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A Note on Cancellation Axioms for Comparative Probability

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Abstract We prove that the generalized cancellation axiom for incomplete comparative probability relations introduced by Ríos Insua (1992) and Alon and Lehrer (2014) is stronger than the standard cancellation axiom for complete comparative probability relations introduced by Scott (1964), relative to their other axioms for comparative probability in both the finite and infinite cases. This result has been suggested but not proved in the previous literature.

Keywords cancellation axioms · comparative probability · qualitative probability · incomplete relations

Let $\succsim$ be a binary relation on an algebra Σ of events over a nonempty state space S. The intended interpretation of $E \succsim F$ is that event E is *at least as likely as* event F. Say that a pair of sequences $\langle E_1, \ldots, E_k \rangle$ and $\langle F_1, \ldots, F_k \rangle$ of events is *balanced* iff for all $s \in S$, the cardinality of $\{i \mid s \in E_i\}$ is equal to the cardinality of $\{i \mid s \in F_i\}$. Consider the following axioms on $\succsim$:

Reflexivity – for all $E \in \Sigma$, $E \succsim E$.
Completeness – for all $E, F \in \Sigma$, $E \succsim F$ or $F \succsim E$.
Positivity – for all $E \in \Sigma$, $E \succsim \emptyset$.
Non-triviality – it is not the case that $\emptyset \succsim S$.

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Finite Cancellation (FC) – for all balanced pairs of sequences $\langle E_1, \ldots, E_n, X\rangle$ and $\langle F_1, \ldots, F_n, Y\rangle$ of events from Σ, if $E_i \succsim F_i$ for all i, then $Y \succsim X$.

Generalized Finite Cancellation (GFC) – for all balanced pairs of sequences

$$\langle E_1, \ldots, E_n, \underbrace{X, \ldots, X}_{r \text{ times}}\rangle \text{ and } \langle F_1, \ldots, F_n, \underbrace{Y, \ldots, Y}_{r \text{ times}}\rangle$$

of events from Σ, if $E_i \succsim F_i$ for all i, then $Y \succsim X$.

FC was introduced by Scott (1964) as a reformulation of axioms from Kraft et al. (1959). For a finite state space, Scott showed that **FC**, **Completeness**, **Positivity**, and **Non-triviality** are necessary and sufficient for the existence of an additive probability measure μ on Σ such that for all $E, F \in \Sigma$:

$$E \succsim F \text{ iff } \mu(E) \geq \mu(F).$$

GFC was introduced by Ríos Insua (1992) and again by Alon and Lehrer (2014).[1] For a finite state space, both papers showed that **GFC**, **Reflexivity**, **Positivity**, and **Non-triviality** are necessary and sufficient for the existence of a nonempty set $\mathcal{P}$ of additive probability measures on Σ such that for all $E, F \in \Sigma$:

$$E \succsim F \text{ iff for all } \mu \in \mathcal{P},\ \mu(E) \geq \mu(F).$$

Clearly **GFC** implies **FC**, and assuming **Completeness** ($X \succsim Y$ or $Y \succsim X$), **FC** implies **GFC**. In the papers by Ríos Insua (p. 89) and Alon and Lehrer (p. 481), it is suggested but not proved that **GFC** is stronger than **FC** for *incomplete* relations, i.e., relative to **Reflexivity**, **Positivity**, and **Non-triviality**.[2] The following establishes the correctness of their claim.

Proposition 1 Let $S = \{a, b, c, d\}$ and define $\succsim$ such that for all $E, F \subseteq S$, $E \succsim F$ iff one of the following holds:

(i) $E \supseteq F$;
(ii) $\{a, c\} \subseteq E$ and $F \subseteq \{b, d\}$;
(iii) $\{a, d\} \subseteq E$ and $F \subseteq \{b, c\}$.

Then $\succsim$ satisfies **Reflexivity**, **Positivity**, **Non-triviality**, and **FC**, but not **GFC**.

Proof **Reflexivity**, **Positivity**, and **Non-triviality** are obvious. To see that **GFC** fails, note that $\langle \{a, c\}, \{a, d\}, \{b\}, \{b\}\rangle$ and $\langle \{b, d\}, \{b, c\}, \{a\}, \{a\}\rangle$ are balanced, so with $\{a, c\} \succsim \{b, d\}$ from (ii) and $\{a, d\} \succsim \{b, c\}$ from (iii), **GFC** requires $\{a\} \succsim \{b\}$, which is not permitted by (i)-(iii).

[1] We adopt Alon and Lehrer's name for the **GFC** axiom and Ríos Insua's equivalent formulation of the axiom.

[2] In correspondence with the authors of both papers, we verified that proving the claim was an open problem.

To see that **FC** holds, assume that $\langle E_1, \dots, E_n, X\rangle$ and $\langle F_1, \dots, F_n, Y\rangle$ are balanced and $E_i \succsim F_i$ for all i. By (i)-(iii), if $a \in F_i$, then $a \in E_i$. Thus, by the balancing assumption, there is at most one j such that $a \in E_j$ and $a \notin F_j$ (in which case $a \in Y$). Suppose there is no such j. Then by (i)-(iii) and the assumed relationships, $E_i \supseteq F_i$ for all i, which with the balancing assumption implies $Y \supseteq X$ and hence $Y \succsim X$ by (i). Suppose there is one such j, say $j = 1$. Then by (i)-(iii) and the assumed relationships, $E_i \supseteq F_i$ for all $i > 1$. If $E_1 \supseteq F_1$, then by the argument above, we have $Y \succsim X$. Otherwise, $E_1 \not\supseteq F_1$, and so the reason for $E_1 \succsim F_1$ is (ii) or (iii). If it is (ii), then since $E_i \supseteq F_i$ for all $i > 1$, the balancing assumption implies $\{a, c\} \subseteq E_1 - F_1 \subseteq Y$ and $X \subseteq F_1 - E_1 \subseteq \{b, d\}$. Thus, by (ii), $Y \succsim X$. The case for (iii) is similar. □

Alon and Lehrer (2014) also considered the case where the state space S may be infinite. The representation theorem in this case requires one additional axiom, analogous to Savage's (1954, §3.3) axiom P6, but for incomplete relations. For $A, B \in \Sigma$, let $A \succ\!\succ B$ iff there is a finite partition $\{G_1, \dots, G_r\}$ of S such that $A - G_i \succsim B \cup G_j$ for all i and j. The additional axiom is:

Non-atomicity – if $A \not\succsim B$ then there is a finite partition of B, $\{B_1, \dots, B_m\}$, such that for all i, $B_i \succ\!\succ \varnothing$ and $A \not\succsim B - B_i$.

Alon and Lehrer (2014) showed that **GFC**, **Reflexivity**, **Positivity**, **Non-triviality**, and **Non-atomicity** are necessary and sufficient for the existence of a nonempty, compact,[3] and uniformly strongly continuous[4] set $\mathcal{P}$ of finitely additive probability measures on Σ such that for all $E, F \in \Sigma$:

$$E \succsim F \text{ iff for all } \mu \in \mathcal{P},\ \mu(E) \geq \mu(F).$$

In the case where Σ is a σ-algebra, Alon and Lehrer (2014) showed that we may replace 'finitely additive' with 'countably additive' in the previous result if we add the axiom:

Monotone Continuity – for any sequence $E_1 \supseteq E_2 \supseteq \dots$ with $\bigcap_n E_n = \emptyset$ and any $F \succ\!\succ \emptyset$, there is some n_0 such that for all $n > n_0$, $F \succsim E_n$.

In this setting, we again show that **GFC** is stronger than **FC**, now relative to **Reflexivity**, **Positivity**, **Non-triviality**, **Non-atomicity**, and **Monotone Continuity**.

To prepare for Proposition 2, let $S = \{a, b, c, d\} \times [0, 1]$. Given $E \subseteq S$ and $x \in \{a, b, c, d\}$, let E_x be the fiber $\{y \in [0, 1] \mid \langle x, y\rangle \in E\}$ over x. Let Σ be the σ-algebra consisting of sets $E \subseteq S$ where E_x is Lebesgue measurable for each $x \in \{a, b, c, d\}$. Let $\mu_x(E) = \mu(E_x)$, where μ is the Lebesgue measure on

[3] By *compact*, Alon and Lehrer mean that $\mathcal{P}$ is weak* compact, i.e., compact in the space of pointwise convergence.

[4] A set $\mathcal{P}$ of measures is uniformly strongly continuous iff both of the following hold:

1. for all $\mu, \mu' \in \mathcal{P}$ and $B \in \Sigma$, $\mu(B) > 0$ iff $\mu'(B) > 0$;
2. for all $\epsilon > 0$, there is a finite partition $\{G_1, \dots, G_r\}$ of S such that for all j, $\mu(G_j) < \epsilon$ for all $\mu \in \mathcal{P}$.

the interval $[0,1]$. A *weight function* w on $\{a,b,c,d\}$ is an assignment, to each $x \in \{a,b,c,d\}$, of a value $w_x \in [1,2]$. Define $\succsim$ such that for all $E, F \in \Sigma$, $E \succsim F$ iff one of the following holds:

(i) for all weight functions w,

$$\sum_x w_x \mu_x(E) \geq \sum_x w_x \mu_x(F),$$

where the sum is taken over $x \in \{a,b,c,d\}$;

(ii) for the weight function $w^{(ii)}$ which gives a and c weight 2 and b and d weight 1,

$$\sum_x w_x^{(ii)} \mu_x(E) - \sum_x w_x^{(ii)} \mu_x(F) \geq 2;$$

(iii) for the weight function $w^{(iii)}$ which gives a and d weight 2 and b and c weight 1,

$$\sum_x w_x^{(iii)} \mu_x(E) - \sum_x w_x^{(iii)} \mu_x(F) \geq 2.$$

Note that in (i), it suffices to take $w_x = 1$ when $\mu_x(E) \geq \mu_x(F)$ and $w_x = 2$ when $\mu_x(F) > \mu_x(E)$. One can view $\succsim$ as just defined as a modification of the relation from Proposition 1 by assigning measures in $[0,1]$ to each of a, b, c, and d. For example, from (i), (ii), and (iii) above we can derive three particular cases (i'), (ii'), and (iii') below under which $E \succsim F$. These correspond to (i), (ii), and (iii) from Proposition 1:

(i') for all $x \in \{a,b,c,d\}$, $\mu_x(E) \geq \mu_x(F)$;
(ii') $\mu_a(E) = \mu_c(E) = 1$ and $\mu_a(F) = \mu_c(F) = 0$;
(iii') $\mu_a(E) = \mu_d(E) = 1$ and $\mu_a(F) = \mu_d(F) = 0$.

The relation $\succsim'$ given by (i'), (ii'), and (iii') does not satisfy **Non-atomicity**. The relation $\succsim$ adds to $\succsim'$ the ability to "exchange" measure from some $x \in \{a,b,c,d\}$ to some other y, but at an exchange rate of two to one. This exchange is required to get **Non-atomicity**. So, for example, by (i) we have $\{a\} \times [0,1] \succsim \{b\} \times [0, \frac{1}{2}]$ because we can exchange one measure of a for half a measure of b.

This relation $\succsim$ separates **FC** and **GFC** relative to the other axioms.

Proposition 2 The relation $\succsim$ on Σ satisfies **Reflexivity**, **Positivity**, **Non-triviality**, **Non-atomicity**, **Monotone Continuity**, and **FC**, but not **GFC**.

Proof **Positivity** and **Non-triviality** are obvious. **Reflexivity** follows from (i). To see that **Monotone Continuity** holds, consider a sequence $E_1 \supseteq E_2 \supseteq \ldots$ with $\bigcap_n E_n = \emptyset$ and any $F \succ\succ \emptyset$. Recall that $F \succ\succ \emptyset$ means that there is a finite partition $\{G_1, \ldots, G_r\}$ of S such that $F - G_i \succsim G_j$ for all i and j. Since the partition is finite, we can pick G_k such that $\sum_x \mu_x(G_k) > 0$. Now given $F - G_k \succsim G_k$, this holds for reason (i), (ii), or (iii). In each case, it follows that $\sum_x \mu_x(F) > 0$. Then since $\lim_{n\to\infty} \sum_x \mu_x(E_n) = 0$, there is an n_0 such that $\sum_x w_x \mu_x(F) > \sum_x w_x \mu_x(E_{n_0})$ for any set of weights in $[1,2]$, so $F \succsim E_{n_0}$ by (i), which clearly implies that for all $n > n_0$, $F \succsim E_n$.

To see that **GFC** fails, note that the example from Proposition 1 still works, replacing $\{a\}$ by $\{a\} \times [0,1]$, $\{b\}$ by $\{b\} \times [0,1]$, and so on.

That $\succsim$ satisfies **Non-atomicity** and **FC** is the content of the next two lemmas which complete the proof of the proposition.

Lemma 1 The relation $\succsim$ on Σ satisfies **Non-atomicity**.

Proof Suppose that $E \not\succsim F$. By (i), there is a weight function w such that

$$\sum_x w_x \mu_x(E) < \sum_x w_x \mu_x(F).$$

Let $\epsilon_1 > 0$ be such that

$$\sum_x w_x \mu_x(F) - \sum_x w_x \mu_x(E) > \epsilon_1.$$

By (ii), we have

$$\sum_x w_x^{(ii)} \mu_x(E) - \sum_x w_x^{(ii)} \mu_x(F) < 2.$$

(Recall that $w_x^{(ii)}$ and $w_x^{(iii)}$ are the weight functions defined in (ii) and (iii) respectively.) Let $\epsilon_2 > 0$ be such that

$$\sum_x w_x^{(ii)} \mu_x(E) - \sum_x w_x^{(ii)} \mu_x(F) < 2 - \epsilon_2.$$

Similarly, by (iii),

$$\sum_x w_x^{(iii)} \mu_x(E) - \sum_x w_x^{(iii)} \mu_x(F) < 2.$$

Let $\epsilon_3 > 0$ be such that

$$\sum_x w_x^{(iii)} \mu_x(E) - \sum_x w_x^{(iii)} \mu_x(F) < 2 - \epsilon_3.$$

Now since $E \not\succsim F$, for some x, $\mu_x(F) > 0$, so we can partition F into finitely many sets F_i such that

$$0 < 2 \sum_x \mu_x(F_i) < \epsilon_j$$

for each $j = 1, 2, 3$. Fix F_i; we must show that $F_i \succ\succ \varnothing$ and that $E \not\succsim F - F_i$.

We start by showing that $F_i \succ\succ \varnothing$. Partition S into finitely many sets $\{G_1, \ldots, G_r\}$ such that

$$5 \sum_x \mu_x(G_j) < \sum_x \mu_x(F_i).$$

for each $j = 1, \ldots, r$. Thus for any weight function v, and j_1, j_2,

$$\begin{aligned}\sum_x v_x \mu_x(F_i - G_{j_1}) &\geq \sum_x \mu_x(F_i) - 2\sum_x \mu_x(G_{j_1}) \\ &> \frac{3}{5}\sum_x \mu_x(F_i) \\ &> 2\sum_x \mu_x(G_{j_2}) \\ &\geq \sum_x v_x \mu_x(G_{j_2})\end{aligned}$$

and so $F_i - G_{j_1} \succsim G_{j_2}$ by (i). Thus $F_i \succ\succ \varnothing$.

Now we show that $E \not\succsim F - F_i$. Let w be the weight from above for which we chose ϵ_1. Then

$$\begin{aligned}&\sum_x w_x \mu_x(F - F_i) - \sum_x w_x \mu_x(E) \\ &\geq \sum_x w_x \mu_x(F) - \sum_x w_x \mu_x(E) - \sum_x w_x \mu_x(F_i) \\ &> \epsilon_1 - \sum_x w_x \mu_x(F_i) \\ &\geq \epsilon_1 - 2\sum_x \mu_x(F_i) \\ &> 0\end{aligned}$$

and hence (i) does not imply $E \succsim F - F_i$. Also,

$$\begin{aligned}&\sum_x w_x^{(ii)} \mu_x(E) - \sum_x w_x^{(ii)} \mu_x(F - F_i) \\ &\leq \sum_x w_x^{(ii)} \mu_x(E) - \sum_x w_x^{(ii)} \mu_x(F) + 2\sum_x \mu_x(F_i) \\ &< 2 - \epsilon_2 + \epsilon_2 \\ &= 2\end{aligned}$$

and hence (ii) does not imply $E \succsim F - F_i$. The case of (iii) is similar. Thus $E \not\succsim F - F_i$. □

Lemma 2 The relation $\succsim$ on Σ satisfies **FC**.

Proof To see that **FC** holds, assume that $\langle E_1, \ldots, E_n, X\rangle$ and $\langle F_1, \ldots, F_n, Y\rangle$ are balanced and $E_i \succsim F_i$ for all i. By the balancing assumption, for all $x \in \{a, b, c, d\}$, we have

$$\mu_x(X) + \sum_{i=1}^{n} \mu_x(E_i) = \mu_x(Y) + \sum_{i=1}^{n} \mu_x(F_i).$$

First, suppose that for all i, $E_i \succsim F_i$ by (i). Let w be a weight function. Then from the balancing assumption,

$$\sum_x w_x \mu_x(Y) - \sum_x w_x \mu_x(X) = \sum_{i=1}^n \sum_x w_x(\mu_x(E_i) - \mu_x(F_i)).$$

Each summand $\sum_x w_x(\mu_x(E_i) - \mu_x(F_i))$ is non-negative, so

$$\sum_x w_x \mu_x(Y) \geq \sum_x w_x \mu_x(X)$$

and hence $Y \succsim X$.

Now suppose that for some j, say $j = 1$, (i) is not satisfied. Then $E_1 \succsim F_1$ by either (ii) or (iii). Suppose that it is by (ii) (the case of (iii) is similar). Then

$$\sum_x w_x^{(ii)} \mu_x(E_1) - \sum_x w_x^{(ii)} \mu_x(F_1) \geq 2.$$

Consider the sum

$$\sum_x w_x^{(ii)} \mu_x(Y) - \sum_x w_x^{(ii)} \mu_x(X) = \sum_{i=1}^n \sum_x w_x^{(ii)}(\mu_x(E_i) - \mu_x(F_i)).$$

For $i = 1$, $\sum_x w_x^{(ii)}(\mu_x(E_i) - \mu_x(F_i)) \geq 2$. For each other i, we will show that $\sum_x w_x^{(ii)}(\mu_x(E_i) - \mu_x(F_i)) \geq 0$. We have three cases. If $E_i \succsim F_i$ by (i), then $\sum_x w_x^{(ii)}(\mu_x(E_i) - \mu_x(F_i)) \geq 0$. If $E_i \succsim F_i$ by (ii), then $\sum_x w_x^{(ii)}(\mu_x(E_i) - \mu_x(F_i)) \geq 2$. Finally, suppose that $E_i \succsim F_i$ by (iii), so we have that

$$\sum_x w_x^{(iii)} \mu_x(E_i) - \sum_x w_x^{(iii)} \mu_x(F_i) \geq 2.$$

Then

$$\begin{aligned}\sum_x w_x^{(ii)} \mu_x(E_i) - \sum_x w_x^{(ii)} \mu_x(F_i) = \sum_x w_x^{(iii)} \mu_x(E_i) - \sum_x w_x^{(iii)} \mu_x(F_i) \\ + (\mu_c(E_i) - \mu_c(F_i)) - (\mu_d(E_i) - \mu_d(F_i)).\end{aligned}$$

Since the measures μ_c and μ_d take values in $[0, 1]$, we get

$$\sum_x w_x^{(ii)} \mu_x(E_i) - \sum_x w_x^{(ii)} \mu_x(F_i) \geq 0.$$

This completes the third case. So we have shown that for this particular weight function $w^{(ii)}$, we have

$$\sum_x w_x^{(ii)} \mu_x(Y) - \sum_x w_x^{(ii)} \mu_x(X) \geq 2.$$

Hence $Y \succsim X$ by (ii). This completes the proof of the lemma. □

Thus, in both the finite and infinite cases, **GFC** is stronger than **FC** relative to the other axioms for comparative probability without **Completeness**. As Fine (1973) remarks about **Completeness**, "The requirement that all events be comparable is not insignificant and has been denied by many careful students of probability including Keynes and Koopman" (p. 17). In light of Propositions 1 and 2, those sympathetic to the denial of **Completeness** have reason to expand the study of cancellation axioms for comparative probability beyond the standard focus on **FC** to include a study of **GFC** as well.

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