A Semantic Hierarchy for Intuitionistic Logic

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Abstract

Brouwer's views on the foundations of mathematics have inspired the study of intuitionistic logic, including the study of the intuitionistic propositional calculus and its extensions. The theory of these systems has become an independent branch of logic with connections to lattice theory, topology, modal logic, and other areas. This paper aims to present a modern account of semantics for intuitionistic propositional systems. The guiding idea is that of a hierarchy of semantics, organized by increasing generality: from the least general Kripke semantics on through Beth semantics, topological semantics, Dragalin semantics, and finally to the most general algebraic semantics. While the Kripke, topological, and algebraic semantics have been extensively studied, the Beth and Dragalin semantics have received less attention. We bring Beth and Dragalin semantics to the fore, relating them to the concept of a nucleus from pointfree topology, which provides a unifying perspective on the semantic hierarchy.

Keywords: intuitionistic logic, intermediate logics, Kripke semantics, Beth semantics, topological semantics, algebraic semantics, Heyting algebra, locale, nucleus

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Contents

1 Introduction

Luitzen Egbertus Jan Brouwer (1881–1966) was one of the great mathematicians of the 20th century. The present volume is dedicated to his numerous contributions and the subsequent developments they have inspired. Not only did Brouwer prove fundamental results in several areas of mathematics, but also he advocated a revolutionary view of the nature of mathematics itself—Brouwer's *intuitionism*. The intuitionistic philosophy of mathematics demands a radical revision of the accepted canons of mathematical reasoning, as formalized by classical logic. Consequently, intuitionism has led to a significant strand in the study of foundations of mathematics and formal logic over the last century. Other papers in this volume analyze aspects of intuitionistic mathematics. The focus of the present paper is on the analysis of intuitionistic logic—in particular, the intuitionistic logic of propositions and its extensions. Intuitionistic systems have proved to be a rich source for both proof-theoretic and semantic studies. In this paper, we aim to present a modern account of a hierarchy of different semantics for intuitionistic propositional systems.[1]

[1]For earlier surveys covering semantics for intuitionistic systems, see, e.g., Ruitenburg 1991, van Dalen 2001, 2002, Bezhanishvili and de Jongh 2006, Moschovakis 2009, and for treatments in books, see, e.g., Rasiowa and Sikorski 1963, Gabbay 1981, Goldblatt 1984, Dragalin 1988, Troelstra and van Dalen 1988a,b, Chagrov and Zakharyaschev 1997, Dummett 2000,

1.1 Background

In 1927, the Dutch Mathematical Association published a prize question calling for the construction of a formal calculus to codify patterns of reasoning used in Brouwer's intuitionistic mathematics (see Troelstra 1990). In 1928, the prize was awarded to Brouwer's former student, Arend Heyting. In the first part of his prize-winning paper, Heyting [1930] proposed a formal calculus governing the following "four basic concepts":

$$\begin{array}{rl} p \to q & \text{“from } p \text{ follows } q\text{”} \\ p \wedge q & \text{“}p \text{ and } q\text{”} \\ p \vee q & \text{“}p \text{ or } q\text{”} \\ \neg p & \text{“not } p\text{”.} \end{array}$$

These four concepts were taken by Whitehead and Russell to be of "fundamental importance" in their treatment of classical propositional logic in *Principia Mathematica* [Whitehead and Russell, 1910, p. 6]. According to Heyting (see van Atten 2014), he arrived at his calculus by going through the axioms and theorems of *Principia Mathematica* and including only the intuitionistically acceptable ones in a set of independent axioms, shown in Figure 1.1.

$$\begin{array}{ll} ((p \to q) \wedge (q \to r)) \to (p \to r) & \\ q \to (p \to q) & \\ (p \wedge (p \to q)) \to q & \\ & \\ p \to (p \wedge p) & p \to (p \vee q) \\ (p \wedge q) \to (q \wedge p) & (p \vee q) \to (q \vee p) \\ (p \to q) \to ((p \wedge r) \to (q \wedge r)) & ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r) \\ & \\ \neg p \to (p \to q) & \\ ((p \to q) \wedge (p \to \neg q)) \to \neg p & \end{array}$$

Figure 1.1: Heyting's axioms.

The set of theorems of Heyting's calculus is the smallest set of formulas that contains Heyting's axioms and is closed under the rules of *modus ponens* (if φ and $\varphi \to \psi$ are theorems, so is ψ) and uniform substitution (if φ is a theorem, then so is any ψ obtained by uniformly substituting formulas for the propositional letters in φ). This logic has come to be called the *intuitionistic propositional calculus* (IPC). Heyting proved that the axioms in Figure 1.1 are independent—none is derivable from the others—and stated that, in contrast to classical logic, in intuitionistic logic none of the connectives $\to$, $\wedge$, $\vee$, or $\neg$ is definable in terms of the others (as was proved in Wajsberg 1938, McKinsey 1939). Heyting also extended IPC to the *intuitionistic predicate calculus* (IQC) with the quantifiers $\forall$ ("for all") and $\exists$ ("there exists"), which forms the logical basis for formalized intuitionistic arithmetic, analysis, and set theory (see Troelstra and van Dalen 1988a,b).

An alternative tradition to the formalization of intuitionistic logic, starting with Kolmogorov [1925], leads to a weaker logical calculus, now known as *minimal calculus* [Johansson, 1937].[2] The distinguishing feature

Humberstone 2011.

[2]Kolmogorov's [1925] propositional calculus is in fact equivalent to the implication-negation fragment of minimal calculus (see Plisko 1988).

of the minimal calculus is that the formula $\neg p \to (p \to q)$, corresponding to the principle *ex falso quodlibet*, is not a theorem. Though the historical debate over the intuitionistic acceptability of *ex falso quodlibet* is interesting, here we focus only on Heyting's formalization of intuitionistic propositional logic as IPC.

Modern treatments of intuitionistic logic often take as the basic connectives only $\to$, $\wedge$, $\vee$, and the propositional constant $\bot$. The connective $\neg$ is then defined by $\neg\varphi := \varphi \to \bot$. A modern axiomatization of IPC in this language is given in Figure 1.2 [Chagrov and Zakharyaschev, 1997].

$$q \to (p \to q)$$
$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

$(p \wedge q) \to p$	$p \to (p \vee q)$
$(p \wedge q) \to q$	$q \to (p \vee q)$
$p \to (q \to (p \wedge q))$	$(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

$$\bot \to p$$

Figure 1.2: A modern axiomatization of IPC.

From IPC one obtains a system equivalent to the *classical propositional calculus* (CPC) used in *Principia* by adding any of the following axioms:

- $p \vee \neg p$ (excluded middle);
- $\neg\neg p \to p$ (double negation elimination);
- $((p \to q) \to p) \to p$ (Peirce's law).[3]

A close connection between IPC and CPC was discovered by Glivenko [1929], who was in communication with Heyting.[4] Using a set of axioms that he considered intuitionistically admissible, Glivenko showed that a formula φ is classically provable iff $\neg\neg\varphi$ is intuitionistically provable. In modern terminology, φ is a theorem of CPC iff $\neg\neg\varphi$ is a theorem of IPC. As a corollary, Glivenko showed that $\neg\varphi$ is a theorem of CPC iff $\neg\varphi$ is a theorem of IPC. Another corollary, observed by Gödel [1933a], is that if a formula φ contains only the connectives $\wedge$ and $\neg$, then φ is a theorem of CPC iff φ is a theorem of IPC (also see Kleene 1952, § 81).[5]

While adding certain classical tautologies such as the principle of excluded middle to IPC yields CPC, this is not the case for all classical tautologies.[6] The resulting logic may be intermediate in strength between IPC and CPC. Examples of intermediate logics include:

$$\begin{aligned} \mathsf{KC} &= \mathsf{IPC} + \neg p \vee \neg\neg p; \\ \mathsf{LC} &= \mathsf{IPC} + (p \to q) \vee (q \to p). \end{aligned}$$

The study of these and other intermediate logics may be viewed as a study of the classification of classically valid principles in terms of their interdeducibility in intuitionistic logic [Hosoi, 1967a].

[3] According to Mints [2006, p. 701], "Russell anticipated intuitionistic logic by clearly distinguishing propositional principles implying the law of the excluded middle from remaining valid principles. In fact, he states what was later called Peirce's law."

[4] See van Atten 2014, § 4 for historical details and discussion of Glivenko's [1928] earlier paper containing a set of axioms for intuitionistic propositional logic weaker than IPC.

[5] Since in classical logic conjunction and negation are sufficient to define classical disjunction and implication, the corollary just cited led Łukasiewicz [1952] to view IPC as an *extension* of CPC with two new connectives, intuitionistic $\to$ and $\vee$. For a critique of this view, see Humberstone 2011, p. 305.

[6] For a characterization of those axioms whose addition to IPC yields CPC, see § 2.1.

Brouwer objected to certain classical principles, such as excluded middle, as involving a commitment to the *solvability of all mathematical problems* [Brouwer, 1908, 1923]. If p represents some unsolved problem of mathematics, Brouwer took an assertion of $p \vee \neg p$ as a commitment to the solvability of the given problem. From a classical perspective, excluded middle says nothing about the solvability of problems; it is traditionally accepted on the basis that every proposition is either true or false, which in the classical view has nothing to do with what humans or machines can solve (unless of course the proposition concerns solvability). Intuitionists reject this classical conception of transcendent truth and understand logical principles such as those above differently. The intuitionistic view is often explained in terms of what it takes to *prove* a mathematical statement formed from others using the logical operators, resulting in what has become known as the Brouwer-Heyting-Kolmogorov (BHK) interpretation of logical operators. In the version presented by Troelstra and van Dalen, the conditions of proof are as follows:

(H1) A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

(H2) A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

(H3) A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

(H4) Absurdity $\bot$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction. [Troelstra and van Dalen, 1988a, p. 9]

According to this interpretation, to claim that excluded middle is a logical law is to claim that for any mathematical statement, either we can prove the statement or find a method of transforming any hypothetical proof of it into a proof of a contradiction. This is a bold claim for which we seem to lack adequate justification.

Are the theorems of IPC precisely the principles justified by the BHK interpretation? Unfortunately, there is no way to prove that they are or are not, since the BHK interpretation is an informal explanation (e.g., what is a "construction"?), rather than a mathematically defined *formal semantics* for Heyting's language.[7] Our interest in this paper is in such formal semantics.

1.2 Formal Semantics

The BHK interpretation has inspired interesting formal semantics, such as Kleene's realizability [Kleene, 1945] and Medvedev's finite problems [Medvedev, 1962], but the resulting logics are stronger than IPC (see Rose 1953, Medvedev 1962, 1963, 1966, Skvortsov 1979, Chagrov and Zakharyaschev 1997, § 2.9, Dummett 2000, § 6.1, and Plisko 2009).[8] For discussion of how the BHK interpretation relates to a number of formal semantics, see Artemov and Beklemishev 2005, § 11.

In the case of classical logic, a formal semantics for CPC is easily defined, using a notion of satisfaction in the style of Tarski [1933]. Given a valuation function v assigning to each propositional letter $p, q, \dots$ a value 0 or 1 (false or true), one recursively defines a relation $\models_v$ of satisfaction of formulas:

- $\not\models_v \bot$;
- $\models_v p$ iff $v(p) = 1$;

[7]Compare this with the case of Church's Thesis identifying intuitively computable functions with recursive functions as formally defined by Gödel (for discussion, see Kleene 1952, § 60).

[8]There are also proof-theoretical and type-theoretical takes on BHK, building on the Curry-Howard correspondence (see, e.g., Sorensen and Urzyczyn 2006).

- $\models_v \varphi \wedge \psi$ iff $\models_v \varphi$ and $\models_v \psi$;
- $\models_v \varphi \vee \psi$ iff $\models_v \varphi$ or $\models_v \psi$;
- $\models_v \varphi \to \psi$ iff $\not\models_v \varphi$ or $\models_v \psi$;
- $\models_v \neg\varphi$ iff $\not\models_v \varphi$.

A formula φ is *valid* according to this Tarskian semantics if $\models_v \varphi$ for all valuations v. One can then prove that the valid formulas are exactly the theorems of CPC (see, e.g., Johnstone 1987, Ch. 2).

In this paper, we survey and study a family of formal semantics for IPC and its extensions. Rather than trying to formalize the BHK interpretation, these semantics exploit a remarkable connection between IPC and topology. As Rasiowa and Sikorski [1963] put it, it is amazing that the intuitionists' "philosophical ideas concerning the notion of existence in mathematics have led to the creation of formalized logical systems which, from the mathematical point of view, proved to be equivalent to the theory of lattices of open subsets of topological spaces" (pp. 8–9). All of the semantics we will study are either special cases or generalizations of the topological interpretation of intuitionistic logic. The semantics are:

- Kripke semantics;
- Beth semantics;
- Topological semantics;
- Dragalin semantics;
- Algebraic semantics.

The order of this list is not arbitrary. One of the main themes of this paper is that the five semantics form a strict hierarchy in terms of generality:

$$\text{Kripke} < \text{Beth} < \text{Topological} < \text{Dragalin} < \text{Algebraic}.$$

On one level, the first four semantics work in a similar way: they use special structures to produce algebras, known as Heyting algebras, in which the theorems of IPC are valid. The hierarchy above indicates that Beth frames can produce all the algebras that Kripke frames can and more; topological spaces can produce all the algebras that Beth frames can and more; and Dragalin frames can produce all the algebras that topological spaces can and more, but not all Heyting algebras.

1.3 The Modal Perspective

The Kripke, topological, and algebraic semantics are the most widely used semantics for IPC and intermediate logics. These semantics may be seen as closely related to Gödel's [1933b] translation of IPC into the classical modal logic $\mathsf{S4}$. The language of modal logic extends that of propositional logic with a unary operator $\Box$, which admits many interpretations (see, e.g., van Benthem 2010). To motivate Gödel's translation, $\Box\varphi$ is understood as φ *is provable* (to be distinguished from: provable in a particular formal system). The modal logic $\mathsf{S4}$ extends CPC with the axioms

- $\Box p \to p$,

- $\Box p \to \Box\Box p$, and
- $\Box(p \to q) \to (\Box p \to \Box q)$,

while closing under the rules of modus ponens, substitution, and "necessitation": if φ is a theorem, so is $\Box\varphi$. Arguably these principles are reasonable in light of the reading of $\Box$ as absolute provability (cf. Orlov 1928). Different readings of $\Box$ would motivate different axioms. For example, for the reading of $\Box\varphi$ as φ *is necessarily true*, the standard system is the logic $\mathsf{S5}$ that extends $\mathsf{S4}$ with the axiom $\neg\Box p \to \Box\neg\Box p$.

There are a number of translations t from the propositional language to the modal language such that φ is a theorem of IPC iff $t(\varphi)$ is a theorem of $\mathsf{S4}$, as proved by McKinsey and Tarski [1948]. One such translation is defined recursively as follows:

- $t(\bot) = \bot$;
- $t(p) = \Box p$ for p a propositional letter;
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$;
- $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$;
- $t(\varphi \to \psi) = \Box(t(\varphi) \to t(\psi))$.

For example, $p \vee q$ translates to $\Box p \vee \Box q$ (either p is provable or q is provable). Using this translation, any formal semantics for the modal logic $\mathsf{S4}$ may be converted into a formal semantics for IPC: in the most direct way, the same models may be used, and the semantic value assigned to a propositional formula φ in the intuitionistic semantics may be defined as the semantic value assigned to the modal formula $t(\varphi)$ in the modal semantics. Thus, the Kripke semantics for $\mathsf{S4}$ based on preordered sets [Kripke, 1963a,b] may be converted into Kripke semantics for IPC based on preordered sets [Kripke, 1965] (see § 2.3). In the case of topological semantics, of which Kripke semantics is a special case, the topological semantics for IPC came first historically [Stone, 1938, Tarski, 1938] and the topological semantics for $\mathsf{S4}$ later [Tsao-Chen, 1938, McKinsey, 1941]; still, the topological semantics for $\mathsf{S4}$ may be converted into the topological semantics for IPC as above (see § 2.2). Finally, for algebraic semantics, the semantics for $\mathsf{S4}$ based on interior algebras may be converted into the algebraic semantics for IPC based on Heyting algebras.[9] This uses the fact that the open elements of an interior algebra—those a such that $a = \Box a$, where $\Box$ is the interior operator in the interior algebra—form a Heyting algebra, and every Heyting algebra can be represented in this way.

After a survey of these standard semantics in §§ 2.1–2.3, we will turn in § 3 to less standard semantics. Because they are less standard, our emphasis on these semantics requires more motivation.

1.4 The Nuclear Perspective

We begin in § 3.1 with Beth semantics [Beth, 1956, 1964] (see Troelstra and van Ulsen 1999 for some history). Although Beth semantics has been less widely used than the later Kripke semantics for the study of propositional logics, it has been preferred for some applications involving intuitionistic predicate logic (see, e.g., van Dalen 1978, Dummett 2000). In both Beth and Kripke semantics, formulas are evaluated at nodes in a poset,[10] but the definition of when a formula is satisfied at a node are different. One result of this

[9] The algebraic semantics for $\mathsf{S4}$ and IPC were first given using the dual concepts of *closure algebras* [McKinsey and Tarski, 1944] and *Brouwerian algebras* [McKinsey and Tarski, 1946]. The switch to interior algebras and Heyting algebras came with later authors (Rasiowa and Sikorski 1963, Blok 1976, Esakia 1985).

[10] Beth originally used trees, but in § 3.1 we use a more general version of Beth semantics over posets from van Dalen 1984.

is that in Beth semantics for predicate logic, one can associate with each node the same domain of objects, while in Kripke semantics one must allow the domain of objects to grow from a node to one of its successors in order to invalidate the principle

$$\forall x(\varphi \vee \psi) \to (\varphi \vee \forall x\psi),$$

where x is not free in φ, which is not a theorem of intuitionistic predicate logic (cf. Görnemann 1971). This gives Beth semantics an advantage in the formalization of *intuitionistic analysis*, as explained by van Dalen: Beth semantics allows one to use a constant domain of standard natural numbers, whereas in Kripke semantics one would have to allow the first-order domains to grow with non-standard numbers, which would "make the model totally unmanageable" [van Dalen, 1978, p. 1]. Dummett [2000] argues that in light of the ability of Beth semantics to invalidate the formula above in a constant domain model, "the advantage, in supplying a representation of the intended meanings of the intuitionistic logical constants, lies heavily with the Beth trees as against the Kripke trees" (p. 150).[11]

Our interest in Beth semantics in this paper comes not from its advantages for predicate logic, but by an illuminating way of viewing Beth semantics already for propositional logic. As Dragalin [1979, 1988] makes clear, the essence of satisfaction in a Beth model is that the evaluation of a formula does not take place in the Heyting algebra directly supplied by the poset of the model, as in Kripke semantics, but rather in another Heyting algebra formed by the fixpoints of a special operator—a *nucleus*—on the Heyting algebra supplied by the poset.[12] Nuclei on Heyting algebras, especially complete Heyting algebras, have been studied extensively in connection with pointfree topology (see § 3.2 for references), so Dragalin's perspective allows a powerful theory of nuclei to be applied to Beth-like semantics.

In traditional Beth semantics, the nucleus used to evaluate formulas is always defined in the same way, yielding what we call the *Beth nucleus* in § 3.2. But this way of viewing Beth semantics unlocks a door to a more general style of semantics, which we call *nuclear semantics*: take as the basic semantic structures not just a poset, but rather a pair of a poset and a nucleus on the Heyting algebra supplied by the poset. We call such pairs *nuclear frames* [Bezhanishvili and Holliday, 2016]. Dragalin observed that any complete Heyting algebra can be realized as an algebra of fixpoints arising from a nuclear frame (see § 2.1 and § 3.2 for definitions). Thus, nuclear semantics based on posets and nuclei is as general as algebraic semantics based on complete Heyting algebras.

Nuclear semantics has one foot in the world of posets and another foot in the world of algebras. It is therefore natural to ask whether the nucleus in a nuclear frame can be replaced by some more concrete data. Dragalin proposed a way of doing so by essentially generalizing the notion of a "path" that is at the heart of Beth semantics to what we will call a *development*. As explained in § 3.3, Dragalin's semantic structures, which we call *Dragalin frames* [Bezhanishvili and Holliday, 2016], consist of a poset together with a function that assigns to each node in the poset a set of developments coming out of that node. Dragalin showed that a semantics based on these frames is at least as general as topological semantics, in the sense that any topological space can be realized by starting from an appropriate Dragalin frame. Yet topological spaces only give rise to a special class of complete Heyting algebras (see § 2.2). Thus, the question remains of whether Dragalin's development functions are adequate to replace the nuclei in all nuclear frames. In fact,

[11] Dummett's [2000] argument is as follows: "It is true enough that the use of variable domains can be given a sound intuitionistic sense as a representation of quantification over an undecidable species; but the fact that, with the Kripke trees, it is essential to use variable domains in order to falsify our formula makes it appear that this formula is invalid only in view of the possibility of quantifying over an undecidable domain; from an intuitionistic viewpoint, however, this is not so at all – the formula remains just as invalid when we take the variable to be ranging over the natural numbers" (p. 50).

[12] Dragalin [1979, 1988, § 3.2.3] worked with a slightly more general concept of a *completion operator*.

Dragalin's idea is fully successful: the nucleus in a nuclear frame can always be replaced by a development function [Bezhanishvili and Holliday, 2016]. As a consequence, Dragalin provides a semantics based on quite concrete objects that is as general as algebraic semantics based on complete Heyting algebras.

Another concrete realization of nuclear semantics is the semantics of Fairtlough and Mendler [1997], based on what we call *FM-frames*. An FM-frame is simply an enrichment of a Kripke frame with an additional preorder that is a subrelation of the Kripke order (see § 4.7). Surprisingly, this Kripke-style semantics is as general as Dragalin semantics and hence algebraic semantics based on complete Heyting algebras [Bezhanishvili and Holliday, 2016, Massas, 2016, Bezhanishvili et al., 2018], as explained in § 4.7.

semantics	**underlying structure**	**associated nucleus**
Kripke	poset	identity nucleus
Beth	poset	Beth nucleus
Nuclear	poset with nucleus j on upsets	nucleus j
Dragalin	poset with development function	Dragalin nucleus
FM	poset with additional partial order	FM nucleus

Figure 1.3: Preview of the nuclear perspective.

1.5 Comparing Semantics

The varying generality of the different semantics makes a difference in the study of intermediate logics. Shehtman [1977, 1980, 2005] has shown that there are intermediate propositional logics that cannot be characterized by Kripke frames. In fact, there are continuum many such logics [Litak, 2002]. Moreover, some of these Kripke-incomplete logics can be characterized by topological spaces [Shehtman, 2005, § 8]. It is a famous open problem of Kuznetsov [1975] whether every intermediate logic can be characterized by topological spaces. The semantic hierarchy displayed in § 1.2, which we will establish in § 4 of this paper, raises further questions of this kind. For example:

- given the hierarchy and Shehtman's result, we know that either there are intermediate logics that are Beth-complete but Kripke-incomplete or there are intermediate logics that are topologically-complete but Beth-incomplete, but it is unknown which disjunct holds.
- Another natural question, a variation on Kuznetsov's, is whether every intermediate logic can be characterized by complete Heyting algebras, or equivalently—in light of the results mentioned in § 1.4—by Dragalin frames or FM-frames.
- If we consider languages more expressive than the basic propositional language, then differences at the level of logics arise more easily between the different semantics. For example, Nadel [1978] shows that the infinitary intuitionistic propositional logic of Beth frames differs from that of Kripke frames. Other natural extensions to consider include the language of intuitionistic propositional logic with propositional quantifiers and the language of intuitionistic predicate logic.

In addition to comparing semantics by their relative generality, we can compare them along another dimension. We can ask not only how well a formal semantics serves as a mathematical tool for the study of logics, but also whether it reflects any intuitive view of the meaning of the logical connectives (cf. Dummett 2000, p. 256 on "merely algebraic valuation systems" vs. "genuine semantics"). For example, for the modal logic S5 mentioned above, a standard semantics interprets $\Box$ as a kind of quantifier over a nonempty set of points:

$$\Box\varphi \text{ is true at a point } w \text{ iff } \varphi \text{ is true at all points } v \text{ in the set.}$$

While one could view this semantics purely as a tool for proving results about S5, typically it is understood as reflecting an intuitive picture of the meaning of the necessity operator: the points in the set are thought of as *possible worlds* and necessity is analyzed as *truth in all possible worlds*. As we will briefly discuss, several of the intuitionistic semantics can be seen as reflecting a picture not of truth relative to possible worlds, but rather of verifiability relative to information states. (In § 3.2, we will show that the theory of nuclei sheds light on Dummett's explanation of Beth semantics in terms of verification.) We hope that readers will find the semantics discussed in this paper both of mathematical and conceptual interest.

2 Standard Semantics

In this section, we survey the semantics outlined in § 1.3. The algebraic interpretation of the intuitionistic propositional language using Heyting algebras (§ 2.1) is central in our story. Other semantics can be viewed as providing more concrete ways to produce the Heyting algebras in which formulas are evaluated. For topological semantics (§ 2.2), the lattice of open sets of any topological space is a Heyting algebra. For Kripke semantics (§ 2.3), the upward-closed sets of a preordered set form a topology and hence give rise to a Heyting algebra. We discuss how these semantics apply not only to IPC, but also to intermediate logics.

2.1 Algebraic Semantics

Algebraic models of IPC are algebras $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ with three binary operations $\wedge, \vee, \to$ and two constants $0, 1$ satisfying equations that correspond to the axioms of IPC, assuming the definition $\neg a := a \to 0$. Among many equivalent sets of equations, a standard choice (see, e.g., Rasiowa and Sikorski 1963, p. 123, Johnstone 1982, p. 8) is to take the equations for bounded lattices plus the following equations for $\to$:

- $x \to x = 1$;
- $x \wedge (x \to y) = x \wedge y$;
- $(x \to y) \wedge y = y$;
- $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.[13]

Birkhoff [1940, pp. 128–30] showed that algebras for IPC can also be described as algebras $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and $\to$ is a residual of $\wedge$; that is, for all $a, b, x \in A$,

$$a \wedge x \leq b \text{ iff } x \leq a \to b. \tag{1}$$

In other words, $a \to b$ is the maximum element of the set $\{x \in A \mid a \wedge x \leq b\}$. In particular, $\neg a$ is the maximum element of $\{x \in A \mid a \wedge x = 0\}$. From the equivalent equational or order-theoretic definitions above, it follows that the lattice is distributive (see Birkhoff 1967, p. 45, Johnstone 1982, p. 8).

[13] In place of the bounded lattice axioms, one can simply add to the four axioms for $\to$ above the axioms $x \wedge 0 = 0$ and $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ (see Monteiro 1955, Birkhoff 1967, p. 47).

It is now customary to call the algebras defined above *Heyting algebras*. Other names that were used in the past include *Brouwerian logics* [Stone, 1938, Birkhoff, 1940], *Brouwerian algebras* [McKinsey and Tarski, 1946], and *pseudo-Boolean algebras* [Rasiowa and Sikorski, 1963].[14]

A complete lattice A is a Heyting algebra iff it satisfies the join-infinite distributive law [Birkhoff, 1967, p. 128] stating that for all $a \in A$ and $X \subseteq A$,

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}. \tag{2}$$

If (2) holds, then the join of $X := \{x \in A \mid a \wedge x \leq b\}$ belongs to X and is therefore its maximum, so $a \to b$ exists. Conversely, given an arbitrary subset X and $b := \bigvee\{a \wedge x \mid x \in X\}$, for each $x \in X$ we have $a \wedge x \leq b$ and hence $x \leq a \to b$, so $\bigvee X \leq a \to b$ and hence $a \wedge \bigvee X \leq b$. Thus, that $\to$ is a residual of $\wedge$ gives us the $\leq$ direction of (2), while the $\geq$ direction is immediate. By contrast, we are not guaranteed the meet-infinite distributive law with $\vee$ and $\bigwedge$ in place of $\wedge$ and $\bigvee$, so there is an inherent asymmetry between meet and join in Heyting algebras (see McKinsey and Tarski 1946, Appendix). Finally, note that since any finite distributive lattice satisfies (2), it becomes a Heyting algebra with $a \to b := \bigvee\{x \in A \mid a \wedge x \leq b\}$.

Heyting algebras provide an algebraic semantics for IPC as follows.[15] A *valuation* on a Heyting algebra $\mathfrak{A}$ is a function that sends propositional letters to elements of $\mathfrak{A}$, which extends to a function sending propositional formulas to elements of $\mathfrak{A}$ by interpreting the logical connectives $\wedge, \vee, \to$ as the corresponding operations of $\mathfrak{A}$ and interpreting $\bot$ as 0. An algebra $\mathfrak{A}$ *validates* a propositional formula φ, written $\mathfrak{A} \models \varphi$, if every valuation on $\mathfrak{A}$ sends φ to 1. It is then straightforward to see that IPC is sound with respect to the class HA of Heyting algebras: $\mathsf{IPC} \vdash \varphi$ implies $\mathfrak{A} \models \varphi$ for each $\mathfrak{A} \in \mathsf{HA}$. For completeness, if $\mathsf{IPC} \nvdash \varphi$, then the standard Lindenbaum-Tarski construction yields that φ is refuted in the free Heyting algebra on countably many generators (see, e.g., Rasiowa and Sikorski 1963, Ch. IX, § 2).

In fact, IPC is complete with respect to finite Heyting algebras, as first observed by Jaśkowski [1936] (also see Surma et al. 1975). A different proof of this fact was given by McKinsey and Tarski [1946, Thm. 1.11], along the following lines. Suppose $\mathsf{IPC} \nvdash \varphi$. Then there is a valuation v on the Lindenbaum algebra $\mathfrak{A}$ of IPC such that $v(\varphi) \neq 1$. Let

$$S = \{v(\psi) \mid \psi \text{ is a subformula of } \varphi\},$$

and let $\mathfrak{H}$ be the bounded sublattice of $\mathfrak{A}$ generated by S. As is well known, any finitely generated distributive lattice is finite (see, e.g., Grätzer 1978, p. 68), so from the fact that S is finite, we have that $\mathfrak{H}$ is a finite distributive lattice. Hence $\mathfrak{H}$ is a Heyting algebra, where for $a, b \in \mathfrak{H}$,

$$a \to_{\mathfrak{H}} b = \bigvee\{x \in \mathfrak{H} \mid a \wedge x \leq b\} = \bigvee\{x \in \mathfrak{H} \mid x \leq a \to b\}.$$

Clearly $a \to_{\mathfrak{H}} b \leq a \to b$, and indeed $a \to_{\mathfrak{H}} b = a \to b$ provided $a \to b \in \mathfrak{H}$. Thus, if $v_{\mathfrak{H}}$ is a valuation on $\mathfrak{H}$ that agrees with v for all propositional letters appearing in φ, then an obvious induction shows that $v_{\mathfrak{H}}(\psi) = v(\psi)$ for all subformulas ψ of φ, so that $v(\varphi) \neq 1$ implies $v_{\mathfrak{H}}(\varphi) \neq 1$. This shows that IPC is complete with respect to finite Heyting algebras.[16]

It is easy to construct a Heyting algebra that refutes the classical law of excluded middle $p \vee \neg p$. For this

[14] To be precise, Brouwerian logics/algebras are Heyting algebras turned upside down, i.e., co-Heyting algebras.

[15] This is an instance of a more general approach of providing algebraic semantics for non-classical logics (see, e.g., Rasiowa 1974, Galatos et al. 2007).

[16] This is one of the earliest proofs establishing the finite embeddability property for a variety (see, e.g., Galatos et al. 2007, § 6.5).

observe that each bounded chain $(C, \leq)$ is a Heyting algebra $\mathfrak{C} = (C, \wedge, \vee, \to, 0, 1)$, where for $a, b \in C$:

$$\begin{aligned} a \wedge b &= \min\{a, b\}; \\ a \vee b &= \max\{a, b\}; \\ a \to b &= \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\mathfrak{C}_3$ be the three-element chain in Figure 2.1 and let v be a valuation on $\mathfrak{C}_3$ assigning a to p. Then

$$v(p \vee \neg p) = a \vee \neg a = a \vee (a \to 0) = a \vee 0 = a \neq 1,$$

so $p \vee \neg p$ is refuted in $\mathfrak{C}_3$. In fact, the law of excluded middle can be refuted in any Heyting algebra that is not a Boolean algebra. A special feature of $\mathfrak{C}_3$ is that for any classical tautology φ, $\mathsf{IPC} + \varphi = \mathsf{CPC}$ iff φ is not valid in $\mathfrak{C}_3$ (Jankov 1963, 1968b, Troelstra 1965, Hanazawa 1966).

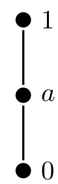

Figure 2.1: The chain $\mathfrak{C}_3$.

While CPC is the logic of the two-element Boolean algebra $\mathfrak{C}_2$, it follows from Gödel 1932 that IPC is not the logic of any single finite Heyting algebra. One way to see this is to consider the following formulas generalizing the law of excluded middle (Maksimova 1972, Chagrov and Zakharyaschev 1997, § 2.5):

- $\mathsf{bd}_1 = p_1 \vee \neg p_1$;
- $\mathsf{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n)$.

A direct calculation shows that bd_n is refuted in the $(n+2)$-element chain $\mathfrak{C}_{n+2}$. Therefore, no bd_n is a theorem of IPC. On the other hand, if $\mathfrak{A}$ is a finite Heyting algebra, then there is a sufficiently large n such that $\mathfrak{A} \models \mathsf{bd}_n$. A similar argument can be given using a sequence of formulas that generalize Peirce's law [Nagata, 1966].

Dummett [1959] showed that the logic of all finite chains is the logic LC, mentioned in § 1.1, that extends IPC with the axiom $(p \to q) \vee (q \to p)$.[17] Later Thomas [1962] axiomatized the logic of each finite chain $\mathfrak{C}_n$ (cf. Hosoi 1966a,b, 1967b). In terms of the bd_n formulas, the logic of $\mathfrak{C}_{n+1}$ is the intermediate logic LC_n that extends LC with bd_n. In fact, Hosoi [1967a] proved that every intermediate logic strictly extending LC is LC_n for some n (cf. Dunn and Meyer 1971).

The set of intermediate logics[18] forms a complete lattice $\Lambda(\mathsf{IPC})$ where the meet of a family of logics is their intersection and the join is the least intermediate logic containing each logic in the family; in fact,

[17] The logic LC is often called the *Gödel-Dummett logic*.

[18] The term *intermediate logic* for logics between IPC and CPC comes from Umezawa 1959. Extensions of IPC (closed under modus ponens and substitution) are also known as *superintuitionistic logics* [Kuznetsov, 1975]. Note that the inconsistent logic is an example of a superintuitionistic logic that is not an intermediate logic. However, this is the only example, since for consistent logics the two notions coincide.

$\Lambda(\mathsf{IPC})$ is a complete Heyting algebra [Hosoi, 1969]. This lattice has a rather complicated structure. Jankov [1968a] proved that there are continuum many intermediate logics.

The lattice $\Lambda(\mathsf{IPC})$ is dually isomorphic to the lattice $\Lambda(\mathsf{HA})$ of nontrivial varieties of Heyting algebras, where we recall that a class of algebras is a *variety* if it is closed under homomorphic images, subalgebras, and products. By the celebrated Birkhoff theorem (see Burris and Sankappanavar 1981, § 11), varieties are exactly the equationally definable classes. The dual isomorphism between $\Lambda(\mathsf{IPC})$ and $\Lambda(\mathsf{HA})$ can be seen by observing the following:

- If L is an intermediate logic, then $\mathsf{Var}(\mathsf{L}) := \{\mathfrak{A} \in \mathsf{HA} \mid \mathfrak{A} \models \mathsf{L}\}$ is a nontrivial variety of Heyting algebras.
- If V is a nontrivial variety of Heyting algebras, then $\mathsf{Log}(\mathsf{V}) := \{\varphi \mid \forall \mathfrak{A} \in \mathsf{V} : \mathfrak{A} \models \varphi\}$ is an intermediate logic.
- If $\mathsf{L} \subseteq \mathsf{L}'$, then $\mathsf{Var}(\mathsf{L}) \supseteq \mathsf{Var}(\mathsf{L}')$, and if $\mathsf{V} \subseteq \mathsf{V}'$, then $\mathsf{Log}(\mathsf{V}) \supseteq \mathsf{Log}(\mathsf{V}')$.
- $\mathsf{Log}(\mathsf{Var}(\mathsf{L})) = \mathsf{L}$ and $\mathsf{Var}(\mathsf{Log}(\mathsf{V})) = \mathsf{V}$.

A convenient way of viewing $\Lambda(\mathsf{IPC})$ is by breaking it into slices in the manner of Hosoi [1967a], as shown in Figure 2.3 below. For $n \geq 1$, let IPC_n be the least intermediate logic containing bd_n. An intermediate logic L belongs to *slice* n if L is between IPC_n and LC_n, and L belongs to *slice* ω if it is between IPC and LC. Using algebraic methods, Hosoi [1967a] proved that these slices partition $\Lambda(\mathsf{IPC})$. Since $\mathsf{IPC}_1 = \mathsf{LC}_1 = \mathsf{CPC}$, slice 1 contains only CPC, while every other slice contains infinitely many logics. Hosoi and Ono [1970] showed that slice 2 is a countable chain of logics

$$\mathsf{LC}_2 \supset \mathsf{Log}(\mathfrak{A}_2) \supset \mathsf{Log}(\mathfrak{A}_3) \supset \cdots \supset \mathsf{IPC}_2$$

where $\mathfrak{A}_n$ is the Heyting algebra obtained by adding a new top element above the 2^n-element Boolean algebra, as in Figure 2.2 below. By contrast, Kuznetsov [1975] observed that for every $n \geq 3$, slice n contains uncountably many logics (cf. Hosoi and Masuda 1993).

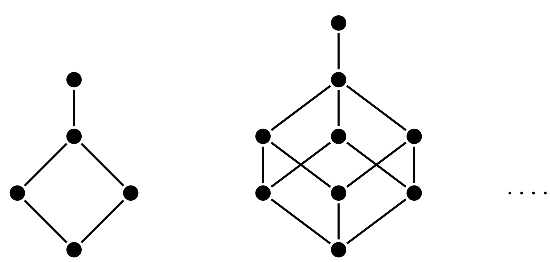

Figure 2.2: The algebras $\mathfrak{A}_2$, $\mathfrak{A}_3$, ...

Although working with Heyting algebras has proven to be a powerful method for establishing properties of IPC and the lattice $\Lambda(\mathsf{IPC})$, one may have the feeling that algebraic "semantics" does not take us very far away from the syntax with which we began. As van Benthem put it, paraphrasing others, one could argue that algebraic semantics is "merely '...syntax in disguise'..." [van Benthem, 2001, p. 358].[19] This is a legitimate

[19] Along similar lines, Grayson [1984] writes: "The interpretation of intuitionistic propositional logic in a...Heyting algebra...can perhaps hardly be counted as "interpreting" at all; it is more a matter of algebraicising logic" (p. 184).

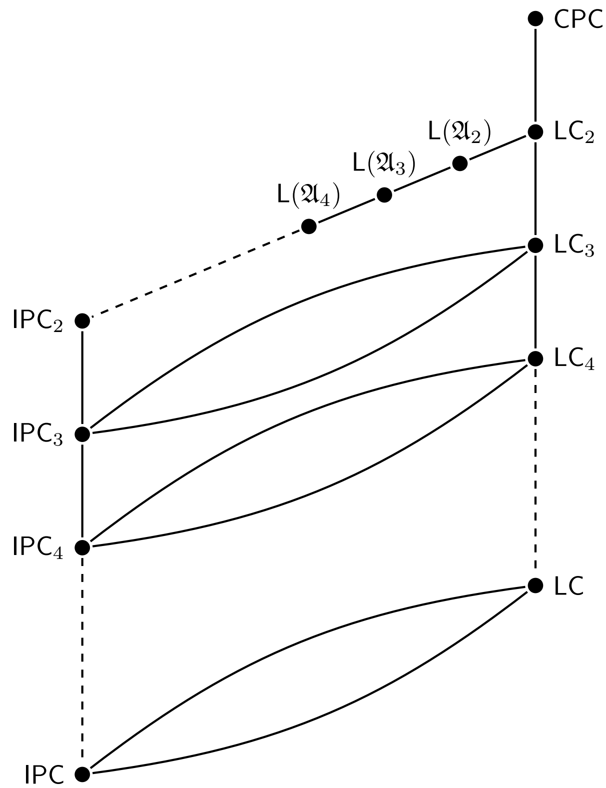

Figure 2.3: The lattice $\Lambda(\mathsf{IPC})$.

objection if all one means by "giving algebraic semantics" is to translate the axioms of IPC into equations defining a class of algebras and then observe that IPC is sound and complete with respect to such algebras. In this case, soundness and completeness is hardly illuminating. By contrast, it is quite illuminating to know that IPC is sound and complete with respect to Heyting algebras defined order-theoretically as above. If we think of the partial order $\leq$ as a relation of entailment between propositions, so $a \leq b$ means that proposition a entails proposition b, then the order-theoretic definition above embodies the following ideas:

- the conjunction $a \wedge b$ is the weakest proposition that entails both a and b (the greatest lower bound);
- the disjunction $a \vee b$ is the strongest proposition that both a and b entail (the least upper bound);
- the implication $a \to b$ is the weakest proposition such that its conjunction with a entails b (the maximum of the set $\{x \in A \mid a \wedge x \leq b\}$);
- 0 is the proposition that entails everything;
- the negation $\neg a = a \to 0$ is the weakest proposition such that its conjunction with a entails everything.

It is rather remarkable that if this is what one assumes about the meaning of conjunction, disjunction, implication, and negation, then the resulting logic is IPC.

2.2 Topological Semantics

2.2.1 Formal Semantics

While algebraic semantics adds much to our understanding of IPC and intermediate logics, it would still be desirable to also have a semantics based on more concrete models. A step in this direction is provided by the topological semantics for IPC, developed by Stone [1938] and Tarski [1938]. A valuation v on a topological space X assigns to propositional letters open subsets of X; the logical constant $\bot$ is interpreted as $\varnothing$; the logical connectives $\wedge, \vee$ are interpreted as set-theoretic intersection and union; and we set

$$v(\varphi \to \psi) = \mathsf{Int}(v(\varphi)^c \cup v(\psi)), \tag{3}$$

where Int is the interior operator and $(\cdot)^c$ is set-theoretic complement. In particular,

$$v(\neg\varphi) = \mathsf{Int}(v(\varphi)^c).$$

We say that a formula φ is *satisfied at* $x \in X$ *under* v, written $x \models_v \varphi$, provided that $x \in v(\varphi)$. It is easy to see, where $\Omega(x)$ is the set of open neighborhoods of x, that:

- $x \not\models_v \bot$;
- $x \models_v \psi \wedge \chi$ iff $x \models_v \psi$ and $x \models_v \chi$;
- $x \models_v \psi \vee \chi$ iff $x \models_v \psi$ or $x \models_v \chi$;
- $x \models_v \psi \to \chi$ iff $\exists U \in \Omega(x)\ \forall y \in U : y \not\models_v \psi$ or $y \models_v \chi$;
- $x \models_v \neg\psi$ iff $\exists U \in \Omega(x)\ \forall y \in U : y \not\models_v \psi$.

If $x \models_v \varphi$ for all $x \in X$ and all valuations v on X, then we say that φ is *valid in* X and write $X \models \varphi$.

For the purposes of topological semantics, we can restrict our attention to T_0-spaces, i.e., spaces in which any two distinct points can be distinguished by an open set. Any space can be turned into a T_0-space that validates the same formulas by identifying points that belong to the same open sets.

It is easy to see that for the law of excluded middle $p \vee \neg p$ to be valid in a topological space X, each closed set must be open, which with T_0-separation yields that every set is open. Thus, for a counterexample to excluded middle we can take X to be the *Sierpiński space* as in Figure 2.4, in which $\{0\}$ is closed and $\{1\}$ is open, and define $v(p) = \{1\}$. Then $0 \not\models_v p$, and since the only open neighborhood of 0 is the whole space, which contains a point satisfying p, we have $0 \not\models_v \neg p$. Hence $0 \not\models_v p \vee \neg p$.

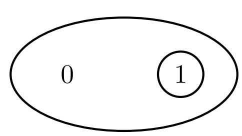

Figure 2.4: The Sierpiński space.

From the perspective of § 2.1, the essence of the topological semantics above is that we are evaluating formulas in the algebra $(\Omega(X), \cap, \cup, \to, \varnothing, X)$ where $\Omega(X)$ is the set of opens of X and

$$\begin{aligned} U \to V &= \bigcup\{W \in \Omega(X) \mid U \cap W \subseteq V\} \\ &= \mathsf{Int}(U^c \cup V). \end{aligned}$$

Since $\Omega(X)$ is a complete lattice, with $\bigvee$ as $\bigcup$ and $\bigwedge$ as $\mathsf{Int}\bigcap$, and since $\cap$ and $\bigcup$ satisfy the join-infinite distributive law in (2), we have that $\Omega(X)$ is a complete Heyting algebra by the reasoning given at the beginning of § 2.1. We follow standard terminology in pointfree topology [Johnstone, 1982] and call complete Heyting algebras *locales*.[20] They provide a pointfree generalization of topological spaces. For a topological space X, we call $\Omega(X)$ the locale of opens of X. Locales that arise from topological spaces in this way are known as *spatial locales*. As a simple example, the locale $\Omega(X)$ of the Sierpiński space X is isomorphic to the three-element chain $\mathfrak{C}_3$ from § 2.1.

Since X and $\Omega(X)$ validate the same formulas, the soundness of IPC with respect to topological semantics follows from soundness with respect to algebraic semantics. For completeness, if $\mathsf{IPC} \nvdash \varphi$, then by § 2.1 there is a Heyting algebra that refutes φ. Stone's [1938] representation of bounded distributive lattices yields that every Heyting algebra embeds in $\Omega(X)$ for some topological space X. Given a Heyting algebra $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$, let X be the set of all prime filters of $\mathfrak{A}$. For $a \in A$, let

$$\beta(a) = \{x \in X \mid a \in x\}. \tag{4}$$

Then $\{\beta(a) \mid a \in A\}$ generates a topology on X such that $\beta : A \to \Omega(X)$ is a Heyting algebra embedding (which is an isomorphism if $\mathfrak{A}$ is finite). Given this embedding, $\mathfrak{A} \nvDash \varphi$ implies $\Omega(X) \nvDash \varphi$, so from the algebraic completeness of IPC we obtain the topological completeness of IPC.

Tarski [1938, 1956] strengthened the topological completeness result for IPC by showing that IPC is the logic of any dense-in-itself metrizable separable space. This implies that IPC is the logic of the real line, the rationals, the Cantor space, or any Euclidean space. Rasiowa and Sikorski [1963] showed that separability could be dropped from Tarski's assumptions, but their proof (unlike Tarski's) requires a nontrivial use of the axiom of choice.

Natural topological semantics can also be given for intermediate logics. For example, it can be shown that the axiom $\neg p \vee \neg\neg p$ of KC is valid in X iff the closure of each open set is open, which means that the space is *extremally disconnected* (ED) (see, e.g., Johnstone 1982, pp. 101–2). In addition, it can be shown that X validates the axiom $(p \to q) \vee (q \to p)$ of LC iff every subspace of X validates $\neg p \vee \neg\neg p$ (see, e.g., Johnstone 2002, p. 1004, Bezhanishvili et al. 2015, Prop. 3.1). Thus, X validates $(p \to q) \vee (q \to p)$ iff every subspace of X is ED, so X is *hereditarily extremally disconnected* (HED).

As mentioned in § 1, it is a famous open problem of Kuznetsov [1975] whether every intermediate logic is the logic of some class of topological spaces—or equivalently, of spatial locales, which form a special subclass of locales. In fact, it is still unknown whether every intermediate logic is the logic of some class of locales. We will discuss additional questions of this kind in § 4.

2.2.2 Verificationist Interpretation

While one can treat the above formal semantics purely as a mathematical tool for the study of logics, one can also try to understand it as an account of the meaning of the logical connectives in terms of a notion

[20] Locales are also known as *frames* in the pointfree topology literature (but the category of locales is dual to the category of frames). However, in this paper we reserve the term 'frame' for *Kripke frames* (§ 2.3) and generalizations thereof (§ 3).

of *verifiability*, as follows.[21] Think of the points in a topological space X as partial states of information. Such states settle some but not necessarily all questions about how things are, including questions about the possibility of acquiring more information. Consider the set V of information states according to which 2147483647 is a prime number. If one has done a calculation verifying that 2147483647 is prime, then (under some idealization) one knows that whatever additional information one may acquire, one's richer information state will be inside V. In general, given any set U of states, let us say that one has *verified* U iff one knows that whatever additional information one may acquire, one's richer information state will be inside U; and U *is verifiable*, relative to one's current information state, iff it is possible to achieve such verification of U after a finite amount of time, starting from the current information state.

To connect this with topology, let $\mathsf{Int}(U)$ be the set of states in which U is *verifiable*.[22] With this definition, one can try to justify the axioms of the interior operator on a topological space. First, it is clear from the definition that $\mathsf{Int}(U) \subseteq U$. Second, if we assume that it is possible to perform any finite sequence of possible verifications in a finite amount of time, then the Int operator should distribute over finite intersections.[23] Yet we do not assume that it is possible to perform an infinite sequence of verifications in a finite amount of time, so we do not assume distribution over arbitrary intersections (cf. § 2.3). Finally, one could adopt a notion of verification according to which by verifying U, one also verifies that one has verified U, which implies that if it is possible to verify U, then it is also possible to verify *that it is possible to verify* U, so $\mathsf{Int}(U) \subseteq \mathsf{Int}(\mathsf{Int}(U))$.[24] In this way, the axioms of the interior operator may be justified.

Let us now return to topological semantics. According to a basic verificationist view of meaning, meaningful propositions are such that their truth is equivalent to their verifiability, corresponding to sets U such that $U = \mathsf{Int}(U)$. Thus, meaningful propositions correspond to *open sets*; so these are the semantic values of formulas, rather than arbitrary sets as in classical semantics. The semantics of implication—and hence negation—also involves verification: $x \vDash_v \psi \to \chi$ iff in x it is *possible to verify* $v(\psi)^c \cup v(\chi)$, where $v(\varphi)$ is the set of states in which φ is verifiably true. From this perspective, excluded middle is invalid because there may be φ for which it is not possible to verify $v(\varphi)$ or to verify $v(\varphi)^c$. Typical examples arise when φ stands for a statement quantifying over an infinite domain (see Dummett 2000).

2.3 Kripke Semantics

2.3.1 Formal Semantics

The most popular of the standard semantics for intuitionistic propositional logic is the relational semantics [Dummett and Lemmon, 1959, Kripke, 1963b, Grzegorczyk, 1964, Kripke, 1965], which has come to be called Kripke semantics.[25] It is a particular case of topological semantics. Call a topological space X *Alexandroff* if the intersection of any family of opens is again open, or equivalently, if interior distributes over arbitrary

[21] Compare the following account with Vickers 1989, Ch. 2 and references therein. For discussion of the relation between "truth" and "verifiability" for a constructivist, see Dummett 1998. Another conceptual explanation of topological semantics for intuitionistic logic is given by Scott [1968, p. 195]: "One may view a neighborhood of a topological space as a kind of "proof": a proof that a point belongs to a more complicated set because the neighborhood of the point is included in the set" (cf. van Dalen 2002, § 4 for a similar explanation but in terms of "evidence" instead of proof).

[22] In § 2.3, we will consider an alternative definition of $\mathsf{Int}(U)$ as the set of states in which U has *been verified*.

[23] This assumes there is not an exclusion principle by which performing one verification precludes performing another verification that would have otherwise been performable.

[24] To see the implication, observe that if V (for verification) is a monotone operator on a poset such that $Va \leq VVa$ for all elements a, and $\Diamond$ (for possibility) is a monotone operator such that $a \leq \Diamond a$ for all a, then $\Diamond V$ is such that $\Diamond Va \leq \Diamond V \Diamond Va$.

[25] Kripke [1965] was the first to develop the predicate version of this semantics and prove the completeness of intuitionistic predicate logic with respect to it. Relational semantics for intuitionistic logic is closely related to relational semantics for modal logic (see, e.g., Chagrov and Zakharyaschev 1997). For the historical development of relational semantics for modal logic, see, e.g., Goldblatt 2006.

intersections [Alexandroff, 1937]. Then each $x \in X$ has a least open neighborhood U_x, and the topology on X is determined by the specialization preorder given by $x \leq y$ iff $y \in U_x$. Open sets are simply the *upward closed sets* (*upsets*) with respect to $\leq$ (i.e., if $x \in U$ and $x \leq y$, then $y \in U$), and the least open neighborhood U_x is the principal upset $\uparrow x := \{y \in X \mid x \leq y\}$. The closure of $A \subseteq X$ is calculated as

$$\mathsf{Cl}(A) = \downarrow A := \{x \in X \mid \exists a \in A : x \leq a\},$$

and the interior is calculated as

$$\mathsf{Int}(A) = (\downarrow(A^c))^c = \{x \in X \mid \uparrow x \subseteq A\}.$$

A *Kripke frame* for IPC is a preordered set X. We can view it as an Alexandroff space and interpret formulas of IPC in X as in § 2.2.1. In particular, $v(\varphi)$ is an upward closed set, and the clauses for $\rightarrow$ and $\neg$ from § 2.2.1 can now be written as follows, since we may take $U \in \Omega(x)$ to be the least open neighborhood of x, namely $\uparrow x$:

- $x \models_v \psi \rightarrow \chi$ iff $\forall y \geq x : y \not\models_v \psi$ or $y \models_v \chi$;
- $x \models_v \neg\psi$ iff $\forall y \geq x : y \not\models_v \psi$.

Since upsets do not feel the difference between preorders and partial orders, it is customary to only work with partially ordered Kripke frames (which correspond to T_0 Alexandroff spaces). Given a partial order $\leq$, we will sometimes consider its strict part $<$ defined by $x < y$ iff $x \leq y$ and $x \neq y$.

Figure 2.5 shows the simplest Kripke frame refuting $p \vee \neg p$, with $v(p) = \{1\}$. Note that the corresponding topological space is the Sierpiński space considered in § 2.2.1.

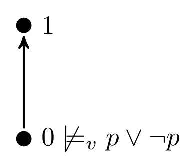

Figure 2.5: The simplest Kripke frame refuting $p \vee \neg p$.

If X is a poset, let $\mathsf{Up}(X)$ be the locale of all upsets, in which $\rightarrow$ is given by

$$U \rightarrow V = \mathsf{Int}(U^c \cup V) = \{x \in X \mid \uparrow x \cap U \subseteq V\}.$$

$\mathsf{Up}(X)$ is just $\Omega(X)$ for X viewed as an Alexandroff space, but we write $\mathsf{Up}(X)$ when thinking of X in terms of its order.

Remark 2.1. The locale $\mathsf{Up}(X)$ is a very special spatial locale. A locale is *completely join-prime generated* if every element is the join of *completely join-prime* elements, i.e., elements p such that $p \leq \bigvee S$ implies $p \leq s$ for some $s \in S$. The completely join-prime elements of $\mathsf{Up}(X)$ are the principal upsets $\uparrow x$, and every element of $\mathsf{Up}(X)$ is a union of principal upsets, so $\mathsf{Up}(X)$ is completely join-prime generated. In fact, a locale L is isomorphic to $\mathsf{Up}(X)$ for some poset X iff L is completely join-prime generated (see, e.g., Davey 1979, Prop. 1.1).

The soundness of IPC with respect to Kripke semantics follows from soundness with respect to topological semantics. For completeness, similar to Stone's topological representation of Heyting algebras, for Kripke semantics we have the following relational representation of Heyting algebras (see, e.g., Fitting 1969, Ch. 1, § 6, Esakia 1985, Ch. III, § 4). Let $\mathfrak{A}$ be a Heyting algebra and X the set of all prime filters of $\mathfrak{A}$. If we order X by inclusion, then X is a poset, and the map β in (4) is a Heyting embedding of $\mathfrak{A}$ into the locale $\mathsf{Up}(X)$ (which is an isomorphism if $\mathfrak{A}$ is finite). Thus, any formula refutable in $\mathfrak{A}$ is refutable in $\mathsf{Up}(X)$, so from the algebraic completeness of IPC we obtain the Kripke completeness of IPC.

Remark 2.2. In the literature on Kripke semantics for intuitionistic predicate or modal logic (see, e.g., Veldman 1976, Fairtlough and Mendler 1997), one also finds the notion of a *fallible* Kripke frame, which in the propositional case is a pair (X, F) where X is a poset and F is a distinguished upset. In a model based on a fallible frame, $v(p)$ is an upset containing F, and $v(\bot) = F$; otherwise the definition of satisfaction is the same. IPC is still sound with respect to this semantics, because the collection $\mathsf{Up}(X)_F$ of all upsets that contain F is still a locale, whose operations are the obvious relativizations of those of $\mathsf{Up}(X)$; and IPC is complete with respect to this semantics because ordinary Kripke frames are the special case wherein $F = \varnothing$.

If we work with Kripke frames instead of topological spaces, then topological properties translate into graph-theoretic properties, which are rather easy to work with. This is one of the reasons for the widespread success of Kripke semantics. For example, it is well known (see, e.g., Chagrov and Zakharyaschev 1997, p. 42) that a Kripke frame X validates the KC axiom $\neg p \vee \neg\neg p$ iff X satisfies the *Church-Rosser property* for all $x, y, z \in X$:

$$\text{if } x \leq y \text{ and } x \leq z, \text{ then } \exists u \in X : y \leq u \text{ and } z \leq u.$$

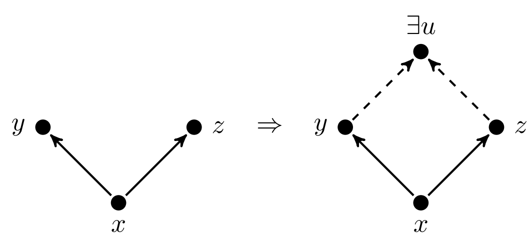

Figure 2.6: The Church-Rosser property.

In addition, a Kripke frame X validates the LC axiom $(p \to q) \vee (q \to p)$ iff X satisfies *upward linearity* for all $x, y, z \in X$:

$$\text{if } x \leq y \text{ and } x \leq z, \text{ then } y \leq z \text{ or } z \leq y.$$

If we view X as an Alexandroff space, then the Church-Rosser property corresponds to X being ED and upward linearity corresponds to X being HED as in § 2.2.1.

A price we pay for the increased concreteness of Kripke semantics is a decrease in the ability to characterize logics. Shehtman [1977] presented an intermediate logic that is not the logic of any class of Kripke frames—in fact, there are continuum many such logics [Litak, 2002]. Moreover, Shehtman [2005, § 8] showed that there are intermediate logics that are Kripke incomplete but topologically complete. However, in contrast to the detailed theory of Kripke-incomplete modal logics that has developed since the 1970s, relatively little is known in general about Kripke-incomplete intermediate logics.

2.3.2 Verificationist Interpretation

In § 2.2.2, we considered an explanation of topological semantics in terms of a notion of truth as verifiability. As noted there, from the fact that for each $i \in I$, a proposition U_i is in principle verifiable in a finite amount of time, it does not follow that $\bigcap_{i \in I} U_i$ is in principle verifiable in a finite amount of time; for we do not assume that any infinite sequence of possible finite verifications is such that it can be performed in finite time. Thus, we could not assume that all our spaces were Alexandroff.

However, let us now consider not what is *verifiable*, but instead what has actually *been verified*. As in § 2.2.2, assume the equivalence that one has verified U iff one knows that whatever additional information one may acquire, one's richer information state will be inside U. In this case, it is more reasonable to accept the following chain of implications:

for each $i \in I$, one has verified U_i;

$\Rightarrow$ for each $i \in I$, one knows that one's richer information states will be inside U_i;

$\Rightarrow$ one knows that one's richer information states will be inside $\bigcap_{i \in I} U_i$;

$\Rightarrow$ one has verified $\bigcap_{i \in I} U_i$.

Thus, if we take the interior of U to be the set of states in which one has verified U, then the interior operator distributes over arbitrary intersections, so we have an Alexandroff space. The preorder $\leq$ defined from this Alexandroff space tells us that $x \leq y$ iff every proposition (open set) that has been verified in state x has also been verified in state y.

This explanation is clearly related to the well-known interpretation of points in a Kripke frame as states of information, suggested by Kripke [1965, § 1.1]. As Dummett [2000] explains it:

> A state of information consists in a knowledge of two things: which of the constituent statements have been verified; and what future states of information are possible. That the constituent statement represented by a sentence letter p has been verified in the state of information represented by a point a is itself represented by the fact that $a \in \phi(p)$ [in our notation: $a \in v(p)$]. That the state of information represented by a may subsequently be improved upon by achieving the state represented by a point b is represented by the fact that $b \leq a$ [in our notation: $a \leq b$].... The extension of the assignment ϕ to a valuation v_ϕ may now be interpreted as supplying inductively defined sense for saying that a complex statement represented by a formula A, has been verified in a state of information represented by a point a. (p. 132–3)

Grzegorczyk [1964] also explained his semantics for intuitionistic propositional logic, which is very similar to Kripke's semantics, in terms of states of information acquired in the course of scientific research (also see Grzegorczyk 1968). These informational explanations of the semantics lead naturally to an analysis from the perspective of epistemic logic, for which we refer to van Benthem 2009, 2017.

Remark 2.3. Interpreting Kripke semantics in terms of verification depends on adopting a notion of "verification" according to which an information state that verifies an implication and verifies the antecedent counts as verifying the consequent. By contrast, a stricter notion of verification may allow for the possibility that one has verified both p and $p \to q$ without having yet verified q, e.g, if one has a concrete proof of p and a method for transforming any proof of p into a proof of q, but one has not actually applied the method to produce a concrete proof of q, as required by the stricter notion of verifying q. In § 3.2.2 we consider a modification of the Kripke clause for implication, due to Dummett, related to this idea about verification.

Remark 2.4. It is natural to ask what relationship Kripke semantics might bear to the BHK interpretation of the logical constants. According to Smoryński [1973], "Kripke's model theory bears no resemblance to intuitionistic reasoning..." (p. 324). By contrast, Humberstone [2011] remarks that it is "hard to escape the feeling that because of its consilience with the informal explanations of the meanings of the connectives offered by intuitionists, the Kripke semantics throws considerable light on their favoured logic" (p. 311). Recently Fine [2014] has argued that Kripke semantics over special Kripke frames can indeed be understood as related to the BHK interpretation. Fine's frames (see Fine 2014, Appendix) are fallible Kripke frames (X, F) as in Remark 2.2 but in which the poset X is a *complete co-Heyting algebra*, i.e., the dual of a complete Heyting algebra. Fine thinks of the join $x \sqcup y$ of x and y as the "fusion" of these states—the least state that includes both x and y. Where $\dot{-}$ is the operation of co-implication, he thinks of $x \dot{-} y$ as the state of "x's leading to y"—the least state whose fusion with x includes y.[26]

Given such a frame, Fine defines the notion of a state *exactly verifying* a formula φ, in the spirit of the BHK interpretation. Each propositional letter p is assigned a subset $E(p) \subseteq X$ thought of as the set of states that exactly verify p, such that $E(p) \supseteq F$. The set $E(p)$ is not required to be an upset, for if x exactly verifies p and $x < y$, then y may have additional content that is irrelevant to p, so y does not *exactly* verify p. Fine then defines $E(\varphi)$ for all formulas φ using operations $\wedge^E$, $\vee^E$, and $\to^E$ on $\wp(X)$ that are defined in the spirit of the BHK interpretation:

- $x \in A \wedge^E B$ iff $x = a \sqcup b$ for some $a \in A$ and $b \in B$;
- $x \in A \vee^E B$ iff $x \in A \cup B$;
- $x \in A \to^E B$ iff $\exists f : A \to B$ with $x = \bigsqcup\{a \dot{-} f(a) \mid a \in A\}$.

In the first two clauses, a state exactly verifies a conjunction iff it is the fusion of exact verifiers for each of the conjuncts, and a state exactly verifies a disjunction iff it exactly verifies one of the disjuncts. The third clause is supposed to relate to the BHK idea that a proof of an implication is a general construction transforming any particular proof of the antecedent to a proof of the consequent.

To relate this notion of exact verification to Kripke semantics, Fine defines the Kripke valuation $v(p)$ of p as $\uparrow E(p)$, so x satisfies p iff x extends some state y that exactly verifies p. Then using the fact that X is a complete co-Heyting algebra, it can be proved by induction that for any formula φ, a state x satisfies φ in the sense of Kripke semantics iff x extends some state y that exactly verifies φ in Fine's BHK-inspired sense. In this way, at least Kripke semantics using fallible Kripke frames based on complete co-Heyting algebras can be understood as related to the BHK interpretation. Fine [2014] then proves the completeness of IPC with respect to the class of such fallible Kripke frames.[27]

While Fine [2014] demonstrates that in a special case of Kripke semantics, satisfaction can be reduced to a notion of exact verification, in § 3.2 we shall see Dummett's [2000] demonstration that in Beth semantics, satisfaction can be reduced to a different notion of verification.

[26]Fine writes '$\to$' for the co-implication, but we reserve '$\to$' for Heyting implication. Since the modern notation '$\leftarrow$' for co-implication clashes with Fine's language of "x leading to y", we use '$\dot{-}$' for co-implication as in McKinsey and Tarski 1946.

[27]Note that IPC is not complete with respect to non-fallible Kripke frames based on co-Heyting algebras (or indeed any poset with a maximum), because such frames satisfy the Church-Rosser property and therefore validate KC.

3 Semantics Based on Nuclei

In this section, we present the semantics outlined in § 1.4. While in surveying the standard semantics above we proceeded from more general to less general semantics, in this section we proceed from less general to more general. Our starting point is Beth semantics (§ 3.1), which shows how posets can give rise to Heyting algebras in a way different than in Kripke semantics. Understanding the essence of how Beth semantics gives rise to Heyting algebras leads to a more general approach that we call nuclear semantics (§ 3.2), which in turn takes a more concrete (but equally general) form in what we call Dragalin semantics (§ 3.3).

3.1 Beth Semantics

3.1.1 Formal Semantics

In Beth's [1956, 1964] original version of his semantics, models are based on finitely branching trees. Kripke [1965] and Dummett [2000] work with a more general version of Beth semantics with models based on arbitrary trees, and van Dalen [1984] works with a still more general version of Beth semantics with models based on arbitrary posets. Requiring that models be based on trees is too restrictive for Beth semantics for intermediate logics (see Example 3.12), so in this paper we follow van Dalen's approach (with a modification).

In Beth semantics, as in Kripke semantics, a model is a poset X together with a valuation v assigning to each propositional letter an upset $v(p)$ of X. The difference between Beth and Kripke semantics lies in the definition of the satisfaction relation. The key concept for defining Beth satisfaction is that of a *path*. Van Dalen [1984] defines a path in a poset to be a maximal chain. As a result, proving certain properties of the semantics (see, e.g., Remark 3.9.1) requires the non-constructive Hausdorff maximality principle stating that every poset contains a maximal chain, which is classically equivalent to the axiom of choice [Rubin and Rubin, 1970, § 4]. In order to avoid reliance on this principle, we relax the notion of path to the following.

Definition 3.1. A *path* in a poset X is a nonempty chain C that is closed under upper bounds: for all $u \in X$, if u is an upper bound of C, then $u \in C$. If C is a path and $x \in C$, then we call C a path *through* x.

The definition of the satisfaction relation $\models$ in Beth semantics is the same as in Kripke semantics for $\bot$, $\wedge$, and $\to$, but differs for propositional letters and disjunctions:

- $x \models_v p$ iff every path through x intersects $v(p)$;
- $x \models_v \varphi \vee \psi$ iff every path through x intersects $\{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$.[28]

Thus, unlike in Kripke semantics, in Beth semantics if a propositional letter will "inevitably" be verified, then it is already satisfied;[29] and if "inevitably" one of the disjuncts of a disjunction will be satisfied, then the disjunction is already satisfied, even if neither of the disjuncts is already satisfied.

Remark 3.2. Given that Kripke semantics and Beth semantics (as well as its generalizations in §§ 3.2–3.3) differ in their interpretations of disjunction, it is noteworthy that any intermediate logic axiomatized by disjunction-free formulas is the logic of a class of finite Heyting algebras [McKay, 1968] and hence of a class of finite Kripke frames.

[28] Beth semantics is often presented using the concept of a *bar for* $x \in X$, which is a $B \subseteq X$ such that every path through x intersects B (see, e.g., Kripke 1965, Dummett 2000). Then $\models_v$ is defined by: $x \models_v p$ iff there is a bar B for x such that $B \subseteq v(p)$, and $x \models_v \varphi \vee \psi$ iff there is a bar B for x such that $B \subseteq \{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$. It is easy to see that the two definitions of $\models_v$ are equivalent.

[29] See below for the distinction between verification ($x \in v(p)$) and satisfaction ($x \models_v p$) in Beth semantics, following Dummett 2000.

The following example shows how calculations in Beth semantics differ from those in Kripke semantics.

Example 3.3.

1. In Beth semantics, every finite poset validates $p \vee \neg p$. For if there is no infinite path, then every path through x contains an endpoint, and each endpoint satisfies p or $\neg p$, whence $x \models_v p \vee \neg p$ for any v.

2. An example of an infinite poset refuting $p \vee \neg p$ is the "Beth comb" in Figure 3.1.

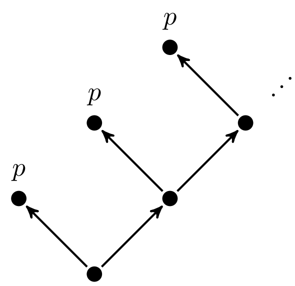

Figure 3.1: The Beth comb.

Let $v(p)$ be the set of all teeth of the comb. The spine of the comb is a path through the root that never intersects $v(p)$, so for every x in the spine, $x \not\models_v p$; but also $x \not\models_v \neg p$ since x can step to a tooth y with $y \models_v p$; thus, $p \vee \neg p$ does not hold at the root.

In trees, the Beth comb is characteristic of refuting $p \vee \neg p$, as in Proposition 3.4 below. Here we take a tree to be a (rooted) poset X in which $\{y \in X \mid y < x\}$ is a finite chain for each $x \in X$.[30] In such trees, every path is countable, and a path has a maximal point iff it is finite.

Proposition 3.4. If X is a tree, then X validates $p \vee \neg p$ iff the Beth comb is not a subposet of X.

Proof. From right to left, if the Beth comb is a subposet, define $v(p)$ to be the upset of the teeth. Then $p \vee \neg p$ is refuted at any point on the spine.

From left to right, suppose X refutes $p \vee \neg p$ with some valuation v. It follows that there is a path C such that $C \cap v(p) = \varnothing$ and $C \subseteq \downarrow v(p)$, which implies that C is infinite. We will take C to be the spine of our Beth comb subposet. Let us write C as $c_0, c_1, c_2, \dots$. Using dependent choice,[31] we construct a function that assigns a tooth to each node of the spine C as follows:

- set $f(c_0)$ to be a $t \in v(p)$ such that $c_0 \leq t$, which exists since $C \subseteq \downarrow v(p)$;
- set $f(c_{n+1})$ to be a $t \in v(p)$ such that $c_{n+1} \leq t$ and for all $m < n+1$:
  - $height(t) > height(f(c_m))$, where $height(x)$ is the cardinality of $\{y \in X \mid y < x\}$, and
  - $\uparrow t \cap \uparrow f(c_m) = \varnothing$.

The existence of such a t is guaranteed by the facts that C is infinite, $C \subseteq \downarrow v(p)$, and X is a tree: simply find a $c_k \in C$ such that $height(c_k) > height(f(c_m))$ for all $m < n+1$ and then a $t \in v(p)$ such that

[30] Later, in Remark 3.15, we will consider a more general notion of tree from set theory.
[31] Here we are working in what Schechter [1996] calls *quasi-constructive mathematics*, defined as "mathematics that permits conventional rules of reasoning plus ZF + DC, but no stronger forms of Choice" (p. 404).

$c_k \leq t$, so $height(t) > height(f(c_m))$. Suppose for contradiction that $\uparrow t \cap \uparrow f(c_m) \neq \varnothing$ for some $m < n+1$. Then since X is a tree and $height(t) > height(f(c_m))$, we have $f(c_m) \leq t$. We also have $c_k \not\leq f(c_m)$ since $height(c_k) > height(f(c_m))$; and $f(c_m) \not\leq c_k$ because with X being a tree and $v(p)$ an upset with $f(c_m) \in v(p)$, from $f(c_m) \leq c_k$ we would have $c_k \in v(p)$, contradicting $C \cap v(p) = \varnothing$. Thus, we have that $f(c_m) \leq t$, $c_k \leq t$, and that $f(c_m)$ and c_k are incomparable, contradicting the assumption that X is a tree. Therefore, $\uparrow t \cap \uparrow f(c_m) = \varnothing$, as desired.

We have shown that for all $c_n, c_m \in C$ with $n \neq m$, we have $c_n \leq f(c_n)$ and $\uparrow f(c_n) \cap \uparrow f(c_m) = \varnothing$. Thus, by restricting the relation $\leq$ to $C \cup \{f(c_n) \mid c_n \in C\}$, we have our Beth comb subposet. ☐

If X is not a tree, then X may refute $p \vee \neg p$ even if the Beth comb is not a subposet of X (see, e.g., Example 4.13). In general, Beth frames validating $p \vee \neg p$ can be characterized as follows.

Proposition 3.5. A poset X validates $p \vee \neg p$ in Beth semantics iff there is no path C and upset L in X such that $C \cap L = \varnothing$ and $C \subseteq \downarrow L$.

In § 3.3, we will prove a more general fact (Proposition 3.42.1) from which Proposition 3.5 follows.

As suggested above, in Beth semantics there is a connection between what will inevitably be satisfied and what is already satisfied. To state this precisely, we introduce the following definition.

Definition 3.6. A subset U of a poset X is *fixed* if for all $x \in X$, we have $x \in U$ iff every path through x intersects U.

It is often helpful to think of this definition in the contrapositive: $x \notin U$ iff there is a path through x that does not intersect U. It is a consequence of the definition of $\models$ above and the assumption that $v(p)$ is an upset that the semantic value of a formula will always be a fixed upset.

Proposition 3.7. For every φ, the set $\{x \in X \mid x \models_v \varphi\}$ is a fixed upset.

Proof. For the atomic case, to see that $\{x \in X \mid x \models_v p\}$ is an upset, suppose $x \models_v p$ and $x \leq y$. If C is a path through y, then $C' := (C \cap \uparrow y) \cup \{x\}$ is a path through both x and y (see Figure 3.2). Then since $x \models_v p$, there is some $z \in C' \cap v(p)$. Either $z \in C$, in which case C intersects $v(p)$ at z, or else $z = x$, in which case C intersects $v(p)$ at y, since $z = x \leq y$ and $v(p)$ is an upset. Thus, C intersects $v(p)$. Since this holds for all paths C through y, we have $y \models_v p$. Now to see that $\{x \in X \mid x \models_v p\}$ is fixed, suppose that every path through x contains a y such that $y \models_v p$. Let C be such a path. Then since C contains a y with $y \models_v p$, it follows that C also contains a $z \in v(p)$. Hence every path through x contains a $z \in v(p)$, whence $x \models_v p$. The rest of the proof is a straightforward induction. Since disjunction is non-standard, we give the proof for this case. An argument similar to that above gives us that $\{x \in X \mid x \models_v \varphi \vee \psi\}$ is an upset. To see that it is fixed, if every path through x contains a y such that $y \models_v \varphi \vee \psi$, then it is easy to see that every path through x intersects $\{z \in X \mid z \models_v \varphi \text{ or } z \models_v \psi\}$, whence $x \models_v \varphi \vee \psi$. ☐

Remark 3.8. In general the concept of a fixed upset is stronger than that of a fixed set:

1. For example, in the two-element tree in Figure 2.5, the set $\{0\}$ is not an upset, but it is a fixed set given our notion of path in Definition 3.1 because 1 is the only element that does not belong to $\{0\}$, and $\{1\}$ is a path through 1 that does not intersect $\{0\}$.

2. Even if paths are defined as maximal chains per tradition, a fixed set still need not be an upset. For example, in the simplest non-tree in Figure 3.3, $\{x\}$ is not an upset, but it is a fixed set, because for every $w \notin \{x\}$ there is a maximal chain, namely $\{r, y, m\}$, that contains w but does not intersect $\{x\}$.

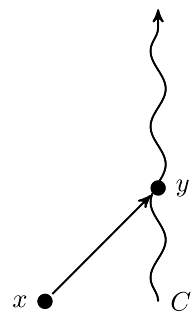

Figure 3.2: Diagram for the proof of Proposition 3.7.

3. However, if paths are defined as maximal chains per tradition and in addition X is a tree, then every fixed set is an upset. For if $x \leq x'$ and C is a maximal chain through x' that does not intersect U, then C is also a maximal chain through x that does not intersect U.

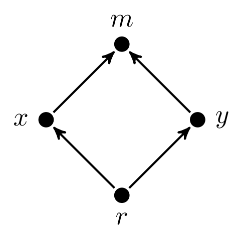

Figure 3.3: The simplest non-tree.

Remark 3.9.

1. If we had defined a path as a *maximal* chain as in van Dalen 1984, then Proposition 3.7 would require a non-constructive proof. Indeed, with that definition of path, Proposition 3.7 is equivalent to the non-constructive Hausdorff maximality principle mentioned above, stating that every poset contains a maximal chain. To see that the Hausdorff maximality principle follows from Proposition 3.7 with paths defined as maximal chains, let X be any poset. Define X' to be the result of adding three new points a, b, and c, and setting

$$\leq' = \leq \cup \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle\} \cup \{\langle x,b\rangle \mid x \in X \cup \{a\}\},$$

so X' is a poset as in Figure 3.4. Define $v(p) = \{c\}$. Hence $v(p)$ is an upset, and $\{a,b\}$ is a maximal chain that does not intersect $v(p)$, so $b \not\models_v p$. Now take any $x \in X$, so $x \leq' b$ by the definition of $\leq'$. Then by Proposition 3.7, $x \not\models_v p$, which with the definition of path as maximal chain implies that there is a maximal chain C such that $x \in C$ and $C \cap v(p) = \varnothing$. Since C is a maximal chain in X' with $x \in C$, it follows by construction of X' that $C \cap X$ is a maximal chain in X, which completes the proof.

2. The principle that every poset contains a maximal chain is equivalent to the principle that every chain in a poset can be extended to a maximal chain. Assuming this principle, we can show that the definition of $\models$ in terms of chains closed under upper bounds is equivalent to that in terms of maximal

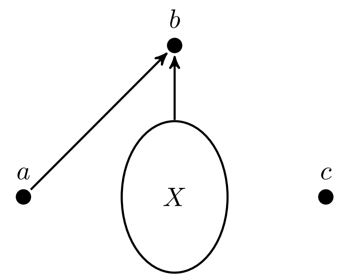

Figure 3.4: Construction of X' for Remark 3.9.1.

chains—hence Proposition 3.7 for our paths implies Proposition 3.7 for van Dalen's paths. It suffices to show that for every $x \in X$ and upset $U \subseteq X$, every chain *closed under upper bounds* that contains x intersects U iff every *maximal* chain that contains x intersects U. The left-to-right implication holds because every maximal chain is closed under upper bounds. From right to left, consider a chain C that is closed under upper bounds and contains x. Extend C to a maximal chain C^*, so by assumption, C^* intersects U at some point y. If $y \in C$, we are done, so suppose $y \notin C$. Then since C is closed under upper bounds, y is not an upper bound of C. Yet since $y \in C^*$, y is comparable with every element of C. Thus, there is some $z \in C$ such that $y \leq z$, which with $y \in U$ implies $z \in U$. Hence C intersects U, which completes the proof.

3. If X is a tree, as in traditional presentations of Beth semantics [Troelstra and van Dalen, 1988b, § 13.1], then the axiom of choice is not required to extend C to the maximal chain C^* in the argument of part 2. Simply take $C^* = {\downarrow}C$.

A crucial fact about fixed upsets is that the collection of all such sets in X becomes a locale with $\wedge$ defined as usual, by $U \wedge V = U \cap V$, but now with

$$\bigvee_{i \in I} U_i = \{x \in X \mid \text{every path through } x \text{ intersects } \bigcup_{i \in I} U_i\}.$$

That this is a locale will follow from a more general result in § 3.2 (see Theorem 3.20). The soundness of IPC with respect to Beth semantics then follows from Proposition 3.7.

Remark 3.10. Unlike in the case of Kripke frames, whose associated locales can be characterized as the completely join-prime generated ones (see Remark 2.1), in the case of Beth frames, a characterization of their associated locales is unknown (see Problem 1 in § 4.2).

For completeness, Beth used his method of *semantic tableaux* to show that any non-theorem of IPC can be refuted in a finitely branching tree according to his semantics. In § 4.1, we go a different route: we show how any poset X can be turned into a poset X_b such that the set of formulas valid in X according to Kripke semantics is exactly the set of formulas valid in X_b according to Beth semantics. This yields the Beth completeness of IPC, given the Kripke completeness of IPC.

The transformation of Kripke frames into Beth frames just mentioned establishes much more: every intermediate logic that is Kripke complete is also Beth complete. However, the properties characterizing posets that validate intermediate axioms in Beth semantics are typically more complex than in Kripke semantics since they have a second-order flavor. For example, de Beer [2012, § 5.3.1–3.2] proves the following.

Proposition 3.11. For any poset X:

1. X validates the axiom $\neg p \vee \neg\neg p$ of KC according to Beth semantics iff there is no path C and sets L and M such that $C \subseteq {\downarrow}L \cap {\downarrow}M$ and ${\uparrow}L \cap {\uparrow}M = \varnothing$;

2. X validates the axiom $(p \to q) \vee (q \to p)$ of LC according to Beth semantics iff there is no path C and subsets L and M such that: $C \subseteq {\downarrow}L \cap {\downarrow}M$; for every $l \in L$, there is a path through l that does not intersect ${\uparrow}M$; and for every $m \in M$, there is a path through m that does not intersect ${\uparrow}L$.

In § 3.3, we will prove a more general fact (Proposition 3.42) from which Proposition 3.11 follows.

Example 3.12. If X is a tree with the Beth comb of Figure 3.1 as a subposet, then X refutes $\neg p \vee \neg\neg p$. To see this, index the teeth of the comb by natural numbers, let L be the set of even numbered teeth, let M be the set of odd numbered teeth, and let C be the spine of the comb. Then $C \subseteq {\downarrow}L \cap {\downarrow}M$ and ${\uparrow}L \cap {\uparrow}M = \varnothing$ (since X is a tree), so X refutes $\neg p \vee \neg\neg p$ by Proposition 3.11.1. It follows, using Proposition 3.4, that if X is a tree that refutes $p \vee \neg p$, then it also refutes $\neg p \vee \neg\neg p$. This shows that for the purposes of studying intermediate logics, we cannot restrict attention to trees in Beth semantics.

Example 3.13. Figure 3.5 shows an elegant example from de Beer [2012, § 5.3.2] of a poset validating KC but not LC according to Beth semantics. To see that the poset has the property for KC in Proposition 3.11, observe that any two nonempty upsets will have a nonempty intersection at some d_i. On the other hand, to see that the poset does not have the property for LC, let $C = \{c_1, c_2, \dots\}$, $L = \{l_1, l_2, \dots\}$, and $M = \{m_1, m_2, \dots\}$. Then $C \subseteq {\downarrow}L \cap {\downarrow}M$, and for every $l \in L$, there is a path, namely L itself, through l that does not intersects ${\uparrow}M$, and similarly for M.

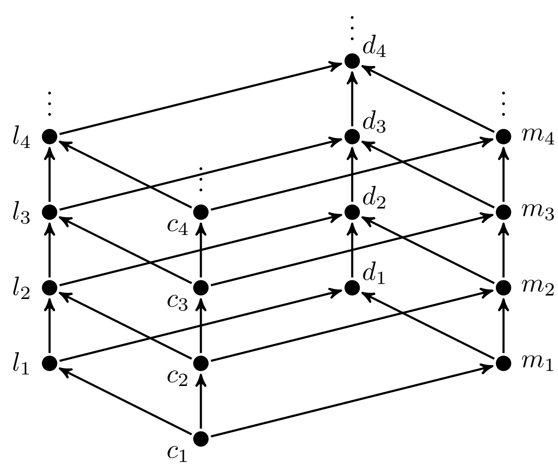

Figure 3.5: A poset validating KC but not LC in Beth semantics.

Due to its more complex definition of satisfaction, Beth semantics has been less usable than Kripke semantics for the study of intermediate logics. However, as noted in § 1.4, Beth semantics has been more usable than Kripke semantics for the purposes of intuitionistic analysis [van Dalen, 1978].

3.1.2 Verificationist Interpretation

We now turn to motivating the definition of satisfaction in Beth semantics. In § 2.3.2, we considered an explanation of Kripke semantics in terms of a notion of truth as verification. Dummett [2000, p. 138–39] explains two related ways to understand Beth semantics, the second of which is more appropriate for us here:

> On this approach, we are distinguishing between the *verification* of an atomic statement in a given state of information, and its being *assertible*; the latter notion is represented by truth at a node, and is defined, for all statements, in terms of the verification of atomic statements. The knowledge that a given atomic statement will be verified within a finite time does not itself constitute a verification of it, but is sufficient ground to entitle us to assert it. (p. 139)

While in Kripke semantics, $x \models_v p$ iff $x \in v(p)$, Dummett suggests that in Beth semantics we can make a distinction: $x \in v(p)$ means that p is verified in x, while $x \models_v p$ means that in x, it is *known* that p *will be verified*. The same idea helps to explain the different treatment of disjunction in Beth semantics vs. Kripke and topological semantics. Assume a constructivist view according to which one has *verified* a disjunction only if one has verified one of the disjuncts. Thus, in Kripke semantics, which is based on what *has been verified*, $x \models p \vee q$ only if $x \models p$ or $x \models q$. It also follows from this view of verification that it is possible to verify a disjunction only if it is possible to verify one of the disjuncts.[32] Thus, in topological semantics, which is based on *what is possible to verify*, $x \models p \vee q$ only if $x \models p$ or $x \models q$. However, it does not follow from the constructivist view of verification that one *knows* that a disjunction will be verified only if one knows of one of the disjuncts that it will be verified; unlike verification, *knowledge* is not assumed to distribute over disjunction. Thus, in Beth semantics, which is based on *knowledge of what will be verified*, it does not hold in general that $x \models p \vee q$ only if $x \models p$ or $x \models q$. Instead, in Beth semantics, $x \models p \vee q$ if it is known that however the future unfolds, one of the disjuncts will be verified (though we may not know which).

Remark 3.14. Just as the Beth semantic clause for $\vee$ is compatible with the idea that *verifying* a disjunction requires verifying a disjunct, it is also compatible with the idea from the BHK interpretation, as presented in § 1.1, that *proving* a disjunction requires proving a disjunct. Beth semantics does not offer an alternative account of verification or proof, but rather an account of the validity of principles of propositional logic in terms of knowledge of what will inevitably be verified. For further discussion of the relation between the BHK interpretation and Beth semantics, see Humberstone 2011, pp. 893–4. The generalizations of Beth semantics in §§ 3.2–3.3 are also compatible with the BHK view of what counts as a proof, though they define validity in terms of additional notions (see the discussion of assertability in § 3.2.2).

Figure 3.6 summarizes the views of $\models$ in Kripke, Beth, and topological semantics that we have discussed (which of course do no exhaust the possible views that one could associate with these semantics—see, e.g., Rabinowicz 1984 for further discussion and subtleties). To compare these views, consider the example discussed by Dummett [1998, p. 123] of whether a certain large number n is prime. A very strict constructivist might only accept that 'n is prime or composite' is true if it has been verified that n is prime or that it is composite. A slightly less strict constructivist might allow that 'n is prime or composite' is true provided we know that it will eventually be verified that n is prime or that it is composite. Finally, a more liberal constructivist might hold, as Dummett suggests, that 'n is prime or composite' is true because there is a decision procedure for primality and hence it is in principle possible to verify that the number is prime or that it is composite, even if we do not know if it will ever be verified.

Remark 3.15. As an aside, let us mention an intriguing variant of Beth semantics. Instead of working with *paths* (chains closed under upper bounds), we could work with the following weaker concept. Let a *trace* in

[32] To see this, note that if V (for verification) is an operator that distributes over join, and $\Diamond$ (for possibility) is an operator that also distributes over join (a standard assumption about possibility), then $\Diamond V$ distributes over join.

semantics	**understanding of** $x \models p$
Kripke	at x, p has been verified
Beth	at x, it is known that p will be verified
Topological	at x, it is possible to verify p

Figure 3.6: Comparing Kripke, Beth, and topological semantics.

a poset X be a nonempty chain C such that for every $x \in C$, either x is maximal (there is no $y \in X$ such that $x < y$) or there is a y such that $x < y \in C$. Clearly every path is a trace. Conversely, every trace is a path if X is a *tree*, understood as a poset in which $\{y \in X \mid y < x\}$ is a finite chain for each $x \in X$.[33] There is a more general notion of a tree used in set theory, in which $\{y \in X \mid y < x\}$ is only required to be well-ordered by $<$ [Jech, 2002, p. 114]. In this case, as in the case of an arbitrary poset, there may be traces that are not paths (see below). Now consider the following modified definition of $\models$ for Beth semantics over posets:

- $x \models_v p$ iff every trace through x intersects $v(p)$;
- $x \models_v \varphi \vee \psi$ iff every trace through x intersects $\{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$.

With this semantics, we can refute $p \vee \neg p$ in the linear order $\omega + 1$ by setting $v(p) = \{\omega\}$ as in Figure 3.7; for the set of natural numbers is a trace (but not a path) that never intersects $v(p)$, so for every n, $n \not\models_v p$; but also $n \not\models_v \neg p$, since n can step to ω and $\omega \models_v p$; thus $p \vee \neg p$ does not hold at any n. It is noteworthy that the chain $\omega + 1$ with $v(p) = \{\omega\}$ is *bisimilar*, in the standard sense from modal logic [Blackburn et al., 2001, § 2.2], to the Beth comb in Figure 3.1 by relating ω to all the teeth of the comb as in Figure 3.8.

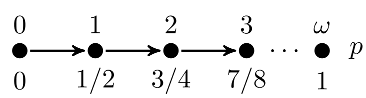

Figure 3.7: Poset refuting $p \vee \neg p$ with the trace semantics.

The distinction between traces and paths can be thought of in terms of the intuitive interpretation of Beth semantics discussed above. If we think of the elements of the poset as *information states*—not associated with any particular times—that one may pass through at different speeds, then the trace picture makes sense: if 0 is one's current information state in the linear order $\omega + 1$, then one possible future is the one in which each day, one's information state increments by one unit, thus never reaching ω. By contrast, suppose we think of the elements of the poset as possible *moments*, at 0 seconds, 1/2 second, 3/4 second, 7/8 second, ..., 1 second from a starting time, as in Figure 3.7. Then any possible future must eventually reach 1, as required by the path picture of Beth semantics. We leave for the future a further comparison of the path and trace versions of Beth semantics.

[33] To see this, suppose u is an upper bound of a trace C but $u \notin C$. Then there is a $y_1 \in C$ such that $y_1 < u$, so y_1 is not maximal. Hence by the definition of a trace, there is a y_2 such that $y_1 < y_2 \in C$, which implies $y_2 < u$, so y_2 is not maximal. In this way we obtain $y_1 < y_2 < y_3 < \cdots < u$ so that $\{y \in X \mid y < u\}$ is infinite.

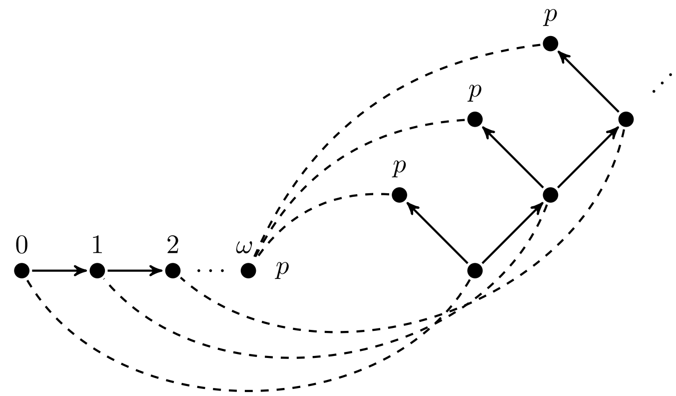

Figure 3.8: A bisimulation between $\omega + 1$ and the Beth comb.

3.2 Nuclear Semantics

3.2.1 Formal Semantics

At the heart of Beth semantics is an operation j_b on the upsets of a poset X defined as follows:

$$j_b U = \{x \in X \mid \text{every path through } x \text{ intersects } U\}. \tag{5}$$

A fixed upset, as in Definition 3.6, may now be equivalently defined as an upset U that is a *fixpoint* of j_b, i.e., $U = j_b U$. In addition, the two key satisfaction clauses in Beth semantics may now be written as:

- $x \models_v p$ iff $x \in j_b v(p)$;
- $x \models_v \varphi \vee \psi$ iff $x \in j_b\{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$;

and Proposition 3.7 now says that $\{x \in X \mid x \models_v \varphi\}$ is a fixpoint of j_b for each formula φ.

It is easy to check that j_b is a *closure operator* on $\mathsf{Up}(X)$, where a closure operator on a poset P is a unary function $\mathbf{c} : P \to P$ satisfying, for all $a, b \in P$:

- $a \leq \mathbf{c}a$ (inflationarity);
- $\mathbf{cc}a \leq \mathbf{c}a$ (idempotence);
- if $a \leq b$, then $\mathbf{c}a \leq \mathbf{c}b$ (monotonicity).

Closure operators play a crucial role in logic, lattice theory, and universal algebra [Wójcicki, 1988, Birkhoff, 1967, Davey and Priestley, 2002, Burris and Sankappanavar, 1981]. The following is well known (see, e.g., Burris and Sankappanavar 1981, § 5).

Theorem 3.16. L is a complete lattice iff it is isomorphic to the lattice of fixpoints ("closed sets") of a closure operator on a powerset.

The operator j_b on $\mathsf{Up}(X)$ is not only a closure operator but also a nucleus on $\mathsf{Up}(X)$.

Definition 3.17. A *nucleus*[34] on a Heyting algebra H is a closure operator $j : H \to H$ that also satisfies, for all $a, b \in H$:

[34]Macnab [1981] uses the term 'modal operator' instead of 'nucleus'.

- $j(a \wedge b) = ja \wedge jb$ (multiplicativity).

A nucleus j is *dense* if $j0 = 0$.

Remark 3.18.

1. The inequality $j(a \wedge b) \leq ja \wedge jb$ (for all $a, b \in H$) is equivalent to monotonicity.

2. Multiplicativity and the properties of the Heyting implication together yield:

- $j(a \to b) \leq ja \to jb$ (distribution over $\to$).

3. Inflationarity, distribution over $\to$, and idempotence together yield:

- $ja \to jb = j(ja \to jb)$.

4. The nucleus j_b on $\mathsf{Up}(X)$ is dense because $j_b\varnothing = \varnothing$.

5. Henceforth we will call j_b the *Beth nucleus*.

Nuclei play an important role in pointfree topology [Simmons, 1978, Johnstone, 1982, Picado and Pultr, 2012], as they characterize sublocales of locales (recall § 2.2.1). For a collection of key facts about nuclei, see the above references as well as Fourman and Scott 1979, Wilson 1994.

Definition 3.19. A *nuclear algebra* is a pair (H, j) of a Heyting algebra H and a nucleus j on H.

The following result is well known (see, e.g., Macnab 1981 or Dragalin 1988, p. 71).

Theorem 3.20. Given a nuclear algebra (H, j), let $H_j = \{a \in H \mid ja = a\}$ be the set of fixpoints of j. Then H_j is a Heyting algebra, called the *algebra of fixpoints* in (H, j), where for $a, b \in H_j$:

- $a \wedge_j b = a \wedge b$;
- $a \to_j b = a \to b$;
- $a \vee_j b = j(a \vee b)$;
- $0_j = j0$.

Moreover, if H is a locale, then so is H_j, where $\bigwedge_j S = \bigwedge S$ and $\bigvee_j S = j(\bigvee S)$. Furthermore, the map $j : H \to H_j$ is a lattice homomorphism (though not necessarily a Heyting algebra homomorphism), and if H is a locale, then j is a $(\wedge, \bigvee)$-homomorphism.

Example 3.21. If j were only a closure operator, but not a nucleus, then H_j would not necessarily be a Heyting algebra. For example, consider the real line $\mathbb{R}$ with its usual topology. Let H be the powerset of the real line, and let $\mathbf{c}$ be the topological closure. Its fixpoints give the closed subsets of the real line, which do not form a Heyting algebra: if $a_1, a_2, \ldots$ is an increasing sequence of reals converging to a and $a < b$, as in Figure 3.9, then we can refute the join-infinite distributive law (2) characterizing complete Heyting algebras by taking x as $[a, b]$ and Y as the family of intervals $[a_i, a_{i+1}]$, so that $x \wedge \bigvee_{\mathbf{c}} Y = \{a\}$ while $\bigvee_{\mathbf{c}}\{x \wedge y \mid y \in Y\} = \varnothing$.

Example 3.22.

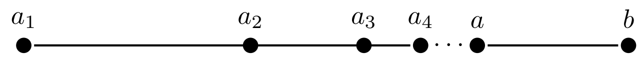

Figure 3.9: Diagram for Example 3.21.

1. An important example of a nucleus on a Heyting algebra is the operation of double negation $\neg\neg$, where $\neg a = a \to 0$ as in § 2.1. For any Heyting algebra H, the algebra $H_{\neg\neg}$ of fixpoints of double negation is a Boolean algebra (see, e.g., Rasiowa and Sikorski 1963, p. 134). This fact may be used to prove Glivenko's theorem mentioned in § 1.1.

2. If H is the locale of opens of a topological space, then the double negation of an open set is the interior of its closure. A set is *regular open* if it is equal to the interior of its closure, so the fixpoints of double negation are the regular open sets of the space. The well-known result that the regular open sets form a complete Boolean algebra (see Tarski 1938, Givant and Halmos 2009, Ch. 10) follows from the facts above that $H_{\neg\neg}$ is a Boolean algebra for any Heyting algebra H and that H_j is a locale for any locale H.

3. More generally, for any $a \in H$, the operation w_a defined by

$$w_a b = (b \to a) \to a$$

(which is $\neg\neg$ when $a = 0$) is a nucleus whose fixpoints form a Boolean algebra. In fact, if j is a nucleus on H, then the algebra H_j of fixpoints is a Boolean algebra iff $j = w_a$ for some $a \in H$ (see, e.g., Johnstone 1982, p. 51).

That H_j is a Heyting algebra for any nuclear algebra (H, j) explains the soundness of IPC with respect to Beth semantics. By Proposition 3.7, the Beth definition of $\models$, and the definition of the operations in H_j above, we can see that what Beth semantics is doing is evaluating formulas in the algebra $\mathsf{Up}(X)_{j_b}$ of fixpoints in the nuclear algebra $(\mathsf{Up}(X), j_b)$. Since the algebra of fixpoints is always a Heyting algebra, IPC is sound with respect to Beth semantics.

By the same reasoning, we can obtain the soundness of IPC with respect to a more general *nuclear semantics* as follows.

Definition 3.23. A *nuclear frame* is a pair (X, j) where X is a poset and j is any nucleus on $\mathsf{Up}(X)$. A nuclear frame is *dense* if j is dense. The *algebra of fixpoints* of a nuclear frame (X, j) is the algebra of fixpoints in the nuclear algebra $(\mathsf{Up}(X), j)$.

A valuation on a nuclear frame assigns to propositional letters elements of $\mathsf{Up}(X)$ as usual, and the definition of $\models$ simply replaces the Beth nucleus j_b with the given nucleus j:

- $x \models_v \bot$ iff $x \in j\varnothing$;
- $x \models_v p$ iff $x \in jv(p)$;
- $x \models_v \varphi \vee \psi$ iff $x \in j\{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$.

Thus, Beth semantics may be regarded as a special case of nuclear semantics. The same is true of Kripke semantics since a Kripke frame X may be regarded as a nuclear frame (X, j_k) where j_k is the identity nucleus on $\mathsf{Up}(X)$ ($j_k U = U$). The nuclear semantic clauses above then reduce to the standard Kripke clauses. From these observations it follows that IPC is complete with respect to nuclear semantics, because it is complete with respect to the class of nuclear Beth frames (X, j_b), by Beth completeness, or the class of nuclear Kripke frames (X, j_k), by Kripke completeness.

As observed in Remark 2.1, only very special locales—the completely join-prime generated ones—arise from Kripke frames. By contrast, for nuclear frames Dragalin [1979, 1988, p. 75] proved the following analogue of Theorem 3.16.

Theorem 3.24 (Dragalin)**.** *L is a locale iff L is isomorphic to the algebra of fixpoints of a dense nuclear frame.*

Proof. (Sketch) The right-to-left direction follows from Theorem 3.20. From left to right, given a locale L, let L_+ be the result of deleting the bottom element from L, and let $(L_+)^\partial$ be the dual poset of L_+ (i.e., with the order reversed);[35] then L can be represented as the algebra of fixpoints of the nuclear algebra $(\mathsf{Up}((L_+)^\partial), j)$ with j defined by:

$$jU = \downarrow \bigvee U,$$

where $\bigvee$ is the join in L, and $\downarrow$ gives the downset in L_+, which is an upset in $(L_+)^\partial$. $\square$

It follows from this representation theorem that if an intermediate logic L is the logic of some class C of locales, according to algebraic semantics, then L is also the logic of a class of nuclear frames, according to nuclear semantics, namely the class of nuclear frames representing the locales in C. Thus, nuclear semantics is as general as algebraic semantics based on locales. In § 3.3 and §§ 4.6–4.7, we will improve this result by providing more concrete representations of nuclear frames.

Remark 3.25. In the setting of nuclear semantics, the difference between intuitionistic and classical logic arises not from different definitions of satisfaction, but rather from different choices of the nucleus j. If we maintain the definition of satisfaction above but restrict attention to nuclear frames (X, j) in which j is the nucleus of double negation, then we obtain a semantics for which CPC is sound and complete. Soundness follows from the fact that the fixpoints of the double negation nucleus form a Boolean algebra as in Example 3.22.1.[36] Completeness follows from the completeness of CPC with respect to the two-element Boolean algebra, which is the algebra of fixpoints in a nuclear frame with one point, wherein the identity nucleus is also the nucleus of double negation. Note that for any nuclear frame, if j is double negation, then the satisfaction clauses above are equivalent to:

- $x \models_v p$ iff $\forall x' \geq x\ \exists x'' \geq x'$: $x'' \in v(p)$;
- $x \models_v \varphi \vee \psi$ iff $\forall x' \geq x\ \exists x'' \geq x'$: $x'' \models_v \varphi$ or $x'' \models_v \psi$.

This definition appears as "weak forcing" in the literature on set theory [Cohen, 1966] and as "possibility semantics" in the literature on modal logic [Humberstone, 1981, Holliday, 2015, 2018].

Remark 3.26.

[35] We reverse the order because we have chosen to work with upsets instead of downsets in posets. If we worked with downsets as in Dragalin 1979, 1988, then there would be no need to reverse the order.

[36] In fact, CPC is sound with respect to the larger class of all nuclear frames of the form (X, w_U) for $U \in \mathsf{Up}(X)$, where w_U is the nucleus in Example 3.22.3.

1. The notion of a nucleus on a locale has been generalized to that of a quantic nucleus on a quantale (see, e.g., Rosenthal 1990). Theorem 3.20 has an analogue in this more general setting, which has been applied to the semantics of substructural logics, as in, e.g., Ono 1993, Sambin 1995. Rumfitt [2012, 2015, § 6] utilizes Sambin 1995 to give a philosophically motivated semantics for intuitionistic logic.

2. Semantics for substructural logics can also be given using an arbitrary closure operator—not necessarily a nucleus—by evaluating formulas as fixpoints of the closure operator and interpreting disjunction by taking the closure of the union [Restall, 2000, § 12.2]. However, note that in light of Example 3.21, we cannot simply replace the nucleus j with an arbitrary closure operator $\mathbf{c}$ in the nuclear semantics above and maintain the soundness of IPC.

Remark 3.27. Given the centrality of nuclei in our story here, it is important to note that there is a formal calculus for reasoning about nuclei: just as IPC is the logic of Heyting algebras, the *propositional lax logic* PLL [Goldblatt, 1981, Fairtlough and Mendler, 1997] is the logic of nuclear algebras. The language of PLL adds a unary connective $\bigcirc$ to the language of IPC, which is interpreted as the nucleus j in a nuclear algebra. The logic PLL is the smallest extension of IPC in this language that contains the axioms $p \to \bigcirc p$, $\bigcirc\bigcirc p \to \bigcirc p$, and $(\bigcirc p \wedge \bigcirc q) \to \bigcirc(p \wedge q)$, and is closed under uniform substitution, modus ponens, and the monotonicity rule: if $\mathsf{PLL} \vdash \varphi \to \psi$, then $\mathsf{PLL} \vdash \bigcirc\varphi \to \bigcirc\psi$.[37] It is straightforward to show that PLL is sound and complete with respect to its algebraic semantics based on nuclear algebras. A number of other, more concrete semantics for PLL have also been proposed [Goldblatt, 1981, Fairtlough and Mendler, 1997, Benton et al., 1998, Alechina et al., 2001, Goldblatt, 2011].

3.2.2 Verificationist Interpretation

Next we discuss connections between nuclei and the theme of verification from § 2.2.2, § 2.3.2, and § 3.1.2. In § 3.1.2, we discussed Dummett's distinction between a propositional letter p being *verified* in a state x and p being *assertible* in x. Beth semantics suggests a particular sufficient condition for the assertibility of p in x: knowledge that p will be verified. To extend this idea to all formulas φ, Dummett [2000, p. 278] defines in Beth models a notion of x *verifying* φ, which we will write as $x \Vdash_v \varphi$:

- $x \Vdash_v p$ iff $x \in v(p)$;
- $x \Vdash_v \varphi \wedge \psi$ iff $x \Vdash_v \varphi$ and $x \Vdash_v \psi$;
- $x \Vdash_v \varphi \vee \psi$ iff $x \Vdash_v \varphi$ or $x \Vdash_v \psi$;
- $x \Vdash_v \varphi \to \psi$ iff $\forall y \geq x$: if $y \Vdash_v \varphi$, then every path through y contains a z with $z \Vdash_v \psi$.

The verification clauses for $\wedge$ and $\vee$ are just as in Kripke semantics. The $\to$ clause says that $\varphi \to \psi$ is verified if we know how, given any verification of φ we might obtain, to obtain a verification of ψ in due time, though perhaps not immediately.[38] Shortly we will see that the formulas that are always verified according to $\Vdash$ are exactly the theorems of IPC. First, let us connect verification with Beth semantics. Where $\models$ is the Beth satisfaction or "assertibility" relation, Dummett [2000, Thm. 7.2] proves the following by induction on φ:

$$x \models_v \varphi \text{ iff every path through } x \text{ intersects } \{y \in X \mid y \Vdash_v \varphi\}. \tag{6}$$

[37] As an admissible rule (given $p \to \bigcirc p$ and modus ponens), we have the rule of necessitation: if $\mathsf{PLL} \vdash \varphi$, then $\mathsf{PLL} \vdash \bigcirc\varphi$.

[38] Cf. Joosten 2006, p. 26: "Once the creative subject knows that $A \to B$, in a future world, where he gets to know A he shall obtain B but possibly at some later time as he might need to perform some calculations."

Thus, as Dummett suggested, in Beth semantics φ is assertible iff it is known that φ will be verified.

One may abstract from this particular view about assertibility and distill a more general view as follows: there is a set $V(\varphi)$ of states in which φ is verified and a set $jV(\varphi)$ of states in which φ is assertible. Whatever one's view of assertibility, verification should be sufficient for assertibility, so j should be inflationary. Moreover, one could reasonably adopt a notion of assertibility according to which if it is assertible that some statement is assertible, then that statement is indeed assertible, so j should be idempotent. Finally, it is also reasonable that a conjunction is assertible iff each conjunct is assertible, so j should be multiplicative. In this way, the axioms of a nucleus—not just a closure operator—are motivated.

Dummett's result (6) can also be explained in terms of nuclei. Dummett's definition of $\Vdash$ uses the standard definitions of $\wedge$ and $\vee$ in $\mathsf{Up}(X)$ but changes the definition of implication. One can easily check that the operation $\to'$ on $\mathsf{Up}(X)$ corresponding to the bulleted clause for implication above is defined by $U \to' V = U \to j_b V$, where $\to$ is the standard implication, defined by $A \to B = \{x \in X \mid {\uparrow}x \cap A \subseteq B\}$, and j_b is the Beth nucleus. More generally, given any nuclear algebra (H, j), we can define a new algebra $D(H, j)$ (in the same signature as H) by changing only the definition of implication to $a \to^j b = a \to jb$, which is equivalent to[39]

$$a \to^j b = ja \to jb.$$

Definition 3.28. We call $D(H, j)$ a *Dummett algebra*.

Dummett's (6) is a consequence of an algebraic fact: j_b is a homomorphism from the Dummett algebra $D(\mathsf{Up}(X), j_b)$, in which the verification clauses evaluate formulas, to the algebra $\mathsf{Up}(X)_{j_b}$ of fixpoints, in which the Beth satisfaction clauses evaluate formulas. This is in turn a consequence of the following more general fact.

Lemma 3.29. If H is a Heyting algebra and j a nucleus on H, then j is a $(\wedge, \vee, \to)$-homomorphism from the Dummett algebra $D(H, j)$ to the algebra H_j of fixpoints.

Proof. That j is a lattice homomorphism was noted in Theorem 3.20. For implication, by the definitions of $\to^j$ and $\to_j$, we have $j(a \to^j b) = j(ja \to jb) = ja \to jb = ja \to_j jb$, where the second equality uses inflationarity from right to left and distribution over $\to$ and idempotence from left to right. $\square$

Above we noted that the theorems of IPC are exactly the formulas that are always verified in Dummett's sense. Completeness follows from Dummett's (6) together with Beth completeness: if $\mathsf{IPC} \nvdash \varphi$, then there is some state in a Beth model that does not satisfy φ, which by (6) implies that there is some path along which φ is never verified. For soundness, below we will prove the more general result that all theorems of IPC evaluate to 1 in every Dummett algebra, so in particular they are verified at all states in all posets.

Remark 3.30. Care is needed if one wishes to define a relation of *consequence* based on Dummett's notion of verification. If one were to define $\Gamma \Vdash \varphi$ iff whenever a state verifies all formulas in Γ, it verifies φ, then the *detachment property* would fail: $\Gamma \Vdash \varphi \to \psi$ would not guarantee that $\Gamma \cup \{\varphi\} \Vdash \psi$. For example, $(p \wedge (p \to q)) \to q$ is always verified, yet a state x may verify $p \wedge (p \to q)$ without verifying q, as the verification of q awaits a later state. The failure of detachment is avoided by defining $\Gamma \Vdash \varphi$ iff whenever x verifies all formulas in Γ, every path through x eventually reaches a state y that verifies φ.

The heart of the issue in Remark 3.30 is that Dummett algebras often fail to be Heyting algebras, as the following example shows.

[39]That $ja \to jb \leq a \to jb$ follows from inflationarity. That $a \to jb \leq ja \to jb$ follows because $a \to jb \leq j(a \to jb) \leq ja \to jjb = ja \to jb$, using inflationarity, distribution over $\to$, and idempotence.

Example 3.31. Consider a two-element poset X and its associated nuclear algebra $(\mathsf{Up}(X), j_b)$ shown in Figure 3.10 with the Beth nucleus indicated by dashed arrows. The associated Dummett algebra $D(\mathsf{Up}(X), j_b)$ is not a Heyting algebra; for we have $\{0,1\} \to^{j_b} \{1\} = \{0,1\} \to j_b\{1\} = \{0,1\} \to \{0,1\} = \{0,1\}$, and yet $\{0,1\} \not\subseteq \{1\}$, so $\to^{j_b}$ violates the residuation condition (1) in the definition of a Heyting algebra. Nonetheless, in this Dummett algebra, as in all Dummett algebras, all theorems of IPC still evaluate to 1.

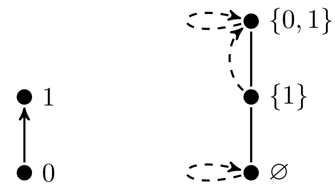

Figure 3.10: A poset X and its associated nuclear algebra $(\mathsf{Up}(X), j_b)$.

Theorem 3.32. $\mathsf{IPC} \vdash \varphi$ iff φ is valid in all Dummett algebras.

To prove Theorem 3.32, we can use the language of lax logic introduced in Remark 3.27. Define a translation $(\cdot)^D$ from the propositional language to the lax language as follows:

- $p^D = p$;
- $(\varphi \wedge \psi)^D = \varphi^D \wedge \psi^D$;
- $(\varphi \vee \psi)^D = \varphi^D \vee \psi^D$;
- $(\varphi \to \psi)^D = \bigcirc\varphi^D \to \bigcirc\psi^D$.

Then the following lemma is immediate from the definitions.

Lemma 3.33. A propositional formula φ is valid in a Dummett algebra $D(H, j)$ iff φ^D is valid in the nuclear algebra (H, j).

To prove Theorem 3.32, we simply combine Lemma 3.33 together with the following.

Lemma 3.34. A propositional formula φ is valid in all Heyting algebras iff φ^D is valid in all nuclear algebras.

Proof. For the right-to-left direction, if φ is refuted in a Heyting algebra, then adding the identity nucleus to this Heyting algebra produces a nuclear algebra that refutes φ^D.

For the left-to-right direction, we prove by induction on propositional formulas φ that if φ^D can be refuted in a nuclear algebra, then φ can be refuted in a Heyting algebra. The base case is obvious since each propositional letter can be refuted in a Heyting algebra, and $\bot$ is refuted in every nontrivial Heyting algebra. The $\wedge$ case is straightforward using the inductive hypothesis. For $\vee$, suppose $(\varphi_1 \vee \varphi_2)^D = (\varphi_1^D \vee \varphi_2^D)$ is refuted in a nuclear algebra (H, j). Hence both φ_1^D and φ_2^D are refuted in (H, j). Then by the inductive hypothesis, there is a Heyting algebra H_1 refuting φ_1 and a Heyting algebra H_2 refuting φ_2. We now apply the standard construction of taking the product of H_1 and H_2 and adding a new top element to refute $\varphi_1 \vee \varphi_2$.

For $\to$, suppose $(\varphi_1 \to \varphi_2)^D = \bigcirc\varphi_1^D \to \bigcirc\varphi_2^D$ is refuted in a nuclear algebra (H, j). Then since $ja \to jb = j(ja \to jb)$, it follows that $\bigcirc(\bigcirc\varphi_1^D \to \bigcirc\varphi_2^D) = \bigcirc(\varphi_1 \to \varphi_2)^D$ is also refuted in (H, j). We will use this to show that $\varphi_1 \to \varphi_2$ is refuted in the fixpoint algebra H_j. To this end, we define a translation $(\cdot)^\sharp$ from the lax language to the lax language and a translation $(\cdot)_\flat$ from the lax language to the propositional language[40]:

$$\begin{array}{ccc@{\qquad\qquad}ccc} p^\sharp & = & \bigcirc p & p_\flat & = & p \\ (\varphi\wedge\psi)^\sharp & = & \varphi^\sharp \wedge \psi^\sharp & (\varphi\wedge\psi)_\flat & = & \varphi_\flat \wedge \psi_\flat \\ (\varphi\to\psi)^\sharp & = & \varphi^\sharp \to \psi^\sharp & (\varphi\to\psi)_\flat & = & \varphi_\flat \to \psi_\flat \\ (\varphi\vee\psi)^\sharp & = & \bigcirc(\varphi^\sharp \vee \psi^\sharp) & (\varphi\vee\psi)_\flat & = & \varphi_\flat \vee \psi_\flat \\ \bot^\sharp & = & \bigcirc\bot & \bot_\flat & = & \bot \\ (\bigcirc\varphi)^\sharp & = & \bigcirc\varphi^\sharp & (\bigcirc\varphi)_\flat & = & \varphi_\flat. \end{array}$$

Now we reason as follows:

1. For any propositional formula ψ and valuation v for (H, j), we have $v((\psi^D)^\sharp) = v(\bigcirc\psi^D)$. This is proved by induction on ψ, using $ja \wedge jb = j(a \wedge b)$ and $j(ja \vee jb) = j(a \vee b)$ in the $\wedge$ and $\vee$ cases, and using idempotence and $ja \to jb = j(ja \to jb)$ in the $\to$ case.

2. For any lax formula χ, if $\chi^\sharp$ is refuted in (H, j), then $(\chi^\sharp)_\flat$ is refuted in H_j. To see this, given a valuation v for (H, j), define a valuation $v^\sharp$ for H_j by setting $v^\sharp(p) = jv(p)$ and extending $v^\sharp$ to complex formulas using the operations of H_j as usual. An easy induction shows that for any lax formula χ, we have $v(\chi^\sharp) = v^\sharp((\chi^\sharp)_\flat)$. Thus, if v refutes $\chi^\sharp$ in (H, j), then $v^\sharp$ refutes $(\chi^\sharp)_\flat$ in H_j.

3. An easy induction shows that for any propositional formula ψ, we have $((\psi^D)^\sharp)_\flat = \psi$

4. Combining steps 1–3, we have that if $\bigcirc\psi^D$ is refuted in (H, j), then ψ is refuted in H_j.

Setting $\psi := \varphi_1 \to \varphi_2$ completes the proof. □

Thus, Theorem 3.32 is proved, showing that Dummett's notion of verification provides another semantics—a kind of hybrid of Kripke and Beth semantics—with respect to which IPC is sound and complete.

Having explained Beth semantics in terms of nuclei, it is important to note that nuclear frames are more general than Beth frames. In one sense, this is obvious: there are many nuclei on the upsets of a poset distinct from the Beth nucleus. Less obviously: not every locale can be realized as the fixpoints of the Beth nucleus on a poset (see § 4.4). Next we will see a way of overcoming these limitations of Beth semantics.

3.3 Dragalin Semantics

Beth semantics differs from nuclear semantics in two respects. First, each poset uniquely determines the Beth nucleus on its upsets, whereas in nuclear frames we can vary the nucleus that we attach to a given poset. Second, the Beth nucleus can be naturally understood with a picture of quantifying over paths of information growth, whereas it is not at all obvious whether all nuclei can be understood in a similar way.

In this section, we will consider a semantics due to Dragalin [1979, 1988] that in a way combines the best of both nuclear and Beth semantics—the flexibility of being able to vary the nucleus attached to a given poset, plus the naturalness of thinking in terms of progressions toward richer information or more refined possibilities. Later in § 4.6 we will show that *all* nuclei on the upsets of a poset can be thought about in this way.

[40] The $(\cdot)_\flat$ translation is also used in Fairtlough and Mendler 1997, Thm. 2.4 (under different notation).

The key to getting from Beth semantics to Dragalin semantics is to liberalize the notion of a path from Definition 3.1 (chain closed under upper bounds) to allow more varied kinds of sets. For instance, the following example shows how we can be more liberal in not requiring the sets to be chains.

Example 3.35. Let us say that a *direction* I in a poset X is a nonempty upward directed set (if $x, y \in I$, then there is a $z \in I$ such that $x \leq z$ and $y \leq z$) that is closed under upper bounds. For $x \in X$, by a *direction through* x we mean a direction I with $x \in I$. Now define a function j_d on $\mathsf{Up}(X)$ by

$$j_d U = \{x \in X \mid \text{every direction through } x \text{ intersects } U\}. \tag{7}$$

It is straightforward to verify that j_d is a nucleus on $\mathsf{Up}(X)$.[41] To see that j_d may differ from the Beth nucleus j_b, let X be the poset of all countable subsets of $\mathbb{R}$ ordered by inclusion. Let U be the upset of all countably infinite sets. For any $r \in \mathbb{R}$, we claim that $\{r\} \in j_b U$ but $\{r\} \notin j_d U$. First, any chain C closed under upper bounds must contain a countably infinite set. For if every set in C were finite, then the union of C would be countable, in which case C would not be closed under upper bounds in X. Thus, every chain closed under upper bounds intersects U, whence $\{r\} \in j_b U$. But now consider the directed set I of all finite sets. This is closed under upper bounds in X, because the union of I is $\mathbb{R}$, which is uncountable and hence not in X. Thus, not every directed set closed under upper bounds intersects U, whence $\{r\} \notin j_d U$.

Paths and directions, as well as the traces of Remark 3.15, are examples of what we will call 'developments'. We will use the letters S and T for developments and the letters s and t for elements of developments, which we call 'stages' of the developments. The idea of Dragalin semantics is to add to a poset X a function D that assigns to each $s \in X$ a set $D(s)$ of developments. With natural constraints on D given in Definition 3.36 below, the pair (X, D) will generate a nucleus on $\mathsf{Up}(X)$ in exactly the way expected from the definitions of j_b and j_d in (5) and (7):

$$j_D U = \{s \in X \mid \text{every development in } D(s) \text{ intersects } U\}. \tag{8}$$

For the following definition, given developments S and T, if $S \subseteq {\downarrow}T$, so $\forall s \in S\, \exists t \in T : s \leq t$ (every stage of development in S is extended by a stage of development in T), then we say that S is *bounded by* T.

Definition 3.36. A *Dragalin frame*[42] is a pair (X, D) where X is a poset and D is a function from X to $\wp(\wp(X))$, called a *Dragalin function*, that satisfies the following properties for all $s, t \in X$:

(1°) $\varnothing \notin D(s)$.

Intuitively: the empty set is not a development of anything.

[41]If $x \leq y$ and I is a direction through y, then $I \cup \{x\}$ is a direction through x, which implies $j_d U \in \mathsf{Up}(X)$ for every $U \in \mathsf{Up}(X)$. Clearly $U \subseteq j_d U$. For $j_d j_d U \subseteq j_d U$, if $x \in j_d j_d U$ and I is a direction through x, then I intersects $j_d U$ at some point y. Thus, I is a direction through y, which with $y \in j_d U$ implies that I intersects U, which shows $x \in j_d U$. For $j_d U \cap j_d V = j_d(U \cap V)$, the right-to-left inclusion is obvious. From left to right, if $x \in j_d U \cap j_d V$ and I is a direction through x, then I intersects U at some point y. Then by directedness, there is a $y' \in I$ such that $y \leq y'$ and $x \leq y'$. Since $U, j_d V \in \mathsf{Up}(X)$, from $y \leq y'$ we have $y' \in U$, and from $x \leq y'$ we have $y' \in j_d V$. Now ${\uparrow}y' \cap I$ is a direction. Directedness is clear. For closure under upper bounds, we claim that if u is an upper bound of ${\uparrow}y' \cap I$, then it is also an upper bound of I. For if $v \in I$, then directedness of I implies there is a $z \in I$ such that $v \leq z$ and $y' \leq z$, so $z \in {\uparrow}y' \cap I$, which implies $z \leq u$ by our choice of u, which with $v \leq z$ implies $v \leq u$, so u is an upper bound of I. Hence $u \in I$, which with $y' \leq u$ gives us $u \in {\uparrow}y' \cap I$. Thus, ${\uparrow}y' \cap I$ is a direction, which with $y' \in U \cap j_d V$ implies that ${\uparrow}y' \cap I$ intersects $U \cap V$. Therefore, I intersects $U \cap V$, so $x \in j_d(U \cap V)$.

[42]Dragalin used the term 'Beth-Kripke frame'. To give due credit to Dragalin, we introduced the term 'Dragalin frame' in Bezhanishvili and Holliday 2016. Note that Dragalin started with a preordered set, but there is no loss of generality in starting with a poset, as one can always take the skeleton of the preorder. Also note that Dragalin worked with downsets in the preorder. Since most of the subsequent literature on intuitionistic semantics works with upsets, we decided to follow the prevailing convention.

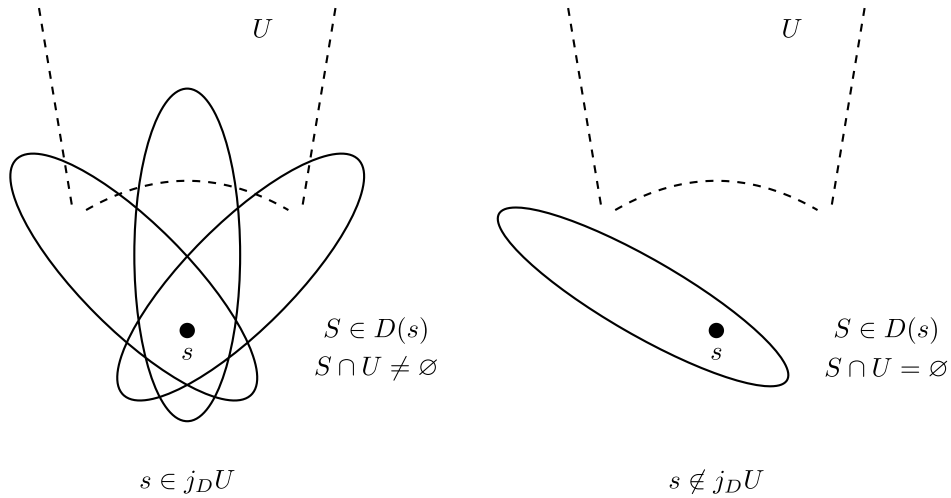

Figure 3.11: The condition for $s \in j_D U$ vs $s \notin j_D U$.

(2°) if $t \in S \in D(s)$, then $\exists x \in S : s \leq x$ and $t \leq x$.

Intuitively: every stage t in a development of s is at least *compatible* with s, in that s and t have a common extension x.

(3°) if $s \leq t$, then $\forall T \in D(t)\ \exists S \in D(s) : S \subseteq {\downarrow}T$.

Intuitively: if at some "future" stage t it will be possible to follow a development T, then it is already possible to follow a development bounded by T.

(4°) if $t \in S \in D(s)$, then $\exists T \in D(t) : T \subseteq {\downarrow}S$.

Intuitively: we "can always stay inside" a development, in the sense that for every stage t in S, we can follow a development T from t that is bounded by S.

A Dragalin frame is *normal* if the set $F = \{s \in X \mid D(s) = \varnothing\}$ of *fallible states* is empty.[43]

The following key lemma is due to Dragalin [1988, pp. 72–3] (cf. Prop. 3.1 of Bezhanishvili and Holliday 2016).

Lemma 3.37. For any Dragalin frame (X, D), the function j_D defined in (8) is a nucleus on $\mathsf{Up}(X)$. Thus, (X, j_D) is a nuclear frame, and hence $\mathsf{Up}(X)_{j_D}$ is a locale. Moreover, j_D is dense iff (X, D) is normal.

Remark 3.38. The conditions of Dragalin frames (in particular, the condition (4°)) are more than one needs to prove Lemma 3.37. In Bezhanishvili and Holliday 2018, we introduce a more general concept of *development frame* in which the conditions on D actually *correspond*, in the precise sense of correspondence theory [van Benthem, 2001], to the axioms of nuclei. We will not need this more general concept here, but it is used in Bezhanishvili and Holliday 2018 to relate frames like Dragalin's to the *localic cover systems* of Goldblatt 2011 and, more generally, to relate Beth-Dragalin style "path" or "development" semantics to Scott-Montague style "neighborhood" semantics [Scott, 1970, Montague, 1970].

[43]Dragalin [1988, p. 73] called this F the set of *strange worlds*.

We can now regard Beth frames as normal Dragalin frames: for any poset X, if for every $s \in X$, $D(s)$ is the set of all paths (resp. directions, traces) through s, then (X, D) is a normal Dragalin frame.

One can motivate still stronger conditions on D than those in Definition 3.36. For example, let a *path starting from s* be a path in $\uparrow s$. It is easy to see that the Beth nucleus can be equivalently defined by

$$j_b U = \{s \in X \mid \text{every path starting from } s \text{ intersects } U\}.$$

Now if for every $s \in X$, $D(s)$ is the set of all paths starting from s, then the following stronger conditions are satisfied:

($2^{\circ\circ}$) if $S \in D(s)$, then $S \subseteq \uparrow s$.

Intuitively: the stages in a development starting from s are extensions of s.

($3^{\circ\circ}$) if $s \leq t$, then $D(t) \subseteq D(s)$.

Intuitively: developments that will be possible to follow at “future” stages are already possible to follow.

($4^{\circ\circ}$) if $t \in S \in D(s)$, then $\exists T \in D(t) : T \subseteq S$.

Intuitively: we “can always stay inside” a development in the stricter sense that for every state t in S, we can follow a development T from t that is included in S.

Similarly, these conditions are satisfied if we define a *direction starting from s* as a direction in $\uparrow s$ and for every $s \in X$, take $D(s)$ to be the set of all directions starting from s (and similarly for traces).

Definition 3.39. A Dragalin frame (resp. Dragalin function) is *standard* if it satisfies ($2^{\circ\circ}$)–($4^{\circ\circ}$).

The following fact is a consequence of Theorem 4.26 in § 4.6 and Lemma 3.37.

Lemma 3.40. For any Dragalin frame (X, D), there is a standard Dragalin frame (X, D') such that $j_D = j_{D'}$.

Although it suffices to restrict attention to standard Dragalin frames, it is useful to have the more general concept of Dragalin frame—both to see that the weaker conditions are sufficient for certain results (e.g., Lemma 3.37) and to easily relate Dragalin frames to other frames, as in the following example.

Example 3.41. Kripke frames may be regarded as normal Dragalin frames (X, D) in which for every $s \in X$, $D(s) = \{\{s\}\}$. Note, though, that Kripke frames so regarded are not standard Dragalin frames, as they satisfy neither ($3^{\circ\circ}$) nor ($4^{\circ\circ}$). Of course, by Lemma 3.40, Kripke frames can be regarded as standard Dragalin frames in a different way (see the proof of Theorem 4.26).

Since for any Dragalin frame (X, D), the function j_D is a nucleus, Dragalin semantics may be defined in exactly the same way as the nuclear semantics of § 3.2, but now using the nucleus j_D defined in (8). Thus, given $s \in X$ and a valuation v assigning elements of $\mathsf{Up}(X)$ to propositional letters, we use the standard clauses for $\wedge$ and $\to$ plus the following:

- $s \models_v \bot$ iff $D(s) = \varnothing$;
- $s \models_v p$ iff every $S \in D(s)$ intersects $v(p)$;
- $s \models_v \varphi \vee \psi$ iff every $S \in D(s)$ intersects $\{x \in X \mid x \models_v \varphi \text{ or } x \models_v \psi\}$.

Since we define $\neg\varphi$ as $\varphi \to \bot$, we have:

- $s \models_v \neg\varphi$ iff for all $t \geq s$, if $t \models_v \varphi$, then $t \in F$,

where F is the set of fallible states in (X, D) as in Definition 3.36.

IPC is sound with respect to Dragalin frames for the same reason it is sound with respect to nuclear frames, namely that we are evaluating formulas in the algebra of fixpoints of a nucleus on $\mathsf{Up}(X)$, and this algebra of fixpoints is always a Heyting algebra (Theorem 3.20). The completeness of IPC with respect to Dragalin frames is immediate from completeness with respect to Kripke frames, given Example 3.41. Moreover, the completeness of IPC with respect to standard Dragalin frames may be obtained from completeness with respect to Beth frames, plus the observation following Remark 3.38 that working with paths *starting from* s is equivalent to working with paths *through* s, and the former give us standard Dragalin frames.

Turning to intermediate logics, the properties characterizing Dragalin frames that validate intermediate axioms generalize the properties characterizing Beth frames that validate the axioms (recall Propositions 3.5 and 3.11).

Proposition 3.42. Let (X, D) be a Dragalin frame and $F = \{s \in X \mid D(s) = \varnothing\}$.

1. (X, D) validates $p \vee \neg p$ iff there is no $s \in X$, development $S \in D(s)$, and upset L such that $S \cap L = \varnothing$ and $S \subseteq {\downarrow}(L \setminus F)$;

2. (X, D) validates $\neg p \vee \neg\neg p$ iff there is no $s \in X$, development $S \in D(s)$, and sets L and M such that $(L \cup M) \cap F = \varnothing$, $S \subseteq {\downarrow}L \cap {\downarrow}M$, and $({\uparrow}L \cap {\uparrow}M) \setminus F = \varnothing$;

3. (X, D) validates $(p \to q) \vee (q \to p)$ iff there is no $s \in X$, development $S \in D(s)$, and sets L and M such that: $S \subseteq {\downarrow}L \cap {\downarrow}M$; for every $l \in L$, there is a development in $D(l)$ that does not intersect ${\uparrow}M$; and for every $m \in M$, there is a development in $D(m)$ that does not intersect ${\uparrow}L$.

Proof. (1) Suppose there are such s, S, and L. Let $v(p) = L$. Then since $S \subseteq {\downarrow}(L \setminus F)$, we have $t \not\models_v \neg p$ for all $t \in S$. In addition, we claim that $t \not\models_v p$ for all $t \in S$. By (4°), $t \in S \in D(s)$ implies that there is a $T \in D(t)$ such that $T \subseteq {\downarrow}S$. Then since $S \cap L = \varnothing$ and $L \in \mathsf{Up}(X)$, it follows that $T \cap L = \varnothing$, which implies $t \not\models_v p$. Thus, $t \not\models_v p$ and $t \not\models_v \neg p$ for all $t \in S$, which with $S \in D(s)$ implies $s \not\models_v p \vee \neg p$.

Conversely, suppose there is a valuation v such that $s \not\models_v p \vee \neg p$. Then there is an $S \in D(s)$ such that for all $t \in S$, we have $t \not\models_v p$ and $t \not\models_v \neg p$, so $t \notin j_D v(p)$ and there is a $t' \geq t$ with $t' \in j_D v(p) \setminus F$. Since this holds for all $t \in S$, we can take $L = j_D v(p)$ to complete the proof.

(2) Suppose there are such s, S, L, and M. Let $v(p) = {\uparrow}L$. First, we will show that $m \models_v \neg p$ for every $m \in M$. If not, then there is an $m' \in {\uparrow}m \setminus F$ such that $m' \models_v p$. Since $m' \notin F$, it follows that there is an $S' \in D(m')$ that intersects $v(p) = {\uparrow}L$ at some point t'. Then by (2°), there is an $x \in S'$ with $m' \leq x$ and $t' \leq x$. Since $m \in M$, from $m \leq m' \leq x$ we have $x \in {\uparrow}M$, and since $t' \in {\uparrow}L$, from $t' \leq x$ we have $x \in {\uparrow}L$. Thus, $x \in {\uparrow}L \cap {\uparrow}M$. Because $({\uparrow}L \cap {\uparrow}M) \setminus F = \varnothing$, it follows that $x \in F$. But by (4°), $x \in S' \in D(m')$ implies $D(x) \neq \varnothing$ and hence $x \notin F$, so we have a contradiction. This shows that $m \models_v \neg p$ for every $m \in M$. Then since $S \subseteq {\downarrow}M$ and $M \cap F = \varnothing$, we have $t \not\models_v \neg\neg p$ for every $t \in S$. Finally, since $S \subseteq {\downarrow}L$ and $L \cap F = \varnothing$, we have $t \not\models_v \neg p$ for every $t \in S$. Therefore, given $S \in D(s)$, we conclude that $s \not\models_v \neg p \vee \neg\neg p$.

Conversely, suppose there is a valuation v such that $s \not\models_v \neg p \vee \neg\neg p$. Then there is an $S \in D(s)$ such that for all $t \in S$, we have $t \not\models_v \neg p$ and $t \not\models_v \neg\neg p$. Thus, $({\uparrow}t \setminus F) \cap \{x \in X \mid x \models_v p\} \neq \varnothing$ and $({\uparrow}t \setminus F) \cap \{x \in X \mid x \models_v \neg p\} \neq \varnothing$. Taking $L = ({\uparrow}t \setminus F) \cap \{x \in X \mid x \models_v p\}$ and $M = ({\uparrow}t \setminus F) \cap \{x \in X \mid x \models_v \neg p\}$, clearly we have $(L \cup M) \cap F = \varnothing$, $S \subseteq {\downarrow}L \cap {\downarrow}M$, and $({\uparrow}L \cap {\uparrow}M) \setminus F = \varnothing$.

(3) If there are such sets, let $v(p) = \uparrow L$ and $v(q) = \uparrow M$. For every $l \in L$, there is a development in $D(l)$ that does not intersect $\uparrow M = v(q)$, so $l \not\models_v q$. Similarly, for every $m \in M$, $m \not\models_v p$. Then since $S \subseteq \downarrow L \cap \downarrow M$, for every $t \in S$, we have $t \not\models_v (p \to q)$ and $t \not\models_v (q \to p)$, which with $S \in D(s)$ implies $s \not\models_v (p \to q) \vee (q \to p)$.

Conversely, if $s \not\models_v (p \to q) \vee (q \to p)$, then there is an $S \in D(s)$ such that for every $t \in S$, we have $t \not\models_v p \to q$ and $t \not\models_v q \to p$. Thus, $t \in \downarrow\{x \in X \mid x \models_v p, x \not\models_v q\}$ and $t \in \downarrow\{x \in X \mid x \models_v q, x \not\models_v p\}$. Let $L = \{x \in X \mid x \models_v p, x \not\models_v q\}$ and $M = \{x \in X \mid x \models_v q, x \not\models_v p\}$, so $S \subseteq \downarrow L \cap \downarrow M$. For each $l \in L$, since $l \not\models_v q$, there is a development in $D(l)$ that does not intersect $v(q)$ and hence does not intersect $\uparrow M$, because $\uparrow M \subseteq v(q)$. Similarly, for each $m \in M$, there is a development in $D(m)$ that does not intersect $\uparrow L$. □

As a direct consequence of Proposition 3.42, we obtain the following.

Corollary 3.43. Let (X, D) be a normal Dragalin frame.

1. (X, D) validates $p \vee \neg p$ iff there is no $s \in X$, development $S \in D(s)$, and upset L such that $S \cap L = \varnothing$ and $S \subseteq \downarrow L$;
2. (X, D) validates $\neg p \vee \neg\neg p$ iff there is no $s \in X$, development $S \in D(s)$, and sets L and M such that $S \subseteq \downarrow L \cap \downarrow M$ and $\uparrow L \cap \uparrow M = \varnothing$;
3. (X, D) validates $(p \to q) \vee (q \to p)$ iff there is no $s \in X$, development $S \in D(s)$, and subsets L and M such that: $S \subseteq \downarrow L \cap \downarrow M$; for every $l \in L$, there is a development in $D(l)$ that does not intersect $\uparrow M$; and for every $m \in M$, there is a development in $D(m)$ that does not intersect $\uparrow L$.

Remark 3.44. Corollary 3.43 generalizes correspondence results for Beth semantics and Kripke semantics.

- If (X, D) is a Beth frame, so $D(s)$ is the set of paths through s, then Corollary 3.43.1 immediately gives us Proposition 3.5 for Beth semantics, and the rest of Corollary 3.43 immediately gives us Proposition 3.11 for Beth semantics.
- If (X, D) is a Kripke frame, so $D(s) = \{\{s\}\}$, then Corollary 3.43.1 says that $p \vee \neg p$ is valid iff the partial order X is discrete (for if $s \leq t$ and $s \neq t$, then we can take $S = \{s\}$ and $L = \uparrow t$); Corollary 3.43.2 says that $\neg p \vee \neg\neg p$ is valid iff for all $s \in X$ and $L, M \subseteq X$, $\{s\} \subseteq \downarrow L \cap \downarrow M$ implies $\uparrow L \cap \uparrow M \neq \varnothing$, which is easily seen to be equivalent to the Church-Rosser property as in § 2.3.1 (by taking L and M to be singletons); and Corollary 3.43.3 says that $(p \to q) \vee (q \to p)$ is valid iff $\{s\} \subseteq \downarrow L \cap \downarrow M$ implies that either there is an $l \in L$ such that $l \in \uparrow M$ or there is an $m \in M$ such that $m \in \uparrow L$, which is easily seen to be equivalent to upward linearity as in § 2.3.1 (by again taking L and M to be singletons).

Not only do Dragalin frames provide a unifying framework that covers both Kripke and Beth frames, but also any topological space can be transformed into a Dragalin frame that produces the same locale. This result, due to Dragalin [1979, 1988, pp. 75–6], will be proved in § 4.5. Then we will go further in § 4.6 and discuss the result from Bezhanishvili and Holliday 2016 that every locale is produced by some Dragalin frame. Thus, Dragalin frames provide a unifying framework of impressive generality.

As a result of the generality of Dragalin frames, we do not think it is possible to give a one-size-fits-all conceptual explanation of all Dragalin frames in the way that we tried to do for Kripke frames, Beth frames, and topological spaces in terms of information and verification. For those Dragalin frames in which each development in $D(s)$ is an upward directed set, an explanation can be given in terms of developments of information. We have $s \in j_D U$ iff it is known that any development of information will eventually reach U.

However, there are other Dragalin frames in which the developments in $D(s)$ are not directed, and they have a natural but different conceptual explanation. For example, the *possibility frames* for classical logic (with modalities) in Humberstone 1981, Holliday 2015, 2018, van Benthem et al. 2016 may be regarded as normal Dragalin frames (with additional modal accessibility relations) in which

$$D(s) = \{\uparrow t \mid t \in \uparrow s\}, \tag{9}$$

so the developments $\uparrow t$ of s may obviously fail to be directed. With this definition of D, we have $s \notin j_D U$ iff there is a $t \geq s$ such that for all $u \geq t$, $u \notin U$, so that j_D is the nucleus of double negation as in Remark 3.25.[44] The idea is that if a partial "possibility" s does not settle the truth of a proposition, then there is a "refinement" t of s that settles the proposition as *false*, so that no further refinement u of t settles the proposition as true. This "possibility semantics" way of thinking is quite different than the information-and-verification way of thinking, and yet both may be fit into Dragalin's framework. For this reason, we doubt that the class of all Dragalin frames has a univocal conceptual interpretation. Yet for the same reason, we may establish results for different kinds of frames in one fell swoop when we work with Dragalin frames.

4 Semantic Hierarchy

Each of the semantics from §§ 2–3 supplies a map σ from a class of structures to the class of Heyting algebras, as shown in Figure 4.1.

semantics	**class of structures**	**map to Heyting algebras**
Kripke	posets	$X \mapsto \mathsf{Up}(X)$
Beth	posets	$X \mapsto \mathsf{Up}(X)_{j_b}$
Topological	spaces	$X \mapsto \Omega(X)$
Nuclear	nuclear frames	$(X, j) \mapsto \mathsf{Up}(X)_j$
Dragalin	Dragalin frames	$(X, D) \mapsto \mathsf{Up}(X)_{j_D}$
Localic	locales	identity
Algebraic	Heyting algebras	identity

Figure 4.1: Maps associating structures with Heyting algebras.

For semantics S and S', we write $S \leq S'$ if every Heyting algebra in the image of σ_S is isomorphic to a Heyting algebra in the image of $\sigma_{S'}$; $S < S'$ if $S \leq S'$ but $S' \not\leq S$; and $S \equiv S'$ if $S \leq S'$ and $S' \leq S$. In this section, we establish that the following semantics form a hierarchy according to the relation $<$:

$$\text{Kripke} < \text{Beth} < \text{Topological} < \text{Dragalin} < \text{Algebraic}.$$

Figure 4.2 lists the results used to establish the non-strict and strict inequalities.

The semantic hierarchy above does not display nuclear or localic semantics because these are in fact equivalent to Dragalin semantics in the sense of the relation $\equiv$. In fact, there is yet another semantics equivalent to these, which we call "FM-semantics" after Fairtlough and Mendler 1997. We will discuss

[44]Possibility frames are not the only normal Dragalin frames that validate classical logic, as shown by Corollary 3.43.1. The classical Dragalin frames are exactly those in which j_D is the nucleus w_U from Example 3.22.3 for some $U \in \mathsf{Up}(X)$. In § 4.6 (proof of Theorem 4.26), we will see how to define the Dragalin function D so that $j_D = w_U$ for any given $U \in \mathsf{Up}(X)$.

semantics	$\leq$	$<$
Kripke vs. Beth	Theorem 4.15	Examples 4.16, 4.17
Beth vs. Topological	Theorem 4.19	Theorem 4.21, Example 4.22
Topological vs. Dragalin	Theorem 4.23	Example 4.24
Dragalin vs. Algebraic	Lemma 3.37	Not every Heyting algebra is complete.

Figure 4.2: Results establishing the semantic hierarchy.

FM-semantics in § 4.7. Thus, we can expand the Dragalin place as follows:

$$\text{Locales} \equiv \text{Nuclear} \equiv \text{Dragalin} \equiv \text{FM}.$$

Figure 4.3 lists the results used to establish these equivalences.

semantics	$\equiv$
Localic vs. Nuclear	Theorem 3.24
Nuclear vs. Dragalin	$\leq$ by Theorem 4.26, $\geq$ by Lemma 3.37
Dragalin vs. FM	$\leq$ by Theorem 4.30, $\geq$ by Lemma 4.28 and Theorem 4.26

Figure 4.3: Results establishing the equivalence of semantics.

One could also add to the equivalences above the cover semantics studied by Goldblatt [2011] (cf. Goldblatt 1981), which is compared to the other semantics in Bezhanishvili and Holliday 2018.

4.1 From Kripke to Beth

Kripke [1965, pp. 108–9] showed how to turn any Kripke frame X into a Beth frame Y in such a way that any valuation refuting a formula φ in X according to Kripke semantics can be transferred to a valuation refuting φ in Y according to Beth semantics. From our perspective, what Kripke showed is how to turn any poset X into a poset Y such that $\mathsf{Up}(X)$ embeds as a Heyting algebra into $\mathsf{Up}(Y)_{j_b}$. Before describing the construction, we will give a sufficient condition for the existence of such an embedding. As in the literature on modal logic, a *p-morphism* from a poset X to a poset X' is a map $f : X \to X'$ such that

$$f[{\uparrow}x] = {\uparrow}f(x),$$

or equivalently, $f^{-1}[{\downarrow}x'] = {\downarrow}f^{-1}(x')$. This is usually expressed as f being order preserving ($x \leq y$ implies $f(x) \leq f(y)$) and satisfying the "back" condition that if $f(x) \leq y'$, then there is a $y \in X$ such that $x \leq y$ and $f(y) = y'$, as shown in Figure 4.4.

Definition 4.1. Given posets X and Y, a *BK-morphism* (for Beth-to-Kripke) from Y to X is a p-morphism such that for all $y \in Y$ and $U \in \mathsf{Up}(X)$, if $f(y) \notin U$, then there is a path C through y such that $f[C] \cap U = \varnothing$ (equivalently, $C \cap f^{-1}[U] = \varnothing$), as in Figure 4.5.

Lemma 4.2. Given posets X and Y, if f is a BK-morphism from Y to X, then f^{-1} is a Heyting homomorphism of $\mathsf{Up}(X)$ to $\mathsf{Up}(Y)_{j_b}$ preserving arbitrary meets and joins. Moreover, if f is onto, then f^{-1} is a Heyting embedding of $\mathsf{Up}(X)$ into $\mathsf{Up}(Y)_{j_b}$.

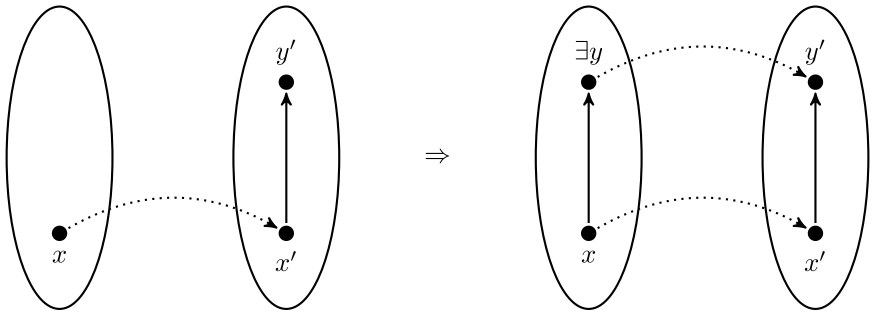

Figure 4.4: p-morphism condition.

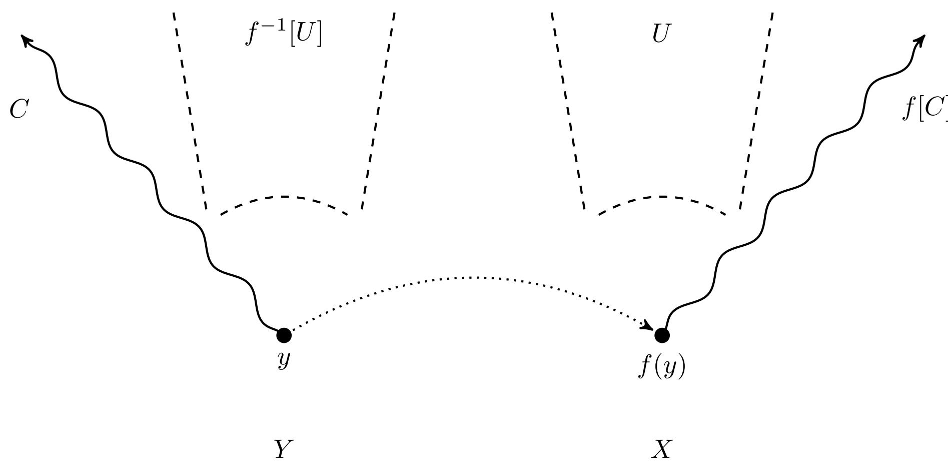

Figure 4.5: BK-morphism from Y to X.

Proof. First, we observe that $f^{-1}[U] = j_b f^{-1}[U]$. The left-to-right inclusion follows from inflationarity, while the right-to-left inclusion follows from the BK property that f satisfies in addition to being a p-morphism. Then to see that f^{-1} preserves joins, recall that the join in $\mathsf{Up}(X)$ is union while the join in $\mathsf{Up}(Y)_{j_b}$ is j_b applied to union, so that

$$\begin{array}{rcl} f^{-1}\left[\bigvee\{U_i \mid i \in I\}\right] & = & f^{-1}\left[\bigcup\{U_i \mid i \in I\}\right] \\ & = & j_b f^{-1}\left[\bigcup\{U_i \mid i \in I\}\right] \\ & = & j_b \bigcup\{f^{-1}[U_i] \mid i \in I\} \\ & = & \bigvee\{f^{-1}[U_i] \mid i \in I\}. \end{array}$$

Since the meet in both $\mathsf{Up}(X)$ and $\mathsf{Up}(Y)_{j_b}$ is intersection, f^{-1} preserves meets; since j_b is a dense nucleus, f^{-1} preserves $\varnothing$; and since f is a p-morphism, f^{-1} preserves implication. Finally, if f is onto and for $U, V \in \mathsf{Up}(X)$, we have $x \in U \setminus V$, then there is a $y \in Y$ such that $f(y) = x \in U \setminus V$, so $f^{-1}[U] \neq f^{-1}[V]$; hence f^{-1} is injective. $\square$

Remark 4.3.

1. The notion of BK-morphism in Definition 4.1 can be seen as a special case of the following more general notion. Given nuclear frames (X, j) and (Y, k), a map $f : Y \to X$ is a *nuclear p-morphism* if it is a p-morphism such that $f^{-1}[jU] = kf^{-1}[U]$. This ensures that f^{-1} is a nucleus-preserving homomorphism from the nuclear algebra $(\mathsf{Up}(X), j)$ to the nuclear algebra $(\mathsf{Up}(Y), k)$; and if f is onto, then f^{-1} is an embedding, in which case $\mathsf{Up}(X)_j$ embeds into $\mathsf{Up}(Y)_k$. Definition 4.1 and Lemma 4.2 are the special case of this where (Y, k) is a Beth frame, so $k = j_b$, and (X, j) is a Kripke frame, so j is the identity nucleus.

2. Another example of a nuclear p-morphism comes from Remark 3.15: the function from the Beth comb to the linear order $\omega + 1$ sending the spine of the comb to the natural numbers and the teeth of the comb to ω as in Figure 3.8 is a nuclear p-morphism from the nuclear frame consisting of the Beth comb with the Beth nucleus onto the nuclear frame consisting of the linear order $\omega + 1$ with the trace nucleus (defined in the same way as the Beth nucleus, but with traces in place of paths). For further discussion of nuclear p-morphisms, see Bezhanishvili and Holliday 2018.

We are now ready to present Kripke's construction of Beth frames from Kripke frames.[45]

Notation 4.4. From now until the end of § 4.2 we will reserve $\leq$ for the standard ordering on the natural numbers, and we will use $\sqsubseteq$ for the partial orders in our Kripke and Beth frames, with $\sqsubset$ defined by $x \sqsubset y$ iff $x \sqsubseteq y$ and $x \neq y$. Unlike in previous sections, we will explicitly display the partial order with each poset.

Definition 4.5. Given a poset $(X, \sqsubseteq)$, we define its *Beth unraveling* $(X_u, \sqsubseteq_u)$ by:

1. X_u is the set of all nonempty finite sequences $\langle x_1, \dots, x_n\rangle$ of elements from X such that for $1 \leq i < n$, x_i is not an endpoint and $x_i \sqsubseteq x_{i+1}$;

2. for all $\sigma, \sigma' \in X_u$, $\sigma \sqsubseteq_u \sigma'$ iff σ is an initial segment of σ'.

Example 4.6. Figure 4.6 shows the Beth unraveling of the two-point Kripke frame refuting $p \vee \neg p$ from Figure 2.5. If we delete the disconnected point $\langle b\rangle$, then the result is isomorphic to the Beth comb refuting $p \vee \neg p$ from Figure 3.1.

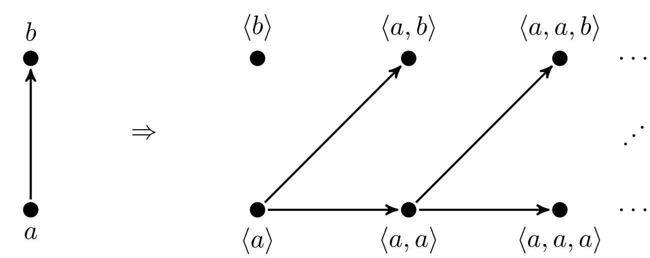

Figure 4.6: Beth unraveling (right) of a Kripke frame (left).

Remark 4.7. The Beth unraveling is infinite if X is not discrete. If one wants a tree, one can define for any $r \in X$ the *Beth unraveling from* r, $(X_u^r, \sqsubseteq_u^r)$, in the same way as in Definition 4.5, except we require that every sequence in X_u^r starts with r (as in Kripke 1965). For simplicity, we will stay with Definition 4.5.

[45] As in Dummett 2000, pp. 139–40, we do not repeat endpoints infinitely as in the definition from Kripke 1965, p. 108.

To apply Lemma 4.2 to Beth unraveling, we use the obvious map from the unraveling of X onto X.

Lemma 4.8. The function $l : X_u \to X$ that sends each sequence σ to the last member of σ is a surjective BK-morphism.

Proof. Clearly $\sigma \sqsubseteq_u \sigma'$ implies $l(\sigma) \sqsubseteq l(\sigma')$ by the definition of $(X_u, \sqsubseteq_u)$. In addition, if $l(\sigma) \sqsubseteq x$, then the concatenation $\sigma^\frown x$ of σ and the sequence $\langle x \rangle$ is such that $\sigma^\frown x \in X_u$, $\sigma \sqsubseteq_u \sigma^\frown x$, and $l(\sigma^\frown x) = x$. Thus, l is a p-morphism. To see that l satisfies the additional condition for a BK-morphism, suppose $l(\sigma) \notin U$. If $l(\sigma)$ is not an endpoint, then $\sigma, \sigma^\frown l(\sigma), \sigma^\frown l(\sigma)^\frown l(\sigma), \dots$ is a path through σ that does not intersect $l^{-1}[U]$. If $l(\sigma)$ is an endpoint, then σ itself provides such a path. Thus, l satisfies the additional condition. Finally, every element of X appears as the last member of a sequence in X_u, so l is surjective. □

Together Lemmas 4.2 and 4.8 imply that Beth unraveling yields the desired embedding of the algebra associated with the Kripke frame into the algebra associated with the Beth frame:

Proposition 4.9. For any poset X, there is a Heyting embedding of $\mathsf{Up}(X)$ into $\mathsf{Up}(X_u)_{j_b}$.

Proposition 4.9 together with the Kripke completeness of IPC gives us the Beth completeness of IPC: by Kripke completeness, any non-theorem of IPC can be refuted in a poset according to Kripke semantics, whence by Proposition 4.9 it can be refuted in a poset according to Beth semantics.

To prove Beth completeness with respect to specific kinds of posets that do not arise as Beth unravelings, we can still use the notion of BK-morphism, as in the proof of the following noteworthy fact.

Theorem 4.10. IPC is complete, according to Beth semantics, with respect to the full countable binary tree.

Proof. Let $\mathcal{T}$ be the full countable binary tree, based on the set of all finite sequences of 0's and 1's. First, we recall that the sequence of *Jaśkowski frames* is defined inductively as follows [Dummett and Lemmon, 1959, p. 258]:

- $\mathfrak{F}_1$ is the one-point poset;
- $\mathfrak{F}_{n+1}$ is obtained by taking the disjoint union of n copies of $\mathfrak{F}_n$ and then adding a new root.

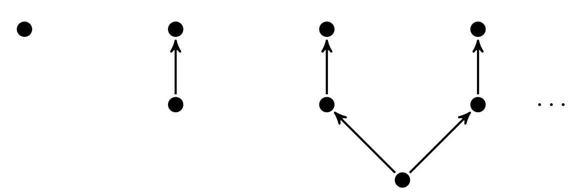

Figure 4.7: Jaśkowski frames.

It is well known that IPC is complete with respect to the class of Jaśkowski frames (see, e.g., Surma et al. 1975 or Dummett 2000, p. 136).[46] Kirk [1979] observed that for every Jaśkowski frame $\mathfrak{F}_n$, there is a surjective p-morphism from $\mathcal{T}$ onto $\mathfrak{F}_n$.[47] To prove the theorem, it suffices to show that these surjective p-morphisms are BK-morphisms and then apply Lemma 4.2.

[46] IPC is also complete with respect to the class of frames defined as follows: $\mathfrak{F}_1$ is the one-point poset; $\mathfrak{F}_{n+1}$ is obtained by taking the disjoint union of $n+1$ copies of $\mathfrak{F}_n$ and then adding a new root (again see Surma et al. 1975 or Dummett 2000, p. 136). Some authors call these frames the "Jaśkowski frames."

[47] More generally, for every finite rooted Kripke frame $\mathfrak{F}$, there is a p-morphism from $\mathcal{T}$ onto $\mathfrak{F}$; this result was proved independently by D. Gabbay, J. van Benthem, and A. G. Dragalin (see Goldblatt 1980, p. 222 and the editor's note on p. 236).

The function mapping every point of $\mathcal{T}$ to the single point of $\mathfrak{F}_1$ is clearly a BK-morphism. For induction, suppose we have a BK-morphism f from $\mathcal{T}$ onto $\mathfrak{F}_n$. Let $\mathfrak{G}_1 = (G_1, \leq_1), \dots, \mathfrak{G}_n = (G_n, \leq_n)$ be the n-copies of $\mathfrak{F}_n$ inside $\mathfrak{F}_{n+1}$, so for $1 \leq i \leq n$, we have a BK-morphism f_i from $\mathcal{T}$ onto $\mathfrak{G}_i$. As usual, we denote the sequence $\underbrace{1 \dots 1}_{n \text{ times}}$ by 1^n. For each $m \geq 1$, let $\mathcal{T}_m$ be the tree of all binary sequences of the form $1^{m-1}0\sigma$ for an arbitrary binary sequence σ, as shown in Figure 4.8. Since the subtree $\mathcal{T}_m$ of $\mathcal{T}$ is isomorphic to $\mathcal{T}$, each BK-morphism f_i above gives us a BK-morphism $f_{i,m}$ from $\mathcal{T}_m$ onto $\mathfrak{G}_i$. We now define a function $f : \mathcal{T} \to \mathfrak{F}_{n+1}$ as follows, where r is the root of $\mathfrak{F}_{n+1}$:

$$f(\sigma) = \begin{cases} r & \text{if } \sigma = 1^k \text{ for some } k \geq 0 \\ f_{i,m}(\sigma) & \text{if } \sigma \in \mathcal{T}_m \text{ and } m \equiv i \pmod{n}. \end{cases}$$

One can easily check that f is a surjective p-morphism. To see that it is a BK-morphism, suppose $U \in \mathsf{Up}(\mathfrak{F}_{n+1})$ and $f(\sigma) \notin U$, so $r \notin U$. We must show there is a path C through σ in $\mathcal{T}$ such that $f[C] \cap U = \varnothing$.

If $f(\sigma) = r$, then $\sigma, \sigma 1, \sigma 11, \dots$ is a path C such that $f[C] \cap U = \{r\} \cap U = \varnothing$.

If $f(\sigma) = f_{i,m}(\sigma)$, then we are given the desired path C by our assumption that $f_{i,m}$ is a BK-morphism. Specifically, $f(\sigma) \notin U \in \mathsf{Up}(\mathfrak{F}_{n+1})$ implies $f_{i,m}(\sigma) \notin U \cap G_i \in \mathsf{Up}(\mathfrak{G}_i)$, which implies there is a path C in $\mathcal{T}_m$ such that $f_{i,m}[C] \cap U \cap G_i = \varnothing$, so $f_{i,m}[C] \cap U = \varnothing$ given $f_{i,m}[C] \subseteq G_i$. Then C is also a path in $\mathcal{T}$, and since $f(\sigma) = f_{i,m}(\sigma)$ implies $f(\tau) = f_{i,m}(\tau)$ for all τ extending σ in $\mathcal{T}$, from $f_{i,m}[C] \cap U = \varnothing$ we conclude $f[C] \cap U = \varnothing$. ☐

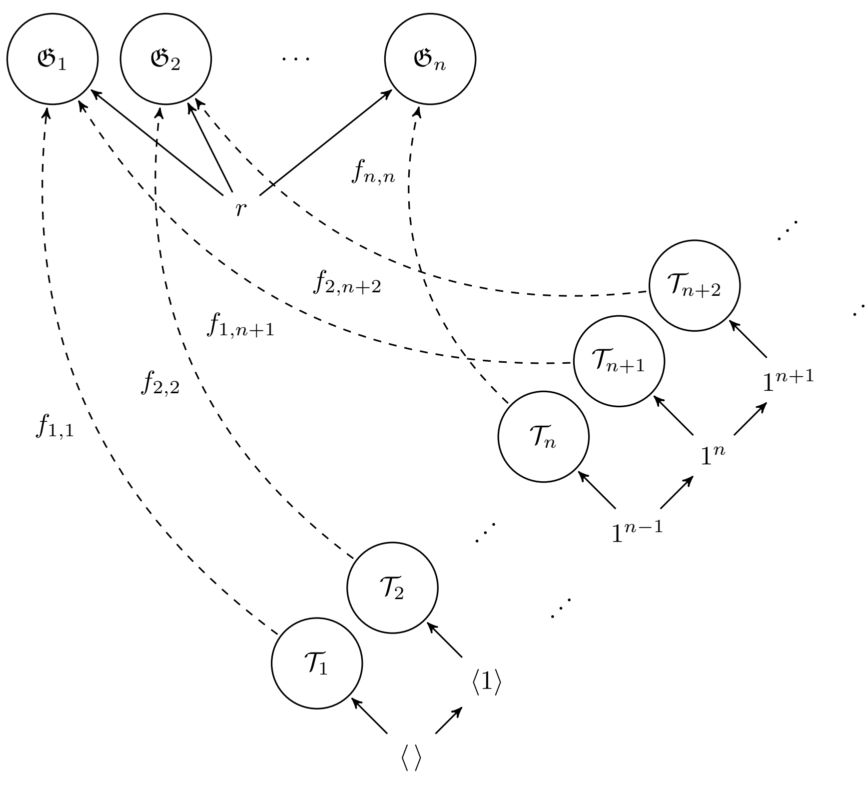

Figure 4.8: Part of Kirk's map from $\mathcal{T}$ onto $\mathfrak{F}_{n+1}$.

Returning to the topic of Beth unraveling, it is important to note that while Proposition 4.9 gives us an embedding of $\mathsf{Up}(X)$ into $\mathsf{Up}(X_u)_{j_b}$, it almost never gives us an isomorphism.

Example 4.11. For the Kripke frame X shown in Figure 4.6, $\mathsf{Up}(X)$ is clearly not isomorphic to $\mathsf{Up}(X_u)_{j_b}$. In fact, $\mathsf{Up}(X_u)_{j_b}$ does not even belong to the variety generated by $\mathsf{Up}(X)$: $\neg p \vee \neg\neg p$ is valid in the Kripke frame, but it can be refuted at $\langle a \rangle$ in the Beth unraveling by making p true at exactly the sequences of odd length ending in b (recall Example 3.12).

We now improve on Proposition 4.9 by showing how to turn any Kripke frame X into a Beth frame Y so that $\mathsf{Up}(X)$ is *isomorphic* to $\mathsf{Up}(Y)_{j_b}$. This has the advantage of showing that not only IPC but in fact any intermediate logic that can be characterized by Kripke frames can also be characterized by Beth frames.

Definition 4.12. Given a poset $(X, \sqsubseteq)$, its *Bethification* $(X_b, \sqsubseteq_b)$ is defined by:

1. X_b is the set of all pairs $\langle x, n \rangle$ where $x \in X$ and $n \in \mathbb{N}$.
2. $\langle x, n \rangle \sqsubseteq_b \langle x', n' \rangle$ iff $[x = x'$ and $n \leq n']$ or $[x \sqsubset x'$ and $n < n']$.

It is easy to see that $\sqsubseteq_b$ is a partial order.

One can think of the second coordinate of each pair as the *time* according to a discrete clock. The definition of $\sqsubseteq_b$ reflects the idea that one may remain at the same state x for all time or one may transition from x to a distinct extension x' of x, which takes time. Whereas a state in the Beth unraveling of a Kripke frame records the exact history of moves through the Kripke frame by which that state was reached, a state in the Bethification records only the "current time" and one's "current location" in the Kripke frame.

Example 4.13. Figure 4.9 shows the Bethification of the same Kripke frame whose Beth unraveling is shown in Figure 4.6. If we are at b at time 1, then we may remain at b at time 2, so we have an arrow from $\langle b, 1 \rangle$ to $\langle b, 2 \rangle$ in the Bethification; by contrast, we had no arrow between the matching states $\langle a, b \rangle$ and $\langle a, a, b \rangle$ in the Beth unraveling, since the second history is not an extension of the first.

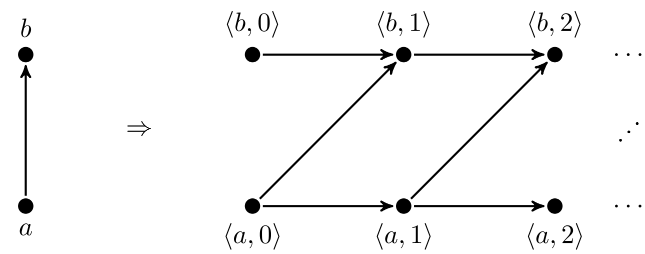

Figure 4.9: Bethification (right) of a Kripke frame (left).

To prove that Bethification gives us the isomorphism between algebras that we want, we first prove the following lemma characterizing the fixpoints of the Beth nucleus in the Bethification.

Lemma 4.14. Let $(X, \sqsubseteq)$ be a poset and $(X_b, \sqsubseteq_b)$ its Bethification. The fixpoints of j_b on $\mathsf{Up}(X_b, \sqsubseteq_b)$ are the $U \in \mathsf{Up}(X_b, \sqsubseteq_b)$ that are *uniform* (in the second coordinate) in the sense that if $\langle x, u \rangle \in U$, then $\langle x, n \rangle \in U$ for all $n \in \mathbb{N}$.

Proof. First, we will show that if $\langle x, u \rangle \in U \in \mathsf{Up}(X_b, \sqsubseteq_b)$, then $\langle x, n \rangle \in j_b U$ for all $n \in \mathbb{N}$; hence if U is a fixpoint, then $\langle x, u \rangle \in U$ will imply $\langle x, n \rangle \in U$ for all $n \in \mathbb{N}$, as desired. Note that in any path through $\langle x, n \rangle$, there is some state $\langle x', n'\rangle \sqsupseteq_b \langle x, n\rangle$ such that $n' > u$, because the second coordinate is increasing along any path. Since $\langle x', n'\rangle \sqsupseteq_b \langle x, n\rangle$, we have $x' \sqsupseteq x$, which with $n' > u$ implies $\langle x', n'\rangle \sqsupseteq_b \langle x, u\rangle$. Then since $\langle x, u\rangle \in U \in \mathsf{Up}(X_b, \sqsubseteq_b)$, we have $\langle x', n'\rangle \in U$. Thus, every path through $\langle x, n\rangle$ intersects U, so $\langle x, n\rangle \in j_b U$.

In the other direction, suppose U is not a fixpoint, so there is some $\langle x, n\rangle \in X \setminus U$ such that $\langle x, n\rangle \in j_b U$, so all paths through $\langle x, n\rangle$ intersect U. Then in particular, the path $\langle x, n\rangle, \langle x, n+1\rangle, \langle x, n+2\rangle, \ldots$ intersects U. Thus, $\langle x, u\rangle \in U$ for some $u \in \mathbb{N}$. Hence U violates the requirement that if $\langle x, u\rangle \in U$, then $\langle x, n\rangle \in U$ for all $n \in \mathbb{N}$. □

We are now ready to prove that Bethification turns any Kripke frame into an "equivalent" Beth frame.

Theorem 4.15. Let $(X, \sqsubseteq)$ be a poset, $(X_b, \sqsubseteq_b)$ its Bethification, and $g : X_b \to X$ be defined by $g(x, t) = x$. Then $g^{-1} : \mathsf{Up}(X, \sqsubseteq) \to \mathsf{Up}(X_b, \sqsubseteq_b)_{j_b}$ is an isomorphism.

Proof. First, it is easy to see that g is a p-morphism from $(X_b, \sqsubseteq_b)$ onto $(X, \sqsubseteq)$. Moreover, it satisfies the additional condition for a BK-morphism from Definition 4.1: if $g(x, n) \notin U$, then $\langle x, n\rangle, \langle x, n{+}1\rangle, \langle x, n{+}2\rangle, \ldots$ is a path that never intersects $g^{-1}[U]$. Thus, by Lemma 4.2, g^{-1} is a Heyting algebra embedding. Finally, to see that g^{-1} is surjective, given any $V \in \mathsf{Up}(X_b, \sqsubseteq_b)_{j_b}$, we have that $g^{-1}[g[V]] = V$ by Lemma 4.14, and $g[V] \in \mathsf{Up}(X, \sqsubseteq)$ since g is a p-morphism. □

Theorem 4.15 gives us the first inequality of the semantic hierarchy:

$$\text{Kripke} \leq \text{Beth}.$$

4.2 Locales from Beth but Not Kripke

In this section, we will show that the above inequality is in fact strict:

$$\text{Kripke} < \text{Beth}.$$

That is, there are Beth frames Y such that $\mathsf{Up}(Y)_{j_b}$ is not isomorphic to $\mathsf{Up}(X)$ for any Kripke frame X.[48] To show this, we recall from Remark 2.1 that $\mathsf{Up}(X)$ is always completely join-prime generated, i.e., every element in $\mathsf{Up}(X)$ is a join of completely join-prime elements. Below we will present posets Y such that $\mathsf{Up}(Y)_{j_b}$ contains no completely join-prime elements, so it cannot be isomorphic to $\mathsf{Up}(X)$ for any poset X.

Example 4.16. As in the proof of Theorem 4.10, consider the full countable binary tree $\mathcal{T}$ viewed as a poset, so that $x \sqsubseteq y$ iff y is a descendant of x. First observe that any principal upset $\uparrow x$ for $x \in \mathcal{T}$ is a fixpoint of the Beth nucleus j_b; for if $y \notin \uparrow x$, then since $\mathcal{T}$ is a binary tree, there is a path through y that never intersects $\uparrow x$, so $y \notin j_b \uparrow x$. Using this fact, we will show that $\mathsf{Up}(\mathcal{T})_{j_b}$ has no completely join-prime elements. Consider any fixpoint U of j_b. First suppose that U is not a principal upset. Since $U = \bigcup\{\uparrow x \mid x \in U\}$, we have $j_b U = j_b \bigcup\{\uparrow x \mid x \in U\}$ and hence $U = \bigvee_{j_b}\{\uparrow x \mid x \in U\}$, which with the non-principalness of U implies that U is not completely join-prime. Next suppose that U is a principal upset $\uparrow x$. In $\mathcal{T}$, x has two children, y and z. It is easy to see that $\uparrow x = j_b(\uparrow y \cup \uparrow z) = \uparrow y \vee_{j_b} \uparrow z$, so U is not join-prime.

[48] López-Escobar [1981] compares Kripke and Beth semantics from a different, categorical perspective. Since he assumes that Kripke models are rooted (in fact, are trees), his example of Beth models with no equivalent Kripke models does not suffice to show that "Kripke < Beth" in our sense. Note that if X is a rooted poset, then the only Boolean algebra that can be realized as $\mathsf{Up}(X)$ is the two-element Boolean algebra. By contrast, any finite Boolean algebra B can be realized as $\mathsf{Up}(Y)_{j_b}$ for an appropriate finite rooted poset Y: if B has n atoms, let Y be the n-fork.

While Example 4.16 shows how a familiar poset X gives rise to a locale $\mathsf{Up}(X)_{j_b}$ with no completely join-prime elements, our next example shows how a familiar locale with no completely join-prime elements arises as $\mathsf{Up}(X)_{j_b}$ for an appropriately chosen poset X.

Example 4.17. The interval $[0,1]$, ordered by the less-than-or-equal-to relation $\leq$, is a locale with no completely join-prime element: for any $r \in [0,1]$, we have $r = \bigvee\{r' \in [0,1] \mid r' < r\}$. To obtain this locale from a Beth frame, consider $(X, \sqsubseteq)$ with $X = \mathbb{R} \times \mathbb{N}$ and $\langle r, t\rangle \sqsubseteq \langle r', t'\rangle$ iff $\langle r, t\rangle = \langle r', t'\rangle$ or both $r < r'$ and $t < t'$. Note that $(X, \sqsubseteq)$ is a poset. We claim that the fixpoints of j_b on $\mathsf{Up}(X, \sqsubseteq)$ are $\varnothing$, sets of the form $\{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$ for some $r \in \mathbb{R}$, and X. Thus, $\mathsf{Up}(X, \sqsubseteq)_{j_b}$ is isomorphic to $[0,1]$.

To prove the claim, we first observe that as in Lemma 4.14, the fixpoints U of j_b on $\mathsf{Up}(X, \sqsubseteq)$ are uniform: if $\langle r, u\rangle \in U$, then $\langle r, t\rangle \in U$ for all $t \in \mathbb{N}$. Other than X and $\varnothing$, there are two kinds of $U \in \mathsf{Up}(X, \sqsubseteq)$ with this property: sets of the form (a) $\{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$ and sets of the form (b) $\{\langle r', t\rangle \mid r < r', t \in \mathbb{N}\}$. Sets of the form (b) are not fixpoints, because every path through $\langle r, 0\rangle$ intersects $\{\langle r', t\rangle \mid r < r', t \in \mathbb{N}\}$ since the first coordinate is increasing along any path (in contrast to Definition 4.12). Yet sets of the form (a) are fixpoints. To see this, given any $\langle r^*, u\rangle \notin \{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$, so $r^* < r$, consider the infinite chain $\langle r^*, 0\rangle$, $\langle r^* + \frac{1}{2}(r - r^*), 1\rangle$, $\langle r^* + \frac{3}{4}(r - r^*), 2\rangle$, ..., $\langle r^* + \frac{2^n-1}{2^n}(r - r^*), n\rangle$, ... This is a maximal chain in $\langle X, \sqsubseteq\rangle$, because if we add any new pair $\langle s, m\rangle$, then $\langle s, m\rangle$ and $\langle r^* + \frac{2^n-1}{2^n}(r - r^*), m\rangle$ are incomparable by definition of $\sqsubseteq$. In addition, this chain never intersects $\{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$, since the first coordinate never reaches r. Thus, $\langle r^*, u\rangle \notin j_b\{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$, which shows that $\{\langle r', t\rangle \mid r \leq r', t \in \mathbb{N}\}$ is a fixpoint. This completes the proof of the claim made in the previous paragraph.

In fact, Example 4.17 suggests the following general result.

Proposition 4.18. Every linearly ordered locale is isomorphic to $\mathsf{Up}(X)_{j_b}$ for some poset X.

Proof. Let $(L, \leq)$ be a linearly ordered locale. We will define a poset $(X, \sqsubseteq)$ with $X = (L \setminus \{0\}) \times \mathbb{N}$. In the definition of $\sqsubseteq$ below, one can think of $(X, \sqsubseteq)$ as representing possible movements *down* $(L, \leq)$ through time, so for $x \neq x'$, we will have $\langle x, t\rangle \sqsubseteq \langle x', t'\rangle$ only if x' is below x in L, and t' is later than t. For $x \in L \setminus \{0\}$, call x a *dense point* if there is a $y \in L$ such that the interval $[y, x]$ in L is dense (i.e., for any $z, u \in [y, x]$ with $z < u$, there is a w such that $z < w < u$). Otherwise call x a *discrete point*. Then we define $\sqsubseteq$ as follows: if x is a dense point, then $\langle x, t\rangle \sqsubseteq \langle x', t'\rangle$ iff either $\langle x, t\rangle = \langle x', t'\rangle$ or $x' < x$ and $t < t'$; if x is a discrete point, then $\langle x, t\rangle \sqsubseteq \langle x', t'\rangle$ iff either $x = x'$ and $t \leq t'$ or $x' < x$ and $t < t'$. Note that this definition of $\sqsubseteq$ combines that of Example 4.17 with that of Definition 4.12.

We claim that the fixpoints of j_b on $\mathsf{Up}(X, \sqsubseteq)$ are $\varnothing$ and sets of the form $\{\langle x', t\rangle \mid x' \leq x, t \in \mathbb{N}\}$ for some $x \in L \setminus \{0\}$. Thus, the algebra $\mathsf{Up}(X, \sqsubseteq)_{j_b}$ of fixpoints will be isomorphic to L.

To prove the claim, we first observe that as in Lemma 4.14, the fixpoints U of j_b on $\mathsf{Up}(X, \sqsubseteq)$ are uniform: if $\langle x, u\rangle \in U$, then $\langle x, t\rangle \in U$ for all $t \in \mathbb{N}$. Uniform upsets in $(X, \sqsubseteq)$ can be associated with downsets in $\langle L, \leq\rangle$, of which there are two kinds: downsets D such that $\bigvee D \in D$, and downsets D such that $\bigvee D \notin D$. Thus, uniform upsets in $(X, \sqsubseteq)$ are, in addition to $\varnothing$, sets of the form (a) $\{\langle x', t\rangle \mid x' \leq x, t \in \mathbb{N}\}$ and sets of the form (b) $\{\langle x', t\rangle \mid x' < x, t \in \mathbb{N}\}$. Now observe that if an $S \subseteq X$ is of form (b) but not (a), then the $x \in L \setminus \{0\}$ that makes S of form (b) must be a dense point. Hence $\langle x, t\rangle \sqsubset \langle x', t\rangle$ only if $x' < x$. Thus, every path through $\langle x, 0\rangle$ intersects $\{\langle x', t\rangle \mid x' < x, t \in \mathbb{N}\}$. This shows that sets of form (b) but not (a) are not fixpoints. On the other hand, sets of form (a) are fixpoints. Take any $\langle x^*, s\rangle \notin \{\langle x', t\rangle \mid x' \leq x, t \in \mathbb{N}\}$. First suppose x^* is a discrete point. Then the chain $\langle x^*, 0\rangle$, $\langle x^*, 1\rangle$, ... is a path that never intersects $\{\langle x', t\rangle \mid x' \leq x, t \in \mathbb{N}\}$. Now suppose that x^* is a dense point. Then there is a y such that $x \leq y < x^*$

and the interval $[y, x^*]$ is dense in L. Hence we can construct a path $\langle x^*, 0\rangle$, $\langle x_1, 1\rangle$, $\langle x_2, 2\rangle$, $\dots$ such that $x < x_{n+1} < x_n < x^*$ for each $n \geq 1$, so the path never intersects $\{\langle x', t\rangle \mid x' \leq x, t \in \mathbb{N}\}$. $\square$

The examples of this section show that it is not difficult to find posets X such that $\mathsf{Up}(X)_{j_b}$ is not isomorphic to $\mathsf{Up}(Y)$ for any poset Y. However, these examples quickly lead to more difficult problems.

Problem 1. Give a characterization of the locales that can be represented as $\mathsf{Up}(X)_{j_b}$ for a poset X.

Problem 2. Is there a variety of Heyting algebras that can be generated by locales of the form $\mathsf{Up}(X)_{j_b}$ but not of the form $\mathsf{Up}(X)$? Equivalently, is there an intermediate logic that is Beth complete but not Kripke complete?

In fact, even the answer to the following is unknown.

Problem 3. Is there an intermediate logic that is not Beth complete?

4.3 From Beth to Spaces

The next piece of the semantic hierarchy is the inequality

$$\text{Beth} \leq \text{Topological}.$$

That is, for any poset X there is a topological space Y such that $\mathsf{Up}(X)_{j_b}$ is isomorphic to the locale $\Omega(Y)$ of opens of Y. Dummett [2000, p. 140] in effect proves the weaker version of this statement with 'embeds into' in place of 'is isomorphic to'. By modifying his construction, we will obtain the stronger statement.

Theorem 4.19. Given a Beth frame X, let Y be the set of all paths in X, and for $U \subseteq X$, let

$$[U] = \{\alpha \in Y \mid \alpha \cap U \neq \varnothing\}.$$

Then the pair (Y, Ω) with $\Omega = \{[U] \mid U \in \mathsf{Up}(X)_{j_b}\}$ is a topological space.[49] Moreover, the function $[\cdot] : \mathsf{Up}(X)_{j_b} \to \Omega(Y)$ is an isomorphism.

Proof. First, the empty set of paths is $[\varnothing]$, and the set Y of all paths is $[X]$. Next, given $[U]$ and $[V]$, observe that $[U] \cap [V] = [U \cap V]$. The right-to-left inclusion is obvious, and for the left-to-right, if a path α intersects U and V, then since U and V are upsets and α is a chain, α must also intersect $U \cap V$. Finally, given a family $\{[U_i]\}_{i \in I}$, we must show that $\bigcup\limits_{i \in I}[U_i] = [V]$ for some fixpoint V of j_b. We claim that

$$\bigcup_{i \in I}[U_i] = [j_b \bigcup_{i \in I} U_i].$$

Suppose α is a path in the left-hand side, so for some $i \in I$, there is an $x \in U_i$ such that α is a path through x. Then since $x \in U_i \subseteq j_b U_i \subseteq j_b \bigcup\limits_{i \in I} U_i$, it follows that α is in the right-hand side. Conversely, suppose α is in the right-hand side, so there is some $x \in j_b \bigcup\limits_{i \in I} U_i$ such that α is a path through x. Since $x \in j_b \bigcup\limits_{i \in I} U_i$, every path through x intersects $\bigcup\limits_{i \in I} U_i$, so in particular, α does, which implies there is some $i \in I$ such that α intersects U_i. Hence there is some $y \in U_i$ such that α is a path through y, which means $\alpha \in [U_i]$, so α is in the left-hand side. Thus, we have shown that Ω is a topology.

[49] The difference between this construction and Dummett's is that Dummett takes $\Omega = \{[U] \mid U \subseteq X\}$.

We now prove that the function $[\cdot]$ is an isomorphism. That $[\cdot]$ is surjective is immediate from the definition of (Y, Ω). Next, for any $U, V \in \mathsf{Up}(X)_{j_b}$, we show that $U \not\subseteq V$ implies $[U] \not\subseteq [V]$. Given $U, V \in \mathsf{Up}(X)_{j_b}$, we have $U = j_b U$ and $V = j_b V$. Then from $U \not\subseteq V$ we have $j_b U \not\subseteq j_b V$, so there is an $x \in j_b U$ with $x \notin j_b V$. It follows that there is a path α through x that does not intersect V, though it does intersect U, so $[U] \not\subseteq [V]$. Thus, $[\cdot]$ is order-reflecting and hence injective. We also showed above that $[\cdot]$ preserves binary meets and hence is order-preserving. Therefore, $[\cdot]$ is an isomorphism. $\square$

4.4 Locales from Spaces but Not Beth

We now add that the inequality of the previous section is strict:

$$\text{Beth} < \text{Topological}.$$

That is, there are topological spaces X such that $\Omega(X)$ cannot be represented as $\mathsf{Up}(Y)_{j_b}$ for any poset Y.

Given a topological space X and $x \in X$, let $\Omega(x) = \{U \in \Omega(X) \mid x \in U\}$. Recall that a *local base* (or *neighborhood base*) of a point x is a $\mathcal{B} \subseteq \Omega(x)$ such that

$$\forall U \in \Omega(x)\ \exists V \in \mathcal{B} : V \subseteq U$$

and that X is *first countable* if each $x \in X$ has a countable local base. This is easily seen to be equivalent to each $x \in X$ having a countable local base linearly ordered by $\subseteq$ (by enumerating the countable base and taking finite intersections). The following notion from Davis 1978 therefore generalizes the notion of first countability.

Definition 4.20. A *lob-space* is a topological space in which each point has a linearly ordered local base.

Theorem 4.21. For any poset X, the locale $\mathsf{Up}(X)_{j_b}$ is isomorphic to the locale of open sets of a lob-space.

Proof. It suffices to show that the topological space (Y, Ω) constructed from X in Theorem 4.19 is a lob-space. Recall that Y is the set of all paths in X, and $\Omega = \{[U] \mid U \in \mathsf{Up}(X)_{j_b}\}$, where $[U] = \{\alpha \in Y \mid \alpha \cap U \neq \varnothing\}$. For any path $\alpha \in Y$, we claim that

$$\mathcal{B} = \{[j_b{\uparrow}x] \mid x \in \alpha\}$$

is a linearly ordered local base of α. Since $j_b{\uparrow}x \in \mathsf{Up}(X)_{j_b}$ and $\alpha \in [j_b{\uparrow}x]$ for $x \in \alpha$, we have $\mathcal{B} \subseteq \Omega(\alpha)$. First, we show that $\mathcal{B}$ is linearly ordered. If $x, x' \in \alpha$, then since α is a path in X, either $x \leq x'$ or $x' \leq x$. Suppose $x \leq x'$. Then ${\uparrow}x \supseteq {\uparrow}x'$, which implies $j_b{\uparrow}x \supseteq j_b{\uparrow}x'$ and hence $[j_b{\uparrow}x] \supseteq [j_b{\uparrow}x']$. This shows that $\mathcal{B}$ is linearly ordered. Next, to see that $\mathcal{B}$ is a local base of α, suppose $[U] \in \Omega(\alpha)$, so $\alpha \cap U \neq \varnothing$. Taking an $x \in \alpha \cap U$, we have ${\uparrow}x \subseteq U$ and hence $j_b{\uparrow}x \subseteq j_b U = U$ since $U \in \mathsf{Up}(X)_{j_b}$. Thus, $[j_b{\uparrow}x] \subseteq [U]$. $\square$

By contrast, there are spatial locales that cannot be represented as $\Omega(Y)$ for any lob-space Y.

Example 4.22. A standard example of a topological space that is not a lob-space is the uncountable product $2^{\mathfrak{c}}$ of the two-element discrete space (see, e.g., Bredon 1993, Problem 8(b), p. 24). Not only is $2^{\mathfrak{c}}$ not a lob-space, but in fact no point in $2^{\mathfrak{c}}$ has a linearly ordered local base. Following the proof in Rüping 2016, consider any $p \in 2^{\mathfrak{c}}$. One of $\{i \in \mathfrak{c} \mid p(i) = 0\}$ and $\{i \in \mathfrak{c} \mid p(i) = 1\}$ is uncountable. Without loss of generality, suppose it is the former, and let $I = \{i \in \mathfrak{c} \mid p(i) = 0\}$. Let S_i be the set of all $x \in 2^{\mathfrak{c}}$ such that $x(i) = 0$. Since the topology in $2^{\mathfrak{c}}$ is the product topology, any basic neighborhood of p is by definition the intersection of finitely many S_i's. Therefore, any open neighborhood of p is a subset of only finitely many S_i's. Suppose for contradiction that p has a linearly order local base $\mathcal{B}$. For each $U \in \mathcal{B}$, let $f(U) = \{i \in I \mid U \subseteq S_i\}$, so $f(U)$ is finite. For $U, V \in \mathcal{B}$, observe that $U \subseteq V$ implies $f(V) \subseteq f(U)$. In addition, $I = \bigcup\{f(U) \mid U \in \mathcal{B}\}$ by the definition of the S_i in terms of p and the definition of f in terms of the S_i. Thus, I is the union of a nested family of finite sets and hence is countable, a contradiction.

Next we claim that $\Omega(2^{\mathfrak{c}})$ is not isomorphic to $\Omega(Y)$ for any lob-space Y. For this, we recall some basic notions relating spaces and locales (see, e.g., Picado and Pultr 2012, Chs. I–II). For a locale L, the space $\mathsf{Sp}(L)$ has as points the meet-prime elements of L and as open sets the sets $\{m \in \mathsf{Sp}(L) \mid m \not\leq a\}$ for $a \in L$. A locale L is spatial iff L is isomorphic to $\Omega(\mathsf{Sp}(L))$. A space X is *sober*, meaning that every meet-prime open set is the complement of the closure of a unique point in X, iff X is isomorphic to $\mathsf{Sp}(\Omega(X))$. Since every Hausdorff space is sober, and $2^{\mathfrak{c}}$ is Hausdorff, it is sober. Now suppose for contradiction that $\Omega(2^{\mathfrak{c}})$ is isomorphic to $\Omega(Y)$ for a lob-space Y. Thus, each $y \in Y$ has a linearly ordered local base, which implies that the corresponding point in $\mathsf{Sp}(\Omega(Y))$, namely $Y \setminus \mathrm{cl}\{y\}$, has a linearly ordered local base. By the assumption that $\Omega(Y)$ is isomorphic to $\Omega(2^{\mathfrak{c}})$, we have that $\mathsf{Sp}(\Omega(Y))$ is homeomorphic to $\mathsf{Sp}(\Omega(2^{\mathfrak{c}}))$, which is in turn homeomorphic to $2^{\mathfrak{c}}$ because $2^{\mathfrak{c}}$ is sober. Then since $\mathsf{Sp}(\Omega(Y))$ has a point with a linearly ordered locale base, so does $2^{\mathfrak{c}}$, contradicting the previous paragraph. This completes the proof that $\Omega(2^{\mathfrak{c}})$ is not isomorphic to $\Omega(Y)$ for any lob-space Y.

Together Theorem 4.21 and Example 4.22 establish the strict inequality that Beth frames cannot give rise to all spatial locales: $\mathsf{Up}(X)_{j_b}$ can always be realized as $\Omega(Y)$ for a lob-space Y (Theorem 4.21), whereas not all spatial locales can be so represented (Example 4.22).

As noted in § 1.5, it is known that there are intermediate logics that are topologically complete but Kripke incomplete [Shehtman, 2005, § 8], or equivalently, that there are varieties of Heyting algebras that can be generated by spatial locales but not by locales of the form $\mathsf{Up}(X)$. Given the inequalities

$$\text{Kripke} \leq \text{Beth} \leq \text{Topological},$$

it follows that Problem 2 or Problem 4 has an affirmative answer.

Problem 4. Is there a variety of Heyting algebras that can be generated by locales of the form $\Omega(X)$ for a topological space X but not of the form $\mathsf{Up}(Y)_{j_b}$ for a poset Y? Equivalently, is there an intermediate logic that is topologically complete but Beth incomplete?

4.5 From Spaces to Dragalin

The penultimate step in establishing our semantic hierarchy is the inequality

$$\text{Topological} \leq \text{Dragalin}.$$

That is, for every topological space X there is a Dragalin frame (Y, D) such that $\Omega(X)$ is isomorphic to $\mathsf{Up}(Y)_{j_D}$. This result was proved by Dragalin [1979, 1988, pp. 75–6] using the construction in Theorem 4.23 below. To keep the paper self-contained, we include a proof of this result.

Theorem 4.23 (Dragalin)**.** Given a topological space (X, Ω), the tuple $(\Omega, \leq, D)$ where $U \leq V$ iff $U \supseteq V$, and

$$D(U) = \{\mathcal{B} \mid \exists x \in U : \mathcal{B} \text{ is a local base of } x \text{ and } \bigcup \mathcal{B} \subseteq U\},$$

is a standard normal Dragalin frame. Moreover, the function $f : \Omega(X) \to \mathsf{Up}(\Omega, \leq)_{j_D}$ given by

$$f(U) = \{V \in \Omega \mid V \subseteq U\}$$

is an isomorphism.

Proof. Since for any point x, the set of all open sets containing x is a local base for x, clearly $D(U) \neq \varnothing$ for each $U \in \Omega$, so the normality condition holds. Next we verify the conditions (1°), (2°°), (3°°), and (4°°) of a standard Dragalin frame:

(1°) $\varnothing \notin D(U)$;

(2°°) if $\mathcal{B} \in D(U)$, then $\mathcal{B} \subseteq {\uparrow}U$;

(3°°) if $U \leq V$, then $D(V) \subseteq D(U)$;

(4°°) if $V \in \mathcal{A} \in D(U)$, then $\exists \mathcal{B} \in D(V)$: $\mathcal{B} \subseteq \mathcal{A}$.

Condition (1°) is immediate because a local base of a point must be nonempty. For (2°°), since ${\uparrow}U = \{V \in \Omega \mid V \subseteq U\}$, the inclusion $\mathcal{B} \subseteq {\uparrow}U$ follows from $\bigcup \mathcal{B} \subseteq U$. For (3°°), suppose $U \leq V$, so $U \supseteq V$. If $\mathcal{B} \in D(V)$, so $\mathcal{B}$ is a local base of a point $x \in V$ with $\bigcup \mathcal{B} \subseteq V$, then we have that $\mathcal{B}$ is a local base of the same point $x \in U$ and $\bigcup \mathcal{B} \subseteq U$, so $\mathcal{B} \in D(U)$. For (4°°), if $V \in \mathcal{A} \in D(U)$, then let $\mathcal{B} = \{V' \in \mathcal{A} \mid V' \subseteq V\}$. Since $\mathcal{A} \in D(U)$, $\mathcal{A}$ is a local base of some point $x \in U$. It is then easy to see that $\mathcal{B}$ is also a local base of x. By definition of $\mathcal{B}$, we also have $\bigcup \mathcal{B} \subseteq V$, so $\mathcal{B} \in D(V)$, and $\mathcal{B} \subseteq \mathcal{A}$, so (4°°) holds.

To prove that $f : \Omega(X) \to \mathsf{Up}(\Omega, \leq)_{j_D}$ is an isomorphism, we show that the elements of $\mathsf{Up}(\Omega, \leq)_{j_D}$ are exactly the principal upsets of $(\Omega, \leq)$ plus $\varnothing$. Then since f sends each U to its principal upset in $(\Omega, \leq)$, it is clear that f is order preserving and reflecting, so it is an isomorphism.

To show that the fixpoints of j_D are exactly the principal upsets, plus $\varnothing$, it suffices to show that for every $\mathcal{U} \in \mathsf{Up}(\Omega, \leq)$, we have $j_D\mathcal{U} = \{U \in \Omega \mid U \subseteq \bigcup \mathcal{U}\}$, for this implies that $\mathcal{U} = j_D\mathcal{U}$ iff $\mathcal{U}$ is principal.

To show that $j_D\mathcal{U} = \{U \in \Omega \mid U \subseteq \bigcup \mathcal{U}\}$, first suppose $U \subseteq \bigcup \mathcal{U}$ and $\mathcal{B} \in D(U)$. Hence $\mathcal{B}$ is a local base of some point $x \in U$. Since $U \subseteq \bigcup \mathcal{U}$ and $x \in U$, there is some $V \in \mathcal{U}$ such that $x \in V$. Then since $\mathcal{B}$ is a local base of x, there is a $V' \in \mathcal{B}$ such that $V' \subseteq V$, which with $V \in \mathcal{U} \in \mathsf{Up}(\Omega, \leq)$ gives us $V' \in \mathcal{U}$. Thus, for every $\mathcal{B} \in D(U)$, there is a $V' \in \mathcal{B} \cap \mathcal{U}$, which means $U \in j_D\mathcal{U}$. Conversely, if $U \not\subseteq \bigcup \mathcal{U}$, then take an $x \in U \setminus \bigcup \mathcal{U}$ and let $\mathcal{B} = \{V \in \Omega \mid x \in V \subseteq U\}$. Then clearly $\mathcal{B} \in D(U)$, but $\mathcal{B} \cap \mathcal{U} = \varnothing$, so $U \notin j_D\mathcal{U}$. ☐

4.6 Locales from Dragalin but Not Spaces

The final step in establishing the semantic hierarchy is to show that the inequality of § 4.5 is strict:

$$\text{Topological} < \text{Dragalin}.$$

Example 4.24. As in the end of § 3.3, for any poset X, we can define a Dragalin frame (X, D) by $D(x) = \{{\uparrow}y \mid y \in {\uparrow}x\}$; j_D is then the nucleus of double negation, so $\mathsf{Up}(X)_{j_D}$ is a complete Boolean algebra. If we take, e.g., X to be the full countable binary tree, then it is easy to check that $\mathsf{Up}(X)_{j_D}$ is atomless. But a complete Boolean algebra is spatial iff it is atomic. Thus, Dragalin frames can produce non-spatial locales.

In fact, something much more general holds:

$$\text{Locales} \equiv \text{Dragalin}.$$

The inequality Locales $\geq$ Dragalin is a consequence of Lemma 3.37, while the inequality Locales $\leq$ Dragalin is a consequence of the following.

Theorem 4.25. For every locale L, there is a standard normal Dragalin frame (X, D) such that L is isomorphic to $\mathsf{Up}(X)_{j_D}$.

To prove the theorem, recall from Theorem 3.24 that for every locale L, there is a dense nuclear frame (X, j) such that L is isomorphic to $\mathsf{Up}(X)_j$. Thus, it suffices to show that for any dense nuclear frame (X, j), there is a standard normal Dragalin frame (X, D) with $j = j_D$. This is a consequence of the following more general result proved in Bezhanishvili and Holliday 2016.

Theorem 4.26. Given any nuclear frame (X, j), there is a standard Dragalin frame (X, D) such that $j = j_D$, and j is dense iff (X, D) is normal.

Proof. (Sketch) As is well known, for any locale L, the collection $N(L)$ of all nuclei on L, ordered by $j \leq k$ iff $ja \leq ka$ for all $a \in L$, is itself a locale, in which meets are computed pointwise (see, e.g., Fourman and Scott 1979, Thm. 2.20, Johnstone 1982, Prop. II.2.5). As observed by Simmons [1978, p. 243], for any nucleus $j \in N(L)$, we have:

$$j = \bigwedge\{w_{ja} \mid a \in L\}, \tag{10}$$

where for $b \in L$, w_b is the nucleus from Example 3.22 defined on L by

$$w_b c = (c \to b) \to b.$$

Thus, given our nuclear frame (X, j), we can assume that the nucleus j on $L = \mathsf{Up}(X)$ can be expressed as a meet of nuclei w_U, for certain $U \in \mathsf{Up}(X)$, as in (10).

Given any one of the nuclei w_U on $\mathsf{Up}(X)$, we define a function $D_U : X \to \wp(\wp(X))$ by

$$D_U(x) = \{{\uparrow}y \setminus U \mid y \in {\uparrow}x \setminus U\}.$$

As shown in Lemma 3.6 of Bezhanishvili and Holliday 2016, D_U is a standard Dragalin function and

$$w_U = j_{D_U}. \tag{11}$$

Next, given any family $\{j_\alpha \mid \alpha \in I\}$ of nuclei on $\mathsf{Up}(X)$, if for each $\alpha \in I$ we have a Dragalin function D_α such that

$$j_\alpha = j_{D_\alpha},$$

then we define a function $D : X \to \wp(\wp(X))$ by

$$D(x) = \bigcup\{D_\alpha(x) \mid \alpha \in I\}. \tag{12}$$

As shown in Lemma 3.7 of Bezhanishvili and Holliday 2016, D is a Dragalin function, which is standard if each D_α is standard, and

$$\bigwedge\{j_\alpha \mid \alpha \in I\} = j_D. \tag{13}$$

Putting everything together, we have:

$$\begin{array}{rcll} j & = & \bigwedge\{w_{jU} \mid U \in \mathsf{Up}(X)\} & \text{by (10)} \\ & = & \bigwedge\{j_{D_{jU}} \mid U \in \mathsf{Up}(X)\} & \text{by (11)} \\ & = & j_D & \text{by (13)}, \end{array}$$

where D is defined from the D_{jU}'s as in (12). Then (X, D) is a standard Dragalin frame, which is normal iff j is dense (see Lemma 3.37). $\square$

4.7 Another Perspective: FM-Semantics

Theorem 4.26 shows that we can always replace the nucleus j in a nuclear frame with a Dragalin function D, while keeping the underlying poset X the same. If we are willing to change X to a "larger" preorder $Y = (Y, \leq_1)$, then in place of a Dragalin function $D : Y \to \wp(\wp(Y))$ we can use something simpler: a second preorder $\leq_2$, which is a subrelation of $\leq_1$. The following structures were introduced by Fairtlough and Mendler [1997] as "constraint models" in their semantics for lax logic (recall Remark 3.27).[50]

Definition 4.27. An *FM-frame* is a tuple $(Y, \leq_1, \leq_2, F)$ where Y is a set, $\leq_1$ and $\leq_2$ are preorders on Y such that $\leq_2$ is a subrelation of $\leq_1$, and $F \in \mathsf{Up}(Y, \leq_1)$. An FM-frame is *normal* if $F = \varnothing$, in which case we may identify the FM-frame with $(Y, \leq_1, \leq_2)$.

The distinguished upset F plays the same role as the set of fallible states in Dragalin frames (Definition 3.36) and fallible Kripke frames (Remark 2.2). The Heyting algebra of interest is then $\mathsf{Up}(Y, \leq_1)_F := \{U \in \mathsf{Up}(Y, \leq_1) \mid F \subseteq U\}$. To extract a nucleus on $\mathsf{Up}(Y, \leq_1)_F$ from the FM-frame, we define for $U \subseteq Y$:

$$\begin{array}{rcl} \Box_1 U & = & \{x \in Y \mid \forall y \geq_1 x : y \in U\} \\ \Diamond_2 U & = & \{x \in Y \mid \exists y \geq_2 x : y \in U\} \\ \Box_1 \Diamond_2 U & = & \{x \in Y \mid \forall y \geq_1 x\ \exists z \geq_2 y : z \in U\}. \end{array}$$

The following result is due to Fairtlough and Mendler [1997, p. 9], who phrase it in terms of the soundness of the logic PLL mentioned in Remark 3.27 (for a proof using the present terminology and notation, see Bezhanishvili and Holliday 2016, Prop. 4.2).

Lemma 4.28. For any FM-frame $(Y, \leq_1, \leq_2, F)$, the operation $\Box_1\Diamond_2$ is a nucleus on the Heyting algebra $\mathsf{Up}(Y, \leq_1)_F$. Moreover, this nucleus is dense if the FM-frame is normal.

It follows from Lemma 4.28 that an FM-frame gives rise to a nuclear algebra $(\mathsf{Up}(Y, \leq_1)_F, \Box_1\Diamond_2)$ and hence a locale $(\mathsf{Up}(Y, \leq_1)_F)_{\Box_1\Diamond_2}$. Thus, FM-frames may be used to give nuclear semantics for the intuitionistic propositional language as in § 3.2. Given a valuation v mapping propositional letters to upsets that include F, the satisfaction clauses besides the Kripke clauses for $\wedge$ and $\to$ are:

- $x \models_v \bot$ iff $x \in \Box_1\Diamond_2 F$;
- $x \models_v p$ iff $x \in \Box_1\Diamond_2 v(p)$;
- $x \models_v \varphi \vee \psi$ iff $x \in \Box_1\Diamond_2\{z \in Y \mid z \models_v \varphi \text{ or } z \models_v \psi\}$.

Remark 4.29.

[50] The name 'FM-frames' is from Bezhanishvili and Holliday 2016.

1. Kripke frames may be regarded as normal FM-frames in which $\leq_2$ is the identity relation, for then $\Box_1\Diamond_2 U = \Box_1 U$ and hence the fixpoint algebra $\mathsf{Up}(Y, \leq_1)_{\Box_1\Diamond_2}$ is simply $\mathsf{Up}(Y, \leq_1)$.

2. The following intuitive explanation of how FM-frames differ from Kripke frames is inspired by Massas 2016, § 5.4.[51] Think of each point x in an FM-frame as a *partial description* of an information state i_x, which one may model as a *partial function* $x : \mathbb{N} \to \{0, 1\}$. Think of $\mathbb{N}$ as coding statements that may or may not be verified in an information state. If $x(n) = 1$, then x reveals that the n-th statement has been verified in i_x; if $x(n) = 0$, then x reveals that the n-th statement has *not* been verified in i_x; and if $x(n)$ is undefined, then x does not reveal whether or not the n-th statement has been verified in i_x. Think of $x \leq_1 y$ as meaning that every fact about what *has been verified* that is revealed by x is maintained by y, so that if $x(n) = 1$, then $y(n) = 1$. Then we will say that y is an *enrichment* of x; this is compatible with some statement going from being *not verified* according to x to being *verified* according to y. By contrast, think of $x \leq_2 y$ as meaning that not only $x \leq_1 y$ but also every fact about what *has not yet been verified* that is revealed by x is maintained by y, so that if $x(n) = 0$, then $y(n) = 0$.[52] Then we will say that y is an *extension* of x; for as partial functions, $x \subseteq y$. This explains why $\leq_2$ may be a proper subrelation of $\leq_1$. For example, if $x = \{\langle n, 1\rangle, \langle m, 0\rangle\}$ and $y = \{\langle n, 1\rangle, \langle m, 1\rangle\}$, then we have $x \leq_1 y$ but $x \not\leq_2 y$. Allowing points to be *partial* descriptions of information states explains why $\leq_2$ need not be the identity relation as in Kripke semantics. For example, if $x = \{\langle n, 1\rangle, \langle m, 0\rangle\}$ while $z = \{\langle n, 1\rangle, \langle m, 0\rangle, \langle q, 1\rangle\}$, then $x \neq z$ but $x \leq_2 z$. Finally, we can use the terminology above to explain satisfaction in FM-semantics: x satisfies p if any enrichment of x (any $y \geq_1 x$) has an extension (a $z \geq_2 y$) according to which p is verified.

FM-frames are related to Dragalin frames by the following result of Bezhanishvili and Holliday 2016.

Theorem 4.30. For any Dragalin frame (X, D), there is an FM-frame $(Y, \leq_1, \leq_2, F)$ such that the nuclear algebras $(\mathsf{Up}(X), j_D)$ and $(\mathsf{Up}(Y, \leq_1)_F, \Box_1\Diamond_2)$ are isomorphic. Moreover, if (X, D) is normal, then $(Y, \leq_1, \leq_2, F)$ is normal.

Proof. (Sketch) We sketch the proof for the case where the Dragalin frame is normal. First, we use the fact that any Dragalin frame can be made *convex*, meaning that for each $S \in D(x)$, $S = {\uparrow}x \cap {\downarrow}S$, by simply replacing each $S \in D(x)$ by ${\uparrow}x \cap {\downarrow}S$ (see Bezhanishvili and Holliday 2016, Prop. 3.13). The transformation of convex normal Dragalin frames into normal FM-frames is similar to the transformation of intuitionistic neighborhood frames into intuitionistic relational frames in Kojima 2012 (cf. the transformation of monotonic neighborhood frames into birelational frames in Kracht and Wolter 1999). Given a convex normal Dragalin frame $(X, \leq, D)$, we define a normal FM-frame $(Y, \leq_1, \leq_2)$ as follows:

- $Y = \{(x, S) \mid x \in X, S \in D(x)\}$;
- $(x, S) \leq_1 (y, T)$ iff $x \leq y$;
- $(x, S) \leq_2 (y, T)$ iff $T \subseteq S$.

To see that $\leq_2$ is a subrelation of $\leq_1$, suppose $(x, S) \leq_2 (y, T)$, so $T \subseteq S$. By convexity, $T \subseteq {\uparrow}y$, so $y \in {\downarrow}T$ since $T \neq \varnothing$ by (1°). Thus, $y \in {\uparrow}y \cap {\downarrow}T = T$. Hence $y \in S$, which with $S = {\uparrow}x \cap {\downarrow}S$ implies $x \leq y$. Therefore, $(x, S) \leq_1 (y, T)$.

[51] For a different intuitive explanation of FM-frames, see Fairtlough and Mendler 2002, p. 70.

[52] This interpretation yields that $\leq_2$ is a partial order. However, this is a harmless assumption, as follows from Theorem 4.33 and Footnote 53.

Define $f : \mathsf{Up}(X, \leq) \to \mathsf{Up}(Y, \leq_1)$ by

$$f(U) = \{(x, S) \mid x \in U, S \in D(x)\}.$$

Then f is an isomorphism between the nuclear algebras $(\mathsf{Up}(X), j_D)$ and $(\mathsf{Up}(Y, \leq_1), \Box_1\Diamond_2)$ (for details, see Bezhanishvili and Holliday 2016, Thm. 4.7). $\square$

We can now relate FM-frames to nuclear frames and locales as follows.

Corollary 4.31.

1. For any nuclear frame (X, j), there is an FM-frame $(Y, \leq_1, \leq_2, F)$ such that the nuclear algebras $(\mathsf{Up}(X), j)$ and $(\mathsf{Up}(Y, \leq_1)_F, \Box_1\Diamond_2)$ are isomorphic. Moreover, if j is dense, then $(Y, \leq_1, \leq_2, F)$ is normal.

2. For every locale L, there is a normal FM-frame $(Y, \leq_1, \leq_2)$ such that L is isomorphic to $\mathsf{Up}(Y, \leq_1)_{\Box_1\Diamond_2}$.[53]

Proof. For part 1, apply Theorems 4.26 and 4.30. For part 2, apply Theorems 4.25 and 4.30. $\square$

The normal FM-frame produced for Corollary 4.31.2 by successively applying the transformations of Theorems 4.25 and 4.30 is a substructure of the following FM-frame, which is related to the representation of (complete) lattices in Urquhart 1978 and Allwein and MacCaull 2001 (for details, see Bezhanishvili et al. 2018).

Definition 4.32. The *canonical FM-frame* of a locale L is the normal FM-frame $(X_L, \leq_1, \leq_2)$ defined as follows, where $\leq$ is the order in L:

1. $X_L = \{(a, b) \in L^2 \mid a \nleq b\}$:

2. $(a, b) \leq_1 (c, d)$ iff $a \geq c$;

3. $(a, b) \leq_2 (c, d)$ iff $a \geq c$ and $b \leq d$.

Using this construction, we can give a direct proof that every locale is representable as the algebra of fixpoints of an FM-frame. This is essentially the approach of Massas [2016, Cor. 6.3.10], except that Massas constructs a smaller substructure of the canonical FM-frame (cf. Bezhanishvili et al. 2018).

Theorem 4.33. If L is a locale, then L is isomorphic to $\mathsf{Up}(X_L, \leq_1)_{\Box_1\Diamond_2}$.

Proof. The elements of the form $(a, 0)$ ordered by $\leq_1$ form a lattice dually isomorphic to $L \setminus \{0\}$. Thus, the principal $\leq_1$-upsets of elements of the form $(a, 0)$, plus $\varnothing$, ordered by $\subseteq$, form a lattice isomorphic to L. Therefore, to prove the theorem it suffices to show that the $\Box_1\Diamond_2$-fixpoints are exactly the principal $\leq_1$-upsets of elements of the form $(a, 0)$, plus $\varnothing$.

First, we show that each principal $\leq_1$-upset $\uparrow_1(a, 0)$ is a $\Box_1\Diamond_2$-fixpoint. Suppose $(c, d) \notin \uparrow_1(a, 0)$, so $c \nleq a$. Then $(c, a) \in X_L$ and $(c, d) \leq_1 (c, a)$. Now consider any $(c', a') \geq_2 (c, a)$, so $c' \nleq a'$ and $a' \geq a$. Then obviously $a \ngeq c'$, so $(a, 0) \nleq_1 (c', a')$. Hence $(c, a) \notin \Diamond_2\uparrow_1(a, 0)$, which with $(c, d) \leq_1 (c, a)$ implies $(c, d) \notin \Box_1\Diamond_2\uparrow_1(a, 0)$.

Conversely, suppose $U = \{(a_i, b_i) \mid i \in I\}$ is a $\Box_1\Diamond_2$-fixpoint. Let $e = \bigvee\{a_i \mid i \in I\}$ (taking the join in L). We claim that $U = \uparrow_1(e, 0)$. Clearly $U \subseteq \uparrow_1(e, 0)$ (remember that $\leq_1$ reverses the order $\leq$). Since

[53] In fact, for part 2 we may take $\leq_1$ and $\leq_2$ to be partial orders (see Proposition 4.5 of Bezhanishvili and Holliday 2016). It is an open question whether a stronger version of part 1 holds in which $\leq_1$ and $\leq_2$ are partial orders.

U is a $\leq_1$-upset, to show $U \supseteq \uparrow_1(e, 0)$ it suffices to show that $(e, 0) \in U$. Since U is a $\Box_1\Diamond_2$-fixpoint, it suffices to show that for any $(a, b) \geq_1 (e, 0)$ there is a $(a', b') \geq_2 (a, b)$ such that $(a', b') \in U$. Consider any $(a, b) \geq_1 (e, 0)$, so $a \not\leq b$ and $a \leq e$. Then for some $i \in I$, we have $a \wedge a_i \not\leq b$. For if $a \wedge a_i \leq b$ for every $i \in I$, then $\bigvee\{a \wedge a_i \mid i \in I\} \leq b$, which by the join-infinite distributive law implies $a \wedge e \leq b$, which with $a \leq e$ implies $a \leq b$, contradicting $a \not\leq b$. Let $j \in I$ be such that $a \wedge a_j \not\leq b$. Let $a' = a \wedge a_j$ and $b' = a \to b$. Since $a' \not\leq b$, we also have $a' \not\leq b'$ and hence $(a', b') \in X_L$. In addition, since $a' \leq a$ and $b' \geq b$, we have $(a', b') \geq_2 (a, b)$. Finally, since $U = \{(a_i, b_i) \mid i \in I\}$ is a $\leq_1$-upset, $a' \leq a_j$ implies $(a', b') \in U$. This completes the proof that $(e, 0) \in U$. $\Box$

In light of the above results (see the summary in Figure 4.3), we arrive at the promised equivalence of semantics:

$$\text{Locales} \equiv \text{Nuclear} \equiv \text{Dragalin} \equiv \text{FM}.$$

As noted at the beginning of § 4, this equivalence also expands to include the cover semantics of Goldblatt 2011 (see Bezhanishvili and Holliday 2018).

The results of this section and § 4.6 lead to open problems parallel to those stated in previous sections.

Problem 5. Is there a variety of Heyting algebras that can be generated by locales but not by spatial locales? Equivalently, is there an intermediate logic that is Dragalin/FM complete but topologically incomplete?

Problem 6. Is every variety of Heyting algebras generated by locales? Equivalently, is every intermediate logic Dragalin/FM complete?

5 Conclusion

In § 1.2 we recalled Rasiowa and Sikorski's [1963] amazement that Brouwer's philosophy of mathematics led to the development of a formal system of intuitionistic logic that was later seen to be deeply connected to topology. In this paper, we hope to have given readers a glimpse of the rich mathematical landscape to which the semantical study of intuitionistic logic leads. Yet there are large parts of the landscape that we have not touched upon here at all. One could continue with a study of semantics for intuitionistic predicate logic and its extensions (see, e.g., Rasiowa and Sikorski 1963, Scott 2008, Gabbay et al. 2009). Or one could follow the connection to modal logic mentioned in the introduction, with a study of intuitionistic-to-modal translations (see, e.g., Chagrov and Zakharyaschev 1992). One could even study a semantic hierarchy parallel to ours in the context of intuitionistic modal logic (see, e.g., Wolter and Zakharyaschev 1999). But there are also quite different directions to explore: for example, the Curry-Howard-Lambek correspondence between intuitionistic logic, simply-typed lambda calculus, and Cartesian closed categories (see, e.g., Lambek and Scott 1986), or higher-order intuitionistic logic as the internal language of toposes (see, e.g., Fourman and Scott 1979, Goldblatt 1984, Lambek and Scott 1986). That intuitionistic logic has proved to be of such mathematical interest, in ways unanticipated by Brouwer, is a testament to the great value of Brouwer's legacy.

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