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Ewald Acceleration for the Dyadic Green's Functions for a Linear Array of Dipoles and a Dipole in a Parallel-Plate Waveguide

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Introduction

We consider the Green function (GF) representation for the field at $\mathbf{r}=\rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{z}}$ radiated by a linear array of point sources at $\mathbf{r}'_n=\mathbf{r}'+nd\hat{\mathbf{z}}$, $n=0,\pm1,\pm2,\ldots$, where $\mathbf{r}'=z'\hat{\mathbf{z}}$ denotes the source in the reference array cell. The array is linearly phased along the z direction with wavenumber k_{z0}, as shown in Fig.1. Due to the symmetry of the problem we use cylindrical coordinates $\mathbf{r}\equiv(\boldsymbol{\rho},z)$, where $\boldsymbol{\rho}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$. Here the hat $\wedge$ denotes unit vectors and we use the $\exp(j\omega t)$ time convention.

The radiated field can be represented as a purely spatial sum of spherical waves or as a purely spectral sum of cylindrical waves. Both these representations are slowly convergent. Furthermore, the purely spatial sum cannot be applied for complex wavenumbers k_{z0} while the purely spectral sum can not be applied when the observation point is on the array axis (i.e., $\rho=0$). The rapidly convergent representation of the scalar field devoid of these disadvantages was obtained in [1,2] using the Ewald method.

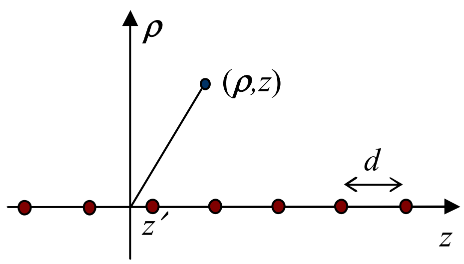


Fig.1. The linear array of dipoles with period d

We provide here the *dyadic* Green's functions (GFs) for a linear array of dipoles in *free space* and for a dipole in a *parallel-plate waveguide*. Both GFs are expressed in a very convenient hybrid series representation involving both spatial and spectral terms, obtained by applying the Ewald method. Only a few terms are sufficient to obtained very high accuracy. This paper thus extends the work in [1,2].

Dyadic Green's function for the linear array of dipoles

The differential electric field produced by a linear array of arbitrary current sources can be represented as

$$d\mathbf{E}(\mathbf{r})=\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}\,dv, \tag{1}$$

where $\mathbf{J}$ is the current density in a volume dv at $\mathbf{r}'$ within the reference unit cell, $\underline{\mathbf{G}}$ is the dyadic GF for the linear array of dipoles represented in terms of the scalar periodic Green's function $G(\mathbf{r},\mathbf{r}')=\sum_{n=-\infty}^{\infty}\exp\left[-jk(\mathbf{r}-\mathbf{r}'_n)-jk_{z0}nd\right]/\left(4\pi|\mathbf{r}-\mathbf{r}'_n|\right)$ (see [1,2]) as

$$\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{1}{j\omega\varepsilon}\left[k^2G(\mathbf{r},\mathbf{r}')\underline{\mathbf{I}}+(\nabla\nabla)G(\mathbf{r},\mathbf{r}')\right], \tag{2}$$

ε is the permittivity of the host medium, $(\nabla\nabla)G(\mathbf{r},\mathbf{r}')$ is the Hessian matrix of G (the matrix of the second derivatives of G with respect to $\mathbf{r}$).
Using the Ewald method the dyadic GF can be represented as the sum of the dyadic spectral and spatial series

$$\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \underline{\mathbf{G}}_{\text{spectral}}(\mathbf{r},\mathbf{r}') + \underline{\mathbf{G}}_{\text{spatial}}(\mathbf{r},\mathbf{r}'), \tag{3}$$

where

$$\underline{\mathbf{G}}_{\text{spectral}}(\mathbf{r},\mathbf{r}') = \frac{1}{j\omega\varepsilon}\left[k^2 G_{\text{spectral}}(\mathbf{r},\mathbf{r}')\underline{\mathbf{I}} + (\nabla\nabla)G_{\text{spectral}}(\mathbf{r},\mathbf{r}')\right], \tag{4}$$

$$\underline{\mathbf{G}}_{\text{spatial}}(\mathbf{r},\mathbf{r}') = \frac{1}{j\omega\varepsilon}\left[k^2 G_{\text{spatial}}(\mathbf{r},\mathbf{r}')\underline{\mathbf{I}} + (\nabla\nabla)G_{\text{spatial}}(\mathbf{r},\mathbf{r}')\right]. \tag{5}$$

and the scalar Ewald terms $G_{\text{spectral}}(\mathbf{r},\mathbf{r}')$ and $G_{\text{spatial}}(\mathbf{r},\mathbf{r}')$ have been derived in [1,2]. Differentiating in cylindrical coordinates leads to

$$(\nabla\nabla)G_{\text{spectral}}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi d}\sum_{q=-\infty}^{\infty} e^{-jk_{zq}(z-z')} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!}(\rho E)^{2p} E_{p+1}\left(\frac{-k_{\rho q}^2}{4E^2}\right)\times \\ \left[\frac{4p(p-1)}{\rho^4}\boldsymbol{\rho}\boldsymbol{\rho} + \frac{2p}{\rho^2}(\underline{\mathbf{I}} - \hat{\mathbf{z}}\hat{\mathbf{z}}) - 2jk_{zq}\frac{p}{\rho^2}(\boldsymbol{\rho}\hat{\mathbf{z}} + \hat{\mathbf{z}}\boldsymbol{\rho}) - k_{zq}^2\hat{\mathbf{z}}\hat{\mathbf{z}}\right], \tag{6}$$

with

$$k_{zq} = k_{z0} + 2\pi q/d, \qquad k_{\rho q} = \sqrt{k^2 - k_{zq}^2}. \tag{7}$$

The sign of the root in (7) must be chosen such that Im $k_{\rho q} \le 0$, assuming that we want proper (exponentially decaying) solutions. E is the Ewald splitting parameter, and $E_p(x)$ is the pth order exponential integral. Differentiation of the spatial term leads to

$$(\nabla\nabla)G_{\text{spatial}}(\mathbf{r},\mathbf{r}') = \frac{1}{8\pi}\sum_{n=-\infty}^{\infty} e^{-jk_{z0}nd}\,\underline{\mathbf{F}}_{spatial,n}, \tag{8}$$

where

$$\underline{\mathbf{F}}_{spatial,n} = \left(\frac{f'(R_n)}{R_n^2} - \frac{f(R_n)}{R_n^3}\right)\underline{\mathbf{I}} + \left(\frac{f''(R_n)}{R_n^3} - \frac{3f'(R_n)}{R_n^4} + \frac{3f(R_n)}{R_n^5}\right)\mathbf{R}_n\mathbf{R}_n, \tag{9}$$

$$\mathbf{R}_n = \boldsymbol{\rho} + (z - z_n')\hat{\mathbf{z}}, \tag{10}$$

$$f(R) = e^{jkR}\operatorname{erfc}[\beta_+(R)] + e^{-jkR}\operatorname{erfc}[\beta_-(R)], \qquad \beta_\pm(R) = ER \pm \frac{jk}{2E}. \tag{11}$$

Dyadic Green's function for a dipole in a parallel-plate waveguide

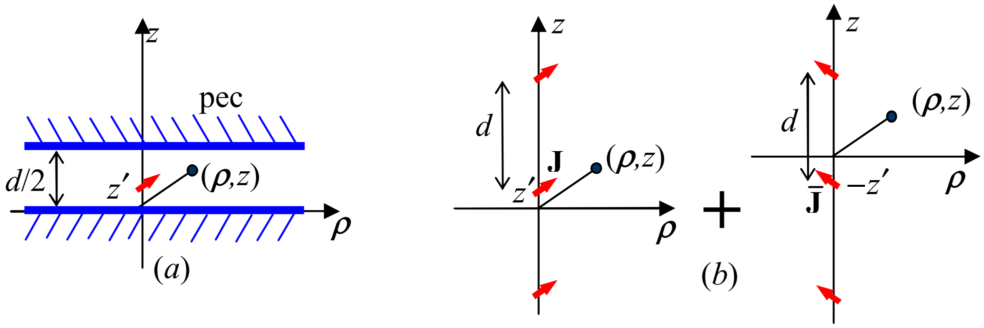


Fig 2. (a) A parallel-plate waveguide with height $d/2$ excited by a dipole at z'. (b) The equivalent problem of two arrays, each with a period d

Consider an electric dipole in a parallel-plate waveguide (PPW) (Fig. 2). It is convenient to denote the distance between the plates of the waveguide by $d/2$. Using the method of reflection and image principle we obtain the equivalent d-periodic array of dipoles with two dipoles per period: current densities $\mathbf{J} = J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}$ at the positions $(z' + nd)\hat{\mathbf{z}}$, and $\bar{\mathbf{J}} = -J_x\hat{\mathbf{x}} - J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}$ at the positions $(-z' + nd)\hat{\mathbf{z}}$, with n an integer. The differential electric field in the PPW from an arbitrary current source within the PPW can then be calculated by the formula

$$d\mathbf{E}(\mathbf{r}) = \underline{\mathbf{G}}(\mathbf{r}, z'\hat{\mathbf{z}}) \cdot \mathbf{J}dv + \underline{\mathbf{G}}(\mathbf{r}, -z'\hat{\mathbf{z}}) \cdot \bar{\mathbf{J}}dv, \tag{12}$$

where $\underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is the same as in (2) with $k_{z0} = 0$. The electric field dyadic GF for an electric current density $\mathbf{J}$ in a volume dv at the position $z'\hat{\mathbf{z}}$ in a PPW can be represented as

$$\underline{\mathbf{G}}_{PPW}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{G}}(\mathbf{r}, z'\hat{\mathbf{z}}) - \underline{\mathbf{G}}(\mathbf{r}, -z'\hat{\mathbf{z}}) \cdot \underline{\boldsymbol{\sigma}}_z . \tag{13}$$

where $\underline{\boldsymbol{\sigma}}_z = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} - \hat{\mathbf{z}}\hat{\mathbf{z}}$ is the operator of reflection with respect to the xy-plane.

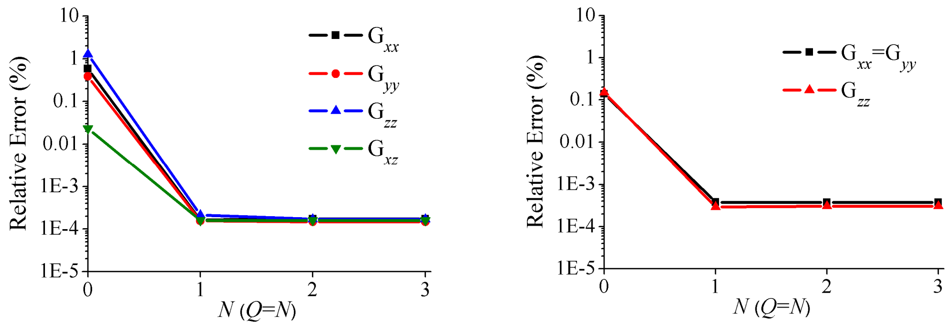
Numerical results: investigation of the convergence rate of the Ewald series

The rate of convergence of the Ewald series for the dyadic GF is illustrated in Fig.3, where the relative error

$$relative\ error\ (\%) = 100 \times \left| G^{Ewald} - G^{exact} \right| / \left| G^{exact} \right| \tag{14}$$

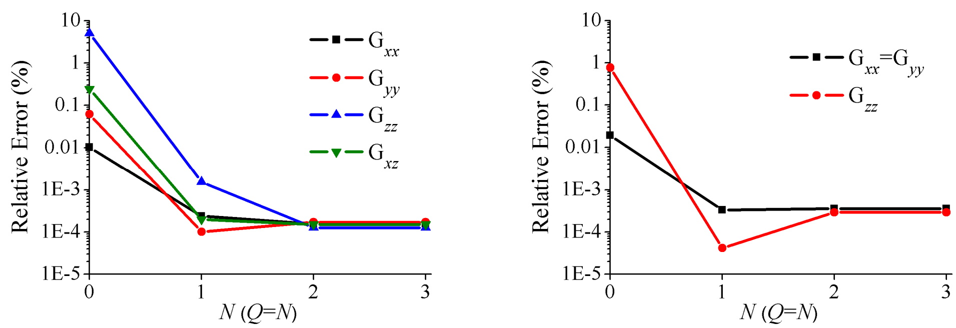
is plotted versus the number $N = Q$, that indicates the total numbers $2N+1 = 2Q+1$ of n- and q-terms in the series (6), (8), for all the nonzero components of the dyad $\underline{\mathbf{G}}$. The reference values $\underline{\mathbf{G}}^{\text{exact}}$ were evaluated in Fig. 3(a) via the pure spectral series when the observation point does not lie on the array axis ($\rho \neq 0$), and in Fig. 3(b) via the pure spatial series when it lies on the axis ($\rho = 0$). A sufficiently large number of terms were used in the pure spectral and spatial series to achieve accuracy up to seven significant figures (2,000 terms in the spectral and 40,000 terms in the spatial series were enough). The period of the array is $d = 0.6\lambda_0$,

where λ_0 is the free space wavelength, the phasing is $k_{z0} = 0.1k$, the $n = 0$ point source is at the coordinate origin, the observation point is at $(x,y,z) = (0.1,0,0.1)\,d$ in Fig. 3(a) and at $(x,y,z) = (0,0,0.1)\,d$ in Fig. 3(b).
The same type of results for the GF of a dipole in a parallel-plate waveguide are shown in Fig.4. In Fig. 4(a), $(x,y,z) = (0.1,0,0.2)\,d$ and $(x',y',z') = (0,0,0.1)\,d$. In Fig. 4(b), $(x,y,z) = (0,0,0.2)\,d$ and $(x',y',z') = (0,0,0.1)\,d$, and the distance between the plates of the waveguide is $d/2 = 0.3\lambda_0$. The relative error of the Ewald representation saturates because of the subroutine of the error function used, whose accuracy is fixed to 6 significant figures [2]. Note that an extremely small number of terms in (6) and (8) is needed to compute the dyadic GF.



(a) (b)

Fig. 3. Convergence of the Ewald dyadic series (4) and (5) for the GF of a linear array of dipole sources in free space, showing the number of needed terms in (6) and (8). See text for the parameters in (a) and (b).



(a) (b)

Fig.4. Convergence of the Ewald dyadic series for GF (13) of the dipole in the parallel-plate waveguide, showing the number of needed terms in (6) and (8). See text for the parameters in (a) and (b).

References

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