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THE SPACE OF CUBIC SPLINES WITH SPECIFIED KNOTS

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ABSTRACT

For a given closed finite interval, which has been partitioned into a finite number of subintervals, we consider all cubic splines[1] which have knots at the endpoints and at each partition point. A finite basic subset may be readily chosen such that any spline in the parent set can be expressed as a unique linear combination of the basic splines.

Any function defined on the interval whose values at the knots and whose first derivatives at the endpoints are known can be approximated by a unique cubic spline, hence as a linear combination of basic splines.

Where a relatively large number of functions are to be approximated, it becomes practical to first compute the basic splines and then determine the required linear combination to approximate each of the functions.

INTRODUCTION

For a given interval, $[a,b]$ with partitioning:

$$a = x_1 < x_2 < \text{----} \; x_n = b,$$

a cubic spline, s (x), defined on $[a,b]$ with knots at the x_i has the following properties:

(I) The function, s, is a polynomial of degree three (or less) in x on each subinterval,

$$[x_i, x_{i+1}]; \; i = 1, n-1$$

(II) The function s has a continuous second derivative on the whole interval $[a,b]$.

Obvious examples of cubic splines are:

$$s = \text{constant (including zero)}$$

$$s = x$$

$$s = x^2$$

$$s = x^3$$

and any linear combination of the above. However these splines do more than meet the requirements (I), (II), they have constant (hence continuous) third derivatives over the whole interval. In general, cubic splines with knots at x_i may have finite discontinuities in the third derivative at interior knots. As an example consider:

$$s = 0 \quad x \le x_2$$

$$s = (x-x_2)^3 \quad x \ge x_2.$$

Any function f (x) defined on $[a,b]$ for which values at the x_i are known and values of its first derivative at a and b are known can be readily approximated by a unique cubic spline with knots at the x_i. The cubic spline fit can be computed directly for the function[1] or where several functions are to be fitted a basic set of splines can be determined and the fitting splines be computed for each function as a unique linear combination of the basis.

<u>Direct Cubic Spline Fit</u>. Suppose we are given a set of points,

$$\left\{x_i, f_i\right\}; \; i = 1, n$$

with the assumption that there is a function , f(x), defined on x_i, x_n such that

$$f\,(x_i) = f_i \quad ; \quad i = 1, n$$

with specified terminal derivatives

$$f'\,(x_1) = f'$$

and

$$f'\,(x_n) = f'_n \,.$$

The cubic spline s which approximates f on $[a,b]$ must be an exact fit of the above data, i.e.

$$s_i = f_i \quad ; \; i = 1, n$$

$$s'_1 = f'_1$$

$$s'_n = f'_n$$

Th$^{\underline{m}}$ 1 There is a unique s which fits f.

Proof:

(a) Existence

From properties (I) and (II) we have

$$s'_{i-1} + 4s'_i + s'_{i+1} = 3\left[\frac{(x_i - x_{i-1})(s_{i+1} - s_i)}{x_{i+1} - x_i} + \frac{(x_{i+1} - x_i)(s_i - s_{i-1})}{x_i - x_{i-1}}\right]$$

for i = 2, n - 1

In terms of the fitting requirements

$$4s'_2 + s'_3 = 3\left[\frac{(x_2 - x_1)(f_3 - f_2)}{x_3 - x_2} + \frac{(x_3 - x_2)(f_2 - f_1)}{x_2 - x_1}\right] - f_1'$$

$$s'_{i-1} + 4s'_i + s'_{i+1} = 3\left[\frac{(x_i - x_{i-1})(f_{i+1} - f_i)}{x_{i+1} - x_i} + \frac{(x_{i+1} - x_i)(f_i - f_{i-1})}{x_i - x_{i-1}}\right]$$

$$s'_{n-2} + 4s'_{n-1} = 3\left[\frac{(x_{n-1} - x_{n-2})(f_n - f_{n-1})}{x_n - x_{n-1}} + \frac{(x_n - x_{n-1})(f_{n-1} - f_{n-2})}{x_{n-1} - x_{n-2}}\right] - f'_n$$

The above linear system of n-2 equations is tridiagonal with diagonal dominance, hence has a unique solution for s'_i ; i = 2, n-1.

(b) Uniqueness

On any subinterval $[x_i, x_{i+1}]$; i = 1, n-1 the values of s_i and s_{i+1} are determined by

$$s_i = f_i \quad \text{and} \quad s_{i+1} = f_{i+1}$$

and the values of s'_i and s'_{i+1} are uniquely determined by the linear system above. These values uniquely determine a cubic segment on $[x_i, x_{i+1}]$. Consequently s is uniquely determined for the whole interval [a,b].

<u>The Cubic Spline Space.</u> Let S be the set of all cubic splines with knots at x_i ; i = 1, n. We have already shown the S is not empty. In fact it contains as a subset the linear space of all polynomials of degree three or less[2]. With the obvious definitions of addition

$$u = s + t \quad : \quad u(x) = s(x) + t(x) \quad ; \quad s,t \in S$$

and scalar multiplication

$$v = as \qquad v(x) = a\, s(x), \qquad s \in S$$

it may be readily shown that S is itself a linear space.

Since S contains a linear space of dimension 4 , its dimension must be greater than or equal 4. Further since any s in S is completely determined by its n-1 cubic segments, the dimension of S is less than or equal 4(n-1). Note that for n=2, the dimension of S is necessarily 4.

The dimension of S is readily established by the following theorem:

Th$^{\underline{m}}$ 2. Any spline s in S is uniquely determined by the value of s at the x_i and s' at x_1 and x_n.

Proof: (Indirect)

Suppose there are two such splines, s and s* satisfying the specified values. Then both s and s* are spline approximations of s and by Th$^{\underline{m}}$ 1 $s = s^*$.

Since n+2 values are necessary and sufficient to determine any s in S, the dimension of S must be n+2.

We now construct a convenient basis $\{t_j\}$; j = 1, n+2 for S. Let t_j be cubic splines in S such that

$$t_j'(x_1) = 0 \quad t_j'(x_n) = 0 \quad t_j(x_i) = \delta_{ij} \quad \text{for } j = 1, n \,;\, i = 1, n$$

$$t'_{n+1}(x_1) = 1 \quad t'_{n+1}(x_n) = 0 \quad t_{n+1}(x_i) = 0 \quad \text{for } i = 1, n$$

$$t'_{n+2}(x_1) = 0 \quad t'_{n+2}(x_n) = 1 \quad t_{n+2}(x_i) = 0 \quad \text{for } i = 1, n.$$

It is obvious that no one of the t_j is a linear combination of the others and, further, that for any s in S specified in accordance with Th$^{\underline{m}}$ 2 we have

$$s(x) = \sum_{i=1}^{n} s(x_i)\, t_i(x) + s'(x_1)\, t_{n+1}(x) + s'(x_n)\, t_{n+2}(x)$$

Practical Applications For any function f defined on [a,b] with terminal first derivatives and function values at the x_i known, the basic splines, t_j , may be generated (this consists of computing $t_j'(x_i)$ for i = 2,n-1) then the fitting spline s for f is determined by

$$s'(x_i) = \sum_{j=1}^{n} f(x_i)\, t_j'(x_i) + f'(x_1)\, t'_{n+1}(x_i) + f'(x_n)\, t'_{n+2}(x_i)$$

for $i = 2, n-1$ together with

$$s'(x_1) = f'(x_1) \quad , \quad s'(x_n) = f'(x_n)$$

and $$s(x_i) = f(x_i) \; ; \; i = 1, n.$$

Obviously this method is not practical for fitting one function, f , or even for fitting n+2 such functions since the generation of the basic splines would require as much computation as the direct computation of the fitting splines and the final vector multiplication would be required. If we assume (perhaps conservatively) that each matrix inversion requires approximately n times as much computation as the vector multiplication, then the basic computation is worthwhile when

$$m \geqslant (n^2+2n) \;/\; (n-1)$$

where m is the number of functions to be fitted.

Methods for obtaining numerical solutions to second-order differential equations in the spline space as linear combinations of basic splines will be discussed in subsequent articles.

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