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Author
Abrahamson, D

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The Case of the Binomial

Dor Abrahamson

The Problem

Consider a penny. It is flipped four times. Now consider two possible outcomes of this experiment:

(a) Heads, Heads, Heads, Heads
(b) Heads, Heads, Tails, Tails

Is one of these two outcomes more likely than the other, or are they equally likely?

This item targets basic knowledge of probability. Namely, it aims to evoke the phenomenon of a random generator (e.g., coins, dice, spinners, etc.) as a context for eliciting and gauging an understanding of randomness, independence, and distribution. Solving this item does not demand any numerical reasoning or arithmetical calculation—one need only grasp the logic of the situation so as to determine the appropriate response. We might thus hope that graduates of the U.S. school system, who have studied at least basic probability concepts, fare well on this simple item. But do they?

According to probability theory, this penny-flipping situation describes a *compound-event* problem, because each trial is defined as a *conjunction* of two or more *singleton* events. Here, each trial is a conjunction of four singleton coin flips. Also, this is a *binomial* situation, because each singleton-event variable—that is, each coin flip—yields one of exactly two possible values (Heads or Tails). In this particular compound-event binomial problem, the two hypothetical experimental outcomes, HHHH and HHTT, are equiprobable, because the conjunction consists of a sequence of *independent events*. That is, the coin has no memory, so each of the four singleton flips is not affected by the result of a preceding flip nor does it affect the result of a subsequent flip. And yet an overwhelming proportion of the adult population chooses option (b), HHTT, as more likely, arguing that it appears to better capture the structure and function of the random generator, that is, its equal numbers of Heads and Tails better represent the essence of a two-sided coin (Tversky & Kahneman, 1974). These findings have been robustly replicated in numerous studies (Jones, Langrall, & Mooney, 2007). Moreover, there is reason to believe that the findings reflect early or even innate reasoning mechanisms. For example, Xu and Vashti (2008) showed eight-month-olds a tub full of balls of two colors from which there issued into a narrow tube a sample of several balls that were thus arranged sequentially. The infants' reactions to different types of samples suggested that they found more interesting those samples whose ratio composition was less representative of the population, for example, a sample of four green balls was more interesting than a sample of two green balls and two blue balls, irrespective of their order.[1]

What are the pedagogical implications of these empirical findings of flawed probabilistic reasoning across the ages? Should we conclude that the subject matter of probability is inherently counterintuitive and that therefore students can at best develop basic fluency in the rote execution of probability solution procedures?

Overview of the Chapter

In the following, I will attempt to cast a more optimistic light on the prospects of a grounded probability education, in which students can make sense of fundamental concepts such as the binomial. I begin by building the following case: (1) There are at least two ways of interpreting inscribed strings such as HHTT; (2) the natural human inclination is to ignore the order of singleton events in conjunctions, for example, to construe a comparison of HHHH and HHTT as though it were a comparison of "four Heads" (hence, 4H) and "two Heads and two Tails in any order" (hence, 2H2T); (3) ignoring the order is the source of participants' canonical error on this item; (4) the inclination to ignore the order results from a tacit perceptual constraint imposed naturally by innate or very early cognitive mechanisms; and (5) once we appreciate that respondents are construing orderless compound events, we may acknowledge the mathematical truth of stating that HHTT is more likely than HHHH (see below).

Yet where would such an exoneration of human intuition leave us in terms of instructional goals? Can students advance in their mathematical learning of probability if they persist to ignore the dimension of order within compound events? Probably not. Attending to the order of singleton events within compound events is absolutely vital for performing *combinatorial analysis*, a key classicist-probability procedure used for establishing anticipated frequencies of random events. Combinatorial analysis is performed by listing all the possible outcomes—that is, building the *sample space* (event space, probability space) of the experiment—and calculating the proportion of *favorable events* out of the entire sample space. For example, there are 16 different ways of arranging four singleton binomial events—TTTT, TTTH, TTHT, THTT, HTTT, TTHH, THTH, HTTH, THHT, HTHT, HHTT, THHH, HTHH, HHTH, HHHT, HHHH (hence, the *elemental events*)—and six of these satisfy the criterion of having two Heads and two Tails in any order (hence, the *aggregate event* 2H2T). Therefore, there is a 6/16 probability (0.375) that an experimental trial will result in 2H2T. However, if students are inclined to aggregate the singletons of compound-event outcomes, how are they to learn the notions of combinatorial analysis and sample space?

This juxtaposition of naïve and professional *orientations of view* (Stevens & Hall, 1998) toward compound-event outcomes—as either aggregate or elemental events—suggests that pedagogical efforts could focus on helping students see mathematical signs such as HHTT in the culturally intended manner. At the same time, the *constructivist* educational perspective (von Glasersfeld, 1987) warns us against forsaking students' natural ways of constructing the world, lest they fail to make sense of mathematical concepts. The current chapter addresses this pedagogical tension between tacit and cultural constructions of mathematical inscriptions. I describe the instructional design for the binomial that we have created in light of this research problem as well as preliminary empirical work for evaluating the potential of our design to foster a deep understanding of the binomial. In turn, I use the empirical results to comment on issues of mathematical epistemology, cognition, and pedagogy.

Background and Motivation: Toward Bridging Tacit and Mathematical Views

A broadly accepted fundamental hypothesis of cognitive development researchers working on the origins and nature of mathematical reasoning is that humans posses domain-specific, ecologically adapted cerebral mechanisms for processing quantitative information (Dahaene, 1997; Devlin, 1999; Gelman & Williams, 1998; Gigerenzer & Brighton, 2009). Specifically, the "ecological hypothesis" implies that in some natural contexts, which mathematicians would model as binomial, attending to outcomes as aggregate, rather than elemental events, is advantageous to human survival. Our agenda has been to identify and simulate these contexts, so that we can build instruction that embraces learners' intuitive reasoning yet steers it toward mathematically normative understandings (Abrahamson, 2009a,b; Clement, Brown, & Zeitsman, 1989; Fischbein, 1975; Smith, diSessa, & Roschelle, 1993; Wilensky, 1997). Thus, students may be able to make sense of the notion of probability rather than just memorize formulas with little if any retention, let alone understanding. We have focused on binomial situations such as the coin-flipping scenario because these situations are both fundamental to the study of probability (Weisstein, 2006) and their learning is well researched (for a review, see Jones et al., 2007). By working in the reciprocal tracks of scholarship and design, the study potentially contributes to developing theoretical models of learning as well as viable instructional materials.

Our emerging conjecture is that, whereas humans possess an ecologically powerful cognitive mechanism that could potentially ground the binomial, its compatibility with mathematical notions depends on the inscriptional forms, discursive framings, and pragmatics that encapsulate formal constructions of mathematical information (cf. Bamberger, 1996; Borovcnik & Bentz, 1991; Gelman & Williams, 1998; Gigerenzer, 1998; diSessa & Wagner, 2005). That is, mathematical semiotic formulations of source phenomena do not necessarily cater to naturalistic constructions of these same phenomena (Bamberger & diSessa, 2003). Further encumbering the adoption of mathematical views, gestalt perception of inscriptions operates temporally a priori to analytic reasoning and is very slow to become entrained (Navon, 1977; Stroop, 1935; Van Dooren, De Bock, Weyers, & Verschaffel, 2004).

We thus face the following pedagogical dilemma. In order to trigger students' tacit knowledge that is relevant to the study of a mathematical concept, we are to create instructional materials aligned with students' ecologically viable perceptions—under such conditions, students' initial judgments are expected to be mathematically correct, if approximate. However, in order to advance from these qualitative judgments to mathematical formulations of the phenomenon, students are to adopt a new, mediated view of the phenomenon, yet such a demand is liable to defeat the entire rationale of recruiting students' tacit knowledge. How, then, might we help students coordinate the tacit and cultural ways of perceiving a certain class of phenomena, so that they can sustain their intuitive views even as they adopt the mathematical formulations inherent to modeling this phenomenon? That is, how might students build meaning for a semiotic artifact, when doing so demands a counterintuitive view of the world?

A Proposed Design for the Binomial

We maintain that students should begin by working with a source phenomenon rather than its semiotic reformulation. Thus, the mathematical validity of students' initial inference regarding quantitative properties of the phenomenon would be objective, not only contingent on interpreting the inference vis-à-vis tacit perceptual construction. Subsequently, students should interact with the mathematical reformulation of the phenomena. Through negotiating tacit and cultural perceptions of these inscriptions, students are to recognize their own perceptual biases. Our agenda has been to engender and then examine students' struggle to coordinate unmediated and mediated perceptions of mathematical situations. We thus wished to articulate epistemological and cognitive aspects of mathematical learning, a developmental process that we perceive as a reconciling of tacit and cultural resources (Abrahamson, 2003, 2004, 2008b, 2009a,b, submitted for review; Abrahamson & Cendak, 2006; Schön, 1981; diSessa, Hammer, Sherin, & Kolpakowski, 1991; Wilson, 1998).

Design-Based Research of Grounded Mathematical Learning

Our scholarly focus on developing theory through developing instructional materials has suggested the suitability of the design-based research approach (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992; Confrey, 2005; Gravemeijer, 1994; Sandoval & Bell, 2004). In this methodological approach, theory and design are mutually informative through iterated cycles of research and development. This relatively new approach is itself still under development and is thus still fraught with methodological challenges relative to more established research practices of the cognitive and learning sciences represented in this volume (Kelly, 2003, 2004). Yet any methodological shortcomings of this approach appear to trade off favorably with the emergence of new theoretical constructs as well as instructional material with potential educational impact (Barab et al., 2007; Edelson, 2002; diSessa & Cobb, 2004). Accordingly, both the theory and the design discussed in this chapter are in progress, yet these tentative results may nevertheless be of value to theoreticians, designers, and instructors.

We have thus designed and built a set of mixed-media instructional materials for the binomial and interviewed grade 4–6 students as they engaged in activities based on these materials (Abrahamson, 2009b; Abrahamson & Cendak, 2006; Abrahamson, Janusz, & Wilensky, 2006; Abrahamson & Wilensky, 2005a,b).

A Design for the Binomial

The learning materials revolved around a single mathematical object, the *4-block*: a 2-by-2 grid of four location "variables" that each can take on one of two color "values," green or blue (see Figure 18.1). The 4-block is instantiated in both substantive

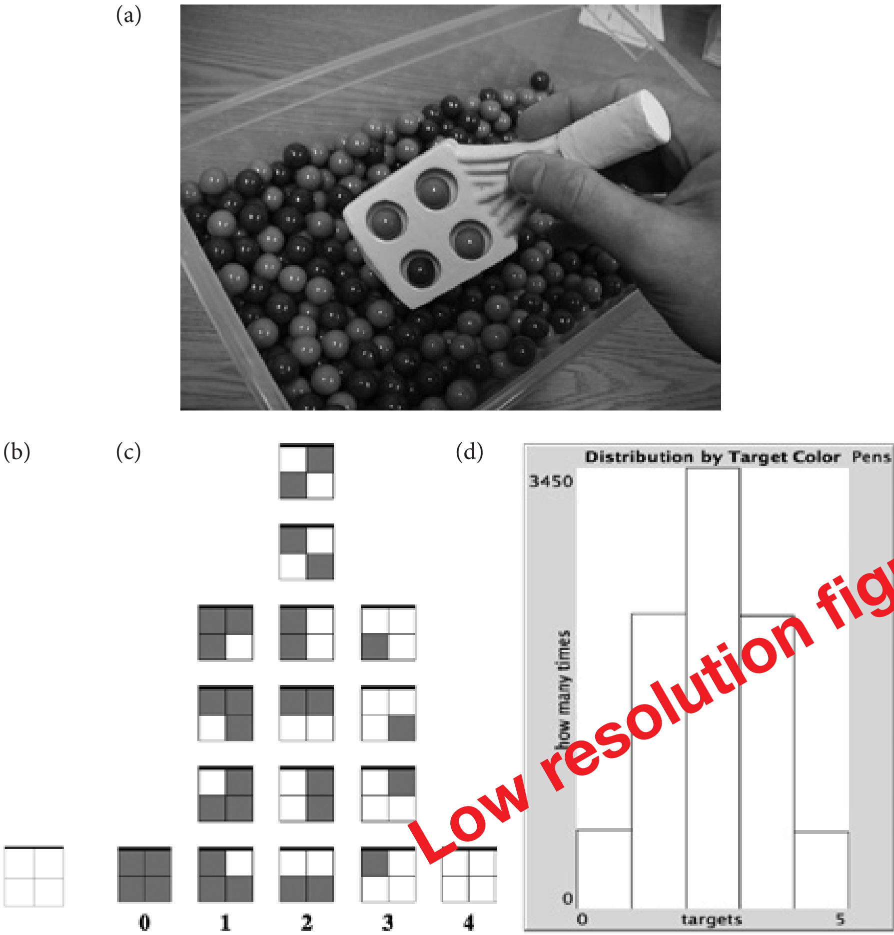

Figure 18.1 (a) The Marble Scooper Bearing a Sample of Four Marbles Drawn from a Box Containing an Equal Number of Green and Blue Marbles. (b) A 4-Block Template for Conducting Combinatorial Analysis of the Experiment. (c) The Combinations Tower Is the Distributed Sample Space, Assembled by Categories into a Pictograph Bar Chart. (d) Histogram from a Computer-based Simulation of the Marbles Experiment.

and virtual environments and is used in both theoretical-probability contexts (figuring out what can be obtained and how often) and empirical-probability contexts (running real or simulated experiments while monitoring cumulative results).

In the first activity (see Figure 18.1a), the student is shown a tub full of hundreds of green and blue marbles of equal numbers. An interviewer demonstrates the operation of a 4-block *marbles scooper*, a utensil for drawing out samples of four marbles. Next, the interviewer asks the participant to predict what will happen when we scoop. This question is phrased ambiguously, so as not to specify interpretation along three dimensions: (a) the "probable" (anticipating the outcome distribution) or the "possible" (creating the sample space); (b) the next scoop or the long term; or (c) an aggregate event, for example, 3g1b, or an elemental event, for example, a scoop that has a blue marble in the bottom right-hand corner, with the rest being green. If a student asks about the color composition of the marbles box, the interviewer replies that there are equal numbers of each color. Once students offer a prediction, we ask them to warrant it. We predicted that these students, who had not studied probability, would articulate their expectations in terms of aggregate, not elemental, events. However, we also hoped that the explicit spatial configuration of the scooper would create an opportunity for the dyad to discuss the dimension of order.

In the second activity, students are provided with a pile of empty 4-block cards (see Figure 18.1b) and a pair of crayons and are asked to create "what we can get" (to perform combinatorial analysis). We thus shift from the "probable" to the "possible." Again, the instructions do not specify whether or not students should attempt to represent only the five aggregate events—no green, 1 green, 2 green, 3 green, or 4 green—or create the entire sample space of 16 elemental events—BBBB; BBBG, BBGB, BGBB, GBBB; GGBB, GBGB, BGBG, BBGG, BGGB, GBBG; GGGB, GGBG, GBGG, BGGG; GGGG. (Note that these sixteen elemental events are equiprobable, as in the four-coin-flips experiment, because there is an equal number of green and blue marbles in the box.) Once the sample space is complete, students are steered to assemble all 16 cards in the form of the *combinations tower*, which organizes the elemental events by their aggregate counts (see Figure 18.1c). If students do not spontaneously offer reflections on this structure, we ask them whether it "tells us" anything about the marbles-scooping experiment. We have been particularly interested whether students would discern connections between the marbles box and the combinations tower, what the nature of these connections would be, and how students would express these connections. (Figure 18.1d hints at the computer-based simulations not discussed in this chapter; see in Abrahamson, 2009a.)

Methods

Participants

Forty-six grades 4–6 students from a private K–8 school in the San Francisco Bay Area (33 percent on financial aid; 10 percent minority students) volunteered to participate in the study. The two grades 4–6 mathematics teachers reported that none of these students had had formal instruction in probability. The teachers collaboratively assigned the pool of volunteering students to three achievement levels ("high," "middle," and "low") within each grade level, on the basis of the students' performance on assessments. From these 3-by-3 grade/performance groups, we selected roughly equal numbers of students, balancing for gender, so that we worked with 28 participants. We preferred selecting students whom the teachers had indicated as typically more disposed to express their thoughts verbally. These more verbose students were distributed roughly uniformly across grade, gender, and achievement level. Whereas this selection as well as the demographics of the school may create methodological blind spots, this initial stage of the research required articulate feedback from the students as they interacted with the designed learning tools. The research team included the author—an experienced interviewer—and a novice researcher under training.

Materials

The study was conducted in the form of semistructured clinical interviews (Ginsburg, 1997; diSessa, 2007). Therefore, in addition to the learning tools (see above), we created a protocol designed to step the participant through a set of key questions, while leaving room for extemporized follow-up questions, based on student response. The protocol was developed iteratively over several pilot interviews that are not reported in this chapter.

Procedure

Over a span of three weeks, the researchers visited the school a total of nine days so as to complete the set of 28 one-to-one interviews. The interviews took place in quiet rooms within the school facilities and lasted on average 56 minutes ($SD = 12$ min.). An audio/video camera recorded all the sessions, and this footage as well as physical and virtual artifacts that the students created and the interviewers' field notes and daily debriefs were all subsequently used for data analysis. At the end of each interview, the interviewer explained to the participant the mathematical formulations of the situations and answered any remaining questions with respect to the content or the interview.

Data Analysis

Audio and video data were studied collaboratively by the design-research team using *microgenetic* qualitative analysis (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) as well as *grounded theory* techniques (Glaser & Strauss, 1967). Through the analysis, we attempted to (a) clarify and interpret students' utterances and multimodal actions; (b) understand student difficulty in coordinating aggregate- and elemental-event constructions of the sample space; (c) capture and articulate the nature of prompts that helped students overcome these difficulties; (d) identify connections students were making between the activities; (e) reveal patterns across students; (f) characterize these patterns in terms of our emergent design framework (Abrahamson & Wilensky, 2007); and (g) identify necessary improvement to the design of objects, activities, and facilitation. As patterns in the data were discerned, we used these as lenses to reexamine earlier data, until we were satisfied that we had built a coherent description of typical learning trajectories through this particular design and that all student confusions were identified and defined.

Results and Discussion

In this section, we evaluate, through the lens of our data, the prospects of creating pedagogical contexts that enable students to coordinate tacit and mathematical constructions of binomial situations.

Eliciting Students' Informal Inference for Binomial Outcome Distribution

All but three of the 28 students working with the marbles box and scooper predicted that the most likely 4-block should be 2g2b. Of these three, a single student (Grade 5 "middle" girl) did not offer any guess, and two students (Grade 4 "high" and "middle" girls) expected the plurality of scoops to have exactly three marbles of either of the two colors—a mathematically correct inference (8/16) that was possibly based on a single sample they had each scooped as they operated the device to learn its mechanism. For the rest of the students, there does not appear to be any definitive relation between the several samples they took and their guesses.

Of the 25 students who initially guessed 2g2b, 23 supported their statement by pointing to the apparently equal distribution of marbles in the bin, which they described as "half–half" or "even," whereas two students initially could not explain their guess. When we pressed students to explain how or why the "half–half"-ness of the marbles bore on the "half–half"-ness of the 4-block, only one student (Grade 6 "high" boy) elaborated, albeit he, too, could not explain *why* the half–half distributive property should carry over from the bin to the scooper (see also Abrahamson, 2007). Unsurprisingly, not a single student generated or even attempted to generate the sample space to support the intuitive inference that 2g2b would be the most frequent outcome.

We concluded that, in line with our design rationale, the marbles-scooping random generator creates for the dyad opportunities for discursive exchanges wherein students can articulate their naïve expectation of outcome frequencies in verbal forms that agree unconditionally with probability theory. Yet, as the following section demonstrates, our participants' verbatim judgments were objectively true only because the protocol had not yet required of them to engage semiotic forms wherein order is relevant.

Insight—Appropriating a Mathematical Artifact to Warrant a Tacit Inference

Asked to build the sample space, each of the participants initially created no more than five cards, which included some permutation on each of 4b, 1g3b, 2g2b, 3g1b, and 4g. As they built this aggregate-event-based sample space, the participants did not refer to the particular location of the singleton colorations within the 4-block, and not one participant suggested how this activity might support their earlier guess as to the most likely event in the scooping experiment. The participants' apparent obliviousness to the particular configuration of the green and blue values in the 4-block variable is striking, given that the 4-block cards do not enable the representation of an aggregate event that is objectively orderless. That is, any coloring-in of, say, 3g1b necessarily instantiates an elemental event—one of the four particular permutations on the 3g1b combination. Yet, just as people tend to compare HHHH and HHTT as aggregate events despite the "glaring" order in these inscriptions, so our participants ignored the 4-blocks' spatial configurations.

By and large, it was only when we guided the participants to expand the sample space so as to include all sixteen elemental events that the dimension of order was first mentioned. Students appeared confused as to why we wished them to build the additional eleven elemental events. Some students argued that they had already shown us all the possible events, so that the cards they were now asked to create were perforce redundant. It was as though we had asked our participants to measure the diameter of a coin or smell it as a means of anticipating its outcomes—the dimension of permutation was perceived as entirely irrelevant to the task at hand.

Yet, once the sample space was completed and, in any case, once the combinations tower was erected, all but one student—the same student who had not initially inferred that the most likely experimental outcome would be 2g2b—eventually offered the insight that 2g2b is the most likely aggregate event because it contains more elemental events than the others. So doing, the participants used the sample space, a mathematical artifact, to support their tacit inference. Yet how was this key insight achieved? Did this insight signal that the students fully and/or stably understood the binomial? Did they map their five-object aggregate view of likelihood onto the five *sets* of elemental events?

In Abrahamson (2009b), I build on cognitivist, sociocultural, and educational-semiotic perspectives in an attempt to explain the participants' insight. In particular, I investigate how the participants came to appropriate the combinations tower, which contains five sets of elemental events, as a *semiotic means of objectifying* (Radford, 2003) their presymbolic sense of outcome distribution, which is couched in five aggregate events. So doing, I draw on *conceptual-blending* theory (Fauconnier & Turner, 2002; Hutchins, 2005), *intuitive-rules* studies (Stavy & Tirosh, 1996), Wilson's (1998) theory of creativity as conjoining disparate ideas, and Peirce's (1931–1958) construct of *generative abduction*. I there argue that students' insight can be characterized as a *semiotic leap*.

A semiotic leap is the discursive action of identifying an available artifact as enabling the objectification of a presymbolic multimodal notion for a phenomenon previously experienced, even though the artifact initially does not appear to couch the phenomenon in the appropriate perceptual unit. The leap is nevertheless motivated by a felt need to communicate an inference, such as because of the pragmatics of a clinical interview.

The semiotic leap does not demarcate complete understanding. On the contrary, the learner has yet to engage in abductive and inductive reasoning in order to rationalize the semiotic artifact as a logical discursive tool. Indeed, when we subsequently asked participants to compare the likelihoods of two 4-block cards, one each from the 3g1b set and the 4g (see Figure 18.2), the participants still demonstrated great difficulty in reconciling the elemental-event perspective, by which the two cards are equiprobable, and the aggregate-event perspective, by which they are heteroprobable. We interpreted this difficulty as marking that the students had not yet adequately recognized their own tendency to shift between elemental and aggregate views of individual 4-blocks: participants thus believed that their inferences were mutually exclusive whereas they were in fact complementary. In a sense, disambiguating the 4-block can be viewed as the pivotal reflective activity in our design. Because in so doing students first appreciate differences between naïve and scientific formulation of compound events and can thus begin to reconcile these formulations.

Conclusions

Learners' explicit commitment to a particular way of seeing the world is essential to the process of building meaning through semiotic objectification (Radford, 2003). Yet, as psychological experiments demonstrate (Tversky & Kahneman, 1974; Xu & Vashti, 2008), once a phenomenon is reified, its tacit and semiotic formulations are liable to be discordant (cf. Bamberger & diSessa, 2003; Sfard, 2007). It is precisely such

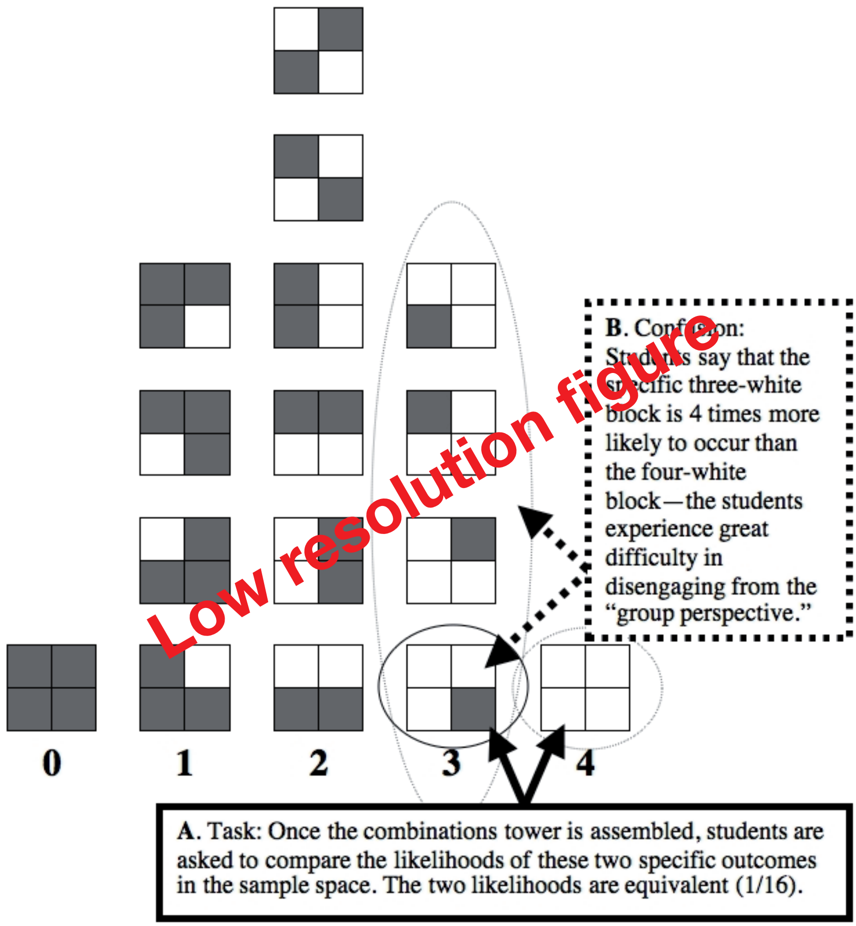

Figure 18.2 Students Confuse Aggregate and Elemental Constructions of a Compound Event.

discord that educational designers and practitioners should help students breach. So doing, students need to make explicit their implicit construction of stimuli.

This process of naming not what one *infers* when one perceives a phenomenon but *how one is perceiving it* is difficult—especially so when privileged forms of perceptions are engaged, because they are inaccessible to cognition (they are "cognitively impenetrable," Pylyshyn, 1973). Thus, for a person to explain how they are seeing the world is contingent on that person realizing *that* they are seeing the world in a particular way rather than another way (Gopnik & Rosati, 2001; Wittgenstein, 1953). And yet the equipmentality of perception is covert and is revealed only in breakdowns (Dreyfus, 1990; Heidegger, 1962; Koschmann, Kuuti, & Hickman, 1998). To bring about such breakdown, a teacher may wish to foster a semiotic conflict, which is revealed to the student as a conflict of inferences drawn for a single referent that is covertly assigned two alternative senses.

Finally, coordinating tacit and mathematical formulations of phenomena involves certain discontinuous logical progressions, wherein multimodal imagistic reasoning precedes logical rationalization. Given this, students may make sense of formal mathematical procedures only retroactively, once the germane conceptual images have been blended and anchored into the semiotic artifacts.

Final Remark and Future Work

What might be realistic objectives for the teaching and learning of probability? Fischbein (1987) maintains that certain mathematical concepts cannot be grasped with *primary intuition*, so that students have to develop alternative, *secondary intuition* that includes the heuristic of inhibiting the primary intuition. Cobb (1989), commenting on Fischbein, thus dubs primary intuition a "double-edged sword." One might assume that attending to order in compound events is a case of secondary intuition that must inhibit the primary intuition of ignoring the order. Yet how might such secondary intuition be fostered?

For "radical constructivists" (e.g., von Glasersfeld, 1987) who would provide students only minimal guidance, the prospect of directly nurturing secondary intuition is anathema. For socio-cultural theorists (Vérillon & Rabardel, 1995; Vygotsky, 1978/1930), however, mediating students' construction of knowledge through guiding their operation of cultural artifacts is humans' fundamental modus operandi, such that acculturating students to perform procedures that foster effective performance is a natural and desirable practice. Somewhere between the constructivists and socio-culturalists are adherents of "realistic mathematics education" (Freudenthal, 1986; Gravemeijer, 1994), who seek to design contexts for students' *guided reinvention* of mathematical concepts. Despite these differences in pedagogical perspectives, though, mathematics-education researchers agree on the merits of learning environments that encourage students to be reflective as they learn to operate mathematical procedures (NCTM, 2006).

Working in an emerging design framework (Abrahamson & Wilensky, 2007), we have been creating instructional materials that leverage students' concept-specific reasoning and then surface productive semiotic conflict between the student and teacher. As they participate in these activities, students experience structured opportunities to acknowledge and then reconcile phenomenologically immediate and semiotically mediated perceptions of situations wherein quantitative properties and relations have been problematized (Abrahamson, 2008a). Optimally, students need not substitute one intuition for another—rather than inhibit their primary intuition, they may sustain and qualify it ("tune it toward expertise," diSessa, 2008).

Equipped with an apparently effective means of demonstrating to students the utility of combinatorial analysis, we have still to demonstrate that future participants in our activities carry on to apply and extend this procedure correctly in situations they identify as cases of the binomial function. Design-based research projects can be very long journeys. Having been entrenched in data analysis for several years, we are now prepared to spiral through new empirical studies of the residual effects our improved activities may engender for student understanding of fundamental notions of probability.

Acknowledgement

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Note

1 Strictly speaking, this and our own proposed probability problem, discussed in this chapter, below, are considered without-replacement hypogeometric approximations of the binomial, because the four singleton events are not truly independent.

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