{\mathbb{CP}^2}(\theta) = \int DA e^{-S_0[A] + i\frac{\theta}{8\pi^2}\int_{\mathbb{CP}^2} F\wedge F} = \sum_{\mathbf{c}_1 = nh} \int_{C_n} [DA]_n e^{-S_0[A] + i\frac{\theta}{2}n^2}$$

where C_n labels the sector of the gauge field configuration space with $\int_h \frac{F}{2\pi} = n$. $Z_{\mathbb{CP}^2}(\theta)$ is therefore periodic in θ with period 4π. (This fact is discussed in detail in [97]; the odd intersection form on $\mathbb{CP}^2$ also plays a role in the discussion of [71].)

Since we know that $Z_{\mathbb{CP}^2}(\tau, BBF)$ is not the same as $Z_{\mathbb{CP}^2}(\tau+1, BBF)$ but that it is the same as $Z_{\mathbb{CP}^2}(\tau+2, BBF)$, it follows that integrating out charged matter which makes the monopole a fermion generates an extra theta term with coefficient 2π (mod 4π), in agreement with previous results [94].

Finally, let us turn to the case of $Z(\tau, FFF)$. By the Improved Postulate 1 we have

$$Z_M(\tau, FFF) = Z_M(\tau+1, FFF)$$

for all 4-manifolds M on which the theory is defined. However, this equation can only be true if M has an even intersection form. If the theory had a bosonic regulator, then we could place it on manifolds with an odd intersection form and no spin structure, such as $\mathbb{CP}^2$. The theory cannot be placed on manifolds with odd intersection form, hence the theory does not have a bosonic regulator[16].

In order for this periodicity in θ

$$Z_{\mathbb{CP}^2}(\tau+1,FFF) \stackrel{!}{=} Z_{\mathbb{CP}^2}(\tau,FFF)$$

to be a consistency condition (that is: its violation is an anomaly) we require that the modular properties of the partition function are determined entirely by the spectrum of electric and magnetic charges. We argue for this claim in a series of comments, which can be regarded as an attempt to make precise the lack of structure in $\mathsf{U}(1)$ gauge theory:

- First, we emphasize that the statistics of particles in *all* charge sectors (q_e,q_m) are fixed by the elementary ones (1,0), (0,1) (the generators of the charge lattice) and the demand that there are no neutral fermions. For example, the spectrum of the FFF theory cannot contain a magnetic-charge-two monopole which is a fermion, because then binding such an object to the (boson) boundstate of two charge (-1) monopoles would produce a neutral fermion.

- In gauge theories with more interesting gauge group or massless matter content, other labels are required to specify the partition function. For example, gauge theories where a 2π-shift of θ produces a different gauge theory were discussed recently in [99]. The new labels there arise from extra topological invariants (beyond the Pontryagin invariant) of gauge

[16]Note than an additional consequence of its lack of spin structure is that $\mathbb{CP}^2$ cannot occur as the boundary of some smooth, compact 5-manifold; it has a non-vanishing Stiefel-Whitney number. See Theorem 4.10 of [98]. This theorem prevents a contradiction with the fact that the partition function of the all-fermion electrodynamics on M *can* be obtained from the *BdC* theory on a space whose boundary is M. Two disjoint copies of $\mathbb{CP}^2$ can occur as the boundary of *e.g.* $\mathbb{CP}^2\times[0,1]$. In this case, the instanton sums in the two copies of all-fermion electrodynamics are correlated by the fact that $B=da_1+da_2$ is flat in the bulk, again avoiding contradiction.

bundles whose structure group (the gauge group) is semisimple but not simply connected (a pedagogical exposition of this subject can be found in §3 of [100]).

- Here we are studying $\mathsf{G} = \mathsf{U}(1)$ where this issue does not arise. That is: The smooth topological data of a line bundle (the structure group is $\mathsf{U}(1)$) on a simply connected manifold is just the first Chern class (for a discussion which makes this clear see *e.g.* page 3 of [97]). Therefore this possibility for modifying the periodicity of theta is not available.

- Another potential source of a theta-dependent phase in the partition function is a possible τ-dependence in the gravitational couplings in the effective action for the gauge fields upon integrating out the gapped charged matter. Such couplings are crucial in computing the partition function of topologically twisted gauge theories [100] on various four-manifolds, and are discussed further in [97]. In that context, such terms produce anomalous factors under the S transformation, but not under the T transformation.

  Further, to see that this is not a meaningful loophole here, we can take the perspective described above: we couple an SPT with $\mathsf{G} = \mathsf{U}(1)$ symmetry (in curved space) to the electromagnetic field. The gravitational effective action for the SPT is completely fixed before the coupling to the EM fields, which is when τ is introduced. Therefore, the τ dependence of the action below the gap is completely fixed by the matter content.

  So the basic question is: what other kinds of UV gerbils can there be in $\mathsf{U}(1)$ gauge theory which might affect the τ-dependence of the partition function? We can see that the answer is ‘none’ as follows.

- Adding fermions restores 2π periodicity of the theta angle. This matches nicely with the fact [19, 20] that the θ angle for a *background* gauge field is only periodic mod 4π in a system made of bosons (since the surface at $\theta = 2\pi$ would have odd-integer quantum Hall response, which is not compatible with bosonic statistics of all neutral excitations). This argument implies that *only* fermions in the charge spectrum can change the periodicity in theta by 2π. But we've already accounted for the fermionic charges.

- As a nice corroboration of our understanding, note that the counting of non-trivial states here is consistent with the counting of $\mathsf{U}(1)$ protected SPT states [20]. In particular, absent time reversal, the three states BBF, FBB, and BFB are smoothly connected.

- Finally, we believe that the argument described here implies that *there is no such thing as 'pure' $\mathsf{U}(1)$ gauge theory*, *i.e.* $\mathsf{U}(1)$ gauge theory without any charged matter at all. [17] From the low energy point of view, the problem with the all-fermion model is simply that the spectrum is duality invariant, and so cannot be rearranged by the Witten effect. The same is true if there are *no* charges, and so we have:

$$Z_M(\tau+1,---) \overset{!}{=} Z_M(\tau,---)$$

(where the dashes emphasize the absence of charged matter). The fact that this demand is violated for $M = \mathbb{CP}^2$ was observed in [97]. We believe that the above argument implies that this failure should be regarded as an inconsistency. We note that there is no known regulator of this model. The $\mathsf{U}(1)$ toric code is described at low energies by electromagnetism coupled to gapped matter with spectrum BBF. Ordinary lattice gauge theory is simply the limit of the toric code where the electric excitations are made infinitely heavy; in particular it still contains gapped magnetic monopole excitations. (A term by which one might try to lift these excitations completely, *e.g.* $\sum_{\text{plaquettes}} \Delta\cdot(\Delta\times a)$, is not single-valued under the equivalence $a_\ell \to a_\ell + 2\pi n_\ell, n_\ell \in \mathbb{Z}$.)

Perhaps there exists a consistent low energy theory where there are only magnetic charges;

[17] In $D=3+1$ except, perhaps, as the boundary of a $D=4+1$ topological phase such as the *BdC* theory without matter.

in that case, we have the condition

$$Z_M(\tau+1, -B-) \stackrel{!}{=} Z_M(\tau, --B)$$

which is not falsified by the lack of a spin structure of M.

The fact that there is an obstruction to a *duality-invariant* regulator of 'pure' electromagnetism was argued in [74] (with hindsight, this result also follows from the calculation of [97]). Here we are making the further claim that there is no regulator at all. The argument above shows that there is no *bosonic* regulator. Many of the other anomalies discussed in this paper may be cured by adding neutral fermions. In this case, it is difficult to see how the addition of gapped, neutral fermionic excitations can help. In particular, the fact that the fermion is neutral means that integrating it out does not generate a theta term. However, the presence of microscopic neutral fermions amounts to a refusal to put the system on a manifold without spin structure, such as $\mathbb{CP}^2$! (Since the fermions are neutral, the existence of a spin$_c$ structure does not help.) So indeed there is no obstruction to a fermionic regulator.

We discuss below in §3.7 the consequences of the analogous line of argument for the all-fermion toric code in $D = 2+1$.

3.5 Coupled Layer Construction

In this section, we describe a 4+1d local lattice model which realizes the continuum model above, using a coupled layer construction (precedents for such an approach include [101, 102, 14, 1, 103]). Like the edge-based proof of bulk non-triviality, the motivation for the layer construction comes from edge physics. If SPTs are only non-trivial because of their edge states, then we should be able to construct interesting SPTs by sewing together pairs of edge states as follows.

First, observe that every short-range entangled state with a non-trivial edge has an inverse short-range entangled state with a non-trivial edge and with the property that the composite short-range entangled state has a trivial edge. In other words, for every non-trivial (anomalous) edge $\mathcal{E}$ there is another non-trivial edge $\mathcal{E}^{-1}$ such that $\mathcal{E} \times \mathcal{E}^{-1} \sim 1$ is trivial. We then imagine a stack of such edges, $(\mathcal{E}_1 \mathcal{E}_1^{-1})...(\mathcal{E}_n \mathcal{E}_n^{-1})$, which can clearly be reduced to a trivial state by pairing $\mathcal{E}_i$ with $\mathcal{E}_i^{-1}$. However, we may also pair $\mathcal{E}_i^{-1}$ with $\mathcal{E}_{i+1}$ in such a way that the edges $\mathcal{E}_1$ and $\mathcal{E}_n^{-1}$ are left un-paired. Assuming interactions are local in the layer index n, these remaining actual edge states cannot be paired with each other and we have produced a non-trivial bulk state. More generally, we may take any lower dimensional "layers" and try to couple them in a similar non-integrable fashion to produce a bulk short-range entangled state with non-trivial edge states.

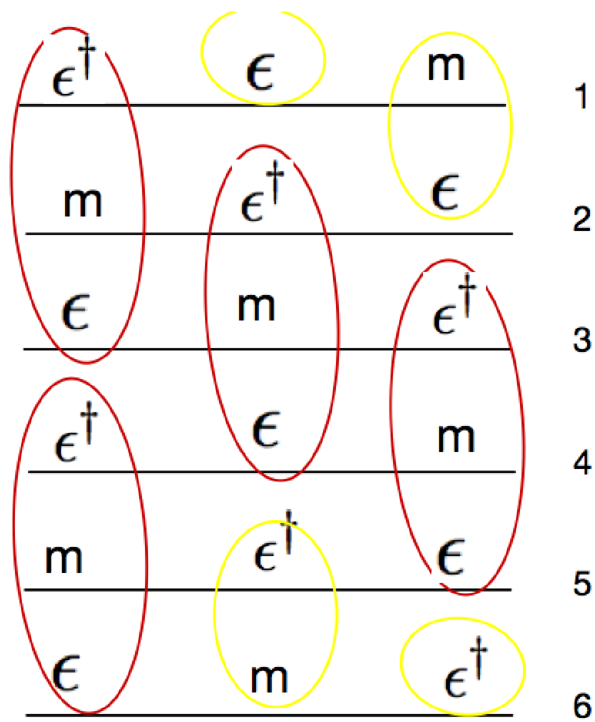

Figure 3.2: A representation of the coupled layer construction, following [1]. The layers are coupled by condensing the objects circled in red.

We make a coupled-layer construction of the all-fermion electrodynamics following (very directly) the one made in [1] for the all-fermion toric code. It produces a trivial bosonic bulk, and the correct edge physics. As an essential part of the construction, we are able to argue that this bosonic bulk is well-described by the *BdC* theory.

The method by which we construct the bulk can be called 'dyon string condensation'. It has a lot in common with the dyon condensation mechanism of statistics transmutation in 3+1 dimensions employed in [104]. The construction can also be regarded as an oblique version of 'deconstruction' of the extra dimension [105]; this will be a useful perspective for understanding the origin of the $B \wedge dC$ term.

First we give a brief summary of the construction:

• Each layer, labelled $i = 1..n$, is ordinary electrodyamics with bosonic charges: the electron and monopole e_i, m_i are gapped bosons. This model is certainly regularizable in 3+1d by itself on an ordinary Hilbert space of bosons on links and sites. Denote the (fermionic) dyon in each layer as ε_i.

• $b_i \equiv \varepsilon_i^\dagger m_{i+1} \varepsilon_{i+2}$ are mutually-local bosons.

• Condensing b_i (obliquely) *confines* the layer gauge fields $a_{i+1}, i+1 = 2...N-1$.

• At the top layer: $m_1\varepsilon_2, \varepsilon_1^\dagger m_1 \varepsilon_2, \varepsilon_1^\dagger$ survive, are fermions, and are the electron, monopole and dyon of a surviving (Coulomb-phase) $\mathsf{U}(1)$ gauge field. A similar statement pertains to the bottom layer.

In the bulk, in the continuum, we will arrive at the claim that this is the *BdC* theory with gapped string matter.

3.5.1 Warmup: deconstruction of lattice electrodynamics

First consider the following toy example, which really is 'deconstruction' of $4+1$d $\mathsf{U}(1) \times \mathsf{U}(1)$ gauge theory on an interval, in the sense of [105]. (A quiver diagram for this construction, more familiar in the high-energy theory literature, appears in Fig. 3.3.) Collocate an even number N of layers of (cubic, say) $3d$ lattices each of which hosts $\mathsf{U}(1)$ lattice gauge theory

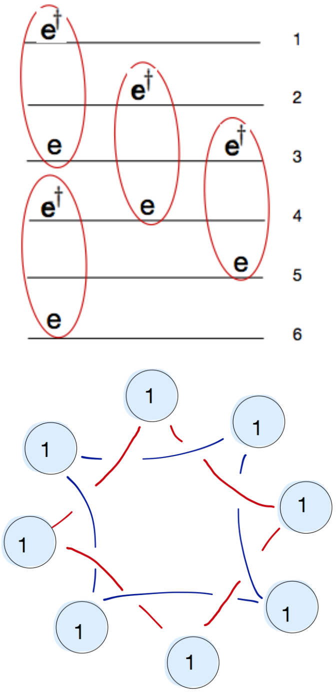

Figure 3.3: Two representations of the (warmup) coupled-layer construction for $D = 4+1$ Maxwell theory with gauge group $\mathsf{U}(1)_o \times \mathsf{U}(1)_e$. The top figure is the direct analog of the previous figure; the bottom is a 'quiver' or 'moose' diagram familiar from the high energy physics literature.

coupled to charge-1 lattice bosons e_i, with arbitrary hopping terms in the three spatial dimensions.

For definiteness, we could consider each layer in the zero-correlation length limit where it is described by a solvable Kitaev-like model with a rotor on each (oriented) link, $[\mathsf{E}_l, \mathbf{a}_{l'}] = \mathbf{i}\delta_{l,l'}$, $\mathsf{E} \in \mathbb{Z}, \mathbf{a} \simeq \mathbf{a} + 2\pi$, with

$$\mathbf{H}_{\text{layer}} = \sum_{+} (\Delta \cdot \mathsf{E})^2 - \sum_{\square} \cos(\Delta \times \mathbf{a}) \,.$$

Δ is a lattice gradient operator. The first sum is over vertices and the second over plaquettes of the square lattice.[106] The charged bosonic matter arises at sites where $0 \neq \Delta \cdot \mathsf{E} \in \mathbb{Z}$.

Couple together the layers by the (completely local and gauge invariant) terms

$$\Delta H = V \sum_x \sum_i (|\tilde{\mathbf{b}}_i(x)|^2 - v^2)^2 \,. \tag{3.9}$$

Here x labels a site of the 3d lattice. Fig. 3.3 shows the case of $N = 6$ layers, with $\tilde{\mathbf{b}}_i, i = 1..4$ circled. Minimizing the potential (3.9) causes $\tilde{\mathbf{b}}_i$ to condense,

$$\tilde{\mathbf{b}}_i \equiv e_i^\dagger e_{i+2} = v e^{\mathbf{i} a_{i,i+2}} , \tag{3.10}$$

higgsing $\prod_i \mathsf{U}(1)_i \to \mathsf{U}(1)_{\text{even}} \times \mathsf{U}(1)_{\text{odd}}$. The phases $a_{i,i+2}$ provide the link variables in the extra dimension. Layers with odd i and even i are decoupled. The result is $4+1d$ Maxwell theory with $\mathsf{G} = \mathsf{U}(1)_{\text{even}} \times \mathsf{U}(1)_{\text{odd}}$, with massless bulk photons. So this is not the bulk state we are looking for, but it will be instructive.

$\mathsf{U}(1)$ lattice theory in 4+1 dimensions should have a kinetic term for the link variables along the extra dimension. This $\mathsf{E}^2_{x,x+\hat{4}}$ term arises as follows. The conjugate variable E to $\mathbf{a}$ arises from the amplitude fluctuations of $\tilde{b}$:

$$\tilde{\mathbf{b}}_l = e^{\mathbf{i}\mathbf{a}_l}(v + \mathsf{E}_l), \quad \tilde{\mathbf{b}}_l^\dagger = (v + \mathsf{E}_l) e^{-\mathbf{i}\mathbf{a}_l} .$$

$$[\tilde{\mathbf{b}}_i^\dagger(x), \tilde{\mathbf{b}}_j(y)] = -\mathbf{i}\delta_{xy}\delta_{ij} \implies [\mathbf{a}_l, \mathsf{E}_{l'}] = -\mathbf{i}\delta_{ll'}.$$

Expanding the condenser term (3.9) about the minimum, $\tilde{\mathbf{b}}^\dagger\tilde{\mathbf{b}} - v^2 = 2v\mathsf{E} + ..$, we find

$$\Delta H = V 4 v^2 \sum_l \mathsf{E}_l^2 + ..$$

The Hamiltonian should also contain terms which suppress flux through plaquettes parallel to the extra dimension: $\sum_{\text{plaquettes} \,\|\, x^4} \cos \Delta \times \mathbf{a}$. These terms arise from microscopic gauge invariant terms including the hopping term for $\tilde{\mathbf{b}}$:

$$\Delta_2 H = -V_2 \sum_{x,i} \sum_{\hat{\mu} \neq \hat{4}} \begin{pmatrix} \tilde{\mathbf{b}}_i(x+\hat{\mu}) e^{\mathbf{i}\int_x^{x+\hat{\mu}} \mathbf{a}_i - \mathbf{i}\int_x^{x+\hat{\mu}} \mathbf{a}_{i+2}} \tilde{\mathbf{b}}_i^\dagger(x) \\ +h.c. \end{pmatrix}$$

where $\mathbf{a}_i$ is the pre-existing gauge field within layer i. Upon condensing the $\tilde{\mathbf{b}}_i$, the new interlayer gauge field $\mathbf{a}_{i,i+2}$ combines with the existing within-layer gauge fields to form a closed Wilson loop in the $\mu 4$ plane for each term in the μ sum.

It will be useful to remind ourselves about magnetic monopoles in $\mathsf{U}(1)$ lattice gauge theory (*e.g.* [107]). A region R of the lattice whose boundary ∂R has $\oint_{\partial R} B = 2\pi g$ contains g magnetic monopoles, $g \in \mathbb{Z}$. This means that the number of monopoles is not conserved on the lattice; for example, consider a region which is a single 3-cell V of the lattice; we may change $\oint_{\partial V} B$ from 0 to 2π without changing anything, since the gauge field is periodic $\mathbf{a} \simeq \mathbf{a} + 2\pi$ and $B = \vec{\nabla} \times \mathbf{a}$.

To make contact with the BdC theory, it will be illuminating to dualize the odd/even gauge fields $a^{o/e}$ to 2-form potentials: $f^{o/e} = da^{o/e} = \star dC^{o/e}$. The action is

$$S = \sum_{\alpha = o,e} \int_{5d} \left(\frac{1}{g_\alpha^2} dC^\alpha \wedge \star dC^\alpha + C^\alpha \wedge \star j_m^\alpha \right).$$

By the Meissner effect, magnetic flux tubes of the broken relative $\mathsf{U}(1)$s collimate the monopoles into *monopole strings*. They must do so, since, by construction, objects magnetically charged under $a^{e/o}$ are minimally coupled to the dual field $C^{e/o}$ and must be strings. States where the total magnetic charge in different layers is not equal do not have finite energy. We sequester a few more details about this to appendix §3.9.

3.5.2 Dyon string condensation in more detail

The actual construction of the nontrivial gapped bulk is as follows. Again each layer is ordinary electrodynamics with bosonic charges. We will call $\varepsilon_i \equiv e_i m_i$ the dyon in each layer, which is a fermion. The object $b_i \equiv \varepsilon_i^\dagger m_{i+1} \varepsilon_{i+2}$ is a boson (two fermions plus one boson, and no net electric charge to produce extra statistics, equals a boson).

The objects b_i $(i = 1..N-2)$, for all i, are mutually local (*i.e.* their charge vectors satisfy $q_i e_j - q_j e_i = 0, \forall i, j = 1..N-2$) under the total $U(1)$ (in particular, they all have $q_e^{Total} = 0, q_m^{Total} = 1$). This means that it is possible to couple the layers so that these objects condense [108, 109, 110].

Explicitly, we can cause them to condense by adding the completely local gauge invariant hamiltonian $\Delta H = V \sum_x \sum_i (|b_i(x)|^2 - v^2)^2$. The phase of the condensate $b_i(x) = ve^{\mathbf{i}a_{i,i+2}}$ is again a link variable along the extra dimension; unlike the simple construction of §3.5.1, the duality frame in which this object is the vector potential rotates as we increase i.

Condensing b_i (obliquely) *confines* the gauge fields in the layers $a_{i+1}, i+1 = 2...N-1$. Objects which are not mutually local with b_i are confined. What's left? We are condensing $N-2$ objects in a theory with gauge group $\mathsf{U}(1)^N$, so two gauge fields remain massless. The charged objects which are mutually local with the condensate and therefore not confined [108, 109, 110] are (just as in the 2d $\mathbb{Z}_2$ case [1]):

- At the top layer : $\varepsilon_1, m_1\varepsilon_2^\dagger$ and their boundstate $\varepsilon_1 m_1 \varepsilon_2^\dagger$ (and powers and products of these) and
- At the bottom layer: $\varepsilon_N, m_{N-1}\varepsilon_N^\dagger, \varepsilon_N m_{N-1}\varepsilon_N^\dagger$ etc.

At the top layer, the objects $\varepsilon_1, m_1\varepsilon_2^\dagger$ are both fermions, and have charge $(q_e, q_m) = (1,1)$ and $(-1,0)$ respectively. The boundstate has charge $(0,1)$ and is therefore also a fermion, by the standard argument reviewed above, because there is still a Maxwell field at the top layer.

To see the full effect of condensing b_i, consider the blue box in the figure at right. Although $\varepsilon_i^\dagger m_{i+1}$ is mutually local with $m_i\varepsilon_{i+1}$, the constituents are not. This has the consequence that condensing b_i *binds* the monopole strings of $a^{e/o}$ to electric flux lines of $a^{o/e}$! This is precisely the effect of the additional term

$$\Delta S = \int \frac{1}{2\pi} C^e \wedge dC^o \equiv \int \frac{1}{2\pi} B \wedge dC$$

in the low-energy description.

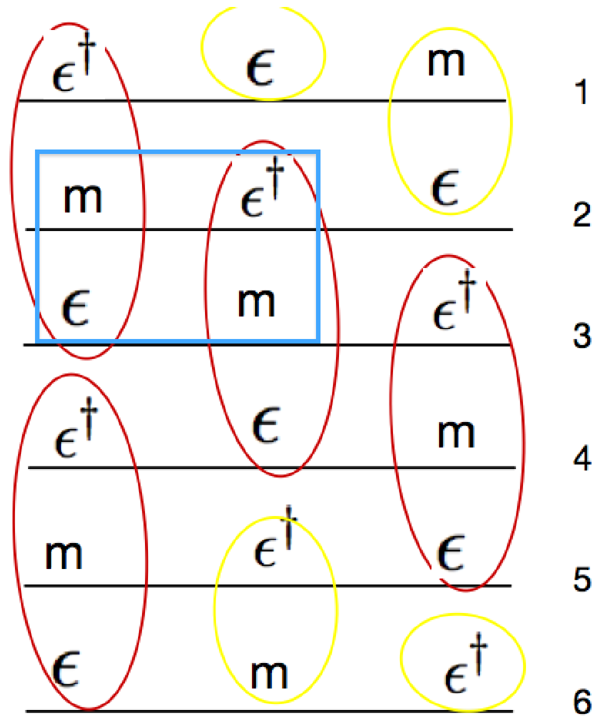

Figure 3.4: Coupled layer construction, the objects in the box are mutually nonlocal.

3.5.3 Alternative description of layer construction

Here we make contact between the coupled layer construction of the previous subsection and the general description (in the section introduction) in terms of coupled layers of $\mathcal{E}$ and $\mathcal{E}^{-1}$ which guarantees the correct edge states.

Again let $\mathcal{E}$ denote a single copy of all-fermion electrodynamics. First we note that the all-fermion electrodynamics is its own inverse: $\mathcal{E} = \mathcal{E}^{-1}$ in the sense that two copies of all-fermion electrodynamics can be regularized in $3+1$ dimensions. More specifically, $\mathcal{E} \times \mathcal{E}$ is deformable (by adding local, gauge-invariant interactions) to ordinary bosonic U(1) gauge theory. To see this[18], let e and $\tilde{e}$ denote the electrons in $\mathcal{E}$ and $\mathcal{E}^{-1}$. Define $b = e\tilde{e}^{\dagger}$, which is a boson. If we condense this boson, we higgs $U(1)_{\mathcal{E}} \times U(1)_{\mathcal{E}^{-1}}$ to the diagonal $\mathsf{U}(1)$ subgroup. The object e is a fermion charged under this gauge group; it is related to $\tilde{e}$ by taking charges from the condensate. We should think of this object as the dyon of ordinary BBF electrodynamics,

[18] An essentially identical argument shows that the all-fermion toric code is its own inverse.

because all of the other excitations which are mutually local with the condensate are bosons:

- $m, \tilde{m}, \varepsilon = em$ and $\tilde{\varepsilon} = \tilde{e}\tilde{m}$ are non-local with respect to the condensate, so they are confined.
- $M \equiv m\tilde{m}^\dagger$ is a boson which differs from e by one unit of electric charge, and so we should think of it as the monopole. It is related by taking stuff from the condensate to $\varepsilon\tilde{\varepsilon}$.
- Adding e to M we get another boson (since we are combining two mutually non-local fermions) $\varepsilon\tilde{m}^\dagger$; apparently we should regard this as the elementary electrically charged boson.

We conclude that $\mathcal{E} \times \mathcal{E}^{-1}$ is separated by simple Higgs transition from the phase $U(1)_{FBB}$, with a propagating photon (if it's in the deconfined phase), and therefore has a $D = 3+1$ regulator.

It is important to note that the remaining electrodynamics still has charged matter which may be condensed to higgs or confine the photon; the choice of whom to condense means that various bulk models are possible.

So, while a single copy of all-fermion electrodynamics cannot be regulated in $3+1$ dimensions, a pair of copies can be so regulated since $U(1)_{FBB}$ can be so regulated and $\mathcal{E} \times \mathcal{E}^{-1} \sim U(1)_{FBB}$. The layer construction in §3.5.2, when applied a slab of finite thickness, provides just such a regulator. As long as the thickness of the slab is not taken to infinity, the two copies of all-fermion electrodynamics can be regarded as living in $3+1$ dimensions.

Further insight into the layer construction is obtained by viewing the construction in terms of a stack of such slabs, where each slab, denoted $(\mathcal{E}\mathcal{E}^{-1})$, hosts two copies of all-fermion electrodynamics, one on the bottom surface and one on the top surface. The stack of slabs is denoted $(\mathcal{E}_1\mathcal{E}_1^{-1})...(\mathcal{E}_n\mathcal{E}_n^{-1})$ where $i = 1, ..., n$ indicates the extra spatial dimension. Pairing up the all-fermion states within each slab produces the trivial bulk state in $4+1$ dimensions. Pairing $\mathcal{E}_i^{-1}$ with $\mathcal{E}_{i+1}$ across neighboring slabs realizes the bulk non-trivial state. This way of thinking about the layer construction realizes the motivating idea given in the section introduction.

To be a little more explicit, condensing only $b_i = (e_{i,\text{top}}\tilde{e}^\dagger_{i,\text{bottom}})$ produces layers of ordinary *FBB* electrodynamics, by the preceding argument. This returns us to the starting point of the layer construction of the previous section. The slabs of *FBB* electrodynamics can then be confined to produce a trivial bulk state.

To produce the non-trivial bulk state, the gluing may be performed by repairing the missing condensates at the top and bottom of Fig. 3.2. In particular, think of each pair $\mathcal{E}_i\mathcal{E}_i^{-1}$ as a copy of Fig. 3.2. At the top we have fermionic charges ε_1 and $m_1\varepsilon_2$; at the bottom we have fermionic charges $\varepsilon^\dagger_{N-1}m_N$ and $\varepsilon^\dagger_N$. If we glue the bottom to the top by condensing

$$b_{N-1} \equiv (\varepsilon^\dagger_{N-1}m_N)\varepsilon_1$$

and

$$b_N \equiv \varepsilon^\dagger_N(m_1\varepsilon_2)$$

then we get the BdC theory rolled up on a circle, *i.e.* the coupled layer construction has translation invariance $i \to i+1$. And in particular, there is no photon in the bulk.

3.5.4 Extension to $D = 3+1$ and derivation of BF theory

The logic by which we inferred the presence of the *BdC* coupling from the coupled layers construction can be applied to the original construction [1] of the $D = 3+1$ boson SPT state with time-reversal symmetry. The string of magnetic excitations is a vortex line; the mutual nonlocality of the constituents of the condensed boson glues this vortex line to the electric flux lines of the other gauge field. The result is that the bulk model contains a term of the form $\frac{1}{2\pi}B\wedge F$. That the bulk theory admits such an effective description is well-known [68]. An implication of this derivation which has not been appreciated to our knowledge is that the all-fermion toric code – when realized on the surface of a bosonic SPT – suffers a fermion-number anomaly, as we discuss in the next section.

3.6 Fermion number anomaly inflow

We will now interpret the obstruction studied here in terms of global anomaly inflow. The only symmetry involved in this system is fermion parity. We emphasize that in the bulk there *are* no fermions; however, the Jackiw-Rebbi effect demonstrates clearly that gauge fields are capable of carrying this quantum number.

In the following we show that the fermion number conservation on the surface of the 4+1d short-range-entangled state constructed in the previous section is violated by high-energy processes.

There is a precedent for such violation of fermion number by quantum gauge theory. The Witten $\mathsf{SU}(2)$ anomaly [111] can be regarded as an anomaly for fermion number: in a Witten-anomalous gauge theory, instanton events create an odd number of fermions and hence violate fermion parity conservation; this is not something we know how to describe with a local field theory.

In the prehistory of SPT physics, a subset of the authors [112] studied a system where the Witten anomaly played a crucial role in preserving the integrity of the classification of statistics of $3+1$d particles. In particular, the Witten anomaly was argued to forbid a gauge theory whose monopoles carry a single majorana zero mode (which monopoles, if they could be deconfined, would enjoy non-Abelian statistics). That paper also described a 4+1d dimensional model whose edge realized such a gauge theory, and therefore could be regarded as exhibiting 'Witten anomaly inflow'.

Fermion number anomaly. The all-fermion electrodynamics, as it arises on a surface of the coupled-layer construction, exhibits crucial differences from an intrinsically 3+1-dimensional system with a bosonic regulator. First of all, note that the slab geometry constructed in §3.5 harbors gauge-invariant states with a single fermionic particle at the top layer[19]. Since all femionic

[19] Here we are assuming that the 3d geometry is noncompact, so that the flux has somewhere to go. If the 3d spatial sections are compact, we cannot have a single string stretching from one end of the slab to the other because

excitations carry some gauge charge (either electric or magnetic) – as they must in a system with a bosonic regulator – there is no state in a putative 3+1 dimensional realization of this form.

Further, the coupled layer construction of §3.5 directly shows that fermion number can be transported across the extra dimension, as follows. Consider a state with an excitation of ε_1, the dyon at the top layer. This excitation can for free absorb bosons from the condensate, which include objects of the form $b_1 = \varepsilon_1^\dagger m_2 \varepsilon_3$. Combining these two objects we get something with the quantum numbers of $m_2\varepsilon_3$. This looks a bit like a bulk fermion excitation, but this object is confined (since it is not mutually local with b_2, which is condensed). Also condensed is $b_3 = \varepsilon_3^\dagger m_4 \varepsilon_5$; adding one of these in, we get $m_2 m_4 \varepsilon_5$. The bottom layer (for argument, we take $N = 6$ layers, as in the figure above) supports a deconfined fermion excitation $\varepsilon_5^\dagger m_6 = f_{\text{bottom}}$. The condensate plus top-layer excitation $f_{\text{top}} = \varepsilon_1$ is related to this by

$$f_{\text{top}} b_1 b_3 = m_2 m_4 m_6 f_{\text{bottom}}^\dagger.$$

With arbitrary (even) N, we have:

$$f_{\text{top}} b_1 b_3 ... b_{N/2} = m_2 m_4 ... m_N f_{\text{bottom}}^\dagger.$$

This equation is understood to be true modulo the creation of neutral excitations (which are all bosonic, by assumption).

This strongly suggests that a monopole string ($m_2 m_4 m_6 ...$) (bosonic, but confined) allows

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of the bulk Gauss law:

$$0 = \frac{\delta S}{\delta B} = \star j + dC + d \star dB \frac{1}{g^2} \tag{3.11}$$

which is a 3-form. If we integrate this over a 3d region Υ at fixed time and codimension 1 in space, we get

$$0 = (\text{ number of strings penetrating the region, counted with orientation}) + \int_{\partial\Upsilon} (C + \star dB/g^2).$$

The last term is the usual Gauss' law term for a 2-form potential, but the important thing is that the dependence on the fields on the RHS of (3.11) is a total derivative. So if there is no boundary of Υ – such as if the whole space is $T^3 \times (0,1)$ and we choose Υ to be the T^3 at some fixed position along the interval – then the net number of strings must be zero.

fermions to tunnel from the top layer to the bottom layer. A quantitative statement to this effect is that there is a nonzero amplitude in the groundstate $|g_s\rangle$ for a pair of fermions to be created at top and bottom, connected by a monopole string:

$$\langle g_s|f_{\text{bottom}}m_2m_4...m_N f^\dagger_{\text{top}}|g_s\rangle = \langle g_s|f_{\text{top}}f^\dagger_{\text{top}}b_1b_3...b_{N/2}|g_s\rangle = v^n\langle g_s|f_{\text{top}}f^\dagger_{\text{top}}|g_s\rangle \neq 0$$

$v^n \sim e^{-L}$ decays exponentially in the thickness of the slab, but this implies a finite tunneling amplitude. (Here $n(N) \equiv \frac{N-2}{2}$.)

Since all fermions are charged either electrically or magnetically (it is ambiguous which should be interpreted as the electron and which as the magnetic monopole), the fermion number anomaly also implies a discrete gauge anomaly. That is, rotating the phase of every fermion by π is part of the $\mathsf{U}(1)$ gauge group (though not only the electric group in any one duality frame). This is similar to Goldstone's understanding of the Witten anomaly [113] (as cited in [114, 115, 116]).

Putting two copies of the system together removes the anomaly. From the point of view above, it is because the monopole strings will reconnect so that they only attach fermions at the same surface. A similar mechanism of reconnection was described in [112].

3.7 Consequences for all-fermion toric code

So far we've discussed bosonic SPTs in $D = 4+1$ with no symmetry, and have briefly mentioned bosonic SPTs in $D = 3+1$ with time-reversal ($\mathcal{T}$) symmetry. In both cases, there is a symmetry-preserving termination which is a gauge theory where all the matter is fermionic. There are many illuminating connections between these two problems. To understand them, we must now discuss the $D = 3+1$ $\mathcal{T}$ invariant SPT [68, 91, 1, 25, 24] in more detail.

Briefly, the bulk $3+1$ dimensional state is a quantum phase of bosons protected by time reversal symmetry. The bulk theory has a surface termination consisting of $2+1$ dimensional $\mathbb{Z}_2$ gauge theory in which the charge, the vortex, and charge-vortex composite are all fermions. As in the case of all-fermion electrodynamics, the statistics of the charge-vortex composite actually follows from those of the charge and the vortex provided there are no gauge-invariant fermions in the spectrum. What does time reversal have to do with such a $2+1$ dimensional state? Naively, the answer is not much: all topological data, e.g., fusion rules, quantum dimensions, braiding phases, etc. are real numbers, so time reversal invariance doesn't seem to provide a constraint on the topological data.

However, there is one piece of topological data which is sensitive to $\mathcal{T}$ and that is the chiral central charge, c_-. Furthermore, in a microscopic bosonic model, the value of c_- is constrained by the topological data. If we have anyon types labeled by a with quantum dimensions d_a and topological spins s_a, then the chiral central charge is determined, $\bmod 8$, by [117, 118, 12]

$$\frac{\sum_a d_a^2 e^{2\pi i s_a}}{\sqrt{\sum_a d_a^2}} = e^{2\pi i c_-/8}. \tag{3.12}$$

In a model of abelian anyons, all $d_a = 1$ and the total quantum dimension, $\mathcal{D} = \sqrt{\sum_a d_a^2}$, is simply the square root of the number of anyon types (including the identity). The fact that the central charge is only determine mod 8 is not an accident [12]. The E_8 state of bosons has no anyonic excitations but has chiral central charge $c_- = 8$, hence we may add layers of the E_8 to any anyon model without changing the anyon content but shifting the chiral central charge by 8.

For the familiar $\mathbb{Z}_2$ gauge theory in which charges and vortices are bosons, we have $a \in \{1, e, m, em\}$, $d_a = 1$, $s_1 = s_e = s_m = 0$, and $s_{em} = 1/2$. Hence (3.12) gives

$$e^{2\pi i c_-/8} = \frac{3 + (-1)}{2} = 1 \tag{3.13}$$

hence $c_- = 0 \bmod 8$. In other words, the minimal $\mathbb{Z}_2$ gauge theory has no chiral edge states. However, if we consider the all-fermion gauge theory, then we find

$$e^{2\pi i c_-/8} = \frac{1 + 3(-1)}{2} = -1 \tag{3.14}$$

hence $c_- = 4 \bmod 8$. Thus the all-fermion gauge theory must have chiral edge states and hence must indeed break $\mathcal{T}$. The reason why this state can be realized in a $\mathcal{T}$-invariant manner at the surface of a $\mathcal{T}$-invariant $3+1$ bulk state is that in this case it is impossible to create an edge for the gauge theory at which the chiral edge states can be exposed.

Now we turn to connections between the system just discussed and the all-fermion electrodynamics in $D = 3+1$. First, suppose all-fermion electrodynamics did have a time reversal symmetric bosonic regulator. Then so does the all-fermion toric code. The argument is as follows. Condense pairs of charges in 3+1d (thereby higgsing the gauge group to $\mathbb{Z}_2$), and place the system on $\mathbb{R}^2 \times S^1$, where the radius of the S^1 is L. The $\mathbb{Z}_2$ topological order implies that states with different $\mathbb{Z}_2$ flux through the circle are split only by an amount of order $E_{\text{flux}} \sim e^{-L|\log t|/\xi}$ where t is a hopping amplitude for $\mathbb{Z}_2$ charged quasiparticles, and ξ is the bulk correlation length. The regime of interest has $L \gg \xi$ (so that our field theory analysis is valid) and $E_{\text{flux}} \gg m_e, m_m$, where m_e and m_m are the rest energies of the electric and magnetic quasiparticle excitations. The result is then the all-fermionic toric code with, by assumption, a time-reversal symmetric bosonic regulator. Assuming that no such regulator exists for the all-fermion toric code, no such regulator can exist for all-fermion electrodynamics. (And as [66] point out, the case with time reversal symmetry is actually the crucial case, in the sense that the SPTness of the state persists even upon breaking time reversal.)

Second, all-fermion electrodynamics does have a time reversal symmetric *fermionic* regulator. Indeed, it is equivalent to BBF electrodynamics by binding the neutral fermion to the electron. (In this case there are particles of both statistics in each charge sector; for purposes of discussion, we label a model by the statistics of the lightest particle in each sector.) Again condense charges and compactify on a circle. This produces a time reversal symmetric fermionic regulator for the all-fermion toric code. And again, we can convert FFF toric code to BBF toric code in the process.

It is instructive to ask what happens to the chiral central charge formula (3.12). The answer is that the formula only applies when the regulator is bosonic. This is crucial because the mod 8 property of the formula relied on the E_8 phase being the simplest phase with chiral edge states and no anyonic excitations. Once we add microscopic fermions, there are simpler chiral states. The simplest is the $p+ip$ state of fermions with $c_- = 1/2$. Hence while the minimal chiral central charge, $c_- = 4$, of the all-fermion gauge theory could not be cancelled with only bosonic short-range entangled states (which can only shift c_- by 8), the minimal central charge of the all-fermion gauge theory can be cancelled by fermionic short-range entangled states (which can shift c_- by $1/2$).

In both cases adding microscopic fermions saves everything, in the sense that all spectra of excitations are adiabatically connected.

Fermion number anomaly. Since the structure of our coupled-layer construction is so similar to that of the $D = 3+1$ beyond-cohomology boson SPT in [1], the same logic applies to that model (removing daggers where necessary since charges are binary). That is, in a slab geometry, a state with a fermion on the top surface can tunnel to a state with a fermion on the bottom surface, because the quasiparticle sectors are related by bosonic operators (some of which are condensed):

$$f_{\text{top}} b_1 b_3 \ldots b_{N/2} = m_2 m_4 \ldots m_N f_{\text{bottom}}.$$

We therefore expect that this bosonic state can transport fermion number between edges.

In this case, the bulk state is protected by time-reversal invariance. Breaking time reversal only at the surface produces a state which is still not edgeable. We give two examples of time-reversal broken surface states momentarily. It will help to see the connection between the fermion number anomaly and the preservation of $\mathcal{T}$ to ask: What happens to the edge if we adiabatically continue the bulk through a $\mathcal{T}$-breaking path to a product state? It is not necessary to have a surface phase transition: Without $\mathcal{T}$, one way to deform the bulk (on a torus, say) to a product state is to open up an array of gapped trivial surfaces (possible because $\mathcal{T}$ is broken) and then expand the intervening vacuum regions to consume the system, following [90]. On a system with boundary, this can be done everywhere except at topologically ordered boundaries which are independently stable. On a slab of finite but large thickness, therefore, in the absence of $\mathcal{T}$, one can disconnect the top from the bottom by cutting open a middle (trivial, gapped) surface, hence ending the fermion tunneling without destroying the surface topological order.

A model with the same spectrum of quasiparticles and braiding statistics *can* be realized intrinsically in $D = 2+1$. For example, it can be obtained from the Kitaev honeycomb model with $\nu = 8$ (see Table 2 of [12]). That model does not preserve time reversal symmetry: the violations of time-reversal symmetry occur at boundaries, where there is a chiral edge spectrum (with $c_L - c_R = 4$). The model at the surface of the boson SPT cannot be put on a space with boundary (since the boundary of a boundary is empty) and is time-reversal invariant. The price for this extra symmetry is that the fermion number is not conserved!

To connect the various phenomena, it is useful to explicitly realize various $\mathcal{T}$ broken surface states starting from the $\mathcal{T}$ invariant all-fermion surface toric code. The basic observation follows from the previous paragraph: given $\mathbb{Z}_2$ charged fermionic matter we may shift the vortex from bosonic to fermionic and vice versa by adding $\nu = \pm 8$ copies of a $p+ip$ state for the charged fermions. Normally in 2+1 dimensions the time-reversal point has an absolute chiral central charge $c_- = 0$ and a bosonic vortex. We can obtain a fermionic vortex and $c_- = 4$ by adding $\nu = 8$ copies of charged fermions in $p+ip$ states. However, on the surface of the $\mathcal{T}$ invariant bosonic SPT, there is a shift in the spectrum so that the $\mathcal{T}$ invariant point has a fermionic vortex. Then we may construct a pair of $\mathcal{T}$ broken surface states which are still topologically ordered by adding $\nu = \pm 8$ copies charged fermions in $p+ip$ states. The system now explicitly breaks time reversal and has a bosonic vortex.

Given a bosonic vortex, we may condense the vortex to destroy the surface topological order. At a domain wall between the two distinct ways to break $\mathcal{T}$ to obtain a bosonic vortex we have $\nu = 16$ Majorana edge modes before condensing the vortex. After condensing the vortex we obtain the edge of E_8 state of bosons [13]. Thus we obtain the same edge physics as the E_8 BF theory discussed in [68]. This analysis provides another route to connect the layer construction to a topological bulk theory via the non-trivial surface, in this case in $3+1$ dimensions. When the surface preserves $\mathcal{T}$ we may interpret the bulk FF term in $3+1$ dimensions as providing a $\mathcal{T}$ invariant regulator for the surface all-fermion toric code.

Again the presence of neutral bulk fermions renders everything trivial. In the presence of microscopic neutral fermions, the bosonic SPT can be deformed into 16 copies of the free fermion topological superconductor, and this in turn is equivalent to nothing [70, 119]. So adding fermions explicitly makes the bulk trivial (in addition to the edge). This picture nicely complements the edge analysis above where we argued that adding fermions effectively changes the minimal chiral central charge one can have without topological order (from $c_- = 8$ to $c_- = 1/2$).

Reality of this phenomenon. We have to ask: Are there real physical systems made just of bosons, with a gap, which can transport fermion number? The $D = 3+1$ boson SPT protected by time-reversal should do so. This makes it even more interesting to try to realize this state in the world.

Finally, we note the following consequence of our claim, given that elementary gauge-neutral fermions have not been observed in nature[20]. Were we to discover a fermionic magnetic monopole in our world, it would imply either[21]:

1. There are microscopic, gauge-neutral fermions. The opposite is conjectured to be true in *e.g.* Ref. [6].

2. We live on the boundary of some higher dimensional space. Boundary theories of 4+1D SPT phases have been suggested in attempts to understand the matter content (and flavor structure) of the standard model [29, 120, 121, 122].

3.8 Lattice bosons for duality-symmetric surface QED

This is a model of bosons. The two-form gauge theory studied in this paper is a model of bosons. Low-energy evidence for this statement is the fact that we did not have to choose a spin structure to put it on an arbitrary 4-manifold. This is in contradistinction to $\mathsf{U}(1)_{k=1}$ CS theory in $D = 2+1$. We note in passing that on a manifold that admits spinors, the intersection form is even ($I(v,v) \in 2\mathbb{Z}$) [53]. (This means that to describe an effective field theory for a *fermionic* SPT state, we should consider the level $k \in \mathbb{Z}/2$.)

High-energy (*i.e.* condensed-matter) evidence for the claim that this is a model of bosons is the following conjecture for a lattice model of bosons which produces this EFT. The Hilbert

[20]Here we mean 'neutral under gauge groups which are unbroken at low energies'; absent discrete gauge symmetries, a right-handed neutrino would falsify this claim.

[21]We must note some uncertainty involving the role of gravity.

space is as follows and is similar to lattice boson constructions of electrodynamics in other dimensions [123, 124, 58, 125].

• Put rotors $e^{\mathbf{i}b_p}$ on the *plaquettes* p of a 4d spatial lattice. (Actually, the model is defined for any 4d simplicial complex. Translation invariance will not play a significant role.) These act as

$$e^{\mathbf{i}b_p}|n_p\rangle = |n_p+1\rangle$$

on states with definite excitation number n_p; we will interpret n_p as a number of (oriented) 'sheets' covering the plaquette.

• Put charge-k bosons $\Phi_\ell = \Phi^\dagger_{-\ell}$ on the *links* ℓ. These satisfy $[\Phi_\ell, \Phi^\dagger_\ell] = 1$. We will say that $\Phi^\dagger_\ell$ creates a string segment, and $\Phi^\dagger_\ell \Phi_\ell$ is the number of (oriented) strings covering the link.

The Hamiltonian is

$$\begin{aligned}\mathbf{H} = \; & -\underbrace{\sum_{\text{links},\ell\in\Delta_1}\Big(\sum_{p\in s(\ell)} n_p - k\Phi^\dagger_\ell\Phi_\ell\Big)^2}_{\mathbf{H}_1,\; textgausslaw.happywhensheetsclose, \text{ or end on strings}} - \underbrace{\sum_{\text{volumes},\, v\in\Delta_3}\prod_{p\in\partial v} e^{\mathbf{i}b_p} + h.c.}_{\mathbf{H}_3\sim B^2\text{, makes sheets hop.}} \\ & - \underbrace{\Gamma\sum_{p\in\Delta_2} n_p^2}_{\mathbf{H}_2\sim E^2\text{. discourages sheets.}} - \underbrace{t\sum_{p\in\Delta_2} e^{\mathbf{i}kb_p}\prod_{\ell\in\partial p}\Phi^\dagger_\ell + h.c.}_{\mathbf{H}_{\text{strings}},\text{ hopping term for matter strings}} + V\left(|\Phi|^2\right)\end{aligned}$$

The subscripts indicate the dimension of the simplices to which the terms are associated. When $\Gamma = 0, V = 0$, these terms all commute. The groundstate for $t > 0$ is described by a soup of oriented closed 2d sheets, groups of k can end on strings.

Now take $V\left(|\Phi|^2\right) = \left(|\Phi|^2 - v^2\right)$. This causes to condense $\Phi_\ell = ve^{\mathbf{i}\varphi_\ell}$, which leads to a 2-form higgs mechanism:

$$\mathbf{H}_{\text{strings}} = -\sum_p tv^4 \cos\left(kb_p - \sum_{\ell\in\partial p}\varphi_\ell\right)$$

On the low-energy manifold of this Hamiltonian, we have

$$\left(e^{\mathbf{i}b_p}\right)^k = \mathbb{1}, \;\; |n_p\rangle \simeq |n_p + k\rangle.$$

This leaves behind k species of (unoriented) sheets.

The groundstates are then described by equal-superposition sheet soup. If the intersection form on the spatial 4-manifold which is triangulated by the simplicial complex has $I = \mathbb{1}$, there are k^{b_2} groundstate sectors. These groundstates represent the algebra of 'tube operators': for any closed union of 2-simplices ω

$$\mathcal{F}_\omega \equiv \prod_{p\in\omega} e^{\mathbf{i}b_p} \qquad\qquad \mathcal{T}_\omega \equiv \prod_{V\in\Delta_3}\prod_{p\in\partial V\cap\omega} n_p$$

$$\mathcal{F}_\omega \mathcal{T}_{\omega'} = e^{2\pi\mathbf{i}I_{\omega\omega'}}\mathcal{T}'_\omega \mathcal{F}_\omega$$

Continuum limit. The higgs mechanism described above leads to $U(1) \overset{\text{higgs}}{\to} \mathbb{Z}_k$ 2-form gauge theory:

$$L = \frac{tv^4}{2}(\mathrm{d}\varphi_1 + kB_2)\wedge\star(\mathrm{d}\varphi_1 + kB_2) + \frac{1}{g^2}\mathrm{d}B_2\wedge\star\mathrm{d}B_2$$

$$\simeq \frac{k}{2\pi}B\wedge\mathrm{d}C + \frac{1}{8\pi tv^4}dC\wedge\star\mathrm{d}C + \frac{1}{g^2}\mathrm{d}B\wedge\star\mathrm{d}B$$

with $\mathrm{d}C \simeq 2\pi t\star(\mathrm{d}\varphi + kB)$. This equivalence is described in [49, 52].

3.9 More details on monopole strings and vortex sheets in 5d abelian gauge theory

Consider a 5d U(1) 1-form gauge field a, with field strength $f = da$. A magnetic excitation with respect to this gauge field has $\oint_{\Sigma_2} f = 2\pi g$, where Σ_2 is a closed 2-surface *surrounding* the object. Such an object is therefore codimension three, and is a string in 4+1 dimensions. The quantity which is localized on the monopole strings is therefore a three-form:

$$\star j_m = \delta^3(\text{monopole strings}) = \star df \equiv dC$$

where C is a two-form.

Suppose we higgs the $U(1)$ gauge field by condensing a charged order parameter field $b \sim ve^{i\phi}$. This adds

$$\delta H = m^2(a + d\phi)^2,$$

so that a eats the phase ϕ, and $m \sim tv$. Topological defects in ϕ, *i.e.* zeros of b around which ϕ winds by 2π, occur at codimension two (since b is a complex function) and in 5d are therefore 2+1-dimensional vortex *sheets*.

These vortex sheets can end on the monopole strings. This is the same fact as the fact that vortex strings can end on magnetic monopoles in 3+1 dimensions. In the higgs phase of a 3+1 dimensional abelian gauge theory, the vortex string provides a means to collimate the magnetic flux coming out of the monopole. The result is the confinement of the magnetic charges; this is a manifestation of the Meissner effect. It is the same in $D = 4+1$, except now it is magnetically charged *strings* which are connected by vortex *sheets*. In the higgs phase, it is energetically favorable for the monopole strings to be connected by such vortex sheets.

The final ingredient in the coupled-layer construction is the fact that the condensate is not purely electric with respect to any individual layer.

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Chapter 4

Non-Relativistic Conformal Field Theories in the Large Charge Sector

4.1 Abstract

We study Schrödinger invariant field theories (nonrelativistic conformal field theories) in the large charge (particle number) sector. We do so by constructing the effective field theory (EFT) for a Goldstone boson of the associated $U(1)$ symmetry in a harmonic potential. This EFT can be studied semi-classically in a large charge expansion. We calculate the dimensions of the lowest lying operators, as well as correlation functions of charged operators. We find universal behavior of three point function in large charge sector. We comment on potential applications to fermions at unitarity and critical anyon systems.

4.2 Introduction and Summary

Symmetry has always been a guiding principle in characterizing physical systems. While weakly coupled field theories are known to be tractable in terms of perturbation theory in coupling, often the strongly coupled ones can only be constrained by symmetry arguments. For example, the physics of low-energy quantum chromo dynamics (QCD) is captured by an effective theory of pions, whose low-energy interactions are fixed by the broken chiral symmetry.

Conformal field theories (CFTs) are especially beautiful examples of how one can leverage the symmetry group. While generically strongly coupled, conformal symmetry almost completely fixes the behavior of correlation functions and gives non-trivial insights into the structure of their Hilbert spaces. In some cases, the conformal bootstrap [126] can provide us with rich physics of such theories entirely based on symmetry principles. However, we are still lacking many concrete calculational tools for these theories. In CFTs with an additional global $U(1)$, recent progress has been made by constructing effective field theories for their large charge (Q) sector. Generically, the large charge sector can be horribly complicated in terms of elementary fields and their interactions, but one can set up a systematic $1/Q$ expansion to probe this strongly coupled regime. This has been useful in finding the scaling of operator dimensions, and many other meaningful physical quantities [127, 128, 129, 130, 131, 132].

In this work, we will be dealing with systems with non relativistic scale and conformal invariance i.e. systems invariant under Schrödinger symmetry. While in CFT, one needs to have a external global symmetry to talk about large charge expansion, the nonrelativistic conformal field theories (NRCFTs) come with a “natural” $U(1)$, the particle number symmetry. The Schödinger symmetry group and its physical consequences have been studied in [133, 134, **?**, 135, 136, 137]. The physical importance of Schrödinger symmetry lies in varied realisation of the symmetry group, starting from fermions at unitarity[138, 139] to examples including spin chain models [140], systems consisting of deuterons [141, 142], ^{133}Cs[143], ^{85}Rb [144],^{39}K [145].

Such theories, similar to CFTs, admit a state-operator correspondence[146, 135] in which the dimensions of operators correspond to energy of a state in a harmonic potential[1]. Specifically, the scaling generator D, which scales $\vec{x} \mapsto \lambda x$ and $t \mapsto \lambda^2 t$ for $\lambda \in \mathbf{R}$ gets mapped to

[1] This state-operator map is different from the one discussed in [137] to explore the neutral sector. In [137], the map is more akin to the $(0+1)$ dimensional CFT.

the Hamiltonian (H_ω) in the harmonic trap i.e. $H_\omega \equiv H + \omega^2 C$ where $C = \frac{1}{2}\int d^d x\, x^2 n(x)$ is the special conformal generator and $n(x)$ is the number density and H is the time translation generator of the Schrödinger group. The parameter ω determines the strength of the potential and plays an analogous role to the radius of the sphere in the relativistic state-operator correspondence.[2].

Given this set up, we consider an operator Φ with large number charge Q. For example, one can think of $\phi^{\frac{N}{2}}$ for $\phi(x) = :\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x):$ in the case of fermions at unitarity in $d = 3$ dimensions. By the state-operator correspondence, the operator is related to a state $|\Phi\rangle$ with finite density of charge (n) in the harmonic trap. There's an energy scale set by the density $\Lambda_{UV} \sim \mu \sim n^{\frac{2}{d}}$, μ being the chemical potential which fixes the total charge to Q. There is also a scale set by the trap $\Lambda_{IR} \sim \omega$ which controls the level spacing of H_ω. The limit of large charge $Q \gg 1$ then implies a parametric separation of these scales. This allows us to set up a perturbatively controlled expansion in $1/Q$ and probe the large charge sector of a theory invariant under Schrödinger symmetry.

In this limit it becomes appropriate to ask, what state of *matter* describes the large charge sector? Such a state with finite density of charge necessarily breaks some of the space-time symmetries e.g. scale transformations, (Galilean) boosts, special-conformal transformations. That these symmetries are spontaneously broken also implies that they must be realized non-linearly in the effective field theory (EFT) describing the large charge sector. We expect the low-energy degrees of freedom to be Goldstones.

One possibility is that the $U(1)$ symmetry remains unbroken. This is the case for a system with a Fermi surface. There the low-energy degrees of freedom would also include fermionic matter in addition to any Goldstones. The simplest candidate EFT, Landau Fermi-Liquid theory, is incompatible with the non-linearly realized Schrödinger symmetry[147] and therefore this is a fairly exotic possibility.

Another possibility is that the $U(1)$ symmetry is also spontaneously broken, leading to

[2]Here and also subsequently, we will be working in non-relativistic "natural" units of $m = \hbar = 1$

superfluid behavior. This has been the case most studied in the literature and seems like the most obvious possibility for a bosonic NRCFT. Additionally, both unitary fermions and the scale invariant anyon gas at large density are suspected to be superfluids. Therefore we focus exclusively on this symmetry breaking pattern.

Summary of Results

We compute the properties of the ground state $|\Phi\rangle$ with finite density of charge, under the assumption it describes a rotationally invariant superfluid, via an explicit path integral representation:

$$\langle\Phi|e^{-H_\omega T}|\Phi\rangle = \int \mathcal{D}\chi\, e^{-S_{eff}[\chi]+\mu\int d^dx\; n(x)} \tag{4.1}$$

where χ is a Goldstone boson describing excitations above the ground state, μ is the chemical potential and $n(x)$ is the number density which is canonically conjugate to χ. This integral can then be computed by saddle point in the large μ limit. The chemical potential μ can then be fixed semi-classically in terms of the charge Q. Thus self-consistently, we are obtaining a large Q expansion. We employ the coset construction to write down the most general effective action for the Goldstone which is consistent with the non-linearly realized Schrödinger symmetry.

- For the case with magnetic vector potential $\vec{A}=0$ (the one that is relevant for the NRCFT in harmonic trap), we find the effective Lagrangian given by

$$\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1 \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X \partial^i X + c_2 \frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 + c_3 \frac{X^{\frac{d}{2}+1}}{X^2}\partial_i\partial^i A_0 + c_4 \frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i\partial^i\chi)^2 \tag{4.2}$$

  where $X = \partial_t\chi - A_0 - \frac{1}{2}\partial_i\chi\partial^i\chi$. However this is not the full set of constraints. It can be shown that imposing 'general coordinate invariance' will reduce the number of independent Wilson coefficients even further[148]. In particular there are the additional constraints: $c_2 = 0$ and $c_3 = -d^2 c_4$. Additionally, in $d = 2$, one can have parity violating operator at this order:

$$c_5 \frac{1}{X} \varepsilon^{ij} (\partial_i A_0)(\partial_j X) \tag{4.3}$$

The details can be found in Section 4.5.

- The dispersion relation of low energy excitation above the ground state is found out to be:

$$\varepsilon(n,\ell) = \pm\omega \left(\frac{4}{d} n^2 + 4n + \frac{4}{d} \ell n - \frac{4}{d} n + \ell \right)^{\frac{1}{2}} \tag{4.4}$$

where ℓ is the angular momentum and n is a non-negative integer and $\varepsilon(n,\ell)$ is the excitation energy. The dispersion determines the low-lying operator dimensions explicitly. Since, $\varepsilon(n=0,\ell=1) = \pm\omega$ and $\varepsilon(n=1,\ell=0) = \pm 2\omega$, they can be identified with two different kinds of descendant operators appearing in the Schrödinger algebra. The details can be found in Section 4.7.2.

- In the leading order in Q, we find the ground state energy i.e. dimension Δ_Q of the corresponding operator Φ:

$$\Delta_Q = \left(\frac{d}{d+1} \right) \xi Q^{1+\frac{1}{d}}, \quad \text{where} \quad \frac{1}{c_0} = \frac{\Gamma(\frac{d}{2}+2)}{\Gamma(d+1)} (2\pi\xi^2)^{\frac{d}{2}}. \tag{4.5}$$

where c_0 is UV parameter of the theory, appearing in the Lagrangian (4.2).

Specifically, we have

$$\Delta_Q = \frac{2}{3} \left(\xi Q^{3/2} \right) + c_1 \frac{4\pi}{3} \xi \left(Q^{\frac{1}{2}} \log Q \right) + O\left(Q^{\frac{1}{2}} \right) \quad \text{for } d = 2. \tag{4.6}$$

$$\Delta_Q = \left(\frac{3}{4} \right) \xi Q^{4/3} - \left(c_1 + \frac{c_3}{2} \right) (3\sqrt{2}\pi^2) \xi^2 Q^{2/3} + O\left(Q^{5/9} \right) \quad \text{for } d = 3. \tag{4.7}$$

The details can be found in Section 4.7.1.

- We find the structure function F appearing in three point function of two operators with large charge Q and $Q+q$ and one operator ϕ_q with small charge q goes as follows:

$$F(v=\mathbf{i}\omega y^2) \propto Q^{\frac{\Delta_\phi}{2d}} \left(1 - \frac{\omega y^2}{2\xi} Q^{-1/d}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{4.8}$$

where y is the insertion point of ϕ_q in the oscillator co-ordinate and Δ_ϕ is the dimension ϕ_q. The details can be found in Section 4.8.2.

4.3 Lightning Review of Schrödinger Algebra

The Schrödinger algebra has been extensively explored in [133, 134, 146, 135, 136, 137]. Here we take the readers through a quick tour of the essential features of Schrödinger algebra, that we are going to use through out this paper. The most important subgroup of Schrödinger group is the Galilean group, generated by time translation generator H, spatial translation generators P_i, rotation generators J_{ij} and boost generators K_i. One can centrally extend this group by appending another $U(1)$ generator N, which generates the particle number symmetry. As a whole, these generators constitute what we call Galilean algebra and they satisfy:

$$\begin{gathered}
[J_{ij},N] = [P_i,N] = [K_i,N] = [H,N] = 0 \\
[J_{ij},P_k] = \mathbf{i}(\delta_{ik}P_j - \delta_{jk}P_i)\,, \\
[J_{ij},K_k] = \mathbf{i}(\delta_{ik}K_j - \delta_{jk}K_i)\,, \\
[J_{ij},J_{kl}] = \mathbf{i}(\delta_{ik}J_{jl} - \delta_{jk}J_{il} + \delta_{il}J_{kj} - \delta_{jl}J_{ki})\,, \\
[P_i,P_j] = [K_i,K_j] = 0\,, \qquad [K_i,P_j] = \mathbf{i}\delta_{ij}N\,, \\
[H,N] = [H,P_i] = [H,J_{ij}] = 0\,, \quad [H,K_i] = -\mathbf{i}P_i\,.
\end{gathered} \tag{4.9}$$

The Galilean group is enhanced to Schrödinger group by appending a scaling generator D

and a special conformal generator C such that they satisfy the following commutator relations:

$$[D,P_i] = \mathbf{i}P_i\,, \quad [D,K_i] = -\mathbf{i}K_i\,, \tag{4.10}$$

$$[D,H] = 2\mathbf{i}H\,, \quad [D,C] = -2\mathbf{i}C\,, [H,C] = -\mathbf{i}D\,, \tag{4.11}$$

$$[J_{ij},D] = 0\,, \quad [J_{ij},C] = 0\,, \quad [N,D] = [N,C] = 0\,. \tag{4.12}$$

The state-operator correspondence for an NRCFT is based on the following definition [135]:

$$|O\rangle \equiv e^{-\frac{H}{\omega}} O^\dagger(0)|0\rangle = O^\dagger\left(-\frac{\mathbf{i}}{\omega},0\right)|0\rangle \tag{4.13}$$

where $O^\dagger$ is a primary operator of number charge $Q_{O^\dagger} = -Q_O \geq 0$. By the Schrödinger algebra, this state satisfies:

$$N|O\rangle = Q_{O^\dagger}|O\rangle \qquad H_\omega|O\rangle = \omega\Delta_O|O\rangle \tag{4.14}$$

where $H_\omega = H + \omega^2 C$ is the Hamiltonian with the trapping potential.

It is natural to define a transformation from Galilean coordinates $x = (t,\vec{x})$ to the "oscillator frame" $y = (\tau,\vec{y})$ where the time translation $\tau \to \tau + a$ is generated by H_ω. Explicitly this is given by

$$\omega\tau = \arctan\omega t\,, \qquad \vec{y} = \frac{\vec{x}}{\sqrt{1+\omega^2 t^2}} \tag{4.15}$$

and allows us to map primary operators and their correlation functions in the oscillator frame to the Galilean frame via the map[135]:

$$\tilde{O}(y) = \left(1+\omega^2 t^2\right)^{\frac{\Delta_O}{2}} \exp\left[\frac{\mathbf{i}}{2} Q_O \frac{\omega^2|\vec{x}|^2 t}{1+\omega^2 t^2}\right] O(x) \tag{4.16}$$

$$O(x) = [\cos(\omega t)]^{\Delta_O} \exp\left[-\frac{\mathbf{i}}{2} Q_O \omega|\vec{y}|^2 \tan(\omega\tau)\right] \tilde{O}(y) \tag{4.17}$$

In this paper, we will be interested in matrix elements of the form:

$$\langle \Phi | \phi_1(y_1) \cdots \phi_n(y_n) | \Phi \rangle \tag{4.18}$$

where $\Phi^\dagger$ is a primary of charge $Q \gg 1$ and ϕ_i are also charged [3] primaries with $q_i \ll Q$.[4]

In the Galilean frame, the general form of a two point function is fixed to be

$$\langle O_1(x_1) O_2(x_2) \rangle = c\,\delta_{\Delta_1,\Delta_2} \delta_{Q_1,-Q_2} \frac{\exp\left[\mathbf{i} Q_2 \frac{|\vec{x}|^2}{2t}\right]}{(t_1 - t_2)^{\Delta_1}} \tag{4.19}$$

where c is a numerical constant, Δ_i is the dimension of the operator O_i, Q_i is the charge of O_i. The symmetry algebra constrains the general form of a three-point function upto a arbitrary function of a cross-ratio v_{ijk} defined below:

$$\begin{aligned} \langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle &\equiv G(x_1; x_2; x_3) \\ &= F(v_{123}) \exp\left[-\mathbf{i} \frac{Q_1}{2} \frac{\vec{x}_{13}^2}{t_{13}} - \mathbf{i} \frac{Q_2}{2} \frac{\vec{x}_{23}^2}{t_{23}} \right] \prod_{i<j} t_{ij}^{\frac{\Delta}{2} - \Delta_i - \Delta_j} \end{aligned} \tag{4.20}$$

where $\Delta \equiv \sum_i \Delta_i$, $x_{ij} \equiv x_i - x_j$, and $F(v_{ijk})$ is a function of the cross-ratio v_{ijk} defined:

$$v_{ijk} = \frac{1}{2} \left(\frac{\vec{x}_{jk}^2}{t_{jk}} - \frac{\vec{x}_{ik}^2}{t_{ik}} + \frac{\vec{x}_{ij}^2}{t_{ij}} \right) \tag{4.21}$$

We note that the three point function becomes zero unless $\sum Q_i = 0$.

[3] The state-operator correspondence breaks down for neutral operators as they actually trivially on the vacuum and their representation theory is not well understood. [137] explores how to circumvent this issue.

[4] Here we point out that if an operator is explicitly written as a function of oscillator co-ordinate, it is to be understood that we have already employed the mapping (4.16). Thus $\phi_i(y_1)$ in (4.18) should technically be written as $\tilde{\phi}_i(y_1)$, albeit we omit "tilde" sign for notational simplicity.

4.4 Lightning Review of Coset Construction

A symmetry is said to be spontaneously broken if the lowest energy state, the ground state, is not an eigenstate of the associated charge. The low-energy effective action, describing the physics above the ground state, is still invariant under the full global symmetry group but the broken subgroup is realized *non-linearly*. Typically this means the effective action describes some number of Goldstones.

The coset construction gives a general method for constructing effective actions with appropriate non-linearly realized symmetry actions. It was developed for internal symmetries by CCZW [149, 150] and later generalized to space-time symmetries[151]. Here we give a nimble review of the method and its application to the superfluid. We refer to the original literature and the recent review [152] for more details. The primary objective of the coset construction is to write down the most general action, invariant under a global symmetry group G but where only the subgroup G_0 is linearly realized. Let us consider a symmetry group which contains the group of translations, generated by P_a. Let us denote the broken generators as X_b corresponding to associated Goldstones $\pi_b(x)$. We denote unbroken generators as T_c.

We can define the exponential map from space-time to the coset space G/G_0

$$U \equiv e^{\mathbf{i}\bar{P}_a x^a} e^{\mathbf{i}X_b \pi^b(x)} \tag{4.22}$$

With this map we can define the 1-form, known as the Maurer-Cartan (henceforth we call it MC) form, on the coset space. Under a G-transformation (4.22) transforms as

$$g : U(x) \to e^{\mathbf{i}\bar{P}_a (x')^a} e^{\mathbf{i}X_b \pi'^b(x')} h(\pi(x), g) \tag{4.23}$$

where $h(\pi(x), g)$ is some element in G_0, determined by the Goldstones and $g \in G$, that "compensates" to bring $U(x)$ back to the form in (4.22). This determines how the Goldstone fields transform[5].

Expanded in a basis of generators the MC form looks like:

$$\Omega \equiv -\mathbf{i}U^{-1}\partial_\mu U \equiv E^a_\mu(\bar{P}_a + (\nabla_a\pi^b)X_b + A^c_a T_c) \tag{4.24}$$

where each of the tensors $\{E^a_\mu,\ \nabla_a\pi^b,\ T_c\}$ is a function of the Goldstone fields π_a. Here E^a_μ is a vierbein, $\nabla_a\pi^b$ are the covariant Goldstone derivatives and A^c_a transforms like a connection.

Several remarks are in order. Once space-time symmetries are broken the quantity $d^d x$ is no longer necessarily a scalar under those transformations. However the quantity $d^d x \det E$ can be used to define an invariant measure for the action. On the other hand, contractions of the objects $\nabla_a\pi^b$, in a way which manifestly preserves the G_0 symmetry, also provides us with G invariants and form the Goldstone part of the effective action. The connection, A^c_a and the vierbein, can be used to define the following "higher" covariant derivative

$$\nabla^H_a \equiv (E^{-1})^\mu_a\partial_\mu + \mathbf{i}A^c_a T_c \tag{4.25}$$

An object like $\nabla^H_a\nabla_b\pi^c$ also transforms covariantly and G_0-invariant contractions with other tensors should be included. The other primary use of (4.25) is for defining covariant derivatives of "matter fields". For example, suppose ψ is a matter field transforming in a k-dimensional linear representation r of G_0 as $\psi \to \psi' = r(h)\psi$. The coset construction provides multiple ways to uplift G_0 representations to full G representations. The one of importance to us is when r appears in the decomposition of a K-dimensional representation R of G. Defining the field $\tilde{\psi} \equiv (\psi,\ 0)$ in the K-dimensional representation, one can show that the field $\Psi = R(\Omega)\tilde{\psi}$ transforms linearly under the full group G. If a subset of the symmetry is gauged then we just covariantly replace $\partial_\mu \to D_\mu = \partial_\mu + \mathbf{i}\bar{A}^d_\mu\bar{T}_d$ in the above. The tensors will then depend on the gauge fields $\bar{A}$ but

[5] For space-time symmetries there's a translation piece even though $\bar{P}_a$ are unbroken. This is because, on coordinates, translations are always non-linearly realized as $x \to (x+a)$

otherwise everything goes through.

One last important aspect of space-time symmetry breaking is that not all the Goldstone bosons are necessarily independent [153]. This occurs when the associated currents differ only by functions of spacetime. A localized Goldstone particle is made by a current times a function of spacetime, so we can not sharply distinguish the resulting particles. This redundancy also appears in the coset construction. Suppose X and X' are two different broken generators in different G_0-multiplets and we denote their associated Goldstone bosons π and π'. Let $\bar{P}_\nu$ be an unbroken translation generator. Let us also assume that there's a non-trivial commutator of the form $[P_\nu, X] \supseteq X'$. One can see, from calculating the Maurer-Cartan form via the BCH identity, that this implies an undifferentiated π in the covariant Goldstone derivative $\nabla_\nu \pi'$. The quadratic term is then $(\nabla_\nu \pi')^2 \sim c^2 \pi^2$; this is an effective mass term for the π Goldstone. Thus we are justified in integrating it out by imposing its equation of motion. A simpler, but equivalent up to redefinitions, constraint is setting $\nabla_\nu \pi' = 0$. This is a covariant constraint, completely consistent with the symmetries. In the literature it is known as an "inverse Higgs constraint" .

4.5 Schrödinger Superfluid from Coset Construction

In this section, we will use the coset construction to construct the most general Goldstone action consistent with the broken symmetries of a rotationally invariant Schrödinger superfluid. For the purpose of determining local properties of the superfluid state in the trap we can first work in the thermodynamic limit defined by $\Lambda_{IR} \sim \omega \to 0$. The symmetry generators are then just those of the usual Schrödinger group.

The superfluid ground state $|\Phi\rangle$ spontaneously breaks the number charge N. As mentioned in the introduction, this state also breaks the conformal generators and boosts. It is simplest to describe such states in the grand canonical ensemble. We remark that in the thermodynamic limit, one can leverage the equivalence between canonical ensemble with fixed chrage and grand canonical ensemble[6]. Thus, in what follows, we define the operator $\bar{H} = H - \mu N$ such that $\bar{H}|\Phi\rangle = 0$. The parameter μ plays the role of a chemical potential; it is a Lagrange multiplier to be determined by the charge density. By assumption, $|\Phi\rangle$ is not an eigenstate of N. It therefore cannot be an eigenstate of H while satisfying $\bar{H}|\Phi\rangle = 0$. The unbroken 'time' translations are therefore generated by $\bar{H}$[154]. The symmetry breaking pattern is then given by:

$$\text{Unbroken: } \{\bar{H} \equiv H - \mu N, P_i, J_{ij}\} \quad \text{Broken: } \{N, K_i, C, D\}, \tag{4.26}$$

for which we can parameterize the coset space as:

$$U = e^{\mathbf{i}\bar{H}t} e^{-\mathbf{i}\vec{P}\cdot\vec{x}} e^{\mathbf{i}\vec{\eta}\cdot\vec{K}} e^{-\mathbf{i}\lambda C} e^{-\mathbf{i}\sigma D} e^{\mathbf{i}\pi N} = e^{\mathbf{i}Ht} e^{-\mathbf{i}\vec{P}\cdot\vec{x}} e^{\mathbf{i}\vec{\eta}\cdot\vec{K}} e^{-\mathbf{i}\lambda C} e^{-\mathbf{i}\sigma D} e^{\mathbf{i}\chi N}. \tag{4.27}$$

Here we use 4 distinct Goldstone fields:

- π is the 'phonon', the Goldstone for the charge. It defines the shifted field $\chi \equiv \pi + \mu t$
- $\vec{\eta}$ is the 'framon', the Goldstone for (Galilean) boosts. It transforms as a vector.
- λ is the 'trapon', the Goldstone for special conformal transformations.
- σ is the 'dilaton', the Goldstone for dilations.

To allow for a background field A_μ, we define the covariant derivative $D_\mu = \partial_\mu + \mathbf{i}A_\mu N$. From this group element we can calculate the MC form:

$$-\mathbf{i}U^{-1}D_\mu U \equiv E_\mu^\nu[\bar{P}_\nu + (\nabla_\nu\eta^i)K_i - (\nabla_\nu\lambda)C - (\nabla_\nu\sigma)D + (\nabla_\nu\pi)Q] \tag{4.28}$$

where $\bar{P}_\mu \equiv (-\bar{H}, \vec{P})$, and we've anticipated the absence of a gauge field for J_{ij}. We remark that the relativistic notation is just for ease of writing; because space and time are treated differently

[6] As a result, one can always view the large charge expansion as a large chemical potential expansion

we have to treat those components of the MC form separately. Explicitly we have the following:

$$E_0^0 = e^{-2\sigma}\,, \quad E_0^i = -\eta^i e^{-\sigma}\,, \quad E_i^0 = 0\,, \quad E_i^j = \delta_i^j e^{-\sigma}\,, \tag{4.29}$$

$$\nabla_0 \eta^j = e^{3\sigma}(\dot{\eta}^j + \vec{\eta}\cdot\vec{\partial}\eta^j)\,, \quad \nabla_i \eta^j = e^{2\sigma}(\partial_i \eta^j - \lambda \delta_i^j)\,, \tag{4.30}$$

$$\nabla_0 \lambda = e^{4\sigma}(\dot{\lambda} + \vec{\eta}\cdot\vec{\partial}\lambda + \lambda^2)\,, \quad \nabla_i \lambda = e^{3\sigma}\partial_i \lambda\,, \tag{4.31}$$

$$\nabla_0 \sigma = e^{2\sigma}(\dot{\sigma} + \vec{\eta}\cdot\vec{\partial}\sigma - \lambda)\,, \quad \nabla_i \sigma = e^{\sigma}\partial_i \sigma\,, \tag{4.32}$$

$$\nabla_0 \pi = e^{2\sigma}(\dot{\chi} - A_0 - \mu e^{-2\sigma} + \vec{\eta}\cdot\vec{\partial}\chi + \frac{1}{2}\eta^2)\,, \quad \nabla_i \pi = e^{\sigma}(\partial_i \chi - A_i + \eta_i)\,, \tag{4.33}$$

which can be used to construct the effective action.

There are 4 commutators that each imply a different constraint

$$[P_i, K_j] = -\mathbf{i}\delta_{ij}N \implies \nabla_i \pi = 0\,, \quad [\bar{H}, D] = -2\mathbf{i}(\bar{H} + \mu N) \implies \nabla_0 \pi = 0\,, \tag{4.34}$$

$$[\bar{H}, C] = -\mathbf{i}D \implies \nabla_0 \sigma = 0\,, \quad [P_i, C] = -\mathbf{i}K_j \delta_{ij} \implies \nabla_i \eta^j = 0\,. \tag{4.35}$$

Imposing them allows everything to be written in terms of a single Goldstone field χ. Upon defining the gauge invariant derivatives:

$$D_t \chi \equiv \partial_t \chi - A_0\,, \quad D_i \chi \equiv \partial_i \chi - A_i\,, \tag{4.36}$$

the simplest pair can be solved as:

$$\nabla_i \pi = 0 \quad \implies \quad \eta_i = -D_i \chi\,, \tag{4.37}$$

$$\nabla_0 \pi = 0 \quad \implies \quad \mu e^{-2\sigma} = D_t \chi - \frac{1}{2} D_i \chi D^i \chi\,. \tag{4.38}$$

The other two involve the trapon λ:

$$\nabla_i \eta^j = 0 \quad \Longrightarrow \quad \lambda \delta_i^j = \partial_i \eta^j = -\partial_i D^j \chi \,, \tag{4.39}$$

$$\nabla_0 \sigma = 0 \quad \Longrightarrow \quad \lambda = \dot{\sigma} + \vec{\eta} \cdot \vec{\partial} \sigma \,, \tag{4.40}$$

which can be written together as:

$$\dot{\sigma} + \vec{\eta} \cdot \vec{\partial} \sigma - \frac{1}{d} \vec{\partial} \cdot \vec{\eta} = -\frac{1}{2} \frac{\partial_0 X}{X} + \frac{1}{2} \frac{D_i \chi \partial^i X}{X} + \frac{1}{d} \partial_i D^i \chi = 0 \,. \tag{4.41}$$

This is simply the leading order equation of motion for χ as we will show below.

The leading order action comes from the vierbein (4.29) which can be expressed with χ as

$$\det E = e^{-(d+2)\sigma} \propto \left(D_t \chi - \frac{1}{2} D_i \chi D^i \chi \right)^{\frac{d}{2}+1} \,. \tag{4.42}$$

Defining the variable X as

$$X = D_t \chi - \frac{1}{2} D_i \chi D^i \chi \,, \tag{4.43}$$

we can write the leading order effective action as

$$S_0 = \int dt d^d x \, c_0 \; O_0 = \int dt d^d x \, c_0 X^{\frac{d}{2}+1} \,, \tag{4.44}$$

where c_0 is a dimensionless constant. The leading order theory (4.44) is time reversal invariant as it acts as:

$$T : \quad t \to -t \,, \quad \pi \to -\pi \,, \quad A_0 \to -A_0 \,. \tag{4.45}$$

Higher derivative terms are constructable from contractions of the following objects:

$$\nabla_0 \eta^i \,, \quad \nabla_0 \lambda \,, \quad \nabla_i \lambda \,, \quad \nabla_i \sigma \,. \tag{4.46}$$

as well as contractions of the 'higher covariants'

$$\nabla_0^H = -e^{2\sigma}\partial_0 + e^{\sigma}\eta^i\partial_i\,, \qquad \nabla_i^H = e^{\sigma}\partial_i\,, \tag{4.47}$$

acting on the tensors (4.46). All of these objects can be expressed in terms of χ by the constraints (4.34) and (4.35). Even though we are interested in large Q expansion eventually, to touch the base with the EFT written in [148], we emphasize that the power counting is done with X, being taken to be $O(p^0)$, which implies that objects like $[(\partial_i\chi)(\partial_i\chi)]^k$, $\partial_t\chi$ and A_0 are also order one. Additional derivatives then increase the dimension. In what follows, the field strengthsE_i and F_{ij} are defined as

$$E_i \equiv \partial_0 A_i - \partial_i A_0 \quad F_{ij} \equiv \partial_i A_j - \partial_j A_i\,. \tag{4.48}$$

At $O(p^2)$ we have following operators:

$$O_1 \equiv \det E\ \nabla_i\sigma\nabla^i\sigma \ \propto\ \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X\partial^i X\,, \tag{4.49}$$

$$O_2 \equiv \det E\ (\nabla_0\eta_i - 2\nabla_i\sigma)^2 \ \propto\ \frac{X^{\frac{d}{2}+1}}{X^3}[E^2 + 2E_iF_{ij}(D_j\chi) + F_{ij}F_{ik}(D_j\chi)(D_k\chi)]\,, \tag{4.50}$$

$$O_3 \equiv \det E\ \nabla_i\sigma(\nabla_0\eta^i - 2\nabla^i\sigma) \ \propto\ \frac{X^{\frac{d}{2}+1}}{X^2}[\partial_i E^i + [\partial_i F_{ij}](D_j\chi) - \frac{1}{2}F_{ij}F^{ij}]\,, \tag{4.51}$$

$$O_4 \equiv \det E\ \nabla_0\lambda \ \propto\ \frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i D^i\chi)^2\,, \tag{4.52}$$

where the second expression of (4.51) is obtained via integration-by-parts and the (4.52) is obtained by a straight forward application of the identity (4.41) and integration-by-parts. These operators were found in reference[148] for $d = 3$ by very different means. Additionally, in $d = 2$, one can construct following parity violating operators at this order:

$$O_5 \equiv \det E\ \varepsilon^{ij}(\nabla_0\eta_i)(\nabla_j\sigma) \propto \frac{X^{\frac{d}{2}+1}}{X^3}\varepsilon^{ij}\left[E_i - F_{jk}(D_k\chi)\right](\partial_j X)\,, \tag{4.53}$$

$$O_6 \equiv \det E\, \varepsilon^{ij}\nabla_i^H(\nabla_0\eta_j - 2\nabla_j\sigma) \propto \frac{X^{\frac{d}{2}+1}}{X^2}\varepsilon^{ij}\partial_i(E_j - F_{jk}(D_k\chi))\,. \tag{4.54}$$

Similarly in $d = 3$ we have ε^{ijk} but that means the parity violating operators will be higher order in the derivative expansion.

4.6 Superfluid Hydrodynamics

In this section, we study the superfluid hydrodynamics. As a warm up, we first consider the fluid without the trap, thus there is no intrinsic length scale associated with such a system. The leading order superfluid Lagrangian is known to take the form [148]:

$$\mathcal{L} = P(X) \tag{4.55}$$

where P stands for 'pressure' as function of the chemical potential μ at zero temperature and X is the same as defined in the previous section. Due to the absence of any internal scale, dimensional analysis dictates that:

$$P = c_0\mu^{\frac{d}{2}+1}\,, \tag{4.56}$$

which we get from (4.44) by evaluating on the groundstate solution $\chi_{cl} = \mu t$. The number density is conjugate to the Goldstone field χ and at leading order is:

$$n \equiv \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = P'(X) = c_0\left(\frac{d}{2}+1\right)X^{\frac{d}{2}}\,. \tag{4.57}$$

One can then define the superfluid velocity in terms of the Goldstone as:

$$v_i \equiv -D_i\pi = -D_i\chi = \eta_i \tag{4.58}$$

where we have used the inverse Higgs constraint (4.37). This gives a simple interpretation of the equation of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} = \partial_t n + \partial_i (n v^i) = 0\,, \tag{4.59}$$

which is the continuity equation of superfluid hydrodynamics. Using equations (4.37), we can write:

$$\partial_\mu n = c_0 \frac{d}{2}\left(\frac{d}{2}+1\right) X^{\frac{d}{2}-1}(\partial_\mu X) = -dn(\partial_\mu \sigma) \qquad \partial_i v^i = -\partial_i D^i \chi = \vec{\partial}\cdot\vec{\eta} \tag{4.60}$$

The equation of motion (4.59) thus comes out to be as follows:

$$\partial_t n + \partial_i (n v^i) = -dn\dot{\sigma} - dn(\vec{\eta}\cdot\vec{\partial}\sigma) + n\vec{\partial}\cdot\vec{\eta} = 0 \tag{4.61}$$

and becomes equivalent to the constraint (4.41). Thus the superfluid EFT is consistent with the symmetry breaking pattern we discussed in the previous section.

4.6.1 Superfluid in a Harmonic Trap

Now we turn on the harmonic trap and study this superfluid EFT in the trapping potential by taking:

$$A_0 = \frac{1}{2}\omega^2 r^2\,, \qquad \vec{A} = 0\,. \tag{4.62}$$

In the presence of a harmonic potential, the ground state density is no longer uniform. The number density is given by the conjugacy relation (4.57) and to leading order is:

$$n(x) = c_0\left(\frac{d}{2}+1\right)(\mu - \frac{1}{2}\omega^2 r^2)^{\frac{d}{2}}\,, \tag{4.63}$$

which is vanishing at the "cloud radius" $R = \sqrt{\frac{2\mu}{\omega^2}}$. This defines an IR cutoff for the validity of our EFT in the trap. Semi-classically, we can fix μ in terms of the number charge Q by imposing[7]:

$$Q = \langle Q|\hat{N}|Q\rangle = \int d^d x \langle Q|n(x)|Q\rangle = \frac{c_0 (2\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)\left(\frac{\mu}{\omega}\right)^d}{\Gamma(d+1)} \implies \frac{\mu}{\omega} \equiv \xi Q^{\frac{1}{d}} \tag{4.64}$$

The naive effective Lagrangian up to next-leading order is then:

$$\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1 \frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X \partial^i X + c_2 \frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 + c_3 \frac{X^{\frac{d}{2}+1}}{X^2}\partial_i \partial^i A_0 + c_4 \frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i \partial^i \chi)^2 \tag{4.65}$$

For $d = 2$ we have an additional parity violating operator at this order:

$$\mathcal{L}_{eff} \ni c_5 \varepsilon^{ij} \frac{(\partial_i A_0)(\partial_j X)}{X} \tag{4.66}$$

However, this is not the full set of constraints. It can be shown that imposing 'general coordinate invariance' will reduce the number of independent Wilson coefficients even further[148]. In particular there are the additional constraints:

$$c_2 = 0 \quad c_3 = -d^2 c_4 \tag{4.67}$$

Obtaining these from the coset construction would require additionally gauging the space-time symmetries [155]. The requirement of gauging the space-time symmetries is expected as a consequence of the number operator being part of the spacetime symmetry algebra and the fact that the number symmetry has been gauged. We leave this refinement for future work. For reasons that will become clear in the next section it is not necessary to work beyond this order in the derivative expansion.

[7]This is equivalent to fixing Q by differentiating the free energy given by the action

4.7 Operator Dimensions

4.7.1 Ground State Energy & Scaling of Operator Dimension

The ground state energy is readily computed by a Euclidean path integral, in the infinite Euclidean time separation, the path integral projects out the ground state, from which one can read off the ground state energy. A nice pedagogical example of this technique can be found in [129] in context of fast spinning rigid rotor. On the other hand, from the state operator correspondence, we know that the ground state energy translated to dimension of the corresponding operator. Thus, equipped with the effective Lagrangian (4.65) obtained, the operator dimensions can be calculated via the path integral (4.1):

$$\lim_{T\to\infty} \langle Q|e^{-H_\omega T}|Q\rangle \sim e^{-S_{eff}[\chi_{cl}]-\mu\int d^Dx\, n(x)} \sim e^{-\Delta_Q \omega T}\,, \tag{4.68}$$

where to leading order we have

$$-S_{eff}[\chi_{cl}] = c_0\Omega_d T\int_0^R dr\, r^{d-1}\left(\mu-\frac{1}{2}\omega^2 r^2\right)^{\frac{d}{2}+1} = c_0\frac{(2\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}{\Gamma(d+2)}\left(\frac{\mu}{\omega}\right)^{d+1}\omega T\,. \tag{4.69}$$

Here, Ω_d is the volume factor. Combining the results of (4.69) and (4.64) then gives the leading order operator dimension:

$$\Delta_Q = \frac{\mu}{\omega}Q - \left(-\frac{S_{eff}}{\omega T}\right) = \frac{d}{d+1}\xi Q^{1+\frac{1}{d}}\,. \tag{4.70}$$

This predicts $\Delta_Q \sim Q^{\frac{3}{2}}$ in $d=2$ and $\Delta_Q \sim Q^{\frac{4}{3}}$ in $d=3$, as in the relativistic case. That these leading order results are finite implies we can trust the EFT prediction. In general, however, the ground state energy in the trap is an infrared (IR) sensitive quantity. This becomes apparent at higher orders in the derivative expansion.

For example, we consider the case of $d=2$. The simplest operator at next leading order is (4.49). To analyze its contribution, define the distance from the cloud s as $r = R - s$. Its contribution to the energy, and hence the operator dimension via (4.69), would go like:

$$\int \mathrm{d}^3x \, \frac{\partial_i X \partial^i X}{X} \sim \int_0^R \mathrm{d}r \, r \frac{\omega^4 r^2}{\mu - \frac{1}{2}\omega^2 r^2} \sim \mu \int \mathrm{d}s \, \frac{1}{s} \,, \tag{4.71}$$

which is log divergent for small s, close to the edge. For $d = 3$, noticed in reference [148], a divergence first appears at next-next leading order associated with the operator:

$$\det E \, (\nabla_i \sigma \nabla^i \sigma)^2 \propto \frac{(\partial_i X \partial^i X)^2}{X^{\frac{7}{2}}} \,. \tag{4.72}$$

This leads to a power-law divergence, implying an even greater sensitivity to IR physics compared to $d = 2$. Ultimately these divergences originate from the breakdown of our EFT as the superfluid gets less dense. This occurs in a small region before the edge of the cloud at radius $R^* \equiv R - \delta$ where δ is roughly the width of this region. Following [148], we can estimate the size of this region as follows. One interpretation of (4.63) is that the chemical potential is now effectively space dependent. At the cutoff radius R^*, there is then an "effective chemical potential"

$$\mu(r) \equiv \mu - \frac{1}{2}\omega^2 r^2 \,, \quad \mu_{eff} \equiv \mu(r = R^*) = \frac{1}{2}\delta(2R - \delta)\omega^2 \approx R\omega^2\delta \,. \tag{4.73}$$

There is a length scale set by μ_{eff} which controls the EFT expansion parameter in this region. Once that length is comparable to the distance δ itself we cannot claim to control the calculation semi-classically. Using (4.73) this gives the estimate scaling:

$$\delta \sim \sqrt{\frac{1}{\mu_{eff}}} \implies \delta \sim \frac{1}{(\omega^2 \mu)^{\frac{1}{6}}} \tag{4.74}$$

We can estimate the contribution of this region to the energy by cutting off the divergent integrals at R^*. For $d=2$ the effective action contains a term:

$$-S_{eff} \ni c_1(2\pi)T\int_0^{R^*} dr\, r\frac{\omega^4 r^2}{\mu-\frac{1}{2}\omega^2 r^2} = 4\pi T\mu c_1\left(\frac{13}{8}-\log\left[\frac{2\mu}{\mu_{eff}}\right]\right)+\cdots \tag{4.75}$$

where the $\cdots$ terms vanish as $\delta\to 0$

Substituting the relations (4.64) and (4.74) gives:

$$\Delta_Q \ni -4\pi\xi Q^{\frac{1}{2}}c_1\left(\frac{13}{8}-\frac{1}{2}\log 2-\frac{1}{3}\log Q-\frac{2}{3}\log\xi\right) \tag{4.76}$$

Changing the cutoff relation (4.74) by a factor can then change the $O(Q^{\frac{1}{2}})$ contribution, but not the logarithmic divergence which is universal. This translates to an uncertainty of order $O(Q^{\frac{1}{2}})$ in the operator dimension in $d=2$. A similar analysis[148] for $d=3$ and (4.72) translates to uncertainty of order $O(Q^{\frac{5}{9}})$.

Unlike $d=2$, the operator (4.49) gives a finite correction to leading order scaling of dimension of operator in $d=3$. This can be found by figuring out the contibution to S_{eff} [see Eq. (4.65)]

$$-S_{eff} \ni c_1\int \mathrm{d}\tau^E \int_0^R \mathrm{d}r\, 4\pi r^2\left(\frac{\omega^4 r^2}{\sqrt{\mu-\frac{1}{2}\omega^2 r^2}}\right) = c_1(3\sqrt{2}\pi^2)\left(\frac{\mu}{\omega}\right)^2\omega T \tag{4.77}$$

Similar contribution[8] comes from (4.51):

$$-S_{eff} \ni c_3\int \mathrm{d}\tau^E \int_0^R \mathrm{d}r\, 4\pi r^2(\omega^2)\left(\mu-\frac{1}{2}\omega^2 r^2\right)^{\frac{1}{2}} = c_3\left(\frac{3\pi^2}{\sqrt{2}}\right)\left(\frac{\mu}{\omega}\right)^2\omega T \tag{4.78}$$

[8]Contribution should have come from (4.50) as well, but as we mentioned earlier, $c_2=0$ [148].

To summarize, using (4.70), we have

$$\Delta_Q = \frac{3}{4}\left(\xi Q^{4/3}\right) - \left(c_1 + \frac{c_3}{2}\right)(3\sqrt{2}\pi^2)\xi^2 Q^{2/3} + O(Q^{\frac{5}{9}}) \quad \text{for } d = 3\,, \tag{4.79}$$

$$\Delta_Q = \frac{2}{3}\left(\xi Q^{3/2}\right) + c_1\frac{4\pi}{3}\xi\left(Q^{\frac{1}{2}}\log Q\right) + O\left(Q^{\frac{1}{2}}\right) \quad \text{for } d = 2\,. \tag{4.80}$$

The Eq. (4.70), (4.79) and (4.80) constitute the main findings of this subsection.

4.7.2 Excited State Spectrum

We can also analyze the low energy excitations above the ground state. These correspond to low lying operators in the spectrum at large charge. To compute their dimension, we expand the leading action (4.44) to quadratic order in fluctuations π about the semi-classical saddle, $\chi = \mu t + \pi$. The spectrum of π can then be found by linearizing the equation of motion (4.59):

$$\ddot{\pi} - \frac{2}{d}\left(\mu - \frac{1}{2}\omega^2 r^2\right)\partial^2\pi + \omega^2 \vec{r}\cdot\vec{\partial}\pi = 0 \tag{4.81}$$

Expanding $\pi(t,x) = e^{\mathbf{i}\varepsilon t} f(r) Y_\ell$ where Y_ℓ is a spherical harmonic, one can show (4.81) reduces to a hypergeometric equation. Details can be found in Appendix A. The dispersion relation is given by:

$$\varepsilon(n,\ell) = \pm\omega\left(\frac{4}{d}n^2 + 4n + \frac{4}{d}\ell n - \frac{4}{d}n + \ell\right)^{\frac{1}{2}} \tag{4.82}$$

where ℓ is the angular momentum and n is a non-negative integer. In the NRCFT state-operator correspondence, there are two different operators which generate descendants. In the Galilean frame, these are the operators $\vec{P}$ and H. While $\vec{P}$ raises the dimension by 1 and carries angular momentum, acting by H raises the dimension by 2 and carries no angular momentum. In the oscillator frame, this corresponds to:

$$\vec{P}_\pm = \frac{1}{\sqrt{2\omega}}\vec{P} \pm \mathbf{i}\sqrt{\frac{\omega}{2}}\vec{K} \qquad L_\pm = \frac{1}{2}\left(\frac{1}{\omega}H - \omega C \pm \mathbf{i}D\right) \tag{4.83}$$

which then satisfy

$$[H_\omega, \vec{P}_\pm] = \pm\omega\vec{P}_\pm \quad [H_\omega, L_\pm] = \pm 2\omega L_\pm \tag{4.84}$$

One can check by equation (4.82) that $\varepsilon(n=0,\ell=1) = \pm\omega$ and $\varepsilon(n=1,\ell=0) = \pm 2\omega$. This allows us to identify these Goldstone modes with the descendant operators in (4.83) as $\pi_{(n=0,\ell=1)} \sim P_\pm$ and $\pi_{(n=1,\ell=0)} \sim L_\pm$. The other modes generate distinct primaries and descendants, including higher spin. We remark that in a strict sense, the above is the leading order result for the difference in dimensions between low-lying operators in this sector and the dimension of the ground state found in the previous section. It is also subject to corrections suppressed in $1/Q$ from subleading operators and loop effects.

4.8 Correlation Functions

In a relativistic CFT, the form of two and three point correlators is entirely fixed by symmetry. However, the four-point function depends on two conformally invariant cross ratios of the coordinates. The Schrödinger symmetry is less constraining, as there exists an invariant cross ratio even for a three-point function. This implies only the two-point functions of (number) charged operators is completely determined by symmetry.

4.8.1 Two Point Function

Following [129], we start with analyzing two point function. In path integral approach, when the in and out states are well separated in time, we have

$$\langle \Phi_Q, \tau_2 | e^{-H_\omega(\tau_2^{(E)} - \tau_1^{(E)})} | \Phi_Q, \tau_1 \rangle = e^{-\Delta_O(\tau_2^{(E)} - \tau_1^{(E)})} \tag{4.85}$$

where $\tau^{(E)}$ is the Euclideanized oscillator time. This is obtained from τ by doing Wick rotation i.e. $\tau^{(E)} = \mathbf{i}\tau$. This is evidently consistent with (4.108) upon doing the Wick rotation and taking

$(\tau_2^{(E)} - \tau_1^{(E)}) \to \infty$. One subtle remark is in order: the Hamiltonian H_ω generates the time (τ) translation in oscillator frame. Thus the states prepared by path integration corresponds to operators in oscillator frame.

4.8.2 Three Point Function

We consider the matrix element that defines the simplest charged[9] three-point function

$$\langle \Phi_{Q+q} | \phi_q(y) | \Phi_Q \rangle \tag{4.86}$$

where ϕ_q is a light charged scalar primary with charge q and both of Φ_Q and Φ_{Q+q} has $O(1)$ dimension, given by Δ_Q and Δ_{Q+q}. By assumption, ϕ_q transforms in a linear representation R of the unbroken rotation group. To enable calculation in our EFT, we can extend this to a linear representation of the full Schroedinger group using the Goldstone fields. In what follows, we take ϕ_q as the "dressed" operator[129]:

$$\phi_q(y) = R\left[e^{\mathbf{i}\vec{K}\cdot\vec{\eta}} e^{-\mathbf{i}\lambda C} e^{-\mathbf{i}\sigma D} e^{\mathbf{i}\chi N}\right] \hat{\phi}_q \tag{4.87}$$

where, by the assumption of ϕ_q being a scalar primary, is trivially acted on by $\vec{K}$ and C. This, combined with (4.37) gives

$$\phi_q = c_q X^{\frac{\Delta_\phi}{2}} e^{\mathbf{i}\chi q} \tag{4.88}$$

[9]The additional charge of $\langle \Phi |$ is required for the correlator to be overall neutral and therefore non-vanishing.

where c_q is a constant, which depends on UV physics. Upon evaluating (4.86) semi-classically about the saddle we found before, the leading order result for the correlator comes out to be:

$$\begin{aligned}\langle \Phi_{Q+q}(\tau_2)|\phi_q(\tau,\vec{y})|\Phi_Q(\tau_1)\rangle &= c_q\left(\mu-\frac{1}{2}m\omega^2y^2\right)^{\frac{\Delta_\phi}{2}} e^{\mathbf{i}\mu q(\tau-\tau_2)}e^{-\mathbf{i}\Delta_Q(\tau_2-\tau_1)}\\ &= c_q\mu^{\frac{\Delta_\phi}{2}}\left(1-\frac{y^2}{R^2}\right)^{\frac{\Delta_\phi}{2}} e^{\mu q\tau^{(E)}}e^{\omega\left(-\Delta_{Q+q}\tau_2^{(E)}+\Delta_Q\tau_1^{(E)}\right)} \qquad (4.89)\end{aligned}$$

where we have used the following identity, which can be derived using the leading order operator dimension (4.70) and (4.64):

$$\frac{\Delta_{Q+q}-\Delta_Q}{q} = \alpha_0\left(1+\frac{1}{d}\right)Q^{\frac{1}{d}} + O\left(\frac{1}{Q}\right) \approx \frac{\partial\Delta_Q}{\partial Q} = \frac{\mu}{\omega} \qquad (4.90)$$

as expected since μ is a chemical potential and $\omega\Delta_Q$ is the energy. We note that the operator insertion should be away from the edge of the cloud $|y-R|\gg\delta$, where δ is the cut-off imposed to keep the divergences coming from the $y\to R$ limit at bay.

Now we use (the details can be found in appendix [4.11.1])

$$\begin{aligned}\lim_{\tau_2^{(E)}\to\infty}\frac{1}{(1+\omega^2t_2^2)^{\Delta_{Q+q}/2}}\exp\left(-\omega\Delta_{Q+q}\tau_2^{(E)}\right) &= 2^{-\Delta_{Q+q}}\omega^{\Delta_{Q+q}/2},\\ \lim_{\tau_1^{(E)}\to-\infty}\frac{1}{(1+\omega^2t_1^2)^{\Delta_Q/2}}\exp\left(\omega\Delta_Q\tau_1^{(E)}\right) &= 2^{-\Delta_Q}\omega^{\Delta_Q/2},\end{aligned}$$

to write down the correlator in terms of operators in Galilean frame (we repeat that the path intergral in oscillator frame prepares a state corresponding to operator in oscillator frame):

$$\langle \Phi_{Q+q}(\mathbf{i}/\omega)|\phi_q(\tau,\vec{y})|\Phi_Q(-\mathbf{i}/\omega)\rangle = c_q\mu^{\frac{\Delta_\phi}{2}}\left(1-\frac{y^2}{R^2}\right)^{\frac{\Delta_\phi}{2}} e^{\mu q\tau^{(E)}}2^{-\Delta_Q-\Delta_{Q+q}}\omega^{(\Delta_Q+\Delta_{Q+q})/2}. \quad (4.91)$$

This can be matched onto the three point function, which is constrained by Schrödinger algebra:

$$\langle\Phi_{Q+q}|\phi_q(\tau,\vec{y})|\Phi_Q\rangle = F(v)\exp\left(\frac{q}{2}\omega y^2\right)(2)^{\Delta_\phi}\left(\frac{\mathbf{i}\omega}{2}\right)^{\frac{\Delta}{2}} e^{-i\omega\left(\Delta_Q-\Delta_{Q+q}\right)\tau}. \tag{4.92}$$

The appendix [4.11.2] has the necessary details. Now, upon comparing (4.92) and (4.91), we deduce the universal behavior of $F(v)$ in the large charge sector:

$$F(v=\mathbf{i}\omega y^2) \propto Q^{\frac{\Delta_\phi}{2d}}\left(1-\frac{\omega y^2}{2\xi}Q^{-1/d}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{4.93}$$

which can be rewritten as following, using (4.44):

$$F(v=\mathbf{i}\omega y^2) \propto \Delta_Q^{\frac{\Delta_\phi}{2(d+1)}}\left(1-\frac{\omega y^2}{2\xi}\left(\frac{d+1}{d\xi}\Delta_Q\right)^{-\frac{1}{d+1}}\right)^{\frac{\Delta_\phi}{2}} e^{-\frac{1}{2}q\omega y^2} \tag{4.94}$$

The (4.93) and (4.94) are the main results of this subsection. This shows the universal scaling behavior of the structure function F in the large charge sector.

4.9 Conclusions and Future Directions

We have studied the large charge (Q) sector of theories invariant under Schrödinger group. We have employed coset construction to write down an effective field theory (EFT) describing the large Q sector in any arbitrary dimension $d \geq 2$ assuming superfluidity and rotational invariance. The effective Lagrangian is given by

$$\mathcal{L}_{eff} = c_0 X^{\frac{d}{2}+1} + c_1\frac{X^{\frac{d}{2}+1}}{X^3}\partial_i X\partial^i X + c_2\frac{X^{\frac{d}{2}+1}}{X^3}(\partial_i A_0)^2 + c_3\frac{X^{\frac{d}{2}+1}}{X^2}\partial_i\partial^i A_0 + c_4\frac{X^{\frac{d}{2}+1}}{X^2}(\partial_i\partial^i\chi)^2$$

where $X = \partial_t\chi - A_0 - \frac{1}{2}\partial_i\chi\partial^i\chi$ and χ is the Goldstone excitation of the superfluid ground state. We emphasize that the general co-ordinate invariance, as discussed in [148] will put more constraints on the Wilson coefficients, we leave that as a future project. The EFT is then studied perturbatively as an expansion in $1/Q$. This is to be contrasted with the EFT written down in [148]. While EFT in [148] is controlled by small momentum parameter, ours is controlled by $1/Q$ expansion, which enables us to probe and derive universal results and scaling behaviors in large Q sector. In particular, when Q is very large, we find the scaling behavior of operator dimension with charge, consistent with that found very recently in [156]. We also find that in the large charge sector, structure function of three point correlator has a universal behavior. Last but not the least we derived the dispersion relation for the low energy excitation over this state with large Q and identify the two different kind of descendents as two different modes of excitations. A summary of the results can be found in the introduction.

The theory of conformal, and even superconformal, anyons has been studied before in great detail [157, 158, 146, 159]. In these systems there exists a simple n-particle operator $O = (\Phi^\dagger)^n$ whose dimension is given as

$$\Delta_O = n + n(n-1)\theta \tag{4.95}$$

where θ is the statistics parameter that arises from the Chern-Simons term of level k as $\theta = \frac{1}{2k}$ for bosonic theories. For large k relative to n, close to the bosonic limit, this is known to be the ground state in the trap. It is known as the "linear solution" in the literature due to the linear dependence on θ. For the superconformal theories it is a BPS operator and the dimension (4.95) is exact. A state corresponding to such an operator is not a superfluid and our theory cannot capture the physics of the system in that regime. However, it is known there is a level crossing for smaller k where the ground state corresponds to an operator whose dimension is not protected by the BPS bound. For those operators the classical dimension scales as $n^{\frac{3}{2}}$, in agreement with our results. We are then led to believe the effective field theory we've constructed may apply to anyon NRCFTs in that regime.

Another family of NRCFTs can be defined by the holographic constructions of McGreevy, Balasubramanian[160] and Son[161]. It would be interesting to study these on the gravitational side in the large charge limit, as there might exist a regime where both the EFT and gravity descriptions are valid. The analog of this for the relativistic case was carried out recently[162].

One can envision to extend our results in several ways. One possible extension of these results would be to study operators with large spin as well as charge. If the superfluid EFT remains valid, for sufficiently large spin, one naively expects such operators correspond to vortex configurations in the trap. This was studied in CFT_3, where multiple distinct scaling regimes were shown to exist [163]. Moreover, one can generalize these results to NRCFTs with a larger internal global symmetry group or study systems where the symmetry breaking pattern is different. Potentially interesting examples include “chiral” superfluids [164], where the rotational symmetry is additionally broken by the superfluid order parameter, or the vortex lattice [165] where the translation symmetry is spontaneously broken.

4.10 Appendix A: Phonons in the Trap

We are solving equation (4.81) in the range of $r \in [0, R]$ where $R^2 = \frac{2\mu}{\omega^2}$ is the cloud radius. Inserting $\pi \propto e^{\mathbf{i}\varepsilon t} f(r) Y_\ell$ and expanding in spherical coordinates:

$$-\frac{\omega^2}{d}(R^2 - x^2)[\partial_r^2 f + \frac{(d-1)}{r}\partial_r f - \frac{1}{r^2}\ell(\ell + d - 2)f] + \omega^2 r \partial_r f = \varepsilon^2 f \tag{4.96}$$

Defining the dimensionless variables $x \equiv \frac{r}{R}$ and $\lambda \equiv \frac{\varepsilon}{\omega}$ and changing variables to $z = x^2$

$$-\frac{1}{d}(1-z)[4z\partial_z^2 f + 2\partial_z f + 2(d-1)\partial_z f - \frac{1}{z}\ell(\ell + d - 2)f] + 2z\partial_z f = \lambda^2 f \tag{4.97}$$

Equation (4.97) is a hypergeometric equation with two independent solutions

$$f(z) \sim c_1 z^{\frac{\ell}{2}} \, {}_2F_1(\alpha_-, \alpha_+, \gamma, z) + c_2 z^{\frac{1}{2}(2-d-\ell)} \, {}_2F_1(\alpha', \beta', \gamma', z) \tag{4.98}$$

Our solution should be valid on the interval $z \in [0,1]$ where it should be regular and finite at both $z = 0$ and $z = 1$. Regularity at the origin kills the second solution immediately.

Therefore we have:

$$f(z) \sim c_1 z^{\frac{\ell}{2}} \, {}_2F_1(\alpha_-, \alpha_+, \gamma, z) \tag{4.99}$$

where $\gamma = \ell + \frac{d}{2}$, $\alpha_\pm = \frac{1}{2}(\ell + d - 1) \pm \kappa$, and $\kappa = \frac{1}{2}(1 - 2d + d^2 - 2\ell + \ell d + \ell^2 + d\lambda^2)^{\frac{1}{2}}$

The function ${}_2F_1(\alpha_-, \alpha_+, \gamma, z)$ is finite at $z = 1$ under one of the following possibilities:

1. The values $\alpha_+ + \alpha_- < \gamma$ for any value of the arguments
2. If either $\alpha_\pm$ is equal to a non-positive integer

To see this, we use the following identity and regularity of ${}_2F_1$ around $(1-z) = 0$:

$$\begin{aligned} {}_2F_1(\alpha_-, \alpha_+, \gamma, z) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha_+ - \alpha_-)}{\Gamma(\gamma - \alpha_-)\Gamma(\gamma - \alpha_+)} {}_2F_1(\alpha_-, \alpha_+, \alpha_- + \alpha_+ + 1 - \gamma, 1 - z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha_+ + \alpha_- - \gamma)}{\Gamma(\alpha_-)\Gamma(\alpha_+)} (1-z)^{\gamma - \alpha_- - \alpha_+} {}_2F_1(\gamma - \alpha_-, \gamma - \alpha_+, 1 + \gamma - \alpha_- - \alpha_+, 1 - z) \end{aligned} \tag{4.100}$$

$${}_2F_1(\alpha_-, \alpha_+, \gamma, z \sim 1) \sim \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha_+ - \alpha_-)}{\Gamma(\gamma - \alpha_-)\Gamma(\gamma - \alpha_+)} + \frac{\Gamma(\gamma)\Gamma(\alpha_+ + \alpha_- - \gamma)}{\Gamma(\alpha_-)\Gamma(\alpha_+)} (1-z)^{\gamma - \alpha_- - \alpha_+} \tag{4.101}$$

We can check explicitly that $\alpha_+ + \alpha_- = \ell + d - 1 \geq \gamma$ for $d \geq 2$, where the superfluid groundstate is possible. Therefore option (1) is ruled out.

Define $\alpha_- = -n$ where n is a non-negative integer.

The relation above implies $\alpha_+ = (\ell + d - 1) - \alpha_- = \ell + d + n - 1$

Consider the explicit product:

$$\alpha_+ \alpha_- = \frac{1}{4}(\ell + d - 1 + 2\kappa)(\ell + d - 1 - 2\kappa) = \frac{d}{4}(\ell - \lambda^2) \tag{4.102}$$

Substituting the integer relations for $\alpha_\pm$ turns equation (4.102) into a quadratic equation which can be solved for λ as:

$$\lambda^2 = \frac{1}{d}(4n^2 + 4dn + 4\ell n - 4n + d\ell) \tag{4.103}$$

which yields the dispersion (4.82)

4.11 Appendix B: Correlation Functions in Oscillator Frame

4.11.1 Two point function

In Galilean frame the two point function is given by

$$\left\langle \mathcal{O}(t = -\mathbf{i}/\omega)\mathcal{O}^\dagger(t = \mathbf{i}/\omega)\right\rangle = c\left(-\frac{2\mathbf{i}}{\omega}\right)^{-\Delta_\mathcal{O}} \tag{4.104}$$

Now we know

$$\begin{aligned}\left\langle \mathcal{O}(t = -\mathbf{i}/\omega)\mathcal{O}^\dagger(t = \mathbf{i}/\omega)\right\rangle &= \lim_{\substack{t_0\to\mathbf{i}\omega \\ \tau_0\to -\mathbf{i}\infty}} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_\mathcal{O}}}\left\langle \mathcal{O}(\tau = \tau_0)\mathcal{O}^\dagger(\tau = -\tau_0)\right\rangle \\ &= c\lim_{\tau_0\to\mathbf{i}\infty} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_\mathcal{O}}}\left(\frac{1}{\sin^2(2\omega\tau_0)}\right)^{\Delta_\mathcal{O}/2} \\ &= c(2\mathbf{i})^{\Delta_\mathcal{O}} \lim_{\tau_0^{(E)}\to\infty} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_\mathcal{O}}}\exp\left(-2\omega\Delta_\mathcal{O}\tau_0^{(E)}\right)\end{aligned} \tag{4.105}$$

where $\omega t_0 = \tan(\omega\tau_0)$. Comparing (4.104) and (4.105), we obtain an identity:

$$\lim_{\substack{t_0\to\mathbf{i}\omega \\ \tau_0^{(E)}\to\infty}} \frac{1}{(1+\omega^2 t_0^2)^{\Delta_\mathcal{O}/2}}\exp\left(-\omega\Delta_\mathcal{O}\tau_0^{(E)}\right) = 2^{-\Delta_\mathcal{O}}\omega^{\Delta_\mathcal{O}/2} \tag{4.106}$$

where we have $\omega t = \tan(\omega\tau)$ and $\tau^{(E)} = \mathbf{i}\tau$. We note that $t = \pm\frac{\mathbf{i}}{\omega}$ corresponds to Oscillator frame

Euclidean time $\tau_E = \mp\infty$, this follows from

$$\omega t = \tan\left(-\mathbf{i}\omega\tau_E\right) \tag{4.107}$$

Thus the operators are inserted at infinitely past and future Euclidean time.

In the oscillator frame, we have

$$\langle O(\tau_1)O^\dagger(\tau_2)\rangle = c\left[1+\tan^2(\omega\tau_1)\right]^{\frac{\Delta_O}{2}}\left[1+\tan^2(\omega\tau_2)\right]^{\frac{\Delta_O}{2}}(\tan(\omega\tau_1)-\tan(\omega\tau_2))^{-\Delta_O},$$

which can be simplied into

$$\langle O(\tau_1)O^\dagger(\tau_2)\rangle = c\left[\sin(\omega(\tau_1-\tau_2))\right]^{-\Delta_O}, \tag{4.108}$$

using the identity

$$\frac{[1+\tan^2(\omega\tau_1)][1+\tan^2(\omega\tau_2)]}{[\tan(\omega\tau_1)-\tan(\omega\tau_2)]^2} = \frac{1}{\sin^2(\omega(\tau_1-\tau_2))}. \tag{4.109}$$

4.11.2 Three point function

In the Galilean frame, the general form of a three-point function is fixed to be:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle \equiv G(x_1;x_2;x_3) = F(v_{123})\exp\left[-\mathbf{i}\frac{Q_1}{2}\frac{\vec{x}_{13}^2}{t_{13}}-\mathbf{i}\frac{Q_2}{2}\frac{\vec{x}_{23}^2}{t_{23}}\right]\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j} \tag{4.110}$$

where $\Delta \equiv \sum_i \Delta_i$, $x_{ij} \equiv x_i - x_j$, and $F(v_{ijk})$ is a function of the cross-ratio v_{ijk} defined:

$$v_{ijk} = \frac{1}{2}\left(\frac{\vec{x}_{jk}^2}{t_{jk}} - \frac{\vec{x}_{ik}^2}{t_{ik}} + \frac{\vec{x}_{ij}^2}{t_{ij}}\right) \tag{4.111}$$

The matrix element (4.86) defines a 3-point function in this frame via (4.16) and (4.13)

$$
\begin{aligned}
\langle\Phi_{Q+q}|\phi_q(\tau,\vec{y})|\Phi_Q\rangle &= (1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\exp\left[\frac{\mathbf{i}}{2}q\frac{x^2\omega^2 t}{1+\omega^2t^2}\right]G\left(-\frac{\mathbf{i}}{\omega},0\,;t,\vec{x};\ \frac{\mathbf{i}}{\omega},0\right)\\
&= F(v)(1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\exp\left[\frac{\mathbf{i}}{2}q\frac{x^2\omega^2 t}{1+\omega^2t^2}\right]\exp\left[-\frac{iqx^2}{2\left(t-\frac{i}{\omega}\right)}\right]\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j}\\
&= F(v)\exp\left[\frac{q}{2}\frac{\omega x^2}{1+\omega^2t^2}\right](1+\omega^2t^2)^{\frac{\Delta_\phi}{2}}\prod_{i<j}t_{ij}^{\frac{\Delta}{2}-\Delta_i-\Delta_j}\\
&= F(v)\exp\left[\frac{q}{2}\frac{\omega x^2}{1+\omega^2t^2}\right](2)^{\frac{1}{2}(-\Delta_{Q+q}+\Delta_\phi-\Delta_Q)}(\mathbf{i}\omega)^{\frac{\Delta}{2}}\left(\frac{1-\mathbf{i}\omega t}{1+\mathbf{i}\omega t}\right)^{\frac{\Delta_Q-\Delta_{Q+q}}{2}}\\
&= F(v)\exp\left(\frac{q}{2}\omega y^2\right)(2)^{\Delta_\phi}\left(\frac{\mathbf{i}\omega}{2}\right)^{\frac{\Delta}{2}}e^{-i\omega\left(\Delta_Q-\Delta_{Q+q}\right)\tau}
\end{aligned}
$$

where

$$
v=\frac{1}{2}\left(\frac{x^2}{t-\frac{\mathbf{i}}{\omega}}+\frac{x^2}{-\frac{\mathbf{i}}{\omega}-t}\right)=\frac{\mathbf{i}\omega x^2}{1+\omega^2t^2} \tag{4.112}
$$

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Chapter 5

The Spinful Large Charge Sector of Non-Relativistic CFTs: From Phonons to Vortex Crystals

5.1 Abstract

We study operators in Schrödinger invariant field theories (non-relativistic conformal field theories or NRCFTs) with large charge (particle number) and spin. Via the state-operator correspondence for NRCFTs, such operators correspond to states of a superfluid in a harmonic trap with phonons or vortices. Using the effective field theory of the Goldstone mode, we compute the dimensions of operators to leading order in the angular momentum L and charge Q. We find a diverse set of scaling behaviors for NRCFTs in both $d = 2$ and $d = 3$ spatial dimensions. These results apply to theories with a superfluid phase, such as unitary fermions or critical anyon systems.

5.2 Introduction and Summary

Superfluid states of matter are one of most fundamental examples of spontaneous symmetry breaking and appear in countless systems from Helium-4 [166, 167, 168, 169] to neutron stars [170]. Superfluidity is also a possibility for finite density states of scale invariant critical systems [171]. Recently this observation has been used to perform explicit calculations of relativistic conformal field theory (CFT) data, despite strong coupling[127, 128, 129, 130, 131, 172]. The key idea behind this is the fact that the large charge operators of the CFT correspond to finite density states on the sphere, which spontaneously break the conformal invariance and $U(1)$ corresponding to the charge. Superfluid phenomenology then becomes relevant for describing the large charge sectors of these CFTs. For example, another hallmark of superfluidity is the formation of vortices upon insertion of angular momentum. Therefore states with vortices correspond to large charge operators with spin, and calculating the energy of these vortices reveal the spinning operator spectrum in CFT [163].

However, many interesting critical systems do not possess Lorentz symmetry. This includes ultracold fermi gases at "unitarity", where observation of vortex lattices is perhaps the most dramatic evidence for a superfluid ground-state in a system which exhibits an emergent scale invariance[173]. At this critical point the system has a non-relativistic conformal symmetry, or Schrödinger symmetry. This symmetry algebra plays a pivotal role in understanding numerous physical systems[1]. Examples include the aforementioned "fermions at unitarity"[138, 139], as well as systems comprised of deuterons [141, 142], ^{133}Cs[143], ^{85}Rb [144],^{39}K [145], and various spin chain models [140]. There has been significant progress in understanding the consequences of Schrödinger symmetry and its realization in field theory[133, 134, 146, 135, 136, 137, 176]. These non-relativistic conformal field theories (NRCFTs) admit a state-operator correspondence akin to their relativistic cousins. Operators with "particle number" charge are related to states in a

[1]It is important to mention that Schrödinger symmetry is not simply the non-relativistic limit of the conformal symmetry but rather an entirely distinct algebra [174, 175].

harmonic potential.[146] This has been exploited to calculate the energies of few-body quantum mechanics systems in a harmonic trap. This correspondence also implies a way that the spectrum of NRCFTs can be determined. The operators with large charge correspond to finite density states in the trap. These states of matter sometimes admit a simple effective field theory description, enabling semi-classical calculations controlled in the large charge limit [177, 156].

The simplest and most physically relevant possibility is that of a superfluid ground-state, which is the situation we will explore here.[2] Extending upon the results of [163, 177], we study NRCFT operators which have both large charge and spin. Such operators correspond to either phonon or vortex excitations of the superfluid. We then compute the leading order scaling of their dimensions $\Delta_{Q,L}$ as functions of their angular momentum L and number charge Q and find a diverse range of behaviors in $d = 2$ and $d = 3$.[3]

Trailer of the Results:

We compute the leading scaling dimension $\Delta_{Q,L}$ of spinning operators of a non-relativistic conformal field theory as a function of $U(1)$ charge Q and angular momentum L in the large charge limit. The answers are determined up to a single Wilson coefficient c_0 in the EFT description. We leverage the state operator correspondence to arrive at the result that depending on the range of angular momentum, the spinning operators correspond to different excitation modes of the superfluid. For a smaller range of angular momentum, we find that they correspond to phonon with angular momentum L. As we increase the angular momentum, we pass through a regime where a single vortex becomes energetically favorable. If we further increase the angular momentum, multiple vortices develop and the superfluid exhibits an effective "rigid body motion" where we can neglect the discrete nature of the vortices.

[2]It should be emphasized that this is not the only possibility. Ultimately the question "Given this NRCFT, what state of matter describes its large charge sector?" depends on the NRCFT, which we treat as UV physics. However we expect our results to be valid for a wide set of NRCFTs, including some of physical relevance such as unitary fermions [173].

[3]We note that d refers to the spatial dimension and we reserve spacetime dimension as $D = d + 1$.

In $d=2$, the leading behavior has 3 regimes and is given as follows:

$$d=2 \quad \Delta_{Q,L}=\begin{cases}\sqrt{L}+\Delta_Q & 0<L\leq Q^{1/3}\\ \sqrt{\frac{c_0\pi}{2}}\sqrt{L}\log L+\Delta_Q & Q^{1/3}<L\leq Q\\ \sqrt{\frac{9c_0\pi}{2}}\left(\frac{L^2}{Q^{3/2}}\right)+\Delta_Q & Q<L<Q^{3/2}\end{cases} \tag{5.1}$$

where $\Delta_Q=\frac{2}{3}\left(\frac{1}{\sqrt{2\pi c_0}}\right)Q^{3/2}$ is the contribution from ground state energy in $d=2$.

In $d=3$ dimensions, we have 4 regimes, given by:

$$d=3 \quad \Delta_{Q,L}=\begin{cases}\sqrt{L}+\Delta_Q & 0<L\leq Q^{2/9}\\ \alpha\left(\frac{L}{Q^{1/9}}\right)+\Delta_Q & Q^{2/9}<L\leq Q^{1/3}\\ \left(\frac{5\pi^4c_0}{8\sqrt{2}}\right)^{1/3}L^{2/3}\log L+\Delta_Q & Q^{1/3}<L\leq Q\\ \frac{1024}{25}\left(\frac{32c_0^2}{25\pi^4}\right)^{1/6}\left(\frac{L^2}{Q^{4/3}}\right)+\Delta_Q & Q<L<Q^{4/3}\end{cases} \tag{5.2}$$

where $\Delta_Q=\frac{3}{2}\frac{1}{\sqrt{2\pi}}\left(\frac{6}{15\sqrt{\pi}c_0}\right)^{1/3}Q^{4/3}$ is the contribution from ground state energy in $d=3$ and α is an undetermined $O(1)$ coefficient. We make two remarks at this point. The first one is that while for $d=2$, the transition happens from a single phonon regime to vortex regime at $L\sim Q^{1/3}$, for $d=3$, there is a regime $Q^{2/9}\leq L\leq Q^{3/9}$, where neither vortex nor the single phonon solution gives the lowest energy. It is a cross-over describing the physics of a vortex string forming near the boundary of the trap where our EFT is strongly coupled. The only well defined configuration in this angular momentum regime contains multiple phonons, and we determine the scaling from that. The second remark is that the EFT description breaks down whenever $\Delta_{Q,L} - \Delta_Q \sim \Delta_Q$, so we can not probe operators with larger angular momentum with this method.

The rest of the paper is organized as follows. We briefly review the superfluid hydrodynamics and large charge NRCFT in section 5.3. The section 5.4 details out the contribution coming from phonons and derives the regime where it is energetically favorable to have them. Subsequently, we discuss the single vortex in $d = 2$ and $d = 3$ in section 5.5. The multi-vortex and rigid body motion is elucidated in section 5.6 followed by a brief conclusion and future avenues to explore in section 5.7. Some of our results and validity regimes are more apparent in dual frame using particle-vortex duality which we elaborate on in appendix 5.8. The appendix 5.9 contains a contour integral useful for calculating interaction energy of multiple vortices in $d = 3$.

5.3 The set up: Superfluid Hydrodynamics and Large Charge NRCFT

In this section we briefly review the superfluid hydrodynamics in the Hamiltonian formalism, specialized to the case of a Schrödinger invariant system in a harmonic potential $A_0 = \frac{1}{2}\omega^2 r^2$. All of our results will be to leading order in the derivative expansion. For a more in-depth review of the formalism, we refer to [148, 177, 156].

The low-energy physics of a superfluid is determined by a single Goldstone field χ. The leading order Lagrangian determines the pressure of the system:

$$\mathcal{L} = c_0 X^{\frac{d+2}{2}} \equiv P(X) \qquad X \equiv \partial_0\chi - A_0 - \frac{1}{2}(\partial_i\chi)^2 \tag{5.3}$$

The number density and superfluid velocity are defined respectively as:

$$n = \frac{\partial L}{\partial \dot{\chi}} = c_0 \left(\frac{d}{2} + 1 \right) X^{\frac{d}{2}} \qquad v_i = -\partial_i \chi \tag{5.4}$$

The action (5.3) has a $U(1)$ symmetry of $\chi \to \chi + c$ whose current can be written as:

$$j^\mu = \left(n, n v^i \right) \tag{5.5}$$

The Hamiltonian density comes out to be:

$$\mathcal{H} = n\dot{\chi} - \mathcal{L} = n \left(X + A_0 + \frac{1}{2} v^2 \right) - P(X) \tag{5.6}$$

Now, using the thermodynamic relation $nX - P(X) \equiv \varepsilon(n)$: we can simplify (5.6) and express the Hamiltonian as:

$$H = \int \mathrm{d}^d x \, \mathcal{H} \qquad \mathcal{H} = \frac{1}{2} n v^2 + \varepsilon(n) + n A_0 \tag{5.7}$$

.

Note that the presence of the harmonic trap implies the density is non-uniform and vanishes at radius $R_{TF} = \sqrt{\frac{2\mu}{\omega^2}}$. For most values of r the density is large and varies slowly compared to the UV length scale $\frac{1}{\sqrt{\mu}}$. However, the large charge expansion begins to break down at $R^* = R_{TF} - \delta$ where $\delta \sim \frac{1}{(\omega^2 \mu)^{\frac{1}{6}}}$[177, 148]. There is a boundary layer of thickness δ where the superfluid effective field theory (EFT) cannot be trusted as it is no longer weakly coupled. At leading order in the derivative expansion this does not affect the observables but leads to divergences at higher orders.[4]

Given this set up, the ground-state at finite density corresponds to the classical solution of

[4]These are UV divergences which can be canceled by counter-terms localized at this edge, as suggested by Simeon Hellerman in a private communication.

$\chi_{cl} = \mu t$. The number charge of this configuration is determined from μ by:

$$Q \equiv \int \mathrm{d}^d x\, n_{cl}(x) = c_0 \left(\frac{d}{2}+1\right) \int \mathrm{d}^d x\; (\mu - A_0)^{\frac{d}{2}} = \frac{1}{\xi}\left(\frac{\mu}{\omega}\right)^d \tag{5.8}$$

where $\frac{1}{c_0} = \frac{\Gamma(\frac{d}{2}+2)}{\Gamma(d+1)}(2\pi\xi^2)^{\frac{d}{2}}$. We can then compute the ground-state energy as function of Q using (5.7):

$$E_Q = \int \mathrm{d}^d x\, [\varepsilon(n_{cl}) + n_{cl} A_0] = \omega\xi \left(\frac{d}{d+1}\right) Q^{\frac{d+1}{d}} \tag{5.9}$$

Via the state-operator correspondence of NRCFTs, this semi-classical calculation determines the dimension of a charged scalar operator to leading order in Q as $\Delta_Q = \frac{E_Q}{\omega}$. In particular, we have obtained [177]:

$$\Delta_Q = \begin{cases} \frac{2}{3}\xi Q^{3/2} & \text{for} \quad d=2 \\ \frac{3}{4}\xi Q^{4/3} & \text{for} \quad d=3 \end{cases} \tag{5.10}$$

In this work, we'll be interested in excited state configurations which carry some angular momentum. These will correspond to spinful operators in the large charge sector of the NRCFTs which the superfluid EFT describes. The simplest of these excitations are phonons; smooth solutions of the equation of motion with $\chi_{cl} = \mu t + \pi$. Expanding π in modes $\pi_{n,\ell}$, the Hamiltonian can be written to leading order in the derivative expansion as:

$$H = H_0 + \sum_{n,\ell} \omega(n,\ell) \pi^\dagger_{n,\ell} \pi_{n,\ell} + \cdots \tag{5.11}$$

where $\omega(n,\ell)$ is the dispersion relation for phonons:

$$\omega(n,\ell) = \omega \left(\frac{4}{d} n^2 + \left(4 - \frac{4}{d}\right) n + \frac{4}{d} n\ell + \ell\right)^{\frac{1}{2}} \tag{5.12}$$

for n is a positive integer and ℓ is the total angular momentum. The phonon wavefunctions are given as $f_{n,\ell} \sim (\frac{r}{R_{TF}})^{\frac{\ell}{2}} G_{n,\ell}(r) Y_\ell$ where $G_{n,\ell}$ is a hypergeometric function[5] and Y_ℓ is a spherical harmonic. A state with M phonon modes of $\{n=0,\ell=1\}$ can be identified as the descendant operator $\vec{\partial}^M O_Q$ with dimension $\Delta_Q + M$. Additionally, NRCFTs have another generator of descendants ∂_t which corresponds to the phonon with $\{n=1,\ell=0\}$. States that can be created by adding phonons with other values of n and ℓ correspond to distinct primaries [177].[6]

The other configuration of a superfluid that can support angular momentum is a vortex, which gives rise to a singular velocity field of the condensate. This is a distinct semi-classical saddle point which is not simply related to the ground state. It must therefore correspond to a unique set of spinful charged operators present in all NRCFTs whose scalar large charge sector is described by the superfluid EFT.

These two excitations, phonons and vortices, are the configurations of the superfluid we know support angular momentum. In the rest of the paper we answer the question, what is the lowest energy configuration of the superfluid for a given angular momentum? By answering this and using the superfluid EFT defined above we compute the scaling behavior of operators carrying charge and angular momentum.

5.4 Phonons

The simplest excited state(s) with angular momentum are phonons. From the dispersion (5.12), we can see that the lowest energy configuration with angular momentum L is a single phonon with $n=0$ and $\ell = L$. This is known as a "surface mode" as the wavefunction is nodeless

[5]In particular $G_{n,\ell}(r) = {}_2F_1(\alpha_-, \alpha_+, \gamma, \frac{r}{R_{TF}})$ where $\gamma = \ell + \frac{d}{2}$ and $\alpha_\pm$ are defined in Appendix A of [177]

[6]They are primary as they are by construction annihilated by the lower operators K and C which correspond to $\pi_{n=0,\ell=1}$ and $\pi_{n=1,\ell=0}$ respectively.

and supported mostly at the end of the trap. The energy cost of this single phonon is given by[7]:

$$\Delta E = \omega L^{\frac{1}{2}} . \tag{5.13}$$

However the validity of (5.12) rests on the assumption that the phonon modes do not carry large amounts of momentum. In particular, the surface mode wavefunction has $f_\ell \sim (\frac{r}{R})^{\frac{\ell}{2}} Y_\ell$ which for large ℓ is increasingly concentrated at the edge of the trap. Once the support of the phonon wavefunction is mostly within the boundary region of thickness δ, we can no longer trust the solution or the dispersion (5.12). This occurs when $\frac{R_{TF}}{\ell}$ becomes comparable to δ [178]. This yields a maximum angular momentum for phonons: $\ell_{max} \sim Q^{\frac{2}{3d}}$.

Thus we have the following scalings for operator dimensions:

$$d = 2 \quad \Delta_{Q,L} = L^{\frac{1}{2}} + \Delta_Q \qquad 0 < L \leq Q^{\frac{1}{3}} \tag{5.14}$$

$$d = 3 \quad \Delta_{Q,L} = L^{\frac{1}{2}} + \Delta_Q \qquad 0 < L \leq Q^{\frac{2}{9}} \tag{5.15}$$

where Δ_Q is the operator dimension determined from (5.9).

We can also consider multi-phonon configurations and ask ourselves whether it is energetically favorable to have a single phonon rather than multi phonon configuration, given total angular momentum. In order to answer this, we assume that phonon interactions are negligible, suppressed to leading order in the Q-expansion, so the energy and angular momentum of multiple phonons add linearly. In particular, suppose we have N_γ phonons, each carrying angular momentum ℓ. The energy and angular momentum to leading order is:

$$\Delta E = \omega N_\gamma \ell^{\frac{1}{2}} \qquad L = N_\gamma \ell \tag{5.16}$$

[7]Note this is parametrically lower in energy than in the relativistic case studied in [163], as the phonon spectrum on the sphere is $\varepsilon(\ell) = \sqrt{\frac{1}{2}\ell(\ell+1)}$.

This tells us that for a given angular momentum L, it is energetically favourable to have a single phonon carrying the entire angular momentum rather than multiple phonons carrying it altogether.[8]

As we'll see below, naively a single phonon of $\ell = L$ would always be the most energetically favorable configuration per angular momentum. However the cutoff of $\ell_{max} \sim Q^{\frac{2}{3d}}$ means we cannot trust this conclusion beyond $L = \ell_{max}$. Multi-phonon configurations are in principle valid for larger values of L.[9] The most energetically favorable of which has N_γ phonons with $\ell = \ell_{max}$, which gives the scaling:

$$\Delta E = \omega L \ell_{max}^{-1/2} \qquad \ell_{max} \sim Q^{\frac{2}{3d}} \tag{5.19}$$

where we cannot determine the dimensionless coefficient from ℓ_{max} as it depends on how we regulate the cutoff region of size δ. Nevertheless, the linear scaling in L means we can compare to other configurations such as vortices. In particular, we will arrive at the conclusion that whenever $L \geq Q^{1/3}$, the minimum energy configuration with a given angular momentum starts to be attained by vortex solutions.

For $d = 2$, the transition happens from a single phonon regime to vortex regime at $L \sim Q^{1/3}$, while for $d = 3$, there is a regime $Q^{2/9} \leq L \leq Q^{1/3}$, which is inaccessible by both the vortex string

[8] One can also arrive at the same conclusion by considering N_γ phonons, each carrying angular momentum $\vec{\ell}_i$. The energy and angular momentum to leading order is then given by:

$$\Delta E = \omega \sum_i |\vec{\ell}_i|^{\frac{1}{2}} \qquad L = \left| \sum_i \vec{\ell}_i \right| \tag{5.17}$$

We have

$$\Delta E = \omega \sum_i |\vec{\ell}_i|^{\frac{1}{2}} = \omega \sqrt{\sum |\vec{\ell}_i| + \sum \sqrt{|\vec{\ell}_i||\vec{\ell}_j|}} \geq \omega \sqrt{\left|\sum \vec{\ell}_i\right| + \sum \sqrt{|\vec{\ell}_i||\vec{\ell}_j|}} \geq \omega\sqrt{L} \tag{5.18}$$

Hence, the minimum value is obtained when all the $l_i = 0$ except one i.e. we land up with single phonon case. On the other hand, if all the $\vec{\ell}_i$'s are along same direction, then using Cauchy-Schwartz inequality, one can obtain $\Delta E \leq \omega (N_\gamma)^{1/2} \sqrt{L}$, which implies that the energy would be maximized if each phonon carries angular momentum of L/N_γ.

[9] N_γ cannot be made arbitrarily large as the assumption that phonon interactions are suppressed breaks down.

and the single-phonon configurations. The most energetically favorable configuration, consistent within the leading order EFT analysis, is therefore the multi-phonon configuration above with a macroscopic number of phonons $N_\gamma \sim Q^{\frac{1}{9}}$ at the upper bound $L \sim Q^{\frac{1}{3}}$.

This would imply the following scaling for the operator dimension:

$$d = 3 \quad \Delta_{Q,L} = \alpha L Q^{-1/9} + \Delta_Q \qquad Q^{2/9} < L \leq Q^{1/3} \tag{5.20}$$

where α is an unknown order one coefficient.

However the exact nature of this state appears to be related to UV physics of how a vortex string configuration forms from surface mode phonons in the boundary region of the condensate, which is inaccessible within our formalism. We therefore cannot give a full accounting of this regime of angular momentum. Beyond $L > Q^{\frac{1}{3}}$ we can be confident the lowest energy configuration is a vortex, as we'll now discuss.

5.5 Single Vortex in the Trap

A vortex is a configuration of the superfluid with a singular velocity field carrying angular momentum. The singular nature arises because of the relation (5.4) implying that v_i is necessarily irrotational except due to defects in the field χ; configurations where $\int_C d\chi = 2\pi s$ for some integer s. In $d = 2$ these are particle like excitations while in $d = 3$ they correspond to strings, these will be the dimensions we focus on in this work. In fact this language can be made precise via particle-vortex duality, where vortices are "charged" objects under some dual gauge field. Adapting this duality to the Schrödinger invariant superfluid has been done in Appendix 5.8 but it is inessential for describing the leading order results.

The simplest configuration in the trap is a single static vortex for which the condensate order parameter changes by only 2π.[10] The approximate velocity profile v_i of such a configuration

[10]This is in contrast to the CFT case where a minimum of two vortices are needed on the sphere to ensure

is:

$$v_i = \frac{\varepsilon^{ij}(r_j - R_j)}{(\vec{r}-\vec{R})^2} \tag{5.21}$$

where r is the radial coordinate in $d=2$ or the axial coordinate in $d=3$, and $\vec{R}$ is the location of the vortex and we assume that the vortex is stretched along the z axis.

The presence of the vortex changes the semi-classical number density, making it singular at $r=R$. Before that point the density vanishes, implying a short distance cutoff for the superfluid EFT. This is the 'vortex core size' a whose scaling dimension we can determine as follows.

One interpretation of the non-uniform density (5.8) is that the effective chemical potential is distance dependent. In the presence of a vortex at $\vec{r}=\vec{R}$ it is given as:

$$\mu_{eff}(\vec{r}) \equiv \mu - \frac{1}{2}\omega^2 r^2 - \frac{1}{2}\frac{1}{|\vec{r}-\vec{R}|^2} \tag{5.22}$$

This determines a locally varying UV length scale $\frac{1}{\sqrt{\mu_{eff}}}$. The EFT, which is controlled in the limit of large density, becomes strongly coupled at the length a when $a \sim \frac{1}{\sqrt{\mu_{eff}}}$. Solving this equation for a gives the scaling relations[11]

$$d=2 \quad a \sim \frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}}} \qquad d=3 \quad a \sim \frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}-\frac{z^2}{R_{TF}^2}}} \tag{5.23}$$

Near the center of the trap, a is on order the UV length scale $\frac{1}{\sqrt{\mu}}$. However as the vortex approaches the boundary of the trap, either in its placement $\vec{R}$ or along the length of the vortex string in $d=3$, the fact the density is depleted due to the trap implies the cutoff near the vortex string must happen sooner [179]. As mentioned previously, the EFT is already strongly coupled in the boundary region of size δ. Therefore the largest placements of the vortex we can confidently study have $R = R_{TF} - \delta$ where the core size scales as $a \sim \frac{1}{(\mu\omega^2)^{\frac{1}{3}}}$ which is still parametrically

compatibility with the Gauss law.

[11]This is an equivalent condition to cutting off the theory when the velocity field sourced by the vortex becomes comparable to the local speed of sound in the superfluid $c_s^2 \sim \frac{\partial P}{\partial n} \sim X$.

suppressed in μ.

Regulating this divergence as described above, the correction to the semi-classical number density due to the vortex is subleading in μ and therefore negligible for leading order results. This implies the dominant contribution to the energy of a vortex configuration comes from the kinetic energy of the velocity field.

The velocity field (5.21) does not define a stationary flow in the sense that $\partial_i(nv^i) \neq 0$ because of the inhomogeneity of the density. This inhomogeneity will cause the vortex to precess in a circle [180]. However since the density varies slowly, as previously discussed, the correction to the velocity field due to this is suppressed in the large-charge expansion. Using particle-vortex duality, this is equivalent to the assumption that particle sourcing the gauge field in dual description has suppressed velocity, hence we are effectively dealing with an electrostatic scenario. The details are relegated to the appendix 5.8, in particular, the discussion after (5.66).

We remark that in dual frame, the cloud boundary is like a conductor, hence the tangential electric field should be vanishing. This means the velocity field of the vortex should be such that there is no radial outflow of particles out of the trap. Given this condition, one might worry that the velocity field above does not vanish at the boundary R_{TF}. However, since we require the normal component of the flow to vanish at the boundary i.e. $\hat{N}\cdot(n\vec{v}) = 0$ where $\vec{N}$ is a vector normal to the trap at boundary, the inhomogeneity of the superfluid comes to rescue and the condition is trivially met by the vanishing of the density $n(x)$ at R_{TF} [181]. [12]

In what follows, we will be evaluating the energy and angular momentum of vortex configurations in $d = 2$ and $d = 3$ spatial dimensions.

[12] This is generically known as a “soft boundary”. Had we been dealing with homogenous fluid with non vanishing density at boundary, we ought to consider a mirror vortex configuration to ensure the imposition of $\hat{N}\cdot(n\vec{v}) = 0$, this is just like considering the mirror charge while solving for electric in the presence of a conductor. Regardless, such modifications to the velocity field in the boundary region of the inhomogeneous condensate give suppressed corrections to our leading order results below.

5.5.1 Single vortex in $d = 2$

Let's first work in $d = 2$ with the velocity field given by (5.21). The difference in energy between the vortex state and the ground state can then be computed from the kinetic energy of Hamiltonian (5.7) as:

$$\Delta E = \int \mathrm{d}^2x \, \frac{1}{2} n v^2 = c_0 \mu \int \mathrm{d}^2x \, \left(1 - \frac{r^2}{R_{TF}^2}\right) \frac{1}{(\vec{r} - \vec{R})^2} \tag{5.24}$$

As mentioned, there is a divergence at $r = R$ which we will regulate by assuming a vortex core size of $a(R) \sim \frac{1}{\sqrt{\mu_{eff}}}$ where $\mu_{eff} = \mu\left(1 - \frac{R^2}{R_{TF}^2}\right)$. Evaluating the integral (5.24) gives:

$$\Delta E = 2c_0\pi\mu\left(1 - \frac{R^2}{R_{TF}^2}\right)\left[\log\left(\frac{R_{TF}}{2a(R)}\right) + \frac{1}{2}\log\left(1 - \frac{R^2}{R_{TF}^2}\right) - 1\right] + c_0\pi\mu + O(a) \tag{5.25}$$

We can also compute the angular momentum via the integral:

$$\vec{L} = \int \mathrm{d}^2x \, n\vec{v} \times \vec{r} \tag{5.26}$$

For our configuration the angular momentum is entirely in the $\hat{z}$ direction with magnitude:

$$L = 4\pi c_0 \mu \int_R^{R_{TF}} \mathrm{d}r \, r \left(1 - \frac{r^2}{R_{TF}^2}\right) = 2\pi c_0 \frac{\mu^2}{\omega^2}\left(1 - \frac{R^2}{R_{TF}^2}\right)^2 \tag{5.27}$$

where we've used $\oint_r \vec{v} \cdot d\vec{\ell} = 2\pi$ for a circle centered at the origin of radius $r > R$, and otherwise vanishes.

As one can see, it is energetically favorable for the vortex to appear at the edge of the cloud $R \approx R_{TF}$. However we cannot trust the solution in the regime of low density near there for reasons previously discussed. Therefore the largest distance the vortex can be where we have confidence in the validity of the semi-classical approximation is $R^* = R_{TF} - \delta$. This gives a minimum angular momentum, of the vortex configuration $L_{min} \sim Q^{\frac{1}{3}}$. The largest value of the angular momentum occurs when the vortex is in the center at $R = 0$ with $L_{max} \sim Q$.

Combining these results gives the leading order expressions for the operator dimensions in terms of L and Q as:

$$d = 2 \quad \Delta_{Q,L} = \sqrt{\frac{c_0 \pi}{2}} \sqrt{L} \log L + \Delta_Q \quad Q^{\frac{1}{3}} < L \leq Q \tag{5.28}$$

5.5.2 Single vortex in $d = 3$

Let's consider the case of $d = 3$ now. The minimal energy excitation is a single vortex string. The string must necessarily break the spherical symmetry of the trap. We will consider the string being streched along the z-axis, ensuring that all the angular momentum is $L = L_z$.[13]

The energy of the vortex string again comes from the kinetic energy and can be evaluated as:

$$\Delta E = \int d^3x \, \frac{1}{2} n v^2 = \int_{-Z(R)}^{Z(R)} dz \, T(z,R) \tag{5.29}$$

where $T(z,R)$ is the tension of the string and $Z(R) = R_{TF} \sqrt{\left(1 - \frac{R^2}{R_{TF}^2}\right)}$ defines the integration bound along the length of the string.

The tension can be computed via a similar integral in $d = 2$ as:

$$\begin{aligned} T(z,R) &= \frac{1}{2} \int_0^{r(z)} dr r \int_0^{2\pi} d\phi \, n(r,z) \frac{1}{(\vec{r} - \vec{R})^2} \\ &= \pi n(R,z) \left[\log\left(\frac{r(z,R)}{a(z,R)}\right) - \log\left(1 + \frac{r(z)}{r(z,R)}\right) \right] + \cdots \end{aligned} \tag{5.30}$$

where $\cdots$ refer to the non logarithmic pieces. Here $n(r,z) = \frac{5}{2} c_0 \mu^{\frac{3}{2}} \left(1 - \frac{1}{R_{TF}^2}(r^2 + z^2)\right)^{\frac{3}{2}}$ is the number density, $r(z) = R_{TF} \sqrt{1 - \frac{z^2}{R_{TF}^2}}$ is the radial (radius in cylindrical co-ordinate) size of the trap at a height z and $r(z,R) = R_{TF} \sqrt{1 - \frac{z^2 + R^2}{R_{TF}^2}}$. Integrating the leading logarithmic piece along

[13] A curved string will generically have to be longer in order to carry the same angular momentum, as parts of the velocity field it sources will cancel against each other. The longer strings will be energetically more expensive, making the straight line configuration energetically favorable to leading order.

the string length gives the energy:

$$\Delta E = \int_{-Z(R)}^{Z(R)} \mathrm{d}z \, \pi n(R,z) \log\left(\frac{r(z,R)}{a(z,R)}\right)$$
$$= \frac{15}{16}\pi^2 c_0 \mu^{3/2} R_{TF} \left(1 - \frac{R^2}{R_{TF}^2}\right)^2 \left[\log\left(1 - \frac{R^2}{R_{TF}^2}\right) + \log\left(R_{TF}\sqrt{\mu}\right)\right] \tag{5.31}$$

Evaluating the angular momentum of this configuration is similar to $d = 2$ and yields:

$$L = \int_{-Z(R)}^{Z(R)} \mathrm{d}z \int_{R}^{r(z)} \mathrm{d}r \, r \left[\frac{5}{2} c_0 \mu^{\frac{3}{2}} \left(1 - \frac{r^2 + z^2}{R_{TF}^2}\right)^{\frac{3}{2}}\right] = \frac{5\pi c_0}{8\sqrt{2}} \left(\frac{\mu}{\omega}\right)^3 \left(1 - \frac{R^2}{R_{TF}^2}\right)^3 \tag{5.32}$$

Again the lowest allowed value of the angular momentum occurs for a vortex at $R^* = R_{TF} - \delta$ and scales as $L_{min} \sim Q^{\frac{1}{3}}$ while the maximum occurs at $R = 0$ with $L_{max} \sim Q$.

Together these results imply the scaling:

$$d = 3 \quad \Delta_{Q,L} = \left(\frac{5\pi^4 c_0}{8\sqrt{2}}\right)^{1/3} L^{2/3} \log L + \Delta_Q \qquad Q^{\frac{1}{3}} < L \leq Q \tag{5.33}$$

This determines the leading order dimension for the operator which creates the vortex string but we can also study the spectrum of operators above it. For example, the presence of a vortex string along the $\hat{z}$-direction should split the phonon m degeneracy in (5.12). Treating this perturbatively, such a splitting is suppressed in the charge[14] Q [179].

Besides phonons, there are unique excitations of the vortex string related to displacements of position. These are known as "Kelvin modes" and they define another set of low-lying operators above the one which created the vortex string. These modes are basically the radial displacement of the vortex core from the original axis. For long wavelength modes $ka(z,R) \ll 1$ and in the regime where $z \ll R_{TF}$, we can effectively assume that density is uniform[15]. Under

[14]One could also consider the energy of a vortex-phonon configuration. The "interaction energy" between the two is given as $\int \mathrm{d}^d x \, \vec{v}_{vortex} \cdot \vec{\partial}\pi$ which is also suppressed in the Q expansion.

[15]For work going beyond this approximation in non-uniform condensates, see [182].

this assumption, followed by considering a situation where the amplitude of the displacement is small, we have the standard result quoted in superfluid literature i.e. $\omega(k) \approx \frac{1}{2}k^2 \log \frac{1}{|k|a(R)}$, where $a(R) = \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{1-\frac{R^2}{R_{TF}^2}}}$ is the vortex core size via (5.23). We remark that the boundary conditions on the string should quantize $k \sim \frac{n}{R_{TF}}$, so there is an approximate continuum of such operators above the gap to create a single vortex string. The spacing of these modes and exact dimensions are only visible at higher orders in the Q expansion.

5.6 Multi-Vortex Profile

Consider a collection of N_v vortices at locations $\vec{R}_i$ with winding numbers s_i. The velocity field of such a contribution is additive and described by:

$$\vec{v} = \sum_i \vec{v}_i = \sum_i s_i \frac{\hat{z} \times (\vec{r} - \vec{R}_i)}{|\vec{r} - \vec{R}_i|^2} \implies \nabla \times \vec{v} = \sum_i s_i \delta(\vec{r} - \vec{R}_i) \tag{5.34}$$

Because the angular momentum is linear in the velocity field, this implies the total angular momentum of the system is given by the sum of the individual ones:

$$L = \sum_i L_i = \begin{cases} 2\pi c_0 \left(\frac{\mu}{\omega}\right)^2 \sum_i s_i \left(1 - \frac{R_i^2}{R_{TF}^2}\right)^2 & d = 2 \\ \frac{5\pi c_0}{8\sqrt{2}} \left(\frac{\mu}{\omega}\right)^3 \sum_i s_i \left(1 - \frac{R_i^2}{R_{TF}^2}\right)^3 & d = 3 \end{cases} \tag{5.35}$$

For vortices far from the boundary, where $\frac{R_{TF}-R_i}{R_{TF}} \sim O(1)$ (as opposed to Q suppressed number), we have that $L_i \sim s_i Q$.

We can compute the energy of a generic multi-vortex configuration explicitly from this velocity field (5.34). The energy breaks up into single-vortex contributions and pair-wise interaction energies:

$$\Delta E = \frac{1}{2} \int \mathrm{d}^d x \, n v^2 = \sum_i E_i + \sum_{i \neq j} \sum_j E_{ij} \tag{5.36}$$

where the single vortex energy is already computed as

$$E_i = \frac{1}{2}\int \mathrm{d}^d x\, n v_i^2 = \begin{cases} \omega\sqrt{\frac{c_0\pi}{2}} s_i^2 \sqrt{L_i}\log L_i & d=2 \\ \omega\left(\frac{5\pi^4 c_0}{8\sqrt{2}}\right)^{1/3} s_i^2 L_i^{2/3}\log L_i & d=3 \end{cases} \tag{5.37}$$

and E_{ij} is the interaction energy given by:

$$E_{ij} = \int \mathrm{d}^d x\, n v_i \cdot v_j \tag{5.38}$$

In $d=2$ this integral evaluates to:

$$\begin{aligned} \frac{E_{ij}}{s_i s_j} = {} & \pi c_0 \mu \left(1 - \frac{\vec{R}_i\cdot\vec{R}_j}{R_{TF}^2}\right)\log\left(\frac{R_{TF}^4 + R_i^2 R_j^2 - 2R_{TF}^2 \vec{R}_i\cdot\vec{R}_j}{(\vec{R}_i - \vec{R}_j)^4}\right) \\ & - 2\pi c_0 \mu \frac{1}{R_{TF}^2}\left(R_i^2 + R_j^2 - R_{TF}^2\right) \\ & + 2\pi c_0 \mu \frac{1}{R_{TF}^2}|\vec{R}_i\times\vec{R}_j| \arctan\left(\frac{|\vec{R}_i\times\vec{R}_j|}{R_{TF}^2 - \vec{R}_i\cdot\vec{R}_j}\right) \end{aligned} \tag{5.39}$$

where $\vec{R}_i$ and $\vec{R}_j$ are the positions of the vortex pair with $R_j > R_i$ assumed without loss of generality. To leading order in the charge and small vortex separation this simplifies to:

$$E_{ij} \sim s_i s_j \mu \log\frac{R_{TF}}{|\vec{R}_i - \vec{R}_j|} + \ldots \tag{5.40}$$

This piece is the result of the singular nature of the vortices and describes their interaction. The analogous result of (5.39) for $d=3$ is not analytically tractable, but the leading interaction piece in the charge and small vortex separation is given by:

$$E_{ij} \sim s_i s_j \mu^{\frac{3}{2}} R_{TF}\log\frac{R_{TF}}{|\vec{R}_i - \vec{R}_j|} + \cdots \tag{5.41}$$

One can extract several physical features of the multivortex profile using the expressions above for the energy. First of all, the minimum energy configurations per angular momentum will have $s_i = 1$ for every vortex as the energy scales quadratically in the charge but the angular momentum only scales linearly. The angular momentum for the entire configuration then scales as $L \sim N_v Q$ assuming $\frac{R_{TF}-R_i}{R_{TF}} \sim O(1)$. Secondly, we remark that the logarithmic terms (5.40) and (5.41) imply that the minimal energy configuration will generically be a triangular array of vortices[183, 184]. Empirically this structure persists as the number of vortices is made large, even in the presence of a harmonic trap[173].

In principle the energy, and therefore the operator dimension, should be found by fixing the angular momentum and varying over the positions R_i to find the minimum energy configuration. However, for $N_v \sim O(1)$ the interaction is negligible and the energy will scale as $E \sim N_v E_v$ where E_v is the energy of a single vortex placed in the center of the trap. To consider L parametrically larger than Q we must consider $N_v \gg 1$. While we cannot exactly analyze (5.36) in this limit, we are justified in approximating the vortex density as a continuous quantity, corroborated by the fact that in this limit the interaction energy dominates and has terms which go as $N_v^2 \mu^{\frac{d}{2}} R_{TF}^{d-2} \sim L^2/I$, where I is the moment of inertia, given later by Eq. (5.47).

Continuum Approximation: We can take advantage of the fact the vortices are dense to coarse grain (5.34) and replace it with a continuous velocity field which satisfies:

$$\oint_C \vec{v} \cdot \mathrm{d}\vec{\ell} = 2\pi N_v(C) \tag{5.42}$$

where $N_v(C)$ is the number of vortices in the area enclosed by the curve C. Let L be the angular momentum (to be precise the z component of the angular momentum) of the configuration. We take a variational approach, minimizing the energy over smooth v with fixed L. To this end, define:

$$E_\Omega = \frac{1}{2}\int \mathrm{d}^d x\, nv^2 - \Omega\left(\int \mathrm{d}^d x\, n\,(\vec{r}\times\vec{v})\cdot\hat{z} - L\right) \tag{5.43}$$

$$= \frac{1}{2}\int \mathrm{d}^d x\, n\,(v - \Omega\hat{z}\times\vec{r})^2 - \frac{\Omega^2}{2}\int \mathrm{d}^d x\, nr^2 + \Omega L \tag{5.44}$$

where Ω is a Lagrange multiplier to fix the angular momentum. From (5.43), we can see that the minimum energy velocity field is that of a rotating rigid body with uniform vortex density:

$$\vec{v} = \Omega\hat{z}\times\vec{r} \implies \Delta E = \frac{\Omega^2}{2}\int \mathrm{d}^d x\, nr^2 = \frac{\Omega^2}{2} I \tag{5.45}$$

where I is the moment of inertia of the condensate, computed from the density as:

$$I = \int \mathrm{d}^d x\, n(r) r^2 \,. \tag{5.46}$$

and Ω can be determined via its relation to L as $\Omega = \frac{L}{I}$. Now the moment of inertia I evaluates to

$$I = \begin{cases} \frac{4}{3}\pi c_0 \frac{\mu^3}{\omega^4} = \frac{1}{\omega}\left(\frac{2}{3}\frac{1}{\sqrt{2\pi c_0}} Q^{\frac{3}{2}}\right) & d = 2 \\ \\ \frac{5\pi^2}{8\sqrt{2}} c_0 \frac{\mu^4}{\omega^5} = \frac{1}{\omega}\left(\frac{25}{1024}\left(\frac{25\pi^4}{32 c_0^2}\right)^{1/6} Q^{\frac{4}{3}}\right) & d = 3 \end{cases} \tag{5.47}$$

Using (5.42) and (5.45) we can also determine that the angular momentum of the configuration scales as $L \sim N_v Q$ as expected from (5.35). Consequently, the energy is that of a rigid body with angular momentum L and is given by:

$$\Delta E = \frac{L^2}{2I}\,, \tag{5.48}$$

Notice that this leading order result is independent of the trap and the inhomogeneity of the density. Corrections will arise from the inhomogeneity of the trap and the discreteness of the vortices, but they are subleading in N_v and suppressed in R_{TF} [180]. Indeed, that there are terms in the energy which scale as N_v being neglected is visible in (5.36).

We remark that there are constraints of the vortex density of the system. The vortex spacing λ should be larger than the vortex core size i.e. $\lambda \gg a \sim \frac{1}{\sqrt{\mu_{eff}}}$. Beyond this limit we expect interactions to be strong and the EFT description to break down [185]. Now, in a scenario where we have multiple vortices, a rough estimation yields that

$$N_v \sim \frac{R_{TF}^2}{\lambda^2} \sim \begin{cases} \sqrt{Q\ell} & d=2 \\ (Q\ell)^{1/3} & d=3 \end{cases} \tag{5.49}$$

where ℓ is the typical angular momentum of a vortex in the multivortex configuration. Thus in $d = 2$, the maximum angular momentum configuration that one can reach within the validity of the EFT corresponds to a maximum density of $N_v \sim Q$. Physically this means most of the vortices are near the center and $\ell \sim Q$ and the total angular momentum $L \sim Q^2$. For $d = 3$ this corresponds to $N_v \sim Q^{2/3}$ which is less than Q because the vortices are extended objects and the total angular momentum amounts to $L \sim Q^{5/3}$. But our EFT breaks down before this. Using particle vortex duality as in 5.8, one can see that the EFT breaks down when the electric field becomes comparable to magnetic field. This means that the EFT breaks down when the contribution coming from rigid body rotation becomes comparable to Δ_Q. Hence, the maximum angular momentum that can be attained within the validity of our EFT is $L \sim Q^{3/2}$ in $d = 2$ and $L \sim Q^{4/3}$ in $d = 3$.

These determine the absolute limits on the angular momentum accessible within our EFT and together with (5.48) and (5.47) imply the following operator dimension scaling:

$$d = 2 \quad \Delta_{Q,L} = \sqrt{\frac{9c_0\pi}{2}} \left(\frac{L^2}{Q^{3/2}}\right) + \Delta_Q \qquad Q < L < Q^{3/2} \tag{5.50}$$

$$d = 3 \quad \Delta_{Q,L} = \frac{1024}{25} \left(\frac{32c_0^2}{25\pi^4}\right)^{1/6} \left(\frac{L^2}{Q^{4/3}}\right) + \Delta_Q \qquad Q < L < Q^{4/3} \tag{5.51}$$

The above constitute the main results of this section.

5.7 Conclusions and Future Directions

To summarize, we have calculated how the dimensions of operators in NRCFTs scale with number charge Q and spin L in the limit of $Q \gg 1$ via the state-operator correspondence. The NRCFTs under consideration exist in $d = 2$ and $d = 3$ and by assumption are described by the superfluid EFT. This allows for explicit calculations by studying phonon and vortex configurations of the superfluid. We expect applicability of our result to "fermions at unitarity" and certain conformal anyon theories, as well any other NRCFT with this symmetry breaking behavior in its large charge sector[146, 158, 157, 159]. In fact the superfluid state of unitary fermions in a harmonic trap has been experimentally observed, including the formation of vortices [173]. It may be possible to verify these scalings in specific models such as those described in [158, 157] from the quantum mechanics of anyons in a harmonic trap.[186]

The most direct extension of these results would be to go to beyond the leading order scaling. To do so would require reasoning about the divergences associated with the vortex core, the size and structure of which is entirely determined by UV physics. It should be possible to regulate such divergences by considering operators localized on the vortex. Such a procedure in the relativistic effective string theory was worked out in [187, 188] and the effective string theory of vortex lines in superfluids was explored in [189]. A similar analysis has also been applied to divergences of the superfluid EFT near R_{TF}, associated with the dilute regime of size δ [190].

It would be especially interesting to study other possible symmetry breaking patterns, such as those relevant for chiral superfluids [164]. As mentioned in this large angular momentum regime the vortices are arranged as a triangular lattice. Deformations of this vortex lattice are a novel excitation in this limit, known as 'Tkachenko modes' [191]. Presumably these excited states would correspond to a tower of low-lying operators above the operator which creates the vortex-lattice. However, treatment these modes and corrections to the results (5.50) and (5.51) would require us to think about a new EFT which captures the spontaneous breaking of spatial symmetry by the vortex lattice. This EFT has been worked out by [165] and may be adaptable to the Schrödinger invariant case in a trap. Especially interesting would be systems with a Fermi surface, however such a critical state must necessarily be a non-fermi liquid following the results of [147].

While our EFT is not valid at larger angular momentum[16], it is interesting to ask if there is an analog of the large spin expansion when $L \sim \Delta$ for NRCFTs [192]. The techniques for NRCFT bootstrap are not well developed, but see ref [135]. It is interesting to note that unlike in CFT, there is no unitary bound restricting $L \leq \Delta$ as spin can be treated as an internal degree of freedom.

Another interesting direction would be to consider correlation functions of charged spinning operators in these NRCFTs. The universal scaling of the 3-point function and higher are all explicitly calculable within this EFT, as was done for scalar charged operators in [177]. In relativistic CFTs this was worked out in [163, 129] for certain operators. We leave this and other questions for future work.

5.8 Appendix A: Particle-Vortex Duality

Here we briefly review the particle vortex duality in nonrelativistic set up. The aim of the appendix is to cast the vortex dynamics in terms of an electrostatic (in $d = 3$ the gauge field is 2

[16]A similar issue occurs in this approach to CFTs [163]

form field, hence we coin the term "gaugostatic") problem, leveraging the duality. The idea is to solve the gaugostatic problem to figure out the field strength, which in turn gives us the velocity profile of the vortex, again using the dictionary of duality.

We consider the leading order superfluid Lagrangian in the presence of a potential $A_0 = \frac{1}{2}\omega^2 r^2, A_i = 0$:

$$\mathcal{L} = c_0 X^{\frac{d+2}{2}} \equiv P(X) \qquad X \equiv \partial_0\chi - A_0 - \frac{1}{2}(\partial_i\chi)^2 \tag{5.52}$$

The number density and superfluid velocity are defined respectively as:

$$n = \frac{\partial L}{\partial\dot{\chi}} = c_0\left(\frac{d}{2}+1\right)X^{\frac{d}{2}} \qquad v_i = -\partial_i\chi \tag{5.53}$$

The action (5.52) has a $U(1)$ symmetry of $\chi \to \chi + c$ whose current can be written as:

$$j^\mu = \left(n, nv^i\right) \tag{5.54}$$

For simplicity and physical relevance, we'll focus on the cases of $d = 2$ and $d = 3$. In $d = 2$, we can define:

$$j^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu a_\rho = \frac{1}{2}\varepsilon^{\mu\nu\rho} f_{\nu\rho} \tag{5.55}$$

for a one-form gauge field a_μ and field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. This relates the superfluid variables (5.53) to the dual electric and magnetic fields as:

$$n = \varepsilon^{ij}\partial_i a_j \equiv b \qquad v^i = \frac{\varepsilon^{ij} f_{0j}}{b} \equiv \frac{\varepsilon^{ij} e_j}{b} \tag{5.56}$$

Similarly, in $d = 3$ we'll define the current in terms of a dual two-form gauge field $B_{\mu\nu}$

$$j^\mu = \varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} = \frac{1}{3}\varepsilon^{\mu\nu\rho\sigma} G_{\nu\rho\sigma} \tag{5.57}$$

where $G_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the three-form field strength. The superfluid variables are then expressible as:

$$n = \varepsilon^{ijk}\partial_i B_{jk} \equiv Y \qquad v^i = \frac{1}{3}\frac{\varepsilon^{ijk}G_{0jk}}{Y} \tag{5.58}$$

Vortices act as sources for the gauge fields and couple minimally as:

$$d = 2: \quad J^V_\mu a^\mu \qquad d = 3: \quad \frac{1}{4}J^V_{\mu\nu}B^{\mu\nu} \tag{5.59}$$

To implement the duality transformation, we note that internal energy $\varepsilon(n)$ is given by $nX - P(X)$ and so we can rewrite the (5.52) as

$$\mathcal{L} = nX - \varepsilon(n) = n\left(\dot{\chi} - A_0 - \frac{1}{2}(\partial_i\chi)(\partial^i\chi)\right) - \varepsilon(n) \tag{5.60}$$

$$= \frac{1}{2}nv^2 - \varepsilon(n) + n\left(\dot{\chi} + v^i\partial_i\chi\right) \tag{5.61}$$

where we have used $v_i = -\partial_i\chi$ and n is understood as a function of χ and its derivatives.

The internal energy is given by:

$$d = 2: \quad \varepsilon(n) = \frac{1}{4c_0}n^2 \qquad d = 3: \quad \varepsilon(n) = \frac{3}{5}\left(\frac{2}{5c_0}\right)^{\frac{2}{3}} n^{\frac{5}{3}} \tag{5.62}$$

Using the relation (5.56) we can express the Lagrangian in $d = 2$ as:

$$\mathcal{L} = \frac{1}{2}\frac{e^2}{b} - \frac{1}{4c_0}b^2 - bA_0 \tag{5.63}$$

This equation describes a kind of non-linear electrodynamics with a modified Gauss law:

$$\partial_i\left(\frac{e^i}{b}\right) = J^V_0 \tag{5.64}$$

Similarly the Lagrangian in $d=3$ is given via (5.58) as:

$$\mathcal{L}=\frac{1}{9}\frac{G_{0ij}{G_0}^{ij}}{Y}-\frac{3}{5}\left(\frac{2}{5c_0}\right)^{\frac{2}{3}}Y^{\frac{5}{3}}-YA_0 \tag{5.65}$$

with a "Gauss law" of:

$$\partial^i\left(\frac{G_{0ij}}{Y}\right)=J^V_{j0} \tag{5.66}$$

Now consider a motion of charged particle under the gauge field, sourced by J^V . In what follows, we will show that to leading order we can treat this as a "gaugostatic" problem and the velocity V_i of the charged particle is negligible. If V_i is negligible, one can potentially drop the kinetic term in the Lagrangian. As a result, the equation of motion for the particle turns out to be the one where there is no Lorentz force acting on the particle. This implies that V_i is of the same order as $|e|/b$ (in $d=3$, this is $\frac{\sqrt{G_{0ij}{G_0}^{ij}}}{Y}$). For self consistency, we need to ensure V_i is very small i.e. the ratio $|e|/b$ is very small. This helps us to render the problem of vortex dynamics into a problem of "gaugostatics". In order to do that, we linearize (5.63) and (5.65) around parametrically large magnetic field b and Y and we see that the coupling goes as b in $d=2$ and in $d=3$, this goes like Y. Hence the electric field strength $|e|$ in $d=2$ and $\sqrt{G_{0ij}{G_0}^{ij}}$ in $d=3$ goes like $\sqrt{b}$ and $\sqrt{Y}$ respectively and we have

$$\sqrt{V_iV^i}\sim\frac{|e|}{b}\sim\frac{1}{\sqrt{Q}},\text{in}\, d=2 \tag{5.67}$$

$$\sqrt{V_iV^i}\sim\frac{\sqrt{G_{0ij}{G_0}^{ij}}}{Y}\sim\frac{1}{\sqrt{Q}},\text{in}\, d=3 \tag{5.68}$$

Thus it is self consistent to assume that the charged particle is just drifting without any Lorentz force acting on it.

5.9 Appendix B: A Contour integral

This appendix contains the evaluation of contour integrals, needed to figure out the vortex interaction energy in the multivortex scenario. In $d = 2$, the vortex interaction energy goes like

$$\int \mathrm{d}r\, rn(r) \int \mathrm{d}\theta\, \vec{v}_i \cdot \vec{v}_j \tag{5.69}$$

where as for $d = 3$, we have an extra integral along the z axis and r becomes the radius in cylindrical coordinate. In both cases, the θ integral can be done using contour integral and expressing $\vec{v}_i$ in terms of complex variables given by

$$v_i = \frac{i}{\bar{z} - \bar{z}_i}\,,\, v_i^* = \frac{-i}{z - z_i} \tag{5.70}$$

Hence the integral evaluates to

$$I = \int \mathrm{d}\theta\, \vec{v}_i \cdot \vec{v}_j = Re\left(\int \mathrm{d}z\, \frac{-i}{z} v_i v_j^*\right) \tag{5.71}$$

$$= Re\left(\int \mathrm{d}z\, \frac{-i}{z} \frac{1}{(\bar{z} - \bar{z}_i)(z - z_j)}\right) \tag{5.72}$$

Now we note that $z\bar{z} = r^2$ and $z_i\bar{z}_i = R_i^2$ to rewrite the integral in following manner:

$$I = Re\left(\int \mathrm{d}z\, -i\frac{z_i}{(r^2 z_i - R_i^2 z)(z - z_j)}\right) = Re\left(\int \mathrm{d}z\, \frac{-i}{-R_i^2} \frac{z_i}{\left(z - \frac{r^2}{R_i^2} z_i\right)(z - z_j)}\right) \tag{5.73}$$

The poles are located at $z = z_j\,, z = \frac{r^2}{R_i^2} z_i$ i.e. they lie on the circle of radius $|z_j| = R_j$ and $|\frac{r^2}{R_i^2} z_i| = \frac{r^2}{R_i}$.

Without loss of generality, we consider $R_i < R_j$. Now there can be three scenarios:

1. $r < R_i < R_j$ implies $\frac{r^2}{R_i} < r < R_j$, hence the pole at $z = \frac{r^2}{R_i^2} z_i$ is picked, answer is

$$I = Re\left(-2\pi \frac{1}{r^2 - z_j z_i^*}\right) = -\frac{2\pi}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left(r^2 - R_i R_j \cos(\phi)\right) \tag{5.74}$$

2. $R_i < R_j < r$ implies $R_j < r < \frac{r^2}{R_i}$, hence the pole at $z = z_j$ is picked. and the answer is

$$I = \frac{2\pi}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left(r^2 - R_i R_j \cos(\phi)\right) \tag{5.75}$$

3. $R_i < r < R_j$ implies $r < R_j$ and $r < \frac{r^2}{R_i}$, so none of the poles is picked, the answer is 0.

Summing up we can write

$$I = \frac{\pi \left(r^2 - R_i R_j \cos(\phi)\right)}{r^4 + R_j^2 R_i^2 - 2r^2 R_i R_j \cos(\phi)} \left[sgn(r - R_i) + sgn(r - R_j)\right] \tag{5.76}$$

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Bibliography

[1] Chong Wang and T. Senthil. Boson topological insulators: A window into highly entangled quantum phases. *Phys. Rev. B*, 87:235122, Jun 2013.

[2] Holger Bech Nielsen and M. Ninomiya. Absence of Neutrinos on a Lattice. 1. Proof by Homotopy Theory. *Nucl.Phys.*, B185:20, 1981.

[3] Holger Bech Nielsen and M. Ninomiya. Absence of Neutrinos on a Lattice. 2. Intuitive Topological Proof. *Nucl.Phys.*, B193:173, 1981.

[4] Holger Bech Nielsen and M. Ninomiya. No Go Theorem for Regularizing Chiral Fermions. *Phys.Lett.*, B105:219, 1981.

[5] D. Friedan. A PROOF OF THE NIELSEN-NINOMIYA THEOREM. *Commun.Math.Phys.*, 85:481–490, 1982.

[6] Xiao-Gang Wen. *Quantum Field Theory of Many-Body Systems*. Oxford Univ. Press, Oxford, 2004.

[7] Xiao-Gang Wen. Topological order: from long-range entangled quantum matter to an unification of light and electrons. 2012.

[8] Michael Levin and Xiao-Gang Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96(11):110405, Mar 2006.

[9] Alexei Kitaev and John Preskill. Topological entanglement entropy. *Phys. Rev. Lett.*, 96(11):110404, Mar 2006.

[10] T. Grover, A. M. Turner, and A. Vishwanath. Entanglement entropy of gapped phases and topological order in three dimensions. *Phys. Rev.B*, 84(19):195120, November 2011.

[11] Ari M. Turner and Ashvin Vishwanath. Beyond Band Insulators: Topology of Semi-metals and Interacting Phases. 2013.

[12] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2 – 111, 2006.

[13] Alexei Kitaev. Toward topological classification of phases with short-range entanglement. *unpublished*, 2011.

[14] Yuan-Ming Lu and Ashvin Vishwanath. Theory and classification of interacting 'integer' topological phases in two dimensions: A Chern-Simons approach. *Phys.Rev.*, B86:125119, 2012.

[15] Jay Sau, Brian Swingle, and T. Senthil. unpublished. 2013.

[16] Alexei Kitaev. Periodic table for topological insulators and superconductors. *AIP Conf.Proc.*, 1134:22–30, 2009.

[17] Shinsei Ryu, Andreas P. Schnyder, Akira Furusaki, and Andreas W.W. Ludwig. Topological insulators and superconductors: Tenfold way and dimensional hierarchy. *New J.Phys.*, 12:065010, 2010.

[18] L. Fidkowski and A. Kitaev. Effects of interactions on the topological classification of free fermion systems. *Phys. Rev. B*, 81(13):134509, April 2010.

[19] T. Senthil and Michael Levin. Integer quantum Hall effect for bosons: A physical realization. *Phys.Rev.Lett.*, 110:046801, 2013.

[20] Ashvin Vishwanath and T. Senthil. Physics of three dimensional bosonic topological insulators: Surface Deconfined Criticality and Quantized Magnetoelectric Effect. *Phys. Rev. X 3,*, 011016, 2013.

[21] Chong Wang and T. Senthil. Boson topological insulators: A window into highly entangled quantum phases. 2013.

[22] Cenke Xu and T. Senthil. Wave Functions of Bosonic Symmetry Protected Topological Phases. 2013.

[23] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen. Symmetry protected topological orders and the group cohomology of their symmetry group. *ArXiv e-prints*, June 2011.

[24] Max A. Metlitski, C.L. Kane, and Matthew P. A. Fisher. Bosonic topological insulator in three dimensions and the statistical Witten effect. 2013.

[25] F.J. Burnell, Xie Chen, Lukasz Fidkowski, and Ashvin Vishwanath. Exactly Soluble Model of a 3D Symmetry Protected Topological Phase of Bosons with Surface Topological Order. 2013.

[26] Xiao-Gang Wen. Classifying gauge anomalies through SPT orders and classifying anomalies through topological orders. 2013.

[27] Jr. Callan, Curtis G. and Jeffrey A. Harvey. Anomalies and Fermion Zero Modes on Strings and Domain Walls. *Nucl.Phys.*, B250:427, 1985.

[28] David B. Kaplan. A Method for simulating chiral fermions on the lattice. *Phys.Lett.*, B288:342–347, 1992.

[29] David B. Kaplan. Chiral Symmetry and Lattice Fermions. 2009.

[30] Xiao-Liang Qi, Taylor Hughes, and Shou-Cheng Zhang. Topological Field Theory of Time-Reversal Invariant Insulators. *Phys.Rev.*, B78:195424, 2008.

[31] Stanley Deser, R. Jackiw, and S. Templeton. Topologically Massive Gauge Theories. *Annals Phys.*, 140:372–411, 1982.

[32] Albert S. Schwarz. The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants. *Lett.Math.Phys.*, 2:247–252, 1978.

[33] Albert S. Schwarz. The Partition Function of a Degenerate Functional. *Commun.Math.Phys.*, 67:1–16, 1979.

[34] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun.Math.Phys.*, 121:351, 1989.

[35] Shmuel Elitzur, Gregory W. Moore, Adam Schwimmer, and Nathan Seiberg. Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory. *Nucl.Phys.*, B326:108, 1989.

[36] Gary T. Horowitz. Exactly Soluble Diffeomorphism Invariant Theories. *Commun.Math.Phys.*, 125:417, 1989.

[37] Matthias Blau and George Thompson. Topological Gauge Theories of Antisymmetric Tensor Fields. *Annals Phys.*, 205:130–172, 1991.

[38] Gary T. Horowitz and Mark Srednicki. A QUANTUM FIELD THEORETIC DESCRIPTION OF LINKING NUMBERS AND THEIR GENERALIZATION. *Commun.Math.Phys.*, 130:83, 1990.

[39] Edward Witten. AdS / CFT correspondence and topological field theory. *JHEP*, 9812:012, 1998.

[40] Gregory W. Moore. Anomalies, Gauss laws, and Page charges in M-theory. *Comptes Rendus Physique*, 6:251–259, 2005.

[41] Dmitriy Belov and Gregory W. Moore. Conformal blocks for AdS(5) singletons. 2004.

[42] Sean A. Hartnoll. Anyonic strings and membranes in AdS space and dual Aharonov-Bohm effects. *Phys.Rev.Lett.*, 98:111601, 2007.

[43] Greg Moore. Minicourse of three lectures on Generalized Abelian Gauge Theories, Self-Duality, and Differential Cohomology, at the Simons Center Workshop on Differential Cohomology, Simons Center for Geometry and Physics, Stonybrook. 2011.

[44] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. Heisenberg Groups and Noncommutative Fluxes. *Annals Phys.*, 322:236–285, 2007.

[45] David J. Gross and Hirosi Ooguri. Aspects of large N gauge theory dynamics as seen by string theory. *Phys.Rev.*, D58:106002, 1998.

[46] Edward Witten. Baryons and branes in anti-de Sitter space. *JHEP*, 9807:006, 1998.

[47] S. Shatashvili. unpublished. 90s.

[48] Edward Witten. On flux quantization in M theory and the effective action. *J.Geom.Phys.*, 22:1–13, 1997.

[49] Juan Martin Maldacena, Gregory W. Moore, and Nathan Seiberg. D-brane charges in five-brane backgrounds, Appendix A. *JHEP*, 0110:005, 2001.

[50] Dmitriy Belov and Gregory W. Moore. Classification of Abelian spin Chern-Simons theories. 2005.

[51] Dmitriy Belov and Gregory W. Moore. Holographic Action for the Self-Dual Field. 2006.

[52] T. H. Hansson, V. Oganesyan, and S. L. Sondhi. Superconductors are topologically ordered. *Annals of Physics*, 313:497–538, October 2004.

[53] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Univ. Press, Oxford, 1990.

[54] S.M. Kravec, John McGreevy, and Brian Swingle. work in progress. 2014.

[55] C.L. Kane, Matthew P.A. Fisher, and J. Polchinski. Randomness at the edge: Theory of quantum Hall transport at filling nu = 2/3. *Phys.Rev.Lett.*, 72:4129–4132, 1994.

[56] C.L. Kane and Matthew P.A. Fisher. Edge state transport. 1994.

[57] John H. Schwarz and Ashoke Sen. Duality symmetric actions. *Nucl.Phys.*, B411:35–63, 1994.

[58] Michael A. Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B*, 71(4):045110, Jan 2005.

[59] Davide Gaiotto. N=2 dualities. *JHEP*, 1208:034, 2012.

[60] M. Levin and Z.-C. Gu. Braiding statistics approach to symmetry-protected topological phases. *Phys. Rev. B*, 86(11):115109, September 2012.

[61] Luis Alvarez-Gaume and Edward Witten. Gravitational Anomalies. *Nucl. Phys.*, B234:269, 1984.

[62] S. Deser. No local Maxwell duality invariance. *Class.Quant.Grav.*, 28:085009, 2011.

[63] Claudio Bunster and Marc Henneaux. Sp(2n,R) electric-magnetic duality as off-shell symmetry of interacting electromagnetic and scalar fields. *PoS*, HRMS2010:028, 2010.

[64] Alberto Saa. Local electromagnetic duality and gauge invariance. *Class.Quant.Grav.*, 28:127002, 2011.

[65] M. Barkeshli and X.-G. Wen. Effective field theory and projective construction for Z_k parafermion fractional quantum Hall states. *Phys. Rev. B*, 81(15):155302, April 2010.

[66] C. Wang, A. C. Potter, and T. Senthil. Classification of Interacting Electronic Topological Insulators in Three Dimensions. *Science*, 343:629–631, February 2014.

[67] X.-G. Wen and A. Zee. Classification of abelian quantum hall states and matrix formulation of topoological fluids. *Phys. Rev. B*, 46:2290–2301, 1992.

[68] Ashvin Vishwanath and T. Senthil. Physics of three-dimensional bosonic topological insulators: Surface-deconfined criticality and quantized magnetoelectric effect. *Phys. Rev. X*, 3:011016, Feb 2013.

[69] E. Witten. Fermion Path Integrals And Topological Phases. *ArXiv e-prints*, August 2015.

[70] Alexei Kitaev. On the classification of short-range entangled states. *unpublished*, 2013.

[71] Anton Kapustin. Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology. 2014.

[72] A Yu Kitaev. Unpaired majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131, 2001.

[73] T. Senthil. Symmetry Protected Topological phases of Quantum Matter. 2014.

[74] S. M. Kravec and John McGreevy. Gauge theory generalization of the fermion doubling theorem. *Phys. Rev. Lett.*, 111:161603, Oct 2013.

[75] Gerard 't Hooft. Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking. *NATO Adv.Study Inst.Ser.B Phys.*, 59:135, 1980.

[76] Juven Wang, Luiz H. Santos, and Xiao-Gang Wen. Bosonic Anomaly, Induced Fractional Quantum Number and Degenerate Zero Modes - the anomalous edge physics of Symmetry Protected Topological States. 2014.

[77] Anton Kapustin. Bosonic Topological Insulators and Paramagnets: a view from cobordisms. 2014.

[78] Anton Kapustin and Ryan Thorngren. Anomalies of discrete symmetries in various dimensions and group cohomology. 2014.

[79] Anton Kapustin and Ryan Thorngren. Anomalies of discrete symmetries in three dimensions and group cohomology. 2014.

[80] Chong Wang, Andrew C. Potter, and T. Senthil. Gapped symmetry preserving surface state for the electron topological insulator. *Phys. Rev. B*, 88:115137, Sep 2013.

[81] Parsa Bonderson, Chetan Nayak, and Xiao-Liang Qi. A time-reversal invariant topological phase at the surface of a 3D topological insulator. *J.Stat.Mech.*, 2013:P09016, 2013.

[82] L. Fidkowski, X. Chen, and A. Vishwanath. Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model. *Physical Review X*, 3(4):041016, October 2013.

[83] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski. Anomalous Symmetry Fractionalization and Surface Topological Order. *ArXiv e-prints*, March 2014.

[84] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher. A symmetry-respecting topologically-ordered surface phase of 3d electron topological insulators. *ArXiv e-prints*, June 2013.

[85] Andrea Amoretti, Alessandro Braggio, Giacomo Caruso, Nicola Maggiore, and Nicodemo Magnoli. Holography in flat spacetime: 4D theories and electromagnetic duality on the border. *JHEP*, 1404:142, 2014.

[86] Kenneth G. Wilson. Confinement of Quarks. *Phys.Rev.*, D10:2445–2459, 1974.

[87] A. Yu. Kitaev. Fault tolerant quantum computation by anyons. *Annals Phys.*, 303:2–30, 2003.

[88] Michael Levin and Xiao-Gang Wen. Quantum ether: photons and electrons from a rotor model. *Phys.Rev.*, B73:035122, 2006.

[89] Ryan Thorngren. Framed Wilson Operators on the Boundaries of Novel SPT Phases. 2014.

[90] B. Swingle and J. McGreevy. Renormalization group constructions of topological quantum liquids and beyond. *ArXiv e-prints*, July 2014.

[91] Cenke Xu and T. Senthil. Wave functions of bosonic symmetry protected topological phases. *Phys. Rev. B*, 87:174412, May 2013.

[92] R. Jackiw and C. Rebbi. Spin from Isospin in a Gauge Theory. *Phys.Rev.Lett.*, 36:1116, 1976.

[93] R. Jackiw and C. Rebbi. Solitons with Fermion Number 1/2. *Phys. Rev.*, D13:3398–3409, 1976.

[94] Edward Witten. Dyons of Charge $\frac{e\theta}{2\pi}$. *Phys.Lett.*, B86:283–287, 1979.

[95] Joseph Maciejko, Xiao-Liang Qi, Andreas Karch, and Shou-Cheng Zhang. Fractional topological insulators in three dimensions. *Phys.Rev.Lett.*, 105:246809, 2010.

[96] Brian Swingle, Maissam Barkeshli, John McGreevy, and T. Senthil. Correlated Topological Insulators and the Fractional Magnetoelectric Effect. *Phys.Rev.*, B83:195139, 2011.

[97] Edward Witten. On S duality in Abelian gauge theory. *Selecta Math.*, 1:383, 1995.

[98] John W Milnor and James D Stasheff. *Characteristic classes,* volume 76 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1974.

[99] Ofer Aharony, Nathan Seiberg, and Yuji Tachikawa. Reading between the lines of four-dimensional gauge theories. *JHEP*, 1308:115, 2013.

[100] Cumrun Vafa and Edward Witten. A Strong coupling test of S duality. *Nucl.Phys.*, B431:3–77, 1994.

[101] J. T. Chalker and P. D. Coddington. Percolation, quantum tunnelling and the integer Hall effect. *Journal of Physics C Solid State Physics*, 21:2665–2679, May 1988.

[102] P. Hosur, S. Ryu, and A. Vishwanath. Chiral topological insulators, superconductors, and other competing orders in three dimensions. *Phys. Rev. B*, 81(4):045120, January 2010.

[103] T. Neupert, C. Chamon, C. Mudry, and R. Thomale. Wire deconstructionism and classification of topological phases. *ArXiv e-prints*, March 2014.

[104] Max A. Metlitski, C. L. Kane, and Matthew P. A. Fisher. Bosonic topological insulator in three dimensions and the statistical witten effect. *Phys. Rev. B*, 88:035131, Jul 2013.

[105] Nima Arkani-Hamed, Andrew G. Cohen, and Howard Georgi. (De)constructing dimensions. *Phys.Rev.Lett.*, 86:4757–4761, 2001.

[106] Xiao-Gang Wen. Quantum field theory of many-body systems. 2004.

[107] Tom Banks, R. Myerson, and John B. Kogut. Phase Transitions in Abelian Lattice Gauge Theories. *Nucl.Phys.*, B129:493, 1977.

[108] Gerard 't Hooft. On the Phase Transition Towards Permanent Quark Confinement. *Nucl.Phys.*, B138:1, 1978.

[109] Gerard 't Hooft. A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories. *Nucl.Phys.*, B153:141, 1979.

[110] Gerard 't Hooft. Topology of the Gauge Condition and New Confinement Phases in Nonabelian Gauge Theories. *Nucl.Phys.*, B190:455, 1981.

[111] Edward Witten. An SU(2) anomaly. *Phys. Lett.*, B117:324–328, 1982.

[112] John McGreevy and Brian Swingle. Non-Abelian statistics versus the Witten anomaly. *Phys.Rev.*, D84:065019, 2011.

[113] Jeffrey Goldstone. unpublished. 1984.

[114] S. Elitzur and V. P. Nair. NONPERTURBATIVE ANOMALIES IN HIGHER DIMENSIONS. *Nucl. Phys.*, B243:205, 1984.

[115] S. P. de Alwis. ON THE RELATION BETWEEN GLOBAL AND U(1) ANOMALIES. *Phys. Rev.*, D32:2837, 1985.

[116] Frans R. Klinkhamer. Another look at the SU(2) anomaly. *Phys. Lett.*, B256:41–42, 1991.

[117] Karl-Henning Rehren. BRAID GROUP STATISTICS AND THEIR SUPERSELECTION RULES. *in* The algebraic theory of superselection sectors*, D. Kastler (ed.) Proceedings Palermo 1989*, 1989.

[118] J. Frohlich and F. Gabbiani. Braid statistics in local quantum theory. *Rev.Math.Phys.*, 2:251–354, 1991.

[119] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath. Interaction effects on 3D topological superconductors: surface topological order from vortex condensation, the 16 fold way and fermionic Kramers doublets. *ArXiv e-prints*, June 2014.

[120] David B. Kaplan and Sichun Sun. Spacetime as a topological insulator: Mechanism for the origin of the fermion generations. *Phys.Rev.Lett.*, 108:181807, 2012.

[121] Xiao-Gang Wen. A lattice non-perturbative definition of an $SO(10)$ chiral gauge theory and its induced standard model. *Chin.Phys.Lett.*, 30:111101, 2013.

[122] Y.-Z. You, Y. BenTov, and C. Xu. Interacting Topological Superconductors and possible Origin of $16n$ Chiral Fermions in the Standard Model. *ArXiv e-prints*, February 2014.

[123] Franz J. Wegner. Duality in generalized ising models and phase transitions without local order parameters. *Journal of Mathematical Physics*, 12(10):2259–2272, 1971.

[124] T. Senthil and O. Motrunich. Microscopic models for fractionalized phases in strongly correlated systems. *Phys. Rev. B*, 66:205104, Nov 2002.

[125] C. W. von Keyserlingk, F. J. Burnell, and S. H. Simon. Three-dimensional topological lattice models with surface anyons. *Phys. Rev. B*, 87(4):045107, January 2013.

[126] Riccardo Rattazzi, Vyacheslav S. Rychkov, Erik Tonni, and Alessandro Vichi. Bounding scalar operator dimensions in 4D CFT. *JHEP*, 12:031, 2008.

[127] Simeon Hellerman, Domenico Orlando, Susanne Reffert, and Masataka Watanabe. On the cft operator spectrum at large global charge. *Journal of High Energy Physics*, 2015(12):1–34, 2015.

[128] Simeon Hellerman, Nozomu Kobayashi, Shunsuke Maeda, and Masataka Watanabe. A note on inhomogeneous ground states at large global charge. *arXiv preprint arXiv:1705.05825*, 2017.

[129] Alexander Monin, David Pirtskhalava, Riccardo Rattazzi, and Fiona K Seibold. Semiclassics, goldstone bosons and cft data. *Journal of High Energy Physics*, 2017(6):11, 2017.

[130] Debasish Banerjee, Shailesh Chandrasekharan, and Domenico Orlando. Conformal dimensions via large charge expansion. *Physical review letters*, 120(6):061603, 2018.

[131] Anton de la Fuente. The large charge expansion at large n. *arXiv preprint arXiv:1805.00501*, 2018.

[132] Baur Mukhametzhanov and Alexander Zhiboedov. Analytic Euclidean Bootstrap. 2018.

[133] Thomas Mehen, Iain W. Stewart, and Mark B. Wise. Conformal invariance for nonrelativistic field theory. *Phys. Lett.*, B474:145–152, 2000.

[134] Yusuke Nishida and Dam Thanh Son. Unitary Fermi gas, epsilon expansion, and nonrelativistic conformal field theories. *Lect. Notes Phys.*, 836:233–275, 2012.

[135] Walter D Goldberger, Zuhair U Khandker, and Siddharth Prabhu. Ope convergence in non-relativistic conformal field theories. *Journal of High Energy Physics*, 2015(12):1–31, 2015.

[136] Siavash Golkar and Dam T. Son. Operator Product Expansion and Conservation Laws in Non-Relativistic Conformal Field Theories. *JHEP*, 12:063, 2014.

[137] Sridip Pal. Unitarity and universality in nonrelativistic conformal field theory. *Phys. Rev.*, D97(10):105031, 2018.

[138] C. A. Regal, M. Greiner, and D. S. Jin. Observation of Resonance Condensation of Fermionic Atom Pairs. *Phys. Rev. Lett.*, 92:040403, 2004.

[139] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle. Condensation of Pairs of Fermionic Atoms near a Feshbach Resonance. *Phys. Rev. Lett.*, 92:120403, 2004.

[140] Xiao Chen, Eduardo Fradkin, and William Witczak-Krempa. Gapless quantum spin chains: multiple dynamics and conformal wavefunctions. *J. Phys.*, A50(46):464002, 2017.

[141] David B. Kaplan, Martin J. Savage, and Mark B. Wise. A New expansion for nucleon-nucleon interactions. *Phys. Lett.*, B424:390–396, 1998.

[142] David B. Kaplan, Martin J. Savage, and Mark B. Wise. Two nucleon systems from effective field theory. *Nucl. Phys.*, B534:329–355, 1998.

[143] C. Chin, V. Vuleti, A. J. Kerman, and S. Chu. High precision Feshbach spectroscopy of ultracold cesium collisions. *Nucl. Phys.*, A684:641–645, 2001.

[144] J. L. Roberts, N. R. Claussen, James P. Burke, Chris H. Greene, E. A. Cornell, and C. E. Wieman. Resonant Magnetic Field Control of Elastic Scattering in Cold R-85b. *Phys. Rev. Lett.*, 81:5109–5112, 1998.

[145] T Loftus, CA Regal, C Ticknor, JL Bohn, and Deborah S Jin. Resonant control of elastic collisions in an optically trapped fermi gas of atoms. *Physical review letters*, 88(17):173201, 2002.

[146] Yusuke Nishida and Dam T Son. Nonrelativistic conformal field theories. *Physical Review D*, 76(8):086004, 2007.

[147] Ira Z Rothstein and Prashant Shrivastava. Symmetry realization via a dynamical inverse higgs mechanism. *Journal of High Energy Physics*, 2018(5):14, 2018.

[148] DT Son and M Wingate. General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary fermi gas. *Annals of Physics*, 321(1):197–224, 2006.

[149] Sidney R. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. 1. *Phys. Rev.*, 177:2239–2247, 1969.

[150] Curtis G. Callan, Jr., Sidney R. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. 2. *Phys. Rev.*, 177:2247–2250, 1969.

[151] VI Ogievetsky. Nonlinear realizations of internal and space-time symmetries, in proceedings of x-th winter school of theoretical physics in karpacz. *Universitas Wratislaviensis, Wroclaw, Poland*, 1974.

[152] Luca V Delacrétaz, Solomon Endlich, Alexander Monin, Riccardo Penco, and Francesco Riva. (re-) inventing the relativistic wheel: gravity, cosets, and spinning objects. *Journal of High Energy Physics*, 2014(11):8, 2014.

[153] Ian Low and Aneesh V Manohar. Spontaneously broken spacetime symmetries and goldstones theorem. *Physical review letters*, 88(10):101602, 2002.

[154] Alberto Nicolis, Riccardo Penco, Federico Piazza, and Rachel A Rosen. More on gapped goldstones at finite density: more gapped goldstones. *Journal of High Energy Physics*, 2013(11):55, 2013.

[155] Tomáš Brauner, Solomon Endlich, Alexander Monin, and Riccardo Penco. General coordinate invariance in quantum many-body systems. *Physical Review D*, 90(10):105016, 2014.

[156] Samuel Favrod, Domenico Orlando, and Susanne Reffert. The large-charge expansion for Schrödinger systems. 2018.

[157] Nima Doroud, David Tong, and Carl Turner. The conformal spectrum of non-abelian anyons. *SciPost Physics*, 4(4):022, 2018.

[158] Nima Doroud, David Tong, and Carl Turner. On superconformal anyons. *Journal of High Energy Physics*, 2016(1):138, 2016.

[159] R. Jackiw and So-Young Pi. Selfdual Chern-Simons solitons. *Prog. Theor. Phys. Suppl.*, 107:1–40, 1992. [,465(1991)].

[160] Koushik Balasubramanian and John McGreevy. Gravity duals for non-relativistic CFTs. *Phys. Rev. Lett.*, 101:061601, 2008.

[161] Dam T Son. Toward an ads/cold atoms correspondence: a geometric realization of the schroedinger symmetry. *Physical Review D*, 78(4):046003, 2008.

[162] Orestis Loukas, Domenico Orlando, Susanne Reffert, and Debajyoti Sarkar. An ads/eft correspondence at large charge. *arXiv preprint arXiv:1804.04151*, 2018.

[163] Gabriel Cuomo, Anton de la Fuente, Alexander Monin, David Pirtskhalava, and Riccardo Rattazzi. Rotating superfluids and spinning charged operators in conformal field theory. *Phys. Rev.*, D97(4):045012, 2018.

[164] Carlos Hoyos, Sergej Moroz, and Dam Thanh Son. Effective theory of chiral two-dimensional superfluids. *Phys. Rev.*, B89(17):174507, 2014.

[165] Sergej Moroz, Carlos Hoyos, Claudio Benzoni, and Dam Thanh Son. Effective field theory of a vortex lattice in a bosonic superfluid. 2018.

[166] Russell J Donnelly. *Quantized vortices in helium II*, volume 2. Cambridge University Press, 1991.

[167] SA Vitiello, L Reatto, GV Chester, and MH Kalos. Vortex line in superfluid he 4: A variational monte carlo calculation. *Physical Review B*, 54(2):1205, 1996.

[168] Gerardo Ortiz and David M Ceperley. Core structure of a vortex in superfluid he 4. *Physical review letters*, 75(25):4642, 1995.

[169] S Giorgini, J Boronat, and J Casulleras. Vortex excitation in superfluid 4 he: A diffusion monte carlo study. *Physical review letters*, 77(13):2754, 1996.

[170] Gordon Baym, Christopher Pethick, and David Pines. Superfluidity in neutron stars. *Nature*, 224(5220):673, 1969.

[171] Sean A Hartnoll, Andrew Lucas, and Subir Sachdev. Holographic quantum matter. *arXiv preprint arXiv:1612.07324*, 2016.

[172] Daniel Jafferis, Baur Mukhametzhanov, and Alexander Zhiboedov. Conformal Bootstrap At Large Charge. *JHEP*, 05:043, 2018.

[173] Martin W Zwierlein, Jamil R Abo-Shaeer, Andre Schirotzek, Christian H Schunck, and Wolfgang Ketterle. Vortices and superfluidity in a strongly interacting fermi gas. *Nature*, 435(7045):1047, 2005.

[174] Andrew Kobach and Sridip Pal. Conformal Structure of the Heavy Particle EFT Operator Basis. *Phys. Lett.*, B783:311–319, 2018.

[175] Malte Henkel and Jeremie Unterberger. Schrodinger invariance and space-time symmetries. *Nucl. Phys.*, B660:407–435, 2003.

[176] Keisuke Ohashi, Toshiaki Fujimori, and Muneto Nitta. Conformal symmetry of trapped Bose-Einstein condensates and massive Nambu-Goldstone modes. *Phys. Rev.*, A96(5):051601, 2017.

[177] S. M. Kravec and Sridip Pal. Nonrelativistic Conformal Field Theories in the Large Charge Sector. *JHEP*, 02:008, 2019.

[178] Christopher J Pethick and Henrik Smith. *Bose–Einstein condensation in dilute gases*. Cambridge university press, 2008.

[179] GM Bruun and L Viverit. Vortex state in superfluid trapped fermi gases at zero temperature. *Physical Review A*, 64(6):063606, 2001.

[180] Daniel E Sheehy and Leo Radzihovsky. Vortices in spatially inhomogeneous superfluids. *Physical Review A*, 70(6):063620, 2004.

[181] Andrew J Groszek, David M Paganin, Kristian Helmerson, and Tapio P Simula. Motion of vortices in inhomogeneous bose-einstein condensates. *Physical Review A*, 97(2):023617, 2018.

[182] Alexander L Fetter. Kelvin mode of a vortex in a nonuniform bose-einstein condensate. *Physical Review A*, 69(4):043617, 2004.

[183] VK Tkachenko. On vortex lattices. *Sov. Phys. JETP*, 22(6):1282–1286, 1966.

[184] LJ Campbell and Robert M Ziff. Vortex patterns and energies in a rotating superfluid. *Physical Review B*, 20(5):1886, 1979.

[185] Nigel R Cooper. Rapidly rotating atomic gases. *Advances in Physics*, 57(6):539–616, 2008.

[186] R Chitra and Diptiman Sen. Ground state of many anyons in a harmonic potential. *Physical Review B*, 46(17):10923, 1992.

[187] Simeon Hellerman and Ian Swanson. Boundary operators in effective string theory. *Journal of High Energy Physics*, 2017(4):85, 2017.

[188] Simeon Hellerman and Ian Swanson. String theory of the regge intercept. *Physical review letters*, 114(11):111601, 2015.

[189] Bart Horn, Alberto Nicolis, and Riccardo Penco. Effective string theory for vortex lines in fluids and superfluids. *Journal of High Energy Physics*, 2015(10):153, 2015.

[190] Simeon Hellerman. Private Communication, 2019.

[191] VK Tkachenko. Elasticity of vortex lattices. *Soviet Journal of Experimental and Theoretical Physics*, 29:945, 1969.

[192] Luis F Alday, Agnese Bissi, and Tomasz Lukowski. Large spin systematics in cft. *Journal of High Energy Physics*, 2015(11):101, 2015.