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by

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Professor Daniel I. Tataru, Chair
Professor Edgar Knobloch
Professor Maciej R. Zworski

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Abstract

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We consider the behavior of nonlinear KdV-type equations that admit quasilinear dynamics in the sense that the nonlinear flow cannot be simply treated as a perturbation of the linear flow, even for small initial data.

We treat two problems in particular. First we study the local dynamics of KdV-type equations with nonlinearities involving two spatial derivatives. A key obstruction to well-posedness arises from the Mizohata condition. This leads to an additional integrability requirement for the solution in the absence of a suitable null structure. In this case we prove local well-posedness for large, low-regularity data in translation-invariant spaces.

Second we explore the global dynamics of the modified Korteweg de-Vries equation. We establish modified asymptotic behavior without relying on the integrable structure of the equation. This approach has the advantage that it can be used for a wide class of short-range perturbations of the mKdV. To give a thorough description of the asymptotic behavior we prove an asymptotic completeness result that relates mKdV solutions to the 1-parameter family of solutions to the Painlevé II equation.

To Reesha.

Contents

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Chapter 1

Introduction

The Korteweg-de Vries equation (KdV),

$$u_t + \tfrac{1}{3}u_{xxx} = (u^2)_x, \tag{1.1}$$

is a $1+1$-dimensional model of long dispersive waves. It was derived in 1877 by Boussinesq [11], and again in 1895 by Korteweg and de Vries [91], as a model for the surface height of a canal. The KdV arises as an asymptotic limit of numerous dispersive systems and, together with its generalizations, has a wide range of physical applications including fluid mechanics, plasma physics and nonlinear optics.

Solutions to the corresponding linear equation,

$$u_t + \tfrac{1}{3}u_{xxx} = 0, \tag{1.2}$$

have the property that waves at different frequencies travel at different velocities. As a consequence, linear solutions tend to spread out or *disperse* leading to both pointwise and space-time averaged decay (see §1.2). For this reason we refer to the KdV as a *dispersive equation*. In order to study nonlinear equations we look to balance the linear dispersion against any potentially harmful nonlinear dynamics.

In this thesis we will be concerned with two different generalizations of the KdV equation. These generalizations have the common feature that solutions exhibit *quasilinear behavior* in the sense that nonlinear solutions cannot be treated simply as a perturbation of a solution to the linear equation, even for arbitrarily small initial data.

Derivative KdV-type equations. In Chapter 2, we consider the local well-posedness of equations of the form

$$u_t + \tfrac{1}{3}u_{xxx} = F(u, u_x, u_{xx}), \tag{1.3}$$

with initial data in low regularity spaces. This type of model arises in the context of wave propagation in elastic media [82, 92].

For linear KdV solutions, the rough high frequencies travel faster than the smooth low frequencies. This leads to a local smoothing effect, first observed by Kato [68]. The key difficulty in proving local well-posedness for (1.3) is then to obtain enough smoothing from the linear operator to compensate for two derivatives falling at high frequency in the nonlinearity. An obstruction arises from the *Mizohata condition*: for a linear equation of the form

$$u_t + \tfrac{1}{3}u_{xxx} + a(x)u_{xx} = f \tag{1.4}$$

to be well-posed in Sobolev spaces, the coefficient a must satisfy an additional L^1-type integrability condition (see §1.3). One approach to this problem is to consider spatially localized initial data. However, as spatial translation is a symmetry of (1.3), it is more natural to consider initial data in translation-invariant spaces.

In Chapter 2 we prove local well-posedness for (1.3) by imposing a translation-invariant l^1-type summability condition on the initial data.[1] Further, in the case that the nonlinearity does not contain a term of the form uu_{xx} we take advantage of a null structure in the nonlinearity to relax this summability condition and prove local well-posedness in Sobolev spaces.

The mKdV. In Chapters 3 and 4, we consider the asymptotic behavior of the modified KdV (mKdV) equation

$$u_t + \tfrac{1}{3}u_{xxx} = \sigma(u^3)_x, \tag{1.5}$$

where the *focusing* case is given by $\sigma = -1$ and the *defocusing* case by $\sigma = +1$. Like the KdV, the mKdV arises as a model for long dispersive waves in many physical contexts. However, one of the most remarkable properties about the mKdV is its relation to the KdV via the *Miura map* [120]. If u is a sufficiently regular solution to the defocusing mKdV, then $v = \mathbf{M}[u]$ is a solution to the KdV, where

$$\mathbf{M}[u] = \sqrt{\tfrac{3}{2}}u_x + \tfrac{3}{2}u^2. \tag{1.6}$$

This map can be shown to be invertible provided the KdV solution contains no soliton components (see §1.5) and hence defocusing mKdV solutions essentially describe the dispersive part of KdV solutions.

When considering the existence of global solutions, a major obstruction comes from parallel resonant interactions: linear waves with the same velocity that interact nonlinearly and feed back into the system. While the mKdV has a large collection of such interactions, it also possesses a null structure that leads to the existence of global solutions. However, the presence of these "bad" nonlinear interactions leads to a logarithmic divergence between the phase of mKdV solutions and of linear KdV solutions. This is known as *modified asymptotic behavior.*

[1] Chapter 2 is similar to the author's previously published work [48, 51].

In Chapter 3 we prove global existence and derive modified asymptotics for solutions to the mKdV with sufficiently small, smooth and spatially localized initial data, without making use of the completely integrable structure.[2] A key advantage of our robust method is that it can also handle (non-integrable) short range perturbations of the mKdV. In Chapter 4 we consider the reciprocal problem: given a suitable asymptotic profile, can we construct a solution to the mKdV matching the asymptotic behavior at infinity? Together these results give us an asymptotic completeness result for the mKdV.

1.1 Notation, definitions and elementary estimates

In this section we briefly collect some notation, definitions and estimates used throughout this thesis.

Basic notation. Given two quantities A, B we will write $A \lesssim B$ if there exists some constant $C > 0$ so that $A \leq CB$ and write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. If $C = C(k)$ we will write $A \lesssim_k B$. We write $A \ll B$ if $A \lesssim B$ and the constant is sufficiently small.

We denote the sets of integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. If $E \subset \mathbb{R}^d$ we denote the indicator function of the set E by $\mathbf{1}_E$. We denote the Euclidean norm by $|\cdot|$ and define the bracket $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We use the notation $x_\pm = \frac{1}{2}(|x| \pm x)$. If X is a normed space we denote its norm by $\|\cdot\|_X$.

We use $\partial_x u$, u_x and u' to denote a (partial) derivative in the variable x and use the notation $D = -i\partial_x$. We define the inverse derivative[3]

$$\partial_x^{-1} u = \frac{1}{2}\left(\int_{-\infty}^{x} u(y)\, dy - \int_{x}^{\infty} u(y)\, dy \right). \tag{1.7}$$

We say a function is localized at scale $\ell > 0$ if for all $k \geq 0$,

$$|f(x)| \lesssim_k \langle \ell^{-1} x \rangle^{-k}.$$

We say that a function is smooth on a scale $\lambda > 0$ if for all $k \geq 0$,

$$|\partial_x^k f(x)| \lesssim_k \lambda^k.$$

If X is a normed space and $I \subset \mathbb{R}$ is an interval, we denote the space of continuous functions $f\colon I \to X$ by $C(I; X)$ equipped with the sup norm. We use the notation C^k to denote k-continuously differentiable functions; $C^\infty = \bigcap C^k$ to denote smooth functions; C_0^∞ to denote smooth, compactly supported functions; $\mathcal{S}$ to denote Schwartz functions. We define the space of ruled functions $\mathcal{R}(I; X)$ to consist of $f\colon I \to X$ such that for every $t \in I$ both the left and right limits at t exist.

[2]Chapters 3 and 4 are similar to the author's work [49], which has been submitted for publication.

[3]This definition of ∂_x^{-1} corresponds to the Fourier multiplier $(i\xi)^{-1}$, where the integral is interpreted in a principal value sense.

Lebesgue spaces. For $1 \le p < \infty$ we use $L^p(\mathbb{R};\mathbb{F})$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) to denote the space of Lebesgue-measurable functions $f\colon \mathbb{R} \to \mathbb{F}$ such that

$$\|u\|_{L^p}^p = \int |u(x)|^p \, dx < \infty,$$

with the usual modification for $p = \infty$. We will typically omit the domain and codomain when they are evident. We denote the L^2-inner product by

$$\langle u, v \rangle = \int u(x)\bar{v}(x)\, dx.$$

The Fourier transform. We define the Fourier transform of a function $u \in \mathcal{S}(\mathbb{R})$ by

$$\hat{u}(\xi) = \int u(x) e^{-ix\xi}\, dx,$$

with inverse given by

$$\check{u}(x) = \frac{1}{2\pi} \int u(\xi) e^{ix\xi}\, d\xi.$$

We recall Plancherel's Theorem,

$$\langle u, v \rangle = \frac{1}{2\pi} \langle \hat{u}, \hat{v} \rangle. \tag{1.8}$$

Given a measurable function $a\colon \mathbb{R} \to \mathbb{C}$ we define the Fourier multiplier

$$a(D)u = \frac{1}{2\pi} \int a(\xi)\hat{u}(\xi) e^{ix\xi}\, d\xi.$$

We define the linear KdV propagator by

$$S(t)u = \frac{1}{2\pi} \int \hat{u}(\xi) e^{i(\frac{1}{3}t\xi^3 + x\xi)}\, d\xi. \tag{1.9}$$

Sobolev spaces. We define the homogeneous and inhomogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$, $H^s(\mathbb{R})$ to be the completion of the Schwartz functions under the norms

$$\|u\|_{\dot{H}^s} = \||D|^s u\|_{L^2}, \qquad \|u\|_{H^s} = \|\langle D\rangle^s u\|_{L^2},$$

and the weighted Sobolev space $H^{s,\sigma}(\mathbb{R})$ with norm

$$\|u\|_{H^{s,\sigma}}^2 = \|\langle D\rangle^s u\|_{L^2}^2 + \|\langle x\rangle^\sigma u\|_{L^2}^2.$$

We will frequently make use of the 1-dimensional Sobolev estimate

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}}. \tag{1.10}$$

The Littlewood-Paley projections. Let $\psi \in C_0^\infty$ be a real-valued, even function so that $0 \leq \psi \leq 1$, ψ is identically 1 on $[-1,1]$ and supported in $(-2,2)$. For $N \in 2^{\mathbb{Z}}$, we define the Littlewood-Paley projections

$$P_{\leq N} = \psi(N^{-1}D), \qquad P_{>N} = 1 - P_{\leq N}, \qquad P_N = P_{\leq N} - P_{\leq \frac{N}{2}}.$$

We also define the projections to positive and negative frequencies by $P_\pm = \mathbf{1}_{(0,\infty)}(\pm D)$. We will commonly write $u_N = P_N u$ and similarly for the other projections. If $R \notin 2^{\mathbb{Z}}$, we will use $P_{\leq R}$ to denote the sum of dyadic frequencies $\leq R$.

We recall Bernstein's inequality for $1 \leq p \leq q \leq \infty$,

$$\|P_{\leq N} u\|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N} u\|_{L^p}, \tag{1.11}$$

and the behavior of the projections with respect to derivatives,

$$\||D|^s P_{\leq N} u\|_{L^p} \lesssim N^s \|P_{\leq N} u\|_{L^p}, \qquad N^s \|P_{>N} u\|_{L^p} \lesssim \||D|^s P_{>N} u\|_{L^p}.$$

We note that by Plancherel's Theorem (1.8) we have

$$\|u\|_{L^2}^2 \sim \sum_N \|u_N\|_{L^2}^2, \qquad \|u\|_{H^s}^2 \sim \|u_{\leq 1}\|_{L^2}^2 + \sum_{N>1} N^{2s} \|u_N\|_{L^2}^2.$$

We define the Besov spaces $B^{s,p}_q$ with norm

$$\|u\|_{B^{s,p}_q}^q = \|u_{\leq 1}\|_{L^p}^q + \sum_{N>1} N^{qs} \|u_N\|_{L^p}^q,$$

with the usual modification for $q = \infty$.

We recall the *Littlewood-Paley trichotomy*: given two functions u and v we may decompose their product at a given output frequency N as

$$P_N(uv) = \sum_{(N_1,N_2) \in \mathcal{N}} P_N(P_{N_1} u P_{N_2} v),$$

and may decompose

$$\mathcal{N} = \mathcal{N}_{\text{high}-\text{low}} \cup \mathcal{N}_{\text{low}-\text{high}} \cup \mathcal{N}_{\text{high}-\text{high}},$$

where we define the sets of high-low interactions, low-high interactions and a high-high interactions by

$$\begin{aligned}
\mathcal{N}_{\text{high}-\text{low}} &= \{\tfrac{N}{4} \leq N_1 \leq 4N, \quad N_2 < \tfrac{N}{4}\}, \\
\mathcal{N}_{\text{low}-\text{high}} &= \{N_1 < \tfrac{N}{4}, \quad \tfrac{N}{4} \leq N_2 \leq 4N\}, \\
\mathcal{N}_{\text{high}-\text{high}} &= \{\tfrac{N_1}{4} \leq N_2 \leq 4N_1, \quad N_1, N_2 \geq \tfrac{N}{4}\}.
\end{aligned}$$

Throughout this thesis we will frequently make use of functions that are localized in both space and frequency on the scale of uncertainty. In this case we may commute the localization up to rapidly decaying tails:

Lemma 1.1. *Let $1 \leq p \leq \infty$ and $\chi \in \mathcal{S}(\mathbb{R})$ be localized at scale ~ 1 in space and frequency and for a given spatial scale $\ell > 0$, let $\chi_\ell(x) = \chi(\ell^{-1}x)$. Then for any function $u \in \mathcal{S}(\mathbb{R})$ and any $k \geq 0$, we have the estimate*

$$\|(1 - \tilde{P}_N)(\chi_\ell P_N u)\|_{L^p} \lesssim_k \langle \ell N \rangle^{-k} \|P_N u\|_{L^p}, \tag{1.12}$$

where $\tilde{P}_N = P_{\frac{N}{4} \leq \cdot \leq 4N}$ satisfies $\tilde{P}_N P_N = P_N$.

The spaces U^p_S and V^p_S. The U^p_S and V^p_S spaces provide an elegant framework in which to treat the local well-posedness theory for the KdV family of equations (and many other equations), especially at critical regularities. These spaces were originally introduced in the context of dispersive PDE by Tataru in unpublished work on the wave maps equation and in the work of Koch and Tataru [88] on the cubic nonlinear Schrödinger equation (NLS). In this section we briefly recall their definitions and basic properties. For a detailed introduction we refer the reader to [90].

Let $I = [a, b) \subset \mathbb{R}$ where $-\infty \leq a < b \leq \infty$ and let $\tau = \{a = t_0 < t_1 < \cdots < t_{n+1} = b\}$ be a partition of I. We define a p-atom to be a step function

$$a(t) = \sum_{j=1}^{n} \phi_j \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\sum_j \|\phi_j\|_{L^2}^p \leq 1$. We then take U^p to be the atomic space consisting of functions $u \colon I \to L^2$ such that

$$\|u\|_{U^p} = \inf \left\{ \sum_j |\lambda_j| : u = \sum_j \lambda_j a_j,\ a_j \text{ are atoms} \right\} < \infty.$$

We note that if $u \in U^p$ then $u(a) = 0$ and u is right-continuous. If $I \subset I' = [A, B)$, we may always extend $u \in U^p$ to I' by taking $u(t) = 0$ for $t < a$ and $u(t) = \lim_{s \uparrow b} u(s)$ for $t \geq b$.

We define the space V^p to be the completion of the ruled functions $\mathcal{R}(I; L^2)$ under the norm

$$\|u\|_{V^p}^p = \sup_\tau \left(\sum_{j=1}^{n-1} \|u(t_{j+1}) - u(t_j)\|_{L^2}^p + \|v(t_n)\|_{L^2}^p \right).$$

We note that if $u \in V^p(I)$ then we may extend it by zero to $V^p(I')$ for $I \subset I'$. We denote the subspace $V^p_{\mathrm{rc}} \subset V^p$ to consist of right-continuous functions $v \in V^p$ so that $\lim_{t \downarrow a} v(t) = 0$.

We define the space DU^p of distributional derivatives of U^p-functions with the induced norm. We then have the following embeddings of the U^p and V^p spaces:

Proposition 1.2. *For $1 \leq p < q < \infty$ we have the embeddings*

$$U^p \subset U^q, \qquad V^p \subset V^q, \tag{1.13}$$

$$U^p \subset V^p_{\mathrm{rc}} \subset U^q. \tag{1.14}$$

Further, with respect to the usual L^2-duality we have $(DU^p)^* = V^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

When using the U^p and V^p spaces to study PDE, it is useful to work with *adapted spaces*. If $S(t)$ is the linear KdV propagator defined as in (1.9) we may define the adapted space $U^p_S = \{S(-t)u \in U^p\}$ with norm

$$\|u\|_{U^p_S} = \|S(-t)u\|_{U^p},$$

and similarly for V^p_S, DU^p_S. We will consider these spaces to be defined on the interval $I = [-\infty, T)$ and extend solutions on $[0, T)$ to I by zero.

We define the space l^2V^2 with norm

$$\|u\|^2_{l^2V^2} = \sum_N \|P_N u\|^2_{V^2},$$

and have the estimate [100, Lemma 4.11],

$$\|u\|_{V^2} \lesssim \|u\|_{l^2V^2}. \tag{1.15}$$

1.2 The linear KdV equation

In this section we discuss some properties of (real or complex-valued) solutions to the linear KdV equation,

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = f \\ u(0) = u_0. \end{cases} \tag{1.16}$$

We refer to the case $f \equiv 0$ as the *homogeneous* equation and $f \not\equiv 0$ as the *inhomogeneous* equation.

Wave packets. We first discuss *wave packets* for the linear KdV. These are approximate solutions, localized on the scale of uncertainty in both space and frequency. They not only provide the intuition for many of the techniques and ideas used in this thesis, but are also used explicitly in Chapter 3.

Given square integrable initial data $u_0 \in L^2(\mathbb{R})$ we may write a solution to the homogeneous linear equation as a superposition of linear waves

$$u(t, x) = S(t)u_0 = \frac{1}{2\pi}\int \hat{u}_0(\xi)e^{i(\frac{1}{3}t\xi^3 + x\xi)}\, d\xi.$$

More generally we may write solutions[4] to the inhomogeneous linear equation using the Duhamel formula,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\, ds. \tag{1.17}$$

[4]We will be exclusively concerned with strong solutions to PDE in this thesis. See [139, §3.2] for a discussion of the relevant definitions and alternative types of solution.

The *uncertainty principle* states that if a function is localized at scale ℓ in physical space, then it cannot be localized at scale smaller than ℓ^{-1} in Fourier space and vice versa. More precisely,

$$\|u\|_{L^2}^2 \lesssim \|xu\|_{L^2}\|\xi\hat{u}\|_{L^2}.$$

Given a length scale $\ell > 0$, we may write $u_0 \in L^2(\mathbb{R})$ as a superposition (see for example [89, 142]),

$$u_0(x) = \frac{1}{2\pi}\int a(x_0,\xi_0)\Psi_{x_0,\xi_0}(x)\,dx_0 d\xi_0,$$

where the Ψ_{x_0,ξ_0} are localized about the point (x_0,ξ_0) in phase space[5] at scale $\sim \ell$ in physical space and $\sim \ell^{-1}$ in Fourier space. Applying the linear propagator, the solution to the homogeneous linear equation may be written as

$$u(t) = \frac{1}{2\pi}\int a(x_0,\xi_0)S(t)\Psi_{x_0,\xi_0}\,dx_0 d\xi_0.$$

As a consequence, in order to understand the behavior of solutions to the linear KdV, it suffices to consider solutions with initial data given by

$$u_0(x) = \chi(\ell^{-1}(x-x_0))e^{i(x-x_0)\xi_0},$$

where $\chi \in \mathcal{S}(\mathbb{R})$ is localized near $x = 0$ at scale ~ 1 in space and frequency. By the translation invariance of the linear KdV operator, we may assume that $x_0 = 0$. The corresponding solution to the homogeneous linear KdV equations is then given by

$$u(t,x) = \frac{1}{2\pi}\int \ell\hat{\chi}(\ell(\xi-\xi_0))e^{i(\frac{1}{3}t\xi^3 + x\xi)}\,d\xi.$$

Linearizing the phase about $\xi = \xi_0$ we have

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\int \ell\hat{\chi}(\ell(\xi-\xi_0))e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0 + (x+t\xi_0^2)(\xi-\xi_0))}\,d\xi + O(t|\xi_0|\ell^{-2} + t\ell^{-3}) \\ &= \chi(\ell^{-1}(x+t\xi_0^2))e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)} + O(t|\xi_0|\ell^{-2} + t\ell^{-3}). \end{aligned}$$

For timescales $\Delta t \ll T$ we see that the solution u behaves like the *wave packet* approximate solution

$$u_{\mathrm{wp}}(t,x) = \chi(\ell^{-1}(x+t\xi_0^2))e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)},$$

provided we choose the scale $\ell > 0$ so that

$$\ell \approx \max\{T^{\frac{1}{3}}, T^{\frac{1}{2}}|\xi_0|^{\frac{1}{2}}\}.$$

We note that u_{wp} is localized on the ray $\{x + t\xi_0^2 = 0\}$ that corresponds to the Hamiltonian flow[6] through the point $(0,\xi_0)$ in phase space associated to the Hamiltonian $H(\xi) = -\frac{1}{3}\xi^3$.

[5]Phase space is considered to be the cotangent bundle $T^*\mathbb{R} = \mathbb{R}^2$ endowed with the canonical symplectic form $\omega = d\xi \wedge dx$.

[6]We recall that given a function $H\colon T^*\mathbb{R} \to \mathbb{R}$, the associated Hamiltonian flow is given by $(\dot{x}(t),\dot{\xi}(t)) = \nabla_\omega H(x(t),\xi(t))$, where $\nabla_\omega H = H_\xi\partial_x - H_x\partial_\xi$ is the symplectic gradient. For more details see [139, §1.4].

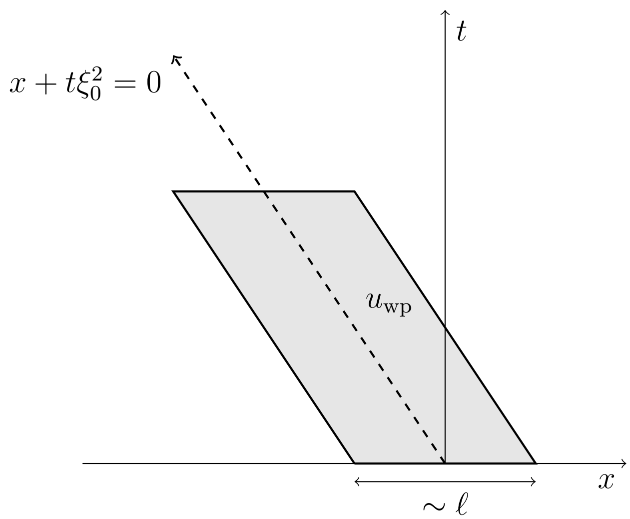

Figure 1.1: The wave packet approximation at $(0, \xi_0)$.

When we consider nonlinear equations with quasilinear dynamics, we do not necessarily expect a solution with wave packet initial data to resemble the linear wave packet approximate solution. However, under suitable conditions we may hope that they will remain close in some sense. In Chapter 2 we use the fact that on timescales $T \sim 1$, the linear wave packet at dyadic frequency $N \geq 1$ is localized in an interval of length N^2. In particular, even if the nonlinear flow of the wave packet initial data deviates from the linear flow, there is some hope that it will still remain well-localized inside this interval. In Chapters 3 and 4 we only expect the nonlinear flow to deviate from the linear flow by a logarithmic phase correction. In Chapter 3 we use the wave packets explicitly, testing our solution against wave packets to construct an asymptotic ODE that gives rise to the logarithmic phase correction. In Chapter 4 we make use of the scale ℓ associated to the wave packets to construct a suitable approximate solution to the mKdV.

The Airy functions. In order to understand the dispersive properties of the linear KdV equation, we first consider the behavior of the fundamental solution,

$$u(t,x) = t^{-\frac{1}{3}} \operatorname{Ai}(t^{-\frac{1}{3}} x),$$

where we define the Airy function $\operatorname{Ai}(x)$ as an oscillatory integral,

$$\operatorname{Ai}(x) = \frac{1}{2\pi} \int e^{i(\frac{1}{3}\xi^3 + x\xi)} \, d\xi = \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3}\xi^3 + x\xi) \, d\xi.$$

We may then write the linear propagator (1.9) as a convolution,

$$S(t)u = \int t^{-\frac{1}{3}} \operatorname{Ai}(t^{-\frac{1}{3}}(x-y)) u(y) \, dy. \tag{1.18}$$

We also define the Airy function

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \sin(\tfrac{1}{3}\xi^3 + x\xi)\,d\xi + \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{3}\xi^3 + x\xi}\,d\xi,$$

and observe that $\{\mathrm{Ai}(x), \mathrm{Bi}(x)\}$ form a linearly-independent set of solutions to the *Airy equation*

$$y''(x) - xy(x) = 0, \tag{1.19}$$

with Wronskian

$$\mathrm{Ai}(x)\,\mathrm{Bi}'(x) - \mathrm{Ai}'(x)\,\mathrm{Bi}(x) = \frac{1}{\pi}.$$

Using stationary phase and steepest descent, we may then prove the following estimates for the Airy functions [128].

Lemma 1.3. *We have the estimates*

$$|\,\mathrm{Ai}(x)| \lesssim \langle x\rangle^{-\frac{1}{4}} e^{-\frac{2}{3}x_+^{\frac{3}{2}}}, \qquad |\,\mathrm{Ai}'(x)| \lesssim \langle x\rangle^{\frac{1}{4}} e^{-\frac{2}{3}x_+^{\frac{3}{2}}}, \tag{1.20}$$

$$|\,\mathrm{Bi}(x)| \lesssim \langle x\rangle^{-\frac{1}{4}} e^{\frac{2}{3}x_+^{\frac{3}{2}}}, \qquad |\,\mathrm{Bi}'(x)| \lesssim \langle x\rangle^{\frac{1}{4}} e^{\frac{2}{3}x_+^{\frac{3}{2}}}. \tag{1.21}$$

Further, we have the asymptotics as $x \to -\infty$,

$$\mathrm{Ai}(x) = \pi^{-\frac{1}{2}}|x|^{-\frac{1}{4}}\cos(-\tfrac{2}{3}|x|^{\frac{3}{2}} + \tfrac{\pi}{4}) + O(|x|^{-\frac{7}{4}}), \tag{1.22}$$

$$\mathrm{Ai}'(x) = \pi^{-\frac{1}{2}}|x|^{\frac{1}{4}}\sin(-\tfrac{2}{3}|x|^{\frac{3}{2}} + \tfrac{\pi}{4}) + O(|x|^{-\frac{5}{4}}), \tag{1.23}$$

$$\mathrm{Bi}(x) = -\pi^{-\frac{1}{2}}|x|^{-\frac{1}{4}}\sin(-\tfrac{2}{3}|x|^{\frac{3}{2}} + \tfrac{\pi}{4}) + O(|x|^{-\frac{7}{4}}), \tag{1.24}$$

$$\mathrm{Bi}'(x) = \pi^{-\frac{1}{2}}|x|^{\frac{1}{4}}\cos(-\tfrac{2}{3}|x|^{\frac{3}{2}} + \tfrac{\pi}{4}) + O(|x|^{-\frac{5}{4}}), \tag{1.25}$$

and as $x \to +\infty$,

$$\mathrm{Ai}(x) = \frac{1}{2}\pi^{-\frac{1}{2}}|x|^{-\frac{1}{4}}e^{-\frac{2}{3}|x|^{\frac{3}{2}}} + O(|x|^{-\frac{7}{4}}e^{-\frac{2}{3}|x|^{\frac{3}{2}}}), \tag{1.26}$$

$$\mathrm{Ai}'(x) = -\frac{1}{2}\pi^{-\frac{1}{2}}|x|^{\frac{1}{4}}e^{-\frac{2}{3}|x|^{\frac{3}{2}}} + O(|x|^{-\frac{5}{4}}e^{-\frac{2}{3}|x|^{\frac{3}{2}}}), \tag{1.27}$$

$$\mathrm{Bi}(x) = \pi^{-\frac{1}{2}}|x|^{-\frac{1}{4}}e^{\frac{2}{3}|x|^{\frac{3}{2}}} + O(|x|^{-\frac{7}{4}}e^{\frac{2}{3}|x|^{\frac{3}{2}}}), \tag{1.28}$$

$$\mathrm{Bi}'(x) = \pi^{-\frac{1}{2}}|x|^{\frac{1}{4}}e^{\frac{2}{3}|x|^{\frac{3}{2}}} + O(|x|^{-\frac{5}{4}}e^{\frac{2}{3}|x|^{\frac{3}{2}}}). \tag{1.29}$$

Dispersive estimates. As a consequence of the formula (1.18) and Young's inequality for convolutions we have the dispersive estimates

$$\|S(t)u\|_{L^\infty} \lesssim t^{\frac{1}{3}}\|u\|_{L^1}, \tag{1.30}$$

$$|S(t)u| \lesssim t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}u\|_{L^1}, \qquad |\partial_x S(t)u| \lesssim t^{-\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}u\|_{L^1}. \tag{1.31}$$

For non-localized data, the bound (1.30) is somewhat naïve and we may recover better dispersive estimates by considering the oscillatory integral,

$$|D|^{\frac{1}{2}+i\sigma}\operatorname{Ai}(x) = \frac{1}{2\pi}\int |\xi|^{\frac{1}{2}+i\sigma}e^{i(\frac{1}{3}t\xi^3+x\xi)}\,d\xi.$$

By stationary phase (see for example [90]) we have the estimate,

$$\left||D|^{\frac{1}{2}+i\sigma}\operatorname{Ai}(x)\right| \lesssim \langle\sigma\rangle. \tag{1.32}$$

We then define the family of operators T_ζ on the strip $\Omega = \{0 < \operatorname{Re}\zeta < 1\} \subset \mathbb{C}$ by

$$T_\zeta u = e^{\zeta^2}|D|^{\frac{\zeta}{2}}S(t)u.$$

Using Plancherel's Theorem (1.8) for the L^2 estimate and the formula (1.18) with the improved bound (1.32) we have

$$\|T_{i\sigma}u\|_{L^2} \lesssim e^{-\sigma^2}\|u\|_{L^2}, \qquad \|T_{1+i\sigma}u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}e^{-\sigma^2}\langle\sigma\rangle\|u\|_{L^1}, \qquad \sigma\in\mathbb{R}.$$

By Stein's complex interpolation theorem, we then have the dispersive estimates [78],

$$\||D|^{\frac{1}{2}-\frac{1}{r}}S(t)u\|_{L^r} \lesssim t^{\frac{1}{r}-\frac{1}{2}}\|u\|_{L^{r'}}, \qquad 2\leq r\leq\infty. \tag{1.33}$$

Strichartz estimates. As it is most natural to consider initial data in L^2-based spaces, in order to study the dispersive properties for non-localized initial data we must relax the pointwise bounds and instead look for space-time averaged decay. By using a TT^* argument with the dispersive estimate (1.33) we may derive the following Strichartz estimates for solutions to (1.16):

Lemma 1.4 ([78])**.** *Suppose u is a solution to* (1.16) *on an interval* $0\in I\subset\mathbb{R}$ *and* (q_j, r_j) *satisfy the admissibility criteria*

$$\frac{2}{q_j}+\frac{1}{r_j}=\frac{1}{2}, \qquad 2\leq r_j\leq\infty. \tag{1.34}$$

Then we have the Strichartz estimate

$$\|u\|_{L^\infty_t L^2_x} + \||D|^{\frac{1}{q_1}}u\|_{L^{q_1}_t L^{r_1}_x} \lesssim \|u_0\|_{L^2} + \||D|^{-\frac{1}{q_2}}f\|_{L^{q_2'}_t L^{r_2'}_x}, \tag{1.35}$$

where $\frac{1}{q_j}+\frac{1}{q_j'}=1=\frac{1}{r_j}+\frac{1}{r_j'}$.

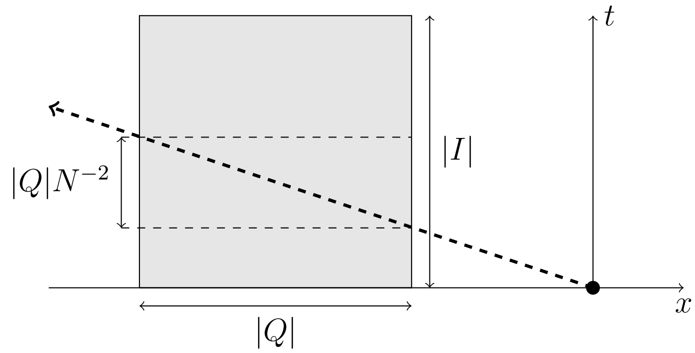

Figure 1.2: Local smoothing for a linear KdV wave packet at frequency $N \gg |I|^{\frac{1}{2}}|Q|^{-\frac{1}{2}}$.

Local smoothing estimates. Unfortunately the Strichartz estimates are insufficient to prove local well-posedness for equations with derivative nonlinearities. Instead we must take advantage of the local smoothing properties of the linear KdV flow, originally observed by Kato [68]. If we consider a time interval $I \subset \mathbb{R}$ and a spatial interval $Q \subset \mathbb{R}$ then wave packets at frequency N will be well localized inside the interval for a time of at most $|Q|N^{-2}$ (see Figure 1.2). In particular, taking $u_N = P_N u$ to be localized at dyadic frequency $N \geq 1$, we have the local energy decay estimate (see [79, Remark 3.7])

$$\sup_{\substack{Q\subset\mathbb{R}\\ |Q|\geq|I|}} \left(|Q|^{-\frac{1}{2}}\|S(t)P_N u\|_{L^2_{t,x}(I\times Q)}\right) \lesssim N^{-1}\|P_N u\|_{L^2}. \tag{1.36}$$

A simple proof of this may be obtained by applying Plancherel's Theorem in the t-variable to get

$$\|\partial_x S(t)u\|_{L^\infty_x L^2_t} \sim \|e^{ix\xi}\hat{u}(\xi)\|_{L^\infty_x L^2_\xi} \sim \|u\|_{L^2}.$$

More generally, we have a family of local smoothing estimates for solutions to the linear KdV equation:

Lemma 1.5 ([74, 80])**.** *If u is a solution to* (1.16) *on an interval* $0 \in I \subset \mathbb{R}$ *and* (q_j, r_j) *are admissible in the sense of* (1.34) *then we have the local smoothing estimate*

$$\|u\|_{L^\infty_t L^2_x} + \||D|^{1-\frac{5}{q_1}}u\|_{L^{q_1}_x L^{r_1}_t} \lesssim \|u_0\|_{L^2} + \||D|^{\frac{5}{q_2}-1}f\|_{L^{q_2'}_x L^{r_2'}_t}, \tag{1.37}$$

where $\frac{1}{q_j} + \frac{1}{q_j'} = 1 = \frac{1}{r_j} + \frac{1}{r_j'}$.

U^p_S **and** V^p_S **estimates.** As the estimates of Lemmas 1.4 and 1.5 apply to U^p_S-atoms, we have the following lemma as a straightforward corollary:

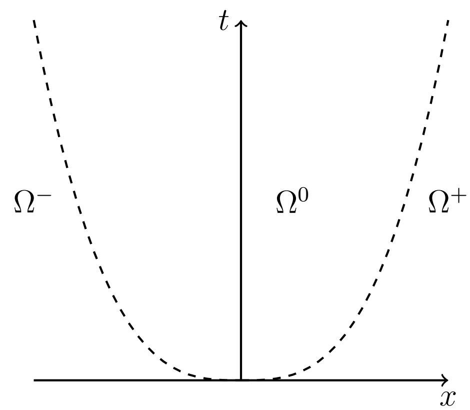

Figure 1.3: Asymptotic regions for the homogeneous linear KdV as $t \to +\infty$.

Lemma 1.6 ([16, Corollaries 3.5, 3.6])**.** *If* $I = [0,T) \subset \mathbb{R}$ *and* (q,r) *are admissible in the sense of* (1.34) *then we have the estimates*

$$\||D|^{\frac{1}{q}}u\|_{L_t^qL_x^r} \lesssim \|u\|_{U_S^q}, \qquad \left\|\int_0^t S(t-s)F(s)\,ds\right\|_{V_S^{q'}} \lesssim \||D|^{-\frac{1}{q}}F\|_{L_t^{q'}L_x^{r'}}, \tag{1.38}$$

$$\||D|^{1-\frac{5}{q}}u\|_{L_x^qL_t^r} \lesssim \|u\|_{U_S^{\min\{q,r\}}}, \quad \left\|\int_0^t S(t-s)F(s)\,ds\right\|_{V_S^{\max\{q',r'\}}} \lesssim \||D|^{1-\frac{5}{q}}F\|_{L_x^{q'}L_t^{r'}}. \tag{1.39}$$

Asymptotic behavior of linear solutions. We now consider the asymptotic properties of solutions to the homogeneous linear KdV equation with real-valued initial data $u_0 \in \mathcal{S}(\mathbb{R})$. The behavior as $t \to +\infty$ may be roughly divided into an oscillatory region $\Omega^- = \{t^{-\frac{1}{3}}x \to -\infty\}$, a self-similar region $\Omega^0 = \{t^{-\frac{1}{3}}|x| \lesssim 1\}$ and a rapidly decaying region $\Omega^+ = \{t^{-\frac{1}{3}}x \to +\infty\}$ (see Figure 1.3).

In the oscillatory region Ω^- we may apply stationary phase to get

$$u(t,x) = \pi^{-\frac{1}{2}}t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}}\,\mathrm{Re}\left(e^{-\frac{2}{3}it^{-\frac{1}{2}}|x|^{\frac{3}{2}}+i\frac{\pi}{4}}\hat{u}_0(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})\right) + O(t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{7}{4}}). \tag{1.40}$$

In the self-similar region Ω^0 we may use the representation of the linear propagator (1.18) and the estimates for the Airy function of Lemma 1.3 to show that

$$u(t,x) = t^{-\frac{1}{3}}\,\mathrm{Ai}(t^{-\frac{1}{3}}x)\int u_0\,dy + O(t^{-\frac{2}{3}}). \tag{1.41}$$

In the rapidly decaying region we may repeatedly integrate by parts in the formula (1.9) to get

$$u(t,x) = O(t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-k}). \tag{1.42}$$

1.3 The Mizohata condition

In this section we discuss a necessary condition for the well-posedness of a linear KdV-type equation of the form

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} + au_{xx} = f \\ u(0) = u_0, \end{cases} \tag{1.43}$$

where $a = a(x) \in C^\infty(\mathbb{R})$ satisfies $|\partial_x^k a| \lesssim_k 1$.

As the au_{xx} term has fewer derivatives than the u_{xxx} term then, at least for small a, one might hope treat the solution of (1.43) as a perturbation of (1.16). In this case we consider a wave packet approximate solution initially localized near the point $(0, \xi_0)$ in phase space of the form

$$u_{\mathrm{wp}}(t, x) = \chi(\ell^{-1}(x + t\xi_0^2))e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)},$$

where $\chi \in C_0^\infty(\mathbb{R})$ and $\ell > 0$. We calculate

$$(\partial_t + \tfrac{1}{3}\partial_x^3 + a\partial_x^2)u_{\mathrm{wp}} = \left(\ell^{-3}\frac{1}{3}\chi''' + i\ell^{-2}\xi_0\chi'' + a\ell^{-2}\chi'' + 2i\xi_0\ell^{-1}a\chi' - \xi_0^2 a\chi\right)e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)}.$$

By choosing a suitable length scale $\ell = \ell(T, \xi_0, a) > 0$, all of these terms will be $O(T^{-1})$ except for $-\xi_0^2 a\chi e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)}$. However, we may remove this term by modifying the phase and taking

$$u_{\mathrm{app}}(t, x) = \chi(\ell^{-1}(x + t\xi_0^2))e^{i(\frac{1}{3}t\xi_0^3 + x\xi_0)}e^{\int_x^{x + t\xi_0^2} a(y)\,dy}. \tag{1.44}$$

In order for this phase correction to be well-defined for $t > 0$, we must have that

$$\sup_{x_1 \leq x_2} \mathrm{Re} \int_{x_1}^{x_2} a(y)\,dy < \infty, \tag{1.45}$$

with a corresponding condition for $t < 0$. If this condition fails, then we may exploit the unbounded exponential growth of an approximate solution of the form (1.44) to show that no uniform estimates can possibly hold for (1.43) on any time interval $[0, T]$ and hence equation (2.1) is ill-posed. This argument originally appeared in work of Mizohata [123] on the Schrödinger equation and can be shown to be both necessary and sufficient for the L^2-well-posedness of (1.43) [3, 141].

1.4 The gKdV equations

In this section we discuss properties of the generalized KdV (gKdV) family of equations,

$$u_t + \tfrac{1}{3}u_{xxx} = \sigma(u^p)_x, \tag{1.46}$$

where $\sigma = \pm 1$ and $p \geq 2$ is an integer. When p is odd, we distinguish between the defocusing $\sigma = +1$ and focusing $\sigma = -1$ cases. When p is even, solutions for $\sigma = -1$ are given by $-u$ where u is a solution for $\sigma = +1$.

Symmetries. The gKdV equation (1.46) is invariant under the following symmetries:

- *Translation.* For $t_0, x_0 \in \mathbb{R}$,
$$u(t,x) \mapsto u(t-t_0, x-x_0).$$
- *Scaling.* For $\lambda > 0$,
$$u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x).$$
- *Reversal.*
$$u(t,x) \mapsto u(-t,-x).$$
- *Reflection (p odd).*
$$u(t,x) \mapsto -u(t,x).$$
- *Galilean invariance ($p=2$).* For $c \in \mathbb{R}$,
$$u(t,x) \mapsto u(t, x-ct) - \frac{\sigma c}{2}.$$

Conserved quantities. Smooth solutions to the gKdV (1.46) have the following conserved quantities:

$$M[u] = \int u\, dx, \tag{1.47}$$

$$E[u] = \int u^2\, dx, \tag{1.48}$$

$$H[u] = \int \left(u_x^2 + \frac{6\sigma}{p+1} u^{p+1} \right) dx. \tag{1.49}$$

In the case of the KdV ($p=2$) and mKdV ($p=3$) there are an infinite number of higher order conservation laws (see §1.5).

Hamiltonian structure. We may formally consider the homogeneous Sobolev space $X = \dot{H}^{-\frac{1}{2}}$ of real-valued tempered distributions to be an infinite dimensional symplectic manifold with symplectic form

$$\omega(u,v) = 6 \int u \partial_x^{-1} v\, dx,$$

where we consider ∂_x^{-1} as the map $\partial_x^{-1} \colon \dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}$.

The energy H is a densely defined operator on X and hence we may define the corresponding Hamiltonian vector field $\nabla_\omega H \colon X \to TX$ by

$$\omega(u, (\nabla_\omega H)_v) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} H(v + \epsilon u).$$

Formally integrating by parts,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} H(v+\epsilon u) = 6\int \frac{1}{3}u_x v_x + \sigma u v^p \, dx = \omega(u, -(\frac{1}{3}v_{xx} - \sigma v^p)_x),$$

and hence the gKdV equation may be seen as the flow associated to the vector field $\nabla_\omega H$.

Local and global well-posedness of the gKdV equations. The local and global well-posedness of the gKdV is an extensively studied topic. In Table 1.1 we briefly summarize the best known local and global well-posedness results and refer the reader to [94] for a more extensive bibliography.

We note that the scaling-critical Sobolev space for the gKdV is $\dot{H}^{s_c}$ where

$$s_c = \frac{1}{2} - \frac{2}{p-1}.$$

Heuristically, we expect to have well-posedness for initial data $u_0 \in H^s$ whenever $s \geq s_c$ and ill-posedness whenever $s < s_c$.

Table 1.1: Cauchy theory for the gKdV equations.

p	**Locally well-posed**	**Globally well-posed**
2	$s \geq -\frac{3}{4}$ [15, 72]	$s \geq -\frac{3}{4}$ [17, 46, 84]
3	$s \geq \frac{1}{4}$ [80]	$s \geq \frac{1}{4}$ [17, 46, 84]
4	$s \geq -\frac{1}{6}$ [42, 140]	$s \geq -\frac{1}{6}$ [140] (small data) $s > -\frac{1}{42}$ [44] (large data)
5	$s \geq 0$ [80]	$s \geq 0$ [32] (defocusing) $s \geq 0$ [80] (focusing, small data)
≥ 6	$s \geq s_c$ [80]	$s \geq s_c$ [80] (small data)

Remark 1.7. In the case of the KdV and mKdV the solution map fails to be uniformly continuous for $s < -\frac{3}{4}$ and $s < \frac{1}{4}$ respectively [15, 77]. A priori bounds in lower regularity Sobolev spaces have been obtained for the KdV [12, 99], the mKdV [16] and for the mKdV in non-L^2-based spaces closer to the critical scaling [43, 45].

The critical result of Tao [140] for the case $p = 4$ was established in the homogeneous space $\dot{H}^{-\frac{1}{6}}$. A more refined statement was also proved by Koch-Marzuola [87]. The mass critical $p = 5$ result [80] builds on the result of [83]. We note that more refined well-posedness results in critical Besov spaces are available for $p \geq 5$ [125, 138].

Solitons, kinks and breathers. A key property of the gKdV equations is the existence of a number of non-dispersive travelling wave solutions. The most famous of these is the soliton, originally observed Russell [132]. Considering the focusing case $\sigma = -1$ in (1.46), solitons take the form

$$u(t,x) = Q_c(x - x_0 - ct), \qquad c > 0, \quad x_0 \in \mathbb{R},$$

where $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$ and

$$Q(x) = \left(\tfrac{p+1}{2} \operatorname{sech}^2\left(\tfrac{\sqrt{3}(p-1)}{2} x\right)\right)^{\frac{1}{p-1}}$$

is a solution to the equation,

$$\tfrac{1}{3} Q_{xx} + Q^p = Q.$$

More generally there exist *multi-soliton* solutions that behave as a sum of N solitons (see for example [20, 105, 119, 121]). In the integrable cases of the KdV and mKdV we may even find explicit formulae for these multi-soliton solutions using the inverse scattering method (see §1.5).

The loosely worded *soliton resolution conjecture* states that for generic data we expect solutions to the gKdV to decompose asymptotically into a radiation component and a sum of solitons. The inverse scattering transform (see §1.5) provides results of this form for the KdV [34, 135], but not for the non-integrable cases. However, soliton resolution-type results have been proved for a handful of other non-integrable equations (see for example [18, 33, 69–71]).

As a first step towards understanding solutions from generic initial data, a vast amount of work has been done to understand the stability of solitons (see for example the survey articles [111, 143]). In particular, solitons are known to be orbitally stable in the mass-subcritical case $p < 5$ [9, 14, 144] and unstable in both the mass-critical $p = 5$ [101] and mass-supercritical $p > 5$ [9, 41] cases. Further, we see that soliton solutions propagate from left to right whereas, as discussed in §1.2, the radiation component propagates from right to left. Due to the separation between the radiation and soliton parts of the solution, solitons and multi-solitons can be shown to be asymptotically stable in the mass subcritical case [7, 12, 39, 106–110, 114, 117, 124, 129]. More recently a significant amount of work has been done to understand the the blow-up dynamics near the soliton in the critical and supercritical cases, see for example [86, 102–104, 111, 113] and references therein.

We do not have spatially localized soliton solutions for the defocusing mKdV ($\sigma = +1$). However, there does exist a family of travelling wave solutions known as *kinks*. These solutions take the form

$$u(t,x) = R_c(x + ct - x_0), \qquad c > 0, \quad x_0 \in \mathbb{R},$$

where $R_c(x) = \sqrt{c} R(\sqrt{c}x)$ and

$$R(x) = \tanh\left(\sqrt{\tfrac{3}{2}} x\right)$$

is a solution to the equation

$$\tfrac{1}{3}R_{xx} + R = R^3.$$

We note that $\lim_{x\to\pm\infty} R(x) = \pm 1$ and hence kinks are not in L^2. There are several results on the orbital and asymptotic stability of kinks and multi-kinks [12, 117, 126, 127, 146]. Most remarkably, kink solutions to the defocusing mKdV may be mapped to soliton solutions of the KdV using the Miura map (see §1.5). This fact has been exploited to establish stability results for KdV solitons at low regularity from the corresponding result for mKdV kinks [12, 117].

Perhaps the most exotic known class of non-dispersive solutions to the focusing mKdV is the two-parameter family of breather solutions,

$$u(t,x) = 2\sqrt{\tfrac{2}{3}}\beta \operatorname{sech}(\beta(x+\gamma t))\frac{\cos(\alpha(x+\delta t)) - \frac{\beta}{\alpha}\sin(\alpha(x+\delta t))\tanh(\beta(x+\gamma t))}{1 + \frac{\beta^2}{\alpha^2}\sin^2(\alpha(x+\delta t))\operatorname{sech}^2(\beta(x+\gamma t))},$$

where $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ and

$$\delta = \tfrac{1}{3}\alpha^2 - \beta^2, \qquad \gamma = \alpha^2 - \tfrac{1}{3}\beta^2.$$

Breather solutions are periodic in time and localized in space. In the limiting case $\alpha = 0$ we recover a 2-soliton solution to the mKdV known as a *double pole*. Breather solutions were used by Kenig-Ponce-Vega [77] to prove the solution map for to the focusing mKdV fails to be uniformly continuous for $s < \frac{1}{4}$. The orbital stability of breather solutions to the mKdV has been established by Alejo and Muñoz [6, 127].

Self-similar solutions and the Painlevé II equation. We can look to construct *self-similar solutions* to the gKdV equations by taking

$$u(t,x) = t^{-\frac{2}{3(p-1)}} Q(t^{-\frac{1}{3}}x),$$

where $Q(y)$ is a solution to the ODE

$$Q_{yyy} - yQ_y - \tfrac{2}{p-1}Q = 3p\sigma Q^{p-1}Q_y.$$

We observe that such a solution is invariant under the gKdV scaling symmetry, hence the terminology "self-similar." These self-similar solutions arise in the asymptotic region connecting oscillatory behavior to rapidly decaying behavior and can play a role in the analysis of blow-up behavior (see for example [10, 23, 24, 39, 86, 112]).

For the mKdV, $Q(y)$ must solve the Painlevé II equation,

$$Q_{yy} - yQ = 3\sigma Q^3. \tag{1.50}$$

A self-similar solution to the KdV (with $\sigma = -1$) may be found by simply applying the Miura map to the defocusing ($\sigma = +1$) mKdV self-similar solution to get

$$v(t,x) = t^{-\frac{2}{3}}\left(\sqrt{\tfrac{3}{2}}Q_x(t^{-\frac{1}{3}}x) + \tfrac{3}{2}Q(t^{-\frac{1}{3}}x)^2\right).$$

In Chapter 4, a key object of study will be the one-parameter family of solutions to (1.50) we boundary conditions at $+\infty$ given by

$$Q(y;W) \sim q_\sigma(W)\,\mathrm{Ai}(y), \qquad y \to +\infty, \tag{1.51}$$

where for $W \in \mathbb{R}$ we define

$$q_\sigma(W) = \operatorname{sgn} W \sqrt{\frac{2\sigma}{3}\left(1 - e^{-\frac{3\sigma}{2}W^2}\right)}. \tag{1.52}$$

The following result of Deift and Zhou gives the asymptotic behavior of these solutions (also see §4.A).

Theorem 1.8 (Deift-Zhou [25, Theorems 1.14, 1.19])**.** *Given $W \in \mathbb{R}$ (sufficiently small if $\sigma = -1$) there exists a unique solution $Q(y;W)$ to* (1.50) *with the boundary conditions* (1.51) *such that*

$$Q(y;W) = \begin{cases} \pi^{-\frac{1}{2}}|y|^{-\frac{1}{4}}\,\mathrm{Re}\left(e^{-\frac{2}{3}i|y|^{\frac{3}{2}}+i\frac{\pi}{4}+\frac{3i\sigma}{4\pi}W^2\log|y|^{\frac{3}{2}}+i\sigma\theta(W^2)}W\right) + O(|y|^{-\frac{5}{4}}\log|y|), & y \to -\infty,\\ q_\sigma(W)\,\mathrm{Ai}(y) + O(|y|^{-\frac{1}{4}}e^{-\frac{4}{3}y^{\frac{3}{2}}}), & y \to +\infty, \end{cases}$$

where we define

$$\theta(W^2) = \tfrac{9\log 2}{4\pi}W^2 - \arg\Gamma\left(\tfrac{3i}{4\pi}W^2\right) - \tfrac{\pi}{2},$$

and Γ is the Gamma function.

Derivation of the KdV from the Euler equations. In this section we outline a derivation of the KdV equation from the Euler equations. We note that there are several methods to obtain the KdV as an asymptotic limit in this context and we refer the reader to [65, 118] for more details. We consider an inviscid, irrotational, incompressible fluid in $\mathbb{R}^2$ lying in the domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 : -h_0 < y < h(t,x)\}$$

between a fixed, flat base at $y = -h_0$ and a free surface $y = h(t,x)$, where $h_0 > 0$ is the depth of the stationary fluid.

The fluid may be described by the velocity field $\mathbf{u}$ and the pressure p. We assume the fluid has constant density $\rho = 1$ and take g to be the gravitational constant. The motion of the fluid in Ω is then described by the Euler equation,

$$D_t\mathbf{u} = -\nabla p - \begin{bmatrix} 0 \\ g \end{bmatrix}, \tag{1.53}$$

where the material derivative is defined by $D_t = \partial_t + \mathbf{u}\cdot\nabla$. We assume that our fluid is incompressible,

$$\operatorname{div}\mathbf{u} = 0, \tag{1.54}$$

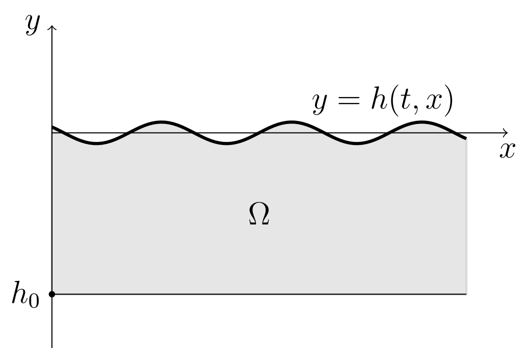

Figure 1.4: The fluid domain.

irrotational,

$$\operatorname{curl}\mathbf{u} = 0, \tag{1.55}$$

and there is no surface tension,

$$p(t, x, h(t, x)) = \text{constant}. \tag{1.56}$$

If $F(t, x) = 0$ describes a surface of the fluid, then we require that $D_t F = 0$. This gives us the boundary conditions,

$$\begin{cases} u_2 = h_t + u_1 h_x, & \text{for } y = h, \\ u_2 = 0, & \text{for } y = -h_0, \end{cases} \tag{1.57}$$

where $\mathbf{u} = (u_1, u_2)$.

From (1.55) we may find a potential Φ so that $\mathbf{u} = \nabla\Phi$. From (1.54) we see that Φ must solve Laplace's equation in Ω,

$$\Delta\Phi = 0, \quad \text{for } x \in \Omega. \tag{1.58}$$

From the Euler equation (1.53), constant pressure condition (1.56) and boundary conditions (1.57), we write the boundary conditions as

$$\begin{cases} \Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gh = 0, & \text{for } y = h, \\ \Phi_y = h_t + \Phi_x h_x, & \text{for } y = h, \\ \Phi_y = 0, & \text{for } y = -h_0. \end{cases} \tag{1.59}$$

Taking a to be a typical amplitude and ℓ to be a typical wavelength of the surface wave, we make the dimensionless rescaling

$$\Phi(t,x,y) \mapsto \frac{h_0}{\ell a\sqrt{gh_0}}\Phi\left(\frac{\ell}{\sqrt{gh_0}}t, \ell x, h_0 y\right), \qquad h(t,x) \mapsto \frac{1}{a}h\left(\frac{\ell}{\sqrt{gh_0}}t, \ell x\right).$$

Defining the dimensionless parameters $\epsilon = h_0^{-1}a$, $\delta = \ell^{-1}h_0$, we may write the system of equations (1.58), (1.59) as

$$\text{(1.60)} \qquad \begin{cases} \delta^2\Phi_{xx} + \Phi_{yy} = 0, & -1 < y < \epsilon h, \\ \Phi_t + \frac{1}{2}\epsilon(\Phi_x^2 + \delta^{-2}\Phi_y^2) + h = 0, & y = \epsilon h \\ \Phi_y = \delta^2(h_t + \epsilon\Phi_x h_x), & y = \epsilon h, \\ \Phi_y = 0, & y = -1. \end{cases}$$

Given any $\delta > 0$, the KdV equation will arise an asymptotic approximation to the equation for the height of the free surface as $\epsilon \to 0$ in a certain region of space-time. To see this we first consider slow spatial and temporal scales by rescaling

$$(t,x) \mapsto \frac{\delta}{\sqrt{\epsilon}}(t,x), \qquad \Phi \mapsto \frac{\sqrt{\epsilon}}{\delta}\Phi.$$

The rescaled system is then given by

$$\text{(1.61)} \qquad \begin{cases} \epsilon\Phi_{xx} + \Phi_{yy} = 0, & -1 < y < \epsilon h, \\ \Phi_t + \frac{1}{2}(\epsilon\Phi_x^2 + \Phi_y^2) + h = 0, & y = \epsilon h, \\ \Phi_y = \epsilon(h_t + \epsilon\Phi_x h_x), & y = \epsilon h, \\ \Phi_y = 0, & y = -1. \end{cases}$$

Inspired by the the first component of (1.61), we consider an expansion

$$\Phi(t,x,y) = \sum_{j=0}^{\infty} \epsilon^j \phi_j(t,x,y),$$

and use the boundary condition at $y = -1$ to get

$$\text{(1.62)} \qquad \partial_x^2\phi_j + \partial_y^2\phi_{j+1} = 0, \qquad \partial_y\phi_j|_{y=-1} = 0.$$

As a consequence we have $\phi_0(t,x,y) = \phi_0(t,x)$.

The leading order terms on the free surface $y = \epsilon h$ as $\epsilon \to 0$ are then given by

$$\partial_t\phi_0 + h = 0, \qquad \partial_y\phi_1|_{y=\epsilon h} = \partial_t h.$$

We may solve (1.62) to get $\partial_y \phi_1|_{y=\epsilon h} = -(1+\epsilon h)\partial_x^2 \phi_0$ and hence to leading order as $\epsilon \to 0$ we have

$$\partial_t \phi_0 + h = 0, \qquad \partial_x^2 \phi_0 + \partial_t h = 0. \tag{1.63}$$

Combining these we have a linear wave equation for ϕ_0,

$$\partial_t^2 \phi_0 - \partial_x^2 \phi_0 = 0.$$

Using d'Alembert's formula ϕ_0 may be written as a sum of a wave that propagates to the right at unit speed and a wave that propagates to the left at unit speed. For spatially localized initial data and large times, we expect the interactions between the right-moving and left-moving components of the surface wave to be of a much lower order as $\epsilon \to 0$. Indeed, a rigorous proof of this was given by Schneider and Wayne [133, 134]. Without loss of generality we may then restrict our attention to the right-travelling wave by considering a moving frame of reference,[7] taking ϕ_0, h to be functions of $(T, X) = (\epsilon t, x - t)$. The corresponding approximation for the left-travelling wave may be recovered by applying an identical analysis in the frame of reference $(\tilde{T}, \tilde{X}) = (\epsilon t, x + t)$.

In the right-moving frame, the leading terms as $\epsilon \to 0$ in (1.61) on the free surface $y = \epsilon h$ are given by,

$$\begin{gathered} \epsilon \partial_T \phi_0 - \partial_X \phi_0 - \epsilon \partial_X \phi_1 + \tfrac{1}{2}\epsilon(\partial_X \phi_0)^2 + h = 0, \\ (1+\epsilon h)\partial_X^2 \phi_0 + \epsilon \partial_X^2 \phi_1 + \tfrac{1}{3}\epsilon \partial_X^4 \phi_0 + \epsilon \partial_T h - \partial_X h + \epsilon \partial_X \phi_0 \partial_X h = 0, \end{gathered}$$

where we have used that $\partial_y \phi_2|_{y=\epsilon h} = -(1+\epsilon h)\partial_X^2 \phi_1|_{y=\epsilon h} - \frac{1}{3}(1+\epsilon h)^3 \partial_X^4 \phi_0$, which again follows from (1.62). To cancel the ϕ_1 term we differentiate the first equation in X and add it the second equation to get

$$\epsilon \partial_T h + \epsilon \partial_T \partial_X \phi_0 + \epsilon h \partial_X^2 \phi_0 + \tfrac{1}{3}\epsilon \partial_X^4 \phi_0 + \epsilon \partial_X \phi_0 \partial_X h + \epsilon \partial_X \phi_0 \partial_x^2 \phi_0 = 0.$$

Further, from (1.63) we have that $h = \partial_X \phi_0 - \epsilon \partial_T \phi_0$ and hence to leading order in ϵ,

$$2h_T + \tfrac{1}{3} h_{XXX} + 3hh_X = 0,$$

which gives us the KdV equation.

1.5 The Miura map and complete integrability

In this section we discuss one of the most remarkable properties of the KdV and mKdV: that they are completely integrable. To simplify the constants we consider the rescaled equations,

$$u_t + u_{xxx} = 6uu_x, \qquad v_t + v_{xxx} = 6\sigma v^2 v_x, \tag{1.64}$$

[7] The introduction of the additional slow time $T = \epsilon t$ arises from the need to eliminate secular terms in the asymptotic expansion. See [118, Chapter 10] for more details.

where $\sigma = \pm 1$. Under this rescaling, the Miura map is given by

$$\mathbf{M}[v] = v_x + v^2. \tag{1.65}$$

Taking $u = \mathbf{M}[v]$, we calculate

$$u_t + u_{xxx} - 6uu_x = (\partial_x + 2v)(v_t + v_{xxx} - 6v^2 v_x),$$

so if v solves the defocusing ($\sigma = +1$) mKdV, then $u = \mathbf{M}[v]$ is indeed a solution to the KdV.

Generalizations of the Miura map. The Miura map proves to be an extremely powerful tool for relating properties of the mKdV to properties of the KdV. When applying this idea, there are several generalizations of the Miura map that arise.

We have a complexified version

$$u = iv_x - v^2, \tag{1.66}$$

which maps solutions v to the focusing ($\sigma = -1$) mKdV to complex-valued solutions u of the KdV. This is the original form of the Miura map appearing in [120] and was used in [77] to transfer ill-posedness results for the focusing mKdV in Sobolev spaces to ill-posedness results for the KdV. Conversely, as every mKdV solution may be mapped to a KdV solution by either (1.65) or (1.66), well-posedness for the KdV in H^s may be used to prove well-posedness for the mKdV in H^{s+1} [17, 46, 84].

Another generalization appearing in Miura's original paper [120] is known as the *Gardner transform* and includes an additional linear term,

$$u = -w + \epsilon w_x + \epsilon^2 w^2. \tag{1.67}$$

This relates a solution u to the KdV to a solutions w of the *Gardner equation*,

$$w_t + w_{xxx} + 6(w - \epsilon^2 w^2)w_x = 0. \tag{1.68}$$

A variation of this transformation was used in [15] to prove local well-posedness for the KdV in $H^{-\frac{3}{4}}$.

We may further transform solutions w to the Gardner equation (1.68) into solutions v to the defocusing mKdV by taking,

$$v(t,x) = \epsilon w(t, x + \tfrac{3}{2\epsilon^2}t) - \tfrac{1}{2\epsilon}. \tag{1.69}$$

We note that this map affects that behavior of solutions as $x \to \pm\infty$ by a constant.

Under the rescaling (1.64), we may write the KdV soliton solution as $u(t,x) = Q_c(x - ct)$ and mKdV kink solution as $v(t,x) = R_c(x + \frac{c}{2}t)$, where

$$Q_c(x) = -\tfrac{c}{2}\operatorname{sech}^2\left(\tfrac{\sqrt{c}}{2}x\right), \qquad R_c(x) = \tfrac{\sqrt{c}}{2}\tanh\left(\tfrac{\sqrt{c}}{2}x\right).$$

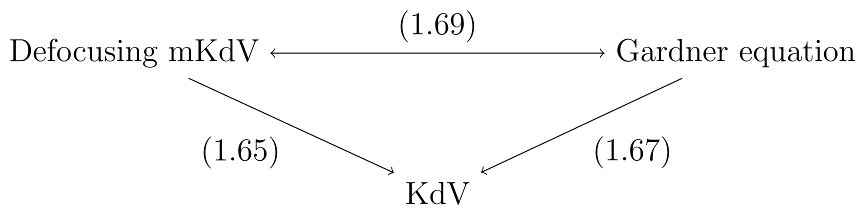

Figure 1.5: Maps between solutions to the KdV, mKdV and Gardner equation.

If we compose the Miura map (1.65) with a Galilean shift by taking

$$u(t,x) = v_x(t, x - \tfrac{3}{2}ct) + v(t, x - \tfrac{3}{2}ct) - \tfrac{1}{4}c, \tag{1.70}$$

then the kink $v(t,x) = R_c(x + \frac{c}{2}t)$ gets mapped to the zero solution $u(t,x) = 0$ and the anti-kink $v(t,x) = -R_c(x + \frac{c}{2}t)$ gets mapped to the soliton $u(t,x) = Q_c(x - ct)$. This relationship is used in [12] to establish a priori bounds and asymptotic stability of the soliton for the KdV in H^{-1}.

There exists a soliton solution to the Gardner equation given by $w(t,x) = W_{c,\epsilon}(x - ct)$, where

$$W_{c,\epsilon}(x) = \frac{\frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}x)}{1 + \sqrt{c}\epsilon\tanh(\frac{\sqrt{c}}{2}x)}, \qquad 0 < c\epsilon^2 < 1.$$

Under the Gardner transform (1.67), the Gardner soliton $w(t,x) = W_{c,\epsilon}(x - ct)$ is mapped to the KdV soliton $u(t,x) = Q_c(x - ct)$ [7, Appendix A]. Under the map (1.69), the Gardner soliton is mapped to

$$v(t,x) = -\tfrac{\sqrt{c}}{2}\tanh\Big(\tfrac{\sqrt{c}}{2}\big(x + (\tfrac{3}{2\epsilon^2} - c)t\big)\Big) + \frac{c\epsilon^2 - 1}{2\epsilon(1 + \sqrt{c}\epsilon\tanh(\frac{\sqrt{c}}{2}(x + (\frac{3}{2\epsilon^2} - c)t)))},$$

which approaches the mKdV anti-kink $v(t,x) = -R_c(x + \frac{1}{2}ct)$ as $\epsilon \to \frac{1}{\sqrt{c}}$. As a consequence of these relations, this family of transformations (see Figure 1.5) has several applications in understanding the behavior of travelling wave solutions to the KdV and mKdV (see for example [7, 12, 117, 126, 127, 146]).

An infinite number of conserved quantities for the KdV. A trick of Miura, Gardner and Kruskal [122] allows us to use the Gardner transform (1.67) to generate an infinite number of conserved quantities for the KdV. If w is a solution to the Gardner equation (1.68), we may formally expand w as a power series in ϵ to get

$$w(t,x,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j w_j(t,x).$$

If u is a solution to the KdV defined as in (1.67) then it must be independent of ϵ so

$$w_0 = -u, \qquad w_{j+2} = \partial_x w_{j+1} + \sum_{k+l=j} w_k w_l.$$

We observe that if w is a sufficiently regular solution of (1.68), then $\partial_t \int w(t,x,\epsilon)\,dx = 0$, so for all $j \geq 0$,

$$\partial_t \int w_j(t,x)\,dx = 0.$$

Each of the w_{2j+1} is a divergence and hence this integral vanishes. However, the w_{2j} give rise to an infinite sequence of conservation laws for the KdV,

$$\begin{aligned} w_0 &= -u, \\ w_2 &= -u_{xx} + u^2, \\ w_4 &= -u_{xxxx} + 5u_x^2 + 6uu_{xx} - 2u^3, \\ &\ldots. \end{aligned}$$

Lax pairs. Viewing the Miura map (1.65) as a Ricatti equation for v, we may linearize it by making a change of variables $v = \frac{\varphi_x}{\varphi}$ to get a linear Schrödinger equation,

$$\mathbf{H}_u \varphi = 0, \qquad \mathbf{H}_u = -\partial_x^2 + u.$$

Lax [93] showed that the eigenvalues of $\mathbf{H}_u$ are integrals (invariant functions) for the KdV equation. Given an eigenfunction φ satisfying

$$\mathbf{H}_u \varphi = \lambda \varphi, \tag{1.71}$$

we may define a skew-adjoint operator,

$$\mathbf{B}_u \varphi = (2u + 4\lambda)\varphi_x - (u_x - \gamma)\varphi,$$

where $\gamma \in \mathbb{C}$ is an arbitrary constant. We then impose a time evolution on the eigenfuntions by

$$\varphi_t = \mathbf{B}_u \varphi. \tag{1.72}$$

Differentiating (1.71) in time and assuming the compatibility condition $\varphi_{xxt} = \varphi_{txx}$ we get the *Lax equation*,

$$(\partial_t \mathbf{H}_u + [\mathbf{H}_u, \mathbf{B}_u])\varphi = \lambda_t \varphi.$$

We observe that

$$\begin{aligned} \partial_t \mathbf{H}_u &= u_t, \\ [\mathbf{H}_u, \mathbf{B}_u] &= u_{xxx} - 6uu_x + 4u_x(\mathbf{H}_u - \lambda), \end{aligned}$$

and hence the eigenvalues satisfy the *isospectral condition* $\lambda_t = 0$ if and only if u solves the KdV equation. The operators $\mathbf{H}_u, \mathbf{B}_u$ are known as a *Lax pair*.

This idea was generalized by Zakharov-Shabat [145] and Ablowitz-Kaup-Newell-Segur [2] by considering the system,

$$\begin{cases} \psi_x = \mathbf{X}_u \psi \\ \psi_t = \mathbf{T}_u \psi, \end{cases} \tag{1.73}$$

where ψ is a vector-valued function and $\mathbf{X}_u, \mathbf{T}_u$ are matrices depending on a scalar function u and a spectral parameter $k \in \mathbb{C}$. Again assuming a compatibility condition, $\psi_{tx} = \psi_{xt}$, we have the equation

$$(\partial_t \mathbf{X}_u - \partial_x \mathbf{T}_u + [\mathbf{X}_u, \mathbf{T}_u])\psi = 0. \tag{1.74}$$

We may use the ZS-AKNS system (1.73) to recover the KdV Lax pair (1.71), (1.72) for $\lambda = k^2$ and $\gamma = 0$ by taking,

$$\psi = \begin{bmatrix} \varphi_x - ik\varphi \\ \varphi \end{bmatrix},$$
$$\mathbf{X}_u = \begin{bmatrix} -ik & u \\ 1 & ik \end{bmatrix}, \qquad \mathbf{T}_u = \begin{bmatrix} -4ik^3 - 2iku + u_x & 4k^2u + 2iku_x + 2u^2 - u_{xx} \\ 4k^2 + 2u & 4ik^3 + 2iku - u_x \end{bmatrix}.$$

However, we may also obtain a Lax pair for the both the focusing and defocusing mKdV by taking

$$\mathbf{X}_u = \begin{bmatrix} -ik & u \\ \sigma u & ik \end{bmatrix}, \qquad \mathbf{T}_u = \begin{bmatrix} -4ik^3 - 2i\sigma ku^2 & 4k^2u + 2iku_x + 2\sigma u^3 - u_{xx} \\ 4\sigma k^2 u + 2i\sigma ku_x + 2u^3 - \sigma u_{xx} & 4ik^3 + 2i\sigma ku^2 \end{bmatrix}.$$

The inverse scattering method for the KdV. In this section we briefly outline the inverse scattering method of solution for the KdV. The method for the mKdV is similar, using the ZS-AKNS system (1.73) instead of the Schrödinger equation (1.71). This method originated in work of Gardner-Greene-Kruskal-Miura on the KdV [36], Zakharov-Shabat on the cubic NLS [145] and Ablowitz-Kaup-Newell-Segur on the sine-Gordon and mKdV [2]. Subsequently numerous authors have developed and adapted these ideas to other contexts. We refer the reader to the book [1] and the recent survey article [85] for more details. In order to justify the various calculations, we will assume that our solution $u(t) \in \mathcal{S}(\mathbb{R})$ although weaker conditions may be assumed.

We start by ignoring the dependence of u on t. Taking $\lambda = k^2$ we find functions

$$\begin{aligned} &m_-(x,k) \sim 1, && n_-(x,k) \sim e^{2ikx}, && x \to -\infty, \\ &m_+(x,k) \sim e^{2ikx}, && n_+(x,k) \sim 1, && x \to +\infty, \end{aligned}$$

such that $\{m_-(x,k)e^{-ikx}, n_-(x,k)e^{-ixk}\}$, $\{m_+(x,k)e^{-ixk}, n_+(x,k)e^{-ixk}\}$ form two sets of linearly independent solutions to (1.71).

From the linear independence, for each k we may find $a(k), b(k), \tilde{a}(k), \tilde{b}(x)$ such that

$$\begin{aligned} m_-(x,k) &= a(k)n_+(x,k) + b(k)m_+(x,k), \\ n_-(x,k) &= -\tilde{a}(k)m_+(x,k) + \tilde{b}(k)n_+(x,k). \end{aligned} \tag{1.75}$$

We define the reflection coefficients

$$\rho(k) = \frac{b(k)}{a(k)}, \qquad \tilde{\rho}(k) = \frac{\tilde{b}(k)}{\tilde{a}(k)},$$

and transmission coefficients

$$\tau(k) = \frac{1}{a(k)}, \qquad \tilde{\tau}(k) = \frac{1}{\tilde{a}(k)}.$$

From the asymptotic behavior and symmetries of (1.71) we see that

$$\begin{aligned} m_+(x,k) = n_+(x,-k)e^{2ikx}, &\qquad n_-(x,k) = m_-(x,-k)e^{2ikx}, \\ \tilde{a}(k) = -a(-k) = -\bar{a}(\bar{k}), &\qquad \tilde{b}(k) = b(-k) = \bar{b}(\bar{k}). \end{aligned}$$

As a consequence, we may rewrite (1.75) as

$$\frac{m_-(x,k)}{a(k)} = n_+(x,k) + \rho(k)n_+(x,-k)e^{2ikx}. \tag{1.76}$$

We may show [1, Lemma 2.2.1] that m_-, a are analytic (in k) in the upper half plane $\{\operatorname{Im} k > 0\}$ and both $m_-(k), a(k) \to 1$ as $|k| \to \infty$ in the upper half plane. Similarly n_+ is analytic in the lower half plane $\{\operatorname{Im} k < 0\}$ and $n_+(k) \to 1$ as $|k| \to \infty$ in the lower half plane. The function $a(k)$ has at most a finite number of simple zeros at $k = i\kappa_1, \dots, i\kappa_N$ [1, Lemma 2.2.2] so we may define the *norming constants* $C_1, \dots, C_N$ such that in a neighborhood of $i\kappa_j$,

$$m_-(x,k) = \frac{C_j n_+(x,-i\kappa_j)e^{-2\kappa_j x}}{k - i\kappa_j} + \text{analytic}.$$

We then define the *scattering data* by

$$\mathbf{S} = \{\rho(k), a(k), \kappa_1, \dots, \kappa_N, C_1, \dots, C_N\}.$$

We will refer to the map $u \mapsto \mathbf{S}$ as the *direct scattering problem* and the map $\mathbf{S} \mapsto u$, which may be constructed by solving the Riemann-Hilbert problem (1.76), as the *inverse scattering problem*.

In order to use the direct and inverse scattering problems to solve the KdV, we consider the time evolution of $\mathbf{S}$. Taking $\varphi = m_- e^{-ikx}$ in (1.72) we have

$$\partial_t m_- = (2u + 4k^2)\partial_x m_- - (2iku + 4ik^3 + u_x - \gamma)m_-.$$

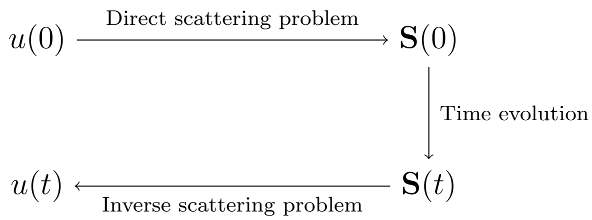

Figure 1.6: The inverse scattering method.

Taking the limit as $x \to -\infty$ and using that $u, u_x \to 0$, $m_- \to 1$ as $x \to -\infty$, we must have $\gamma = 4ik^3$. Taking the limit as $x \to +\infty$ and using (1.75) we get

$$a_t + b_t e^{2ikx} = 8ik^3 b e^{2ikx},$$

and hence

$$a(t,k) = a(0,k), \qquad \rho(t,k) = e^{8itk^3}\rho(0,k).$$

We note that the inverse scattering transform has diagonalized the nonlinear KdV flow in the same way that the Fourier transform diagonalizes the linear KdV flow! Further, as a is t-invariant, $\kappa_1, \dots, \kappa_N$ must also be t-independent. A similar calculation gives us the time dependence of the norming constants to be

$$C_j(t) = e^{8t\kappa_j^3} C_j(0).$$

The inverse scattering method may now be used to solve the KdV by first solving the direct scattering problem, then applying the time evolution to the scattering data and finally solving the inverse scattering problem to recover the solution at time t (see Figure 1.5).

Inverting the Miura map. Taking $c > 0$ we may choose scattering data $\mathbf{S}(0)$ so that the reflection coefficient is given by

$$\rho(0,k) = \begin{cases} 0, & k \in \mathbb{R}, \\ \frac{\sqrt{c}}{k - i\sqrt{\frac{c}{4}}}, & \operatorname{Im} k > 0, \end{cases}$$

so a has a unique simple zero at $k = i\kappa_1 = i\sqrt{\frac{c}{4}}$ with norming constant $C_1 = \sqrt{c}$. The explicit solution may be computed to be the soliton,

$$u(t,x) = -\frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right).$$

In this way the zeros of a correspond to the soliton components of the solution u.

As the Miura map acts on solutions to the defocusing mKdV, which does not have soliton solutions, we expect the range of the Miura map to only contain purely dispersive solutions to the KdV. Indeed, the range of the Miura map was characterized by Kappeler, Perry, Shubin and Topalov as follows.

Theorem 1.9 ([66, Theorem 1.2]). *Let $s \geq 0$ and $u \in H^{s-1}(\mathbb{R})$ be real-valued. Then $u = \mathbf{M}[v]$ for some real-valued $v \in H^s(\mathbb{R})$ if and only if*

(i) $\mathbf{H}_u \geq 0$,

(ii) We may find functions $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ such that $u = f_x + g$.

As part of a proof of a priori bounds for the KdV in $H^{-1}(\mathbb{R})$, Buckmaster and Koch [12] were able to improve this result and show that when $\mathbf{H}_u$ has negative spectrum the Miura map may be inverted to give a perturbation of a kink solution to the mKdV.

Theorem 1.10 ([12, Proposition 6]). *Let $\lambda > 0$ and $u \in H^{-1}(\mathbb{R})$ be real-valued. Then,*

(i) The ground state energy of $\mathbf{H}_u$ for $u \in H^{-1}(\mathbb{R})$ is $-\lambda^2$ if and only if there exists $v \in L^2(\mathbb{R}) - \lambda \tanh(\lambda x)$ such that $\mathbf{M}[v] = u + \lambda^2$.

(ii) The spectrum of $\mathbf{H}_u$ is contained in the interval $(-\lambda^2, \infty)$ if and only if there exists $v \in L^2(\mathbb{R}) + \lambda \tanh(\lambda x)$ with $\mathbf{M}[v] = u + \lambda^2$.

In Chapter 3 we prove modified asymptotics for solutions to the mKdV with small, smooth, spatially localized initial data. Naïvely we might hope to be able to extend this result to the KdV by simply inverting the Miura map for sufficiently "well-behaved" initial data. However, the following result of Damanik, Killip and Simon [22] shows us that smallness alone cannot be sufficient to guarantee that $\mathbf{H}_u \geq 0$ and hence rule out the presence of solitons.

Theorem 1.11 ([22, Theorem 5]). *Suppose that $u \in L^2_{\text{loc}}(\mathbb{R})$ and $\mathbf{H}_{\pm u} \geq 0$, then $u \equiv 0$.*

As a consequence, we see that given any non-zero initial data $u_0 \in L^2_{\text{loc}}$ either $\mathbf{H}_{u_0}$ or $\mathbf{H}_{-u_0}$ must fail to satisfy condition *(i)* of Theorem 1.9, regardless of the size of u_0. However, by combining smallness of the initial data with a positivity requirement, we can obtain a sufficient condition. More precisely we have the following result.

Theorem 1.12. *Let $\sigma > \frac{3}{2}$ and $C > 0$. Then there exists $\epsilon = \epsilon(\sigma, C) > 0$ so that for any real-valued $u \in H^{0,\sigma}(\mathbb{R})$ satisfying*

$$\|u\|_{H^{0,\sigma}} \leq \epsilon, \qquad \int u\,dx \geq C\epsilon, \tag{1.77}$$

we have $\mathbf{H}_u \geq 0$ and hence u is in the range of the Miura map restricted to $H^1(\mathbb{R})$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ and without loss of generality assume that $\|\varphi'\|_{L^2} = 1$, so

$$\langle \mathbf{H}_u\varphi, \varphi\rangle = 1 + \int u(x)|\varphi(x)|^2\, dx.$$

Applying the Cauchy-Schwarz inequality we have,

$$\left|\int_0^x \varphi'(y)\, dy\right| \leq |x|^{\frac{1}{2}}\|\varphi'\|_{L^2} \leq \langle x\rangle^{\frac{1}{2}}.$$

We now define the constant $M = \frac{\int u(x)\, dx}{2\int |u(x)|\, dx}$. By the Cauchy-Schwarz inequality, $\int |u(x)|\, dx \leq \|\langle x\rangle^{-\sigma}\|_{L^2}\|u\|_{H^{0,\sigma}}$, so from (1.77) we have

$$M \geq \frac{C}{2\|\langle x\rangle^{-\sigma}\|_{L^2}}.$$

Writing $\varphi(x) = \varphi(0) + \int_0^x \varphi'(y)\, dy$ we may then estimate

$$\begin{aligned}
\int u(x)|\varphi(x)|^2\, dx &\geq \int u(x)|\varphi(0)|^2 - 2\int |u(x)||\varphi(0)|\langle x\rangle^{\frac{1}{2}}\, dx - \int u(x)\langle x\rangle\, dx \\
&\geq |\varphi(0)|^2\left(\int u(x)\, dx - M\int |u(x)|\, dx\right) - \left(1 + \frac{1}{M}\right)\int |u(x)|\langle x\rangle\, dx \\
&\geq \frac{1}{2}|\varphi(0)|^2\int u(x)\, dx - \left(1 + \frac{1}{M}\right)\int |u(x)|\langle x\rangle\, dx \\
&\geq -\epsilon\left(1 + \frac{2\|\langle x\rangle^{-\sigma}\|_{L^2}}{C}\right)\|\langle x\rangle^{1-\sigma}\|_{L^2}.
\end{aligned}$$

Choosing $\epsilon = \epsilon(\sigma, C) > 0$ sufficiently small we have

$$\int u(x)|\varphi(x)|^2\, dx \geq -1,$$

so $\langle \mathbf{H}_u\varphi, \varphi\rangle \geq 0$. As $u \in L^1$ it satisfies the hypothesis of Theorem 1.9 and hence lies in the range of the Miura map restricted to H^1. $\square$

Chapter 2

Local well-posedness for derivative KdV-type equations

2.1 Introduction

In this chapter we consider local well-posedness for equations of the form

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = F(u, u_x, u_{xx}) \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where F is a constant coefficient polynomial of degree $m \geq 2$ with no linear or constant terms. For simplicity we only present our results for real-valued functions $u\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$. However, it will be clear from the proof that our results also hold for complex-valued functions, see Remark 2.5.

The natural setting for questions of well-posedness are the Sobolev spaces $H^s(\mathbb{R})$. However, if F is a polynomial containing a term of the form uu_{xx} and we project to a dyadic frequency $N \geq 1$, we have the equation

$$(\partial_t + \tfrac{1}{3}\partial_x^3)u_N = u_{\ll N}\partial_x^2 u_N + \text{better terms}.$$

Due to the Mizohata condition (see §1.3), this equation will fail to be well-posed unless $u_{\ll N}$ has some additional integrability. Indeed, an ill-posedness result in H^s was proved by Pilod [131].

One way to address this difficulty is to consider weighted spaces. Kenig-Ponce-Vega proved local well-posedness for small data in [76] and arbitrary data in [73] using the weighted space $H^{s,\sigma}(\mathbb{R})$ for sufficiently large $s, \sigma > 0$. Replacing weighted L^2-spaces with weighted Besov spaces, Pilod [131] proved local well-posedness for certain quadratic nonlinearities with small initial data in the space $H^s(\mathbb{R}) \cap B_2^{s-2,2}(\mathbb{R}, x^2dx)$ where $s > \frac{9}{4}$.

As spatial translation is a symmetry of equation (2.1), it is natural to look for a solutions in translation invariant spaces. By replacing weighted spaces with a spatial summability condition, Marzuola-Metcalfe-Tataru [115] proved a small data result for quasilinear Schrödinger equations with initial data in a translation invariant space $l^1H^s \subset H^s$.

In this chapter we adapt their approach to equation (2.1) and prove low regularity local well-posedness for initial data in a similar subspace of H^s. Further, as (2.1) is linear in u_{xxx} we are able to extend our result to handle large data using similar ideas to Bejenaru and Tataru [8].

As the need for additional integrability is solely due to bilinear interactions, as in [73, 76, 79, 116], we should expect to be able to remove the summability condition and prove local well-posedness for initial data in H^s whenever F contains no quadratic terms. However, as only terms of the form uu_{xx} are truly problematic, we are also able to remove the spatial summability condition for quadratic nonlinearities that do not contain a uu_{xx}-type term.

Statement of results. In order to state the results, we first define the spaces l^pH^s that are the natural adaptation of the corresponding spaces defined in [115, 116] to the KdV setting. For each dyadic $N \geq 1$ we take a partition $\mathcal{Q}_N$ of $\mathbb{R}$ into intervals of length N^2 and an associated locally finite, smooth partition of unity

$$1 = \sum_{Q \in \mathcal{Q}_N} \chi_Q^p,$$

where we assume $\chi_Q \sim 1$ on Q. For a Lebesgue-type space S we define the space $l^p_N S$ by

$$\|u\|^p_{l^p_N S} = \sum_{Q \in \mathcal{Q}_N} \|\chi_Q u\|^p_S.$$

We then define the space l^pH^s with norm

$$\|u\|^2_{l^pH^s} = \|P_{\leq 1} u\|^2_{l^p_1 L^2} + \sum_{N>1} N^{2s} \|P_N u\|^2_{l^p_N L^2}.$$

We note that $l^1H^s \subset l^2H^s = H^s$ and for $s > 1$ we have $l^1H^s \subset L^1$.

Our first result handles the most general case when F may contain terms of the form uu_{xx}.

Theorem 2.1. *For $s > \frac{9}{2}$, equation* (2.1) *is locally well-posed in l^1H^s on the time interval $[0,T]$ where $T = e^{-C(\|u_0\|_{l^1H^s})}$.*

Our second result handles the case that F contains no terms of the form uu_{xx}. In this case we may obtain well-posedness in Sobolev spaces.

Theorem 2.2. *Suppose F contains no terms of the form uu_{xx}. Then, for $s > \frac{9}{2}$, equation* (2.1) *is locally well-posed in H^s on the time interval $[0,T]$ where $T = e^{-C(\|u_0\|_{H^s})}$.*

Remark 2.3. We take the definition of "well-posedness" to be the existence and uniqueness of a solution $u \in l^pX^s \subset C([0,T], l^pH^s)$ and Lipschitz continuity of the solution map, $l^pH^s \ni u_0 \mapsto u \in C([0,T], l^pH^s)$.

Remark 2.4. We note that although the equation (2.1) behaves quasilinearly, it is linear in u_{xxx} and hence we are able to prove Lipschitz dependence on the initial data. This is in contrast to the case of quasilinear Schrödinger equations considered in [115, 116] where continuous dependence on the initial data is all that can be expected.

Remark 2.5. Our results extend to the case of complex valued functions $u\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$ without modification. In this case we may also take F to depend on $\bar{u}, \bar{u}_x$.

For sufficiently small initial data (see Theorems 2.16, 2.17), our results hold without modification for vector-valued functions $u\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}^k$ and we may also allow F to depend on $\bar{u}_{xx}$.

Remark 2.6. As a consequence of our approach, we are able to obtain significantly more refined regularity results for specific nonlinearities. We summarize these improved results in §2.A. In the case of quadratic nonlinearities involving two derivatives with which we are most concerned Theorem 2.1 holds with $s > \frac{5}{2}$ for $F = uu_{xx}$ and Theorem 2.2 holds with $s > \frac{7}{2}$ for $F = u_xu_{xx}$ and $s > \frac{9}{2}$ for $F = u_{xx}^2$.

Outline of the proof. We briefly outline the proof of Theorems 2.1 and 2.2. For small data we first prove linear, bilinear and trilinear estimates for solutions in a suitable subspace $l^pX^s \subset C([0,1]; H^s)$. Our method is similar to Marzuola-Metcalfe-Tataru [115, 116], using local energy decay spaces similar to those suggested by Kenig-Ponce-Vega [79]. We then use the contraction principle to complete the proof.

For large data we might naïvely hope to simply rescale the initial data and then apply the small data techniques. However, as we are working with inhomogeneous spaces, after rescaling we are still left with a large low frequency component. As the low frequency component of the data is essentially stationary on a unit time interval however, we use a similar argument to Bejenaru-Tataru [8] and freeze it at $t = 0$. We then rewrite (2.1) as an equation for the evolution of the small high frequency component of the form

$$(\partial_t + \partial_x^3 + a(x)\partial_x^2)v = \tilde{F}(x, v, v_x, v_{xx}), \tag{2.2}$$

and prove estimates for the corresponding linear equation of the form

$$(\partial_t + \partial_x^3 + a(x)\partial_x^2)v = f. \tag{2.3}$$

The Mizohata condition (1.45) suggests the term $a(x)\partial_x^2 v$ will not be perturbative, so we include this in the principal part and remove it by means of a gauge transform.

For Theorem 2.2 the l^2-summation is insufficient to estimate quadratic terms involving u_{xx}. However, as we are assuming that there are no uu_{xx}-type terms, we may remove these terms by means of a quadratic correction in the spirit of the normal form method of Shatah [136].

We note that the proof presented in this chapter slightly simplifies the author's previously published work [48, 51]. First, we use a slightly different rescaling that is adapted to the spaces rather than the nonlinearities. Second, in the proof of Theorem 2.2 we use a normal form instead of a paradifferential decomposition and gauge transform as in [48]. The normal form is essentially the first two terms of the Taylor expansion of the exponential gauge used in the original article.

Further questions. We conclude this introduction with further questions motivated by this work.

As in [115, 116] our small data result may be extended to smooth F that behaves quadratically for Theorem 2.1 or cubically for Theorem 2.2 near $(u, u_x, u_{xx}) = 0$. However, it is not clear that the gauge transform may be extended so straightforwardly in the large data case. Similarly, it would be of interest to extend Theorems 2.1 and 2.2 to systems of equations for large data. This would allow us to handle the nonlinearity $F = u\bar{u}_{xx}$ for large, complex-valued initial data. This problem has been considered by Kenig-Staffilani [81] for initial data in weighted spaces.

Another problem would be to consider genuinely a quasilinear version of equation (2.1) of the form

$$u_t + a(u, u_x, u_{xx})u_{xxx} = F(u, u_x, u_{xx}).$$

Using similar ideas to [115, 116] one would expect to be able to extend Theorems 2.1 and 2.2 to this case for small initial data. Local well-posedness for initial data in weighted spaces at high regularities has also been established [3, 13, 21]. For large data, recent results for the quasilinear NLS in translation-invariant spaces have been announced by Marzuola, Metcalfe and Tataru and it is likely that similar techniques might apply to the KdV setting.

A further question would be to whether one might obtain sharper well-posedness results for specific nonlinearities. While we are able to significantly relax the regularity assumptions for certain nonlinearities (see §2.A) it is likely that by assuming additional structure one could lower the threshold still further.

2.2 Function spaces

In this section we outline the construction and basic properties of the function spaces needed in the proof of Theorems 2.1 and 2.2. We consider time-dependent function spaces to be defined on the unit time interval $[0, 1]$.

Elementary estimates. We may replace the spatial partition of unity by a frequency localized version up to rapidly decaying tails. For $q \in [1, \infty]$ and $1 \le s \le r \le \infty$ we then have the following version of the Bernstein inequality (1.11),

$$\|P_N u\|_{l^p_N L^q_t L^r_x} \lesssim N^{\frac{1}{s}-\frac{1}{r}} \|P_N u\|_{l^p_N L^q_t L^s_x}. \tag{2.4}$$

In order to produce both the linear and nonlinear estimates we will need to change the scale of the l^p-summation. The following lemma gives us the estimates required to do this:

Lemma 2.7. *For $1 \leq p \leq q \leq \infty$ we have the estimate*

$$\|u\|_{l^p_N L^q} \lesssim \begin{cases} N^{\frac{2}{q}-\frac{2}{p}} M^{\frac{2}{p}-\frac{2}{q}} \|u\|_{l^p_M L^q}, & N \leq M, \\ \|u\|_{l^p_M L^q}, & N > M. \end{cases} \tag{2.5}$$

For $1 \leq q \leq p \leq \infty$ we have corresponding dual the estimate

$$\|u\|_{l^p_N L^q} \lesssim \begin{cases} \|u\|_{l^p_M L^q}, & N \leq M, \\ N^{\frac{2}{q}-\frac{2}{p}} M^{\frac{2}{p}-\frac{2}{q}} \|u\|_{l^p_M L^q}, & N > M. \end{cases} \tag{2.6}$$

Proof. It suffices to consider the estimate (2.5) as (2.6) follows from duality. Using the embedding $l^p \subset l^q$, we have

$$\|u\|_{l^p_N L^q} \sim \|u\|_{l^p_N l^q_M L^q} \lesssim \|u\|_{l^p_N l^p_M L^q} \sim \|u\|_{l^p_M l^p_N L^q}.$$

If $N \leq M$ we over-count when we change scale, so applying Hölder's inequality to the summation in N we get

$$\|u\|_{l^p_M l^p_N L^q} \lesssim \left(\frac{M^2}{N^2}\right)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{l^p_M l^q_N L^q} \lesssim \left(\frac{M^2}{N^2}\right)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{l^p_M L^q}.$$

If $N > M$ we are simply subdividing the scale N intervals, so we may estimate

$$\|u\|_{l^p_M l^p_N L^q} \lesssim \|u\|_{l^p_M L^q}.$$

□

The solution space $l^p X^s$. In view of the local energy decay estimate (1.36) and recalling that for $Q \in \mathcal{Q}_M$, $|Q| = M^2$, we define the local energy space X with norm

$$\|u\|_X = \sup_{\substack{M \geq 1 \\ M \in 2^{\mathbb{Z}}}} \sup_{Q \in \mathcal{Q}_M} M^{-1} \|u\|_{L^2_{t,x}([0,1]\times Q)}.$$

We then define our solution space $l^p X^s \subset C([0,1], l^p H^s)$ with norm

$$\|u\|^2_{l^p X^s} = \|P_{\leq 1} u\|^2_{l^p_1 X_1} + \sum_{N>1} N^{2s} \|P_N u\|^2_{l^p_N X_N},$$

where we define

$$\|u\|_{X_N} = \|u\|_{L^\infty_t L^2_x} + N\|u\|_X.$$

The inhomogeneous space l^pY^s. We define a Y-atom to be a function a with $\operatorname{supp} a \subset [0,1]\times Q$ where $Q \in \mathcal{Q}_M$ for some $M \geq 1$ such that $\|a\|_{L^2_{t,x}([0,1]\times Q)} \lesssim M^{-1}$. We then define the atomic space Y with norm

$$\|f\|_Y = \inf\left\{\sum |\lambda_j| : f = \sum \lambda_j a_j,\ a_j \text{ atoms}\right\}.$$

We note that with respect to the usual L^2-duality, $Y^* = X$ [115, Proposition 2.1]. We then define the space l^pY^s with norm

$$\|f\|^2_{l^pY^s} = \|P_{\leq 1} f\|^2_{l^p_1 Y_1} + \sum_{N>1} N^{2s} \|P_N f\|^2_{l^p_N Y_N},$$

where we define

$$\|f\|_{Y_N} = \inf_{f=f_1+f_2} \left\{ \|f_1\|_{L^1_t L^2_x} + N^{-1}\|f_1\|_Y \right\}.$$

In order to take advantage of the local smoothing effects we will use the following estimate for the Y_N space:

Lemma 2.8. *For $N \geq M$ we have the estimate*

$$\|f\|_{l^p_N Y_N} \lesssim N^{1-\frac{2}{p}} M^{\frac{2}{p}-1} \|f\|_{l^p_M L^2_{t,x}}. \tag{2.7}$$

Proof. We first change summation scale to get

$$\|f\|_{l^p_N Y_N} \lesssim N^{1-\frac{2}{p}} M^{\frac{2}{p}-2} \|f\|_{l^p_M Y}.$$

If $Q \in \mathcal{Q}_M$ then $a_Q = M^{-1}\|f\|^{-1}_{L^2_{t,x}} \chi_Q f$ is a Y-atom and hence $\|a_Q\|_Y \leq 1$. As a consequence

$$\|\chi_Q f\|_Y \leq M \|\chi_Q f\|_{L^2_{t,x}}.$$

The estimate (2.7) then follows from summation over $Q \in \mathcal{Q}_M$. □

2.3 Nonlinear estimates

In this section we prove a number of nonlinear estimates for the spaces l^1H^s, l^pX^s and l^pY^s.

Bilinear estimates. We first consider bilinear estimates for the initial data space l^pH^s.

Proposition 2.9. *For $p = 1, 2$ and $s > \frac{1}{2}$, the space $l^p H^s$ is an algebra,*

$$\|uv\|_{l^p H^s} \lesssim \|u\|_{l^p H^s} \|v\|_{l^p H^s}, \tag{2.8}$$

and for $\alpha > s$ we have the estimate

$$\|uv\|_{l^p H^s} \lesssim \|u\|_{B^{\alpha,\infty}_{\infty}} \|v\|_{l^p H^s}. \tag{2.9}$$

Proof. Considering the Littlewood-Paley trichotomy (see §1.1) it suffices to consider high-low, low-high and high-high bilinear interactions.

A(i). Algebra estimate: high-low interactions. We estimating the low frequency term in L^∞ and then apply Bernstein's inequality. Using that $s > \frac{1}{2}$ we may then sum the low frequencies using the Cauchy-Schwarz inequality,

$$\begin{aligned} \|u_N v_{\ll N}\|_{l^p H^s} &\lesssim \sum_{M \ll N} \|u_N\|_{l^p H^s} \|v_M\|_{L^\infty} \\ &\lesssim \sum_{M \ll N} M^{\frac{1}{2}} \|u_N\|_{l^p H^2} \|u_M\|_{l^p_M L^2} \\ &\lesssim \|u_N\|_{l^p H^s} \|v\|_{l^p H^s}, \end{aligned}$$

The estimate for the high-low interactions then follows from summation in N. The symmetric low-high interactions are similar.

A(ii). Algebra estimate: high-high interactions. We first use Bernstein's inequality at the low frequency N, then change summation scale and sum the high comparable frequencies using the Cauchy-Schwarz inequality,

$$\begin{aligned} \|P_N(u_{\gtrsim N} u_{\gtrsim N})\|_{l^p H^s} &\lesssim N^s \|P_N(u_{\gtrsim N} u_{\gtrsim N})\|_{l^p_N L^2} \\ &\lesssim \sum_{M_1 \sim M_2 \gtrsim N} N^{s+\frac{1}{2}} \|u_{M_1} u_{M_2}\|_{l^p_{M_1} L^1} \\ &\lesssim \sum_{M_1 \sim M_2 \gtrsim N} N^{s+\frac{1}{2}} \|u_{M_1}\|_{l^p_{M_1} L^2} \|u_{M_2}\|_{L^2} \\ &\lesssim N^{\frac{1}{2}-s} \|u\|_{l^p H^s} \|u\|_{l^p H^s}. \end{aligned}$$

The estimate then follows from summation in N, using that $s > \frac{1}{2}$.

B(i). Besov space estimate: high-low interactions. As we are considering an asymmetric estimate, we must place u into L^∞ and v into L^2. We then change summation scale and sum in the low frequencies using the Cauchy-Schwarz inequality to get

$$\|u_N v_{\ll N}\|_{l^p H^s} \lesssim \sum_{M \ll N} N^s \|u_N\|_{L^\infty} \|v_M\|_{l^p_N L^2} \lesssim N^{s-\alpha} \|u\|_{B^{\alpha,\infty}_{\infty}} \|v\|_{l^p H^s}.$$

We may then use that $\alpha > s$ to sum in N.

B(ii). Besov space estimate: low-high interactions. This estimate is similar to the high-low interactions, placing u into L^∞ and v into L^2 to get

$$\|u_{\ll N} v_N\|_{l^p H^s} \lesssim \sum_{M \ll N} \|u_M\|_{L^\infty} \|v_N\|_{l^p H^s} \lesssim \|u\|_{B^{\alpha,\infty}_\infty} \|v_N\|_{l^p H^s}.$$

B(iii). Besov space estimate: high-high interactions. We estimate similarly, considering

$$\begin{aligned}\|P_N(u_{\gtrsim N} v_{\gtrsim N})\|_{l^p H^s} &\lesssim \sum_{M_1 \sim M_2 \gtrsim N} N^s \|u_{M_1}\|_{L^\infty} \|v_{M_2}\|_{l^p_N L^2} \\ &\lesssim \sum_{M_2 \gtrsim N} N^{s+1-\frac{2}{p}} M_2^{\frac{2}{p}-1-\alpha-s} \|u\|_{B^{\alpha,\infty}_\infty} \|v_{M_2}\|_{l^p H^s} \\ &\lesssim N^{-\alpha} \|u\|_{B^{\alpha,\infty}_\infty} \|v\|_{l^p H^s},\end{aligned}$$

where we have used that $\alpha + s > 2s > \frac{2}{p} - 1$ in the last inequality. The estimate then follows from summation in N. □

Next we prove bilinear estimates for the spaces $l^p X^s$ and $l^p Y^s$.

Proposition 2.10. *For $p = 1, 2$ we have the following estimates.*

A. Algebra estimate. If $s > \frac{1}{2}$ then $l^p X^s$ is an algebra,

$$\|uv\|_{l^p X^s} \lesssim \|u\|_{l^p X^s} \|v\|_{l^p X^s}. \tag{2.10}$$

B. Bilinear $X \times X \to Y$ estimate. If $\alpha, \beta \geq s - \frac{2}{p}$ and $\alpha + \beta > s + \frac{1}{2}$,

$$\|uv\|_{l^p Y^s} \lesssim \|u\|_{l^p X^\alpha} \|v\|_{l^p X^\beta}. \tag{2.11}$$

C. Besov space estimates. For $\alpha > s$ and $s > \frac{1}{2}$, we have the estimates

$$\|uv\|_{l^p X^s} \lesssim \|u\|_{B^{\alpha+1,\infty}_\infty} \|v\|_{l^p X^s}, \tag{2.12}$$

$$\|uv\|_{l^p Y^s} \lesssim \|u\|_{B^{\alpha+2-\frac{2}{p},\infty}_\infty} \|v\|_{l^p Y^s}. \tag{2.13}$$

Proof. We again use the Littlewood-Paley trichotomy and consider the high-low, low-high and high-high interactions.

A(i). Algebra estimate: high-low interactions. We proceed similarly to the proof of Proposition 2.9, estimating the low frequency term in L^∞, then applying Bernstein's inequality and summing using the Cauchy-Schwarz inequality using that $s > \frac{1}{2}$,

$$\begin{aligned}\|u_N v_{\ll N}\|_{l^p X^s} &\lesssim \sum_{M \ll N} \|u_N\|_{l^p X^s} \|v_N\|_{L^\infty_{t,x}} \\ &\lesssim \sum_{M \ll N} M^{\frac{1}{2}} \|u_N\|_{l^p X^s} \|v_N\|_{L^\infty_t L^2_x} \\ &\lesssim \|u_N\|_{l^p X^s} \|v\|_{l^p X^s},\end{aligned}$$

We may then sum in N to prove the estimate for the high-low interactions. The symmetric low-high interactions are similar.

A(ii). Algebra estimate: high-high interactions. We first use Bernstein's inequality at the low frequency N, then change summation scale and summing the comparable high frequencies using the Cauchy-Schwarz inequality to get,

$$\begin{aligned}
\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^pX^s} &\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+\frac{1}{2}}\|u_{M_1}\|_{l^p_N X_N}\|v_{M_2}\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+\frac{1}{2}-\frac{2}{p}}M_1^{\frac{2}{p}}\|u_{M_1}\|_{l^p_{M_1}X_{M_1}}\|v_{M_2}\|_{l^p_{M_2}X_{M_2}}\\
&\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+\frac{1}{2}-\frac{2}{p}}M_1^{\frac{2}{p}-2s}\|u_{M_1}\|_{l^pX^s}\|v_{M_2}\|_{l^pX^s}\\
&\lesssim N^{\frac{1}{2}-s}\|u\|_{l^pX^s}\|v\|_{l^pX^s},
\end{aligned}$$

where we have used that $s > \frac{1}{p}$ in the last inequality. We may then sum in N whenever $s > \frac{1}{2}$ to complete the estimate.

B(i). Bilinear $X \times X \to Y$ estimates: high-low interactions. In order to take advantage of the local energy decay spaces, we will estimate the product uv in the Y-space using the estimate (2.7). We then place the high frequency term into the local energy space X and use Bernstein's inequality at low frequency to get

$$\begin{aligned}
\|u_N v_{\ll N}\|_{l^pY^s} &\lesssim \sum_{M\ll N} N^{s+1-\frac{2}{p}}M^{\frac{2}{p}-1}\|u_N v_M\|_{l^p_M L^2_{t,x}}\\
&\lesssim \sum_{M\ll N} N^{s+1-\frac{2}{p}}M^{\frac{2}{p}-1}\|u_N\|_{l^\infty_M L^2_{t,x}}\|v_M\|_{l^p_M L^\infty_{t,x}}\\
&\lesssim \sum_{M\ll N} N^{s-\frac{2}{p}}M^{\frac{2}{p}+\frac{1}{2}}\|u_N\|_{X_N}\|v_M\|_{l^p_M L^\infty_t L^2_x}\\
&\lesssim \sum_{M\ll N} N^{s-\frac{2}{p}-\beta}M^{\frac{2}{p}+\frac{1}{2}-\alpha}\|u_N\|_{l^pX^\alpha}\|v_M\|_{l^pX^\beta}.
\end{aligned}$$

Using Minkowski's inequality to exchange the order of summation, we may first sum over $N \gg M$ using that $\beta > s - \frac{2}{p}$ and then sum in M using the Cauchy-Schwarz inequality and that $\alpha + \beta > s + \frac{1}{2}$.

B(ii). Bilinear $X \times X \to Y$ estimates: high-high interactions. We again look to take advantage of the local energy decay spaces by estimating uv in Y using (2.7) with $N = M$. We then use Bernstein's inequality at the low frequency N, change summation scale and use the Cauchy-Schwarz inequality in the comparable high frequencies to get

$$\begin{aligned}\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^pY^s} &\lesssim \sum_{M_1\sim M_2\gtrsim N} N^s\|P_N(u_{M_1}v_{M_2})\|_{l^p_N L^2_{t,x}} \\ &\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+1-\frac{2}{p}}M_1^{\frac{2}{p}-1}\|P_N(u_{M_1}v_{M_2})\|_{l^p_{M_1}L^2_tL^1_x} \\ &\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+\frac{3}{2}-\frac{2}{p}}M_1^{\frac{2}{p}-1}\|u_{M_1}\|_{l^p_{M_1}L^\infty_tL^2_x}\|v_{M_2}\|_{l^\infty_{M_1}L^2_{t,x}} \\ &\lesssim \sum_{M_1\sim M_2\gtrsim N} N^{s+\frac{3}{2}-\frac{2}{p}}M_1^{\frac{2}{p}-1}\|u_{M_1}\|_{l^p_{M_1}L^\infty_tL^2_x}\|v_{M_2}\|_{l^p_{M_1}X_{M_1}} \\ &\lesssim N^{s+\frac{1}{2}-\alpha-\beta}\|u\|_{l^1X^\alpha}\|u\|_{l^1X^\beta}.\end{aligned}$$

Finally we may sum in N using that $\alpha+\beta > s+\frac{1}{2}$ to complete the estimate.

C(i). Besov estimates: High-low interactions. As we are once again considering an asymmetric estimate, we place u into L^∞, change summation scale and then sum using the Cauchy-Schwarz inequality to get

$$\begin{aligned}\|u_Nv_{\ll N}\|_{l^pX^s} &\lesssim \sum_{M\ll N} N^s\|u_N\|_{L^\infty}\|v_M\|_{l^p_NX_N} \\ &\lesssim \sum_{M\ll N} N^{s+1}M^{-1}\|u_N\|_{L^\infty}\|v_M\|_{l^p_MX_M} \\ &\lesssim N^{s-\alpha}\|u\|_{B^{\alpha+1,\infty}_\infty}\|v\|_{l^pX^s}.\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}\|u_Nv_{\ll N}\|_{l^pY^s} &\lesssim \sum_{M\ll N} N^s\|u_N\|_{L^\infty}\|v_M\|_{l^p_NY_N} \\ &\lesssim \sum_{M\ll N} N^{s+2-\frac{2}{p}}M^{\frac{2}{p}-2}\|u_N\|_{L^\infty}\|v_M\|_{l^p_MY_M} \\ &\lesssim N^{s-\alpha}\|u\|_{B^{\alpha+2-\frac{2}{p},\infty}_\infty}\|v\|_{l^pY^s}.\end{aligned}$$

We may then sum in N the estimates whenever $\alpha > s$.

C(ii). Besov estimates: Low-high interactions. Estimating the low frequency term in L^∞ and summing using the Cauchy-Schwarz inequality we have

$$\|u_{\ll N}v_N\|_{l^pX^s} \lesssim \sum_{M\ll N}\|u_M\|_{L^\infty}\|v_N\|_{l^pX^s} \lesssim \|u\|_{B^{\alpha+1,\infty}_\infty}\|v_N\|_{l^pX^s}.$$

Similarly we may estimate,

$$\|u_{\ll N}v_N\|_{l^pY^s} \lesssim \sum_{M\ll N}\|u_M\|_{L^\infty}\|v_N\|_{l^pY^s} \lesssim \|u\|_{B^{\alpha+2-\frac{2}{p},\infty}_\infty}\|v_N\|_{l^pY^s}.$$

The estimates then follow from summation in N.

C(iii). Besov estimates: High-high interactions. Again we estimate u in L^∞, change summation scale and sum using the Cauchy-Schwarz inequality to get

$$\begin{aligned}\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^pX^s} &\lesssim \sum_{M_1\sim M_2\gtrsim N} M_2^{\frac{2}{p}}N^{s-\frac{2}{p}}\|u_{M_1}\|_{L^\infty}\|u_{M_2}\|_{l^p_M X_M}\\ &\lesssim N^{-1-\alpha}\|u\|_{B^{\alpha+1,\infty}_\infty}\|u\|_{l^pX^s}\end{aligned}$$

where we have used that $s+\alpha > 2s > \frac{2}{p}-1$ in the second inequality.

Proceeding similarly, we have

$$\begin{aligned}\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^pY^s} &\lesssim \sum_{M_1\sim M_2\gtrsim N} M_2^{3-\frac{2}{p}}N^{s+\frac{2}{p}-3}\|u_{M_1}\|_{L^\infty}\|u_{M_2}\|_{l^p_{M_2}Y_{M_2}}\\ &\lesssim N^{\frac{2}{p}-2-\alpha}\|u\|_{B^{\alpha+2-\frac{2}{p},\infty}_\infty}\|u\|_{l^pY^s}\end{aligned}$$

where we have used that $s+\alpha > 2s > 1$ in the second inequality.

The estimates then follow from summation in N

□

As a corollary to the proof of Proposition 2.10, we have the following frequency localized bilinear estimates in the case $p=2$.

Corollary 2.11. *For $s>\frac{1}{2}$ and $\alpha+\beta > s+\frac{1}{2}$ we have the following estimates.*

A. Frequency localized algebra estimates.

$$\|u_{\ll N}v_N\|_{l^2X^s} \lesssim \|u\|_{l^2X^\alpha}\|v_N\|_{l^2X^s}, \qquad \alpha > \tfrac{1}{2}, \tag{2.14}$$

$$\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^2X^s} \lesssim N^{s+\frac{1}{2}-\alpha-\beta}\|u\|_{l^2X^\alpha}\|v\|_{l^2X^\beta}. \tag{2.15}$$

B. Frequency localized $X\times X\to Y$ estimates.

$$\|u_{\ll N}v_N\|_{l^2Y^s} \lesssim \|u\|_{l^2X^\alpha}\|v_N\|_{l^2X^{s-1}}, \qquad \alpha > \tfrac{3}{2}, \tag{2.16}$$

$$\|P_N(u_{\gtrsim N}v_{\gtrsim N})\|_{l^2Y^s} \lesssim N^{s+\frac{1}{2}-\alpha-\beta}\|u\|_{l^2X^\alpha}\|v\|_{l^2X^\beta}. \tag{2.17}$$

Trilinear estimates. As the $p=2$ case of the bilinear estimate (2.11) cannot handle terms with two derivatives at high frequency, we require an improved trilinear estimate for Theorem 2.2.

Proposition 2.12. *If $\alpha,\beta,\gamma \geq s-2$, $\alpha+\beta+\gamma > s+1$ and $\alpha+\beta, \beta+\gamma, \gamma+\alpha > s-\frac{1}{2}$ then*

$$\|uvw\|_{l^2Y^s} \lesssim \|u\|_{l^2X^\alpha}\|v\|_{l^2X^\beta}\|w\|_{l^2X^\gamma}. \tag{2.18}$$

We note that with respect to the usual L^2-duality, $(l^2_N Y_N)^* = l^2_N X_N$. In order to prove Proposition 2.12 we first prove the following lemma, which will allow us to prove trilinear estimates by duality.

Lemma 2.13. *If $N_1 \leq N_2 \leq N_3 \leq N_4$, we have the estimate*

(2.19)
$$\int u_{N_1} v_{N_2} w_{N_3} z_{N_4}\, dxdt \lesssim N_1^{\frac{3}{2}} N_2^{\frac{3}{2}} N_3^{-1} N_4^{-1} \|u_{N_1}\|_{l^2_{N_1} X_{N_1}} \|v_{N_2}\|_{l^2_{N_2} X_{N_2}} \|w_{N_3}\|_{l^2_{N_3} X_{N_3}} \|z_{N_4}\|_{l^2_{N_4} X_{N_4}}.$$

Proof. We will aim to place the highest frequencies N_3, N_4 into the local energy space X by introducing a partition of unity at the scale of the lowest frequency N_1. We then use Bernstein's inequality in the the low frequencies N_1, N_2 and then changing summation scale using to get

$$\begin{aligned}\int u_{N_1} v_{N_2} w_{N_3} z_{N_4}\, dxdt &= \sum_{Q \in \mathcal{Q}_{N_1}} \int \chi_Q u_{N_1} \chi_Q v_{N_2} w_{N_3} z_{N_4}\, dxdt \\ &\lesssim \|u_{N_1}\|_{l^2_{N_1} L^\infty_{t,x}} \|v_{N_2}\|_{l^2_{N_1} L^\infty_{t,x}} \|w_{N_3}\|_{l^\infty_{N_1} L^2_{t,x}} \|z_{N_4}\|_{l^\infty_{N_1} L^2_{t,x}} \\ &\lesssim N_1^{\frac{5}{2}} N_2^{\frac{1}{2}} N_3^{-1} N_4^{-1} \|u_{N_1}\|_{l^2_{N_1} L^\infty_t L^2_x} \|v_{N_2}\|_{l^2_{N_1} L^\infty_t L^2_x} \|w_{N_3}\|_{X_{N_3}} \|z_{N_4}\|_{X_{N_4}} \\ &\lesssim N_1^{\frac{3}{2}} N_2^{\frac{3}{2}} N_3^{-1} N_4^{-1} \|u_{N_1}\|_{l^2_{N_1} X_{N_1}} \|v_{N_2}\|_{l^2_{N_2} X_{N_2}} \|w_{N_3}\|_{l^2_{N_3} X_{N_3}} \|z_{N_4}\|_{l^2_{N_3} X_{N_4}}.\end{aligned}$$

□

Proof of Proposition 2.12. We consider a sum of terms of the form $P_N(u_{N_1} v_{N_2} w_{N_3})$ and by symmetry we may assume that $1 \leq N_1 \leq N_2 \leq N_3$. We will argue by duality, using Lemma 2.13 to produce frequency localized bounds in $l^2 Y_N$.

We note that the integral in (2.19) vanishes unless the two largest frequencies are comparable. As such we may divide the proof into two cases, the first when $N \gtrsim N_3$ and hence $N \sim N_3$, and the second when $N \ll N_3$ and hence $N_2 \sim N_3$.

A. Output high: $N \gtrsim N_3$. In this case we must have $N \sim N_3$. By duality and symmetry in the highest frequency terms in (2.19), we have the estimate

$$\begin{aligned}\|P_N(u_{N_1} v_{N_2} w_{N_3})\|_{l^2 Y^s} &\lesssim N_1^{\frac{3}{2}-\alpha} N_2^{\frac{3}{2}-\beta} N_3^{-1-\gamma} N^{s-1} \|u_{N_1}\|_{l^2 X^\alpha} \|v_{N_2}\|_{l^2 X^\beta} \|w_{N_3}\|_{l^2 X^\gamma} \\ &\lesssim N_1^{\frac{3}{2}-\alpha} N_2^{s-\frac{1}{2}-\beta-\gamma} \|u_{N_1}\|_{l^2 X^\alpha} \|v_{N_2}\|_{l^2 X^\beta} \|w_{N_3}\|_{l^2 X^\gamma},\end{aligned}$$

where we have used that $\gamma > s-2$ and $N, N_3 \gtrsim N_2$. We first sum in $N_2 \geq N_1$ using the Cauchy-Schwarz inequality and that $\beta + \gamma > s - \frac{1}{2}$ to get

$$\sum_{N_2} \|P_N(u_{N_1} v_{N_2} w_{N_3})\|_{l^2 X^s} \lesssim N_1^{s+1-\alpha-\beta-\gamma} \|u_{N_1}\|_{l^2 X^\alpha} \|v\|_{l^2 X^\beta} \|w_{N_3}\|_{l^2 X^\gamma}.$$

Next we sum in N_1, again using the Cauchy-Schwarz inequality and that $\alpha+\beta+\gamma > s+1$,

$$\sum_{N_1} N_1^{s+1-\alpha-\beta-\gamma}\|u_{N_1}\|_{l^2X^\alpha}\|v\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma} \lesssim \|u\|_{l^2X^\alpha}\|v\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma}.$$

Finally we sum in $N \sim N_3$ to complete the estimate.

B. Output low: $N \ll N_3$. In this case we must have $N_2 \sim N_3$. Again using duality and symmetry in the lowest order terms in (2.19), we have

$$\|P_N(u_{N_1}v_{N_2}w_{N_3})\|_{l^2X^s} \lesssim N_1^{\frac{3}{2}-\alpha}N^{\frac{3}{2}+s}N_3^{-2-\beta-\gamma}\|u_{N_1}\|_{l^2X^\alpha}\|v_{N_2}\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma}.$$

Using Minkowski's inequality to exchange the order of summation, we first sum in $N \ll N_3$ to get

$$\begin{aligned}\left(\sum_N \|P_N(u_{N_1}v_{N_2}w_{N_3})\|_{l^2X^s}^2\right)^{\frac{1}{2}} &\lesssim N_1^{\frac{3}{2}-\alpha}N_3^{s-\frac{1}{2}-\beta-\gamma}\|u_{N_1}\|_{l^2X^\alpha}\|v_{N_2}\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma}\\ &\lesssim N_1^{s+1-\alpha-\beta-\gamma}\|u_{N_1}\|_{l^2X^\alpha}\|v_{N_2}\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma},\end{aligned}$$

where we have used that $N_3 \geq N_1$ and that $\beta+\gamma > s-\frac{1}{2}$ in the second inequality. Using the Cauchy-Schwarz inequality we may then sum in $N_1 \leq N_2$ using that $\alpha+\beta+\gamma > s+1$ to get

$$\sum_{N_1} N_1^{s+1-\alpha-\beta-\gamma}\|u_{N_1}\|_{l^2X^\alpha}\|v_{N_2}\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma} \lesssim \|u\|_{l^2X^\alpha}\|v_{N_2}\|_{l^2X^\beta}\|w_{N_3}\|_{l^2X^\gamma}.$$

Finally we sum in the comparable frequencies $N_2 \sim N_3$ using the Cauchy-Schwarz inequality to complete the estimate. □

2.4 Linear estimates

In this section we prove linear estimates for solutions in the space l^1X^s.

The linear KdV. First we consider the linear KdV equation

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = f \\ u(0) = u_0, \end{cases} \tag{2.20}$$

and have the following well-posedness result for (1.16) that we prove in a similar way to the [115, Proposition 4.1].

Proposition 2.14. *Let $s \geq 0$ and $p \in \{1, 2\}$. If $u_0 \in l^p H^s$ and $f \in l^p Y^s$ there exists a unique solution $u \in l^p X^s$ to the linear KdV (2.20) satisfying the estimate*

$$\|u\|_{l^p X^s} \lesssim \|u_0\|_{l^p H^s} + \|f\|_{l^p Y^s}. \tag{2.21}$$

Proof. It suffices to prove the a priori estimate (2.21). We first consider the frequency localized equation

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)u_N = f_N \\ u_N(0) = u_{0N}. \end{cases}$$

For the energy component of the X_N norm we take $T \in (0, 1]$ and consider

$$\begin{aligned} \|u_N(T)\|_{L_x^2}^2 &= \|u_{0N}\|_{L^2}^2 + \int_0^T \partial_t(\|u_N\|_{L^2}^2)\,dt \\ &\leq \|u_{0N}\|_{L^2}^2 + 2\langle u_N, f_N\rangle_{t,x} \\ &\leq \|u_{0N}\|_{L^2}^2 + 2\|u_N\|_{X_N}\|f_N\|_{Y_N}, \end{aligned}$$

where we have used that $(Y_N)^* = X_N$ in the final inequality. Taking the supremum over $T \in [0, 1]$ we have

$$\|u_N\|_{L_t^\infty L_x^2}^2 \lesssim \|u_{0N}\|_{L^2}^2 + \|u_N\|_{X_N}\|f_N\|_{Y_N}. \tag{2.22}$$

For the local energy component we use a positive commutator argument: for each dyadic $M \geq 1$ and $Q \in \mathcal{Q}_M$ we construct a self-adjoint operator $\mathbf{A}$ such that

(A1) $\|\mathbf{A}u_N\|_{L_x^2} \lesssim \|u_N\|_{L_x^2}$,

(A2) $\|\mathbf{A}u_N\|_X \lesssim \|u_N\|_X$,

(A3) $N^2 M^{-2}\|u_N\|_{L_{t,x}^2([0,1]\times Q)}^2 \lesssim \langle[\frac{1}{3}\partial_x^3, \mathbf{A}]u_N, u_N\rangle_{t,x} + \|u_N\|_{L_{t,x}^2}^2$.

Suppose that such an operator exists, then

$$\partial_t\langle u_N, \mathbf{A}u_N\rangle = 2\langle f_N, \mathbf{A}u_N\rangle + \langle[\partial_x^3, \mathbf{A}]u_N, u_N\rangle.$$

Integrating in time over the interval $[0, 1]$ we may then use (A1)–(A3) to get

$$N^2 M^{-2}\|u_N\|_{L_{t,x}^2([0,1]\times Q)}^2 \lesssim \|u_{N0}\|_{L^2}^2 + \|u_N\|_{L_t^\infty L_x^2}^2 + \|u_N\|_{X_N}\|f_N\|_{Y_N}.$$

Taking the supremum over $M \geq 1$ and using (2.22) we then have

$$N^2\|u_N\|_X^2 \lesssim \|u_{N0}\|_{L^2}^2 + \|u_N\|_{X_N}\|f_N\|_{Y_N}. \tag{2.23}$$

We now construct the operator $\mathbf{A}$. By translation invariance we may assume that the interval $Q = [-\frac{1}{2}M^2, \frac{1}{2}M^2]$. Let $\psi \in \mathcal{S}(\mathbb{R})$ be a real-valued function ~ 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and localized at frequency $\lesssim 1$. We then take $a \in C^\infty$ to be an antiderivative of ψ^2 and rescale by taking $a_M(x) = a(M^{-2}x)$. We define $\mathbf{A}$ to be multiplication by a_M, which evidently satisfies the properties (A1) and (A2). To prove (A3) we simply integrate by parts to get

$$\begin{aligned}\langle[\tfrac{1}{3}\partial_x^3, \mathbf{A}]u_N, u_N\rangle &= \tfrac{1}{3}\langle\partial_x^3 a u_N, u_N\rangle - \langle\partial_x a \partial_x u_N, \partial_x u_N\rangle \\ &= \tfrac{1}{3}\langle\partial_x^3 a u_N, u_N\rangle + M^{-2}\langle\psi(M^{-2}x)^2\partial_x u_N, \partial_x u_N\rangle \\ &\gtrsim N^2 M^{-2}\|u_N\|^2_{L^2_{t,x}([0,1]\times Q)} - O(\|u_N\|^2_{L^2}).\end{aligned}$$

Combining the estimates (2.22) and (2.23) we then have

$$\|u_N\|^2_{X_N} \lesssim \|u_{N0}\|^2_{L^2} + \|f_N\|^2_{Y_N}. \tag{2.24}$$

In order to prove the estimate with l^p_N summation, we take $Q \in \mathcal{Q}_{KN}$ for some large fixed dyadic $K \gg 1$. We then take $\chi_Q \in \mathcal{S}(\mathbb{R})$ to be spatially localized on Q up to rapidly decaying tails and localized at frequency $\lesssim (KN)^{-2}$. We then have the equation for $\chi_Q u_N$,

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)(\chi_Q u_N) = \chi_Q f_N + [\frac{1}{3}\partial_x^3, \chi_Q]u_N \\ \chi_Q u_N(0) = \chi_Q u_{0N}. \end{cases}$$

From the estimate (2.24) we have

$$\|\chi_Q u_N\|^2_{X_N} \lesssim \|\chi_Q u_{0N}\|^2_{L^2} + \|\chi_Q f_N\|^2_{Y_N} + \|[\tfrac{1}{3}\partial_x^3, \chi_Q]u_N\|^2_{Y_N}.$$

To estimate the commutator term we use the localization of χ_Q, u_N to estimate

$$\sum_{Q\in\mathcal{Q}_{KN}} \|[\tfrac{1}{3}\partial_x^3, \chi_Q]u_N\|^p_{L^1_t L^2_x} \lesssim K^{-2p}\|u_N\|^p_{l^p_{KN} L^\infty_t L^2_x}.$$

Choosing $K \gg 1$ to be sufficiently large, independent of the size of N, we have

$$\|u_N\|_{l^p_{KN} X_N} \lesssim \|u_{0N}\|_{l^p_{KN} L^2} + \|f_N\|_{l^p_{KN} Y_N}.$$

Finally we may argue as in Lemma 2.7 to change scale, which gives us the estimate (2.21). □

The large data equation. In order to handle large data we consider the linear equation

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} + a u_{xx} = f \\ u(0) = u_0. \end{cases} \tag{2.25}$$

For $p \in \{1, 2\}$ let $N_0 \sim 1$ be a fixed dyadic integer and define the space $Z \subset l^p L^2 \cap \partial_x L^\infty$ to consist of functions $a = a(x)$ localized at frequencies $\leq N_0$ and satisfying

$$\|a\|_Z = \|a\|_{l^p L^2} + \|\partial_x^{-1} a\|_{L^\infty} < \infty,$$

where we define ∂_x^{-1} as in (1.7).

Proposition 2.15. *Suppose that* $s > \frac{3}{2}$, $p \in \{1, 2\}$ *and* $a \in Z$ *satisfies the estimate*

$$\|a\|_Z \leq K_0.$$

Then there exists a constant $C_* = C_*(s, p, N_0) \gg 1$ *so that whenever*

$$\|a_x\|_Z \leq e^{-C_*(1+K_0)},$$

there exists a unique solution $u \in l^1 X^s$ *to* (2.25) *satisfying the estimate*

$$\|u\|_{l^p X^s} \lesssim e^{CK_0}(\|u_0\|_{l^p H^s} + \|f\|_{l^p Y^s}), \tag{2.26}$$

where the constants depend only on s, p, N_0.

Proof. We will consider the case $p = 1$ as the case $p = 2$ is similar. We take $\Psi = \partial_x^{-1} a$ and calculate

$$e^{\Psi}(\partial_t + \tfrac{1}{3}\partial_x^3 + a\partial_x^2)(e^{-\Psi} w) = w_t + \tfrac{1}{3} w_{xxx} - (a_x + a^2) w_x + (\tfrac{2}{3} a^3 - \tfrac{1}{3} a_{xx}) w.$$

As a consequence, we expect that the solution u to (2.25) may be well-approximated by $v = e^{-\Psi} w$ where w is a solution to the equation

$$\begin{cases} w_t + \frac{1}{3} w_{xxx} = e^{\Psi} f \\ w(0) = e^{\Psi} u_0. \end{cases} \tag{2.27}$$

Using the Besov space estimates (2.9) and (2.13), for an integer $k \in (s, s+1]$ we have

$$\|e^{\Psi} u_0\|_{l^1 H^s} \lesssim \|e^{\Psi}\|_{B^{k,\infty}_\infty} \|u_0\|_{l^1 H^s}, \qquad \|e^{\Psi} f\|_{l^1 Y^s} \lesssim \|e^{\Psi}\|_{B^{k,\infty}_\infty} \|f\|_{l^1 Y^s}.$$

As a is localized at frequencies $\leq N_0$,

$$\|e^{\Psi}\|_{B^{k,\infty}_\infty} \lesssim \|e^{\Psi}\|_{C^k} \lesssim e^{\|a\|_Z} \langle \|a\|_{C^{k-1}} \rangle^{k-1} \lesssim e^{CK_0},$$

where the constants depend only on s, p, N_0. From Proposition 2.14, we may then find a solution w to (2.27) so that

$$\|w\|_{l^1 X^s} \lesssim e^{CK_0} \left(\|u_0\|_{l^1 H^s} + \|f\|_{l^1 Y^s} \right).$$

Taking $v = e^{-\Psi} w$ we have

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3 + a\partial_x^2) v = f - \big((a_x + a^2) v_x + (\frac{1}{3} a_{xx} + a a_x + \frac{1}{3} a^3) v \big) \\ v(0) = u_0, \end{cases}$$

and estimating similarly,

$$\|v\|_{l^1 X^s} \lesssim e^{CK_0} \left(\|u_0\|_{l^1 H^s} + \|f\|_{l^1 Y^s} \right).$$

We now estimate the each of the error terms using the bilinear estimate (2.11), the Besov estimate (2.9), the frequency localization of a and Bernstein's inequality (1.11). For the first term we have

$$\|a_x v_x\|_{l^1Y^s} \lesssim \|a_x\|_{l^1X^s}\|v_x\|_{l^1X^{s-1}} \lesssim e^{-C_*(1+K_0)}\|v\|_{l^1X^s},$$

and for the second term,

$$\|a^2 v_x\|_{l^1Y^s} \lesssim \|a\|_{B^{\alpha,\infty}_\infty}\|a\|_{l^1X^s}\|v_x\|_{l^1X^{s-1}} \lesssim e^{-C_*(1+K_0)}K_0\|v\|_{l^1X^s}$$

The remaining terms may be estimated similarly to get

$$\|(a_x+a^2)v_x + (\tfrac{1}{3}a_{xx} + aa_x + \tfrac{1}{3}a^3)v\|_{l^1Y^s} \lesssim e^{-C_*(1+K_0)}(1+K_0)\|v\|_{l^1X^s}$$

where the constant depends only on s, p, N_0.

We now construct a solution to (2.25) by iteration. We define $v^{(0)} = v$ and for $k \geq 1$ take

$$f^{(k)} = (a_x + a^2)v_x^{(k-1)} + (\tfrac{1}{3}a_{xx} + aa_x + \tfrac{1}{3}a^3)v^{(k-1)},$$

where $v^{(k)} = e^{-\Psi}w^{(k)}$ and $w^{(k)}$ is the solution to

$$\begin{cases} w_t^{(k)} + \frac{1}{3}w_{xxx}^{(k)} = e^{\Psi}f^{(k)} \\ w^{(k)}(0) = 0. \end{cases}$$

We observe that

$$\|f^{(k)}\|_{l^1Y^s} \lesssim e^{-C_*(1+K_0)}(1+K_0)\|v^{(k-1)}\|_{l^1L^2},$$

and estimating as before,

$$\|v^{(k)}\|_{l^1X^s} \lesssim e^{(C-C_*)(1+K_0)}\|v^{(k-1)}\|_{l^1X^s}.$$

For $C_* \gg 1$ sufficiently large we have

$$\|v^{(k)}\|_{l^1X^s} \leq \tfrac{1}{2}\|v^{(k-1)}\|_{l^1X^s},$$

and hence $u = \sum_k v^{(k)}$ converges in l^1X^s to a solution to (2.25).

To prove uniqueness, suppose that $u_0 = 0 = f$. Taking $w = e^{\Psi}u$ we have

$$\begin{cases} w_t + \frac{1}{3}w_{xxx} = (a_x + a^2)w_x + (\frac{1}{3}a_{xx} - \frac{2}{3}a^3)w \\ w(0) = 0. \end{cases}$$

Estimating as above we have

$$\|w\|_{l^1X^s} \lesssim e^{-C_*(1+K_0)}(1+K_0)\|w\|_{l^1X^s},$$

so choosing $C_* \gg 1$ sufficiently large, we obtain the estimate $\|w\|_{l^1X^s} \leq \frac{1}{2}\|w\|_{l^1X^s}$ and hence $u = w = 0$. □

2.5 Small data

In this section we prove versions of Theorems 2.1 and 2.2 for sufficiently small initial data. Our proof will rely on a contraction principle argument using the linear and nonlinear established in §2.3 and §2.4.

A small data version of Theorem 2.1 We start by considering the case that F may contain a term of the form uu_{xx}. We will assume our nonlinearity may be written as

$$F(u, u_x, u_{xx}) = \sum_{2\leq|\alpha|\leq m} c_\alpha u^{\alpha_0} u_x^{\alpha_1} u_{xx}^{\alpha_2},$$

where the coefficient $c_{(1,0,1)} \neq 0$.

Theorem 2.16. *Suppose F contains a term of the form uu_{xx}, then there exists $\sigma_1 = \sigma_1(F) \in [\frac{5}{2}, \frac{9}{2}]$ and $\epsilon = \epsilon(s, F) > 0$ sufficiently small that if $s > \sigma_1$ and $\|u_0\|_{l^1H^s} \leq \epsilon$, equation (2.1) is locally well-posed in l^1H^s on the time interval $[0, 1]$ and the solution satisfies*

$$\|u\|_{l^1X^s} \lesssim \epsilon.$$

Proof. We will use the contraction principle in the ball $B \subset l^1X^s$ of radius $M\epsilon$. For $u \in B$, let $w = \mathcal{T}(u)$ be the solution to the linear equation

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)w = F(u) \\ w(0) = u_0. \end{cases}$$

It then suffices to show that $\mathcal{T} \colon B \to B$ is a contraction for sufficiently large $M > 0$ and sufficiently small $\epsilon > 0$.

We first estimate the nonlinear term F using the bilinear estimates of Proposition 2.10. Provided $s > \frac{5}{2}$, we have

$$\|uu_{xx}\|_{l^1Y^s} \lesssim \|u\|_{l^1X^s}\|u_{xx}\|_{l^1X^{s-2}} \lesssim \|u\|_{l^1X^s}^2.$$

Choosing sufficiently large $\sigma_1 \in [\frac{5}{2}, \frac{9}{2}]$ (see §2.A) and estimating similarly we have

$$\|F(u)\|_{l^1Y^s} \lesssim (1 + \|u\|_{l^1X^s}^{m-2})\|u\|_{l^1X^s}^2.$$

Applying identical estimates to the difference we have,

$$\|F(u_1) - F(u_2)\|_{l^1Y^s} \lesssim (\|u_1\|_{l^1X^s} + \|u_2\|_{l^1X^s})(1 + \|u_1\|_{l^1X^s}^{m-2} + \|u_2\|_{l^1X^s}^{m-2})\|u_1 - u_2\|_{l^1X^s}.$$

Applying the linear estimate (2.21) we see that for sufficiently large M_0 and small $\epsilon > 0$, the map $\mathcal{T} \colon B \to B$ is a contraction. By the contraction principle we have the existence of a unique solution and that the solution map is Lipschitz. □

A small data version of Theorem 2.2 We now suppose that F contains no uu_{xx} term and prove an analogous result. The key difficulty here is that we cannot estimate quadratic terms with two derivatives at high frequency. To circumvent this problem we make use of a normal form correction to upgrade the bad quadratic interactions to cubic and higher order ones, which may then be estimated using the trilinear estimates of Proposition 2.12.

Theorem 2.17. *Suppose F does not contain a term of the form uu_{xx}. Then, there exists $\sigma_2 = \sigma_2(F) \in [\frac{1}{2}, \frac{9}{2}]$ and $\epsilon = \epsilon(s, F) > 0$ sufficiently small so that if $s > \sigma_2$ and $\|u_0\|_{H^s} \le \epsilon$, equation* (2.1) *is locally well-posed in H^s on the time interval* $[0, 1]$ *and the solution satisfies the estimate*

$$\|u\|_{l^2X^s} \lesssim \epsilon.$$

Proof. Once again we will use the contraction principle in a ball $B \subset l^2X^s$ of radius $M\epsilon$ for sufficiently large M and small $\epsilon > 0$.

We first decompose our nonlinearity into the bad quadratic terms involving u_{xx} and the remaining good quadratic terms in u, u_x and cubic and higher order terms,

$$F(u, u_x, u_{xx}) = C_1 u_x u_{xx} + C_2 u_{xx}^2 + F_0(u, u_x, u_{xx}).$$

Choosing $\sigma_2 \in [\frac{1}{2}, \frac{9}{2}]$ sufficiently large (see §2.A) we may use Propositions 2.10 and 2.12 to estimate the good terms,

$$\|F_0(u)\|_{l^2Y^s} \lesssim (1 + \|u\|_{l^2X^s}^{m-2})\|u\|_{l^2X^s}^2,$$
$$\|F_0(u_1) - F_0(u_2)\|_{l^2Y^s} \lesssim (\|u_1\|_{l^1X^s} + \|u_2\|_{l^2X^s})(1 + \|u_1\|_{l^2X^s}^{m-2} + \|u_2\|_{l^2X^s}^{m-2})\|u_1 - u_2\|_{l^2X^s}.$$

In order to remove the quadratic terms involving u_{xx}, we define a bilinear operator

$$\mathbf{B}[u, v] = \tfrac{1}{2} C_1 uv + 2\mathbf{T}_{u_x} v, \tag{2.28}$$

where we define the paraproduct

$$\mathbf{T}_u v = \sum_{N>4} P_N(u_{<\frac{N}{4}} v).$$

When we apply the linear operator to $\mathbf{B}[u, u]$ we recover the bad quadratic terms and an error term,

$$(\partial_t + \tfrac{1}{3}\partial_x^3)\mathbf{B}[u, u] = C_1 u_x u_{xx} + C_2 u_{xx}^2 + F_1(u),$$

where

$$F_1(u) = C_2(2\mathbf{T}_{u_{xx}} u_{xx} - u_{xx}^2) + C_1 uF + 2C_2(\mathbf{T}_{u_{xxx}} u_x + \mathbf{T}_{\partial_x F} u + \mathbf{T}_{u_x} F).$$

We now estimate the error terms in l^2Y^s. For the first term we write

$$u_{xx}^2 - 2\mathbf{T}_{u_{xx}} u_{xx} = P_{\le 4}(u_{xx}^2) + \sum_{N>4} P_N\left((\partial_x^2 u_{\ge \frac{N}{4}})^2\right).$$

Using the bilinear estimate (2.11) and taking $\sigma_2 = \frac{9}{2}$ if $C_2 \neq 0$, we have

$$\|P_{\leq 4}(u_{xx}^2)\|_{l^2Y^s} \lesssim \|u_{xx}^2\|_{l^2Y^0} \lesssim \|u_{xx}\|_{l^2X^1}\|u_{xx}\|_{l^2X^1} \lesssim \|u\|_{l^2X^s}^2.$$

Using the frequency localized bilinear estimate (2.17), we have

$$\|P_N((\partial_x^2 u_{\geq \frac{N}{4}})^2)\|_{l^2Y^s} \lesssim N^{s+\frac{1}{2}}\|\partial_x^2 u_{>\frac{N}{4}}\|_{l^2X^0}^2 \lesssim N^{\frac{9}{2}-s}\|u\|_{l^2X^s}^2,$$

which may be summed in N when $s > \frac{9}{2}$. We may estimate the remaining quadratic term similarly, using the frequency localized bilinear estimate (2.16),

$$\|\mathbf{T}_{u_{xxx}} u_x\|_{l^2Y^s} \lesssim \|u\|_{l^2X^s}^2.$$

The terms $uF, \mathbf{T}_{\partial_x F} u, \mathbf{T}_{u_x} F$ are all cubic and higher order, so we may use the algebra estimate (2.10) and the trilinear estimate (2.18) to get

$$\|uF\|_{l^1Y^s} + \|\mathbf{T}_{\partial_x F} u\|_{l^1Y^s} + \|\mathbf{T}_{u_x} F\|_{l^1Y^s} \lesssim (1 + \|u\|_{l^2X^s}^{m-2})\|u\|_{l^2X^s}^3.$$

Defining $F_2 = F_0 - F_1$ we then have the equation

$$(\partial_t + \tfrac{1}{3}\partial_x^3)(u - \mathbf{B}[u,u]) = F_2(u),$$

where F_2 satisfies similar estimates to F_0. Further, using the algebra estimate (2.10) and the frequency localized algebra estimate (2.14), we see that

$$\|\mathbf{B}[u,u]\|_{l^2X^s} \lesssim \|u\|_{l^2X^s}^2,$$
$$\|\mathbf{B}[u_1,u_1] - \mathbf{B}[u_2,u_2]\|_{l^2X^s} \lesssim (\|u_1\|_{l^2X^s} + \|u_2\|_{l^2X^s})\|u_1 - u_2\|_{l^2X^s}.$$

We now take $w = \mathcal{T}(u)$ to be the solution to

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)(w - \mathbf{B}[u,u]) = F_2(u) \\ w(0) = u_0. \end{cases}$$

Choosing $M > 0$ sufficiently large and $\epsilon > 0$ sufficiently small, we may apply Proposition 2.14 to show that $\mathcal{T}\colon B \to B$ is a contraction. Applying the contraction principle we may complete the proof. □

2.6 Proof of Theorem 2.1

To complete the proof of Theorem 2.1 it remains to consider the case of large data. First we rescale the solution so that the high-frequency component of the initial data is small. We then linearize about the large low frequency component argue using the contraction principle with the linear estimates of Proposition 2.15 and the nonlinear estimate of §2.3.

Rescaling. As we are considering generic polynomial nonlinearities, there is no natural scaling associated with the problem. However, due to the natural scaling of the spaces and the fact that we are primarily concerned with the uu_{xx} nonlinearity, we will use the L^1-adapted scaling

$$u_\lambda(t,x) = \lambda u(\lambda^3 t, \lambda x), \qquad u_{0\lambda}(x) = \lambda u_0(\lambda x),$$

where we assume that $\lambda \in 2^{\mathbb{Z}}$ and $0 < \lambda \ll 1$. We define the low and high frequency components of the rescaled initial data to be

$$v_0^{\mathrm{low}} = P_{\leq 1} u_{0\lambda}, \qquad v_0^{\mathrm{high}} = P_{>1} u_{0\lambda},$$

and have the following estimates for the rescaled initial data:

Lemma 2.18. *If $s > 1$, $\lambda \in 2^{\mathbb{Z}}$ and $0 < \lambda \ll 1$, we have the estimates*

$$\|v_0^{\mathrm{low}}\|_{l^1 L^2} \lesssim \|u_0\|_{l^1 H^s}, \qquad \|v_0^{\mathrm{high}}\|_{l^1 H^s} \lesssim \lambda^{s-1} \|u_0\|_{l^1 H^s}. \tag{2.29}$$

For $s \notin \mathbb{Z}$, we have the estimates

$$\|\partial_x^k v_0^{\mathrm{low}}\|_{l^1 L^2} \lesssim \lambda^{\min\{k, s-1\}} \|u_0\|_{l^1 H^s}, \tag{2.30}$$

$$\|\partial_x^k v_0^{\mathrm{low}}\|_{L^\infty} \lesssim \lambda^{\min\{k+1, s+\frac{1}{2}\}} \|u_0\|_{l^1 H^s}. \tag{2.31}$$

Proof. Rescaling, we have

$$\|\partial_x^k v_0^{\mathrm{low}}\|_{l^1 L^2} \lesssim \lambda^{k+\frac{1}{2}} \|\partial_x^k P_{\leq \lambda^{-1}} u_0\|_{l^1_{\lambda^{\frac{1}{2}}} L^2} \lesssim \lambda^k \|P_{\leq 1} u_0\|_{l^1_1 L^2} + \sum_{1 < N \leq \lambda^{-1}} \lambda^k N^{k+1} \|P_N u_0\|_{l^1_N L^2}.$$

The first part of (2.29) and the estimate (2.30) then follow by summation. We note that if $k = s-1$ we may estimate similarly, but have a logarithmic loss in (2.30).

For the second part of (2.29) we proceed similarly to get

$$\|v_0^{\mathrm{high}}\|_{l^1 H^s}^2 \lesssim \sum_{N > \lambda^{-1}} \lambda^{1+2s} N^{2s} \|P_N u_0\|_{l^1_{\lambda^{\frac{3}{2}} N} L^2}^2 \lesssim \lambda^{2(s-1)} \|u_0\|_{l^1 H^s}^2.$$

For (2.31) we simply use Bernstein's inequality to get

$$\|\partial_x^k v_0^{\mathrm{low}}\|_{L^\infty} \lesssim \lambda^{k+1} \|\partial_x^k P_{\leq \lambda^{-1}} u_0\|_{L^\infty} \lesssim \lambda^{\min\{k, s-\frac{1}{2}\}+1} \|u_0\|_{l^1 H^s}.$$

□

The high frequency evolution. We now linearize about the large low frequency component of the rescaled initial data and consider the evolution of the small high-frequency component. First we define $v = u - v_0^{\text{low}}$, which satisfies the equation

$$\begin{cases} v_t + \frac{1}{3}v_{xxx} = \tilde{F}(x,v) \\ v(0) = v_0^{\text{high}}, \end{cases} \tag{2.32}$$

where

$$\tilde{F}(x,v) = -\tfrac{1}{3}\partial_x^3 v_0^{\text{low}} + \sum_{\substack{2\le|\alpha|\le m\\ \beta\le\alpha}} \lambda^{4-|\alpha|-\alpha_1-2\alpha_2} c_{\alpha\beta}(v_0^{\text{low}})^{\alpha_0-\beta_0}(\partial_x v_0^{\text{low}})^{\alpha_1-\beta_1}(\partial_x^2 v_0^{\text{low}})^{\alpha_2-\beta_2} v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}.$$

We then peel off the linear terms in v_{xx} that we expect to be non-perturbative due to the Mizohata condition,

$$\tilde{F}(x,v) = -a(x)v_{xx} + G(x,v),$$

where we define

$$a(x) = \sum_{2\le|\alpha|\le m} \lambda^{4-|\alpha|-\alpha_1-2\alpha_2} C_\alpha (v_0^{\text{low}})^{\alpha_0}(\partial_x v_0^{\text{low}})^{\alpha_1}(\partial_x^2 v_0^{\text{low}})^{\alpha_2-1},$$

and the linear (in v) part of $G(x,v)$ depends only on v, v_x.

We note that a is localized at frequencies $\le N_0 \sim 1$ where $N_0 = N_0(F)$ and as $s > \frac{5}{2}$ we have $l^1H^s \subset L^1$. As a corollary to Lemma 2.18 we have the following estimates for a:

Corollary 2.19. *Suppose that $s > \sigma_1$ where σ_1 is defined as in Theorem 2.16, then*

$$\|a\|_Z \lesssim \|a\|_{l^1H^s} \lesssim \|u_0\|_{l^1H^s}\langle\|u_0\|_{l^1H^s}\rangle^{m-2}, \tag{2.33}$$

$$\|a_x\|_Z \lesssim \|a_x\|_{l^1H^s} \lesssim \lambda\|u_0\|_{l^1H^s}\langle\|u_0\|_{l^1H^s}\rangle^{m-2}. \tag{2.34}$$

Completing the proof. We choose $C_* > 0$ and take $\lambda \in 2^{\mathbb{Z}}$ so that

$$0 < \lambda \le e^{-C_*\langle\|u_0\|_{l^1H^s}\rangle^{m-1}}.$$

By choosing $C_* \gg 1$ to be sufficiently large a will satisfy hypothesis of Proposition 2.15.

For $\mu > 0$ we then look to solve (2.32) using the contraction principle in the ball

$$B = \{v \in l^1X^s : \|v\|_{l^1X^s} \le \lambda^\mu\|u_0\|_{l^1H^s}\} \subset l^1X^s.$$

Given $v \in B$, let $w = \mathcal{T}(v)$ be the solution to

$$\begin{cases} w_t + \frac{1}{3}w_{xxx} + aw_{xx} = G(x,v) \\ w(0) = v_0^{\text{high}}. \end{cases} \tag{2.35}$$

The existence of a solution to (2.1) is then a consequence of the following Proposition:

Proposition 2.20. *There exists* $s_1 = s_1(F) \in [\frac{5}{2}, \frac{9}{2}]$ *so that if* $s > s_1$ *then for a suitable choice of* $\mu = \mu(s, F) > 0$ *and for* $C_* = C_*(s, F) \gg 1$ *chosen sufficiently large,* $\mathcal{T}\colon B \to B$ *is a contraction.*

Proof. Using the linear estimate (2.26) it will suffice to prove the appropriate bounds for the nonlinear term G. We start by choosing $s_1 \geq \sigma_1(\beta)$ where $\sigma_1(\beta)$ is defined as in Theorem 2.16 for the nonlinearity $v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}$ and the constant $c_{\alpha\beta}$ appearing in the definition of $\tilde{F}$ is nonzero.

Using the estimate for the rescaled initial data (2.30),

$$\|\partial_x^3 v_0^{\text{low}}\|_{l^1 Y^s} \lesssim \|\partial_x^3 v_0^{\text{low}}\|_{l^1 H^s} \lesssim \lambda^{\min\{3, s-1\}} \|u_0\|_{l^1 H^s},$$

with a loss of $\log|\lambda|$ if $s = 4$.

The remaining terms in G may be written in the form

$$G_{\alpha\beta} = \lambda^{4 - |\alpha| - \alpha_1 - 2\alpha_2} c_{\alpha\beta} (v_0^{\text{low}})^{\alpha_0 - \beta_0} (\partial_x v_0^{\text{low}})^{\alpha_1 - \beta_1} (\partial_x^2 v_0^{\text{low}})^{\alpha_2 - \beta_2} v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}.$$

Case 1: $|\beta| = 0$. Here we estimate one term in $l^1 L^2$ and the rest in L^∞ using the low frequency estimates (2.30) and (2.31). This gives us

$$\|G_{\alpha\beta}\|_{l^1 Y^s} \lesssim \lambda^3 \|u_0\|_{l^1 H^s}^{|\alpha|}.$$

Case 2: $|\beta| = 1$. We recall that we have place all the linear terms involving v_{xx} into the principal part of the equation, so we must have $\beta_2 = 0$. We then use the bilinear estimate (2.11) to place one low frequency term in $l^1 L^2$ and the Besov space estimate (2.13) to place the rest into L^∞. This gives us

$$\|G_{\alpha\beta}\|_{l^1 Y^s} \lesssim \lambda \|u_0\|_{l^1 H^s}^{|\alpha| - 1} \|v\|_{l^1 X^s}.$$

Case 3: $|\beta| \geq 2$. Here we first estimate all the low frequency terms in L^∞ using (2.13) to get

$$\|G_{\alpha\beta}\|_{l^1 Y^s} \lesssim \lambda^{4 - |\beta| - \beta_1 - 2\beta_2} \|u_0\|_{l^1 H^s}^{|\alpha| - |\beta|} \|v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}\|_{l^1 Y^s}.$$

We may then use Proposition 2.10 to get

$$\begin{aligned} \|G_{\alpha\beta}\|_{l^1 Y^s} &\lesssim \lambda^{4 - |\beta| - \beta_1 - 2\beta_2} \|u_0\|_{l^1 H^s}^{|\alpha| - |\beta|} \|v\|_{l^1 X^s}^{|\beta|} \\ &\lesssim \lambda^{4 - |\beta| - \beta_1 - 2\beta_2} \lambda^{\mu(|\beta| - 1)} \|u_0\|_{l^1 H^s}^{|\alpha| - 1} \|v\|_{l^1 H^s}. \end{aligned}$$

By choosing s_1 sufficiently large (see §2.A) we may choose $\mu \in (0, s-1)$ so that

$$\max\left\{1 + \frac{2\beta_2 + \beta_1 - 3}{|\beta| - 1} : G_{\alpha\beta} \not\equiv 0\right\} < \mu < \min\{3, s-1\}. \tag{2.36}$$

Applying the linear estimate (2.26) we have

$$\begin{aligned}\|T(v)\|_{l^1X^s} &\lesssim e^{C\langle\|u_0\|_{l^1H^s}\rangle^{m-1}}\left(\|v_0^{\mathrm{high}}\|_{l^1H^s} + \|G(x,v)\|_{l^1Y^s}\right)\\ &\lesssim e^{C\langle\|u_0\|_{l^1H^s}\rangle^{m-1}}\big(\lambda^{\min\{3,s-1\}}\|u_0\|_{l^1H^s}(1+\|u_0\|_{l^1H^s}^{m-1})\\ &\quad +\lambda^{\sigma}\|u_0\|_{l^1H^s}(1+\|u_0\|_{l^1H^s}^{m-2})\|v\|_{l^1X^s}\big)\,,\end{aligned}$$

for some $\sigma\in(0,1)$. By choosing $C_*>0$ sufficiently large (and hence λ sufficiently small) we have

$$\|\mathcal{T}(v)\|_{l^1X^s} \leq \lambda^{\mu}\|u_0\|_{l^1H^s}.$$

Applying identical estimates to the difference, we have

$$\|\mathcal{T}(v_1)-\mathcal{T}(v_2)\|_{l^1X^s} \lesssim e^{C\langle\|u_0\|_{l^1H^s}\rangle^{m-1}}\lambda^{\sigma}\|u_0\|_{l^1H^s}(1+\|u_0\|_{l^1H^s}^{m-2})\|v_1-v_2\|_{l^1X^s},$$

and hence for $C_*>0$ sufficiently large, $\mathcal{T}$ is a contraction. □

Using the contraction principle we may find a solution to the equation (2.35). Adding the initial data and rescaling we have a solution $u\in C([0,T];l^1H^s)$ to (2.1) where the time of existence $T=e^{-C\langle\|u_0\|_{l^1H^s}\rangle^{m-1}}$ and the solution satisfies the estimate

$$\sup_{t\in[0,T]}\|u(t)\|_{l^1H^s} \leq e^{C_1\|u_0\|_{l^1H^s}\langle\|u_0\|_{l^1H^s}\rangle^{m-2}}\|u_0\|_{l^1H^s}.$$

To prove Lipschitz dependence on the initial data for the original equation (2.1), we take two initial data $u_0^{(1)},u_0^{(2)}\in l^1H^s$. We then rescale both initial data according to the same choice of λ so that the rescaled solutions lie in l^1X^s. We then estimate the difference as in the small data Theorem 2.16 to show that

$$\|u_\lambda^{(1)}-u_\lambda^{(2)}\|_{l^1X^s} \lesssim C(\|u_{0\lambda}^{(1)}\|_{l^1H^s},\|u_{0\lambda}^{(2)}\|_{l^1H^s})\|u_{0\lambda}^{(1)}-u_{0\lambda}^{(2)}\|_{l^1H^s}.$$

Reversing the rescaling, we have that the solution map is locally Lipschitz as a map into $C([0,T];l^1H^s)$.

2.7 Proof of Theorem 2.2

The proof of Theorem 2.2 is similar to Theorem 2.1, although as in the small data case of Theorem 2.17 we will need to make use of a normal form correction to remove the quadratic nonlinearities involving two derivatives.

Rescaling and the high frequency evolution. As $l^2H^s=H^s$ we use the L^2-adapted scaling,

$$u_\lambda(t,x)=\lambda^{\frac12}u(\lambda^3t,\lambda x),\qquad u_{0\lambda}(x)=\lambda^{\frac12}u_0(\lambda x).$$

Again we define the low and high frequency parts of the initial data to be

$$v_0^{\mathrm{low}}=P_{\leq1}u_{0\lambda},\qquad v_0^{\mathrm{high}}=P_{>1}u_{0\lambda},$$

and have the following estimates for the rescaled initial data, which simply follow from the scaling of H^s:

Lemma 2.21. *If $s > \frac{1}{2}$, $\lambda \in 2^{\mathbb{Z}}$ and $0 < \lambda \ll 1$, we have the estimates*

$$\|v_0^{\text{low}}\|_{L^2} \lesssim \|u_0\|_{H^s}, \qquad \|v_0^{\text{high}}\|_{H^s} \lesssim \lambda^s \|u_0\|_{H^s}, \tag{2.37}$$

$$\|\partial_x^k v_0^{\text{low}}\|_{L^2} \lesssim \lambda^{\min\{k,s\}} \|u_0\|_{H^s}, \tag{2.38}$$

$$\|\partial_x^k v_0^{\text{low}}\|_{L^\infty} \lesssim \lambda^{\min\{k+\frac{1}{2},s\}} \|u_0\|_{H^s}. \tag{2.39}$$

Next we linearize about the low frequency part of the initial data by defining $v = u - v_0^{\text{low}}$ to get the equation

$$\begin{cases} v_t + \frac{1}{3} v_{xxx} + a(x) v_{xx} = G(x, v) \\ v(0) = v_0^{\text{high}}, \end{cases} \tag{2.40}$$

where

$$G(x, v) = -\tfrac{1}{3} \partial_x^3 v_0^{\text{low}} + \sum \lambda^{\frac{7}{2} - \frac{1}{2}|\alpha| - \alpha_1 - 2\alpha_2} c_{\alpha\beta} (v_0^{\text{low}})^{\alpha_0 - \beta_0} (\partial_x v_0^{\text{low}})^{\alpha_1 - \beta_1} (\partial_x^2 v_0^{\text{low}})^{\alpha_2 - \beta_2} v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2},$$

contains no terms linear terms in v_{xx} and

$$a(x) = \sum_{2 \le |\alpha| \le m} \lambda^{\frac{7}{2} - \frac{1}{2}|\alpha| - \alpha_1 - 2\alpha_2} C_\alpha (v_0^{\text{low}})^{\alpha_0} (\partial_x v_0^{\text{low}})^{\alpha_1} (\partial_x^2 v_0^{\text{low}})^{\alpha_2 - 1}.$$

We observe that a is again localized at frequencies $\le N_0 \sim 1$ where the constant N_0 depends only on F.

Next we observe that the coefficient a only contains linear terms with a derivative or cubic and higher order terms. In particular, the antiderivative $\partial_x^{-1} a$ is well-defined and lies in L^2. Using the estimates for the rescaled initial data Lemma 2.21, we have the following estimates for a:

Corollary 2.22. *Suppose that $s > \sigma_2$ where σ_2 is defined as in Theorem 2.17 and $0 < \lambda \ll 1$, then*

$$\|a\|_Z \lesssim \|u_0\|_{H^s} \langle \|u_0\|_{H^s} \rangle^{m-2}, \tag{2.41}$$

$$\|a_x\|_Z \lesssim \lambda \|u_0\|_{H^s} \langle \|u_0\|_{H^s} \rangle^{m-2}. \tag{2.42}$$

The normal form. As in Theorem 2.17, in order to handle quadratic terms involving two derivatives we make use of a quadratic correction. We start by removing the bad quadratic terms from G,

$$G(x, v) = C_1 \lambda^{-\frac{1}{2}} v_x v_{xx} + C_2 \lambda^{-\frac{3}{2}} v_{xx}^2 + G_0(x, v),$$

and as in (2.28) we define a bilinear operator by

$$\mathbf{B}[u,v] = \tfrac{1}{2}\lambda^{-\frac{1}{2}}C_1 uv + 2\lambda^{-\frac{3}{2}}C_2\mathbf{T}_{u_x}v.$$

We calculate

$$(\partial_t + \tfrac{1}{3}\partial_x^3 + a\partial_x^2)\mathbf{B}[v,v] = C_1\lambda^{-\frac{1}{2}}v_x v_{xx} + C_2\lambda^{-\frac{3}{2}}v_{xx}^2 + G_1(x,v),$$

where

$$\begin{aligned} G_1(x,v) = {} & C_2\lambda^{-\frac{3}{2}}(2\mathbf{T}_{v_{xx}}v_{xx} - v_{xx}^2) + C_1\lambda^{-\frac{1}{2}}(vG + av_x^2) \\ & + 2C_2\lambda^{-\frac{3}{2}}\left(\mathbf{T}_{G_x}v + a\mathbf{T}_{v_{xxx}}v - \mathbf{T}_{(av_{xx})_x}v + \mathbf{T}_{v_x}G \right. \\ & \left. + a\mathbf{T}_{v_x}v_{xx} - \mathbf{T}_{v_x}(av_{xx}) + \mathbf{T}_{v_{xxx}}v_x + 2a\mathbf{T}_{v_{xx}}v_x\right). \end{aligned}$$

Taking $G_2 = G_0 - G_1$ we then have the equation

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3 + a\partial_x^2)(v - \mathbf{B}[v,v]) = G_2(x,v) \\ v(0) = v_0. \end{cases}$$

Completing the proof. Once again we take $0 < \lambda \leq e^{-C_*\langle\|u_0\|_{H^s}\rangle^{m-1}}$, where $C_* \gg 1$ is sufficiently large that a satisfies the hypothesis of Proposition 2.15. For a suitable choice of $\mu \in (0,s)$, we look to solve (2.40) using the contraction principle in a ball

$$B = \{v \in l^2X^s : \|v\|_{l^2X^s} \leq \lambda^\mu\|u_0\|_{H^s}\} \subset l^2X^s.$$

Given $v \in B$ we take $w = \mathcal{T}(v)$ be a solution to

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)(w - \mathbf{B}[v,v]) = G_2(x,w) \\ w(0) = v_0^{\mathrm{high}}. \end{cases} \tag{2.43}$$

We then have the following analogue of Proposition 2.20:

Proposition 2.23. *There exists $s_2 = s_2(F) \in [\frac{1}{2},\frac{9}{2}]$ so that if $s > s_2$, then for a suitable choice of $\mu = \mu(s,F) > 0$ and $C_* = C_*(s,F) \gg 1$ sufficiently large, $\mathcal{T}\colon B \to B$ is a contraction.*

Proof. Using the linear estimate (2.26), it suffices to prove the appropriate nonlinear estimates for G, $\mathbf{B}$. As in Proposition 2.20, we choose $s_2 \geq \sigma_2(\beta)$ where $\sigma_2(\beta)$ is defined as in Theorem 2.17 for the nonlinearity $v^{\beta_0}v_x^{\beta_1}v_{xx}^{\beta_2}$ where an expression of this form appears in the rescaled version of F.

A. Estimates for G_0. Using (2.38) we have

$$\|\partial_x^3 v_0^{\mathrm{low}}\|_{l^2Y^s} \lesssim \|\partial_x^3 v_0^{\mathrm{low}}\|_{L^2} \lesssim \lambda^{\min\{3,s\}}\|u_0\|_{H^s}.$$

The remaining terms in G_0 are of the form

$$G_{\alpha\beta} = \lambda^{\frac{7}{2}-\frac{1}{2}|\alpha|-\alpha_1-2\alpha_2} c_{\alpha\beta}(v_0^{\text{low}})^{\alpha_0-\beta_0}(\partial_x v_0^{\text{low}})^{\alpha_1-\beta_1}(\partial_x^2 v_0^{\text{low}})^{\alpha_2-\beta_2} v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}.$$

Case 1: $|\beta| = 0$. Here we estimate one term in L^2 and the rest in L^∞ using the low frequency estimates (2.38) and (2.39). We then have

$$\|G_{\alpha\beta}\|_{l^2Y^s} \lesssim \lambda^3 \|u_0\|_{H^s}^{|\alpha|}.$$

Case 2: $|\beta| = 1$. As we have removed all the linear terms involving v_{xx}, we must have $\beta_2 = 0$. We then use the bilinear estimate (2.11) to place one low frequency term in L^2 and the Besov space estimate (2.13) to place the rest into L^∞. This gives us

$$\|G_{\alpha\beta}\|_{l^2Y^s} \lesssim \lambda^{\frac{3}{2}} \|u_0\|_{H^s}^{|\alpha|-1} \|v\|_{l^2X^s}.$$

Case 3a: $|\alpha| = |\beta| = 2$. As we have removed all the quadratic terms in v involving two derivatives with the normal form, we again must have $\beta_2 = 0$. We then use the bilinear estimate (2.11) to get

$$\|G_{\alpha\beta}\|_{l^2Y^s} \lesssim \lambda^{\frac{7}{2}-\frac{1}{2}|\beta|-\beta_1-2\beta_2} \|v\|_{l^2X^s}^2 \lesssim \lambda^{\frac{7}{2}-\frac{1}{2}|\beta|-\beta_1-2\beta_2} \lambda^{\mu(|\beta|-1)} \|u_0\|_{H^s} \|v\|_{l^2X^s}.$$

Case 3b: $|\alpha| > |\beta| = 2$. Here we use the trilinear estimate (2.18) to place one low frequency term in L^2 and the Besov space estimate (2.13) to place the rest into L^∞ to get

$$\|G_{\alpha\beta}\|_{l^2Y^s} \lesssim \lambda^{3-\frac{1}{2}|\beta|-\beta_1-2\beta_2} \|u_0\|_{H^s}^{|\alpha|-2} \|v\|_{l^2X^s}^2 \lesssim \lambda^{3-\frac{1}{2}|\beta|-\beta_1-2\beta_2} \lambda^{\mu(|\beta|-1)} \|u_0\|_{H^s}^{|\alpha|-1} \|v\|_{l^2X^s}.$$

Case 4: $|\beta| \geq 3$. We estimate all the low frequency terms in L^∞ using the Besov space estimate (2.13) and use the trilinear estimate (2.18) and algebra estimate (2.10) for the v terms to get

$$\|G_{\alpha\beta}\|_{l^2Y^s} \lesssim \lambda^{\frac{7}{2}-\frac{1}{2}|\beta|-\beta_1-2\beta_2} \lambda^{\mu(|\beta|-1)} \|u_0\|_{H^s}^{|\alpha|-1} \|v\|_{l^2X^s}.$$

By choosing s_2 sufficiently large (see §2.A) we may find $\mu \in (0, s)$ so that

$$\max\left\{\frac{2\beta_2+\beta_1-3}{|\beta|-1} + 1 : G_{\alpha\beta} \not\equiv 0\right\} < \mu < \min\{3, s\}, \tag{2.44}$$

which suffices to give the estimates

$$\|G_0(v)\|_{l^2Y^s} \lesssim \lambda^{\min\{3,s\}} \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^{m-1}) + \lambda^\sigma \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^{m-2}) \|v\|_{l^2X^s},$$
$$\|G_0(v_1) - G_0(v_2)\|_{l^2Y^s} \lesssim \lambda^\sigma \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^{m-2}) \|v_1 - v_2\|_{l^2X^s},$$

for some $\sigma > 0$.

B. Estimates for G_1. Next we note that from the definition 2.44 we may take $\mu > 1$ if $C_1 \neq 0$ and $\mu > 2$ if $C_2 \neq 0$. We may then use the algebra estiamte (2.10), Besov estimate (2.12) and L^∞ estimate (2.39) for u_0^{low} to estimate

$$\|G_1\|_{l^2X^{s-2}} \lesssim \lambda^{\min\{3,s\}} \|u_0\|_{H^s} + \lambda^\sigma \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^{m-2}) \|v\|_{l^2X^s},$$

for some $\sigma > 0$.

Estimating as in Theorem 2.17 we have

$$\|\lambda^{-\frac{3}{2}}(v_{xx}^2 - \mathbf{T}_{v_{xx}} v_{xx})\|_{l^2Y^s} \lesssim \lambda^{-\frac{3}{2}}\|v\|_{l^2X^s}.$$

The remaining terms in G_1 are either quadratic in v involving at most one derivative at high frequency, for which we may use the frequency localized estimate (2.16) or cubic and higher in u_0^{low}, v for which we may use the trilinear estimate (2.18) and algebra estimate (2.10). As a consequence we have the estimate

$$\begin{aligned}\|G_1(x,v)\|_{l^2Y^s} &\lesssim \lambda^\sigma \|u_0\|_{H^s}(1+\|u_0\|_{H^s}^{m-1})\|v\|_{l^2X^s},\\ \|G_1(x,v_1) - G_1(x,v_2)\|_{l^2Y^s} &\lesssim \lambda^\sigma \|u_0\|_{H^s}(1+\|u_0\|_{H^s}^{m-1})\|v_1 - v_2\|_{l^2X^s}\end{aligned}$$

C. Estimates for $\mathbf{B}$. Estimating as in Theorem 2.17, we have the estimates

$$\begin{aligned}\|\mathbf{B}[v,v]\|_{l^2X^s} &\lesssim \lambda^{\frac{1}{2}}\|u_0\|_{H^s}\|v\|_{l^2X^s},\\ \|\mathbf{B}[v_1,v_1] - \mathbf{B}[v_2,v_2]\|_{l^2X^s} &\lesssim \lambda^{\frac{1}{2}}\|u_0\|_{H^s}\|v_1 - v_2\|_{l^2X^s},\end{aligned}$$

where again we have used that $\mu > 1$ if $C_1 \neq 0$ and $\mu > 2$ if $C_2 \neq 0$.

By choosing $C_* \gg 1$ sufficiently large we may now use the linear estimate (2.26) to show that $\mathcal{T}\colon B \to B$ is a contraction. □

To complete the proof of Theorem 2.2, we may apply the contraction principle to prove the existence of a solution. As for Theorem 2.1, we may then use the estimates of Theorem 2.17 to prove Lipschitz dependence on the initial data.

2.A Refined regularities

In this appendix we briefly outline the improved regularities in the case of specific nonlinearities.

Small Data. Suppose that

$$F(u, u_x, u_{xx}) = \sum_{2\leq|\alpha|\leq m} c_\alpha u^{\alpha_0} u_x^{\alpha_1} u_{xx}^{\alpha_2}. \tag{2.45}$$

For Theorem 2.16 we define $\sigma_1(F)$ as in Table 2.1 and for Theorem 2.17 we define $\sigma_2(F)$ as in Table 2.2.

Large Data. In the large data case of Theorem 2.1, we take F as in (2.45) and for each $2 \leq |\alpha| \leq m$ such that $c_\alpha \neq 0$, we consider all multi-indices $|\beta| \geq 2$ such that $\beta \leq \alpha$. We then define $\sigma_1(\beta)$ as in Table 2.1 to correspond to the nonlinearity $v^{\beta_0} v_x^{\beta_1} v_{xx}^{\beta_2}$ and take $s_1 \geq \max_\beta \sigma_2(\beta)$. Due to the rescaling, as in (2.36), we also require that

$$s_1 \geq 2 + \max_\beta \frac{2\beta_2 + \beta_1 - 3}{|\beta| - 1}.$$

The large data case of Theorem 2.2 is similar, taking $s_2 \geq \max_\beta \sigma_2(\beta)$, where by convention we take $\sigma_2(1,0,1) = \frac{5}{2}$. We also once again have a scaling condition,

$$s_2 \geq 1 + \max_\beta \frac{2\beta_2 + \beta_1 - 3}{|\beta| - 1}.$$

Table 2.1: Refined regularities for Theorem 2.16.

σ_1	**F contains terms of the form**	
$\frac{1}{2}$	u^2	
1	u^{α_0}	$\alpha_0 \geq 3$
$\frac{3}{2}$	$u^{\alpha_0} u_x$	
2	$u^{\alpha_0} u_x^{\alpha_1}$	$\alpha_0 \geq 1$
$\frac{5}{2}$	$u^{\alpha_0} u_x^{\alpha_1} u_{xx}$	$\alpha_0 \geq 1$
	$u_x^{\alpha_1}$	
3	$u^{\alpha_0} u_x^{\alpha_1} u_{xx}^{\alpha_2}$	$\alpha_0 \geq 1$
$\frac{7}{2}$	$u_x^{\alpha_1} u_{xx}^{\alpha_2}$	$\alpha_1 \geq 1$
$\frac{9}{2}$	$u_{xx}^{\alpha_2}$	

Table 2.2: Refined regularities for Theorem 2.17.

σ_2	**F contains terms of the form**	
$\frac{1}{2}$	u^{α_0}	
1	$u^{\alpha_0} u_x$	$\alpha_0 \geq 2$
$\frac{3}{2}$	$u^{\alpha_0} u_x^{\alpha_1}$	$\alpha_0 \geq 1$
	$u^{\alpha_0} u_x^{\alpha_1} u_{xx}$	$\alpha_0 \geq 2$
2	$u^{\alpha_0} u_x^{\alpha_1} u_{xx}$	$\alpha_0 \geq 1$
	$u_x^{\alpha_1}$	$\alpha_1 \geq 3$
$\frac{5}{2}$	$u^{\alpha_0} u_x^{\alpha_1} u_{xx}^{\alpha_2}$	$\alpha_0 + \alpha_1 \geq 2$
	$u u_{xx}^{\alpha_2}$	$\alpha_2 \geq 2$
3	$u_x u_{xx}^{\alpha_2}$	$\alpha_2 \geq 2$
$\frac{7}{2}$	$u_x u_{xx}$	
	$u_{xx}^{\alpha_2}$	$\alpha_2 \geq 3$
$\frac{9}{2}$	u_{xx}^2	

Chapter 3

Modified asymptotics for the mKdV

3.1 Introduction

In this chapter we consider the long-time behavior of solutions $u\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ to the Cauchy problem

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = \sigma(u^3)_x \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $\sigma = \pm 1$ and u_0 is sufficiently small, smooth and spatially localized data.

If we treat the nonlinear solution as a small perturbation of the linear solution then it is reasonable to expect that the linear pointwise decay (see §1.2) is still valid and hence for initial data of size $0 < \epsilon \ll 1$ and for times $t \geq 1$,

$$|u| \lesssim \epsilon t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}, \qquad |u_x| \lesssim \epsilon t^{-\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}.$$

In particular, for any reasonable Sobolev-type norm $\|\cdot\|_X$, this decay allows us to bound nonlinearity as

$$\|(u^3)_x\|_X \leq \|uu_x\|_{L^\infty}\|u\|_X \lesssim \epsilon^2 t^{-1}\|u\|_X,$$

which just fails to be integrable in time. As a consequence, in order to establish global bounds and asymptotic behavior we must analyze the nonlinear interactions more carefully.

The first consideration is that of four-wave resonances: when linear waves may combine nonlinearly to create another linear wave feeding back into the system. Resonances of this form will correspond to solutions to the system

$$\begin{cases} \xi_1^3 + \xi_2^3 + \xi_3^3 = \xi_0^3 \\ \xi_1 + \xi_2 + \xi_3 = \xi_0, \end{cases}$$

where ξ_1, ξ_2, ξ_3 represent the input frequencies and ξ_0 the output frequency. An algebraic manipulation shows that this condition is equivalent to

$$(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_3 + \xi_1) = 0,$$

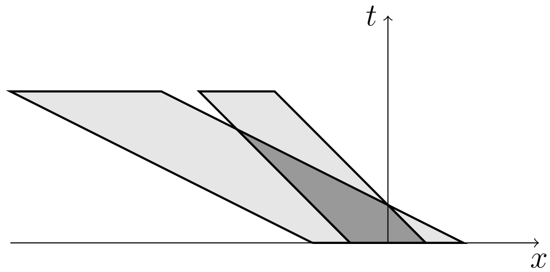

Figure 3.1: Short interaction time for transversal wave packets.

and hence resonant interactions occur whenever a pair of input frequencies sum to zero.

The second consideration is the direction in which linear waves may travel. From the Hamiltonian flow for the linear KdV (see §1.2) we see that for initial data localized in phase space at $(0, \xi_0)$, linear solutions are localized along the ray

$$\Gamma_v = \{x = tv\}$$

where the velocity $v = -\xi_0^2$. If linear waves interact transversally, we can hope to gain additional decay from the short interaction time (see Figure 3.1). However, given any output frequency $\xi_0 \in \mathbb{R}$, there exist parallel resonant interactions,

$$\{\xi_1, \xi_2, \xi_3\} = \{\xi_0, \xi_0, -\xi_0\},$$

so we must look for some additional structure to close the argument.

To understand the null structure in the nonlinearity that allows us to remove these parallel resonant interactions, we project to positive frequencies $\sim N$ and using that ∂_x behaves like multiplication by iN,

$$(\partial_t + \tfrac{1}{3}\partial_x^3)u_{N,+} \approx 3\sigma i N |u_{N,+}|^2 u_{N,+} + \text{lower order terms.}$$

As $3\sigma N|u_{N,+}|^2$ is real-valued, the leading order term may then be removed by means of a bounded gauge transform. Due to the non-integrable pointwise decay the phase will grow logarithmically, leading to modified asymptotics.

As the mKdV is completely integrable, global existence and asymptotic behavior has been studied using inverse scattering techniques such as in the work of Deift and Zhou [24] and references therein. A natural question to ask is whether it is possible to study the asymptotic behavior of the mKdV without relying on the completely integrable structure. Hayashi and Naumkin [57, 58] were able to prove global existence and derive modified asymptotics in a neighborhood of a self-similar solution for small initial data with errors bounded in L^p for $4 < p \leq \infty$ without relying on the complete integrability. In this chapter we present a significant improvement by establishing modified scattering for small initial data with errors bounded in $L^2 \cap L^\infty$. We also derive the leading asymptotic in the oscillatory region and use slightly weaker assumptions on the initial data. Some similar results have been recently obtained by Germain-Pusateri-Rousset [39] using a different method en route to studying modified asymptotics in a neighborhood of a soliton.

A key advantage of our robust approach is that our method also works for short-range perturbations of the form

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = (\sigma u^3 + F(u))_x \\ u(0) = u_0, \end{cases} \tag{3.2}$$

where $F \in C^2(\mathbb{R})$ satisfies

$$|F(u)| = O(|u|^p), \qquad |u| \to 0, \quad p > 3, \tag{3.3}$$

with some minor modifications if $p \in (3, \frac{7}{2})$ (see [49]). These perturbations are not known to be integrable and to the author's knowledge there are no other results on the asymptotic behavior of solutions. However, in the case of the defocusing cubic nonlinear Schrödinger equation, Deift and Zhou [27] were able to prove modified asymptotics for certain short-range perturbations using inverse scattering techniques. As an interesting corollary they showed that this allows the construction of action-angle variables for the perturbed equation. It is likely that their techniques may be able to be adapted to handle certain nonlinearities in (3.2).

In the related case of the cubic nonlinear Schrödinger equation on $\mathbb{R}$, modified asymptotics have been proved without inverse scattering techniques using both spatial methods [98] and Fourier methods [52, 67]. In this chapter we use the *method of testing by wave packets*, originally developed in the work of Ifrim and Tataru on the $1d$ cubic NLS [61] and $2d$ water waves [62, 63] and in joint work with the author, adapted to the KP-I equation [50]. This robust approach to proving global existence and studying asymptotic behavior essentially interpolates between the previously used spatial methods [95–98, 137] and Fourier methods [37, 38, 40, 52, 54, 56–59, 64, 67]. We also mention the semi-classical methods of Delort [28–31] and Alazard-Delort [4, 5].

Statement of results. Our first result gives the existence of global solutions satisfying the linear pointwise decay for small, smooth and spatially localized initial data.

Theorem 3.1. *There exists $\epsilon > 0$ so that for all $u_0 \in H^{1,1}$ satisfying*

$$\|u_0\|_{H^{1,1}} \leq \epsilon, \tag{3.4}$$

there exists a unique global solution u to (3.1) *with $S(-t)u \in C(\mathbb{R}; H^{1,1})$ so that for $t \geq 1$ and a.e. $x \in \mathbb{R}$ the solution satisfies the pointwise estimates*

$$|u(t,x)| \lesssim \epsilon t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}, \qquad |u_x(t,x)| \lesssim \epsilon t^{-\frac{2}{3}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}. \tag{3.5}$$

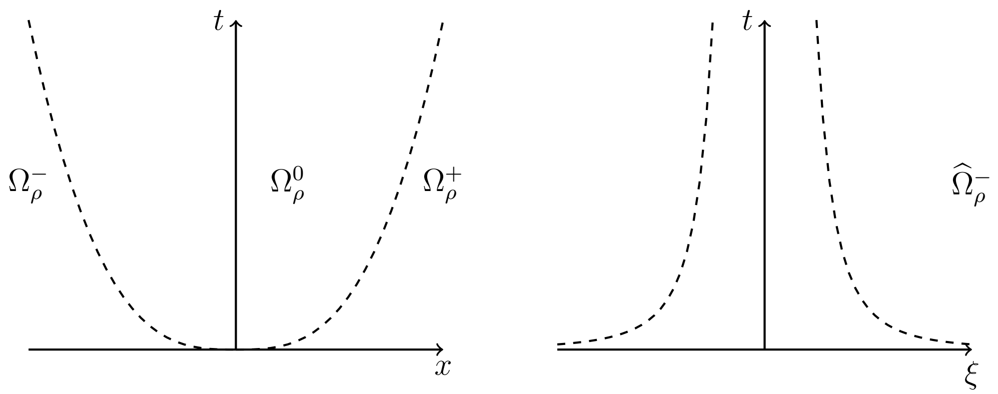

Figure 3.2: Asymptotic regions for the mKdV as $t \to +\infty$.

Next we consider the asymptotic behavior of these solutions. For $\rho \geq 0$ we define the oscillatory, self-similar and decaying regions of physical space,

$$\Omega_\rho^- = \{x < 0 : t^{-\frac{1}{3}}|x| \gtrsim t^{2\rho}\}, \quad \Omega_\rho^0 = \{x \in \mathbb{R} : t^{-\frac{1}{3}}|x| \lesssim t^{2\rho}\}, \quad \Omega_\rho^+ = \{x > 0 : t^{-\frac{1}{3}}|x| \gtrsim t^{2\rho}\}.$$

We also define a region of Fourier space corresponding to the oscillatory region,

$$\widehat{\Omega}_\rho^- = \{\xi \in \mathbb{R} : t^{\frac{1}{3}}|\xi| \gtrsim t^{\rho}\}.$$

We then have the following asymptotics for the solution u of Theorem 3.1:

Theorem 3.2. *If $u_0 \in H^{1,1}$ satisfies* (3.4)*, then the solution u to* (3.1) *satisfies the following asymptotics as $t \to +\infty$.*

(i) Oscillatory region. There exists a unique (complex-valued) continuous function W satisfying,

$$W(\xi) = \overline{W}(-\xi), \qquad W(0) = \int u_0 \, dx,$$

such that for $C > 0$ sufficiently large,

$$\|W\|_{H^{1-C\epsilon^2,1} \cap L^\infty} \lesssim \epsilon, \tag{3.6}$$

and in the oscillatory region Ω_ρ^-, for any $\rho \geq 0$,

$$\begin{aligned} u(t,x) = \pi^{-\frac{1}{2}} t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} \operatorname{Re}\left(e^{-\frac{2}{3}it^{-\frac{1}{2}}|x|^{\frac{3}{2}} + i\frac{\pi}{4} + \frac{3i\sigma}{4\pi}|W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})|^2 \log(t^{-\frac{1}{2}}|x|^{\frac{3}{2}})} W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}}) \right) \\ + \mathbf{err}_x, \end{aligned} \tag{3.7}$$

where the error satisfies the estimates

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{3}{8}} \mathbf{err}_x\|_{L^\infty(\Omega_\rho^-)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}|x|)^{\frac{1}{4}} \mathbf{err}_x\|_{L^2(\Omega_\rho^-)} \lesssim \epsilon. \tag{3.8}$$

(ii) Oscillatory region in Fourier space. In the corresponding frequency region $\widehat{\Omega}_\rho^-$*, for any* $\rho \geq 0$ *we have*

$$\hat{u}(t,\xi) = e^{\frac{1}{3}it\xi^3 + \frac{3i\sigma \operatorname{sgn}\xi}{4\pi}|W(\xi)|^2 \log(t\xi^3)} W(\xi) + \mathbf{err}_\xi, \tag{3.9}$$

where the error satisfies

$$\|(t^{\frac{1}{3}}|\xi|)^{\frac{1}{4}} \mathbf{err}_\xi\|_{L^\infty(\widehat{\Omega}_\rho^-)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{\frac{1}{3}}|\xi|)^{\frac{1}{2}} \mathbf{err}_\xi\|_{L^2(\widehat{\Omega}_\rho^-)} \lesssim \epsilon. \tag{3.10}$$

(iii) Self-similar region. There exists a solution $Q(y)$ *to the Painlevé II equation*

$$Q_{yy} - yQ = 3\sigma Q^3, \tag{3.11}$$

satisfying

$$|Q(y)| \lesssim \epsilon, \tag{3.12}$$

so that in the self similar region Ω_ρ^0 *for* $0 \leq \rho \leq \frac{1}{3}(\frac{1}{6} - C\epsilon^2)$*, we have the estimates*

$$\|u - t^{-\frac{1}{3}} Q(t^{-\frac{1}{3}}x)\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon t^{-\frac{1}{2}(\frac{5}{6} - C\epsilon^2)}, \qquad \|u - t^{-\frac{1}{3}} Q(t^{-\frac{1}{3}}x)\|_{L^2(\Omega_\rho^0)} \lesssim \epsilon t^{-\frac{2}{3}(\frac{5}{12} - C\epsilon^2)}. \tag{3.13}$$

(iv) Decaying region. In the decaying region Ω_ρ^+*, for any* $\rho \geq 0$ *we have the estimates*

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}x)^{\frac{3}{4}} u\|_{L^\infty(\Omega_\rho^+)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}x) u\|_{L^2(\Omega_\rho^+)} \lesssim \epsilon. \tag{3.14}$$

Remark 3.3. As (1.5) has time reversal symmetry given by

$$u(t,x) \mapsto u(-t,-x),$$

we get corresponding asymptotics as $t \to -\infty$.

Remark 3.4. We note that the estimates (3.7) and (3.13) are both valid in the overlapping region,

$$\mathcal{O}^- = \{x < 0 : 1 \lesssim t^{-\frac{1}{3}}|x| \leq t^{\frac{2}{3}(\frac{1}{6} - C\epsilon^2)}\},$$

and similarly the estimates (3.13) and (3.14) in the overlapping region,

$$\mathcal{O}^+ = \{x > 0 : 1 \lesssim t^{-\frac{1}{3}}|x| \leq t^{\frac{2}{3}(\frac{1}{6} - C\epsilon^2)}\}.$$

From the estimate (3.14) we see that $|u| \to 0$ as $t^{-\frac{1}{3}}x \to +\infty$. Comparing this to (3.13) in the overlapping region $\mathcal{O}^+$, we see that the solution Q to the Painlevé II (3.11) must satisfy

$$Q(y) \sim Q_0 \operatorname{Ai}(y), \qquad y \to +\infty,$$

for some $Q_0 \in \mathbb{R}$. Comparing the asymptotics for Q given in Theorem 1.8 to the asymptotics (3.7) and (3.13) in the overlapping region $\mathcal{O}^-$, we see that $Q_0 = q_\sigma(W(0))$, where q_σ is defined as in (1.52).

Remark 3.5. The loss of regularity of W in Theorem 3.2 can be compared to the similar results [61, 63]. Indeed, as the direct scattering problem for the cubic NLS and mKdV is the same, we expect the correspondence between the W of Theorem 3.2 and u_0 to be the same as in [61, Theorem 1]. From the inverse scattering theory, we expect this loss of regularity to be logarithmic in nature (see for example [24, 27]).

Outline of the proof. We start by giving a brief outline of the proof of Theorem 3.1. The asymptotics of Theorem 3.2 will arise as a consequence of the proof.

In order to control the spatial localization of solutions we look to control the L^2-norm of

$$Lu = S(t)xS(-t)u = (x - t\partial_x^2)u.$$

However, the operator L does not behave well with respect to the nonlinearity, so as in [55, 56, 58] we instead work with

$$\Lambda u = \partial_x^{-1}(3t\partial_t + x\partial_x + 1)u,$$

and observe that if u is a solution to (3.1) then

$$\Lambda u = Lu + 3t\sigma u^3. \tag{3.15}$$

As $3t\partial_t + x\partial_x + 1$ generates the mKdV scaling symmetry

$$u(t,x) \mapsto \lambda u(\lambda^3 t, \lambda x), \qquad u_0(x) \mapsto \lambda u_0(\lambda x), \tag{3.16}$$

the function $v = \Lambda u$ satisfies the linearized equation

$$\begin{cases} v_t + \frac{1}{3}v_{xxx} = 3\sigma u^2 v_x, \\ v(0) = xu_0. \end{cases} \tag{3.17}$$

For a large fixed constant $M_0 \geq 2$ we define the space X with norm

$$\|u\|_X^2 = \|u\|_{H^1}^2 + \langle t\rangle^{-2\delta}\|\Lambda u\|_{L^2}^2, \tag{3.18}$$

where

$$\delta = 3M_0^2\epsilon^2. \tag{3.19}$$

We then have the following local well-posedness result that can be proved as in [75, 80]:

Theorem 3.6. *If $u_0 \in H^{1,1}$ satisfies* (3.4) *then there exists $T = T(\epsilon) \to \infty$ as $\epsilon \to 0$ and a unique solution $u \in C([0,T]; X)$ to* (3.1) *such that*

$$\sup_{t\in[0,T]} \|u(t)\|_X \leq 2\epsilon. \tag{3.20}$$

Further, the solution map $u_0 \mapsto u(t)$ is locally Lipschitz.

The proof of Theorem 3.1 will take the form of a bootstrap estimate. Using the local well-posedness result, for $\epsilon > 0$ sufficiently small we can find $T > 1$ and a unique solution $u \in C([0,T];X)$ to (1.5). We then make the bootstrap assumption that u satisfies the linear pointwise estimate

$$\sup_{t\in[1,T]}\left(\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}u\|_{L^\infty} + \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}u_x\|_{L^\infty}\right) \leq M_0\epsilon \tag{3.21}$$

and show that under this assumption, for $\epsilon > 0$ sufficiently small, we have the energy estimate

$$\sup_{t\in[0,T]}\|u\|_X \lesssim \epsilon, \tag{3.22}$$

with a constant independent of M_0 and T.

Next we use these energy estimates to prove initial pointwise bounds that reduce closing the bootstrap to proving pointwise bounds for frequency localized pieces of u along the rays of the Hamiltonian flow Γ_v. To control the pointwise behavior of solutions along these rays we test the solution against a wave packet solution Ψ_v adapted to the ray Γ_v, by defining

$$\gamma(t,v) = \int u(t,x)\bar{\Psi}_v(t,x)\,dx. \tag{3.23}$$

We then reduce closing the bootstrap estimate (3.21) to proving global bounds for γ. To derive these bounds, we show that γ satisfies the asymptotic ODE

$$\dot{\gamma}(t,v) = 3i\sigma t^{-1}|\gamma(t,v)|^2\gamma(t,v) + \text{error}.$$

The logarithmic correction to the phase then arises as a consequence of solving this ODE.

Further questions. The development of robust techniques for understanding the asymptotic behavior of solutions has recently become a topic of much interest, motivated in part by trying to prove global existence for quasilinear equations arising in the study of water waves [4, 5, 38, 60, 62–64]. As such there are numerous related models to which the techniques developed in this chapter may be readily applied.

A key open problem is to extend these small data techniques to the large data setting where one must account for the existence of traveling wave solutions. Recently Germain, Pusateri and Rousset [39] have considered asymptotics in a neighborhood of the soliton using the *method of space-time resonances* due to Germain-Masmoudi-Shatah [37, 38, 40]. It is likely that our approach could be adapted to yield some similar results. Further, as discussed in §1.5, solitons must be accounted for if we are to extend Theorem 3.2 to the KdV for generic small initial data.

A further problem would be to try and improve the polynomial loss of regularity between the initial data for the PDE and the initial data for the modified scattering state W. The results of Deift and Zhou [26] using the inverse scattering transform suggest that this should in fact be a logarithmic loss. A similar loss of regularity is seen in [50, 61] and the non-integrable cases of the water waves [62, 63]. It would be of significant interest to see if any insight gained from the integrable cases could be applied in the non-integrable cases.

3.2 Energy estimates

We first derive energy estimates for u under the the bootstrap assumption (3.21). Our argument is similar to Hayashi-Naumkin [55, 56, 58].

Proposition 3.7. *For $\epsilon > 0$ chosen sufficiently small and $t \in [0, T]$ we have the energy estimates*

$$\|u\|_{H^1} \lesssim \epsilon, \tag{3.24}$$

$$\|\Lambda u\|_{L^2} \lesssim \epsilon \langle t \rangle^{\delta}, \tag{3.25}$$

where δ is defined as in (3.19) *and the constants are independent of M_0, T.*

Proof. From the conservation of mass (1.48) we have

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \leq \epsilon. \tag{3.26}$$

Similarly, from the conservation of energy (1.49) we have

$$\|\partial_x u(t)\|_{L^2}^2 + \tfrac{3\sigma}{2}\|u(t)\|_{L^4}^4 = \|\partial_x u_0\|_{L^2}^2 + \tfrac{3\sigma}{2}\|u_0\|_{L^4}^4. \tag{3.27}$$

Applying the Sobolev estimate (1.10), for any $\theta > 0$,

$$\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2} \lesssim \theta^{-1}\|u\|_{L^2}^6 + \theta\|u_x\|_{L^2}^2 \lesssim \theta^{-1}\epsilon^4\|u\|_{L^2}^2 + \theta\|u_x\|_{L^2}^2.$$

For $\theta > 0$ chosen sufficiently small we may use (3.26) and (3.27) to show that

$$\|u(t)\|_{H^1} \sim \|u_0\|_{H^1} \lesssim \epsilon, \tag{3.28}$$

where the constants are independent of M_0.

If $v = \Lambda u$, then from the estimate (3.20) we have

$$\sup_{t \in [0,1]} \|v(t)\|_{L^2_x} \lesssim \epsilon.$$

For $t \geq 1$ we first use (3.21) to show that

$$\|u u_x\|_{L^\infty} \leq M_0^2 \epsilon^2 t^{-1}, \quad t \geq 1. \tag{3.29}$$

We then use the linearized equation (3.17) and integration by parts to estimate,

$$\partial_t \|v\|_{L^2}^2 = 6\sigma \int u^2 v_x v \, dx = -6\sigma \int u u_x v^2 \, dx \leq 6{M_0}^2 \epsilon^2 t^{-1} \|v\|_{L^2}^2.$$

The estimate (3.25) then follows from Gronwall's inequality.

□

In order to control the pointwise behavior of solutions for times $t \geq 1$, we define the norm

$$\|u\|_{X_1}^2 = \|Lu\|_{L^2}^2 + \|t^{\frac{1}{3}} \langle t^{\frac{1}{3}} D_x \rangle^{-1} u\|_{L^2}^2.$$

Our reason for using X_1 is that it is well adapted to the mKdV scaling (3.16). We note that we have the compatibility estimate

$$\|u\|_{X_1}^2 \sim \|u_{\leq t^{-\frac{1}{3}}}\|_{X_1}^2 + \sum_{N > t^{-\frac{1}{3}}} \|u_N\|_{X_1}^2. \tag{3.30}$$

We then have the following corollary to Proposition 3.7:

Corollary 3.8. *For $\epsilon > 0$ sufficiently small and $t \in [1, T]$, we have the estimate*

$$\|u\|_{X_1} \lesssim \epsilon t^{\frac{1}{6}}. \tag{3.31}$$

Proof. From the local well-posedness estimate (3.20) and the bootstrap assumption (3.21) we have the estimate

$$\|u\|_{L^p} \lesssim M_0 \epsilon \langle t \rangle^{\frac{1}{3p} - \frac{1}{3}}, \qquad p \in (4, \infty]. \tag{3.32}$$

From the equation (3.15) and the energy estimate (3.25),

$$\|Lu\|_{L^2} \lesssim \|\Lambda u\|_{L^2} + 3t\|u\|_{L^6}^3 \lesssim \epsilon \langle t \rangle^{\delta} + (M_0 \epsilon)^3 \langle t \rangle^{\frac{1}{6}}.$$

This gives the first part of (3.31), provided $\epsilon = \epsilon(M_0) > 0$ is chosen sufficiently small that $\delta \in (0, \frac{1}{6}]$.

For the second component of the norm we make a self-similar change of variables, defining

$$U(t, y) = t^{\frac{1}{3}} u(t, t^{\frac{1}{3}} y). \tag{3.33}$$

We observe that U satisfies the equation

$$\begin{cases} \partial_t U = \frac{1}{3} t^{-1} \partial_y (yU - U_{yy} + 3\sigma U^3) \\ U(1, y) = u(1, y), \end{cases} \tag{3.34}$$

and undoing the rescaling,

$$\|yU - U_{yy} + 3\sigma U^3\|_{L_y^2} = t^{-\frac{1}{6}} \|\Lambda u\|_{L_x^2}.$$

Applying the energy estimate (3.25), we then have

$$\partial_t \|\langle D_y \rangle^{-1} U\|_{L_y^2} \lesssim t^{-1} \|yU - U_{yy} + 3\sigma U^3\|_{L_y^2} \lesssim t^{-\frac{7}{6}} \|\Lambda u\|_{L_x^2} \lesssim \epsilon t^{\delta - \frac{7}{6}}.$$

At time $t = 1$, we have the bound

$$\|\langle D_y \rangle^{-1} U(1)\|_{L^2} \lesssim \|u(1)\|_{L^2} \lesssim \epsilon.$$

For $\epsilon > 0$ chosen sufficiently small we may then integrate in time to get

$$\|\langle D_y \rangle^{-1} U(t)\|_{L_y^2} \lesssim \epsilon.$$

The estimate (3.31) then follows from undoing the rescaling (3.33). □

3.3 Initial pointwise bounds

In this section we prove a number of estimates for u that will allow us to reduce closing the bootstrap estimate (3.21) to considering the behavior of u along the rays Γ_v for $|v| \gtrsim t^{\frac{2}{3}}$. Our argument is similar to [50, 60, 62, 63].

Let $t \geq 1$ be fixed. We first decompose u into a piece on which L acts hyperbolically and piece on which it acts elliptically. Let $\psi \in C_0^\infty$ be a non-negative function, identically 1 on $[-1, 1]$ and supported in $(-2, 2)$. Let $\nu \gg 1$ be a fixed parameter and define

$$\chi(x) = \psi(\nu^{-1}x) - \psi(\nu x), \qquad \chi^{\text{hyp}} = \mathbf{1}_{(-\infty,0)}\chi, \qquad \chi^{\text{ell}} = 1 - \chi^{\text{hyp}}.$$

We rescale for dyadic $N \in 2^{\mathbb{Z}}$ by defining $\chi_N(x) = \chi(t^{-1}N^{-2}x)$, and similarly χ_N^{hyp}, χ_N^{ell}.

For each $N > t^{-\frac{1}{3}}$, we decompose u_N as

$$u_N = u_{N,+}^{\text{hyp}} + u_{N,-}^{\text{hyp}} + u_N^{\text{ell}},$$

where $u_{N,\pm}^{\text{hyp}} = \chi_N^{\text{hyp}} P_\pm u_N$. We then define the hyperbolic parts of u by

$$u_\pm^{\text{hyp}} = \sum_{N > t^{-\frac{1}{3}}} u_{N,\pm}^{\text{hyp}},$$

and use this to decompose u,

$$u = u_+^{\text{hyp}} + u_-^{\text{hyp}} + u^{\text{ell}}.$$

We note that $u^{\text{hyp}} = u_+^{\text{hyp}} + u_-^{\text{hyp}} = 2\,\mathrm{Re}(u_+^{\text{hyp}})$ is supported in the oscillatory region $\Omega_0^- = \{t^{-\frac{1}{3}}x < -\nu^{-1}\}$.

In the region Ω_0^-, the symbol of L factorizes as

$$x - t\xi^2 = -(|x|^{\frac{1}{2}} \mp it^{\frac{1}{2}}\xi)(|x|^{\frac{1}{2}} \pm it^{\frac{1}{2}}\xi),$$

and hence we define operators associated to this factorization,

$$L_\pm = |x|^{\frac{1}{2}} \pm it^{\frac{1}{2}}\partial_x.$$

We note that L_- is elliptic on positive frequencies and L_+ is elliptic on negative frequencies.

The main result of this section is the following proposition giving pointwise bounds on the hyperbolic and elliptic parts of u.

Proposition 3.9. *For $t \in [1, T]$ we may decompose u into a hyperbolic part u^{hyp} supported in Ω_0^- and an elliptic part u^{ell} so that,*

$$u = u^{\text{hyp}} + u^{\text{ell}},$$

and have the estimates,

$$\|t^{\frac{1}{6}}\langle t^{-\frac{1}{3}}x\rangle u^{\text{ell}}\|_{L^2} \lesssim \epsilon, \qquad \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{3}{4}} u^{\text{ell}}\|_{L^\infty} \lesssim \epsilon, \qquad \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}} u_x^{\text{ell}}\|_{L^\infty} \lesssim \epsilon, \tag{3.35}$$

$$\|t^{\frac{1}{3}} u^{\text{hyp}}\|_{L^\infty} \lesssim \epsilon, \qquad \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{2}} u_x^{\text{hyp}}\|_{L^\infty} \lesssim \epsilon. \tag{3.36}$$

The key component in the proof of Proposition 3.9 will be the following elliptic estimates.

Lemma 3.10. *For $t \in [1, T]$ we have the estimates*

$$\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|u_{\leq t^{-\frac{1}{3}}}\|_{X_1}, \tag{3.37}$$

$$\|(|x| + tN^2)u_N^{\mathrm{ell}}\|_{L^2} \lesssim \|u_N\|_{X_1}, \qquad N > t^{-\frac{1}{3}}, \tag{3.38}$$

$$\|(|x|^{\frac{1}{2}} + t^{\frac{1}{2}}N)L_\pm u_{N,\pm}^{\mathrm{hyp}}\|_{L^2} \lesssim \|u_N\|_{X_1}, \qquad N > t^{-\frac{1}{3}}. \tag{3.39}$$

Proof.

A. Low frequencies. We first produce bounds for the low frequency component $u_{\leq t^{-\frac{1}{3}}}$. As the Fourier multiplier $\langle t^{\frac{1}{3}}D\rangle^{-1}$ behaves like multiplication by a constant at frequencies $\leq t^{-\frac{1}{3}}$, we have

$$\|t^{\frac{1}{3}}u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|t^{\frac{1}{3}}\langle t^{\frac{1}{3}}D_x\rangle^{-1}u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|u_{\leq t^{\frac{1}{3}}}\|_{X_1}. \tag{3.40}$$

Further, due to the localization,

$$\|t\partial_x^2 u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|t^{\frac{1}{3}}u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|u_{\leq t^{\frac{1}{3}}}\|_{X_1}.$$

We may then use the operator L to estimate,

$$\|xu_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|Lu_{\leq t^{-\frac{1}{3}}}\|_{L^2} + \|t\partial_x^2 u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim \|u_{\leq t^{\frac{1}{3}}}\|_{X_1}. \tag{3.41}$$

The estimate (3.37) then follows from (3.40) and (3.41).

B. Elliptic region. Let $N > t^{-\frac{1}{3}}$ and recall that $t \geq 1$. By rescaling under the mKdV scaling (3.16), it will suffice to prove estimates for the case $N = 1$.

We first decompose

$$\chi_1^{\mathrm{ell}} = \chi_1^{\mathrm{in}} + \chi_1^{\mathrm{out}} + \chi_1^{\mathrm{mid}},$$

where we define

$$\chi_1^{\mathrm{in}}(t,x) = \psi(\nu t^{-1}x), \qquad \chi_1^{\mathrm{out}}(t,x) = 1 - \psi(\nu^{-1}t^{-1}x), \qquad \chi_1^{\mathrm{mid}}(t,x) = \chi_1^{\mathrm{hyp}}(t,-x).$$

We observe that the functions $\chi_1^{\mathrm{in}}, \chi_1^{\mathrm{mid}} \in C_0^\infty$ and $\chi_1^{\mathrm{out}} \in C^\infty$ are supported in the sets $\{|x| < 2\nu^{-1}t\}$, $\{\nu^{-1}t < x < 2\nu t\}$ and $\{|x| > \nu t\}$ respectively.

B(i). Inner region. We first observe that $\chi_1^{\mathrm{in}}u_1$ is localized at frequencies ~ 1 up to rapidly decaying tails. More precisely, applying the estimate (1.12) we have

$$\begin{aligned} t\|\chi_1^{\mathrm{in}}u_1\|_{L^2} &\lesssim t\|P_{\frac{1}{4}\leq\cdot\leq 4}(\chi_1^{\mathrm{in}}u_1)\|_{L^2} + t\|(1 - P_{\frac{1}{4}\leq\cdot\leq 4})(\chi_1^{\mathrm{in}}u_1)\|_{L^2} \\ &\lesssim_k t\|\partial_x^2 P_{\frac{1}{4}\leq\cdot\leq 4}(\chi_1^{\mathrm{in}}u_1)\|_{L^2} + t\langle\nu^{-1}t\rangle^{-k}\|u_1\|_{L^2} \\ &\lesssim t\|\chi_1^{\mathrm{in}}\partial_x^2 u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1}. \end{aligned}$$

As $|x| \ll t$ in the support of χ_1^{in}, we may then estimate

$$\begin{aligned} t\|\chi_1^{\text{in}} u_1\|_{L^2} &\lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1} + \|x\chi_1^{\text{in}} u_1\|_{L^2} \\ &\lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1} + \nu^{-1} t\|\chi_1^{\text{in}} u_1\|_{L^2}. \end{aligned}$$

For $\nu \gg 1$ sufficiently large, we may absorb the final term into the left hand side to get

$$t\|\chi_1^{\text{in}} u_1\|_{L^2} \lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1}.$$

Finally, as

$$\|x\chi_1^{\text{in}} u_1\|_{L^2} \lesssim \nu^{-1} t\|\chi_1^{\text{in}} u_1\|_{L^2},$$

we have the estimate

$$\|(|x| + t)\chi_1^{\text{in}} u_1\|_{L^2} \lesssim t\|\chi_1^{\text{in}} u_1\|_{L^2} \lesssim \|u\|_{X_1}.$$

B(ii). Outer region. Similarly, we observe that $\chi_1^{\text{out}} u_1$ is localized at frequencies ~ 1 up to rapidly decaying tails, so

$$\|\chi_1^{\text{out}} \partial_x^2 u_1\|_{L^2} \lesssim \|\partial_x^2(\chi_1^{\text{out}} u_1)\|_{L^2} + \|[\chi_1^{\text{out}}, \partial_x^2]u_1\|_{L^2} \lesssim \|\chi_1^{\text{out}} u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1}.$$

Proceeding as for the inner region we may then estimate,

$$\begin{aligned} \|x\chi_1^{\text{out}} u_1\|_{L^2} &\lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + \|t\chi_1^{\text{out}} \partial_x^2 u_1\|_{L^2} \\ &\lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1} + t\|\chi_1^{\text{out}} u_1\|_{L^2} \\ &\lesssim \|(x - t\partial_x^2)u_1\|_{L^2} + C(\nu)\|u_1\|_{X_1} + \nu^{-1}\|x\chi_1^{\text{out}} u_1\|_{L^2}. \end{aligned}$$

The final term may again be absorbed into the left hand side for sufficiently large $\nu \gg 1$. As $t \ll |x|$ on the support of χ_1^{out}, we then have

$$\|(|x| + t)\chi_1^{\text{out}} u_1\|_{L^2} \lesssim \|x\chi_1^{\text{out}} u_1\|_{L^2} \lesssim \|u\|_{X_1}.$$

B(iii). Middle region. We now ignore the dependence of the constants on ν. Integrating by parts we have

$$\begin{aligned} &\|\chi_1^{\text{mid}} x u_1\|_{L^2}^2 + \|\chi_1^{\text{mid}} t\partial_x^2 u_1\|_{L^2}^2 + 2t\int (\chi_1^{\text{mid}})^2 x(\partial_x u_1)^2\, dx \\ &= \|(x - t\partial_x^2)u_1\|_{L^2}^2 + 2t\int \partial_x \left((\chi_1^{\text{mid}})^2\right) u_1^2\, dx + t\int \partial_x^2 \left((\chi_1^{\text{mid}})^2\right) x u_1^2\, dx \\ &\lesssim \|u_1\|_{X_1}^2. \end{aligned}$$

Once again we see that $\chi_1^{\text{mid}} u_1$ is localized at frequencies ~ 1 up to rapidly decaying tails. Using this localization, we then have

$$\begin{aligned} \|(|x| + t)\chi_1^{\text{mid}} u_1\|_{L^2}^2 &\lesssim \|\chi_1^{\text{mid}} x u_1\|_{L^2}^2 + \|\chi_1^{\text{mid}} t\partial_x^2 u_1\|_{L^2}^2 + \|u_1\|_{X_1}^2 + 2t\int (\chi_1^{\text{mid}})^2 x(\partial_x u_1)^2\, dx \\ &\lesssim \|u_1\|_{X_1}^2 \end{aligned}$$

C. Hyperbolic region. We note that $u^{\text{hyp}}_{N,-} = \overline{u^{\text{hyp}}_{N,+}}$ so it suffices to consider positive frequencies and again by scaling, it will suffice to consider the case $N = 1$. We define

$$f_{1,+} = L_+ u^{\text{hyp}}_{1,+}.$$

As $f_{1,+}$ is supported away from $x = 0$ and localized at frequencies ~ 1 up to rapidly decaying tails, we may use the estimate (1.12) to show that

$$\|(1 - P_{\frac{1}{4}\le\cdot\le 4} P_+)\partial_x^\alpha (|x|^\beta f_{1,+})\|_{L^2} \lesssim_k t^{-k}\|u_1\|_{X_1}. \tag{3.42}$$

Integrating by parts, we have the identity

$$\||x|^{\frac{1}{2}} f_{1,+}\|_{L^2}^2 + t\|\partial_x f_{1,+}\|_{L^2}^2 = \|L_- f_{1,+}\|_{L^2}^2 + 4t\,\mathrm{Im}\int (|x|^{\frac{1}{4}} f_{1,+})\partial_x\overline{(|x|^{\frac{1}{4}} f_{1,+})}\,dx. \tag{3.43}$$

Using the estimate (3.42), we have

$$t^{\frac{1}{2}}\|f_{1,+}\|_{L^2} \lesssim t^{\frac{1}{2}}\|\partial_x f_{1,+}\|_{L^2} + \|u_1\|_{X_1}, \qquad \|L_- f_{1,+}\|_{L^2} \lesssim \|Lu_1\|_{L^2} + \|u_1\|_{X_1},$$
$$4t^{\frac{1}{2}}\,\mathrm{Im}\int (|x|^{\frac{1}{4}} f_{1,+})\partial_x\overline{(|x|^{\frac{1}{4}} f_{1,+})}\,dx \lesssim \|u_1\|_{X_1},$$

where the last estimate uses that $|x|^{\frac{1}{4}} f_{1,+}$ is localized to positive frequencies up to rapidly decaying tails. Combining these estimates with the identity (3.43), we have the estimate

$$\|(|x|^{\frac{1}{2}} + t^{\frac{1}{2}}) f_{1,+}\|_{L^2}^2 \lesssim \||x|^{\frac{1}{2}} f_{1,+}\|_{L^2}^2 + t\|\partial_x f_{1,+}\|_{L^2}^2 + \|u_1\|_{X_1}^2 \lesssim \|u_1\|_{X_1}^2,$$

which completes the proof of (3.39). □

Proof of Proposition 3.9. We first consider the estimates (3.35) for the elliptic part u^{ell} of u. The L^2 bound simply follows from the energy estimate (3.31) and the elliptic estimates (3.37) and (3.38).

For the L^∞ bound, we first consider the region Ω_0^0. Applying Bernstein's inequality (1.11) we have

$$\|t^{\frac{1}{3}} u^{\text{ell}}\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1}{6}}\|t^{\frac{1}{3}} u_{\le t^{-\frac{1}{3}}}\|_{L^2} + \sum_{N > t^{-\frac{1}{3}}} t^{-\frac{2}{3}} N^{-\frac{3}{2}}\|tN^2 u_N^{\text{ell}}\|_{L^2}.$$

The first term may be controlled by (3.37). For the second term we use the elliptic bound (3.38) and then sum in N using the Cauchy-Schwarz inequality to get

$$\|t^{\frac{1}{3}} u^{\text{ell}}\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1}{6}}\|u\|_{X_1} \lesssim \epsilon.$$

For the corresponding bound for u_x^{ell} we estimate similarly, applying Bernstein's inequality (1.11) and using the frequency localization to get

$$\|t^{\frac{2}{3}} u_x^{\text{ell}}\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1}{6}}\|t^{\frac{1}{3}} u_{\le t^{-\frac{1}{3}}}\|_{L^2} + \sum_{N > t^{-\frac{1}{3}}} t^{-\frac{1}{3}} N^{-\frac{1}{2}}\|tN^2 u_N^{\text{ell}}\|_{L^2}.$$

Applying the elliptic estimates (3.37), (3.38) and summing in N gives us the bound

$$\|t^{\frac{2}{3}} u_x^{\mathrm{ell}}\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1}{6}} \|u\|_{X_1} \lesssim \epsilon.$$

To prove the L^∞ bounds for u^{ell} in $\mathbb{R}\backslash\Omega_0^0$ we take dyadic $M \geq t^{-\frac{1}{3}}$ and consider each region $\{|x| \sim tM^2\}$ separately. Let $\chi_M \in C_0^\infty$ be supported in the set $\{|x| \sim tM^2\}$ as in Lemma 3.10. From (1.12), $\chi_M u_N^{\mathrm{ell}}$ is localized at frequency $\lesssim N$ for $N \leq M$ up to rapidly decaying tails of size $O((tM^2N)^{-k})$. From Bernstein's inequality, we then have

$$\|\chi_M u^{\mathrm{ell}}\|_{L^\infty} \lesssim t^{-\frac{1}{6}} \|\chi_M u_{\leq t^{-\frac{1}{3}}}\|_{L^2} + \sum_{t^{-\frac{1}{3}} < N \leq M} N^{\frac{1}{2}} \|\chi_M u_N^{\mathrm{ell}}\|_{L^2} + \sum_{N > M} N^{\frac{1}{2}} \|u_N^{\mathrm{ell}}\|_{L^2} + t^{-1} M^{-\frac{3}{2}} \|u\|_{X_1}.$$

For the first term we use the low frequency estimate (3.37) and that $M \geq t^{-\frac{1}{3}}$ to estimate

$$t^{-\frac{1}{6}} \|\chi_M u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim t^{-\frac{7}{6}} M^{-2} \|x u_{\leq t^{-\frac{1}{3}}}\|_{L^2} \lesssim t^{-1} M^{-\frac{3}{2}} \|u\|_{X_1}$$

For the second term we use the elliptic estimate (3.38) and then sum using the Cauchy-Schwarz inequality to get

$$\sum_{t^{-\frac{1}{3}} < N \leq M} N^{\frac{1}{2}} \|\chi_M u_N^{\mathrm{ell}}\|_{L^2} \lesssim \sum_{t^{-\frac{1}{3}} < N \leq M} t^{-1} N^{\frac{1}{2}} M^{-2} \|x u_N^{\mathrm{ell}}\|_{L^2} \lesssim t^{-1} M^{-\frac{3}{2}} \|u\|_{X_1}.$$

Similarly for the third term we have

$$\sum_{N > M} N^{\frac{1}{2}} \|u_N^{\mathrm{ell}}\|_{L^2} \lesssim \sum_{N > M} t^{-1} N^{\frac{3}{2}} \|t N^2 u_N^{\mathrm{ell}}\|_{L^2} \lesssim t^{-1} M^{-\frac{3}{2}} \|u\|_{X_1}.$$

From the energy estimate (3.31), we then have

$$\|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{3}{4}} u^{\mathrm{ell}}\|_{L^\infty(|x| \sim tM^2)} \lesssim t^{-\frac{1}{6}} \|u\|_{X_1} \lesssim \epsilon.$$

The second part of (3.35) follows from taking the supremum over M.

For the third part of (3.35) we estimate similarly to get

$$\begin{aligned} \|t^{\frac{2}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}} u_x^{\mathrm{ell}}\|_{L^\infty(|x| \sim tM^2)} &\lesssim t^{-\frac{1}{2}} M^{-\frac{3}{2}} \|x u_{\leq t^{-\frac{1}{3}}}\|_{L^2} + \sum_{t^{-\frac{1}{3}} < N \leq M} t^{-\frac{1}{6}} N^{\frac{3}{2}} M^{-\frac{3}{2}} \|x u_N^{\mathrm{ell}}\|_{L^2} \\ &\quad + \sum_{N > M} t^{-\frac{1}{6}} M^{\frac{1}{2}} N^{-\frac{1}{2}} \|t N^2 u_N^{\mathrm{ell}}\|_{L^2} + t^{-\frac{1}{6}} \|u\|_{X_1} \\ &\lesssim t^{-\frac{1}{6}} \|u\|_{X_1}. \end{aligned}$$

For the hyperbolic bound (3.36) we apply the Sobolev estimate (1.10) to $e^{\frac{2}{3} i t^{-\frac{1}{2}} |x|^{\frac{3}{2}}} u_{N,+}^{\mathrm{hyp}}$ and then use (3.39) to get

$$\begin{aligned} \|t^{\frac{1}{3}} u_{N,+}^{\mathrm{hyp}}\|_{L^\infty} &\lesssim t^{\frac{1}{12}} \|u_{N,+}^{\mathrm{hyp}}\|_{L^2}^{\frac{1}{2}} \|L_+ u_{N,+}^{\mathrm{hyp}}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim t^{-\frac{1}{6}} \|t^{\frac{1}{2}} N L_+ u_{N,+}^{\mathrm{hyp}}\|_{L^2} + t^{-\frac{1}{6}} N^{-1} \|u_N\|_{L^2} \\ &\lesssim t^{-\frac{1}{6}} \|u_N\|_{X_1}. \end{aligned}$$

Summing over $N > t^{-\frac{1}{3}}$ and using that the $u_{N,\pm}^{\text{hyp}}$ have almost disjoint spatial supports, we have

$$\|t^{\frac{1}{3}} u_{+}^{\text{hyp}}\|_{L^\infty} \lesssim t^{-\frac{1}{6}} \|u\|_{X_1}.$$

The first part of (3.36) then follows from the energy estimate (3.31).

For the second part of (3.36) we may use that $u_{N,+}^{\text{hyp}}$ is localized in the spatial region $x \sim -tN^2$ to estimate,

$$\|t^{\frac{2}{3}} \langle t^{-\frac{1}{3}} x\rangle^{-\frac{1}{2}} \partial_x u_{N,+}^{\text{hyp}}\|_{L^\infty} \lesssim \|t^{\frac{1}{3}} u_{N,+}^{\text{hyp}}\|_{L^\infty} + t^{-\frac{1}{6}} \|u_N\|_{X_1}.$$

The estimate then follows from the first part of (3.36).

□

3.4 Testing by wave packets

Construction of the wave packets. Let $\chi \in C_0^\infty(\mathbb{R})$ be a real-valued function localized in both space and frequency near 0 at scale ~ 1. To simplify the calculations, we will normalize $\int \chi = 1$. We define a wave packet adapted to the ray $\Gamma_v = \{x = tv\}$ by

$$\Psi_v(t,x) = e^{i\phi} \chi(\lambda(x - tv)), \tag{3.44}$$

where the phase and scale are defined by

$$\phi(t,x) = -\frac{2}{3} t^{-\frac{1}{2}} |x|^{\frac{3}{2}} + \frac{\pi}{4}, \qquad \lambda(t,v) = t^{-\frac{1}{2}} |v|^{-\frac{1}{4}}.$$

We define the set $\mathbf{\Omega}_\rho^- = \{v < 0 : t^{-\frac{2}{3}} |v| \gtrsim t^{2\rho}\}$ such that Ψ_v is supported on Ω_ρ^- whenever $v \in \mathbf{\Omega}_\rho^-$.

As discussed in §1.2, we expect that Ψ_v will be a good good approximate solution on timescales $\Delta t \ll t$. In particular, for $v \in \mathbf{\Omega}_0^-$ we have

$$(\partial_t + \tfrac{1}{3}\partial_x^3)\Psi_v = t^{-1}\tilde{\Psi}_v - \frac{1}{4} i t^{-\frac{1}{2}} |x|^{-\frac{3}{2}} \Psi_v, \tag{3.45}$$

where

$$\tilde{\Psi}_v = \lambda^{-1} e^{i\phi} \partial_x \left(\frac{1}{2} \lambda (x - tv)\chi + i\lambda^2 t^{\frac{1}{2}} |x|^{\frac{1}{2}} \chi' + \frac{1}{3} t \lambda^3 \chi'' \right), \tag{3.46}$$

has similar localization to Ψ_v and hence $(\partial_t + \frac{1}{3}\partial_x^3)\Psi_v = O(t^{-1})$. Crucially, we note that $\tilde{\Psi}_v$ has some additional divergence structure.

By construction Ψ_v is localized at frequency $\xi_v = \sqrt{|v|}$. In fact we have the following lemma that demonstrates that Ψ_v is also a good approximate solution in Fourier space:

Lemma 3.11. *For $t \geq 1$ and $v \in \mathbf{\Omega}_0^-$*

$$\hat{\Psi}_v(t,\xi) = \pi^{\frac{1}{2}}\lambda^{-1}\chi_1(\lambda^{-1}(\xi - \xi_v))e^{\frac{1}{3}it\xi^3}, \tag{3.47}$$

where $\xi_v = \sqrt{|v|}$, and $\chi_1 \in \mathcal{S}(\mathbb{R})$ is localized at scale 1 in space and frequency satisfying

$$\int \chi_1(\xi) = 1 + O\left((t^{\frac{2}{3}}|v|)^{-\frac{3}{4}}\right). \tag{3.48}$$

Proof. We consider the Taylor approximation of ϕ at $x = tv$,

$$\phi(t,x) = \tfrac{1}{3}t\xi_v^3 + x\xi_v + \frac{\pi}{4} - \frac{1}{4}(\lambda(x-tv))^2 + R(\lambda(x-tv), t^{\frac{2}{3}}|v|),$$

where

$$R(x,y) = -\int_0^1 \frac{y^{-\frac{3}{4}}x^3(1-h)^2}{8|y^{-\frac{3}{4}}xh - 1|^{\frac{3}{2}}}\,dh$$

is well defined for $x \in \operatorname{supp}\Psi_v$ whenever $v \in \mathbf{\Omega}_0^-$. Taking $\chi_2(x) = \chi(x)e^{iR(x,t^{\frac{2}{3}}|v|)}$, we may write

$$\begin{aligned}
\hat{\Psi}_v(t,\xi) &= e^{\frac{1}{3}it\xi_v^3 + i\frac{\pi}{4}}\int e^{-\frac{1}{4}i(\lambda(x-tv))^2}\chi_2(\lambda(x-tv))e^{-ix(\xi-\xi_v)}\,dx \\
&= \lambda^{-1}e^{\frac{1}{3}it\xi_v^3 + i\frac{\pi}{4}}e^{it\xi_v^2(\xi-\xi_v)}\int e^{-\frac{1}{4}ix^2}\chi_2(x)e^{-i\lambda^{-1}x(\xi-\xi_v)}\,dx \\
&= \pi^{-\frac{1}{2}}\lambda^{-1}e^{\frac{1}{3}it\xi_v^3}e^{it\xi_v^2(\xi-\xi_v)}\int e^{i(\lambda^{-1}(\xi-\xi_v)-\eta)^2}\hat{\chi}_2(\eta)\,d\eta \\
&= \pi^{-\frac{1}{2}}\lambda^{-1}e^{\frac{1}{3}it\xi^3}e^{-it(\xi-\xi_v)^3}\int e^{-2i\lambda^{-1}(\xi-\xi_v)\eta}e^{i\eta^2}\hat{\chi}_2(\eta)\,d\eta.
\end{aligned}$$

In order to write this in the form (3.47), we define

$$\chi_1(\xi) = \pi^{-1}e^{-\frac{1}{3}it\lambda^3\xi^3}\int e^{-2i\xi\eta}e^{i\eta^2}\hat{\chi}_2(\eta)\,d\eta.$$

As $e^{\frac{1}{3}it\lambda^3\xi^3} = 1 + O((t^{\frac{2}{3}}|v|)^{-\frac{3}{4}}\xi^3)$ and $\chi_2 \in \mathcal{S}(\mathbb{R})$ we have

$$\int \chi_1 = \hat{\chi}_2(0) + O((t^{\frac{2}{3}}|v|)^{-\frac{3}{4}}),$$

and similarly, as $e^{iR(x,y)} = 1 + O(y^{-\frac{3}{4}}x^3)$,

$$\hat{\chi}_2(0) = 1 + O((t^{\frac{2}{3}}|v|)^{-\frac{3}{4}}),$$

which gives us (3.48). □

Testing by wave packets. In order to understand the behavior of the solution u along the ray Γ_v we test it against the wave packet Ψ_v by defining

$$\gamma(t,v) = \int u(t,x)\bar{\Psi}_v(t,x)\,dx. \tag{3.49}$$

As a consequence of the pointwise bounds of Proposition 3.9 and the frequency localization of Ψ_v in Lemma 3.11 we may replace u by the hyperbolic part at frequencies $\sim \xi_v$ in the definition of γ up to a rapidly decaying error:

Lemma 3.12. *For $t \in [1,T]$, $v \in \mathbf{\Omega}_0^-$ and $k \geq 0$, we have*

$$\left|\gamma(t,v) - \int w_{v,+}(t,x)\chi(\lambda(x-tv))\,dx\right| \lesssim_k \epsilon(t^{\frac{2}{3}}|v|)^{-k}, \tag{3.50}$$

where we define

$$w_{v,+}(t,x) = e^{-i\phi}\sum_{N\sim\xi_v} u_{N,+}^{\text{hyp}}. \tag{3.51}$$

Proof. We define the Fourier multiplier $\zeta_v \in C^\infty$ localizing to frequencies $\sim \xi_v$ by

$$\zeta_v(D) = \sum_{N\sim\xi_v} P_N P_+.$$

We observe that $\zeta_v(D)\Psi_v$ is spatially localized on the set $\{\lambda|x-tv| \lesssim 1\}$ up to rapidly decaying tails at scale $|v|^{-\frac{1}{2}}$. In particular,

$$\|\chi_{\{\lambda|x-tv|\gg 1\}}\zeta_v(D)\Psi_v\|_{L^1} \lesssim_k t^{\frac{1}{3}}(t^{\frac{2}{3}}|v|)^{-k}.$$

As Ψ_v is localized in Fourier space at frequency ξ_v, from (3.47) we then have

$$\begin{aligned}\|(1-\zeta_v(D))\Psi_v\|_{L^1} &\leq \|\chi_{\{\lambda|x-tv|\lesssim 1\}}(1-\zeta_v(D))\Psi_v\|_{L^1} + \|\chi_{\{\lambda|x-tv|\gg 1\}}\zeta_v(D)\Psi_v\|_{L^1}\\ &\lesssim_k \lambda^{-1}\|(1-\zeta_v(\xi))\hat{\Psi}_v\|_{L^1_\xi} + t^{\frac{1}{3}}(t^{\frac{2}{3}}|v|)^{-k}\\ &\lesssim_k t^{\frac{1}{3}}(t^{\frac{2}{3}}|v|)^{-k}.\end{aligned}$$

From the initial pointwise bounds (3.35) and (3.36) we have

$$\left|\gamma(t,v) - \sum_{N\sim\xi_v}\int u_{N,+}(t,x)\overline{\Psi}_v(t,x)\,dx\right| \lesssim \|\zeta_v(D)\Psi_v\|_{L^1}\|u\|_{L^\infty} \lesssim_k \epsilon(t^{\frac{2}{3}}|v|)^{-k}.$$

□

Energy estimates for γ. We may consider γ to be a function of $\xi_v = \sqrt{|v|}$ and by a slight abuse of notation take $\widehat{\Omega}_\rho^- \subset \mathbb{R}_+$ so that $\xi_v \in \widehat{\Omega}_\rho^-$ if and only if $v \in \mathbf{\Omega}_\rho^-$. The energy estimates for u then lead to the following energy estimates for γ:

Lemma 3.13. *For $t \in [1, T]$ we have the energy estimates*

$$\|\gamma\|_{H^{0,1}_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon, \tag{3.52}$$

$$\|\partial_{\xi_v}\gamma - 3t\xi_v^{-1}\partial_t\gamma\|_{L^2(\widehat{\Omega}_0^-)} \lesssim \epsilon t^\delta, \tag{3.53}$$

where $\delta > 0$ defined as in (3.19).

Proof. We first show that,

$$\left\| \int f(t,x)\chi(t^{-\frac{1}{2}}\xi_v^{-\frac{1}{2}}(x + t\xi_v^2))\, dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_\rho^-)} \lesssim \|f\|_{L^2(\Omega_\rho^-)}. \tag{3.54}$$

Making an affine change of variables, we have

$$\int f(t,x)\chi(t^{-\frac{1}{2}}\xi_v^{-\frac{1}{2}}(x + t\xi_v^2))\, dx = \int t^{\frac{1}{2}}\xi_v^{\frac{1}{2}} f(t, t^{\frac{1}{2}}\xi_v^{\frac{1}{2}}x - t\xi_v^2)\chi(x)\, dx.$$

We then define a nonlinear change of variables by

$$\xi_v \mapsto q = t^{\frac{1}{2}}\xi_v^{\frac{1}{2}}x - t\xi_v^2,$$

and calculate

$$t^{-\frac{1}{3}}q = -(t^{\frac{1}{3}}\xi_v)^2\left(1 - (t^{\frac{1}{3}}\xi_v)^{-\frac{3}{2}}x\right), \qquad \frac{dq}{d\xi_v} = -2t\xi_v\left(1 - \frac{1}{4}(t^{\frac{1}{3}}\xi_v)^{-\frac{3}{2}}x\right).$$

If $\xi_v \in \widehat{\Omega}_\rho^-$, then $t^{\frac{1}{3}}\xi_v \gtrsim t^\rho \geq 1$. Provided χ is supported in a sufficiently small neighborhood of the origin we have

$$-t^{-\frac{1}{3}}q \gtrsim t^{2\rho}, \qquad \left|\frac{dq}{d\xi_v}\right| \gtrsim t\xi_v,$$

which gives us (3.54).

As a consequence of (3.54), we have the estimate

$$\|\gamma\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \|u\|_{L^2}.$$

We calculate

$$\xi_v\Psi_v = -i\partial_x\Psi_v + \lambda\tilde{\Psi}_v,$$

where

$$\tilde{\Psi}_v(t,x) = e^{i\phi}\left(\lambda^{-1}(\xi_v - t^{-\frac{1}{2}}|x|^{\frac{1}{2}})\chi(\lambda(x - tv)) + i\chi'(\lambda(x - tv))\right)$$

has similar localization to Ψ_v. Integrating by parts in the first term and using (3.54), we have

$$\|\xi_v \gamma\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \|u\|_{H^1}.$$

We now turn to the estimate (3.53). We observe that $(3t\partial_t + x\partial_x)\Psi_v = \xi_v \partial_{\xi_v} \Psi_v$, so integrating by parts we have

$$\partial_{\xi_v}\gamma - 3t\xi_v^{-1}\partial_t \gamma = \int \Lambda u \, \xi_v^{-1} \partial_x \bar{\Psi}_v \, dx.$$

We calculate

$$\xi_v^{-1}\partial_x \Psi_v(t,x) = \left(\xi_v^{-1} \lambda \chi'(\lambda(x+tv)) + it^{-\frac{1}{2}}|x|^{\frac{1}{2}} \xi_v^{-1} \chi(\lambda(x+tv)) \right) e^{i\phi}$$

and observe that this has the same localization as Ψ_v. From the estimate (3.54), we then have

$$\|\partial_{\xi_v}\gamma - 3t\xi_v^{-1}\partial_t \gamma\|_{L^2(\widehat{\Omega}_0^-)} \lesssim \|\Lambda u\|_{L^2}.$$

□

Reduction of pointwise estimates to wave packets. Due to the localization of Ψ_v, we expect γ to measure u^{hyp} along the ray Γ_v. Using the pointwise bounds of Proposition 3.9, the following lemma allows us to reduce closing the bootstrap estimate (3.21) to proving

$$\|\gamma\|_{L^\infty_v(\mathbf{\Omega}_0^-)} \lesssim \epsilon \tag{3.55}$$

with a constant independent of M_0 and T.

Proposition 3.14. *For $t \in [1,T]$ we have the following estimates.*

A. Physical-space estimates.

$$\left\| t^{\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{\frac{3}{8}} \left(P_+ u(t,x) - t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} e^{i\phi} \gamma(t, t^{-1}x) \right) \right\|_{L^\infty(\Omega_0^-)} \lesssim \epsilon, \tag{3.56}$$

$$\left\| t^{\frac{2}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{8}} \left(P_+ u_x(t,x) - it^{-\frac{2}{3}} (t^{-\frac{1}{3}}|x|)^{\frac{1}{4}} e^{i\phi} \gamma(t, t^{-1}x) \right) \right\|_{L^\infty(\Omega_0^-)} \lesssim \epsilon, \tag{3.57}$$

$$\left\| t^{\frac{1}{6}} (t^{-\frac{1}{3}}|x|)^{\frac{1}{4}} \left(P_+ u(t,x) - t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} e^{i\phi} \gamma(t, t^{-1}x) \right) \right\|_{L^2(\Omega_0^-)} \lesssim \epsilon. \tag{3.58}$$

B. Fourier-space estimates.

$$\|(t^{\frac{1}{3}}\xi)^{\frac{1}{4}} (\hat{u}(t,\xi) - \pi^{-\frac{1}{2}} e^{\frac{1}{3}it\xi^3} \gamma(t, -\xi^2))\|_{L^\infty_\xi(\widehat{\Omega}_0^-)} \lesssim \epsilon, \tag{3.59}$$

$$\|t^{\frac{1}{6}} (t^{\frac{1}{3}}\xi)^{\frac{1}{2}} (\hat{u}(t,\xi) - \pi^{-\frac{1}{2}} e^{\frac{1}{3}it\xi^3} \gamma(t, -\xi^2))\|_{L^2_\xi(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.60}$$

Proof.
A. Physical-space estimates. For the L^2 bound (3.58), from the elliptic estimate (3.35) and the estimate (3.50), it suffices to show that

$$\left\| \lambda^{-2} w_{v,+}(t,tv) - \lambda^{-1} \int w_{v,+}(t,x)\chi(\lambda(x-tv))\,dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon t^{\frac{1}{6}}.$$

As $\int \chi = 1$ we have

$$\begin{aligned}
&\lambda^{-1} w_{v,+}(t,tv) - \int w_{v,+}(t,x)\chi(\lambda(x-tv))\,dx \\
&\quad= \int \left(w_{v,+}(t,tv) - w_{v,+}(t,x)\right)\chi(\lambda(x-tv))\,dx \\
&\quad= -\int\int_0^1 (\partial_x w_{v,+})(t, x-(x-tv)h)(x-tv)\chi(\lambda(x-tv))\,dhdx.
\end{aligned}$$

From the definition (3.51) we see that

$$\partial_x w_{v,+} = e^{-i\phi} \sum_{N\sim\xi_v} L_+ u^{\text{hyp}}_{N,+},$$

so we may apply the hyperbolic estimate (3.36) with the convolution estimate (3.54) to prove (3.58).

For the L^∞ estimate (3.56) we proceed similarly, using (3.35) and (3.50) to reduce the estimate to showing that

$$\left\| \lambda^{-\frac{1}{2}} \int \left(w_{v,+}(t,tv) - w_{v,+}(t,x)\right)\chi(\lambda(x-tv))\,dx \right\|_{L^\infty_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon t^{\frac{1}{6}}.$$

From the hyperbolic bound (3.36) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\lambda^{-\frac{1}{2}}|(w_{v,+}(t,tv) - w_{v,+}(t,x))| &\lesssim (t^{\frac{1}{3}}\xi_v)^{\frac{1}{4}}\|\partial_x w_{v,+}\|_{L^2}|x-tv|^{\frac{1}{2}} \\
&\lesssim \epsilon t^{\frac{1}{6}}\lambda^{\frac{3}{2}}|x-tv|^{\frac{1}{2}}.
\end{aligned}$$

The estimate then follows from the localization of χ.

For the remaining estimate (3.57), we first use the localization estimate (3.50) and then use that $w_{v,+}$ is localized at frequencies $\sim \xi_v$ to reduce to the estimate to (3.56).

B. Fourier-space estimates. We use the formula (3.47) for the Fourier transform of Ψ_v and the estimate (3.48) to get

$$\begin{aligned}
e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - \pi^{-\frac{1}{2}}\gamma(t,v) &= \pi^{-\frac{1}{2}} \int \left(e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi)\right)\lambda^{-1}\chi_1(\lambda^{-1}(\xi-\xi_v))\,d\xi \\
&\quad + O\left((t^{\frac{1}{3}}\xi_v)^{-\frac{3}{2}} e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v)\right).
\end{aligned}$$

For the difference we have

$$e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi) = -i(\xi_v-\xi)\int_0^1 e^{-\frac{1}{3}it\eta^3}\widehat{(Lu)}(t,h(\xi_v-\xi)+\xi)\,dh.$$

For the error terms we have

$$\|t^{\frac{1}{6}}(t^{\frac{1}{3}}\xi_v)^{-1}e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v)\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim t^{-\frac{1}{6}}\|t^{\frac{1}{3}}\langle t^{\frac{1}{3}}D_x\rangle^{-1}u\|_{L^2}.$$

The estimate (3.60) then follows from the energy estimate (3.31).

For (3.59) we use that

$$|e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi)| \lesssim \|Lu\|_{L^2}|\xi_v-\xi|^{\frac{1}{2}},$$

and estimate similarly. □

3.5 Global existence.

The asymptotic ODE. In order to prove (3.55), we fix $v \in \mathbf{\Omega}_0^-$ and consider the ODE satisfied by γ,

$$\dot{\gamma}(t,v) = \sigma\langle (u^3)_x, \Psi_v\rangle + \langle u, (\partial_t + \tfrac{1}{3}\partial_x^3)\Psi_v\rangle. \tag{3.61}$$

Our goal is to show that we may integrate $\dot{\gamma}$ in time and hence prove a uniform pointwise bound for γ. To do this, we first prove the following lemma:

Lemma 3.15. *For $t \in [1,T]$ and $\epsilon > 0$ sufficiently small, we have the estimates*

$$\|t(t^{\frac{2}{3}}|v|)^{\frac{1}{8}}(\dot{\gamma} - 3i\sigma t^{-1}|\gamma|^2\gamma)\|_{L_v^\infty(\mathbf{\Omega}_0^-)} \lesssim \epsilon, \tag{3.62}$$

$$\|t^{\frac{7}{6}}(t^{\frac{1}{3}}|\xi_v|)^{\frac{1}{2}}(\dot{\gamma} - 3i\sigma t^{-1}|\gamma|^2\gamma)\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.63}$$

Proof. We use **err** to denote error terms that satisfy the estimates

$$\|t(t^{\frac{2}{3}}|v|)^{\frac{1}{8}}\mathbf{err}\|_{L^\infty(\mathbf{\Omega}_0^-)} \lesssim \epsilon, \qquad \|t^{\frac{7}{6}}(t^{\frac{1}{3}}\xi_v)^{\frac{1}{2}}\mathbf{err}\|_{L^2(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.64}$$

We first integrate by parts to get

$$\int (u^3)_x\bar{\Psi}_v\,dx = 3i\int t^{-\frac{1}{2}}|x|^{\frac{1}{2}}u^3\bar{\Psi}_v\,dx - 3\int u^3\lambda e^{-i\phi}\chi'(\lambda(x-tv))\,dx.$$

Using the bootstrap assumption (3.21) and elliptic estimates (3.35), we have

$$3i\int t^{-\frac{1}{2}}|x|^{\frac{1}{2}}u^3\bar{\Psi}_v\,dx - 3\int u^3\lambda e^{-i\phi}\chi'(\lambda(x-tv))\,dx = 3i\xi_v\int (u^{\text{hyp}})^3\bar{\Psi}_v\,dx + \mathbf{err}.$$

Using the spatial localization of the frequency localized pieces of u^{hyp} and of Ψ_v, we have

$$3i\xi_v \int (u^{\text{hyp}})^3 \bar{\Psi}_v \, dx = \sum_{N \sim \xi_v, \pm} 3i\xi_v \int (u^{\text{hyp}}_{N,\pm})^3 \bar{\Psi}_v \, dx + \mathbf{err}.$$

However, as $u^{\text{hyp}}_{N,\pm}$ is localized at frequency $\sim \pm N$ up to rapidly decaying tails and Ψ_v is localized at frequency $+\xi_v$, we may use the frequency localization of Ψ_v and estimate as in (3.50) to remove the terms $(u^{\text{hyp}}_{N,+})^3$, $(u^{\text{hyp}}_{N,-})^3$ and $|u^{\text{hyp}}_{N,+}|^2 u^{\text{hyp}}_{N,-}$ to get

$$\sum_{N \sim \xi_v, \pm} 3i\xi_v \int (u^{\text{hyp}}_{N,\pm})^3 \bar{\Psi}_v \, dx = 3i\xi_v \int |w_{v,+}|^2 w_{v,+} \chi \, dx + \mathbf{err}.$$

Using the Cauchy-Schwarz inequality and the hyperbolic estimate (3.39), we have

$$|w_{v,+}(t,x) - w_{v,+}(t,tv)| \lesssim \sum_{N \sim \xi_v} \|L_+ u^{\text{hyp}}_{N,+}\|_{L^2} |x - tv|^{\frac{1}{2}} \lesssim \epsilon (t^{\frac{2}{3}} |v|)^{-\frac{3}{8}} \lambda^{\frac{1}{2}} |x - tv|^{\frac{1}{2}}.$$

We may then use this to replace two of the $w_{v,+}(t,x)$ terms by $w_{v,+}(t,tv)$ up to error terms,

$$3i\xi_v \int |w_{v,+}|^2 w_{v,+} \chi \, dx = 3i\xi_v |w_{v,+}(t,tv)|^2 \int w_{v,+} \chi \, dx + \mathbf{err}.$$

Finally we may estimate $w_{v,+}(t,tv)$ by $t^{-\frac{1}{2}} \xi_v^{-\frac{1}{2}} \gamma(t,v)$ as in (3.56) to get

$$\begin{aligned} 3i\xi_v |w_{v,+}(t,tv)|^2 \int w_{v,+} \chi \, dx + \mathbf{err} &= 3it^{-1} |\gamma(t,v)|^2 \int w_{v,+} \chi \, dx + \mathbf{err} \\ &= 3it^{-1} |\gamma|^2 \gamma + \mathbf{err}. \end{aligned}$$

For the linear terms we recall (3.45),

$$(\partial_t + \tfrac{1}{3} \partial_x^3) \Psi_v = t^{-1} \lambda^{-1} e^{i\phi} \partial_x \tilde{\chi} - \frac{1}{4} i t^{-\frac{1}{2}} |x|^{-\frac{3}{2}} \Psi_v,$$

where $\tilde{\chi}$ has the same localization as χ. For the first term we use the frequency localization of Ψ_v as in (3.50), integrate by parts and use the hyperbolic estimate (3.39) to get

$$t^{-1} \lambda^{-1} \int u \, \overline{(e^{i\phi} \partial_x \tilde{\chi})} \, dx = -t^{-1} \lambda^{-1} \int \partial_x w_{v,+} \bar{\tilde{\chi}} \, dx + \mathbf{err} = \mathbf{err}.$$

For the second term, we may simply use the localization and the hyperbolic estimate (3.36) to get

$$\frac{1}{4} i \int u t^{-\frac{1}{2}} |x|^{-\frac{3}{2}} \bar{\Psi}_v \, dx = \mathbf{err}.$$

□

Closing the bootstrap. We now use Lemma 3.15 to solve the ODE (3.61) and prove the pointwise bound estimate (3.55).

We first consider bounds for the initial data. For fixed v, we define the time at which the ray Γ_v enters the region Ω_0^- by

$$t_0(v) = \max\{1, C|v|^{-\frac{3}{2}}\}.$$

For velocities $|v| \geq C^{\frac{2}{3}}$, the ray lies inside Γ_v for all times $t \geq t_0(v) = 1$. We may then use the formula (3.47) for the Fourier transform of Ψ_v, the Sobolev estimate (1.10) applied to $e^{-\frac{1}{3}it\xi^3}\hat{u}(1,\xi)$ and the energy estimate (3.20) to get the initial estimate

$$|\gamma(1,v)| \lesssim \|\hat{u}(1)\|_{L^\infty} \lesssim \|u(1)\|_{L^2}^{\frac{1}{2}} \|Lu(1)\|_{L^2}^{\frac{1}{2}} \lesssim \epsilon. \tag{3.65}$$

For velocities $0 < |v| < C^{\frac{2}{3}}$, the ray Γ_v lies inside the self-similar region up to time $t = t_0(v)$. At time $t = t_0(v)$ we may use (3.50) to reduce to frequencies $\sim \xi_v \sim t_0(v)^{-\frac{1}{3}}$, then apply Bernstein's inequality (1.11) and the energy estimate (3.31) to get

$$|\gamma(t_0,v)| \lesssim t_0^{\frac{1}{6}} \sum_{N \sim t_0^{-\frac{1}{3}}} \|u_N(t_0)\|_{L^2} + \epsilon \lesssim \epsilon. \tag{3.66}$$

To complete the proof we observe that from (3.62), for $v \in \mathbf{\Omega}_0^-$, we have

$$\partial_t \left(e^{-3i\sigma \int_{t_0}^t |\gamma(s)|^2 \frac{ds}{s}} \gamma \right) = \mathbf{err}. \tag{3.67}$$

For each $v \in \mathbf{\Omega}_0^-$ we may then integrate from $t_0(v)$ to T with the initial bounds (3.65) and (3.66) to prove the estimate (3.55). Choosing M_0 sufficiently large and then $\epsilon > 0$ sufficiently small, we may close the bootstrap estimate and complete the proof of Theorem 3.1.

3.6 Asymptotic behavior

Asymptotic behavior in the oscillatory region. Integrating (3.67) from t to ∞, there exists a measurable function $A\colon (0,\infty) \to \mathbb{C}$ satisfying $|A| \lesssim \epsilon$ so that for $t \geq 1$,

$$e^{-3i\sigma \int_{t_0}^t |\gamma(s,v)|^2 \frac{ds}{s}} \gamma(t,v) = A(\xi_v) + t\,\mathbf{err}. \tag{3.68}$$

We observe that

$$\left|\partial_t \left(\int_{t_0}^t |\gamma(s,v)|^2 \frac{ds}{s} - |A(\xi_v)|^2 \log(t\xi_v^3) \right)\right| \lesssim t^{-1} \left||\gamma(t,v)|^2 - |A(\xi_v)|^2\right| \lesssim \epsilon^2 t^{-1} (t^{\frac{2}{3}}|v|)^{-\frac{1}{8}},$$

and hence there exists $B(\xi_v) \in \mathbb{R}$ so that

$$\left| \int_{t_0}^t |\gamma(s,v)|^2 \frac{ds}{s} - |A(\xi_v)|^2 \log(t\xi_v^3) - B(\xi_v) \right| \lesssim \epsilon^2 (t^{\frac{2}{3}}|v|)^{-\frac{1}{8}}. \tag{3.69}$$

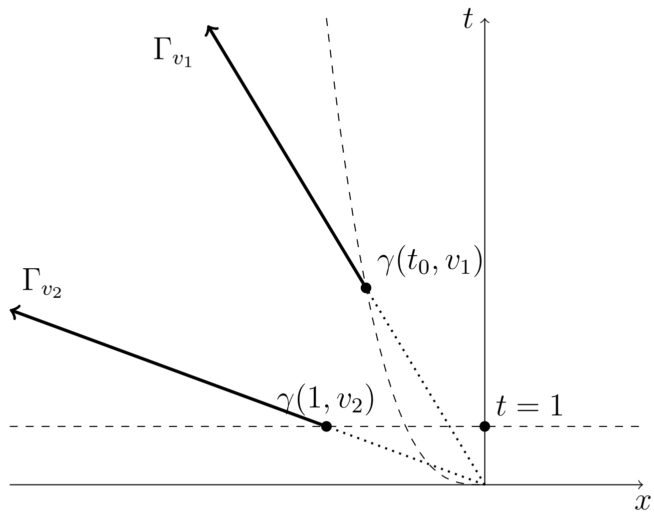

Figure 3.3: Solving the asymptotic ODE for $|v_1| \ll 1$ and $|v_2| \gtrsim 1$.

We then define $W(\xi_v) = (2\pi)^{\frac{1}{2}} A(\xi_v) e^{3i\sigma B(\xi_v)}$, and extend W to $\mathbb{R}$ by defining

$$W(-\xi_v) = \overline{W}(\xi_v), \qquad W(0) = \int u_0\, dx.$$

From the pointwise bound (3.55) for γ we have $\|W\|_{L^\infty} \lesssim \epsilon$ and by the energy estimate (3.52) and Fatou's Lemma we have

$$\|W\|_{H^{0,1}} \lesssim \epsilon. \tag{3.70}$$

As a consequence of the estimates (3.68) and (3.69), we have

$$\|(t^{\frac{2}{3}} v)^{\frac{1}{8}} (\gamma(t,v) - (2\pi)^{-\frac{1}{2}} W(\xi_v) e^{\frac{3i\sigma}{4\pi} |W(\xi_v)|^2 \log(t\xi_v^3)})\|_{L_v^\infty(\boldsymbol{\Omega}_0^-)} \lesssim \epsilon, \tag{3.71}$$

$$\|t^{\frac{1}{6}} (t^{\frac{2}{3}} v)^{\frac{1}{4}} (\gamma(t,v) - (2\pi)^{-\frac{1}{2}} W(\xi_v) e^{\frac{3i\sigma}{4\pi} |W(\xi_v)|^2 \log(t\xi_v^3)})\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.72}$$

Combining these estimates with Proposition 3.14 we obtain the asymptotics (3.7), (3.9).

To complete the analysis in the oscillatory region, it remains to prove $W \in H^{1-C\epsilon^2}$. We start by defining the phase $\Phi = 3\sigma |\gamma(t,v)|^2 \log(t\xi_v^3)$ and the region

$$\widehat{\Omega}_*^- = \widehat{\Omega}_{1/2}^- \backslash \widehat{\Omega}_{1/6}^- = \{t^{-\frac{1}{6}} \lesssim \xi_v \lesssim t^{\frac{1}{6}}\}.$$

From (3.72) we have the estimate,

$$\left\| e^{-i\Phi} \gamma(t,v) - (2\pi)^{-\frac{1}{2}} W(\xi_v) \right\|_{L^2_{\xi_v}(\widehat{\Omega}_*^-)} \lesssim \epsilon t^{-\frac{1}{4}} (1 + \log t).$$

Using the estimate (3.63), we have

$$\begin{aligned} &e^{i\Phi}\partial_{\xi_v}\left(e^{-i\Phi}\gamma(t,v)\right) \\ &\quad = \partial_{\xi_v}\gamma - 9i\sigma\xi_v^{-1}|\gamma|^2\gamma - 3i\sigma\partial_{\xi_v}(|\gamma|^2)\gamma\log(t\xi_v^3) \\ &\quad = (\partial_{\xi_v}\gamma - 3t\xi_v^{-1}\dot\gamma) - 6i\sigma\operatorname{Re}\left((\partial_{\xi_v} - 3t\xi_v^{-1}\dot\gamma)\bar\gamma\right)\gamma\log(t\xi_v^3) + t\xi_v^{-1}\left(1+|\gamma|^2\log(t\xi_v^3)\right)\mathbf{err} \end{aligned}$$

From the energy estimate (3.53), we then have

$$\left\|\partial_{\xi_v}\left(e^{-\Phi}\gamma(t,v)\right)\right\|_{L^2_{\xi_v}(\widehat\Omega_*^-)} \lesssim \epsilon t^\delta(1+\log t).$$

We may then interpolate between these bounds (see §3.A) to get

$$\|W\|_{H^{1-C\epsilon^2}} \lesssim \epsilon. \tag{3.73}$$

Asymptotic behavior in the self-similar region. To complete the proof of Theorem 3.2, it remains to show that the leading asymptotics in the region Ω_0^0 are given by a solution to the Painlevé II equation (3.11). To do this, we will work with the self-similar change of variables defined in (3.33). We will identify $\Omega_\rho^0 = \{t^{-\frac{1}{3}}|x| \lesssim t^{2\rho}\} = \{|y| \lesssim t^{2\rho}\}$ under this change of coordinates.

Let $\rho > 0$ and $C \gg 1$. From the equation (3.34) for U, the energy estimate (3.25), Bernstein's inequality (1.11) and the elliptic estimate (3.38), we have

$$\begin{aligned} \|\partial_t P_{\leq Ct^\rho}U\|_{L^\infty(\Omega_\rho^0)} &\lesssim t^{\frac{\rho}{2}}\|P_{\leq Ct^\rho}\partial_t U\|_{L^2} + t^{-1}\|P_{\sim Ct^\rho}U\|_{L^\infty(\Omega_\rho^0)} \\ &\lesssim \epsilon t^{\frac{3}{2}\rho+\delta-\frac{7}{6}} + \epsilon t^{\frac{\rho}{2}-\frac{5}{6}}\sum_{N\sim Ct^{\rho-\frac{1}{3}}}\|u_N^{\mathrm{ell}}\|_{L^2} \\ &\lesssim \epsilon t^{-\min\{\frac{1}{6}-\delta-\frac{3}{2}\rho,\frac{3}{2}\rho\}-1}. \end{aligned}$$

From the elliptic estimate (3.38) we also have

$$\|P_{>Ct^\rho}U\|_{L^\infty(\Omega_\rho^0)} \lesssim \sum_{N>Ct^{\rho-\frac{1}{3}}} t^{\frac{1}{6}}N^{\frac{1}{2}}\|u_N^{\mathrm{ell}}\|_{L^2} \lesssim \epsilon t^{-\frac{3}{2}\rho}.$$

Choosing $0<\rho<\frac{2}{3}(\frac{1}{6}-\delta)$, for almost every $y\in\mathbb{R}$ we may then define $Q(y) = \lim_{t\to\infty} U(t,y)$ such that

$$\|Q\|_{L^\infty} \lesssim \epsilon, \qquad \|U-Q\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon t^{-\min\{\frac{1}{6}-\delta-\frac{3}{2}\rho,\frac{3}{2}\rho\}}.$$

We recall that

$$\|yU - U_{yy} + 3\sigma U^3\|_{L^2_y} = t^{-\frac{1}{6}}\|\Lambda u\|_{L^2_x} \lesssim \epsilon t^{\delta-\frac{1}{6}},$$

so taking the limit as $t\to\infty$ we see that

$$Q_{yy} - yQ = 3\sigma Q^3.$$

3.A An interpolation estimate.

In this appendix we give a proof of the interpolation estimate needed in (3.73). Variations on this result are used in [50, 61, 63].

We first note that if $w = w(x)$ and for all $t \geq 1$,

$$w \in t^{-\alpha} L^2(\mathbb{R}) + t^{\delta} H^1(\mathbb{R}), \tag{3.74}$$

then by real interpolation $w \in H^s$ for all $s \in [0, 1 - \frac{\delta}{\alpha+\delta})$. Our goal is to extend this to the case that we only have this representation in some time-dependent set

$$\Omega = \{x \in \mathbb{R} : \tfrac{1}{2} t^{-2\beta} \leq |x| \leq 2t^{2\beta}\}.$$

Lemma 3.16. *Let $0 < \delta \ll \alpha, \beta$, $w \in C(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$ satisfy*

$$\|w\|_{L^\infty \cap H^{0,1}} \lesssim 1,$$

and for all $t \geq 1$ suppose that there exists $u(t) \in H^1(\Omega)$ such that

$$\|u(t)\|_{L^\infty \cap L^2(\Omega)} \lesssim 1, \qquad \|u(t)\|_{H^1(\Omega)} \lesssim t^{\delta}, \qquad \|u(t) - w\|_{L^2(\Omega)} \lesssim t^{-\alpha}.$$

Then $w \in H^s$ for $s \in [0, 1 - \frac{\delta}{\min\{\alpha,\beta\}+\delta})$.

Proof. We will show that (3.74) holds (with α replaced by $\min\{\alpha, \beta\}$) by explicitly constructing an extension v of u. By taking real and imaginary parts, it suffices to assume that u, w are real-valued. Further, we may decompose u, w into even and odd parts and consider each of these separately.

We first consider the case that u, w are even. For smooth χ identically 1 on $[-1, 1]$ and supported in $(-2, 2)$ we freeze u in the region $\{|x| < t^{-2\beta}\}$ by defining

$$v(t, x) = \chi(2t^{-2\beta} x) u(t, x) \mathbf{1}_{\{|x| \geq t^{-2\beta}\}} + u(t, t^{-2\beta}) \mathbf{1}_{\{|x| < t^{-2\beta}\}},$$

and observe that

$$\begin{aligned} \|v - w\|_{L^2} &\lesssim \|u - w\|_{L^2(\Omega)} + \|w\|_{L^2(|x| < t^{-2\beta})} + \|w\|_{L^2(|x| > t^{2\beta})} + \|u(t, t^{-2\beta})\|_{L^2(|x| < t^{-2\beta})} \\ &\lesssim t^{-\alpha} + t^{-\beta} \|w\|_{L^\infty} + t^{-2\beta} \|w\|_{H^{0,1}} + t^{-\beta} \|u\|_{L^\infty(\Omega)} \\ &\lesssim t^{-\min\{\alpha,\beta\}}. \end{aligned}$$

Further, we have the estimates,

$$\|v\|_{L^2} \lesssim \|u\|_{L^2(\Omega)} + \|u(t, t^{-2\beta})\|_{L^2(|x| < t^{-2\beta})} \lesssim 1, \qquad \|v_x\|_{L^2} \lesssim \|u_x\|_{L^2(\Omega)} + t^{-2\beta} \|u\|_{L^2(\Omega)} \lesssim t^{\delta},$$

so $v \in t^{\delta} H^1(\mathbb{R})$ and hence w satisfies (3.74).

Second we consider the case that u, w are odd. We extend u to $\mathbb{R}$ by zero and take our extension to be

$$v(t, x) = \chi(t^{2\beta} x)(w_{\leq t^{\delta}} - u_{\leq t^{\delta}}) + \chi(2t^{-2\beta} x) u.$$

We then have

$$\begin{aligned}\|v-w\|_{L^2} &\lesssim \|u-w\|_{L^2(\Omega)} + \|\chi(t^{2\beta}x)w_{>t^\delta}\|_{L^2} + \|\chi(t^{2\beta}x)u_{>t^\delta}\|_{L^2} + \|(1-\chi(2t^{-2\beta}x))w\|_{L^2} \\ &\lesssim t^{-\alpha} + t^{-\beta}\|w\|_{L^\infty} + t^{-\beta}\|u\|_{L^\infty} + t^{-2\beta}\|w\|_{H^{0,1}} \\ &\lesssim t^{-\min\{\alpha,\beta\}}.\end{aligned}$$

Using that $w_{\leq t^\delta}, u_{\leq t^\delta}$ are odd, we may apply the classical Hardy inequality [47] to get

$$\begin{aligned}\|t^\beta \chi'(t^\beta x) w_{\leq t^\delta}\|_{L^2} \lesssim \|\partial_x w_{\leq t^\delta}\|_{L^2} \lesssim t^\delta, \\ \|t^\beta \chi'(t^\beta x) u_{\leq t^\delta}\|_{L^2} \lesssim \|\partial_x u_{\leq t^\delta}\|_{L^2} \lesssim t^\delta.\end{aligned}$$

As a consequence $v \in t^\delta H^1(\mathbb{R})$ and again w satisfies (3.74)

□

Chapter 4

Asymptotic completeness for the mKdV

4.1 Introduction

In this chapter we consider the asymptotic completeness problem for the mKdV: given a suitable asymptotic profile u_{asymp}, can we find a solution u to the mKdV so that $u_0 \in H^{1,1}$ and the leading asymptotics of u agree with u_{asymp} as $t \to +\infty$? More precisely, we look to solve the problem

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = \sigma(u^3)_x \\ \lim\limits_{t\to+\infty} \|u(t) - u_{\text{asymp}}(t)\|_S = 0, \end{cases} \tag{4.1}$$

where the norm

$$\|u\|_S = \|u\|_{L^2} + \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}u\|_{L^\infty}.$$

Hayashi-Naumkin [53] showed that under strong conditions on the data, including that it has mean zero, it is possible to find such a solution. The result presented in this chapter greatly improves this by considering a much larger class of data, including those with non-trivial mean. In the case of the gKdV, where solutions scatter to free solutions, asymptotic completeness was established by Côte [19] and refined by Farah-Pastor [35]. Similar results have also been obtained for the cubic NLS, see for example [61, 98] and references therein.

As mentioned in §1.4, a key object of study will be the one-parameter family of solutions $Q(y;W)$ to the Painlevé II equation

$$\begin{cases} Q_{yy} - yQ = 3\sigma Q^3 \\ Q(y;W) \sim q_\sigma(W)\,\mathrm{Ai}(y), \qquad y \to +\infty, \end{cases} \tag{4.2}$$

where $q_\sigma(W)$ is defined as in (1.52). Comparing the asymptotics for the Painlevé II of Theorem 1.8 to the asymptotics for solutions to the mKdV established in Theorem 3.2, we see that a suitable candidate for u_{asymp} is given by

$$u_{\text{asymp}}(t,x) = t^{-\frac{1}{3}} Q(t^{-\frac{1}{3}}x; W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})), \tag{4.3}$$

where we assume that W is a real-valued, even function.

Statement of results. We define the space Y of real-valued even functions with norm

$$\|W\|_Y = \|\langle D\rangle^{C\epsilon^2} W\|_{H^{1,1}} \tag{4.4}$$

and then have the following asymptotic completeness result.

Theorem 4.1. *There exist $\epsilon, C > 0$ so that for all $W \in Y$ satisfying*

$$\|W\|_Y \leq \epsilon, \tag{4.5}$$

there exists a unique $u_0 \in H^{1,1}$ satisfying

$$\|u_0\|_{H^{1,1}} \lesssim \epsilon. \tag{4.6}$$

such that the corresponding solution $S(-t)u \in C(\mathbb{R}; H^{1,1})$ to the mKdV scatters to u_{asymp} in the sense of (4.1)*.*

Remark 4.2. Similar to Theorem 3.2 we have an $O(\epsilon^2)$ loss of regularity between W and u. As our approximate solution will not have a conserved energy, we require additional regularity for zW as well.

Outline of the proof. In order to prove Theorem 4.1 we use an approach similar to [61] and replace u_{asymp} by a regularized version u_{app}, where the regularization is on the scale of the wave packets. The approximate solution then satisfies the equation

$$(\partial_t + \tfrac{1}{3}\partial_x^3)u_{\text{app}} = \sigma(u_{\text{app}}^3)_x + f. \tag{4.7}$$

We will prove the existence of a solution to (4.1) on the the interval $[1, \infty)$ satisfying

$$\|u(1)\|_X \lesssim \epsilon,$$

where the space X is defined as in (3.18). We may then extend it to $[0, \infty)$ by applying the local well-posedness result Theorem 3.6 backwards in time on the interval $[0, 1]$.

If we define $v = u - u_{\text{app}}$, the equation (4.1) becomes

$$\begin{cases} (\partial_t + \frac{1}{3}\partial_x^3)v = \mathbf{N}(u_{\text{app}}, v) - f \\ \lim\limits_{t\to+\infty} v(t) = 0, \end{cases} \tag{4.8}$$

where the nonlinear term

$$\mathbf{N}(u_{\text{app}}, v) = \sigma\left((v + u_{\text{app}})^3 - u_{\text{app}}^3\right)_x. \tag{4.9}$$

For $\delta = C\epsilon^2$, where $C > 0$ is as in the definition of the Y-norm, we define the norms

$$\|v\|_Z = \sup_{T\geq 1}\left\{T^{\frac{1}{3}+\frac{\delta}{3}}\|v\|_{L_T^\infty L_x^2} + T^{\frac{1}{4}+\frac{\delta}{3}}\|v\|_{L_x^4 L_T^\infty} + T^{\frac{\delta}{3}}\|v_x\|_{L_T^\infty L_x^2}\right\},$$

$$\|v\|_{\tilde{Z}} = \sup_{T\geq 1}\left\{\frac{T^{\frac{\delta}{3}}}{1+\epsilon^2\log T}\|v\|_{L_T^\infty L_x^2}\right\},$$

where we use the notation $L_T^p = L^p([T, 2T])$ and the supremum is taken over dyadic $T \geq 1$. We then look to solve (4.8) using the contraction principle in the ball

$$Z_\epsilon = \{v : \|v\|_Z + \|Lv\|_{\tilde{Z}} \leq B\epsilon\}, \tag{4.10}$$

where $L = x - t\partial_x^2$ is defined as in Chapter 3.

Further questions. As we use the 1-parameter family of real-valued solutions to the Painlevé II as our asymptotic object, we are restricted to considering real-valued W. This leaves a small gap between Theorems 3.1 and 4.1. It would be of significant interest to try and extend Theorem 4.1 to handle W satisfying $W(z) = \overline{W}(-z)$ in order to complete the picture of the small data asymptotics.

As discussed in §1.4, the Painlevé II also gives rise to a self-similar solution to the KdV. A further question would be whether a similar construction would give an asymptotic completeness result for the KdV. The result of Deift Venakides and Zhou [23] shows that there is a "collisionless shock region" between the self-similar and oscillatory region in this case, so it is possible that a different asymptotic profile would have to be used.

Another natural extension would be to consider (4.1) with the asymptotic profile $u_{\text{asymp}}+v$ where v is a kink or soliton solution to the mKdV. Results of this form for the gKdV, where u_{asymp} is simply a linear wave, have been established by Côte [19]. Modified asymptotics in a neighborhood of the soliton have been proved for suitable initial data [39, 124], but the author is unaware of any work on the asymptotic completeness problem.

4.2 Construction of the approximate solution

Regularization of W. We start by dyadically decomposing

$$W(z) = \sum_{N\in 2^{\mathbb{Z}}} W_N(z), \qquad W_N = P_N W.$$

We then take $\chi \in C^\infty(\mathbb{R})$ to be smooth on scale ~ 1 and to satisfy $\chi(z) \equiv 1$ for $|z| \geq 1$, $\chi \equiv 0$ for $|z| \leq \frac{1}{2}$. For each $N > 1$ we define the function

$$\chi_N(t, z) = \chi(N^{-2}t^{\frac{2}{3}}\langle t^{\frac{1}{3}}z\rangle),$$

We observe that $\chi_N \equiv 1$ for frequencies $N \leq t^{\frac{1}{3}}$ and $\chi_N = 1$ is supported on the set $|z| > tN^{-2}$ for frequencies $N > t^{\frac{1}{3}}$. We then define a regularized version of W by

$$\mathcal{W}(t,z) = \sum_{N \leq t} \chi_N(t,z) W_N(z). \tag{4.11}$$

By construction, the map

$$x \mapsto \mathcal{W}(t, t^{-\frac{1}{2}}|x|^{\frac{1}{2}})$$

is smooth on the scale of the wave packets on $\mathbb{R}\backslash\{0\}$. However, to ensure that u_{app} is a good approximation on $\mathbb{R}$ we require additional smoothing at $x = 0$. To do this we take an even function $\zeta \in C^\infty(\mathbb{R})$ so that $\zeta(y) = |y|^{\frac{1}{2}}$ for $|y| \geq 1$, $\zeta(0) = 0$ and $\zeta'(y) \neq 0$ for $y \neq 0$. We then define the approximate solution u_{app} by

$$u_{\text{app}}(t,x) = t^{-\frac{1}{3}} Q\left(t^{-\frac{1}{3}}x; \mathcal{W}\left(t, t^{-\frac{1}{3}}\zeta(t^{-\frac{1}{3}}x)\right)\right). \tag{4.12}$$

Remark 4.3. We note that in defining $\mathcal{W}$ we have introduced an additional regularization by only selecting frequencies $\leq t$. This is merely a technical assumption, and may be removed by assuming additional decay for W and slightly modifying the Z-spaces, for example by requiring that $\|\langle y\rangle \log\langle y\rangle D^\delta W\|_{L^2} \lesssim \epsilon$.

Estimates for $\mathcal{W}$. By construction, $\mathcal{W}(t,z)$ is localized at frequencies $\lesssim t^{\frac{1}{3}}\langle t^{\frac{1}{3}}z\rangle^{\frac{1}{2}}$. As a straightforward consequence of this localization, we have the following Lemma:

Lemma 4.4. *For $t \geq 1$ we have the following estimates.*

A. Estimates for $\mathcal{W} = \mathcal{W}(t, t^{-\frac{1}{3}}\zeta(t^{-\frac{1}{3}}x))$.

$$\begin{gathered} \|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\partial_x\mathcal{W}\|_{L^2} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\delta}\partial_x^k\mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad k \geq 2, \\ \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\log\langle t^{-\frac{1}{3}}x\rangle\partial_x\mathcal{W}\|_{L^2} \lesssim \delta^{-1}\epsilon(1+\epsilon^2\log t). \end{gathered} \tag{4.13}$$

$$\begin{gathered} \|t^{-1}(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})\mathcal{W}\|_{L^2} \lesssim \epsilon, \\ \|t^{-1}(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+1+\delta}\partial_x^k\mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad k \geq 1. \end{gathered} \tag{4.14}$$

$$\begin{gathered} \|\mathcal{W}\|_{L^\infty} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k\mathcal{W}\|_{L^\infty} \lesssim \epsilon, \qquad k \geq 1. \end{gathered} \tag{4.15}$$

B. Estimates for $\mathcal{W}_t = \mathcal{W}_t(t, t^{-\frac{1}{3}}\zeta(t^{-\frac{1}{3}}x))$.

$$\begin{gathered} \|t(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\delta}\partial_x^k\mathcal{W}_t\|_{L^2} \lesssim \epsilon, \qquad k \geq 0, \\ \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+1+\delta}\partial_x^k\mathcal{W}_t\|_{L^2} \lesssim \epsilon, \qquad k \geq 0. \end{gathered} \tag{4.16}$$

C. Estimates for $W - \mathcal{W}$.

$$\begin{aligned} \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{\delta}(W-\mathcal{W})\|_{L^2} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{\frac{1}{2}+\delta}(W-\mathcal{W})\|_{L^\infty} \lesssim \epsilon. \end{aligned} \tag{4.17}$$

Proof. We define the regions $\Omega_0^-, \Omega_0^0, \Omega_0^+, \widehat{\Omega}_0^-$ as in Chapter 3 and consider the regions $\Omega_0^- \cup \Omega_0^+$ and Ω_0^0 separately.

For $|y| \gtrsim 1$ we have $\zeta(y) = |y|^{\frac{1}{2}}$, so making a simple change of variables and using the frequency localization of the χ_N we have

$$\begin{aligned} \|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\mathcal{W}\|_{L^2_x(\Omega_0^-\cup\Omega_0^+)} &\lesssim \|\mathcal{W}\|_{L^2_z} \lesssim \|W\|_{L^2}, \\ \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\partial_x\mathcal{W}\|_{L^2_x(\Omega_0^-\cup\Omega_0^+)} &\lesssim \|\mathcal{W}\|_{H^1_z} \lesssim \|W\|_{H^1}. \end{aligned}$$

Next we consider

$$\begin{aligned} \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\log\langle t^{-\frac{1}{3}}x\rangle\partial_x\mathcal{W}\|_{L^2(\Omega_0^-\cup\Omega_0^+)} &\lesssim \|\log\langle t^{\frac{1}{3}}z\rangle\partial_z\mathcal{W}\|_{L^2_z} \\ &\lesssim \|W\|_{H^1_z}(1+\log t) + \|\log\langle z\rangle\partial_z W\|_{L^2_z} \end{aligned}$$

We may interpolate to get the bound $\|\langle z\rangle^{\delta}\partial_z W\|_{L^2} \lesssim \|W\|_Y$, and hence we have the estimate,

$$\|\log\langle z\rangle\partial_z W\|_{L^2} \lesssim \delta^{-1}\|\langle z\rangle^{\delta}\partial_z W\|_{L^2} \lesssim \delta^{-1}\epsilon.$$

We now consider the higher order derivatives. For $k \geq 2$ we differentiate to obtain

$$\partial_x^k\mathcal{W} = \sum_{m=1}^{k} c_{m,k} t^{-\frac{k+m}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{m}{2}-k}(\partial_z^m\mathcal{W})(t, t^{-\frac{1}{2}}|x|^{\frac{1}{2}}),$$

and may then estimate

$$\|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\delta}\partial_x^k\mathcal{W}\|_{L^2_x(\Omega_0^-\cup\Omega_0^+)} \lesssim \sum_{m=1}^{k}\|t^{\frac{1+\delta-m}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1+\delta+2m-3k}{2}}\partial_z^m\mathcal{W}\|_{L^2_z(\widehat{\Omega}_0^-)}.$$

For $m = k$, up to rapidly decaying tails, we have

$$\begin{aligned} &\|(t^{\frac{1}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1}{2}})^{1+\delta-k}\partial_z^k W_{\leq t^{\frac{1}{3}}}\|^2_{L^2(\widehat{\Omega}_0^-)} + \sum_{N>t^{\frac{1}{3}}}\|(t^{\frac{1}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1}{2}})^{1+\delta-k}\partial_z^k(\chi_N W_N)\|^2_{L^2} \\ &\lesssim t^{\frac{2(1+\delta-k)}{3}}\|\partial_z^k W_{\leq t^{\frac{1}{3}}}\|^2_{L^2} + \sum_{N>t^{\frac{1}{3}}} N^{2(1+\delta)}\|W_N\|^2_{L^2} \\ &\lesssim \|W\|^2_{H^{1+\delta}}. \end{aligned}$$

For $1 \leq m < k$ we estimate $\mathcal{W}$ in L^∞ and apply Bernstein's inequality to ensure that we take advantage of the additional regularity of W, even when $m = 1$. Again up to rapidly decaying tails we have

$$\begin{aligned}
&\|t^{\frac{1+\delta-m}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1+\delta+2m-3k}{2}}\partial_z^m W_{\leq t^{\frac{1}{3}}}\|^2_{L^2_z(\widehat{\Omega}_0^-)} + \sum_{M>t^{\frac{1}{3}}} \|t^{\frac{1+\delta-m}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1+\delta+2m-3k}{2}}\partial_z^m(\chi_M W_M)\|^2_{L^2_z(\widehat{\Omega}_0^-)} \\
&\quad\lesssim \|t^{\frac{1+\delta-m}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1+\delta+2m-3k}{2}}\|^2_{L^2(\widehat{\Omega}_0^-)}\|\partial_z^m W_{\leq t^{\frac{1}{3}}}\|^2_{L^\infty} \\
&\qquad + \sum_{N>t^{\frac{1}{3}}} \|t^{\frac{1+\delta-m}{3}}(t^{\frac{1}{3}}|z|)^{\frac{1+\delta+2m-3k}{2}}\|^2_{L^2(|z|\gtrsim t^{-1}N^2)}\|\partial_z^m(\chi_N W_N)\|^2_{L^\infty_z} \\
&\quad\lesssim \|W\|^2_{H^{1+\delta}}.
\end{aligned}$$

The remaining L^2 estimates (4.14) and (4.16) in the region $\Omega_0^- \cup \Omega_0^+$ are similar.

Next we consider the self-similar region Ω_0^0. In this region we only have frequencies $\leq t^{\frac{1}{3}}$ and hence we have,

$$\partial_x^k \mathcal{W} = \sum_{m=1}^{k} c_{m,k} t^{-\frac{1}{3}(k+m)} R(t^{-\frac{1}{3}}x)\partial_z^m \mathcal{W},$$

where R is a smooth, bounded function depending on ζ. Applying the Cauchy-Schwarz and Bernstein inequalities, we then estimate

$$\begin{aligned}
\|t^{-\frac{1}{3}}\mathcal{W}\|_{L^2(\Omega_0^0)} &\lesssim t^{-\frac{1}{6}}\|W_{\leq t^{\frac{1}{3}}}\|_{L^\infty} \lesssim \|W\|_{L^2}, \\
\|t^{\frac{1}{3}}\partial_x \mathcal{W}\|_{L^2(\Omega_0^0)} &\lesssim t^{-\frac{1}{6}}\|\partial_z W_{\leq t^{\frac{1}{3}}}\|_{L^\infty} \lesssim \|W\|_{H^1}, \\
\|t^{\frac{1}{3}(k+\delta)}\partial_x^k \mathcal{W}\|_{L^2(\Omega_0^0)} &\lesssim \sum_{m=1}^{k} t^{\frac{1}{3}(\frac{1}{2}+\delta-m)}\|\partial_z^m W_{\leq t^{\frac{1}{3}}}\|_{L^\infty} \lesssim \|W\|_{H^{1+\delta}}.
\end{aligned}$$

The L^2 estimates (4.14) and (4.16) in the region Ω_0^0 are similar.

We now consider the L^∞ estimate (4.15). By Sobolev embedding we have

$$\|\mathcal{W}\|_{L^\infty} \lesssim \|\mathcal{W}\|_{H^1} \lesssim \|W\|_{H^1}.$$

For the the second part of (4.15), we first observe that

$$\begin{aligned}
&\|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k \mathcal{W}\|_{L^\infty_x} \\
&\quad\lesssim \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k \mathcal{W}\|_{L^\infty_x(\Omega_0^0)} + \sup_{M>t^{\frac{1}{3}}} \|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k \mathcal{W}\|_{L^\infty_x(|x|\sim t^{-1}M^4)}.
\end{aligned}$$

In the self-similar region Ω_0^0 we apply Bernstein's inequality (1.11) to get

$$\|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k \mathcal{W}\|_{L^\infty_x(\Omega_0^0)} \lesssim \sum_{m=1}^{k} t^{\frac{1}{6}+\frac{\delta}{3}-\frac{m}{3}}\|\partial_z^m W_{\leq t^{\frac{1}{3}}}\|_{L^\infty_z} \lesssim \|W_{\leq t^{\frac{1}{3}}}\|_{H^{1+\delta}}.$$

In the region $\Omega_0^- \cup \Omega_0^+$, we consider the sets $\{x \sim t^{-1}M^4\}$ for $M > t^{\frac{1}{3}}$ separately. Using the spatial localization of the χ_N and Bernstein's inequality (1.11), we then have

$$\begin{aligned}\|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{k+\frac{1}{2}+\delta}\partial_x^k \mathcal{W}\|_{L_x^\infty(|x|\sim t^{-1}M^4)} &\lesssim \sum_{k=1}^m t^{k-m} M^{2m-3k+\frac{1}{2}+\delta}\|\partial_z^m W_{\lesssim M}\|_{L_z^\infty} \\ &\lesssim \|W\|_{H^{1+\delta}}.\end{aligned}$$

Taking the supremum over $M > t^{\frac{1}{3}}$ we get (4.15).

For the estimates on the difference (4.17) we write

$$W - \mathcal{W} = \sum_{t^{\frac{1}{3}}<N<t} (1-\chi_N)W_N + W_{>t}.$$

For the first term we may estimate similarly to before, using that $1-\chi_N$ is supported on the set $\{|z| \lesssim t^{-1}N^2\}$. For the second term, we have

$$\begin{aligned}\|(t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}})^{\delta} W_{>t}\|_{L^2} &\lesssim \|t^{\frac{1+\delta}{3}}\langle t^{\frac{1}{3}}z\rangle^{\frac{1+\delta}{2}} W_{>t}\|_{L^2} \\ &\lesssim t^{1+\delta}\|\chi_{\{|z|\le t\}}W_{>t}\|_{L^2} + t^{\delta}\|\chi_{\{|z|>t\}}zW_{>t}\|_{L^2} \\ &\lesssim \|\langle D\rangle^{1+\delta}W\|_{L^2} + \|\langle z\rangle\langle D\rangle^{\delta}W\|_{L^2}.\end{aligned}$$

The L^∞ estimate is similar, using the Sobolev estimate (1.10). □

Estimates for u_{app}. We now look to derive estimates for u_{app}. We first state the following lemma giving estimates for solutions to the Painlevé II equation (4.2). For completeness, we outline the proof in Appendix 4.A.

Lemma 4.5. *Let $|W| \ll 1$ and $Q(y;W)$ be the solution to* (4.2)*. We then have the estimate*

$$|\partial_y^k\partial_w^m Q(y;W)| \lesssim_{k,m} \begin{cases} |W|\langle y\rangle^{-\frac{1}{4}+\frac{k}{2}} e^{-\frac{2}{3}y_+^{\frac{3}{2}}}(1+|W|^2\log\langle y\rangle)^m, & m \text{ even}, \\ \langle y\rangle^{-\frac{1}{4}+\frac{k}{2}} e^{-\frac{2}{3}y_+^{\frac{3}{2}}}(1+|W|^2\log\langle y\rangle)^m, & m \text{ odd}. \end{cases} \tag{4.18}$$

We note that if $|W| \lesssim \epsilon$ and $\delta = C\epsilon^2$ then as a consequence of (4.18), we have the estimate

$$|\partial_y^k\partial_w^m Q(y;W)| \lesssim_{k,m} \begin{cases} |W|\langle y\rangle^{\frac{2k-1+\delta}{4}}, & m \text{ even}, \\ \langle y\rangle^{\frac{2k-1+\delta}{4}}, & m \text{ odd}. \end{cases} \tag{4.19}$$

Using the estimates of Lemmas 4.4 and 4.5 we now prove several estimates for u_{app} and show that it is is a good approximation to u_{asymp} under the S-norm.

Lemma 4.6. *For $t \geq 1$ we have estimates for u_{app}*

$$\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}u_{\mathrm{app}}\|_{L^\infty} \lesssim \epsilon, \qquad \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}(u_{\mathrm{app}})_x\|_{L^\infty} \lesssim \epsilon, \tag{4.20}$$

$$\|u_{\mathrm{app}}\|_{H^1} \lesssim \epsilon, \qquad \|Lu_{\mathrm{app}} + 3\sigma t u_{\mathrm{app}}^3\|_{L^2} \lesssim \epsilon(1+\epsilon^2\log t), \tag{4.21}$$

and estimates for the difference $u_{\mathrm{app}} - u_{\mathrm{asymp}}$

$$\|t^{\frac{1+\delta}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}(u_{\mathrm{app}} - u_{\mathrm{asymp}})\|_{L^2} \lesssim \epsilon, \tag{4.22}$$

$$\|t^{\frac{1}{2}+\frac{\delta}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{3}{8}}e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}\left(u_{\mathrm{app}} - u_{\mathrm{asymp}}\right)\|_{L^\infty} \lesssim \epsilon. \tag{4.23}$$

Further, if $T \geq 1$ is a dyadic integer we have the estimate

$$\|u_{\mathrm{app}}\|_{L^4_x L^\infty_T} \lesssim \epsilon T^{-\frac{1}{4}}. \tag{4.24}$$

Proof. We start by considering (4.20). From the estimate (4.18) for Q and the estimate (4.15) for $\mathcal{W}$, we have

$$\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}u_{\mathrm{app}}\|_{L^\infty} \lesssim \|\mathcal{W}\|_{L^\infty} \lesssim \epsilon.$$

For the second part we differentiate to get

$$\partial_x u_{\mathrm{app}} = t^{-\frac{2}{3}}Q_y(t^{-\frac{1}{3}}x;\mathcal{W}) + t^{-\frac{1}{3}}Q_w(t^{-\frac{1}{3}}x;\mathcal{W})\partial_x\mathcal{W},$$

and then estimate similarly to get,

$$\|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}(u_{\mathrm{app}})_x\|_{L^\infty} \lesssim \|\mathcal{W}\|_{L^\infty} + \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{2}+\frac{\delta}{4}}\partial_x\mathcal{W}\|_{L^\infty} \lesssim \epsilon.$$

For the first part of (4.21) we estimate similarly to (4.20) using the L^2 and L^∞ estimates (4.13) and (4.15) for $\mathcal{W}$ and the estimate (4.19) for Q to get

$$\begin{aligned}\|u_{\mathrm{app}}\|_{L^2} &\lesssim \|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\mathcal{W}\|_{L^2} \lesssim \epsilon,\\ \|(u_{\mathrm{app}})_x\|_{L^2} &\lesssim \|t^{-\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\mathcal{W}\|_{L^2} + \|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}+\frac{\delta}{4}}\partial_x\mathcal{W}\|_{L^2} \lesssim \epsilon + \epsilon t^{-\frac{2}{3}}.\end{aligned}$$

For the second part we use that Q satisfies (4.2) to get

$$Lu_{\mathrm{app}} + 3\sigma t u_{\mathrm{app}}^3 = -2t^{\frac{1}{3}}Q_{wy}\partial_x\mathcal{W} - t^{\frac{2}{3}}Q_w\partial_x^2\mathcal{W} - t^{\frac{2}{3}}Q_{ww}(\partial_x\mathcal{W})^2.$$

Using the estimate (4.18) for Q and (4.13) for $\mathcal{W}$, we have a logarithmic loss arising from the first term,

$$\begin{aligned}\|t^{\frac{1}{3}}Q_{wy}\partial_x\mathcal{W}\|_{L^2} &\lesssim \|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\partial_x\mathcal{W}\|_{L^2} + \|\mathcal{W}\|_{L^\infty}^2\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\log\langle t^{-\frac{1}{3}}x\rangle\partial_x\mathcal{W}\|_{L^2}\\ &\lesssim \epsilon(1+\epsilon^2\log t).\end{aligned}$$

For the remaining terms we may use the estimate (4.19) for Q and the estimate (4.13) for $\mathcal{W}$ to get

$$\|t^{\frac{2}{3}}Q_w\partial_x^2\mathcal{W}\|_{L^2} \lesssim \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}+\frac{\delta}{4}}\partial_x^2\mathcal{W}\|_{L^2} \lesssim \epsilon t^{-\frac{\delta}{3}},$$
$$\|t^{\frac{2}{3}}Q_{ww}(\partial_x\mathcal{W})^2\|_{L^2} \lesssim \|\mathcal{W}\|_{L^\infty}\|t^{\frac{1}{3}}\partial_x\mathcal{W}\|_{L^\infty}\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}+\frac{\delta}{4}}\partial_x\mathcal{W}\|_{L^2} \lesssim \epsilon^3 t^{-\frac{1}{6}-\frac{\delta}{3}}.$$

For (4.22) and (4.23) we write

$$u_{\mathrm{app}} - u_{\mathrm{asymp}} = \int_0^1 t^{-\frac{1}{3}}Q_w(t^{-\frac{1}{3}}x; h\mathcal{W} + (1-h)W)(\mathcal{W} - W)\,dh.$$

We may then estimate using the estimate (4.17) for the difference $W - \mathcal{W}$ and the estimate (4.18) for Q.

To prove (4.24) we take a dyadic partition of unity $1 = \sum \varphi_M^2$ where, for $M \in 2^{\mathbb{Z}}$, $\varphi_M(z)$ is supported in the region $\{z \sim M\}$. Taking l^p to correspond to summation in M,

$$\begin{aligned}\|u_{\mathrm{app}}\|_{L_x^4 L_T^\infty} &\lesssim \left(\sum_M \|\varphi_M(t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{2}})|u_{\mathrm{app}}|\|_{L_x^4 L_T^\infty}^2\right)^{\frac{1}{2}}\\ &\lesssim \|T^{-\frac{1}{3}}\langle T^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\|_{l^\infty L_x^4}\|\mathcal{W}\|_{l^2 L_{T,x}^\infty}\\ &\lesssim T^{-\frac{1}{4}}\|W\|_{H^1},\end{aligned}$$

where the last line follows from the Sobolev embedding (1.10) and the Cauchy-Schwarz inequality. □

4.3 Nonlinear estimates

In this section we prove estimates for nonlinear terms appearing in the equation (4.8) for v. We define the operator

$$\Phi h = \int_t^\infty S(t-s)h(s)\,ds,$$

and using the Duhamel formula (1.17), we may write the solution $v = u - u_{\mathrm{app}}$ to (4.8) as

$$v = \Phi\mathbf{N} - \Phi f,$$

where the nonlinear term $\mathbf{N}$ is defined as in (4.9) and the inhomogeneous term f is defined as in (4.7).

In order to complete the proof of Theorem 4.1 we will show that $\Phi\colon Z_\epsilon \to Z_\epsilon$ is a contraction. To do this we also need to estimate Lv. As in Chapter 3, we again work with a modification,

$$\Gamma v = Lv + 3\sigma t((v + u_{\mathrm{app}})^3 - u_{\mathrm{app}}^3),$$

which satisfies the equation

$$(\partial_t + \tfrac{1}{3}\partial_x^3)\Gamma v = \tilde{\mathbf{N}}(v, u_{\text{app}}) - \tilde{f},$$

where

$$\tilde{\mathbf{N}} = 3\sigma(v + u_{\text{app}})^2(\Gamma v)_x + 3\sigma(v^2 + 2vu_{\text{app}})(Lu_{\text{app}} + 3\sigma t u_{\text{app}}^3)_x, \tag{4.25}$$

$$\tilde{f} = Lf + 9\sigma t u_{\text{app}}^2 f. \tag{4.26}$$

Again using the Duhamel formula, we may write

$$\Gamma v = \Phi\tilde{\mathbf{N}} - \Phi\tilde{f}.$$

For the nonlinear terms $\Phi\mathbf{N}, \Phi\tilde{\mathbf{N}}$ we then have the following estimates.

Lemma 4.7. *Let $T \geq 1$ be a dyadic integer and $v_1, v_2 \in Z_\epsilon$ where Z_ϵ is defined as in* (4.10)*. Then, if $\delta = C\epsilon^2$ for $C > 0$ sufficiently large and $\epsilon > 0$ sufficiently small, we have the estimates*

$$\|v_1 - v_2\|_Z + \|Lv_1 - Lv_2\|_{\tilde{Z}} \sim \|v_1 - v_2\|_Z + \|\Gamma v_1 - \Gamma v_2\|_{\tilde{Z}}, \tag{4.27}$$

$$\|\Phi(\mathbf{N}(u_{\text{app}}, v_1) - \mathbf{N}(u_{\text{app}}, v_2))\|_Z \ll \|v_1 - v_2\|_Z \tag{4.28}$$

$$\|\Phi(\tilde{\mathbf{N}}(u_{\text{app}}, v_1) - \tilde{\mathbf{N}}(u_{\text{app}}, v_2))\|_{\tilde{Z}} \ll \|v_1 - v_2\|_Z + \|Lv_1 - Lv_2\|_{\tilde{Z}} \tag{4.29}$$

Proof. It suffices to consider $v_1 = v$, $v_2 = 0$ as the general case follows by applying identical estimates.

We first note that from the dispersive estimates (1.31), for $t \geq 1$ we have

$$|v| \lesssim t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}S(-t)v\|_{L^1}, \qquad |v_x| \lesssim t^{-\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}S(-t)v\|_{L^1}.$$

As $xS(-t)v = S(-t)Lv$ and $S(t)$ is a unitary operator, we may estimate

$$\begin{aligned}
\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}S(-t)v\|_{L^\infty_T L^1_x} &\lesssim T^{\frac{1}{6}}\|S(-t)v\|_{L^\infty_T L^2_x} + T^{-\frac{1}{6}}\|xS(-t)v\|_{L^\infty_T L^2_x}\\
&\lesssim T^{\frac{1}{6}}\|v\|_{L^\infty_T L^2_x} + T^{-\frac{1}{6}}\|Lv\|_{L^\infty_T L^2_x}\\
&\lesssim T^{-\frac{\delta}{3}}(\|v\|_Z + \|Lv\|_{L^2}).
\end{aligned}$$

As a consequence we have the dispersive estimates

$$\begin{aligned}
\|\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}v\|_{L^\infty_{T,x}} &\lesssim T^{-\frac{1+\delta}{3}}(\|v\|_Z + \|Lv\|_{L^2}),\\
\|\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}v_x\|_{L^\infty_{T,x}} &\lesssim T^{-\frac{2+\delta}{3}}(\|v\|_Z + \|Lv\|_{L^2}).
\end{aligned} \tag{4.30}$$

Using the dispersive estimates (4.30) with the L^∞ estimates (4.20) for $u_{\rm app}$, we then have

$$\begin{aligned}\|Lv - \Gamma v\|_{L^\infty_T L^2_x} &\lesssim T\|(v+u_{\rm app})^3 - u_{\rm app}^3\|_{L^\infty_T L^2_x}\\ &\lesssim T(\|v\|_{L^\infty_{T,x}} + \|u_{\rm app}\|_{L^\infty_{T,x}})^2\|v\|_{L^\infty_T L^2_x}\\ &\lesssim \epsilon^3 T^{-\frac{\delta}{3}}\|v\|_Z,\end{aligned}$$

Provided $\epsilon > 0$ is sufficiently small we obtain (4.27).

Applying the local smoothing estimate (1.37) on the interval $[T,\infty)$ we have

$$\|\Phi\mathbf{N}\|_{L^\infty_T L^2_x} \lesssim \||D|^{-1}\mathbf{N}\|_{L^1_x L^2_t([T,\infty)\times\mathbb{R})}, \qquad \|\partial_x \Phi\mathbf{N}\|_{L^\infty_T L^2_x} \lesssim \|\mathbf{N}\|_{L^1_x L^2_t([T,\infty)\times\mathbb{R})},$$
$$\|\Phi\mathbf{N}\|_{L^4_x L^\infty_T} \lesssim \||D|^{-1}\mathbf{N}\|^{\frac{3}{4}}_{L^1_x L^2_t([T,\infty)\times\mathbb{R})}\|\mathbf{N}\|^{\frac{1}{4}}_{L^1_x L^2_t([T,\infty)\times\mathbb{R})},$$

where the last estimate follows from interpolation. As a consequence, we have the estimate

$$\|\Phi\mathbf{N}\|_Z \lesssim \sup_{T_0\geq 1}\left\{T_0^{\frac{1+\delta}{3}}\sum_{T\geq T_0}\||D|^{-1}\mathbf{N}\|_{L^1_x L^2_T} + T_0^{\frac{\delta}{3}}\sum_{T\geq T_0}\|\mathbf{N}\|_{L^1_x L^2_T}\right\}$$

where we assume T, T_0 are dyadic integers.

Using the $L^4_x L^\infty_T$ estimate (4.24) for $u_{\rm app}$, we bound $|D|^{-1}\mathbf{N}$ in $L^1_x L^2_T$ by placing two terms into $L^4_x L^\infty_T$ and estimating the remaining term in $L^\infty_T L^2_x$ as follows:

$$\begin{aligned}\||D|^{-1}\mathbf{N}\|_{L^1_x L^2_T} &\lesssim \|(v+u_{\rm app})^3 - u_{\rm app}^3\|_{L^1_x L^2_T}\\ &\lesssim T^{\frac{1}{2}}(\|v\|_{L^4_x L^\infty_T} + \|u_{\rm app}\|_{L^4_x L^2_T})^2\|v\|_{L^\infty_T L^2_x}\\ &\lesssim T^{-\frac{1}{3}-\frac{\delta}{3}}(T^{-\frac{\delta}{3}}\|v\|_Z + \epsilon)^2\|v\|_Z.\end{aligned}$$

Similarly, using the H^1 estimate (4.21) for $u_{\rm app}$ we have,

$$\begin{aligned}\|\mathbf{N}\|_{L^1_x L^2_T} &\lesssim \|((v+u_{\rm app})^3 - u_{\rm app}^3)_x\|_{L^1_x L^2_T}\\ &\lesssim T^{\frac{1}{2}}\|v\|_{L^4_x L^\infty_T}(\|v\|_{L^4_x L^\infty_T} + \|u_{\rm app}\|_{L^4_x L^\infty_T})\|\partial_x u_{\rm app}\|_{L^\infty_T L^2_x}\\ &\quad + T^{\frac{1}{2}}(\|v\|_{L^4_x L^\infty_T} + \|u_{\rm app}\|_{L^4_x L^\infty_T})^2\|v_x\|_{L^\infty_T L^2_x}\\ &\lesssim T^{-\frac{\delta}{3}}(T^{-\frac{\delta}{3}}\|v\|_Z + \epsilon)^2\|v\|_Z.\end{aligned}$$

Summing over $T \geq T_0$ and using that $\delta^{-1}\epsilon^2 \ll 1$ we have (4.28).

In order to estimate $\tilde{\mathbf{N}}$, we first decompose

$$\tilde{\mathbf{N}} = \partial_x\tilde{\mathbf{N}}_1 - \tilde{\mathbf{N}}_2,$$

where

$$\tilde{\mathbf{N}}_1 = 3\sigma(v+u_{\rm app})^2\Gamma v + 3\sigma(v^2 + 2vu_{\rm app})(Lu_{\rm app} + 3\sigma t u_{\rm app}^3),$$
$$\tilde{\mathbf{N}}_2 = 6\sigma(v+u_{\rm app})(v+u_{\rm app})_x\Gamma v + 3\sigma(v^2 + 2vu_{\rm app})_x(Lu_{\rm app} + 3\sigma t u_{\rm app}^3).$$

We then use the local smoothing estimate (1.37) to control $\Phi(\partial_x \tilde{\mathbf{N}}_1)$ and the Strichartz estimate (1.35) to control $\Phi\tilde{\mathbf{N}}_2$, to get

$$\|\Phi\tilde{\mathbf{N}}\|_{\tilde{Z}} \lesssim \sup_{T_0\geq 1}\left\{\frac{T_0^{\frac{\delta}{3}}}{1+\epsilon^2\log T_0}\left(\sum_{T\geq T_0}\|\tilde{\mathbf{N}}_1\|_{L^1_xL^2_T}+\sum_{T\geq T_0}\|\tilde{\mathbf{N}}_2\|_{L^1_TL^2_x}\right)\right\}.$$

We estimate $\tilde{\mathbf{N}}_1$ as before by placing two terms into $L^4_xL^\infty_T$ and the remaining term into $L^\infty_TL^2_x$ to get

$$\begin{aligned}\|\tilde{\mathbf{N}}_1\|_{L^1_xL^2_T} &\lesssim T^{\frac{1}{2}}(\|v\|_{L^4_xL^\infty_T}+\|u_{\mathrm{app}}\|_{L^4_xL^\infty_T})^2\|\Gamma v\|_{L^\infty_TL^2_x}\\ &\quad + T^{\frac{1}{2}}\|v\|_{L^4_xL^\infty_T}(\|v\|_{L^4_xL^\infty_T}+\|u_{\mathrm{app}}\|_{L^4_xL^\infty_T})\|Lu_{\mathrm{app}}+3\sigma t u_{\mathrm{app}}^3\|_{L^\infty_TL^2_x}\\ &\lesssim T^{-\frac{\delta}{3}}(1+\epsilon^2\log T)(T^{-\frac{\delta}{3}}\|v\|_Z+\epsilon)^2\|\Gamma v\|_{\tilde{Z}}\\ &\quad + \epsilon T^{-\frac{\delta}{3}}(1+\epsilon^2\log T)(T^{-\frac{\delta}{3}}\|v\|_Z+\epsilon)\|v\|_Z.\end{aligned}$$

For $\tilde{\mathbf{N}}_2$ we use the dispersive estimates (4.30) and the L^∞ estimates (4.20) for u_{app} to place two terms in $L^\infty_{T,x}$ and the remaining term in $L^\infty_TL^2_x$,

$$\begin{aligned}\|\tilde{\mathbf{N}}_2\|_{L^1_TL^2_x} &\lesssim T\|(v+u_{\mathrm{app}})(v+u_{\mathrm{app}})_x\|_{L^\infty_{T,x}}\|\Gamma v\|_{L^\infty_TL^2_x}\\ &\quad + T\|(v^2+2vu_{\mathrm{app}})_x\|_{L^\infty_{T,x}}\|Lu_{\mathrm{app}}+3\sigma t u_{\mathrm{app}}^3\|_{L^\infty_TL^2_x}\\ &\lesssim T^{-\frac{\delta}{3}}(\|v\|_Z+\|Lv\|_{\tilde{Z}}+\epsilon)^2\|\Gamma v\|_{\tilde{Z}}+\epsilon T^{-\frac{\delta}{3}}(1+\epsilon^2\log T)\|v\|_Z(\|v\|_Z+\epsilon).\end{aligned}$$

The estimate (4.29) then follows by summing over dyadic $T\geq T_0$.

□

4.4 Estimates for the inhomogeneous term

To complete the proof of Theorem 4.1 we prove estimates for the inhomogeneous terms Φf and $\Phi\tilde{f}$, defined as in (4.7) and (4.26).

Lemma 4.8. *We have the estimates*

$$\|\Phi f\|_Z \lesssim \epsilon, \qquad \|\Phi\tilde{f}\|_{\tilde{Z}} \lesssim \epsilon. \tag{4.31}$$

Proof. We start by observing that from the local smoothing estimate (1.37), the U^p estimates of Lemma 1.6, the embedding of $V^2_{\mathrm{rc}}\subset U^4$ of Proposition 1.2,

$$\|\Phi h\|_{L^\infty_TL^2} \lesssim \|\Phi h\|_{V^2_S([T,2T))} \lesssim \|h\|_{L^1([T,\infty);L^2)}, \tag{4.32}$$

$$\|\Phi h\|_{L^4_xL^\infty_T} \lesssim \||D|^{\frac{1}{4}}\Phi h\|_{V^2_S([T,2T))} \lesssim \||D|^{\frac{1}{4}}h\|_{L^1([T,\infty);L^2)}. \tag{4.33}$$

Estimating $\|\Phi f\|_{L^\infty_T L^2}$. We will show that

$$\|f\|_{L^2} \lesssim \epsilon t^{-\frac{4+\delta}{3}}, \tag{4.34}$$

and then use (4.32) to prove the desired estimate.

We calculate,

$$\begin{aligned} f = t^{-\frac{1}{3}} Q_w \mathcal{W}_t + & t^{-1} R Q_w \partial_x \mathcal{W} + t^{-\frac{2}{3}} Q_{wy} \partial_x^2 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_w \partial_x^3 \mathcal{W} + 6\sigma t^{-1} Q^2 Q_w \partial_x \mathcal{W} \\ & + t^{-\frac{2}{3}} Q_{wwy} (\partial_x \mathcal{W})^2 + t^{-\frac{1}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^2 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_{www} (\partial_x \mathcal{W})^3, \end{aligned}$$

where $R(y) = \frac{2}{3} y - \frac{\zeta(y)}{3\zeta'(y)}$ vanishes for $|y| \geq 1$. We note that we have used that Q satisfies the Painlevé II equation (4.2), that

$$\partial_t \mathcal{W}(t, t^{-\frac{1}{3}} \zeta(t^{-\frac{1}{3}} x)) = \left(t^{-\frac{2}{3}} R(t^{-\frac{1}{3}} x) - t^{-1} x \right) \partial_x (\mathcal{W}(t, t^{-\frac{1}{3}} \zeta(t^{-\frac{1}{3}} x))) + \mathcal{W}_t(t, t^{-\frac{1}{3}} \zeta(t^{-\frac{1}{3}} x)),$$

and that Q_w satisfies the differentiated Painlevé II equation

$$y Q_w - Q_{yyw} + 9\sigma Q^2 Q_w = 0.$$

To prove (4.34) we now estimate each of the terms in f in L^2 using the estimates for $\mathcal{W}$ of Lemma 4.4 and the estimate (4.19) for Q. For the first term we use the L^2 estimate (4.16) to get

$$\|t^{-\frac{1}{3}} Q_w \mathcal{W}_t\|_{L^2} \lesssim \|(t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x \rangle)^{-\frac{1}{4} + \frac{\delta}{4}} \mathcal{W}_t\|_{L^2} \lesssim \epsilon t^{-\frac{4+\delta}{3}}.$$

For the second term, we will use that R is supported in the region $|y| \lesssim 1$ to first apply the Cauchy-Schwarz inequality and then estimate $\partial_x \mathcal{W}$ in L^∞ using (4.15) to get

$$\|t^{-1} R Q_w \partial_x \mathcal{W}\|_{L^2} \lesssim \|t^{-\frac{5}{6}} \langle t^{-\frac{1}{3}} x \rangle^{-\frac{1}{4} + \frac{\delta}{4}} \partial_x \mathcal{W}\|_{L^\infty} \lesssim \epsilon t^{-\frac{4+\delta}{3}}.$$

For the third and fourth terms we use the estimate (4.13) for $\mathcal{W}$ and the estimate (4.19) for Q to get

$$\begin{aligned} \|t^{-\frac{2}{3}} Q_{wy} \partial_x^2 \mathcal{W}\|_{L^2} &\lesssim \|t^{-\frac{2}{3}} \langle t^{-\frac{1}{3}} x \rangle^{\frac{1+\delta}{4}} \partial_x^2 \mathcal{W}\|_{L^2} \lesssim \epsilon t^{-\frac{4+\delta}{3}}, \\ \|t^{-\frac{1}{3}} Q_w \partial_x^3 \mathcal{W}\|_{L^2} &\lesssim \|t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x \rangle^{-\frac{1}{4} + \frac{\delta}{4}} \partial_x^3 \mathcal{W}\|_{L^2} \lesssim \epsilon t^{-\frac{4+\delta}{3}}. \end{aligned}$$

For the fifth term we use the L^∞ estimate (4.15) for $\mathcal{W}$ and the estimate (4.18) for Q to get

$$\|t^{-1} Q^2 Q_w \partial_x \mathcal{W}\|_{L^2} \lesssim \|t^{-1} \langle t^{-\frac{1}{3}} x \rangle^{-\frac{3}{4} + \frac{\delta}{4}}\|_{L^2_x} \|\mathcal{W}\|_{L^\infty}^2 \|\partial_x \mathcal{W}\|_{L^\infty} \lesssim \epsilon^3 t^{-\frac{4+\delta}{3}}.$$

For the remaining terms we estimate one $\mathcal{W}$ term in L^2 using (4.13) and the remaining terms in L^∞ using (4.15) to get

$$\begin{aligned} \|t^{-\frac{2}{3}} Q_{wwy} (\partial_x \mathcal{W})^2\|_{L^2} &\lesssim \|\mathcal{W}\|_{L^\infty} \|t^{-\frac{2}{3}} \langle t^{-\frac{1}{3}} x \rangle^{\frac{1+\delta}{4}} \partial_x \mathcal{W}\|_{L^\infty} \|\partial_x \mathcal{W}\|_{L^2} \lesssim \epsilon^3 t^{-\frac{3}{2} - \frac{\delta}{3}}, \\ \|t^{-\frac{1}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^2 \mathcal{W}\|_{L^2} &\lesssim \|\mathcal{W}\|_{L^\infty} \|\partial_x \mathcal{W}\|_{L^\infty} \|t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x \rangle^{-\frac{1}{4} + \frac{\delta}{4}} \partial_x^2 \mathcal{W}\|_{L^2} \lesssim \epsilon^3 t^{-\frac{3}{2} - \frac{\delta}{3}}, \\ \|t^{-\frac{1}{3}} Q_{www} (\partial_x \mathcal{W})^3\|_{L^2} &\lesssim \|\partial_x \mathcal{W}\|_{L^\infty}^2 \|t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x \rangle^{-\frac{1}{4} + \frac{\delta}{4}} \partial_x \mathcal{W}\|_{L^2} \lesssim \epsilon^3 t^{-\frac{5+\delta}{3}}. \end{aligned}$$

The estimate for Φf then follows from the estimate (4.32).

Estimating $\|\Phi f\|_{L^4_x L^\infty_T}$. We start by calculating,

$$\begin{aligned} f_x = {} & t^{-\frac{2}{3}} Q_{wy} \mathcal{W}_t + t^{-\frac{1}{3}} Q_{ww} \mathcal{W}_t \partial_x \mathcal{W} + t^{-\frac{1}{3}} Q_w \partial_x \mathcal{W}_t + t^{-\frac{4}{3}} R Q_{wy} \partial_x \mathcal{W} \\ & + t^{-1} R Q_{ww} (\partial_x \mathcal{W})^2 + t^{-\frac{4}{3}} R_y Q_w \partial_x \mathcal{W} + t^{-1} R Q_w \partial_x^2 \mathcal{W} + t^{-1} Q_{wyy} \partial_x^2 \mathcal{W} \\ & + \tfrac{4}{3} t^{-\frac{2}{3}} Q_{wy} \partial_x^3 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_w \partial_x^4 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^3 \mathcal{W} \\ & + 6\sigma t^{-1} Q^2 Q_w \partial_x^2 \mathcal{W} + 6\sigma t^{-\frac{4}{3}} Q^2 Q_{wy} \partial_x \mathcal{W} + 6\sigma t^{-1} Q^2 Q_{ww} (\partial_x \mathcal{W})^2 \\ & + 12\sigma t^{-\frac{4}{3}} Q Q_y Q_w \partial_x \mathcal{W} + 12\sigma t^{-1} Q Q_w^2 (\partial_x \mathcal{W})^2 + t^{-1} Q_{wwyy} (\partial_x \mathcal{W})^2 \\ & + \tfrac{4}{3} t^{-\frac{2}{3}} Q_{wwwy} (\partial_x \mathcal{W})^3 + 4 t^{-\frac{2}{3}} Q_{wwy} \partial_x \mathcal{W} \partial_x^2 \mathcal{W} + 2 t^{-\frac{1}{3}} Q_{www} (\partial_x \mathcal{W})^2 \partial_x^2 \mathcal{W} \\ & + t^{-\frac{1}{3}} Q_{ww} (\partial_x^2 \mathcal{W})^2 + t^{-\frac{1}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^3 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_{wwww} (\partial_x \mathcal{W})^4. \end{aligned}$$

Estimating each term using Lemmas 4.4 and 4.5 as for f, we have

$$\|f_x\|_{L^2} \lesssim \epsilon t^{-1-\frac{\delta}{3}}. \tag{4.35}$$

Interpolating between the bounds (4.34), (4.35) we have the estimate

$$\||D|^{\frac{1}{4}} f\|_{L^2} \lesssim \epsilon t^{-\frac{5}{4}-\frac{\delta}{3}},$$

and estimating using (4.33) we have,

$$\|\Phi f\|_{L^4_x L^\infty_T} \lesssim \||D|^{\frac{1}{4}} \Phi f\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{1}{4}-\frac{\delta}{3}}.$$

Estimating $\|\Phi f_x\|_{L^\infty_T L^2_x}$. Using the estimate (4.35) and integrating in time we have

$$\|f_x\|_{L^1([T,\infty);L^2)} \lesssim \delta^{-1} \epsilon T^{-\frac{\delta}{3}},$$

which is not quite sufficient to prove the estimate for Φf_x as $\delta \sim \epsilon^2$.

Instead we decompose

$$f = g + b,$$

into a good part g and a bad part b, where

$$\begin{aligned} g &= t^{-\frac{2}{3}} Q_{wwy} (\partial_x \mathcal{W})^2 + t^{-\frac{1}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^2 \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_{www} (\partial_x \mathcal{W})^3 \\ & \quad + 6\sigma t^{-1} Q^2 Q_w \partial_x \mathcal{W} + \tfrac{1}{3} t^{-\frac{1}{3}} Q_w \partial_x^3 \mathcal{W} + t^{-1} R Q_w \partial_x \mathcal{W}, \\ b &= t^{-\frac{1}{3}} Q_w \mathcal{W}_t + t^{-\frac{2}{3}} Q_{wy} \partial_x^2 \mathcal{W}. \end{aligned}$$

For $0 < \epsilon \ll 1$ we may estimate using Lemmas 4.4 and 4.5 to get

$$\|g_x\|_{L^2} \lesssim \epsilon t^{-\frac{7}{6}-\frac{\delta}{3}}.$$

For the bad part, we first note that we expect $\mathcal{W}$ to behave like the Fourier transform of $S(-t)u_{\text{app}}$ with respect to localization in space in frequency: we expect frequency localization of $S(-t)u_{\text{app}}$ to correspond to spatial localization of $\mathcal{W}$ and conversely spatial localization of $S(-t)u_{\text{app}}$ to correspond to frequency localization of $\mathcal{W}$. We will use this diagonal relationship to show that we have good bounds for $S(-t)b$ in the space $l^2L^1([T,\infty);L^2)$ where the l^2 summation is with respect to dyadic regions in frequency. We may then use the estimate (1.15) to commute the l^2 summation with the V_S^2 norm.

We first note that we have an improved bound for low frequencies:

$$\|P_{\leq T^{\frac{1}{3}}}\partial_x S(-t)b\|_{L^2} \lesssim T^{\frac{1}{3}}\|b\|_{L^2} \lesssim \epsilon T^{\frac{1}{3}}t^{-\frac{4+\delta}{3}},$$

so integrating in time we have the estimate

$$\|P_{\leq T^{\frac{1}{3}}}\partial_x S(-t)b\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

For dyadic $M > T^{\frac{1}{3}}$ and $t \geq T$ we use the elliptic estimate (3.38) from Chapter 3 to show that $P_M b$ must be localized in the set $\{|x| \sim tM^2\}$,

$$\|P_M b\|_{L^2} \lesssim \|\chi_{\{|x|\sim tM^2\}}P_M b\|_{L^2} + t^{-1}M^{-2}\|Lb\|_{L^2} + t^{-1}M^{-3}\|b\|_{L^2}.$$

We then observe that using the localization estimate (1.12), we may commute the spatial and frequency localization of b up to rapidly decaying tails to get

$$\|P_M b\|_{L^2} \lesssim \|P_M(\chi_{\{|x|\sim tM^2\}}b)\|_{L^2} + t^{-1}M^{-2}\|Lb\|_{L^2} + t^{-1}M^{-3}\|b\|_{L^2}.$$

Next we calculate,

$$\begin{aligned}
Lb = &-9\sigma Q^2 Q_w \mathcal{W}_t - t^{\frac{2}{3}}Q_w\partial_x^2\mathcal{W}_t - 2t^{\frac{1}{3}}Q_{wy}\partial_x\mathcal{W}_t - 2t^{\frac{2}{3}}Q_{ww}\partial_x\mathcal{W}\partial_x\mathcal{W}_t\\
&- 2t^{\frac{1}{3}}Q_{wy}\partial_x\mathcal{W}\mathcal{W}_t - t^{\frac{2}{3}}Q_{www}(\partial_x\mathcal{W})^2\mathcal{W}_t - t^{\frac{2}{3}}Q_{ww}\partial_x^2\mathcal{W}\mathcal{W}_t - t^{\frac{1}{3}}Q_w\partial_x^2\mathcal{W}\\
&- 9\sigma t^{-\frac{1}{3}}Q^2Q_{wy}\partial_x^2\mathcal{W} - 18\sigma t^{-\frac{1}{3}}QQ_wQ_y\partial_x^2\mathcal{W} - 2Q_{wwyy}\partial_x\mathcal{W}\partial_x^2\mathcal{W} - 2Q_{wyy}\partial_x^3\mathcal{W}\\
&- t^{\frac{1}{3}}Q_{wwwy}(\partial_x\mathcal{W})^2\partial_x^2\mathcal{W} - t^{\frac{1}{3}}Q_{wwy}(\partial_x^2\mathcal{W})^2 - 2t^{\frac{1}{3}}Q_{wwy}\partial_x\mathcal{W}\partial_x^3\mathcal{W} - t^{\frac{1}{3}}Q_{wy}\partial_x^4\mathcal{W},
\end{aligned}$$

and estimating using Lemma 4.4 and (4.19), we have

$$\|Lb\|_{L^2} \lesssim \epsilon t^{-1-\frac{\delta}{3}}.$$

As a consequence, we have the estimate

$$\|P_M\partial_x S(-t)b\|_{L^1([T,\infty);L^2)} \lesssim M\|P_M S(-t)(\chi_{\{|x|\sim tM^2\}}b)\|_{L^1([T,\infty);L^2)} + \epsilon T^{-1-\frac{\delta}{3}}M^{-1},$$

where the second term may be summed over dyadic $M > T^{\frac{1}{3}}$.

Estimating as in Lemma 4.4 using (4.19), we have the estimate

$$\|\chi_{\{|x|\sim tM^2\}}b\|_{L^2} \lesssim t^{-\frac{3}{2}-\frac{\delta}{3}}M^{-\frac{1}{2}}\|\chi_{\{|z|\sim M\}}\langle D\rangle^{1+\delta}W\|_{L^2} + \epsilon t^{-2-\frac{\delta}{3}}M^{-\frac{3}{2}}. \tag{4.36}$$

Next we calculate

$$\begin{aligned}\partial_x^2 b = {} & t^{-1}Q_{wyy}\mathcal{W}_t + t^{-\frac{1}{3}}Q_w\partial_x^2\mathcal{W}_t + 2t^{-\frac{2}{3}}Q_{wy}\partial_x\mathcal{W}_t + 2t^{-\frac{1}{3}}Q_{ww}\partial_x\mathcal{W}\partial_x\mathcal{W}_t \\ & + 2t^{-\frac{2}{3}}Q_{wy}\partial_x\mathcal{W}\mathcal{W}_t + t^{-\frac{1}{3}}Q_{www}(\partial_x\mathcal{W})^2\mathcal{W}_t + t^{-\frac{1}{3}}Q_{ww}\partial_x^2\mathcal{W}\mathcal{W}_t \\ & + t^{-\frac{4}{3}}Q_{wyyy}\partial_x^2\mathcal{W} + 2t^{-1}Q_{wwyy}\partial_x\mathcal{W}\partial_x^2\mathcal{W} + 2t^{-1}Q_{wyy}\partial_x^3\mathcal{W} \\ & + t^{-\frac{2}{3}}Q_{wwwy}(\partial_x\mathcal{W})^2\partial_x^2\mathcal{W} + t^{-\frac{2}{3}}Q_{wwy}(\partial_x^2\mathcal{W})^2 + 2t^{-\frac{2}{3}}Q_{wwy}\partial_x\mathcal{W}\partial_x^3\mathcal{W} \\ & + t^{-\frac{2}{3}}Q_{wy}\partial_x^4\mathcal{W},\end{aligned}$$

and estimate

$$\|\partial_x^2(\chi_{\{|x|\sim tM^2\}}b)\|_{L^2} \lesssim t^{-\frac{\delta}{3}}M\|\chi_{\{|z|\sim M\}}\langle D\rangle^\delta W\|_{L^2} + \epsilon t^{-\frac{1}{2}-\frac{\delta}{3}}M^{-\frac{1}{2}}, \tag{4.37}$$

where we have use the fact that $\mathcal{W}$ is localized at frequencies $\leq t$.

Using (4.36), we may estimate

$$\int_{\max\{M,T\}}^\infty M\|P_M(\chi_{\{|x|\sim tM^2\}}b)\|_{L^2}\,dt \lesssim T^{-\frac{\delta}{3}}\|\chi_{\{|z|\sim M\}}\langle D\rangle^{1+\delta}W\|_{L^2} + \epsilon T^{-\frac{1}{2}-\frac{\delta}{3}}M^{-\frac{1}{2}}.$$

If $M \geq T$ we may use the localization and (4.37) to estimate

$$\begin{aligned}\int_T^M M\|P_M(\chi_{\{|x|\sim tM^2\}}b)\|_{L^2}\,dt &\lesssim \int_T^M M^{-1}\|\partial_x^2(\chi_{\{|x|\sim tM^2\}}b)\|_{L^2}\,dt \\ &\lesssim T^{-\frac{\delta}{3}}M\|\chi_{\{|z|\sim M\}}\langle D\rangle^\delta W\|_{L^2} + \epsilon T^{-\frac{1}{2}-\frac{\delta}{3}}M^{-\frac{1}{2}}.\end{aligned}$$

We may then sum in M to get

$$\|b_x\|_{l^2L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

From the estimates (1.15) and (4.32) we may then estimate Φb_x using the embeddings,

$$\|\Phi b_x\|_{L^\infty_T L^2} \lesssim \|\Phi b_x\|_{V^2_S([T,2T))} \lesssim \|\Phi b_x\|_{l^2V^2_S([T,2T))} \lesssim \|b_x\|_{l^2L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

Estimating $\Phi\tilde{f}$. Using the estimate (4.20) for u_{app} and (4.34) for f, we have

$$\|tu_{\text{app}}^2 f\|_{L^2} \lesssim \epsilon^3 t^{-1-\frac{\delta}{3}},$$

and hence

$$\|tu_{\text{app}}^2 f\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

To estimate Lf, we again decompose

$$f = g + b$$

into a good part g and a bad part b defined by

$$g = t^{-\frac{2}{3}}Q_{wwy}(\partial_x \mathcal{W})^2 + t^{-\frac{1}{3}}Q_{ww}\partial_x\mathcal{W}\partial_x^2\mathcal{W} + \tfrac{1}{3}t^{-\frac{1}{3}}Q_{www}(\partial_x\mathcal{W})^3 + 6\sigma t^{-1}Q^2Q_w\partial_x\mathcal{W},$$
$$b = t^{-\frac{1}{3}}Q_w\mathcal{W}_t + t^{-1}RQ_w\partial_x\mathcal{W} + t^{-\frac{2}{3}}Q_{wy}\partial_x^2\mathcal{W} + \tfrac{1}{3}t^{-\frac{1}{3}}Q_w\partial_x^3\mathcal{W}.$$

We may calculate Lg and estimate as before using Lemma 4.4 and (4.19) to get

$$\|Lg\|_{L^2} \lesssim \epsilon^3 t^{-1-\frac{\delta}{3}}.$$

Again using that $\delta \sim \epsilon^2$, we may integrate in time to get

$$\|Lg\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

For the bad part b we will use the diagonal nature of the map from $\mathcal{W} \mapsto u_{\text{app}}$ to relate spatial localization of u_{app} to frequency localization of $\mathcal{W}$. We first note that for $|x| \leq T^{\frac{1}{3}}$ we have the improved estimate

$$\|\chi_{\{|x|\leq T^{\frac{1}{3}}\}}S(-t)Lb\|_{L^2} \lesssim \|\chi_{\{|x|\leq T^{\frac{1}{3}}\}}xS(-t)Lb\|_{L^2} \lesssim T^{\frac{1}{3}}\|b\|_{L^2} \lesssim \epsilon T^{\frac{1}{3}}t^{-\frac{4+\delta}{3}}.$$

Integrating we have the estimate

$$\|\chi_{\{|x|\lesssim T^{\frac{1}{3}}\}}S(-t)Lb\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

Next we use the frequency localization to show that for $j = 0, 1, 2$,

$$\|L^j b\|_{L^2} \lesssim \epsilon t^{-\frac{3}{2}+\frac{j}{3}-\frac{\delta}{3}}\|\langle t^{-\frac{1}{3}}D\rangle^{-1}|D|^{\frac{3}{2}+\delta}W\|_{L^2}.$$

For dyadic $M > T^{\frac{1}{3}}$, we then have

$$\|\chi_{\{|x|\sim M\}}S(-t)Lb\|_{L^2} \lesssim t^{-\delta}\min\{M^{\frac{1}{2}}t^{-\frac{4}{3}}, M^{-\frac{1}{2}}t^{-1}\}\|\langle t^{-\frac{1}{3}}D\rangle^{-1}|D|^{\frac{3}{2}+\delta}W\|_{L^2}.$$

Applying the Cauchy-Schwarz inequality on the intervals $[T, M^3]$ and $[M^3, \infty)$ respectively, we have the estimate

$$\|\chi_{\{|x|\sim M\}}xS(-t)b\|^2_{L^1([T,\infty);L^2)} \lesssim T^{-\frac{2\delta}{3}}\int_T^\infty \min\{M^{\frac{1}{2}}t^{-\frac{3}{2}}, M^{-\frac{1}{2}}t^{-\frac{7}{6}}\}\|\langle t^{-\frac{1}{3}}D\rangle^{-1}|D|^{\frac{3}{2}+\delta}W\|^2_{L^2}\,dt.$$

Summing over dyadic $M > T^{\frac{1}{3}}$ we then have

$$\|xS(-t)b\|^2_{l^2L^1([T,\infty);L^2)} \lesssim T^{-\frac{2\delta}{3}}\int_T^\infty t^{-\frac{4}{3}}\|\langle t^{-\frac{1}{3}}D\rangle^{-1}|D|^{\frac{3}{2}+\delta}W\|^2_{L^2}\,dt.$$

Dyadically decomposing W in frequency, we have

$$\begin{aligned}\int_T^\infty t^{-\frac{4}{3}}\|\langle t^{-\frac{1}{3}}D\rangle^{-1}|D|^{\frac{3}{2}+\delta}W\|^2_{L^2}\,dt &\lesssim \int_T^\infty \sum_N \min\{N^{3+2\delta}t^{-\frac{4}{3}}, N^{1+2\delta}t^{-\frac{2}{3}}\}\|W_N\|^2_{L^2}\,dt \\ &\lesssim \sum_{N\leq t} N^{2(1+\delta)}\|W_N\|^2_{L^2} \\ &\lesssim \epsilon.\end{aligned}$$

Commuting the l^2-summation with the V^2 norm using (1.15), we have

$$\|\Phi Lb\|_{V_S^2([T,2T))} \lesssim \|\Phi Lb\|_{l^2 V_S^2([T,2T))} \lesssim \|Lb\|_{l^2 L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}},$$

which completes the proof of (4.31). □

4.A Properties of the Painlevé II equation

In this appendix we discuss properties of solutions to the Painlevé II equation

$$Q_{yy} - yQ = 3\sigma Q^3. \tag{4.38}$$

We look to prove the existence of a 1-parameter family of classical solutions $Q(y,w)$ with the asymptotic behavior

$$Q(y,w) \sim q_\sigma(w)\,\mathrm{Ai}(y) + O(|y|^{-\frac{1}{4}} e^{-\frac{4}{3}y^{\frac{3}{2}}}), \qquad y \to +\infty,$$

where

$$q_\sigma(w) = \operatorname{sgn} w \left(\tfrac{2\sigma}{3}\left(1 - e^{-\frac{3\sigma}{2}w^2}\right)\right)^{\frac{1}{2}}.$$

We note that q_σ is smooth in w and satisfies the estimate

$$\left|\frac{d^k q_\sigma}{dw^k}\right| \lesssim \begin{cases} |w|, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

Lemma 4.5 will arise as a consequence of the estimates obtained in this appendix.

The inhomogeneous Airy equation. We start by considering the inhomogeneous Airy equation

$$Q_{yy} - yQ = F. \tag{4.39}$$

We may write a solution to (4.39) using the variation of parameters formula as

$$Q(y) = Q(y_0) - \frac{1}{\pi}\int_{y_0}^{y} K(y,z)F(z)\,dz, \tag{4.40}$$

where the kernel

$$K(y,z) = \mathrm{Ai}(y)\,\mathrm{Bi}(z) - \mathrm{Ai}(z)\,\mathrm{Bi}(y). \tag{4.41}$$

From Lemma 1.3, we have the following bounds for the kernel K:

Lemma 4.9. *For K defined as in* (4.41)*, we have the estimates*

$$|K(y,z)| \lesssim \langle y\rangle^{-\frac{1}{4}}\langle z\rangle^{-\frac{1}{4}}\left(e^{\frac{2}{3}(z_+^{\frac{3}{2}}-y_+^{\frac{3}{2}})}+e^{\frac{2}{3}(y_+^{\frac{3}{2}}-z_+^{\frac{3}{2}})}\right), \tag{4.42}$$

$$|K_y(y,z)| \lesssim \langle y\rangle^{\frac{1}{4}}\langle z\rangle^{-\frac{1}{4}}\left(e^{\frac{2}{3}(z_+^{\frac{3}{2}}-y_+^{\frac{3}{2}})}+e^{\frac{2}{3}(y_+^{\frac{3}{2}}-z_+^{\frac{3}{2}})}\right). \tag{4.43}$$

For $y \ll -1$ we will use slightly different linear solutions. We define the complex valued function

$$\mathfrak{Ai}(y) = \frac{\sqrt{\pi}}{2}(\mathrm{Ai}(y) + i\,\mathrm{Bi}(y)). \tag{4.44}$$

From Lemma 1.3, we see the leading term as $y \to -\infty$ is given by

$$\mathfrak{Ai}_0(y) = |y|^{-\frac{1}{4}}e^{-\frac{2}{3}i|y|^{\frac{3}{2}}+i\frac{\pi}{4}}. \tag{4.45}$$

We note that $\mathfrak{Ai}, \bar{\mathfrak{Ai}}$ are a pair of linearly independent solutions to (1.19) and have Wronskian

$$\mathfrak{Ai}(y)\bar{\mathfrak{Ai}}'(y) - \mathfrak{Ai}'(y)\bar{\mathfrak{Ai}}(y) = \frac{1}{2i}$$

We then have the variation of parameters formula

$$Q(y) = Q(y_0) - \mathrm{Im}\int_{y_0}^{y} L(y,z)F(z)\,dz, \tag{4.46}$$

where the Kernel is given by

$$L(y,z) = \mathfrak{Ai}(y)\bar{\mathfrak{Ai}}(z). \tag{4.47}$$

We also define the leading order term by

$$L_0(y,z) = \mathfrak{Ai}_0(y)\bar{\mathfrak{Ai}}_0(z). \tag{4.48}$$

As in Lemma 4.9, we may use Lemma 1.3 to produce the following bounds for the kernel L.

Lemma 4.10. *Let L, L_0 be defined as in* (4.47), (4.48)*. For $y, z \ll -1$, we have the estimates*

$$|L(y,z)| \lesssim \langle y\rangle^{-\frac{1}{4}}\langle z\rangle^{-\frac{1}{4}}, \qquad |L_y(y,z)| \lesssim \langle y\rangle^{\frac{1}{4}}\langle z\rangle^{-\frac{1}{4}}. \tag{4.49}$$

Solution near $+\infty$. Let $M > 0$ be fixed. We now prove the existence of solutions on the interval $[-M, \infty)$ using the contraction principle. We define the Banach space $C_{\to 0}([-M, \infty))$ to consist of continuous functions on $[-M, \infty)$ that vanishing at $+\infty$ equipped with the sup norm. We then define the weighted space $X \subset C_{\to 0}([-M, \infty))$ with norm

$$\|Q\|_X = \|\langle y\rangle^{\frac{1}{4}} e^{\frac{2}{3} y_+^{\frac{3}{2}}} Q\|_{\sup}.$$

We note that the restriction of $q_\sigma(w)\,\mathrm{Ai}(y)$ to $[-M, \infty)$ satisfies the bound

$$\|q_\sigma(w)\,\mathrm{Ai}(y)\|_X \lesssim |q_\sigma(w)| \lesssim |w|.$$

For the kernel K defined as in (4.41), we define the operator

$$\Phi(F)(y) = \frac{1}{\pi}\int_y^\infty K(y, z) F(z)\, dz. \tag{4.50}$$

We then have the following estimates for Φ:

Lemma 4.11. *If $Q_j \in X$, we have the estimates*

$$\|\langle y\rangle^{\frac{7}{4}} e^{\frac{4}{3} y_+^{\frac{3}{2}}} \Phi(Q_1 Q_2 Q_3)\|_{\sup} \lesssim \|Q_1\|_X \|Q_2\|_X \|Q_3\|_X,$$
$$\|\langle y\rangle^{\frac{5}{4}} e^{\frac{4}{3} y_+^{\frac{3}{2}}} \partial_y \Phi(Q_1 Q_2 Q_3)\|_{\sup} \lesssim \|Q_1\|_X \|Q_2\|_X \|Q_3\|_X.$$

Proof. From Lemma 4.9, for all $y \in [-M, \infty)$ and $z \in [-M, \infty)$

$$\begin{aligned} &|\langle y\rangle^{\frac{7}{4}} e^{\frac{4}{3} y_+^{\frac{3}{2}}} K(y, z) Q_1(z) Q_2(z) Q_3(z)| + |\langle y\rangle^{\frac{5}{4}} e^{\frac{4}{3} y_+^{\frac{3}{2}}} K_y(y, z) Q_1(z) Q_2(z) Q_3(z)| \\ &\lesssim \langle y\rangle^{\frac{3}{2}} \langle z\rangle^{-1} \left(e^{\frac{4}{3}(y_+^{\frac{3}{2}} - z_+^{\frac{3}{2}})} + e^{\frac{8}{3}(y_+^{\frac{3}{2}} - z_+^{\frac{3}{2}})} \right) \|Q_1\|_X \|Q_2\|_X \|Q_3\|_X. \end{aligned}$$

We observe that $z \mapsto K(y, z)$ is integrable on $[-M, \infty)$ and by making a suitable change of variables, for $y > 1$ and $k \geq 1$,

$$\left| \int_y^\infty \langle z\rangle^{-1} e^{-\frac{k}{3} z^{\frac{3}{2}}}\, dz \right| \lesssim y^{-\frac{3}{2}} e^{-\frac{k}{3} y^{\frac{3}{2}}}.$$

□

As a consequence, for $|w| \leq \epsilon \ll 1$ we may use the contraction principle in the space $X_\epsilon = \{Q \in X : \|Q\|_X \leq C\epsilon\}$ to prove that there exists a unique solution $Q \in X_\epsilon$ to the integral equation

$$Q(y) = q_\sigma(w)\,\mathrm{Ai}(y) + 3\sigma \Phi(Q^3)(y).$$

Such a solution is then clearly a smooth solution to (4.38) on $[-M,\infty)$. Further, using the differentiated bounds for the first derivative and the equation for higher order derivatives, we have $\|\langle y\rangle^{-\frac{k}{2}}\partial_y^k Q\|_X \lesssim \epsilon$.

Next we consider the differentiated Painlevé II equation

$$Q_{wyy} - yQ_w = 9\sigma Q^2 Q_w. \tag{4.51}$$

We note that this is linear in Q_w and recall that $|q_\sigma'(w)| \lesssim 1$. Using the established bounds for Q, we may now solve this by applying a contraction mapping theorem in a ball $X_1 \subset X$ to the integral equation

$$Q(y) = (q_\sigma)_w(w)\,\mathrm{Ai}(y) + 9\sigma\Phi(Q^2 Q_w),$$

to get a solution on $[-M,\infty)$ satisfying the estimate $\|Q\|_X \lesssim 1$.

Repeating the argument, we can show that Q is smooth in w on $[-M,\infty)$ and have the estimates

$$\|\langle y\rangle^{-\frac{k}{2}}\partial_y^k\partial_w^m Q\|_X \lesssim \begin{cases} \epsilon, & m \text{ even}, \\ 1, & m \text{ odd}. \end{cases}$$

Solution near $-\infty$. We now turn to establishing bounds near $-\infty$. Our approach is similar to [130]. Let $M > 0$ be fixed and let Q be the solution on $(-2M,\infty)$.

We define the coefficient of $\mathfrak{Ai}$ appearing in Q by

$$P(y) = \beta + 3\sigma\int_y^{-M} (Q(z))^3 \bar{\mathfrak{Ai}}(z)\,dz,$$

where $\beta \in \mathbb{C}$ is chosen such that

$$Q(-M) = \mathrm{Im}\left(\beta\mathfrak{Ai}(-M)\right), \qquad Q_y(-M) = \mathrm{Im}\left(\beta\mathfrak{Ai}'(-M)\right).$$

Using variation of parameters (4.46), we may then write

$$Q(y) = \mathrm{Im}\left(P(y)\mathfrak{Ai}(y)\right).$$

From the equation (4.38) we obtain an equation for P,

$$P_y = \frac{3\sigma}{8i}\left(-3|\mathfrak{Ai}|^4|P|^2P + |\mathfrak{Ai}|^2\mathfrak{Ai}^2P^3 + 3|\mathfrak{Ai}|^2\bar{\mathfrak{Ai}}^2|P|^2\bar{P} - \bar{\mathfrak{Ai}}^4\bar{P}^3\right).$$

We observe that only the first term is non-oscillatory and that it may be removed by means of a gauge transform similar to the one used in Chapter 3. We define the gauge

$$\Phi(y) = \frac{9\sigma}{8}\int_y^{-M} |\mathfrak{Ai}(z)|^4|P(z)|^2\,dz,$$

and take $R(y) = P(y)e^{-i\Phi(y)}$. We observe that $R(-M) = P(-M) = \beta$, $|R| = |P|$ and

$$R_y = \frac{3\sigma}{8i}\left(|\mathfrak{Ai}|^2\mathfrak{Ai}^2R^3e^{2i\Phi} + 3|\mathfrak{Ai}|^2\bar{\mathfrak{Ai}}^2|R|^2\bar{R}e^{-2i\Phi} - \bar{\mathfrak{Ai}}^4\bar{R}^3e^{-4i\Phi}\right). \tag{4.52}$$

We now proceed by means of a bootstrap argument. Let $M_0 > 0$ be a large fixed constant and suppose that R satisfies

$$|R(y)| \leq M_0\epsilon. \tag{4.53}$$

As a consequence of (4.52) and Lemma 1.3, we have

$$|R'(y)| \lesssim (M_0\epsilon)^3|y|^{-1}, \tag{4.54}$$

which is clearly not integrable. However, using Lemma 1.3 we may replace $\mathfrak{Ai}$ by $\mathfrak{Ai}_0$ up to integrable errors to get

$$R_y = \frac{3\sigma}{8i}\left(|y|^{-1}e^{2i\phi}R^3e^{2i\Phi} + 3|y|^{-1}e^{-2i\phi}|R|^2\bar{R}e^{-2i\Phi} - |y|^{-1}e^{-4i\phi}\bar{R}^3e^{-4i\Phi}\right) + O((M_0\epsilon)^3|y|^{-\frac{5}{2}}),$$

where $\phi = -\frac{2}{3}|y|^{\frac{3}{2}} + \frac{\pi}{4}$. We observe that for $y < 0$,

$$\frac{1}{i|y|^{\frac{1}{2}}}\frac{d}{dy}\left(e^{i\phi}\right) = e^{i\phi},$$

so using the estimates (4.53), (4.54) and assuming that $M_0\epsilon \ll 1$, we have

$$\begin{aligned}\frac{d}{dy}\left(R + \frac{3\sigma}{8}\left(\frac{1}{2}|y|^{-\frac{3}{2}}e^{2i\phi}R^3e^{2i\Phi} - \frac{3}{2}|y|^{-\frac{3}{2}}e^{-2i\phi}|R|^2\bar{R}e^{-2i\Phi} + \frac{1}{4}|y|^{-\frac{3}{2}}e^{-4i\phi}\bar{R}^3e^{-4i\Phi}\right)\right)\\ = O((M_0\epsilon)^3|y|^{-\frac{5}{2}}).\end{aligned} \tag{4.55}$$

Integrating in y we see that R is continuous and satisfies

$$R(y) = \beta + O((M_0\epsilon)^3|y|^{-\frac{3}{2}}),$$

so for $M_0 > 0$ sufficiently large and $\epsilon = \epsilon(M_0) > 0$ sufficiently small,

$$|R(y)| \leq \tfrac{1}{2}M_0\epsilon, \tag{4.56}$$

which closes the bootstrap estimate.

Using (4.56) and (4.54), we may extend the solution Q to $\mathbb{R}$ and have the estimates

$$|Q(y)| \lesssim \epsilon\langle y\rangle^{-\frac{1}{4}}e^{-\frac{2}{3}y_+^{\frac{3}{2}}}, \qquad |Q_y(y)| \lesssim \epsilon\langle y\rangle^{\frac{1}{4}}e^{-\frac{2}{3}y_+^{\frac{3}{2}}}. \tag{4.57}$$

Solution near $-\infty$ for the differentiated Painlevé II equation. We now consider the differentiated Painlevé II equation (4.51). Let $|w| \leq \epsilon$ be fixed and for $M \gg 1$ solve to find Q_w on $(-2M, \infty)$. We will again proceed via a bootstrap argument but the work we have already done makes the argument rather simpler. Let R be defined as before and note that R_w is well defined near $-M$ and continuous in y. We make the bootstrap assumption that

$$|R_w(y)| \leq M_1. \tag{4.58}$$

As $|\mathfrak{Ai}|^4 \lesssim |y|^{-1}$, we have

$$|\Phi_w| \lesssim M_1 \epsilon^2 \log |y|. \tag{4.59}$$

Applying an identical analysis to the ODE satisfied by R_w, we may integrate in y to get

$$R_w(y) = R_w(-M) + O\left(M_1 \epsilon^2 |y|^{-\frac{3}{2}} (1 + \epsilon^2 \log |y|) \right).$$

Choosing $M_1 > 0$ sufficiently large and $\epsilon = \epsilon(M_1) > 0$ sufficiently small, we then have

$$|R_w(y)| \leq \tfrac{1}{2} M_1,$$

which closes the bootstrap. Using (4.59), we obtain the estimates

$$|Q_w(y)| \lesssim \langle y \rangle^{-\frac{1}{4}} (1 + \epsilon^2 \log\langle y \rangle) e^{-\frac{2}{3} y_+^{\frac{3}{2}}}, \qquad |Q_{wy}(y)| \lesssim \langle y \rangle^{\frac{1}{4}} (1 + \epsilon^2 \log\langle y \rangle) e^{-\frac{2}{3} y_+^{\frac{3}{2}}} \tag{4.60}$$

The argument for higher-order derivatives in w is identical.

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