

UCLA

UCLA Electronic Theses and Dissertations

Title

Expanding Competence: Supporting Students to Engage with Each Other's Mathematical Ideas

Permalink

<https://escholarship.org/uc/item/2s07d7fj>

Author

Johnson, Nicholas Charles

Publication Date

2017

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Expanding Competence:

Supporting Students to Engage with Each Other's Mathematical Ideas

A dissertation submitted in partial satisfaction of the requirements for the degree

Doctor of Philosophy in Education

by

Nicholas Charles Johnson

2017

© Copyright by

Nicholas Charles Johnson

2017

ABSTRACT OF THE DISSERTATION

Expanding Competence:
Supporting Students to Engage with Each Other's Mathematical Ideas

by

Nicholas Charles Johnson

Doctor of Philosophy in Education

University of California, Los Angeles, 2017

Professor Megan Loef Franke, Chair

This case study examined competence in two third-grade classrooms where teachers centered children's mathematical thinking in instructional decision-making. Offering a synthesis of sociocultural characterizations of competence, and drawing from a variety of data sources including classroom video, student work and assessments, and teacher interviews, this study investigated how competence was assigned, the forms of participation assigned competence, and how these particular constructions of competence shaped the mathematics that students learned and the kinds of agency they exercised.

Analyses revealed two predominant ways of participating shared across both classrooms—providing explanation and engaging with others’ mathematical ideas. Constructing competence around and in relation to students’ explanations and engagement with each other’s ideas created space for students to participate and be positioned competently for making a wide variety of contributions, including ways of contributing available when their understanding of the mathematics was still emerging. The two classrooms surfaced and built upon students’ partial understandings in different ways, leveraging students’ explanations and engagement with each other’s ideas to support individuals to take up their space in class discussions while also advancing instructional goals. Analyses of an end-of-year problem solving assessment demonstrated that students developed a strong understanding of multiplication and division, both in relation to their own prior achievement as well as to that of their peers across the state. Thus, both classrooms expanded collective notions of competence, and were successful in supporting students to demonstrate their mathematical learning. The findings of this study illustrate the relational nature of competence, and suggest that examining competence in relation to children’s thinking can provide a potential way to bring together learning about who children are and how they participate, the details of their mathematical thinking, and the contexts in which particular instructional practices are effective.

The dissertation of Nicholas Charles Johnson is approved.

Tyrone C. Howard

Elham Kazemi

Noreen M. Webb

Megan Loef Franke, Committee Chair

University of California, Los Angeles

2017

Table of Contents

CHAPTER ONE: BACKGROUND.....	1
CHAPTER TWO: LITERATURE REVIEW.....	4
EXPLORING ISSUES OF (IN)EQUITY IN THE LEARNING AND TEACHING OF MATHEMATICS	4
CHILDREN’S MATHEMATICAL THINKING AS CENTRAL TO UNDERSTANDING LEARNING AND TEACHING.....	7
A PERSPECTIVE ON LEARNING IN CLASSROOMS	9
CONCEPTUALIZING COMPETENCE.....	15
<i>Competence and supporting students to engage with each other’s ideas</i>	<i>20</i>
<i>Expanding competence, engaging with others’ ideas, and equity.....</i>	<i>21</i>
SITUATING THE CURRENT STUDY.....	23
CHAPTER THREE: METHODS	25
RESEARCH QUESTIONS	27
SETTING AND PARTICIPANTS.....	28
POSITIONING THE RESEARCHER	29
DATA SOURCES	30
RESEARCH DESIGN	32
CODING AND ANALYSIS	33
<i>Constructing competence: An example.....</i>	<i>38</i>
<i>Coding and analysis of teacher interviews, student work and assessments</i>	<i>40</i>
CHAPTER 4: FINDINGS.....	42
ORGANIZING OPPORTUNITIES TO LEARN: TASKS AND STRUCTURES ACROSS TWO CLASSROOMS	42
<i>Warm-up activities</i>	<i>43</i>
<i>Solving story problems.....</i>	<i>44</i>
CONSTRUCTING COMPETENCE IN MS. J’S CLASSROOM.....	51
<i>Doing mathematics</i>	<i>51</i>
<i>Taking up space</i>	<i>55</i>
<i>Teacher support of student participation.....</i>	<i>58</i>

<i>Children’s mathematical thinking</i>	61
<i>The relational nature of competence: Sharing an incomplete idea</i>	63
<i>Taking up opportunities to explain and engage with each other’s ideas without teacher support</i>	68
CONSTRUCTING COMPETENCE IN MS. C’S CLASSROOM	74
<i>Doing mathematics</i>	75
<i>Taking up space</i>	79
<i>Teacher support of student participation</i>	82
<i>Children’s mathematical thinking</i>	85
<i>The relational nature of competence:</i>	86
<i>Supporting students to engage with more sophisticated strategies</i>	86
<i>Status in negotiating productive engagement with each other’s ideas</i>	92
STUDENT ACHIEVEMENT	96
<i>Prior achievement</i>	97
<i>End-of-year problem solving assessment</i>	98
SUMMARY: CONSTRUCTING COMPETENCE ACROSS TWO THIRD-GRADE CLASSROOMS	105
CHAPTER 5: DISCUSSION	111
LEVERAGING THE POTENTIAL OF EXPLANATION AND ENGAGEMENT WITH OTHERS’ IDEAS	112
CHILDREN’S THINKING AS AN ENTRY POINT INTO EXPANDING COMPETENCE	114
THE SOCIAL CONSTRUCTION OF COMPETENCE	116
LIMITATIONS AND FUTURE DIRECTIONS	119
CONCLUSION	122
APPENDICES	123
REFERENCES	137

List of Tables

Table 3.1: Operationalizing competence across four dimensions of classroom activity.....	35
Table 4.1: End-of-Year Problem Solving Assessment Items and Results.....	99

List of Figures

Figure 3.1: Coding classroom interactions in Studiocode.....	36
Figure 4.1: Deyanira’s counting and adding strategies across three sets of numbers within the same story problem.....	48
Figure 4.2: Multidimensional ways of doing mathematics in Ms. J’s classroom.....	52
Figure 4.3: Taking up space: Exercising agency and authority in Ms. J’s classroom.....	55
Figure 4.4: Teacher support of student participation in Ms. J’s classroom.....	59
Figure 4.5: Joanna explains Marisa and Shawna’s strategy.....	60
Figure 4.6: Children’ strategies and high-level explanation and engagement with each other’s ideas in Ms. J’s classroom.....	62
Figure 4.7: Ms. J poses a mental math warm-up activity.....	64
Figure 4.8: Shonda’s initial strategy for $70 \div 2$	71
Figure 4.9: Sonia’s appropriated and adapted strategy.....	72
Figure 4.10: Multidimensional ways of doing mathematics in Ms. C’s classroom.....	76
Figure 4.11 Gerardo circles groups of 10 within his direct modeling by 1s strategy.....	78
Figure 4.12: Taking up space: Exercising agency and authority in Ms. C’s classroom.....	79
Figure 4.13: Melani’s strategy for solving $93 \div 3$	80
Figure 4.14: Teacher support of student participation in Ms. C’s classroom.....	82
Figure 4.15: Children’ strategies and high-level explanation and engagement with each other’s ideas in Ms. C’s classroom.....	85
Figure 4.16: Students’ strategies on the end-of-year assessment in Ms. J’s classroom.....	102
Figure 4.17: Students’ strategies on the end-of-year assessment in Ms. C’s classroom.....	103

Acknowledgements

This research was supported in part by a National Academy of Education/Spencer Dissertation Fellowship. The findings, conclusions, or recommendations expressed in this paper are the author's alone and do not reflect the views or policies of the funding agency.

I am greatly indebted to the students and teachers who made this study possible by sharing their classrooms with our research team; we continue to learn so much from all of you. And my deepest love and appreciation to Kelly, Charlie, Hazel, and Willa. Without you none of this would have been possible, nor worth doing. You are my everything.

Vita

EDUCATION

- 2008 M.A. California State University, Dominguez Hills
Educational Administration & Leadership
- 2000 B.A. University of California, Los Angeles
Economics, Philosophy Minor

PUBLICATIONS

- Carpenter, T. P., Franke, M. L., Johnson, N. C., Turrou, A. C., & Wager, A. A. (2017). *Young children's mathematics: Cognitively guided instruction in early childhood education*. Portsmouth, NH: Heinemann.
- Webb, N. M., Franke, M. L., Ing, M., Turrou, A. C., & Johnson, N. C. (2015). Student participation, teacher instructional practices, and the development of mathematical understanding in the elementary classroom. In R. M. Gillies (Ed.), *Collaborative learning: Developments in research and practice*. New York, NY: Nova Publishers.
- Webb, N. M., Franke, M. L., Ing, M., Johnson, N. C., Turrou, A. C., & Zimmerman, J. (2015). Engaging students in others' mathematical ideas. *Better: Evidence-Based Education*, 7(1).

PRESENTATIONS

- Johnson, N.C. & Turrou, A.G. (2017, June). *Supporting the development of young children's counting*. Presentation at the ninth biennial conference for Cognitively Guided Instruction, Seattle, WA.
- Turrou, A.G. & Johnson, N.C. (2017, June). *Bridging counting and problem solving*. Presentation at the ninth biennial conference for Cognitively Guided Instruction, Seattle, WA.
- Danielson, C., Hintz, A., Smith, T., Franke, M.L., Turrou, A.G. & Johnson, N.C. (2017, June). *Talking math with kids*. Keynote at the ninth biennial conference for Cognitively Guided Instruction, Seattle, WA.
- Johnson, N.C., Chest, A., & Hayward, V. (2017, May). Leveraging partial understandings through supporting students to engage with each other's mathematical ideas. Presentation at Cotsen alumni conference, Downey, CA.
- Johnson, N.C., Franke, M.L., Ing, M., Turrou, A.C., Webb, N.M., & Zimmerman, J. (2017,

- April). *Negotiating multiple mathematical goals: Broadening competence and engaging with each other's mathematical ideas*. Paper presented at the annual meeting of the American Educational Research Association, San Antonio, TX.
- Johnson, N.C. (2017, April). Expanding competence: Creating space for students to engage with each other's mathematical ideas. In G. L. Wimberly & F. J. Levine (Chairs), *Promising Scholarship in Education: NAEd/Spencer Dissertation Fellows and Their Research*. Invited poster session at the annual meeting of the American Educational Research Association, San Antonio, TX.
- Johnson, N.C., Franke, & Turrou, A.C. (2017, April). *Expanding competence: Leveraging mistakes as learning opportunities*. Paper presented at the annual meeting of the National Council of Teachers of Mathematics, San Francisco, CA. Association, San Antonio, TX.
- Johnson, N.C. (2017, March). *Expanding competence: Creating space for students to engage with each other's mathematical ideas*. Presentation at the spring retreat of the National Academy of Education, Washington D.C.
- Johnson, N.C. (2016, November). *Expanding competence: Creating space for students to engage with each other's mathematical ideas*. Poster presented at the fall retreat of the National Academy of Education, Washington D.C.
- Turrou, A.G. & Johnson, N.C. (2016, November). *Extending counting to problem solving in early childhood*. Presentation at the annual conference of the California Mathematics Council (South), Palm Springs, CA.
- Johnson, N.C. & Turrou, A.G. (2016, November). *Supporting the development of young children's counting*. Presentation at the annual conference of the California Mathematics Council (South), Palm Springs, CA.
- Johnson, N.C. (2016, May). *Cognitively Guided Instruction: An introduction*. Presentation at the southern California regional conference for Cognitively Guided Instruction, Manhattan Beach, CA.
- Johnson, N.C., Franke, M.L., Ing, M., Turrou, A.C., Webb, N.M., & Zimmerman, J. (2016, April) *Constructing competence: Creating space for students to engage with each other's mathematical ideas*. Paper presented at the annual meeting of the American Educational Research Association, Washington D.C.
- Johnson, N.C. & Turrou, A.C. (2016, April) *Counting collections with young children: New evidence from research*. Presentation at the annual meeting of the National Council of Teachers of Mathematics, San Francisco, CA.
- Johnson, N.C., McMillan, B., Raygoza, M.C., Franke, M.L., & Turrou, A.C. (2016, January). *Reframing the discourse in early childhood mathematics: Creating learning opportunities with challenging, open-ended tasks*. Poster session at the annual meeting of the Association of Mathematics Teacher Educators, Irvine, CA.

CHAPTER ONE: BACKGROUND

Framing Investigations of Opportunities to Learn

A primary goal for researchers, practitioners, and policy makers is to improve the experiences and educational outcomes for students from nondominant communities.¹ The learning and teaching of mathematics is amongst the most critical areas. Mathematics has long played a role as a gatekeeper into higher education, a range of professional fields, and positions of power (Martin, Gholson, & Leonard, 2010; Stinson, 2004). Limited access to resources and quality learning experiences, as documented by a history of disparate performance on large-scale measures of achievement (Darling-Hammond, 2007; Howard, 2010), have led some to view mathematics learning as nothing less than a civil rights issue (Moses & Cobb, 2001).

While there is much agreement that working toward providing a more equitable distribution of learning opportunities (and outcomes) should be a focus of research and reform efforts, the framing of the challenge itself is instrumental in determining our approach. Varied conceptualizations of the “achievement gap” suggest investigations of different factors related to teaching and learning, and thus different solutions. One perspective commonly attributes observed achievement disparities to differences in race, culture, or social class (e.g. Herrnstein & Murray, 1994). This (often tacit, but sometimes explicit) conflating of association with causality is often present not only within the public discourse, but also within research communities and policy documents. For example, a recent report on early childhood mathematics learning (NRC, 2009) devoted an entire chapter to discussing documented differences associated with socioeconomic status, race/ethnicity, gender, home language, and parents’ schooling, while

¹ I follow Gutiérrez, Morales, & Martinez (2009) in using the term *nondominant* to reflect the ways in which power relations, historically and presently, structure inequitable access to learning opportunities in schools.

giving little attention to research that has examined the ways in which these features are constructed by the systems and processes within and outside of schools. This focus on outcomes that excludes the social and historical contexts that produce difference permeates ongoing discussions of the “achievement gap,” shaping perceptions of who is able (or unable) to succeed at mathematics (Martin, 2009). Such characterizations locate the problem of underachievement within students, their families, cultures, and communities, as opposed to the sociopolitical structures and long histories of disinvestment in urban schools and neighborhoods (Anyon, 2014; Ladson-Billings, 2006). These are often accompanied by a narrow view of mathematics that serves to privilege dominant conceptions of what doing mathematics should look like, and fails to recognize the varied competencies that students display within and outside of school (Taylor, 2009; Wager & Whyte, 2013). These deficit framings often suggest that the role of policy and school reform is to fix students and their families, rather than recognizing and building from the rich understandings and resources children possess when they enter school (Civil & Bernier, 2006; Moll, Amanti, Neff, & Gonzalez, 1992; Turner et al., 2012).

An alternative perspective is offered by scholarship focusing on the “opportunity gap” between students from nondominant communities and their higher-achieving peers (e.g., Carter & Welner, 2013). Rather than viewing inequities as characteristics of students, these scholars interrogate the “historical, economic, socio-political, and moral antecedents of our current educational situation” (Ladson-Billings, 2013, p. 14). Viewing inequitable outcomes as products of systems and processes that have been designed and carried out by people suggests different interventions, and different research foci. Within mathematics education, for example, Flores (2007) compiled research documenting how nondominant student populations are less likely to receive equitable per student funding, less likely to be taught by qualified, experienced teachers,

and more likely to face low expectations in terms of engaging in challenging mathematics.

Taylor (2009) examined children's shopping practices outside of school and uncovered many mathematical competencies children exhibit that may not be captured by traditional measures of mathematics learning. And decades of research have documented links between tracking, achievement, and social and racial stratification, structuring inequitable learning opportunities into schools (e.g., Oakes, 2008). This perspective argues that understanding how and why some students have been provided with inequitable opportunities to learn, and identifying productive ways to disrupt existing patterns of inequity, must involve analyses of both the structures and processes of classrooms, schools, and policy makers.

In this dissertation I focus my investigation of opportunities to learn primarily on students' experiences within classrooms. Specifically, I investigate classroom interactions as they shape what it means to participate in and learn mathematics in school. A part of understanding how children experience classroom processes involves uncovering the ways they are positioned to take up roles as successful knowers and doers of mathematics (or not). Examining what counts as mathematical competence, and the ways in which notions of competent participation are constructed within learning environments, offers possibilities to illuminate the processes by which opportunities to learn are distributed in classrooms and schools.

CHAPTER TWO: LITERATURE REVIEW

A Theoretical Basis for Exploring Competence

This chapter explores the research base that forms the foundation of this study. An investigation of competence is grounded in several overlapping bodies of literature which situate the learning and teaching of mathematics within the larger sociopolitical structures of schooling, as well as societal narratives about who is viewed as mathematically capable. I begin with a discussion of research addressing issues of inequity in mathematics education. This is followed by a discussion of children's mathematical thinking, and a theoretical perspective that foregrounds the social nature of learning. I then explore literature that has examined issues related to competence from a variety of perspectives.

Exploring issues of (in)equity in the learning and teaching of mathematics

A central goal of advancing the learning and teaching of mathematics is to ensure that each student is provided with high-quality learning opportunities. Reform efforts and policy documents are increasingly grounded in goals of promoting “excellence and equity” (NRC, 2009) or “ensuring mathematical success for all” (NCTM, 2014). However, the field of mathematics education continues to grapple with what it would mean to focus on equity, and what might distinguish “talk of equity” (R. Gutiérrez, 2013) and “good teaching” for all (Battey, 2013) from concerted efforts to support the learning of students from historically underserved, nondominant communities. An increasing number of researchers have begun to explore how issues of race, culture, and power influence the teaching and learning of mathematics (e.g. Diversity in Mathematics Education Center for Learning and Teaching (DiME), 2007), and

scholars continue to push the field to consider whose mathematics, language, and knowledge are valued, and whose interests are served by particular policies, practices, and approaches to research (e.g. Civil, 2012; Martin, 2009, 2015). These questions traverse both the ways that students experience mathematics in school, as well as structural and historical factors that shape access to mathematics. Together, these local and global processes construct who is viewed as mathematically capable, and ultimately who is provided with opportunities to learn.

In conceptualizing equity within mathematics education, Gutiérrez (2012) describes four dimensions: access, achievement, identity, and power. Within these four dimensions, access and achievement comprise the dominant axis, or “how well students can play the game called mathematics,” while identity and power form the critical axis, which “acknowledges the position of students as members of a society rife with issues of power and domination” (2012, pp. 20–21). Gutiérrez calls attention to both the connections and tensions that may arise between these four dimensions, observing that attending to equity may involve a shifting focus between different goals in different situations.

In this dissertation, I focus on learning within classroom activity, and necessarily focus my attention on the ways in which opportunities to participate and demonstrate understanding play out in interaction. This entails attending to who gets to talk and how, whose ideas are taken up and by whom, and whose mathematical ideas are the focus of instruction. In considering what it would mean for students to be afforded more equitable opportunities to learn, and how teachers’ practice shapes these learning opportunities, I draw from conceptions of equity offered by Esmonde (2009b) and Hand (2012).

In unpacking what is contained within opportunities to learn, Esmonde describes both students’ “access to mathematical content and discourse practices along with their access to

(positional) identities as knowers and doers of mathematics” (2009b, p. 249). As students engage with content and participate in mathematical practices, they position themselves and are positioned by others in particular ways. For Esmonde, students’ opportunities to learn mathematics cannot be disentangled from who students get to be as they do mathematics, and if that version of themselves is recognized as a valued, competent contributor to the classroom community. Similarly, Hand (2012) conceives of equity in terms of students’ opportunities to “take up space.” Hand stresses that the idea of “taking up space” within mathematical activity does not simply entail that students participate in mathematical practices, but that they are given voice and the means to participate in the creation of a classroom community that reflects both who a student is, and who the individual wishes to become. Essential to providing opportunities for students to take up space is that teachers attend to students’ ways of participating in relation to their identities and cultural practices. For Hand, equitable mathematics instruction involves “supporting dialogic space in classroom interaction, blurring distinctions between mathematical and cultural activity, and reframing the system of mathematics education” (p. 238).

Both Esmonde and Hand’s conceptions of equity recognize that students’ participation in classroom mathematical practices is varied and multifaceted, and that teachers play a critical role in framing students’ efforts to participate and exercise agency in the shaping of the classroom community. In this dissertation I focus on the ways that particular interactions, tasks, and lesson structures open or constrain opportunities to learn mathematics and to take up space within the mathematical work of the classroom. Following Esmonde and Hand, my view of opportunities to learn and take up space goes beyond students’ participation in mathematical practices to also include the in-the-moment ways that status and power play out in the overlapping social and mathematical spaces of classrooms. While I do not directly explore issues of identity

development in this study, I acknowledge that these are of critical importance—at least as important as children’s mathematical achievement. I find it fruitful to consider, however, the ways in which children’s views of themselves (and others) as knowers and doers of mathematics are shaped by whether or not their own mathematical ideas, strategies, and sense-making are given prominence in classrooms. In other words, whether or not one’s ideas and perspectives are engaged with by others is related to whether or not one is positioned as a competent member of the community. I therefore extend the idea of taking up space to include the role of children’s mathematical thinking within classroom activity. I argue that how teachers notice and frame students efforts to participate are shaped in part by their knowledge of students’ mathematical thinking, and by a stance that centers children’s thinking in instructional decisions. In particular, the ways that teachers interpret and respond to student-authored strategies that reflect partial or incomplete understandings shape ideas of what it means to be competent, and therefore who is viewed as competent. Before further detailing competence, I turn to a discussion of children’s thinking, and explore the potential of children’s thinking as an entry point into issues of equity.

Children’s mathematical thinking as central to understanding learning and teaching

Even very young children show remarkable abilities to reason about and engage in sophisticated mathematics. Each child enters school with rich informal understandings and experiences that can form the basis of the mathematics to be learned in early childhood and elementary school. Over 30 years of research and collaboration with teachers has documented the ways in which children draw upon their knowledge of the world to solve a variety of problems, as well as their ability to construct a range of increasingly sophisticated strategies (Carpenter, 1985; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, Fennema,

& Franke, 1996; Carpenter, Fennema, Franke, Levi, & Empson, 2015; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Empson, 1999). This research has also demonstrated the potential for professional development focused on children's mathematical thinking to shift teachers' beliefs and practices, and that there is a relationship between teachers' knowledge of their students' mathematical thinking and student achievement (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996; Peterson, Fennema, Carpenter, & Loef, 1989). Subsequent work has extended the findings of the original Cognitively Guided Instruction project into areas such as multidigit multiplication and division, algebraic thinking, fractions, and early counting (Carpenter, Franke, Johnson, Turrou, & Wager, 2017; Empson & Levi, 2011; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). Children often think about mathematics differently than adults, and viewing mathematics from the child's perspective can open possibilities for teachers to engage in generative learning with both their students and colleagues, (Franke, Carpenter, Levi, & Fennema, 2001; Franke & Kazemi, 2001). As teachers learn about the development of children's thinking, and begin to *listen to* and *learn from* their students, they can begin to reconceptualize their practice to build from children's thinking in their teaching. Attending, interpreting, and responding to the details of children's thinking, and in relation to their participation, is critical in supporting their learning (Jacobs, Lamb, & Philipp, 2010; Wager, 2014).

Collectively, work in Cognitively Guided Instruction illustrates a powerful connection between theory, research, and practice. It also highlights the transformative potential of a focus on children's thinking. Learning about how children make sense of different problem structures, how they initially construct strategies by modeling the action or relations within a problem, and how increasingly sophisticated strategies can be thought of as abstractions of these initial

strategies can help teachers to see students' emerging ideas along a developmental trajectory, and can help them to focus on the details of what students know and can do. Viewing mathematics from a child's perspective and inquiring into their thinking can be an entry point into rethinking what mathematics is, and who can do challenging, meaningful mathematics. This perspective can support teachers to recognize how opportunities to participate in mathematics are structured in classrooms and schools, and to reflect on how one might adapt or push back on these constraints to broaden opportunities for student participation. Of course, learning about, attending to, and creating space for children's thinking to emerge in classrooms alone is not sufficient to ensure equity. However, the principled ideas that underpin learning about children's thinking—that children bring mathematical experiences and knowledge with them to school, that these understandings can be built from, that children's intuitive ideas about mathematics make sense and can be quite sophisticated, and that students' learning can be supported by communicating their ideas and by attending to the ideas of their peers—offer potential to broaden idea of what counts as mathematics and to create space for multiple ways of knowing and doing mathematics to emerge in classrooms.

Investigating how competence is constructed in classrooms that center children's thinking therefore provides an opportunity to examine connections between children's thinking and issues of equity. Such an investigation is built upon a view of learning in that highlights the local construction and adaptation of the practices of classroom communities.

A perspective on learning in classrooms

This dissertation is grounded in a view of learning that centers its social nature. From a sociocultural perspective, examining learning means examining the ways that people come to

participate in the practices of a community. In contrast to cognitive perspectives, in which learning occurs “in the head,” sociocultural perspectives view learning as both taking place and being demonstrated through activity. Rather than as the accumulation and acquisition of knowledge, learning can be conceptualized as participating (Sfard, 1998). Specifically, in this study I draw from a situative perspective (Greeno & MMAP, 1998; Lave, 1988; Lave & Wenger, 1991). Central to this perspective is the idea of legitimate peripheral participation, which recognizes that there are “multiple, varied, more- or less-engaged and -inclusive ways of being located in the fields of participation defined by a community” (Lave & Wenger, 1991, p. 36). Investigating learning therefore requires examining the ways that learners engage in the process of becoming community members, and how their participation shifts both in-the-moment and over time.

Viewing learning as situated emphasizes its dynamic, dialectic character. As newcomers and old-timers, apprentices and masters, or students and teachers engage in practice together, meaning is negotiated. The norms and practices of the community evolve as they are taken up and adapted by varied participants. The learner thus shapes practice, as practice in turn shapes the learner. Likewise, in-the-moment participation shapes and is shaped by the larger histories and structures within which the community is nested. In other words, “learning, thinking, and knowing are relations among people in activity in, with, and arising from the socially and culturally structured world” (Lave & Wenger, 1991, p. 51).

When viewed in this light, people in activity are always engaged in some form of learning, formal or informal, intended or unintended. Learning is both the process and an outcome of “being” in the world. While I focus primarily on interactions within classrooms, and on the learning of mathematics, viewing learning as situated helps us to see how individual

learners bring with them into classrooms varied ways of knowing, doing, and being. How these ways of participating are marginalized or integrated into the norms and practices of the community is central in shaping students' opportunities to learn (Erickson, 1996; K. D. Gutiérrez, Baquedano-López, & Tejeda, 1999; K. D. Gutiérrez, Rymes, & Larson, 1995; Hand, 2010).

Learning mathematics in activity

A situated perspective on the teaching and learning of mathematics requires a focus on the norms and practices of particular classroom communities, and on the ways interactions are mediated by the use of language and tools. A large body of work within mathematics education examines students' participation in mathematical practices, and how classroom norms for doing mathematics shape the roles and positions that are made available to students, and hence their participation. Rather than an exhaustive review of this literature, here I highlight several influential contributions that shape the research questions and methods of this study.

For example, Lampert's (1990) illustration of mathematics teaching and learning in action provided a potential vision for what it might look and sound like for students to engage in problem solving, reasoning, and justification while doing mathematics in school. Lampert argued that a goal of instruction should be to engage students in mathematical practices that bear a closer resemblance to those of communities of mathematicians. This vision of school mathematics required a reorganization of teacher and student roles, particularly on the distribution of mathematical authority. In her teaching, Lampert focused on facilitating classroom discourse, while placing the responsibility for evaluating the underlying logic and "correctness" of a solution or explanation with students. The norms for engaging in mathematical

activity evolved over time through patterns in the ways that students' ideas were made public, and the merits of mathematical ideas are negotiated.

Yackel and Cobb (1996) distinguished between the social and sociomathematical norms of classrooms. Social norms for classroom talk might involve obligations to explain and justify solutions, to indicate agreement or disagreement, or to attempt to make sense of other's ideas. While social norms certainly influence the ways that students participate in mathematical conversation, they are not necessarily specific to mathematics. Sociomathematical norms, on the other hand, involve taken as shared notions of what "counts" as mathematically acceptable. What counts as "enough" when giving an explanation, what counts as a mathematically "different" solution strategy, or whether or not a mathematical representation faithfully reflects the author's idea all involve interpretations of the mathematical aspects of student participation.

Yackel and Cobb's study illustrates that while teachers may (or may not) make explicit normative statements, collective norms are established jointly in interaction. Teachers may initially structure opportunities for participation in particular ways, but what come to be normative ways of doing mathematics are negotiated as students take up these opportunities, which are then interpreted and responded to by other students and the teacher. The teacher nevertheless plays a critical role in the establishment and maintenance of classroom norms.

Kazemi and Stipek (2001) found that teachers who pressed for conceptual thinking in interactions with students shaped norms that fostered mathematical reasoning within explanations, valued mistakes as learning opportunities, supported students to make connections between strategies, and achieved consensus through mathematical argumentation. Collectively, this research provides compelling evidence that attending to the development of particular sociomathematical norms is critical in supporting students' mathematical learning (Cobb,

Stephan, McClain, & Gravemeijer, 2001; Franke, Kazemi, & Battey, 2007; Kazemi & Stipek, 2001; Lampert, 1990, 2003; McClain & Cobb, 2001; Yackel & Cobb, 1996).

However, even in classrooms that establish strong social and sociomathematical norms around explaining and justifying thinking, discussions may still bear a resemblance to aspects of traditional IRE discourse patterns. As students share strategies or respond to questions about mathematical ideas, conversations may flow *through* the teacher. While this mediation may at times be productive in supporting student learning and participation, it can also serve to reinscribe traditional distributions of power, status, and knowledge between students and teachers. For example, McClain and Cobb (2001) noted a distinction between the teacher-student-teacher patterns of turn-taking during whole-class discussions and of instances in which this turn-taking pattern broke down. In their study, contributing to whole group mathematical discussions usually involved hand-raising and recognition from the teacher. Even when students were directed to speak to the entire class, speaking publically usually took the form of addressing the teacher. This suggests that even classrooms that are successful in distributing mathematical authority to students may not necessarily disrupt other power structures within classroom learning environments, particularly who is afforded agency in gaining and holding the floor in conversations. However, in the classroom studied by McClain and Cobb, these patterns broke down in relation to norms associated with engaging in each other's ideas. When students indicated that they didn't understand a peer's explanation and asked clarifying questions, or when they explained why they considered a particular student's explanation to be invalid, they were much more likely to address each other directly. McClain and Cobb's findings, as well as recent findings by Franke, Webb, and colleagues (2015) suggest that the norms associated with

engaging with each other's ideas may present an opportunity to redistribute agency in determining who is given permission to speak, and when.

Recent research has attended to issues of authority, status, and power as they play out in classrooms. In an investigation of how one heterogeneous group of students navigated mathematical discussions, Esmonde and Langer-Osuna (2013) highlighted varied forms of power displayed and contested as a group of students engaged in mathematical activity together. Their analysis demonstrated how students' positions within the social and mathematical spaces of a given classroom community shaped the ways they interacted, the types of agency they displayed, and therefore their opportunities to participate and learn. The authors also provided a nuanced illustration of what it may look and sound like for different students to participate in mathematical practices, and that the ways that students engage in these practices may depend on their preferred styles of interacting and communicating. One student, for example, took up a position of power in critiquing another student's ideas. This work helps us to see that engaging in mathematical practices may not look the same for all students, and that students' opportunities to learn mathematics cannot be disentangled from their multifaceted identities within and outside of classrooms.

To summarize, understanding the learning and teaching of mathematics in classrooms requires attending to the complexities of classroom spaces, and the relational character of what is learned, who learns, and how.

“Together, the teacher, students, and families can shape who learns and how. Who gets to talk, when, about what, who participates and how, the kinds of solutions and questions considered acceptable, and so on are determined by those participating and the experiences they bring to the classroom. Shaping participation is not always accomplished explicitly but can be driven by implicit goals, beliefs, and identities of the teacher, school, and community.” (Franke et al., 2007, p. 238)

A focus on the processes of teaching and learning mathematics in classrooms raises questions about what it would mean for students to be competent in particular classrooms.

Conceptualizing competence

Use of the term “competence” and its related forms is increasingly common in mathematics education as well as the larger research and policy community. Traditionally, competence has been used synonymously with terms such as ability, skill, capability, or proficiency, though competence sometimes holds a broader, more general connotation. A recent publication by the National Research Council explicitly designates “21st Century Competencies” to reflect a “blend of both knowledge and skills” necessary for “success in education, work, and other areas of adult responsibility,” (NRC, 2012, p. 19). Similarly, the Program for International Student Assessment (PISA) is intended to measure and monitor the “key competencies... essential for full participation in society” (Organisation for Economic Co-operation and Development, 2005, p. 3). In its dominant use, competence or competencies are learning goals or outcomes that reflect desired traits or characteristics, to be acquired or possessed by individuals.

Sociocultural views of learning, however, offer a different perspective on what is meant by competence. Rather than a collection of attributes of individuals, competence is viewed in relation to particular ways of participating in the practices of a given community. What counts as competence then is negotiated and co-constructed as teachers and students interact and participate together. Viewing competence as constructed shifts the focus from static qualities of individuals to the dynamic meanings that are created within interactions and over time within a given environment. A particular skill, understanding, or ability may be demonstrated in activity, but whether or not this display is deemed competent is socially determined, shaped by the specific learning environment, even as it is nested within larger sociopolitical structures and histories. Mathematical competence then may take different forms (a correct answer, a detailed explanation), relate to specific behaviors (quiet listening, voicing disagreement), or involve

varied displays of proficiency (persevering through struggle, writing an equation). Competence is constructed, in part, through being assigned (often, but not solely, by the teacher). Competence may be assigned explicitly (e.g., through thanking a student or publicly recognizing an answer as correct) or implicitly (e.g. through asking for further explanation or suggesting that a student confer with a peer). Competent ways of engaging in mathematical activity therefore vary across classrooms and schools, influencing not only students' opportunities to learn mathematical content, but also what they learn about the discipline of mathematics and who they get to be within that discipline. In this dissertation I follow Gresalfi and colleagues' (2009; see also Hand, 2010) conceptualization of competence as a phenomenon that is co-constructed in interaction and over time, and define competent participation as the ways of engaging in mathematical activity that are interpreted by a community to be productive, successful, or desirable.

Gresalfi and colleagues' (2009) framework for examining competence provides a starting point to examine the factors that serve to construct competence, and to consider how notions of competence might be expanded in ways that would open opportunities to participate in mathematics. Their framework highlights the ways that students' roles in and contributions to mathematical activity ultimately shape what students are accountable for learning, to whom they are accountable, and the kinds of agency they exercise within the discipline. Here, what counts as competence is closely linked with what it means to do mathematics within a given classroom community. Hence, "efforts to understand *whether* students are learning should not be attempted without a critical focus on *what* students might be able to learn, given the organization of the classroom settings in which they are participating" (p. 68, emphasis in original). Expanding teachers' and students' views of competence would involve teaching and learning mathematics in ways that incorporate students' mathematical ideas and distribute agency and accountability

for sense-making to students (Lampert, 1990). This conceptualization of expanding competence is closely linked with efforts to support students' participation in mathematical practices. In highlighting distinctions between traditional and "reform" classrooms, however, this initial work does not explore how competent participation might vary even within classrooms that take up current recommendations for mathematics teaching and learning (NCTM, 2014).

Research in Complex Instruction (Cohen & Lotan, 2014) offers possibilities for how teachers might leverage broader notions of competence to address issues of differential status between peers. Boaler and colleagues (e.g., Boaler, 2008; Boaler & Staples, 2008; see also Featherstone et al., 2011) describe the multiple ways that students were able to be successful in doing mathematics at Railside School (e.g., using different strategies to solve problems, asking questions, and using varied mathematical representations), and the teacher's role in strategically assigning competence to address status differentials between peers. This work suggests potential links between broadening opportunities for students' mathematical participation, student achievement, and the development of what Boaler calls "relational equity." However, while this research offers some general examples of what teachers do in assigning competence, the level of analysis in Boaler's work focused primarily on schools and students' ideas of what it meant to do mathematics. Teachers' specific instructional practices and the knowledge, skills, and dispositions that guide their decision-making in assigning competence are not well understood by researchers. These are especially relevant as Complex Instruction holds that competence must be specifically assigned to students' *mathematical* contributions. If teachers are unable to observe or recognize students' varied contributions, some students, especially those whose knowledge and practices have been historically marginalized, may be denied opportunities to participate and learn (see discussion in Esmonde, 2009a, p. 1026).

Expanding notions of competence therefore requires that teachers are able to recognize and interpret the varied ways that students display and communicate their understandings. In an analysis of two mathematical discussions, Moschkovich (2002) illustrates the shortcomings of viewing bilingual students' participation only in terms of vocabulary use or of formal vs. informal registers. Such lenses may overlook the understandings that students display through the use of invented words, gestures, tools, metaphors, or their first language. Moschkovich reminds us that narrow views of competence run the risk of perpetuating deficit perspectives of bilingual students, neglecting the resources that they draw upon in communicating their mathematical understandings.

Students also display varied understandings in the ways that they interpret and solve problems that might inform broader notions of mathematical competence. In examining a teacher's interactions with two struggling first grade students, Empson (2003) illustrates the ways that the teacher uncovered productive mathematical ideas within students' vague or incomplete strategies and later positioned them as competent contributors to whole class discussions. The teacher's ability to build upon struggling students' partial understandings was supported through both the use of tasks that allowed students to marshal their prior and informal knowledge, as well as the teacher's facility in attending, interpreting, and responding to students' mathematical thinking (Jacobs et al., 2010). Related work has described how English learners' participation in discussions was supported through a teacher's use of various positioning moves, such as explicitly validating a student's reasoning, inviting a student to share, justify, or clarify thinking, or inviting peers to respond (Turner, Dominguez, Maldonado, & Empson, 2013). These moves to assign competence to emerging bilingual students' mathematical ideas supported them to take up agentive roles in classroom discussions. This research suggests teachers' notions of

competence may depend in part on their understanding of the details of children's mathematical thinking, and on a stance that centers students' mathematical ideas in instructional decision-making (Carpenter et al., 2015). Expanding competence would require that teachers are able to recognize the mathematical potential in students' informal, partial, or incomplete understandings and build from these in interaction.

Critical perspectives highlight the gendered, raced, and classed nature of participation, and how these can influence the ways in which notions competence are constructed. Martin describes how discussions of the "achievement gap" have served to construct African American children as mathematically illiterate and incompetent. These discourses perpetuate restrictive views of mathematical knowledge that normalize the performance of White students and undervalue children's out-of-school knowledge. According to Martin, "it is very likely that mathematical competencies linked to the cultural contexts and everyday life experiences of African American children are under-assessed and under-valued because their competencies do not fall within dominant views of what counts as mathematical knowledge" (2009, p. 16). These dominant conceptualizations of knowing and doing mathematics overlook the ways students' histories and cultural experiences play a role in how they may display their understandings and engage in classroom discourse and learning (Nasir, Hand, & Taylor, 2008). Indeed, there is growing evidence that individual students' participation varies widely in reform classrooms, and that gendered and raced notions of participation shape students' classroom experiences (Battey, 2013; Esmonde & Langer-Osuna, 2013; Langer-Osuna, 2011; Parks, 2010). Not interrogating how these factors shape notions of competence in classrooms risks reducing the idea of expanding competence to general calls for "good teaching," and may therefore continue to systematically marginalize nondominant student populations.

Competence and supporting students to engage with each other's ideas

Research that examines the ways that teachers support students to participate through engaging with each other's ideas can also inform an expanded conceptualization of how competence is co-constructed within classroom interaction. The work of O'Connor, Michaels, and colleagues presents many in-depth accounts of how teachers orchestrate academically productive discourse in urban classrooms (e.g., Chapin, O'Connor, & Anderson, 2009; Resnick, Michaels, & O'Connor, 2010). In particular, talk moves, such as revoicing (O'Connor & Michaels, 1996), display potential not only to support student learning and participation, but also to disrupt the authority and power structures often associated with traditional initiation-response-evaluation (IRE) discourse patterns. Though the use of these moves by no means guarantees that discussions are productive and equitable (Enyedy et al., 2008), they have potential to function as scaffolds to support students to attend to, make sense of, evaluate, and build upon their peers' contributions. Used effectively, talk moves can provide a multitude of entry points for students to participate in mathematical discourse while simultaneously promoting democratic principles of communication and dialogic classroom space (O'Connor & Michaels, 2007).

The work of Webb, Franke, and colleagues (recently summarized in Webb, Franke, Ing, Turrou, & Johnson, 2015) examines links between student participation in mathematical discussions, achievement, and teachers' instructional practices. Through detailed analyses of classroom interactions, this research has demonstrated that the mathematical potential of discussions is shaped by the ways that teachers invite and follow up on the details of students' ideas, and that the degree to which students explain their mathematical thinking or engage with others' mathematical ideas is linked to student achievement. Recent work has detailed the ways that teachers support students to engage with each other's ideas by providing specific scaffolds

and positioning students as contributors to discussions (Franke et al., 2015; Webb et al., 2014). This work once again suggests that the teachers' role in eliciting and framing students' contributions is central in supporting student learning and participation.

Expanding competence, engaging with others' ideas, and equity

Collectively, these bodies of literature suggest connections between multidimensional views of doing mathematics, knowledge of students and student thinking, recognizing diverse ways of knowing and participating, and supporting students to engage in each other's ideas. Recent scholarship has argued that attending to status and competence as constructed within interactions and across different classrooms can provide an entry point into discussions of equity and opportunities to learn (Hand, Quindel, & Esmonde, 2010; Jackson, 2009). Notions of competence shape what are viewed as assets or deficits, desirable ways of participating, who is viewed as having (or lacking) power, and what counts as mathematics. Learning is integrally linked with being positioned as a competent contributor within the classroom community. Teachers play a central role in supporting learning and participation through the ways that they assign competence, and to what and to whom competence is assigned. As Gloria Ladson-Billings states, "students treated as competent are likely to demonstrate competence." (1997, p. 703). How competence is conceived of and constructed therefore shapes the forms of participation available to students, how power and authority are distributed, and the kinds of agency that students exercise. What counts as competence determines not only what it means to do mathematics in classrooms, but *who* gets to do mathematics. Research that investigates how notions of competence might be expanded could provide a means by which to move beyond broad depictions of instructional practices that promote mathematics learning and success for *all*

to *how* specific supports, lesson structures, and positionings open or constrain opportunities for *particular* students to participate and learn.

Recently, NCTM's *Principles to Actions* provided a set of eight mathematics teaching practices intended to “ensure mathematical success for all” (NCTM, 2014). It is certainly true that, for example, supporting students’ productive struggle, facilitating meaningful mathematical discourse, and eliciting and using evidence of student thinking are widely recognized to be critical components of teaching and learning mathematics with understanding. However, research has only begun to explore what engagement in these practices might look and sound like in diverse classrooms or among populations of students from nondominant communities. The potential for these practices to promote more equitable learning opportunities requires research that examines what is meant by competent ways of participating, how collective understandings of competence are constructed through both classroom interactions as well as larger structures and discourses, and the affordances of varying interpretations of competence for student learning.

Students learn more than mathematics in classrooms; they learn who they are and negotiate who they want to be within the affordances and constraints of the classroom, school, local community and larger society (Cobb & Hodge, 2011). Identifying and distinguishing between the forms of participation deemed competent or incompetent informs our understanding of how classrooms center or marginalize students’ ideas, and broaden or constrain the kinds of agency that they are permitted to display, thereby opening or limiting the ways that students are able to “take up their space” in mathematics classrooms (Hand, 2012). A more nuanced conceptualization of competence could inform the field’s understanding of the range of ways that current reforms are enacted in classrooms. In particular, investigating the co-construction of

competence as students engage with each other's ideas, and relations between competence and other learning outcomes, present a promising avenue for research. This research could continue to enhance our understanding of classroom practice, providing insight into one component of what may be involved in disrupting patterns of inequity in how opportunities to learn are distributed within schools.

Situating the current study

A critical space for investigating the construction of competence is within classroom interactions, both those that involve the teacher as well as those between students when the teacher is not present. While there is a substantial body of literature that has begun to identify teaching practices that support students to participate in mathematical discussions and to engage with others' ideas, this research has not yet explored specifically how notions of competence are constructed within classroom discourse. Supporting *each* student to access and participate productively in classroom discussions remains a challenge for practitioners and researchers. This goal of this dissertation is to bring together research on classroom teaching with research on students' opportunities to learn through a focus on competence. The lens of competence provides a means by which to examine issues of access and equity that move beyond broad depictions of instructional practices that promote mathematics learning and success for all (Martin, 2003, 2015) to how specific supports, lesson structures, and interactions may open or constrain opportunities for particular students to participate and learn. However, there remain many unanswered questions with respect to competence. How is competence assigned or otherwise communicated? What forms of participation are assigned competence by teachers and students? And how do collective understandings of competence within particular classrooms relate to the

mathematics that students learn and the kinds of agency they exercise? This dissertation study explores these questions through an investigation of competence in two third-grade classrooms.

CHAPTER THREE: METHODS

A Multidimensional Analysis of Competence

The purpose of this study is to explore the ways in which teachers and students co-construct notions of competent participation in interaction, and how these local constructions of competence within classrooms shape students' opportunities to learn and engage in mathematics. Previous research suggests that notions of competence are related to what counts as mathematics in classrooms, the ways that teachers elicit, interpret, and respond to student thinking, and the ways that students are positioned as contributors to classroom mathematical activity. This research also suggests that issues of agency, power, and status play out in relation to what it means to be competent in particular classroom environments. Collective notions of competence therefore determine what mathematics is, and in turn *who* gets to do mathematics (Boaler & Greeno, 2000; Jackson, 2009). However, little research has examined the details of how competence is co-constructed within classroom interaction, and relations between what come to be shared understandings of competence, students' opportunities to learn, and outcomes. Of particular interest are classrooms that center children's mathematical thinking and take up many of the current recommendations aimed at advancing the teaching and learning of mathematics in elementary school (NCTM, 2014; NRC, 2001).

The analytical approach undertaken in this study attempts to provide a more cohesive and articulated conception competence. While "competence" is commonly referred to both within and outside of mathematics education, the field lacks a nuanced framework for examining the construct of competence itself.² Such a framework could provide insights into how varied

² The field is indebted to Gresalfi and colleague's influential contributions (2009) that have drawn much-needed attention to competence as a socially constructed phenomenon (see also Greeno, 2011; Hand, 2010)

aspects of competence collectively function to shape students' opportunities to learn, and therefore how teachers and teacher educators might more effectively engage in efforts to disrupt patterns of inequity. A goal of this study is therefore to unpack varied aspects of what is intuitively a familiar phenomenon.

Within classrooms competence is a collective endeavor, but these collective understandings are constructed, negotiated, and displayed during in-the-moment activity. Within interactions, competence is constructed through both *being assigned*, and being assigned *to something*. In other words, whether or not particular ways of participating (e.g., asking questions, correctly solving a problem, attending to the details of someone else's strategy) are constructed as competent is dependent upon competence being communicated (or withheld) in some way, either explicitly or implicitly (e.g., by taking up someone's idea, probing for further explanation, or declining to respond). Consequently, operationalizing competence as evidenced in interaction requires attending to both the forms of participation that are deemed competent, and the ways in which competence is assigned or otherwise communicated. Competence is not assigned in isolation, but in relation to many other overlapping aspects of the classroom activity. These include (but are not limited to) students' previous patterns of participation, social relations between peers, the ways that the teacher has supported students' learning in prior lessons, students' prior mathematical understandings and experiences, and the specific mathematics under consideration in a given moment.

Over time and across the various settings of a classroom (e.g. whole group discussions, independent work) interactions serve to construct collective understandings of what counts as competent mathematical activity. These collective understandings influence the kinds of agency that students display, the role of students' ideas, and the mathematics that is learned. Thus,

understanding students' opportunities to learn in particular classrooms requires both examinations of specific classroom interactions, as well as analyses of general patterns and trends over time.

Research questions

This dissertation presents a case study of competence as constructed in two third-grade classrooms. Drawing from a variety of data sources including classroom video footage, student work and assessments, and teacher interviews, this dissertation examines multiple dimensions of competence related to student participation, agency, teachers' instructional practices, and mathematics learning outcomes. Specifically, this dissertation investigates the following questions:

1. In what ways is competence co-constructed by teachers and students within interaction?
 - In what ways is competence demonstrated, assigned, or otherwise communicated?
 - What forms of participation are constructed as competent?
2. How do collective notions of competence relate to students' opportunities to learn?
 - How do particular constructions of competence shape the ways in which students exercise agency?
 - How do particular constructions of competence shape the mathematics that students learn?

To investigate these questions this study examines interactions across classroom spaces in two third-grade classrooms where teachers supported students to solve problems together and participate in mathematical discussions. A variety of data sources including classroom video,

student work and assessments, and teacher interviews were analyzed with respect to varied dimensions of competence suggested by previous research, and further detailed below.

Setting and participants

The two third-grade classrooms (two teachers and 45 students) studied were from the same school within a small urban school district in southern California. The two classroom teachers, Ms. J and Ms. C³ had taught at the school and at the same grade-level for several years, and had adopted an approach to mathematics instruction focused on children’s mathematical thinking (Carpenter et al., 2015; Carpenter, Franke, & Levi, 2003). The school within which the two classrooms were located served a large proportion of low-income students of color, many of whom were emerging bilingual students. In the previous (2012-2013) school year, 57% of students at the school scored as proficient or advanced on the state mathematics assessment. 89% of students were classified by the state as “socioeconomically disadvantaged”, and 61% were classified as English learners. The ethnic/racial breakdown of school as reported by state accountability measures was 85% Latinx, 6% White, and 5% Black. The demographic composition of the two classrooms studied here reflected that of the school.⁴

Among five third-grade teachers at the school at the time of the study, the two teachers studied here were the only ones implementing a reform-oriented approach to teaching mathematics. In contrast to their colleagues, the teachers’ instruction involved a focus on story problems, supporting students to generate their own strategies, eliciting their ideas, and making

³ All names are pseudonyms.

⁴ I report these school-level data not to make claims about students’ individual identities or previous mathematical understanding, but rather to situate this study within the context of an urban school. In doing so I attempt to call attention to the ways in which local interactions shape and are shaped by the broader structures, histories, and societal discourses within which classrooms and schools are nested (Erickson, 2004; R. Gutiérrez, 2013).

student thinking public. While the two teachers were experienced in centering student thinking in instructional decision-making, had less experience in supporting students to engage with each other's ideas, and were selected in part because of their commitment to developing this aspect of their practice. Upon entering third grade, it is unlikely that students had previously engaged in doing mathematics in these ways. While Ms. J and Ms. C were recognized by their administrator and peers as strong teachers, they were nonetheless expected to adhere to district mandates with respect to scope and sequence and the administration of common periodic assessments.

Positioning the researcher

My interest in children's mathematical thinking and my previous experiences in education inform and shape the approaches used in this study and my research in general. I serve as a member of the UCLA Mathematics Teaching and Learning Research Group with Megan Franke, Noreen Webb, Marsha Ing, Angela Turrou, and other graduate students. As a member of the research team, I participated in the collection of data used in this study. Through this project and additionally through my work as a teacher educator and professional developer, I built relationships with the teachers whose classrooms are studied here, getting to know them as people, working in their classrooms alongside them, and discussing children's thinking in formal and informal settings. While my primary role is that of a researcher, various aspects of my identity and history influence what I attend to and how I interpret the data I choose to collect and analyze. I am a white male, raised in an upper-middle class family in southern California. As a former third-grade teacher who taught in a nearby community, I identify with the many challenges of running a classroom and finding ways to make sure that each child is successful amid a wide array of demands from curricula, standards, assessments, and other structures that

exist within and around urban schools. My work with teachers, students, and my own children adopts a stance that children come to school with rich intuitive mathematical understandings, and that a central part of the work of teaching involves recognizing and building upon children's individual strengths and ideas.

Data sources

The classroom video footage and student work presented in this dissertation study is drawn from a larger data set collected by the UCLA Mathematics Teaching and Learning Research Group during the 2013-2014 school year. Through this project, three classrooms (one fourth-grade and the two third-grade classrooms that are focused upon here) were filmed for multiple days at three different time points during the school year (January, March, and May). On each day of data collection the research team recorded classroom interactions using six iPod touch devices to record six pairs of students and one iPad device to record the teacher with the class or with specific students throughout the entire lesson. This recording system made it possible to identify which student was speaking even during simultaneous pair conversations. Rotating recording across the pairs of students in the classroom made it possible to record all students and the teacher in the class during each time point.

For the purpose of investigating the construction of competence, this study focuses on four days of mathematics lessons in each of the two third-grade classrooms. The four days (two in January and one in March and May respectively) in each classroom were chosen to highlight a range of instructional goals and mathematics content within multiplication and division. The two classrooms were filmed on same day, used same instructional tasks, and implemented a similar lesson structure, consisting of a whole-class warm up activity, followed by the posing of a story

problem, students working in pairs to solve the problem, and a whole-class sharing of solution strategies.

The goal of this study is to investigate the co-construction of competence in relation to teachers' instructional practices and student participation. Therefore, the primary focus of video analyses was footage that tracked whole-class discussions and teachers' conversations with individual students and student pairs as they worked together (approximately 9 hours).

A more thorough understanding of each classroom also required an examination of how students negotiated and took up notions of competence when the teacher was not present. Previous work within the larger project coded and analyzed pair interactions with respect to the nature of their engagement with each other's mathematical ideas. Of particular interest to the current study were interactions that involved mathematical disagreements between partners, and pairs where one student tended to control the pair's work, limiting their partner's opportunities to participate. It was hypothesized that these negotiations would shed light on how students positioned themselves and were positioned by others as knowers and doers of mathematics, and how these shared notions of competence were taken up within these pairs. With this in mind, the entire corpus of student pair data was examined for trends with respect to disagreement and one student controlling the pair's work. This revealed that pair interactions within the first time point contained a larger proportion of instances related to these features. In an effort to incorporate both teacher and student-led interactions into the analysis while also capturing a comprehensive record of each student's participation over the course of a day, only student pair interactions during the first time point were analyzed (2 days per classroom; approximately 22 hours total).

Additional data analyzed included student work, student assessment data, and transcripts from semi-structured interviews with the two teachers. A researcher-created problem solving

assessment was administered to each student near the end of the school year. The problem solving assessment focused on the target content of the lessons filmed (multidigit multiplication and division). The previous year's (second grade) mathematics CST score was also collected for 44 of 45 students to provide a partial record of students' prior achievement in mathematics.

Two rounds of semi-structured interviews were conducted with each teacher during the spring and fall of 2016. The interviews provided an opportunity for the teachers to corroborate, challenge, or expand upon insights related to competence derived from video data. The interviews also offered a window into teachers' interpretations of students' mathematical activity and their own instructional goals and decision-making. Appendix A contains the questions used to guide conversations during the two rounds of interviews.

Research design

This case study of the two third-grade classrooms presents an opportunity to examine similarities and differences across classrooms while many other features of the learning environment overlap. As previously stated, the two classrooms were observed on the same days of the school year, posed the same mathematical tasks, and used similar lesson structures. Furthermore, both teachers had previously participated in preparation and professional development focused on children's mathematical thinking (Carpenter et al., 2015), and had adopted an instructional focus on eliciting student thinking and on engaging students in each other's mathematical ideas. Thus, this is *not* a study comparing traditional and reform classrooms, or effective vs. ineffective teachers; both teachers' instruction embodied many current recommendations of researchers and educators (NCTM, 2014). However, while many surface features of the two classrooms appeared similar, that did not necessarily mean that

competence was constructed in the same ways or resulted in similar outcomes. A goal of the case study was to unpack these features through a multidimensional analysis of competence. Such an analysis might uncover the nature of similarities and differences across classrooms, and reveal how the organizational structures and in-the-moment supports within each classroom functioned together to shape students opportunities to exercise agency and to engage in and learn mathematics.

Coding and analysis

Data analysis investigated 1) the ways in which competence was co-constructed by teachers and students in interaction, and 2) relationships between collective notions of competence and students' opportunities to learn. Analysis proceeded through a series of passes through the video footage, student work, teacher interviews, and student assessment data. Studiocode software (Studiocode Group, 2015) was used to code and analyze video data. Unlike a transcript, coding within Studiocode other contextual features captured by video, such as students' gestures, gaze, facial expressions, intonation, the physical space being occupied, the time of the interaction in relation to the overall lesson, and so on. Studiocode also allows for analyses of different grain sizes, as detailed subsequently.

The initial phase of coding in Studiocode followed Erickson's (1992, 2006) recommendations for interaction analysis using video data. Each day of footage that followed the teacher was in its entirety without stopping. A second viewing identified beginnings, endings, and transitions within the lesson, parsing each lesson into segments consisting of the warm-up activity, the posing of the day's task, students solving the problem with their partners, and a whole-group discussion of solution strategies (Jordan & Henderson, 1995). Studiocode was used

to create an additional unit of analysis by duplicating and then parsing each phase of the lesson into episodes. An episode consisted of the series of interactions related to a particular strategy (Forman & Ansell, 2002). Parsing the lesson into episodes allowed for the creation of a grain size that attended to both the contextual features of the mathematics (the details of students' ideas related to a particular strategy) as well as patterns in teachers' and students' turn-taking and attempts to follow-up on each other's ideas. While there was some variability in the duration of episodes,⁵ this unit of analysis offered the ability to attend to critical features of meaning-making within interactions between people and among people and tasks, tools, and the environment (Hall & Stevens, 2016). One brief episode in Ms. J's classroom was omitted because of inadequate audio. Five episodes in Ms. C's classroom were omitted due to inadequate audio or because of interactions featuring a student whose parents declined participation in the study. Across four days of instruction in each classroom, this process yielded a total of 82 episodes in Ms. J's classroom and 65 episodes in Ms. C's classroom.

In an effort to capture various features of classroom activity identified by the literature, each episode was coded for four dimensions of competence—multidimensional ways of doing mathematics, taking up space, children's mathematical thinking, and teacher support of student participation. Table 3.1 provides a brief description of each of the four dimensions. Within these dimensions, episodes were assigned labels characterizing specific aspects of interactions within each dimension. Labels within each dimension were drawn from the literature, and adapted and refined through multiple passes through the data and in consultation with project members.

Following an initial labeling pass for each dimension, an additional row was added in

Studiocode to allow for a more in-depth labeling of students' explanations and engagement with

⁵ Episodes during warm-up activities tended to be shorter than those during students' problem solving work with partners and the whole-class sharing of solution strategies that followed. See appendix C for a detailed breakdown of lesson phases and episodes for each classroom.

each other’s ideas. Figure 3.1 displays a timeline, video, and coding window used simultaneously to code within Studiocode. See Appendix B for descriptions of the codes, subcodes, and labels used in analysis. All coding was performed by the author under the guidance of the dissertation chair.

Table 3.1: Operationalizing competence across four dimensions of classroom activity

<i>Dimension</i>	<i>Features of classroom activity captured</i>
Multidimensional ways of doing mathematics	This coding dimension captures what students are doing as they participate in classroom mathematical activity, including varied ways of participating highlighted by Complex Instruction (Boaler & Staples, 2008; Featherstone et al., 2011), as well as sociomathematical aspects of participation (Yackel & Cobb, 1996).
Taking up space	This coding dimension tracks the ways in which power, status, and agency play out in interactions. These include how authority and agency are negotiated, whether interactions serve to open or close dialogic space, or to blur boundaries between mathematical and cultural activity (Hand, 2012; O’Connor & Michaels, 2007).
Teacher support of student participation	This coding dimension identifies in-the-moment instructional moves employed by teachers, such as revoicing (O’Connor & Michaels, 1996) invitations to explain or add on, and follow up moves such as probes, scaffolds, and positioning moves (Franke et al., 2015). This dimension also captures teachers’ normative statements and explicit assignments of competence.
Children’s mathematical thinking	This coding dimension captures the details of students’ mathematical ideas (Carpenter et al., 2015) as well as the nature of student explanations and engagement with others’ mathematical ideas (Webb et al., 2014).

Figure 3.1: Coding classroom interactions in Studiocode

The image displays the Studiocode software interface, which is used for coding classroom interactions. It features a video window on the left showing two students in a classroom setting, one pointing at a whiteboard with mathematical equations. The main area is a coding tree with various categories and sub-categories, such as 'Multidimensional Mathematics', 'Taking up space: Power and agency', 'Engagement with others', and 'Children's Math Thinking'. The bottom of the interface shows a timeline and a table of coded interactions.

Time	00:10:00.00	00:20:00.00	00:30:00.00	00:40:00.00	00:50:00.00	01:00:00.00	01:10:00.00	01:20:00.00	01:30:00.00
1 Lesson Phase	Wa	W	3	2 hrs 45 min =	4 Warm Up	3rd task	5	6 Problem Solving	Multiplication 8 x 15; 6 x 16; 4 x 26
2 Episode		3	4	6	7	8	9	10	11 Doe
3 Multidimensional Mathematics		3	4	6	7	8	9	10	11 Ways o
4 Taking up space: Power and age		3	4	6	7	8	9	10	11 blurring
5 EOI & Explanation		3	4	6	7	8	9	10	11 EOI Ex
6 T support		3	4	6	7	8	9	10	11 Invitati
7 Children's Math Thinking		3	4	6	7	8	9	10	11 solution
8 Combined Lesson Phase		3	4	6	7	8	9	10	11 solution
9 Combined Episode		3	4	6	7	8	9	10	11 solution

A goal of analysis was not only to track frequencies of different kinds of participation and teacher support, but also to capture these in relation to each other, and in relation to the mathematical details under examination at that moment. Following the coding and labeling of each dimension of competence, combined episodes and lesson phases were created to create a comprehensive unit of analysis. Matrices within Studiocode were then used to examine frequencies and relations between code labels across dimensions and within lesson phases and episodes. This process was used to analyze all interactions that followed the teachers in their work with students across each day. Studiocode analyses of student pair footage was previously coded according to features of students' engagement with each other's ideas by research team as part of the larger project (Franke, Ing, Johnson, Turrou, & Webb, in press).

In addition to the coding of various dimensions of classroom activity, narrative descriptions of each segment were written within Studiocode. These qualitative descriptions provided an overview of the episode and noted significant moments such as a particular sequence of teacher moves, the details of a mathematical representation, or an important shift in participation. The narratives were then synthesized into analytic memos (Saldaña, 2013) to provide a holistic view of interactions to compliment analyses of patterns within Studiocode. This multidimensional analysis of video footage provided a comprehensive picture of how student participation, agency, teachers' instructional practice, and the details of children's thinking jointly shaped what it meant to do mathematics in each classroom, and what it meant to co-construct collective notions of competence in interaction.

Constructing competence: An example

To illustrate the ways in which competence was constructed within and across dimensions, I provide a brief example and corresponding analysis.⁶ The following episode occurred during a warm-up activity in which students were being asked to consider different possible ways to decompose the number 150. Several other solutions have already been shared and discussed when Gerardo is invited to share and offers “10 times 15” as a possible solution.

Ms. C: Gerardo.
Gerardo: Uh, 10 times 15?
Ms. C: Ah. What do you guys think about that one? [records 10 x 15 on chart]
Students: Yeah/yes
Ms. C: Who can prove it?
Students: [several students raise hands]
Karla: I don't know.
Ms. C: Do you want to ask him?
Karla: [addressing Gerardo] I don't, can you repeat that but in an easier way?
Gerardo: Like, what?
Karla: Like, I don't, like, the 10 times 15? Prove that it's 150.
Gerardo: Cause 10, 20, [holds up a finger for each count] 30, 40, 50, 60, 70, 80, 90, 100, [nods, clears fingers], 110, 120, 130, 140, 150.
Students: ooh
Ms. C: So, what was he counting by?
Students: Tens!
Ms. C: How many times?
Students: 5/15!
Ms. C: Hmmm. Did it equal 150?
Students: Yes
Ms. C: Karla, thanks for asking that question.

Within this episode collective notions of competence are displayed and negotiated as students and teachers participate together in mathematical activity. Gerardo offers his idea of “10 times 15” to the group as a possible way to make 150. Ms. C follows up by inviting the class to agree or disagree with his idea (“What do you guys think?”), and then asks for volunteers to add on to Gerardo’s idea (“Who can prove it?”). Several students raise their hands to respond, while Karla utters “I don't know” in a way that is audible to the class and to the teacher. Rather than calling

⁶ As a way of beginning to develop an approach to capture the co-construction of competence, the warm-up lessons in one of the study classrooms were analyzed as a pilot study. This analysis provided an initial attempt to identify and operationalize varied aspects of competence.

on one of the students who have raised their hands, Ms. C instead takes up Karla's utterance, inviting her to ask Gerardo a question ("Do you want to ask him?"). Karla asks Gerardo to repeat his idea "but in an easier way," to which Gerardo responds with some uncertainty. Karla then makes a more specific request—for him to "prove that it's 150," to which Gerardo is able to respond by articulating a counting by 10s strategy. Ms. C follows up on his explanation by questioning the class about the amount within each count, the number of counts, and the total. Finally, the episode closes as Ms. C thanks Karla "for asking that question."

Examining interactions across the episode reveals the ways in which competence is constructed in relation to many overlapping dimensions of classroom activity. Analyzing this episode from the lens of *multidimensional ways of doing mathematics*, we see competence constructed around participating by providing explanation (Gerardo), voicing confusion (Karla), and engaging with others' mathematical ideas (Karla and the class). Sociomathematically, notions of mathematical correctness and adequate proof are constructed through the initial teacher's initial appeal to the collective for agreement, follow-up press for justification, and later invitations for choral responses to highlight the details of Gerardo's strategy. The lens of *taking up space* illuminates how Karla exercises agency by publicly stating "I don't know," a bid that is taken up by the teacher. Mathematical authority throughout the episode is vested in the collective, rather than in the teacher or textbook. *Teacher support of student participation* is provided through invitational and follow-up moves to engage in others' ideas (e.g., "What do you guys think about that one?" and "what was he counting by?"), and through specific positioning moves ("Do you want to ask him?") and an explicit assignment of competence ("Karla, thanks for asking that question"). Finally, viewing the details of *children's mathematical thinking* reveals that competence is demonstrated through a complete, fully

detailed explanation of a counting by 10s strategy, and by through Karla's engagement with the Gerardo's idea in a detailed way (asking a specific question). Competence, however, is relational in nature. While specific utterances can be mapped onto these individual dimensions of competence, it is only in connection with each other that they play out and are given meaning. Throughout the episode, elements of students' mathematical thinking, their varied ways of participating and exercising agency, the teacher's invitational and supporting moves, and both explicit and tacit assignments of competence serve to co-construct notions of what it means to do mathematics, and what it means to be a competent participant.⁷

Coding and analysis of teacher interviews, student work and assessments

Analysis of student work and end-of-year assessment attended to students' strategy use in solving multidigit multiplication and division problems. Strategies included direct modeling using 1s or 10s, counting/adding, and invented algorithms (Ambrose, Baek, & Carpenter, 2003; Carpenter et al., 2015). Students' solutions were examined both with respect to both the correctness of the answer as well as the validity of the strategy. Teacher interviews were coded with respect to emergent themes within the analysis, including the most prominent ways of participating in each classroom, teachers' support moves, and explicit assignments of competence.

This dissertation study provides a comprehensive analysis of competence as constructed in local classroom settings. Multidimensional analyses of classroom video attended to the ways

⁷ For the purpose of illustrating the coding of dimensions used in Studiocode, the discussion of competence here has not been exhaustive. The analysis might also have noted, for example, 1) the teacher's subtle acknowledgement ("Hmmm") that the class does not have a common understanding of how many times Gerardo counted by ten, 2) a tacit nod towards the sophistication of Gerardo's strategy ("Ah"), and 3) that Gerardo may have, for the benefit of his peers, chosen to explain a less-sophisticated strategy than he actually used; there is some evidence that he drew on an understanding of ten 10s within 100 in order to solve the problem.

in which competence is constructed around and in relation to particular ways of participating, exercising agency, the details of students' mathematical ideas, and teachers' instructional practice. These in-depth analyses of classroom interactions were supported by corresponding analyses of student work, teacher interviews, and an end-of-the-year problem solving assessment. Collectively these analyses detailed the ways in which competence was constructed in two third-grade classrooms.

CHAPTER 4: FINDINGS

Constructing Competence in Two Third-Grade Classrooms

Analyses of classroom interactions in two classrooms revealed that competence was constructed around and in relation to students providing explanations, and students engaging with others' mathematical ideas. The tasks and organizational features of both classrooms created the necessary conditions within which a focus on explanation and engagement with others' ideas expanded opportunities for students to participate and be recognized as competent. In detailing the ways in which competence was constructed, I begin by describing the mathematical tasks and shared organizational features of the two classrooms. I then provide detailed analyses of different dimensions of competence in each classroom. These analyses of competence as evidenced in classroom interactions are followed by an analysis of student achievement. I conclude by summarizing competence as constructed across both classrooms, noting similarities and differences in the ways that each classroom created space for students to be recognized as capable contributors to mathematical activity.

Organizing opportunities to learn: Tasks and structures across two classrooms

Ms. J's and Ms. C's classrooms were purposefully structured to provide opportunities for students to engage with each other as they made use of their intuitive mathematical ideas. These planned classroom features provided an environment within which in-the-moment opportunities were nested. Before detailing the ways in which these opportunities were enacted, I first present an analysis of the overall classroom structures and mathematical tasks used across the two classrooms.

Mathematics lessons on the days observed in both classrooms involved two major components: a warm-up activity followed by the solving of a multiplication or division story problem. The solving of story problems involved multiple phases, including the launch of the problem, solving the problem with a partner, a whole-class sharing of solution strategies, and (sometimes) independently solving the problem again using a different set of numbers. The structure of the lessons observed in many ways reflected lesson formats identified in student-centered or problem-based mathematics classrooms (e.g., Stein, Smith, Henningsen, & Silver, 2009; Van de Walle, 2007). Appendix C provides a detailed breakdown of each day in terms of the amount of time spent across various structures and the number of episodes within a particular phase of the day.

Warm-up activities

On each day observed, mathematics lessons began with a warm-up activity. Student were usually seated on the rug in the front of the room with their math partners, and asked to participate in a number sense routine such as decomposing numbers, mental math, or true/false and open number sentences (Carpenter et al., 2003; Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010). Appendix D lists the warm-up activities used in each classroom for each day. Warm-ups typically did not involve contexts, and students were usually expected to participate by solving mentally or using their fingers as tools.⁸ Warm-ups would begin with the teacher briefly posing the task and then inviting student to take 1-2 minutes of quiet thinking time to generate solutions independently. Students were often invited to generate multiple strategies if possible. Following quiet think time, students were invited to turn and talk with their partners to

⁸ On January 22 and 24 Ms. J posed the warm-up with students seated at tables and indicated that they record their strategies on white boards, but in all other cases in both classrooms students were seated on the rug and not provided with specific tools during warm-up activities.

share their solution strategies (Chapin et al., 2009). The teacher then facilitated a whole-group sharing and discussion of various strategies. As students shared their strategies, the teacher recorded their thinking on the board using mathematical symbols and representations, and followed up on students' explanations by asking clarifying questions, pressing for details, or inviting questions or comments from other students about the strategy. Occasionally, the teacher might invite students to turn and talk with their partner about an idea that emerged within the sharing of a particular strategy. Thus, the overall organizational structure of the warm-up activity was whole-group, but at different moments within this overall structure students were expected to participate by solving independently, talking with their partners, sharing out individually, listening to other students, or chorally responding to the teacher's prompts. Warm-up activities typically made up 10-20 minutes of a day's lesson. As compared with other lesson phases, episodes within warm-up activities tended to be shorter and more frequent, with an average duration of roughly 2 minutes per episode.

Solving story problems

The primary focus of mathematics lessons in both classrooms was the posing, solving, and discussing of story problems involving a range of contexts related to multiplication and division. Following the warm-up activity, while students were still seated on the rug, the teacher would introduce the day's problem. Typically, Ms. J and Ms. C would begin by reading the problem out loud or inviting the class to read the problem together, and then gauge the class' understanding of the story by asking, "What's the story about?" or "What's happening in the story." After eliciting and following up on students' responses, the class would be dismissed to begin working with their partners to solve the problem. The process of launching or unpacking

the day's problem took various forms across the classrooms and days observed, but the overall goal was to support student to understand the problem without taking over too much of the mathematical work or suggesting a particular approach students should use to solve (e.g., Carpenter et al., 2015; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). This often involved supporting students to make sense of the story context by relating it to their own knowledge and experiences, or to engage with the quantities in the problem in relation to its overall structure. On some days (one of the observed days in Ms. J's classroom, and two in Ms. C's classroom) students were provided the opportunity to unpack the day's problem with their partners rather than in the whole group with the assistance of the teacher. Across the two classrooms, posing and unpacking the story problem typically lasted 4-5 minutes a day.

Immediately following the launch phase of the lesson, students worked with a partner (or in a few cases as a trio) to solve the day's problems. While most pairs of students worked together at their tables, students were also free to work on the ground or at the carpet. A variety of tools were available to students for use during problem solving, including base-ten blocks, counters, money, and hundreds charts. While many students made use of these tools, others made use of pencil and paper strategies and representations as well. The day's problem was either printed on a small piece of paper that students were to paste into their math notebooks, or on a full-sized $8\frac{1}{2} \times 11$ paper that was provided for each student to solve and represent their thinking. By the end of the day's lesson, it was expected that each student would have some sort of written representation that reflected how they had solved the problem. Each problem contained three different sets of numbers. For example, the following problem (with different student names) was used in both classrooms on May 28:

Leo is allowed to watch TV for ____ minutes a day. How many minutes of TV can he watch in ____ days?

(15, 8)

(16, 6)

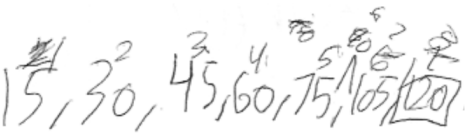
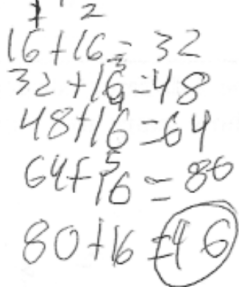
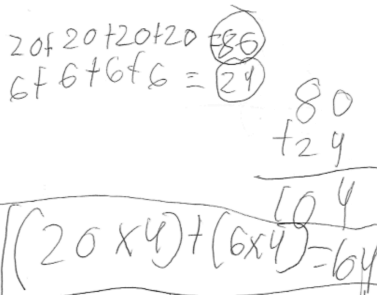
(26, 4)

In the example problem presented, the 15 would be placed into the first blank space, and the 8 into the second blank (thus resulting in a multiplication story problem for 8 groups of 15). Students typically (but not always) began by solving the problem using the first set of numbers. The expectations in both classrooms were that after solving the initial problem, students were to continue working by either 1) solving the same story problem with a different set of numbers, 2) solving the same problem with the same set of numbers using a different strategy, or 3) both. As the year progressed, students were also expected to record number sentences that reflected the strategies that they had used. The third number set was used differently across the two classrooms. In Ms. J's classroom, students were free to begin with a number set of their choosing, with the other number sets available to them whenever they liked. In Ms. C's class, however, the third number set was saved for students to work on at a later point in the lesson, following the whole-class sharing and discussion of solution strategies. The inclusion of multiple number sets provided several different opportunities. First, they served to allow for multiple entry points into the task; at least one of the number sets was usually small enough to make direct modeling by ones an accessible strategy.⁹ Next, the availability of multiple sets of numbers allowed for students to allocate their time during problem solving in various ways to fit their needs; some pairs of students might spend the entirety of problem solving time working through a single number set, while others might complete multiple strategies for two or even three sets of

⁹ Direct modeling strategies involve representing each of the quantities in the problem in a way that is consistent with the problem's overall structure. For example, the TV watching problem above would be directly modeled by making 8 piles of 15 counters (or drawing 8 groups of 15 marks), and then counting the total amount of counters (or marks). Without prior instruction, children naturally use direct modeling strategies to represent the action or relationships within problems, enabling even young children to draw on their understanding of story situations to solve a variety of problems across multiple operations (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, Fennema, Franke, Levi, & Empson, 2015). In Ms. J's and Ms. C's classrooms students were encouraged and supported to develop and make use of their intuitive strategies; thus even the more sophisticated strategies that students used tended to be abstractions of direct modeling. It is important to note that making 15 piles of 8 counters, while a viable strategy for solving the problem, is *not* a direct modeling strategy, and accordingly was rarely observed in the two classrooms studied here.

numbers. Finally, the different number sets provided opportunities for students to make connections between strategies, or nudged students to develop new strategies. For example, in the above problem Deyanira used a skip counting strategy to solve the first number set, used a related adding strategy to solve the second set, but then adapted this strategy to make use of place value in order to solve the third number set, for which skip counting or incremental adding would have been more complicated (see figure 4.1). While students were partnered with another student during problem solving time, this pair work was enacted in different ways. Some pairs worked collaboratively together, jointly constructing solutions, while others took turns leading the problem solving for different strategies or number sets. Still others worked primarily in parallel. Many pairs moved between these varied ways of interacting and working together across the day, and worked differently with different partners. Student pairings tended to remain consistent in the short term (at a given time point in January, March, and May), but across the year all students worked with multiple partners. This paired participation structure within problem solving time was fairly consistent, with two notable exceptions. On one day in Ms. C's class (January 22), problem solving time was split into an independent segment and a partner work segment. On this day students were first expected to begin working on the problem independently, and after about 15 minutes were directed to begin working with their partner by first explaining their strategies to each other. On another day in Ms. J's class (January 24), the teacher interrupted the class' partner work to explicitly direct them to take turns in explaining their strategies to their partners.

Figure 4.1: Deyanira’s counting and adding strategies across three sets of numbers within the same story problem.

First number set	Second number set	Third number set
		

As pairs of students worked on the day’s problem together, the teacher would circulate, monitoring students’ work, and engage with pairs or individual students. Both Ms. J and Ms. C used this time to learn more about their students’ thinking, asking students to explain their ideas to the teacher and to each other, supporting them to make sense of the story, probing the details of their explanations and representations, and occasionally providing additional challenges or adapting the problem to fit a student’s specific needs. Problem solving time tended to last between 20 and 30 minutes per lesson (a mean of 26 and 24 minutes per day for Ms. J and Ms. C, respectively, see appendix C). During this phase of the lesson there were between 5 and 10 episodes per day in which the teacher would engage with a student or pair of students around their work on the problem. While the overall mean episode length in both classrooms was just under 4 minutes, there was quite a range in the duration of episodes during this phase of the lesson. Each classroom contained episodes that were both very brief (less than 30 seconds) and rather extended (up to 9 minutes).

After students worked on the day’s problem with their partners, the class reconvened on the rug to share and discuss solutions to the day’s problem. Like the warm-up, students were

seated with their partners on the rug, and attended to the board (onto which sometimes the document camera was being projected) as their classmates' strategies and representations were discussed. Unlike the warm-up, a student (or pair of students) occupied the physical space at the front of the board or at the document camera. The teacher facilitated the discussion by first inviting a student or pair to share a particular strategy with the class. In all cases observed, the teachers had previously visited with the student or pair during problem solving time; thus the strategies shared had already been intentionally selected (Kazemi & Hintz, 2014). In sharing strategies, the student(s) sometimes would explain their thinking as they re-created the strategy from scratch. On other occasions, a completed strategy was projected onto the board and the rest of the class would try to make sense of how the student had solved the problem before the author of the strategy would explain. Teachers played an active role in facilitating the strategy sharing, at times posing clarifying questions, pausing the discussion to invite further participation from the rest of the class, or supporting students to make connections between the strategies that had been presented. The overall participation structure of the share out was whole-class, but similar to the warm-up, students would sometimes be invited to turn and talk with their partners to respond to the strategy under discussion. Thus the share out involved participating through listening, talking with partners, chorally responding, or responding individually to the whole group. Between 1 and 3 strategies were shared with the class on each day observed. Discussions of strategies were rather extended; episodes during the share out tended to be longer compared with those during the other phases of the lesson (between 6 and 7 minutes). On a per-day basis, the share out typically lasted about 15 minutes.

In Ms. J's classroom, math lessons usually concluded with the sharing and discussing of students' strategies. In Ms. C's classroom, however, a brief independent work time followed the

whole-class discussion. During independent work time students were expected to finish up their work on a previous strategy, or to solve the problem using the third set of numbers. This provided students with the opportunity to “try out” an idea or strategy discussed during the share out, or to continue to use a familiar strategy but apply it to different set of numbers. For example, here is one way Ms. C closed the discussion of strategies and posed directions for what students were to do during independent work time:

So think about what you did, boys and girls, when you were problem solving today, and if you did something similar or different (to the strategies just discussed)? And when you go back to do the third number set, you can borrow a strategy, or you’re going to stick with one that works for you if you think yours is more efficient. So before you get going, make yourself a plan. What is your plan for the third number set; what are you going to do to solve it? When you’re ready you can give me your thumbs up. [Ms. C, January 22, 63:59]

The teacher then circulated and checked in with particular students as they worked independently. Independent work time averaged 6 minutes a day in Ms. C’s classroom.

Independent work time was also included as part of one lesson in Ms. J’ classroom on May 28.

To summarize, the tasks and structures of mathematics lessons in both classrooms created conditions in which teachers could elicit and respond to students’ mathematical thinking.

Teachers posed problems without demonstrating how children were expected to solve, supported students to make sense of the contexts of story problems, provided students with a variety of tools, and encouraged students to use strategies that made sense to them (Carpenter et al., 2015).

The organizational features of the two classrooms also contained a variety of participation structures within which students could begin to communicate their thinking with each other, and within which teachers could begin to listen to and learn about their students’ thinking. Put another way, the tasks and structures of mathematics lessons provided a canvas onto which the shapes, colors, and textures of students’ ideas could be illustrated. These tasks and structures provided a necessary, but insufficient means to leverage student’s mathematical thinking and expand collective notions of competence. In the following sections, I detail the ways in which

competent participation was constructed and negotiated in interactions, and how constructing competence around and in relation to providing explanation and engaging with others' mathematical ideas served to open opportunities for students to contribute in a range of ways that were recognized as competent while engaging in substantive mathematical work.

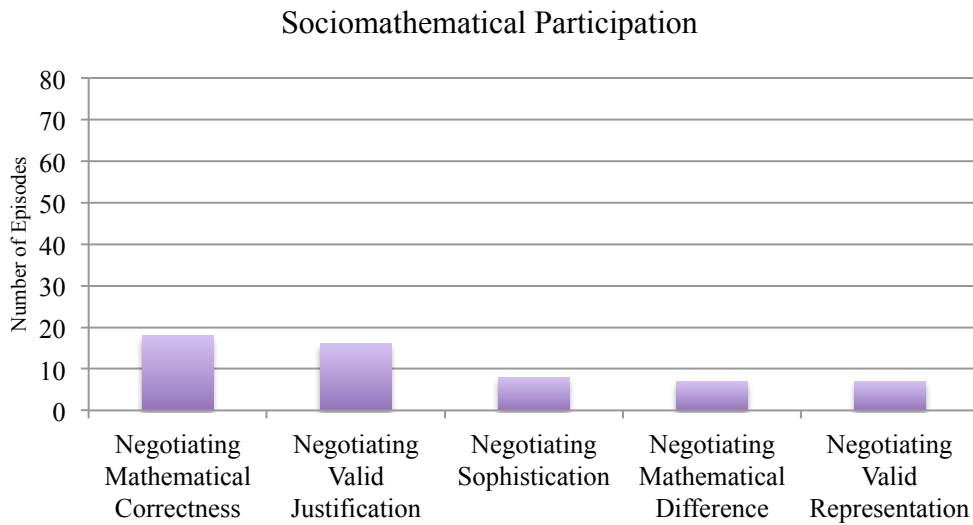
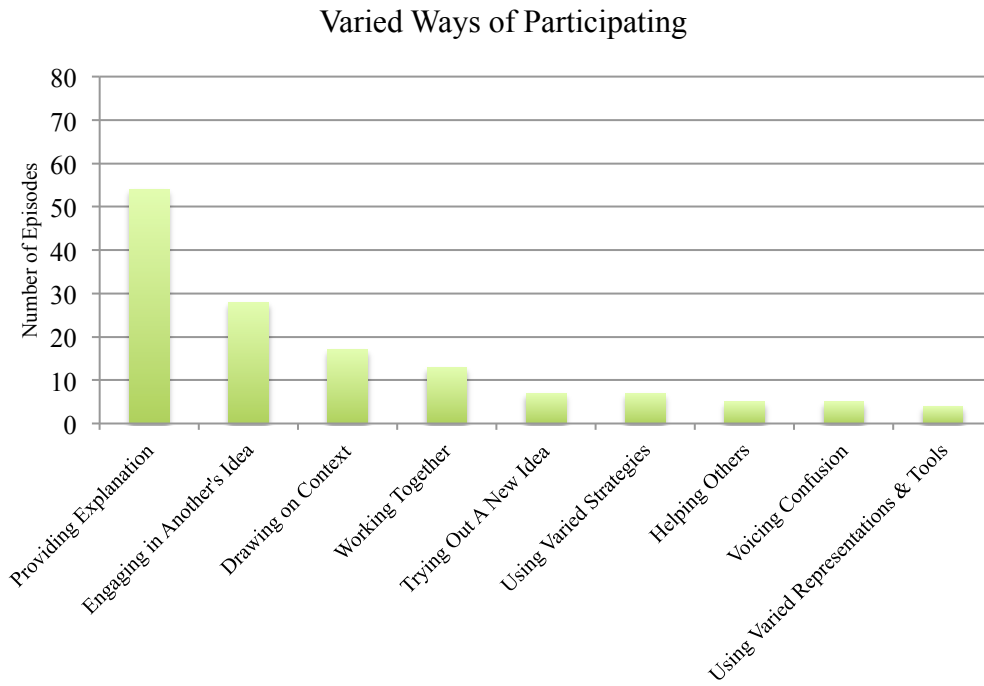
Constructing competence in Ms. J's classroom

Analyses of competence in Ms. J's classroom revealed two predominant ways of participating—providing explanation and engaging with another's mathematical idea. Constructing competence around and in relation to explanation and engagement with each other's ideas created space for students to participate in a range of ways and be recognized as competent while also advancing goals for mathematical learning. In particular, Ms. J consistently positioned students' errors or incomplete ideas as reflecting understanding that could be built upon. In detailing the ways in which competence was constructed in Ms. J's classroom, I first provide frequency data along with brief examples to illustrate each dimension of competence individually. I then provide an extended whole-class example and an example of students' work in pairs to illustrate the overlapping, relational nature of these dimensions.

Doing mathematics

Figure 4.2 displays frequencies for the multidimensional ways that students participated in mathematical activity across 4 days (82 episodes) in Ms. J's classroom. Participation within this dimension was parsed into a) what students were *doing* within an episode as they engaged in mathematics, as well as b) the sociomathematical ideas under negotiation within a given episode.

Figure 4.2: Multidimensional ways of doing mathematics in Ms. J’s classroom



The most frequent ways students participated in Ms. C’s classroom were through providing explanation (54 episodes) and through engaging in another students’ idea (28 episodes). Other

common ways of participating included drawing on the context of the story problem in relation to a strategy (17 episodes) and participating by working together with a peer (13 episodes). In terms of sociomathematical participation, negotiating the correctness of a solution (18 episodes) and working through what counted as a valid justification for a strategy (16 episodes) were common occurrences. The reader is reminded that even those ways of participating that occurred less frequently in proportion to others (for example, there were 5 instances each of helping others or voicing confusion) when viewed on a daily basis are relatively commonplace. In interviews Ms. J spoke to the ubiquity of eliciting student explanations as part of doing mathematics in her classroom.

We don't want kids to think that, oh, the only time she questions me is when I'm wrong. Well no I'm questioning you because I want to understand how you thought about this, and that your strategy is valuable, and so it kind of starts when you are conferencing and you know, you're valuing what they do by questioning them.
[Ms. J, interview 2]

By asking students to provide explanations of their thinking, Ms. J hoped not only to learn about how they had solved a given problem, but also to communicate that she valued and wanted to understand the perspectives and sense-making of individual students.

These multidimensional ways of doing mathematics often operated in concert with one other. For example, determining mathematical correctness most often occurred in combination with providing explanation (13 shared episodes) or engaging with others' ideas (8 shared episodes). The following episode provides an example of how students participated in varied ways as they engaged in mathematics together. Students are sharing their solutions for the warm up task, $75 \text{ min} = 1 \text{ hr } \underline{\quad} \text{ min}$. During turn and talk, Neffy has already explained his answer to his partner, Giancarlo, and Giancarlo is responding to his partner as Ms. J listens in. After listening in, Ms. J calls the class back.

Giancarlo: I think that too because 60 plus 15 equals 75 and one hour is 60 minutes, and if you have 15 that would equal 75, so if you added it is 1 hour 15.

Ms. J: Good explanation you two (addressing to the pair). Alright, and 5, 4, 3 (calling class back and holding up fingers). Okay, really nice job of explaining down here. Um actually, Neffy why don't you go ahead and say what you think it is and explain it.

Neffy: I think it's 1 hour 15 minutes because—

Students: D'oh!/Aw!

Ms. J: Shh listen, listen to his explanation.

Neffy: Because 1 hour is 60 minutes and then you just add 10 that's 1 hour 10 minutes, and if you add a 5 that's 75 minutes and that's 1 hour 15 minutes.

Ms. J: Okay, do you all agree?

Students: Yeah/yes.

Ms. J: Give me a thumbs up if you agree. Okay, does somebody have a different way to explain it? Shauna?

In this episode students participate by providing explanation and by engaging with each other's ideas, and in doing so negotiate the correctness of the explanation that has been put forth. Ms. J actively supports students to participate in these ways, explicitly assigning competence to the pair's explanations ("really nice job of explaining down here"), redirecting the class to attend to Neffy's explanation when several students respond in disappointment when Neffy gives his answer ("listen to his explanation"), and by inviting an additional response that emphasizes the importance of varied explanations ("does somebody have a different way to explain it?"). From a sociomathematical perspective, the mathematical correctness of Neffy's response is determined through his explanation and through his peers' engagement with his idea ("Do you all agree?/Give me a thumbs up if you agree"), rather than by the teacher or answer key.

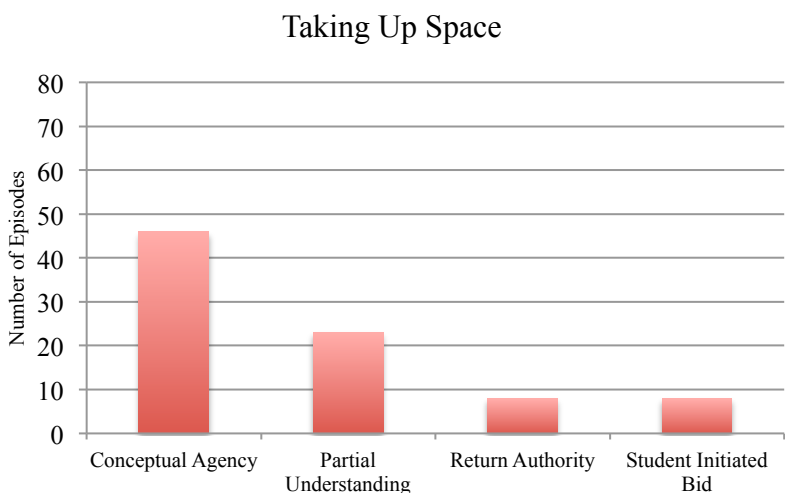
The prevalence and interconnected nature of providing explanation and engaging with other's ideas is further illustrated by examining the proportion of these two ways of participating in relation to the overall corpus of data in Ms. C's classroom. 66% of episodes involved students participating through providing explanations, and 34% of episodes involved students engaging with another student's idea. There was a strong relationship between these two ways of participating; 24% of episodes involved *both* explaining and engaging with another's idea.

Overall, 76% of episodes involved students participating by either providing explanation, engaging in another’s idea, or both.

Taking up space

Interactions in Ms. J’s classroom created space for students to exercise varied forms of agency and authority. Figure 4.3 displays frequencies for the most common ways students “took up their space” in whole-class discussions and in conversation with the teacher.

Figure 4.3: Taking up space: Exercising agency and authority in Ms. J’s classroom



46 episodes involved students taking up or being encouraged to take up positions as agents of their own conceptual understanding. Students made sense of the mathematics they worked on, decided for themselves how problems would be solved, and used a range of strategies and representations of their choosing. For example, in the following episode Ms. J checks in with Shonda, who earlier had expressed confusion about the situation being described in the day’s

story problem when it was first posed to the class. As students begin working, Ms. J approaches Shonda and asks about her plan for getting started.

Ms. J: So what do you think you're going to do to get started on this?
Shonda: I'm thinking that maybe I should draw 70 circles, and then I'm gonna circle, I'm gonna circle half of it, and see how much I get. If it's not equal then I'll try again.
Ms. J: How are you going to know how much half is?
Shonda: I'm gonna split it.
Ms. J: Okay, alright have a go at that and see how it works out.
(T walks away and Shonda starts drawing circles as she counts out loud)
[Ms. J, January 22, 30:45]

Ms. J's phrasing of her question, "what do you think you're going to do" indicates that it is Shonda (as opposed to the teacher or the textbook) who will decide on the approach to solving the problem. Shonda's response indicates that she has an understanding of the problem's structure.¹⁰ After following up on how Shonda would find half of her total, Ms. J encourages her to "have a go at it and see how it works out" and then walks away, communicating her confidence in Shonda's ability to enact her plan and to decide for herself if it worked.

Ms. J's classroom also created space for students' incorrect, incomplete, or emerging understandings of the mathematics to surface and be recognized as competent. 23 episodes involved interactions in which these "partial understandings" surfaced and were built upon. Surfacing students' partial understandings often involved Ms. J first inviting the student to explain their thinking. This allowed Ms. J to identify a productive idea within a student's explanation and to use this as a starting point. Ms. J would often explicitly recognize what was correct about students' ideas, and support them to use their ideas in relation to the problem's context in order to develop a valid strategy. In the following example Ms. J is talking with Jessica and Aleece, who after struggling together to understand the day's story problem have arrived at an (incorrect) solution of 50 and 20 for the amount of seashells that were collected in each of two days. After first eliciting an explanation of what they have done to solve the

¹⁰ The story problem on this day described a partitive division situation for $70 \div 2$.

problem, Ms. J encourages the pair to reread the story and to connect what they have done with the problem's context.

Ms. J: Okay so you're saying that this is day one (points to the 50) and this is day two (points to the 20)? Is it okay that there's different amounts?
Jessica: (shakes head indicating no)
Ms. J: Is there anything in the problem that says—
Jessica: The same number.
Ms. J: Where does it say that?
Jessica: Here (points to paper) "If she collected the same number of seashells..."
Ms. J: Okay, so you're right; these two *do* add up to 70 altogether, but what do they need to be?
Jessica: The same.
Ms. J: Uh huh, so how are you going to do that?

In supporting Jessica and Aleece to make sense of the story in relation to their strategy, Ms. J attends to and acknowledges what is correct about their thinking—that they have a total of 70 seashells for the two days. It is only after she explicitly recognizes this ("so you're right, these two *do* add up to 70 altogether") that she nudges them to revise their strategy in relation to the constraints of the problem (that the same number of seashells were collected on each day). The prevalence of beginning interactions by inviting explanations, even when students had incorrect or incomplete solutions, provided opportunities for students' partial understandings to surface and be recognized.

Two particular ways of taking up space directly related to students engaging with others ideas. Eight episodes involved students attempting to take the floor by asking questions or engaging with the ideas being discussed in some way without being prompted to do so by the teacher. There were also eight episodes in which Ms. J created space for students to maintain authority for how their ideas were taken up by other students. As students responded to each other's contributions, Ms. J would check back with the author of the strategy or question to make sure that their ideas had been interpreted correctly. In the following example, Leo interjects and asks a question as Adam and Sebastian are sharing their strategy with the class.

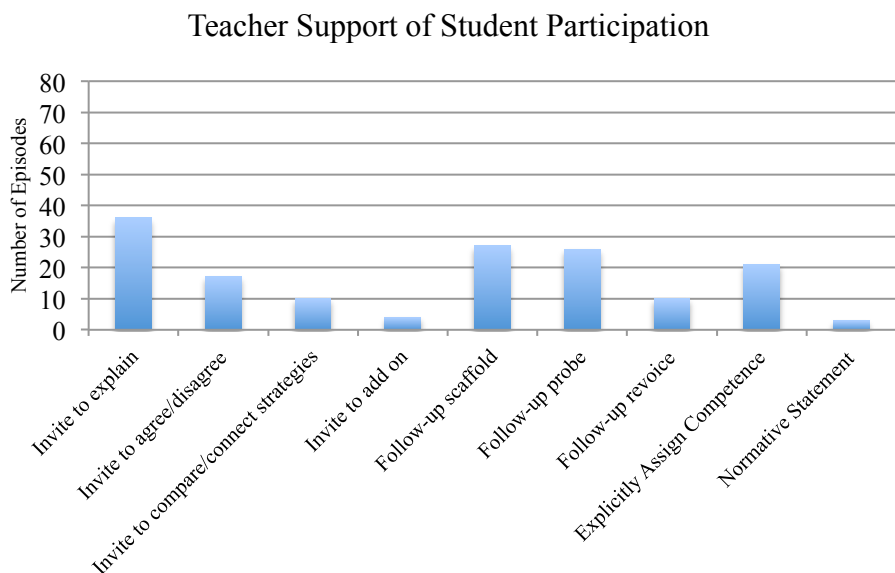
Leo: (interjecting) I have a question where you did the 40, 40 and 20 equals 40?
 Adam: No it's 20 and 20 equals 40.
 Leo: I know. (inaudible) that question.
 Sebastian: What don't you get?
 Leo: How did you get it?
 Students: He added!/We told him!
 Leo: Oh.
 Nico: Leo we got that from timing, when 6 times 4.
 Leo: I didn't hear that question.
 Nico: Leo we answered that one.
 Ms. J: Well Leo was asking how do you know 20 plus 20 is 40. Is that right Leo?
 Leo: Yeah.
 Ms. J: Is that what you're asking?
 Leo: Uh huh.
 Ms. J: Can someone help him with that? Go on Edwin.

In sharing strategies, students were expected to ask questions and receive responses without necessarily filtering their ideas through the teacher. In this example, Leo interjects to ask a question (possibly misspeaking and saying 40 instead of 20) and receives several responses from peers, some of whom misinterpret his question (Nico) or deem his question too obvious (“He added!”). As Leo begins to withdraw (“Oh”), Ms. J steps in, offering her interpretation of Leo’s question and returning the floor to him (“Is that right Leo?/Is that what you’re asking?”). This serves to clarify Leo’s question as well as to legitimize his attempt to receive further explanation of what may have been an inconsequential calculation for others, but was not for Leo. Like several other students, Leo often struggled with issues of place value. By clarifying his question and checking back that her interpretation was correct, Ms. J positioned Leo as a competent participant, ensuring that he was able to take up his space within the strategy discussion.

Teacher support of student participation

Ms. J employed a range of invitational and follow-up moves to uncover the details of students’ ideas and to support them to participate in varied ways. Figure 4.4 displays the most frequently occurring moves employed by Ms. J in interactions with students and with the class.

Figure 4.4: Teacher support of student participation in Ms. J's classroom



Ms. J most frequently supported students to contribute through inviting them to explain, and would often follow-up their explanations with additional probes for detail about the student's idea, or by scaffolding the mathematics (for instance by supporting the student to connect their strategy with the problem's context). While Ms. J would occasionally make normative statements about how students were expected to participate and why, the importance of explanation was most often explicitly communicated through assigning competence to students who provided explanations which made the details of their thinking clear. Almost half (10 of 21) of the explicit assignments of competence made by Ms. J were related to students' explanations, the other two most frequent being assigning competence to partial understandings (4) or to the sophistication of a student's strategy (4).

Students were also invited to explain their peers' ideas. In the following example, after Marisa and Shawna have presented their strategy to the class, Ms. J invites Joanna to provide another explanation of the strategy that has just been shared (see figure 4.5).

Ms. C: How many of you agree with them that also got that answer? (several students raise hands) A few of you, okay. Do you guys understand kind of what their thinking is? If you got 35? Joanna? Are you kind of seeing how they did it? Yeah?

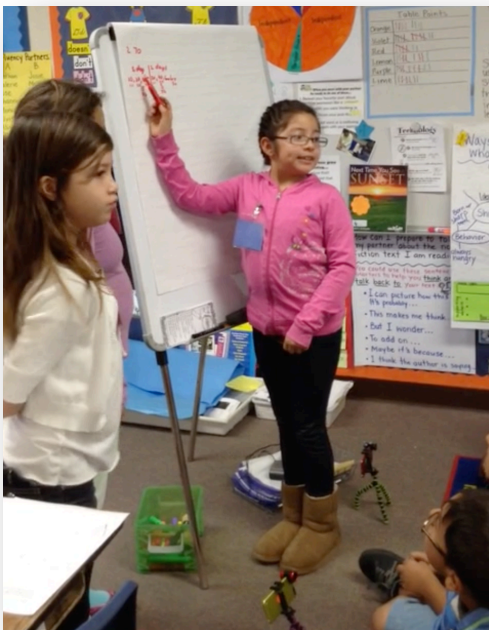
Joanna: (nods yes)

Ms. J: Could you explain it; can you explain what they did? Go ahead. You have a go.

Joanna: Um.

Ms. J: Go ahead and go up there and point as you're explaining.
(Joanna gets up from carpet and moves to board to explain the strategy that has just been shared)
[Ms. J, January 22, 56:44]

Figure 4.5: Joanna explains Marisa and Shawna's strategy



In this example, Ms. J attempts to support the class's understanding of the strategy under discussion by inviting another student to explain the strategy that has just been detailed. The import of explaining another's idea is further emphasized by Ms. J's suggestion that Joanna shift her physical position within the classroom so that she is able to point to what is written on the board as she explains.

In reflecting on her decision-making, Ms. J spoke about how some of her choices in selecting strategies for public discussion served not only mathematical goals, but social goals as well.

I think when you give students who aren't necessarily seen as the strongest mathematicians in the class to go up and share a strategy successfully, and be confident about it when they really understand it I think it's really helpful for them...it gives them a lot of confidence.
[Ms. J, interview 1]

Ms. J was aware of students' social standing in the classroom, and sometimes used the strategy share as a way of positioning students competently to disrupt status differentials related to perceived mathematical ability. She believed that helping students to see themselves as capable knowers and doers of mathematics involved creating opportunities for them to share their successes in the public space of the whole-class discussion.

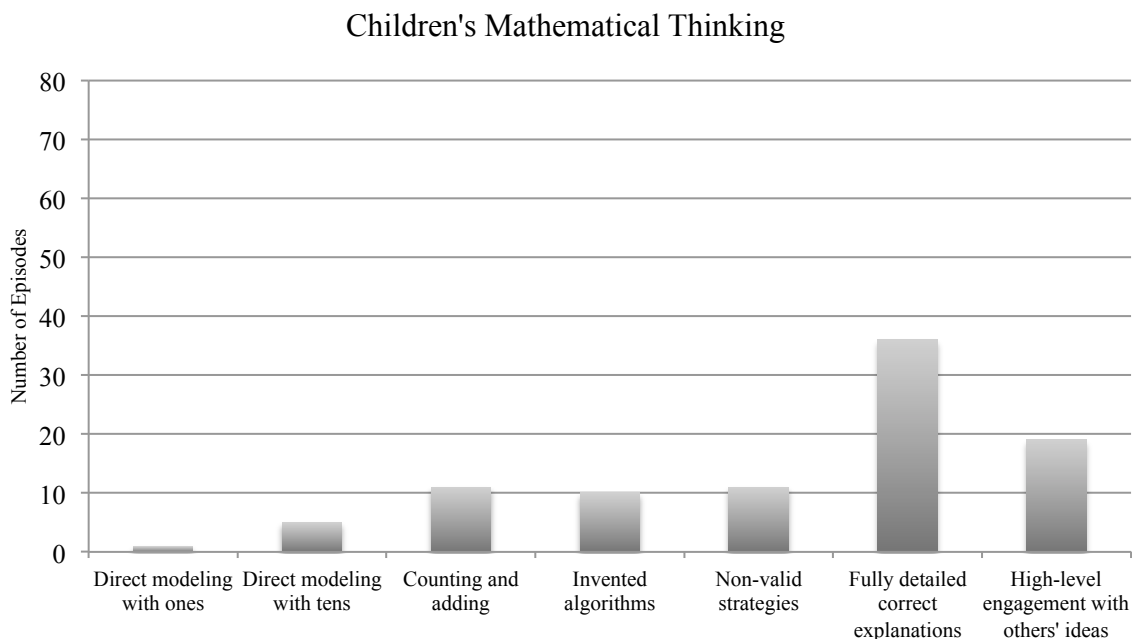
Children's mathematical thinking

As the previous examples have illustrated, students' mathematical thinking was integral to the mathematical work in Ms. J's classroom. Meaning-making within interactions was closely tied to the specific details of students' mathematical ideas, and students generated and used a range of strategies in solving problems. Figure 4.6 displays the frequencies of students' strategies that surfaced in whole class discussions or within individuals' and pairs' interactions with the teacher.¹¹ Figure 4.6 also includes the frequency of episodes in which students provided fully detailed correct explanations, and the frequencies with which students' engagement with each

¹¹ As the focus of the lessons observed across the two classrooms was multiplication and division, for simplicity figures 4.6 and 4.14 omit strategies shared during warm-up activities that were not specifically multiplication or division tasks. Within interactions, students' non-valid strategies were often revised into valid strategies. In these cases, the episode would be labeled for both a non-valid strategy as well as for the type of valid strategy that also emerged. For further elaboration on the details of children's strategies for multiplication and division problems, see *Children's Mathematics* (Carpenter et. al, 2015).

other’s ideas occurred at a high level.¹² Students provided correct, fully detailed explanations in 36 of 82 total episodes involving the teacher, a strikingly large proportion of interactions.¹³

Figure 4.6: Children’ strategies and high-level explanation and engagement with each other’s ideas in Ms. J’s classroom



Students used a wide range of strategies to solve multiplication and division problems, including counting/adding strategies and invented algorithms that were more sophisticated than those called for by grade-level content standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The prevalence of fully detailed

¹² For elaboration on fully detailed, correct and complete explanations see Franke et. al, 2009. For elaboration on high-level student engagement with others’ mathematical ideas, see Webb et al., 2014.

¹³ For comparison, a previous study conducted by Webb and colleagues found that 10 of 59 interactions involving the teacher resulted in students producing fully detailed, correct explanations (Webb et al., 2009).

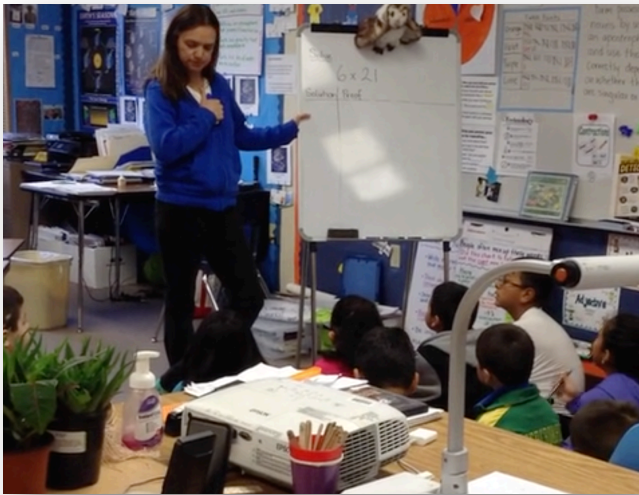
correct explanations suggests that demonstrating competence in terms of providing explanation involved making the details of a strategy explicit, and that competent ways of engaging with others' ideas involved adding on to, challenging, or questioning the details of other students' ideas.

The relational nature of competence: Sharing an incomplete idea

For the purposes of detailing competence across dimensions in Ms. J's classroom, the analysis presented thus far has described each dimension individually, with little description of their overlapping nature. Competent participation, however, was a relational phenomenon, dependent on situationally specific ways that students and the teacher participated together, negotiated ways to take up space, and worked on the details of the mathematics. In this way the various dimensions of competence were mutually constituted. To illustrate the relational nature of competence, I present an extended example in which students' partial understandings are leveraged to support explanation, engagement with each other's ideas, and the advancement of particular mathematical goals.

In the example below, an incomplete strategy is leveraged as a learning opportunity while also positioning the authors of the strategy as competent contributors to the mathematical work of the classroom. The episode takes place within a warm-up activity in which students are asked to mentally solve 6×21 . The teacher, Ms. J, begins by writing the problem on the board and asking the class to try to solve the problem mentally. She then reminds students that they will be "sharing out all of the steps that you took to get to your solution," and to indicate when they have arrived at a solution by putting a thumb up on their chest (figure 4.7). After about 90 seconds of quiet think time, students are invited to turn and talk with their partners to share their thinking.

Figure 4.7: Ms. J poses a mental math warm-up activity



As the class begins to share their strategies with their partners, Ms. J crouches down to listen in on Deyanira and Karly's conversation about the problem. They talk together about how 21 can be broken apart into 20 and 1, but there is some uncertainty about how to use this idea in service of solving the problem—which numbers to add and which to multiply. Ms. J listens, encourages the girls to attend to each other's explanations, and follows up on what they say with several questions and comments. However, she steps away after about two minutes to call the class back together, knowing that Deyanira and Karly have not yet arrived at a solution. The episode below immediately follows, and is the first strategy shared during the warm-up on this day. To support the reader in navigating moments of significance within the transcript, instances in which a critical mathematical idea surfaced are highlighted in blue, and instances in which the floor was shifted to another student have been highlighted in yellow.

1 Ms. J: (calling class back) 5, 4, face the front (counts down on fingers silently 3-2-1). I know a lot of
2 you have solved this and you want to share, but I want to do something with Karly and
3 Deyanira right now 'cause they are so close to getting this, and I want to see if we can help
4 them get from where they're at right now. Okay? Can you do that? Okay so what did we start
5 with? **We started off by doing what, Karly?**
6 Karly: Um, multiplying the 1 and 6?
7 Ms. J: Okay so first of all we split apart?
8 Karly & Deyanira: The 21.
9 Ms. J: We decomposed the 21 into what?
10 Deyanira: The 20
11 Karly: The 20 and the 1.
12 Ms. J: The 20 and 1. Right? You all with me so far?
13 Students: Mhmm.
14 Ms. J: So she's broken apart, kinda like I think that one of you did on Monday. (aside to child who is
15 offering her a marker) That's okay thank you, Edwin. (now addressing class) We've broken
16 apart 21 into 20 and 1. And then you said that you're going to multiply the 1 by?
17 Karly: Um
18 Ms. J: You said I could multiply the one by?
19 Karly: 6.
20 Ms. J: 6. And what would you get?
21 Karly & Deyanira: 6.
22 Ms. J: 1 times 6 is 6.
23 Karly & Deyanira: (laugh)
24 Ms. J: Do we all agree?
25 Students: Yeah
26 Ms. J: Okay, what do you think you're going to do next? Guys be thinking about this so maybe we
27 can ask you. Deyanira?
28 Deyanira: Hmm
29 Ms. J: What do you think?
30 Karly: **Maybe add the twenty to the six... um times, multiply?**
31 Ms. J: Add the 20 to the 6?
32 Karly: No multiply.
33 Ms. J: Multiply the 20 by the 6?
34 Karly: Mmm, yeah.
35 Ms. J: Yeah?
36 Students: No
37 Ms. J: What do you think Deyanira? Should we multiply 20 by 6?
38 Deyanira: No.
39 Ms. J: No? Why not?
40 Deyanira: Umm
41 Ms. J: Is it this 6 that we're multiplying by Karly? (points to the six that is the product of 1×6) Or this
42 one (points to the 6 in the original problem)?
43 Karly: (points to 6 in the original problem)
44 Ms. J: That one. Okay sorry. I'm confusing you. So you're saying multiply 20 by 6.
45 Deyanira: Mhmm
46 Ms. J: And this is the 6 that—boys and girls can you pay attention please? Okay, so we're
47 multiplying, so this 20 by this 6.
48 Karly: (nods)
49 Ms. J: Do you know how to do that?
50 Karly: (nods)
51 Ms. J: Go on then.
52 Karly: **So 20 and then (raises 1 finger) 30 (pauses)**
53 Deyanira: (silently raises 1, 2, 3, 4, 5, 6 fingers)
54 Ms. J: Can you all in your heads multiply 20 by 6? Think about how you're doing it?
55 Karly & Deyanira: (raising fingers to make 6 counts, but not arriving at answer)
56 Ms. J: No? Shall we ask somebody to help us?

57 Karly & Deyanira: (nod) Yeah.
58 Ms. J: Okay, Neffy. Can you have a go? How do you multiply 20 by 6?
59 Neffy: You could just add like 20 and then count by jumping like 20, 40, 60, 80, and then 100, and
60 then a hundred ten—twenty. And then you just add 6. And then—
61 Ms. J: Hold on, hold on just a second cause they just needed help with that part, right? To multiply the
62 20 by 6. Did you see what he just did? Can you try that? Everybody let's do that together. So
63 we're going to put 6 fingers up and count by 20s, right? (aside to Neffy) That was really smart.
64 (to class) Okay ready, go.
65 Students & Ms. J: (chorally) 20, 40, 60, 80, 100, 120.
66 Unknown student: And plus 6!
67 Ms. J: (to Karly) What do you think, does that work?
68 Karly: (nods)
69 Ms. J: Okay. Alright, Deyanira can you finish off the problem? What are you going to do next? So I
70 multiplied the 20 by the 6, I multiplied the 1 by the 6. What am I going to do next?
71 Deyanira: Add both of them?
72 Ms. J: Add them together. And what would I get?
73 Deyanira: 126.
74 Ms. J: 126. Good job you two; that was really nice. Okay, somebody else. Uh, go ahead, Nico

In this episode Karly and Deyanira are supported to share their incomplete strategy, which is then built on by Neffy and the class, and finally completed by Deyanira. The episode begins with Ms. J inviting the class to help Karly and Deyanira, who are “so close to getting this” (line 3). Karly then shares the first step of their strategy, to multiply 1 by 6 (line 6), and Ms. J restates and records this idea on the white board (lines 14-16), and asks if the class agrees (line 24). Ms. J then invites Deyanira into the conversation, asking her what they should do next (line 27). Karly states that she wants to multiply (line 32), but this exchange makes public some of what the pair was unsure about in their strategy—which numbers should be multiplied, and which should be added (lines 30, 32, 38, and 40). This exchange also reveals that some other students in the class share this uncertainty (line 36), possibly influencing Deyanira’s disagreement with the idea of multiplying 20 by 6 (line 38). Ms. J moves the discussion forward by clarifying Karly’s idea (lines 41-44), and then provides time for the pair to try to multiply 6×20 (lines 51-55). While doing so Ms. J also invites the rest of the class to think about how they would use Karly’s idea (line 54). It is only after Karly and Deyanira agree that they would like some help (lines 56-57) that Neffy is asked to share his strategy for how to multiply 6×20 (line 58). After Neffy shares his strategy of counting by 20s (lines 59-60), Ms. J invites the class to try his idea together (lines 62-65). Ms. J then returns to the authors of the strategy, asking Karly if she agrees (line 67) and invites Deyanira to “finish off the problem” (line 69).

Competence is constructed throughout the episode across several overlapping dimensions. Examining the varied ways that students engage in mathematical activity reveals that students participate through helping their classmates (lines 3 and 56), providing explanation (lines 6, 11, 59-60, and 71), and engaging with others’ mathematical ideas (lines 25, 54, 59-60, 65, and 66). Karly and Deyanira take up their space by sharing an incomplete idea (lines 6-21),

and maintain their authorship of the strategy by completing the final step and providing an answer (71 and 73). The teacher plays an active role in supporting student participation by inviting the pair to explain their incomplete strategy (lines 4-5), inviting Neffy to add on (line 58), and in holding off Neffy (line 61) and ignoring another student's interjection (line 66) in order to preserve Deyanira's final turn to finish the strategy. Competence is explicitly assigned to Neffy for adding on and detailing his explanation (line 63), and to Deyanira and Karly for sharing and working through their (now no longer) incomplete idea (line 74). Mathematically, a fully detailed explanation of an invented algorithm for 6×21 is collectively produced and made public, and Karly and Deyanira's strategy has been engaged with at a high level. Throughout the episode, Karly and Deyanira are positioned as capable contributors to the class discussion, even as their mathematical ideas are still emerging.

Taking up opportunities to explain and engage with each other's ideas without teacher support

The analysis presented up to this point has detailed the ways in which competence was constructed in whole-class discussions or in individual and pair conversations with the teacher. On two days of instruction, video footage that tracked pairs of students was also analyzed to investigate how these collective notions of competence were taken up as students worked together when the teacher was not present.

Students engaged with their partners during problem solving work in a range of productive ways. All pairs participated with each other by either providing explanation, engaging with each other's ideas, or both. However, there was a great deal of variability in the ways in which explaining and engaging with each other's ideas were enacted by different pairs of students. Of 11 pairs, five pairs consistently engaged in sustained, synchronous work together,

while another five pairs mostly worked in parallel but would sporadically engage with each other around the mathematics. Only one pair did not engage in mathematics together in a substantive way.¹⁴ Six of the 11 pairs engaged with each other's ideas at high levels, attending to the details of each other's thinking or adding on to and challenging each other's ideas. Four of the pairs showed some engagement with each other's ideas, but did not consistently engage in the details of each other's thinking. Rather, these four pairs primarily engaged with each other by checking in on each other's progress or comparing whether or not their answers agreed without elaborating on their strategies. In nine of 11 pairs a fully detailed explanation of a strategy was articulated (with four of these pairs providing multiple explanations) and in three pairs a strategy was jointly produced. Only one pair, Shonda and Sonia, did not explain or jointly produce a strategy during problem solving. However, as the following example will show, the pair did at times engage in mathematical work together in different ways.

On this day, Shonda and Sonia were attempting to solve a partitive division problem in which 70 seashells were to be split evenly across two days. As shown in a previous example, shortly after launching the problem Ms. J briefly visited Shonda to check in on how she planned to get started on her strategy. In the following excerpt, Shonda is continuing to draw her 70 circles, while Sophia already has an answer, having added 35 and 35 to get 70.¹⁵

Sonia: (taps Shonda on the arm)
Shonda: Stop bothering me, please
Sonia: (continues tapping Shonda)
Shonda: (smiling) What? What, what? Stop it, please!
Sonia: I'm trying to show you! But you don't wanna know.

¹⁴ In this pair (Giancarlo and Vanessa) each student had a different interpretation of problem's structure and used different strategies. The pair did make several attempts to engage with the other, but these attempts did not result in them engaging in mathematical work together. They did, however, explain their strategies out loud to each other after the teacher prompted the class to do so, but did not subsequently engage with each other's explanations.

¹⁵ It is not clear *how* Sonia knew to add 35 and 35; she may have solved mentally by knowing or deriving that half of 70 is 35, or she may have appropriated the answer from listening to another nearby pair of students and justified it by adding 35 and 35.

Shonda: (shifts over to look at Sonia's paper) Uhhhhh... Where did you get the 35?
 Sonia: (looks around; does not respond)
 Shonda: (looks around; returns to her work)
 [Shonda and Sonia, January 22, 32:38]

Sonia, having arrived at an answer, attempts to engage Shonda by playfully tapping her on the arm. Shonda, however, has not yet completed her strategy, and initially rebuffs Sonia's attempts. When Sonia persists, Shonda does momentarily interrupt her work to look over onto Sonia's paper, but does not appear to understand her strategy ("Uhhhhh..."), and asks a specific question about what Sonia has written on her paper ("Where did you get the 35?"). When Sonia does not take up this opportunity to explain what she has done, the pair returns to working in parallel.

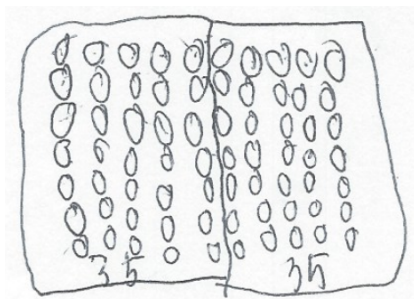
Sonia's participation illustrates a possible tension in constructing collective notions of competence around providing explanation and engaging in each other's ideas. Though Sonia does initiate engagement with Shonda, she is unable (or possibly unwilling) to provide an explanation of her strategy, and the momentary engagement with each other is not sustained. In this instance, as Shonda is unable to make sense of what Sonia has written on her paper, providing explanation serves as a necessary condition for engagement with each other's ideas, a condition which is not met.

The following excerpt takes place approximately two minutes later, just as Shonda has completed her initial strategy of drawing 35 circles (see figure 4.8).

Shonda: 5, 10, 15, 20, 25, 30 35 (counting to herself, then turns to Sonia). You're right! Once I split it I got 35.
 Sonia: Can I see how you did it? (Shonda holds up paper to show Sonia) So how many circles did you draw?
 Shonda: 70. That's going to take a long time.
 Sonia: No it's not (Sonia begins to draw pictures of base 10 rods on her paper).
 Joanna: You know you're copying.
 Sonia: So!
 Shonda: Well she needs help.
 Sonia: Sometimes we could actually copy. To help.
 Shonda: Well she's drawing it a little different. I draw by circles.
 Sonia: So now let me count.
 Shonda: I'm just going to split it, but I'm not going to share the rest with you.
 [Shonda and Sonia, January 22, 34:26]

Once she has completed an initial strategy, Shonda is ready to share her thinking with her partner and initiates engagement with Sonia through agreeing with the answer of 35 that Sonia previously stated. Sonia takes up this offer to engage, asking to see Shonda's paper and asking her about her circles. As Sonia begins to appropriate Shonda's strategy, a student sitting across the table (Joanna) interjects that Sonia is copying, possibly reacting to the fact that neither Sonia nor Shonda have provided a complete explanation of their strategies to each other. Shonda and Sonia protest, asserting that copying is a form of help. Shonda also notes a mathematical difference between her own representation and the one Sonia has begun, in which she is representing units of ten (as opposed to ones, see figure 4.9). A moment later, however, Shonda states that she is not going to share the rest of her strategy, possibly in response to Joanna's accusations about copying, or to Sonia's dismissal of Shonda's earlier comment that it would "take a long time."

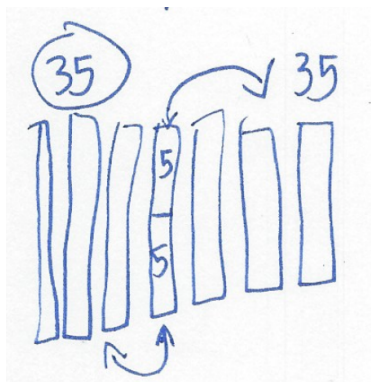
Figure 4.8: Shonda's initial strategy for $70 \div 2$.



In this excerpt different tensions emerge around constructing competence in terms of explaining and engaging with each other's ideas. Sonia has engaged with Shonda's thinking, taking up and adapting Shonda's strategy into a (more sophisticated) direct modeling with tens strategy. However, the legitimacy of Sonia's appropriation of Shonda's strategy is under question. While she has asked Shonda a question about her strategy, Sophia did not extend the

conversation, opting instead to immediately begin writing on her paper. A likely interpretation is that Sophia was able to understand what Shonda had done simply by examining the written representation of her strategy; a complete verbal explanation was unnecessary in this case. While little explanation has taken place, Sophia has successfully engaged with and added on to Shonda's strategy.

Figure 4.9: Sonia's appropriated and adapted strategy



Several minutes later the pair again briefly engage with each other. They have each begun working on the second number set. Shonda is attempting to use a strategy similar to her previous one, and is drawing 75 circles. Sonia has skip counted by 5s to 75, and counted her counts, resulting in an (incorrect) answer of 15.

- Shonda: (counting out loud to herself) 36, 37, 38, 39, 40, 41, 42
 Sonia: 15!
 Shonda: Yahh! (jumping back in her seat) Stop it, please.
 Sonia: I'm not doing anything. I'm done. 15.
 Shonda: I'm on this one (points to the second number set).
 Sonia: I'm on the second one.
 Shonda: Just please stop bothering me I'm trying to figure out the answer. When we're both done, we'll explain.
 Sonia: Sure, whatever.
 [Shonda and Sonia, January 22, 38:52]

Like earlier in the day, Sonia has completed a strategy and attempts to engage Shonda. Shonda resists, exclaiming and jumping back in her seat when she is interrupted, and then explicitly states what she thinks should happen—that they will explain once they both have finished.

Shonda and Sonia illustrate a variety of tensions that emerged as students worked to solve problems, to explain their strategies, and to engage with each other around the mathematics. Shonda clearly articulates that for her, explaining one's thinking requires first that they solve the problem separately. Put another way, demonstrating competence in terms of providing explanation and engaging with others' ideas cannot take place until after time and space have been provided for individual sense-making. Sonia, on the other hand, seems to resist this constraint. Though Sonia does not explicitly state her preferences, it is possible that engaging with each other might involve comparing answers or swapping strategies, which does not necessarily require that each person has completed solving, nor does it require providing detailed explanations of strategies. For Sonia, possibly, engaging with each other's strategies is a means to complete the day's assignment of solving problems and representing your thinking on paper. Despite these tensions, each child did benefit in some way from their engagement with each other. Sonia was able to produce, and add to a valid strategy used by her partner. The benefits for Shonda are perhaps less apparent, but she was able to see her own strategy adapted into a more sophisticated strategy, to note the difference, and to provide some explanation of what she had done to solve the problem.

Ms. J was well aware of the challenges involved in students attempting to work together to solve problems.

It's kind of hard for kids to solve together. You know, to actually—when I first started doing this I was thinking that kids could just sit down and solve the problem together and that would be easy for them but it turns out that that's really kind of hard for kids to engage and really understand what each other are doing while they are solving, because they're still solving it. I mean if I think about if I was trying to solve something and trying to understand what, how somebody else is solving it at the same time that would

actually be quite hard. So we've rethought that, and so oftentimes we'll say as you're solving you're sitting next to your partner with problem solving. And you know you might see something that they did and you can maybe ask them about that, and that's okay, and I want to hear your (voices), you know there definitely should be noise in the room, you don't have to be silently solving. And you might point to something they've done and ask them about it. And that could be if you understand what they've done that could be your second strategy. You know so getting ideas from other students is okay. But you know we don't necessarily have to, our papers don't need to look the same, because we might have solved it differently the first time... We want to avoid that, this student doesn't really have an idea about how to do it but this one's like full steam ahead, so avoiding that "well I'm just going to copy what you did." You know, so try something on your own first, and then see how it compares with what your partner did.
[Ms. J, interview 2]

In interviews, Ms. J spoke of negotiating the same tensions faced by Shonda and Sonia; it was not easy for some students to engage with each other while they were in the process of solving, and that she did not want students to simply copy what each other had written without making sense of the mathematics of a strategy. For Ms. J, solving in parallel presented a possible compromise, giving individuals time to work through their own thinking but also providing opportunities for partners to notice and ask questions about, and potentially try out each other's strategies.

Constructing competence in Ms. C's classroom

Like in Ms. J's classroom, competence in Ms. C's classroom was constructed around providing explanation and engaging with others' mathematical ideas. Constructing competence in relation to these two ways of participating created space for students to contribute in a variety of ways and be recognized as competent. In particular, Ms. C often leveraged students' engagement with each other's ideas to support their emerging understanding of increasingly sophisticated mathematics.

Before discussing similarities and differences across the two classrooms, I present a corresponding analysis of Ms. C's classroom. As in the previous analysis, I first detail each dimension of competence individually, providing overall frequency data and illustrating

particular elements of competence with specific examples. I then provide an extended example to highlight the ways in which different dimensions overlapped and interacted with each other to shape what it meant to engage in mathematics. Finally, I present a detailed example of the efforts of one pair of students to take up and make meaning of collective notions of competence in their work together.

Doing mathematics

Students in Ms. C's classroom participated in a wide range of ways as they engaged in mathematical activity. The central features of student participation were providing explanation (40 episodes) and engaging with others' mathematical ideas (39 episodes). Other common ways of participating included working together (15 episodes), drawing on the context of the story problem (13 episodes), trying out a new idea (11 episodes), voicing confusion about the mathematics under discussion (10 episodes), and using varied representations and tools (10 episodes). As students participated in these ways, they also commonly negotiated ideas of mathematical sophistication (16 episodes), correctness (14 episodes), and the validity of particular representations (14 episodes) or justifications (13 episodes). In interviews Ms. C spoke to her goals of providing a wide range of opportunities for students to participate in mathematics.

I just feel like if they're not participating in some sort of a way, there's no learning happening, and I'm not doing my job, and they're not getting, you know, they're not learning anything. So I think for me it was always around engagement and how can I make sure that all of my kids are doing something. Whether it's obviously at a different level or not...everyone can do something.

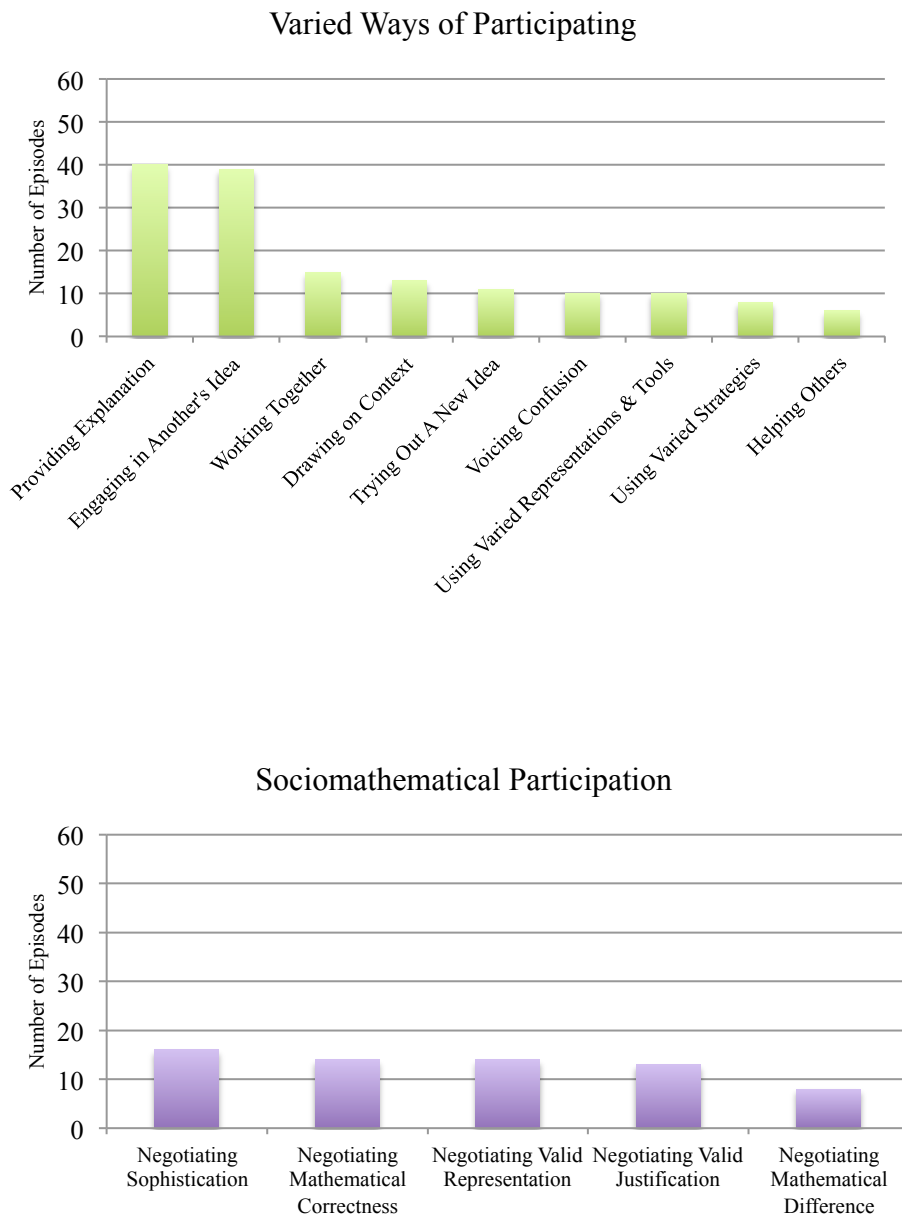
[Ms. C, interview 1]

Ms. C did not expect every student to participate in the same way, but stated that it was important for each student to have some form of engaging in mathematics available to them.

Figure 4.10 displays frequencies for the varied ways that students participated and negotiated sociomathematical ideas across 65 episodes in Ms. C's classroom. The reader is reminded that

even those ways of participating that appear less common in relation to others (e.g., using varied strategies or helping others) are still somewhat frequent when considering that the corpus of data represents a mere four days of mathematics instructional time.

Figure 4.10: Multidimensional ways of doing mathematics in Ms. C’s classroom



Like in Ms. J's classroom, these ways of participating often occurred in combination with each other. In the following example Carlos and Gerardo are in the midst of working together on a strategy to distribute 70 tally marks equally across two groups. Ms. C visits the pair and elicits an explanation of what they are doing to solve the problem. They are dealing out 70 marks one at a time, but on their papers every fifth mark in each group is recorded diagonally, forming a "gate" of five. After clarifying the details of their strategy and what remains to be done, Ms. C picks up on the emerging idea of grouping present within their representation.

Ms. C: So I see that you guys organized this by 5s, but when you are counting it you are just putting it in by 1s, right? How many seashells does Kaya have?

Carlos & Gerardo: 70.

Ms. C: 70. So thinking about that number, is that a lot of seashells or a little?

Carlos: A lot.

Gerardo: Hmm, half little, half (a lot).

Ms. C: So is there something that you could do that is even more efficient than that?

Carlos: Doing it by 10s.

Ms.C: Do you think that would be quicker?

Carlos: Yeah.

Gerardo: Hmm, yeah I think doing it as 10s.

Ms. C: Okay, let's see.

Gerardo: Wait, we gotta restart, no.

Carlos: No, we could do it by 5s, but then we could circle the 10s.

Gerardo: Okay.

Ms. C: That's smart.

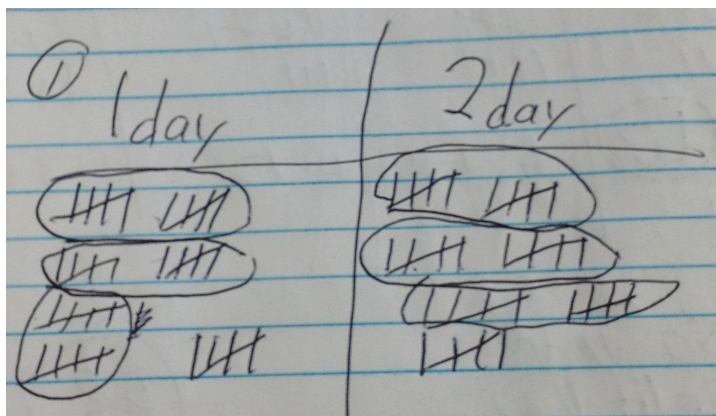
Gerardo: (Begins drawing circles around each two groups of five marks on his paper)
This is a 10, this is a 10, this is a 10.

In this example competence is constructed around "trying out" a new mathematical idea—making groups of ten. In doing so, Carlos and Gerardo participate by working together, providing explanation, and engaging with each other's ideas. Sociomathematical notions of sophistication are also negotiated as the pair works to extend a mathematical idea present in their existing strategy (grouping ones into five) in order to make groups of ten. Checking in with the pair, Ms. C notices that they are beginning to use ideas of grouping in their strategy, and invites them to think about another way that they could reach their total of 70. Carlos quickly seizes the idea of tens, and Ms. C encourages the pair to try it ("Okay, let's see"). However, it is not

immediately clear to Gerardo how to take up this idea in relation to their current strategy.

Gerardo agrees that tens would be “quicker” or “more efficient,” but does not recognize the tens already present within his strategy, stating that they “gotta restart” in order to deal out tens. It is not until Carlos helps him to see the relationship between what they have done and the new idea (“we could do it by fives, but then we could circle the tens”) that Gerardo takes up the idea of groups of ten to complete the strategy (“This is a ten, this is a ten...” see figure 4.11). Several minutes later, the pair will extend this idea to solve the problem using a second strategy—dealing out groups of 10 at a time. This example illustrates how participating through trying out a new idea occurs as Carlos and Gerardo explain and engage with each other. It is not that they are simply invited to try out a “more efficient” strategy; it is that they are invited and supported to make meaning of a more sophisticated idea in relation to what they have already done, and that explaining and engaging with each other are fundamental to their sense-making.

Figure 4.11 Gerardo circles groups of 10 within his direct modeling by 1s strategy



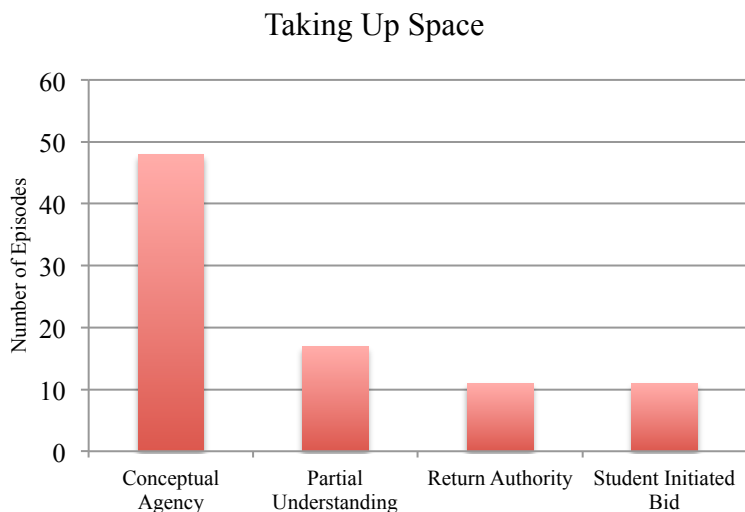
The overall frequencies in figure 4.10 make apparent the prominent role of explaining and engaging with others’ ideas in Ms. C’s classroom, and the above example illustrates the ways in which explaining and engaging with each others’ ideas often occurred in tandem, and in

connection with other ways of participating. Over the four days captured in interactions that followed Ms. C in her work with students, providing explanations and engaging with others' ideas occurred in 62% and 60% of episodes, respectively. Moreover, 48% of episodes involved *both* explaining and engaging with another's idea. Collectively, 74% of episodes involved either students providing explanations, students engaging with others' ideas, or both.

Taking up space

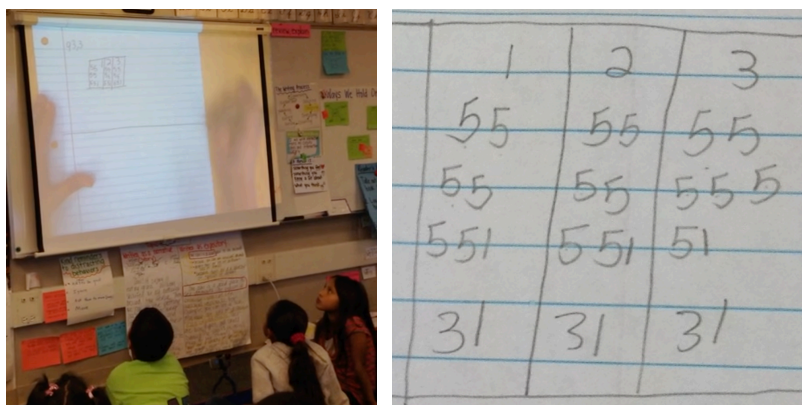
The varied ways of participating available to students enabled them to exercise different forms of agency as they worked on and attempted to make sense of mathematics together (see figure 4.12). Students were consistently positioned as agents of their own conceptual understanding (48 episodes), and were provided opportunities to take up their space in interactions even as their understanding of the mathematical ideas under discussion were incomplete, incorrect, or still emerging (17 episodes).

Figure 4.12: Taking up space: Exercising agency and authority in Ms. C's classroom



In the following example Melani is sharing her strategy with the class. The document camera projects her paper onto the board as she re-enacts her strategy for solving a partitive division story problem for $93 \div 3$. She has drawn three boxes and dealt out groups of five at a time (using the numeral 5), skip counting by fives until she gets to 90. She then deals out one more (using the numeral 1) to each group to reach a total of 93 (see figure 4.13). In the example below, Melani is demonstrating how she then counted up the amount in one group to arrive at an answer of 31. Josue then voices confusion about her strategy.

Figure 4.13: Melani's strategy for solving $93 \div 3$.



- Melani: 5, 10, 15, 20, 25, 30, 31.
 Josue: Why did you put 31?
 Students: Because that's her answer/Because every answer that she got!
 Ms. C: Thanks for asking, Josue.
 Melani: 5, 10, 15, 20, 25, 30, 31.
 (as Melani continues to count, Ms. C walks over and sits next to Josue)
 Ms. C: (whispering to Josue) You still look confused do you want to ask another question? Ask her.
 Melani: Then I put 31 in the box and that was my answer.
 Josue: Mmm (hesitating)
 Ms. C: (whispering to Josue) Go ahead.
 Josue: Why did you did that?
 Ms. C: (addressing Josue but now speaking publicly so that class can hear) So can you be more specific with your question? Did what part?
 Josue: Why did you put one on under line? It's just confusing I don't know why.
 Gerardo: Because it's just like a, uh, the, wait I don't know.
 Ms. C: So who thinks they can explain really quickly what Melani did to see if that can help Josue? You want to try Miguel? Go ahead.
 Miguel: So what she did was that she gave 5 to each box but when she got to 90 she only had 3 left so she split the 3 into ones.

Josue: I get it now.
Ms. C: Are you sure?
Josue: Yeah.
Ms. C: Okay. Thank you. Questions or comments for Melani?
[Ms. C, January 24, 43:00]

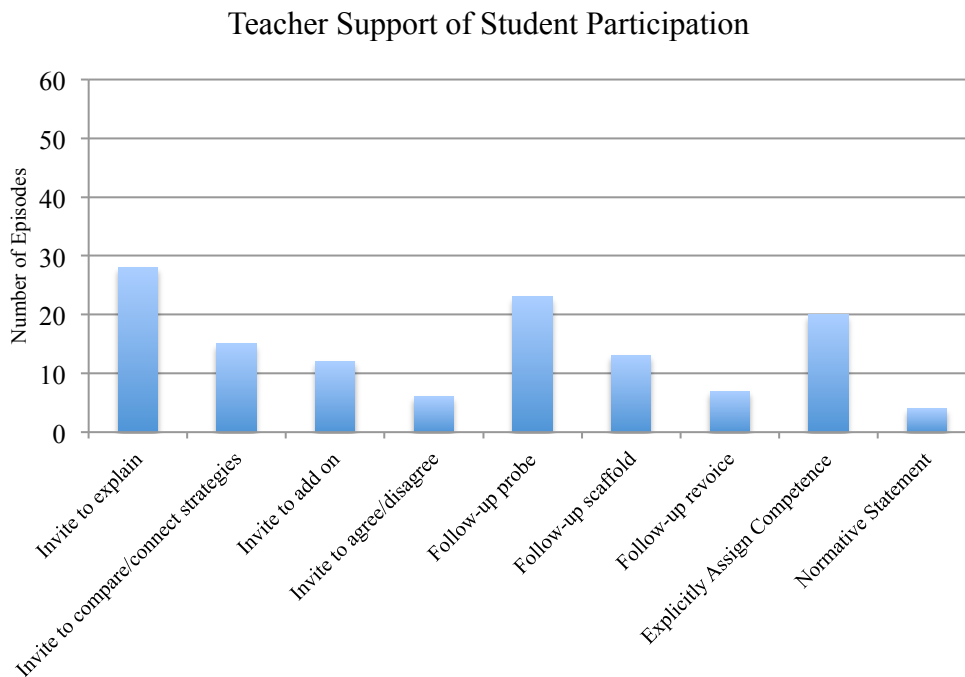
As Melani nears the conclusion of her explanation, Josue attempts to take up his space in the whole class discussion by asking a question about her answer (“Why did you put 31?”). Several other students, possibly voicing impatience with his question, interject and prevent Milani from responding. Ms. C follows up by explicitly thanking Josue for his question. She then shifts her location so that she can sit with Josue and whisper to him, encouraging Josue to ask his question again. When he takes up this opportunity, Ms. C presses him to be more specific with his question. While Josue struggles to do so, his response (“Why did you put one”) suggests that he may be wondering about the shift in Melani’s strategy from dealing out 5s to dealing out 1s. Ms. C then invites further participation from the group, asking who can help Josue by explaining what Melani did. In his explanation, Miguel elaborates on how and why Melani transitioned from dealing out 5s to 1s. Ms. C closes the episode by checking back with Josue and again thanking him for his contribution.

Across the episode Josue takes up his space in several different ways. He interjects to voice confusion about the strategy being shared, is supported to reassert himself in continuing to ask about what for him is unclear about the strategy, is offered help and further clarification from a peer, and is given the authority to decide whether or not his question has been answered satisfactorily. Throughout the episode competence is constructed around voicing confusion and taking up your space when you do not understand a peer’s idea, even if other students might discourage you from doing so. Specifically, interjecting into the discussion by asking a question about Melani’s idea opens opportunities for Josue to take up his space.

Teacher support of student participation

Ms. C played an active role in shaping notions of competence through a range of invitational and in-the-moment follow-up moves. These supports enabled students to make the details of their thinking explicit and public, broadening opportunities for other students to engage with their ideas and for them to be positioned competently. As figure 4.14 shows, Ms. C frequently invited students to explain their ideas, and to compare and connect, add on, or agree or disagree with their peer's thinking. As students shared their ideas, she would follow up in situationally specific ways to probe the details of the mathematics and to support other students to make sense of their peers' ideas.

Figure 4.14: Teacher support of student participation in Ms. C's classroom



The following example takes place approximately two minutes prior to the previous example (with Josue) as Melani is dealing out 5s across three groups. As Melani reaches a critical decision point in her strategy, Ms. C asks her to pause and invites the class to engage with Melani’s strategy by making a prediction about what Melani might do in order to continue to solve the problem.

Melani: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90. And then—
Ms. C: So stop, now what? What did you realize?
Melani: Now I realized that I couldn’t get to 93 so—
Ms. C: (to class) How many (miles) does she have left?
Class: Three. (chorally)
Ms. C: And how many days does she have left?
Class: Three!
Ms. C: So can you predict what she’s going to do with your partner?
(class turns and talks)
Ms. C: (calling class back) 3, 2, 1. Let’s see if you’re right.
Melani: I got the 3 from the 93, then I split the 3 into 3 days.
Students: (whispering) Yes! Yes.
Ms. C: Thumbs up if that’s what you said.
[Ms. C, January 24, 41:12]

It is often challenging for students to transition between skip counting and counting by ones, both to successfully integrate the different counting sequences and to know how and why to do so. Ms. C strategically interjects as Melani is sharing her strategy in an effort to support the rest of the class to engage with this idea. She elicits choral responses from the class to clarify what remains of the total quantity within the context of the story, and then invites students to “predict what she’s going to do” with their partners. A “turn and talk” in this moment provides students the opportunity to offer their predictions to their partners. Following turn and talk, Ms. C returns the floor to Melani to complete her strategy. While in this instance Ms. C follows up in a general way (by asking students to give a “thumbs up” if their thinking was similar to Melani’s), on other occasions he also followed up by inviting specific students to share their predictions with the class. This example illustrates how the teacher’s specific in-the-moment support moves create

opportunities for students to take up competence in relation to engaging with and adding on to a peer's strategy through making a prediction.

Explicit assignments of competence in Ms. C's classroom were most frequently made in relation to students engaging with each other's ideas (10 of 20 instances). Ms. C would also sometimes explicitly assign competence as students provided detailed explanations and/or attempted to take up new or more sophisticated mathematics. Normative statements were less frequent than explicit assignments of competence (four instances over the four days), but each of Ms. C's normative statements specifically related to how students were expected to engage with each other. For example, when Ms. C asked about a number sentence a student had written on his paper, and he responded by saying that it had come from his partner, Ms. C made her expectations for appropriating strategies explicit.

Ms. C: But it's on each of your papers so I'm expecting both of you to understand it, right?
[Ms. C, May 28, 36:48]

Ms. C followed up by continuing to press the pair for details about how the number sentence they had written was related to the strategy they had used to solve the problem. In interviews Ms. C expressed a desire to make her expectations for participation more transparent, and directly connected to students' in-the-moment uptake of particular ways of participating.

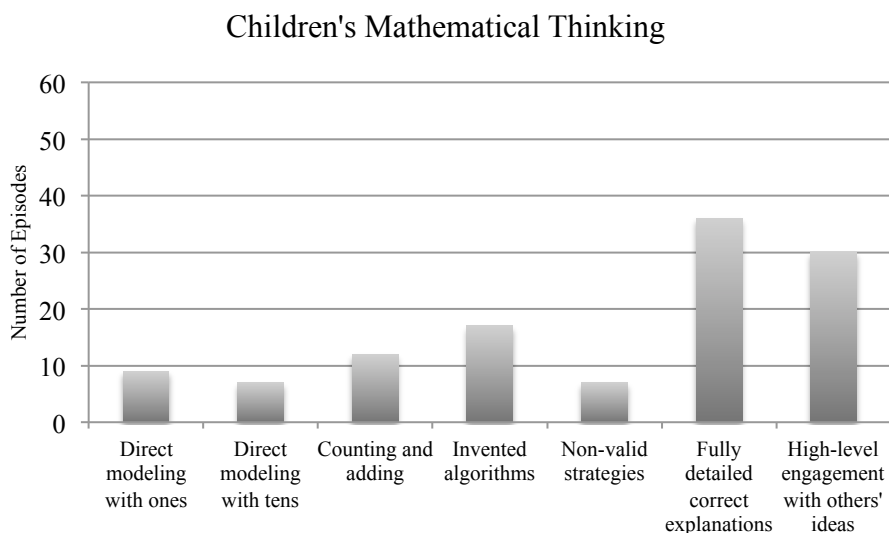
But I also think, which something that I've always told myself like I need to be better at is just say it, like if this is what you like in the classroom why can't you just tell them? Like sometimes I would keep things to myself and going back it's like... I'm just going to tell them, 'hey, I really value when you ask smart, like some questions.' You know I really value or um, 'hey so and so asked a really good question.' And that's the other thing I guess I would do is like 'I really like that question that you asked,' or 'thank you for taking risks.' So like some sort of positive thing mixed in with hey this is what I *do* value, you know in mathematics.
[Ms. C, interview 1]

Ms. C wanted students to take risks and to ask questions, and believed that part of supporting these ways of participating required that she publicly recognize students for these particular kinds of contributions.

Children's mathematical thinking

Like in Ms. J's classroom, the learning and teaching of mathematics in Ms. C's classroom focused on *children's* mathematical thinking. Students were supported to make sense of problems and to come up with strategies that made sense to them. As figure 4.13 illustrates, students used a range of strategies, including counting and adding and invented algorithms that exceeded grade-level expectations.

Figure 4.15: Children's strategies and high-level explanation and engagement with each other's ideas in Ms. C's classroom



It was not enough for students simply to generate strategies to solve problems; interactions focused on making the details of students' ideas explicit, often through providing fully detailed explanations (36 episodes) and through engagement with the details of peers' ideas (30 episodes) by adding on, challenging, or explaining someone else's strategy. Ms. C spoke to the ways that

providing opportunities for students to generate and detail their own strategies supported them to move beyond past struggles and informed her own instructional decision-making.

Well I think just I mean allowing them to come up with their own strategies, it makes them feel comfortable and confident... and not necessarily follow like a set of rules that they have to memorize that they might not have been successful with before. And by seeing the strategies that they come up with you can understand more about what they *can* do, and what they *do* know... they can all do something. And I think you can by knowing what they can do you can push them or you can encourage them.

[Ms. C, interview 1]

Drawing on an in-depth understanding of the development of children's thinking supported Ms. C to see students as capable, and to make decisions about how to build from what they knew. Constructing competence in relation to students' intuitive strategies and sense-making also helped students to, in Ms. C's opinion, build confidence and to feel comfortable in doing mathematics in school.

The relational nature of competence:

Supporting students to engage with more sophisticated strategies

Once again, in order to detail the ways that different aspects of interactions functioned to construct competence, the preceding analysis has presented each dimension individually. Within complex interactions, however, these dimensions interacted with and mutually shaped each other. The ways in which students participated and exercised agency in relation to the teacher's support, and with respect to specific details of the mathematics, collectively constituted what it meant to be a competent participant.

The following extended example illustrates the overlapping nature of these varied dimensions of competence. This episode takes place within a warm up-activity in which students were asked to solve 6×21 . To support the reader, space has been inserted in the transcript to indicate shifts in the participation structure (from whole-class, to turn and talk, and back to

whole-class). Moments in which a critical mathematical idea surfaced are highlighted in blue, and moments in which the floor was shifted have been highlighted in yellow. At this point in the discussion, three correct, complete strategies have been explained and represented,¹⁶ and Miguel is invited to share a different strategy.

¹⁶ Strategies shared and engaged with by the class earlier during the warm-up included a multiplicative strategy that drew on the distributive property $[(6 \times 20) + (6 \times 1)]$, a counting and adding strategy that involved skip counting by 20s and then by 1s, and a strategy that decomposed 21 into two tens and one $[(6 \times 10) + (6 \times 10) + 6]$.

1 Ms. C: Something different? Alright Miguel.
 2 Miguel: So I went 21 plus 21
 3 Ms. C: Okay
 4 Miguel: plus 21
 5 Ms. C: Okay. I'm going to write it over here because I think we might run out (of space). [Ms. C
 6 stands and moves to white board at front of room] So you guys just turn your heads, you don't
 7 need to turn your bodies, okay? So really quickly, 21 plus 21 plus 21 [writes $21 + 21 + 21$ on
 8 white board]
 9 Miguel: Equals 63.
 10 Ms. C: How many 21s did he do?
 11 Students: 3
 12 Ms. C: How many 21s are there?
 13 Students: 6
 14 Ms. C: How many more does he need to do?
 15 Students: 3
 16 Ms. C: So stop. He already has three 21s done. He needs 3 more. What could he do that's really
 17 efficient without having to solve? Turn and tell your partner.

[Students turn and talk with their partners. Ms. C crouches and listens in on several pairs of students (one of which is Karla and Kaya) as they offer their ideas to each other]

18 Karla: [within turn and talk with Kaya] He could just do it like, um, 21 plus 21 plus 21, then he could
 19 put like plus 3 times 21.
 20 Kaya: I know
 21 Karla: And that equals 63, and 63 plus 63 is 126.
 22 Kaya: Or maybe you know how [points to board] he got his answer for 21 times 3? It was 63, so he
 23 would just add another 63, and then he adds those together, and then he shows his answer, and
 24 he adds the rest.
 25 Ms. C: Will you share that with the class?
 26 Kaya: (nods) mmhmm
 27 Ms. C: Awesome.

28 Ms. C: [calling class back] 3, 2, 1. Kaya what do you think he's going to do?
 29 Kaya: You know how he has his, um, answer for 21 times 3?
 30 Ms. C: Oh, hold on; say that again?
 31 Kaya: You know how he has his answer for 21 times 3?
 32 Ms. C: Did he, is that the same as 21 times 3?
 33 Kaya: Yeah
 34 Students: Yes
 35 Ms. C: So can I write that?
 36 Kaya: Yeah
 37 Ms. C: [writes 21×3 underneath $21 + 21 + 21$ on white board] Go on.
 38 Kaya: and it equals 63 so he just add 63 plus 63
 39 Ms. C: Does that make sense?
 40 Students: uh huh/yes
 41 Ms. C: So over here I don't have to solve it again, right? I can just continue it [writes another 21×3].
 42 And he already knows that it equals?
 43 Students: 63.
 44 Ms. C: So I have how many 63s?
 45 Students: Two
 46 Ms. C: Two. Miguel, would you like to, is that what you were gonna to do?
 47 Miguel: Yeah (smiles)
 48 Ms. C: Okay. So you have 63 and 63 so now what?
 49 Miguel: I just do 63 plus 63 equals 126.

50 Ms. C: How did you get 126?
51 Miguel: I got it because since, um, since the 63
52 Ms. C: Yeah
53 Miguel: since 6 plus 6 equals 12.
54 Ms. C: mmhmm
55 Miguel: And so I put the 12 there and then since 3 plus 3 equals 6. And if I went, but if I went 12 plus 6
56 it wouldn't work because the 12 is not in the tens and ones places, the 1 in the one hundreds
57 place.
58 Ms. C: Great so this really means?
59 Miguel: 120 plus 6.
60 Ms. C: 120 plus 6. Cool.
[Ms. C, March 26, 12:45]

In this episode Miguel's idea is leveraged to support students to make connections between strategies and to build toward more sophisticated mathematics. This is made possible by the ways in which providing explanation and engaging with others' ideas operate in relation to students' mathematical thinking and the teacher's support to create varied opportunities for students to participate.

The episode begins as Ms. C invites Miguel to share and explain his strategy. Ms. C records the beginning of Miguel's strategy ($21 + 21 + 21$) and questions the class about what he has done so far in relation to the problem (lines 10-15). At this point Ms. C strategically pauses the discussion and asks the class to consider possibilities for completing the strategy that Miguel has begun ("What could he do that's really efficient without having to solve?"). She then turns the floor over to the class, providing an opportunity for students to share their ideas and predictions with their partner (line 17). During turn and talk, Ms. C crouches down and listens in on several students' conversations, including Kaya who offers to her partner the idea of doubling 63 (lines 22-24). After asking Kaya if she is willing to share her idea publically (line 25), Ms. C calls the class back together and invites Kaya to share her prediction with the class (line 28). Following Kaya's explanation, Ms. C records her idea and elicits choral responses from the class to make the details of the strategy explicit. Ms. C then returns the floor to Miguel to see if Kaya's idea is in alignment with his own (line 46), and to complete the explanation of his strategy, which makes a critical understanding of place value explicit (lines 55-57, 59). In sum, Miguel's strategy is leveraged to support the rest of the class to engage with an ambitious mathematical goal, a multiplicative strategy that makes implicit use of the associative property of multiplication (CCSSM, grade 4). The affordances of explaining and engaging with others' ideas are realized through the teacher's use of specific in-the-moment instructional moves, which

function in concert with student thinking to strategically structure both the learning opportunities made available to students, as well as Ms. C's own opportunities to learn about her students' thinking.

Examining different dimensions of competence across the episode reveals the overlapping ways in which students and the teacher participate together, allowing students to be positioned as competent contributors while they wrestle with sophisticated mathematics. Examining multidimensional aspects of student participation, we see students providing explanations and engaging with others' ideas. Miguel explains his strategy (lines 2, 4, 9 and 49-59), while Kaya (lines 22-24 and 29-38), Karla (lines 18-21) and other students (during turn and talk) add on to Miguel's idea. A sociomathematical lens on participation also reveals aspects of mathematical difference (line 1), mathematical sophistication (lines 16-17 and 29-41) and mathematical representation (lines 37 and 41) under negotiation. Students are provided opportunities to take up space within the discussion of a sophisticated strategy by being offered a chance respond to a question with multiple valid responses ("What *could* he do...?", line 16) and to work through their ideas with a partner during turn and talk. Miguel's authority over the strategy is preserved when Ms. C returns the floor to him, asking if Kaya's prediction matches his own thinking ("Is that what you were gonna do," line 46), and then inviting Miguel to finish off the strategy (line 48). With respect to children's mathematical thinking, a strategy that initially resembles a counting and adding strategy (each of the first 3 groups of 21 is represented individually) is bridged to an invented algorithm (the second group of 3 21s is abstracted). Over the course of the episode, a correct, fully detailed explanation of the strategy is articulated, and Kaya adds on to Miguel's idea in a detailed way. Ms. C uses a range of moves to support student participation, including invitations to explain (lines 1, 28, and 48), invitations to add on (lines

16-17 and 28) and probes for additional detail (line 50). The episode closes with an explicit assignment of competence (“cool” in line 60), although it is not necessarily clear whether this assignment is in reference to Miguel’s explanation, the sophistication of the strategy, students’ general participation within the episode, or to all of these collectively. Taken together, the episode illustrates the ways in which constructing competent participation around and in relation to providing explanation and engaging with others’ ideas allows students to take up positions as capable knowers and doers of mathematics even as their understanding of a sophisticated strategy is still emerging.

Status in negotiating productive engagement with each other’s ideas

Video footage that captured the perspectives and continuous participation of specific pairs of students sheds additional light on what it meant to negotiate and take up the varied ways of demonstrating competence within mathematics lessons in Ms. C’s classroom. All students explained and engaged with each other’s mathematical ideas in some ways, though the level of detail with which they explained their ideas or engaged with their partners’ ideas varied. Of 11 pairs¹⁷ captured at the first time point, seven participated in ongoing, synchronous work together around the mathematics. The other four pairs mostly worked in parallel but would engage sporadically. Six of the 11 pairs engaged by attending to the details of each other’s strategies or by adding on to or challenging each other’s ideas, while the remaining five pairs did engage with each other’s ideas in some way, but not at a high-level. They mostly checked in with what the other was doing or compared their answers with each other without probing what their partner had done. Explanations were common; a fully detailed explanation was provided by at

¹⁷ One group of students on January 22 was a trio rather than a pair.

least one student in nine of the 11 pairs (two pairs provided multiple complete explanations). Six pairs worked together to jointly produce a strategy. There was only one pair that did not explain or jointly produce a strategy in their work together.¹⁸ Different pairs illustrated different tensions that emerged as students attempted to take up opportunities to engage with each other around mathematics, most notably in how they went about resolving disagreements. The following example illustrates both the ways in which pairs sometimes struggled to make sense of each other's ideas, as well as the productive aspects of engagement that occurred even within incorrect or incomplete strategies.

Like several other students on this day, Walter and Josue had struggled to correctly solve the partitive division problem of splitting 70 seashells between two days. The pair had completed solutions separately and were now explaining and comparing their strategies. Walter had correctly interpreted the structure of the problem, but miscounted in distributing his 70 marks across two groups and arrived at an answer of 32. Josue had interpreted the problem to be about adding 70 and 70. He represented his interpretation of the story using two groups of 7 base-ten rods, but struggled to count to find his total. In the following excerpt, Ms. C has stopped by the pair and is attempting to support them to engage with each other. Josue has already explained his strategy, and Walter has just stated his answer of 32.

Josue: Walter, 70 plus 70 is not going to make a lower number. It's gonna make a higher number.
Walter: I know. But I didn't add 70 and 70.
Josue: Actually I did 70, 70 and I keep on struggling, struggling because it's a high number.
Ms. C: Okay hold on now it's his turn, right? So Walter why *didn't* you add them together and what *did* you do? And Josue will listen.
Walter: I tried to do the days, and I tried to go all the way up to 70 and I got um 32.
Ms. C: Did you guys do the same thing?
Josue: (shakes head no)
Walter: No he did something else different than me.

¹⁸ This pair disagreed about whether the problem was asking them to split 70 between two groups or to add 70 and 70. They noticed that they disagreed but struggled to identify or explain what it was that they disagreed about.

Ms. C: Okay so now we need to figure out which one matches the story. So which one do you think matches the story? Going back to here (points to story problem on the paper), who can prove why they're right?

Josue: I could prove!

Ms. C: Why, what does it say?
 (Josue rereads part of story out loud and immediately begins modeling two groups of 70 with base-ten blocks, omitting the last sentence of the story)
 [Josue & Walter, January 22, 32:21]

As the excerpt begins, Josue challenges the logic of Walter's answer ("70 plus 70 is not going to make a lower number"). Walter agrees with Josue's statement, but states (in a somewhat implicit challenge of Josue's premise) that he did not add 70 and 70. Rather than responding to Walter's statement, Josue returns to his own strategy, noting his struggle to count to find his total. At this point, noting that Josue has not taken up Walter's bid to explain his interpretation of the problem, Ms. C redirects Josue to attend to his partner, and asks Walter to detail what he did and why. However, Walter does not sufficiently detail his reasoning, and Ms. C follows up with a normative statement about resolving disagreement by considering strategies in relation to the story context. Josue takes up Ms. C's invitation to prove his idea, but in only rereading part of the story does not engage in a discussion of the problems' structure, returning once again to his own strategy.

As the pair continues their work together, with Josue continues talking about and reenacting his strategy, and Walter asking questions about what he is doing. Josue at times provides responses, although he sometimes talks over Walter as Walter asks additional questions. Seeing that the pair are engaging with each other, Ms. C walks away to monitor other pairs of students. Several minutes later Josue returns to discussing how to add 70 and 70.

Josue: So what the easier way what I did was count it by ones, cause what's more harder counting by tens, *alllll* the way 'till you finish, or counting by ones to make it much easier? 1, 2, 3, 4, 5, 6, 7, 8, 9 (counting squares on first base-ten rod)

Walter: But you—

Josue: (now counting each rod as ten) 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 101, 102, 103, 104. That's right.

Walter: Yeah but look, if you get to 100 you gotta go by 100, by tens again. So you gotta go 110, 120—

Josue: Oh yeah, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100
 Josue & Walter: (together) 110, 120, 130, 140
 Josue: 140!
 [Josue & Walter, January 22, 37:21]

Josue has struggled to count by tens across 100 (“alllll the way ‘till you finish”), and asserts that it is easier to count by ones. Walter attempts to respond to Josue, but Josue continues to talk over him by verbalizing his count, first counting by ones then shifting to count the remaining rods by tens. He again struggles as he crosses 100, counting rods as ones, but in verbalizing his count his struggle is made explicit to his partner. Walter responds by stating that Josue must keep counting by tens, and offers the next counts in the sequence. With this specific suggestion, Josue takes up Walter’s effort to engage (“Oh yeah”) and begins to recount by tens, with Walter joining in his counting once he reaches 100. With Walter’s support, Josue has resolved what for him was an important mathematical challenge. By persevering in asking questions about Josue’s strategy, challenging the details of what he has done, and offering a specific support (an alternative count sequence), Walter is able to take up his space within the pair. The pair has done some significant mathematical work together, albeit in the service of correctly solving a *different* problem.

Later, following the strategy share, Ms. C visits Walter during independent work time to reflect on his strategy. Josue is now sitting at another table as Ms. C and Walter converse privately.

Ms. C: Did this (points to Walter’s original strategy) match one that somebody else did up there?
 Walter: Yeah.
 Ms. C: Yeah so I know that Josue likes to talk, right? That doesn't mean that you're wrong. You've got to stand up for your own thoughts, okay?
 Walter: (nods) Okay.
 Ms. C: So who was right, you or Josue?
 Walter: Josue...
 Ms. C: Does this look like somebody’s?
 Walter: Oh yeah it looks like um, Carlos and Gerardo's.
 Ms. C: mmhmm. So you did the same thing as them, right? So you got to stand up for yourself, okay?
 Walter: Okay.
 [Josue & Walter, January 22, 68:13]

Ms. C checks in Walter to see if he now realizes that his original interpretation of the problem was correct, and encourages him to “stand up for your own thoughts” when working with Josue. In doing so she uncovers that Walter is still somewhat unsure about who was right, and again asks him to compare his strategy with those that were discussed during the share out. After Walter makes this connection, she returns to the idea of maintaining confidence in his ideas. Following this exchange, Ms. C continues to sit with Walter as he completes his strategy, finishing by saying “Good job, you’re almost there.”

Walter and Josue’s work together on this day highlights the ways in which status issues can influence the ways in which students attempt to provide explanations and engage with each other’s ideas. There are both productive and problematic elements of the pair’s efforts to participate on this day. In attending and responding to the details of Jose’s thinking, Walter supports Josue to overcome his struggle to add 70 and 70; the pair jointly produces a complete, correct strategy for counting 14 groups of ten. On the other hand, in accommodating Josue’s insistence to work through his own strategy, Walter abandoned his original (correct) interpretation of the problem.

Student achievement

The previous sections have detailed the ways in which competence in Ms. J’s and Ms. C’s classrooms was constructed around providing explanations, engaging with each other’s ideas, and solving problems in ways that made sense to individual students. Within the larger sociopolitical contexts of schooling, competence is often associated with students’ performance on assessments of mathematics achievement. In order to situate these local constructions of

competence within the larger contexts of schools, I turn now to an analysis of students' performance on two measures of mathematics achievement.

Prior achievement

No pre-test was administered prior to the observations conducted across the two classrooms. However, students' scores on the second grade California Standards Test (CST) provide a partial picture of their prior achievement in mathematics. The CST was a standardized multiple-choice test administered for assessment and accountability reporting to all students in grades 2-11 in the state of California (California Department of Education, 2016). The content tested on the CST was derived from the 1997 California state mathematics standards (California Department of Education, 1997). Scores were available for 44 of 45 students in Ms. J's and Ms. C's classrooms who had participated in the assessment the previous spring during their second-grade year. The CST was not administered during students' third-grade year.¹⁹

Across the two classrooms students' mean scaled second grade CST score was 380 ($\sigma = 58.3$), identical to the statewide average ($\sigma = 83$; California Department of Education, 2014). The incoming mean scores of Ms. J's students ($\bar{x} = 394$; $\sigma = 57.9$) were slightly higher than those of Ms. C's students ($\bar{x} = 365$; $\sigma = 56.3$), but this difference was not statistically significant ($t_{(42)} = -1.692, p = .098$). Thus, students' prior achievement in terms of standardized mathematics test scores was similar across classrooms, and comparable to their peers across the state.

¹⁹ The 2013-2014 school year was a time of transition for large-scale assessment in the state of California. While the Common Core State Standards were adopted by the state board of education in 2010, the California Standards Test (CST) continued to be administered through the spring of 2013. California conducted field testing of newly-developed Smarter Balanced Assessments in the spring of 2014, but student data from these assessments was not reported publicly. Thus, no standardized mathematics achievement data was available for Ms. J's and Ms. C's students for the 2013-2014 school year analyzed in this study.

End-of-year problem solving assessment

At the end of May students in both classrooms took a written problem solving assessment consisting of four questions. The assessment was designed by the research team to capture children's strategies for solving a range of multidigit multiplication and division problems. The mathematics within the problems exceeded grade-level expectations of the Common Core State Standards.²⁰

In many ways, the end-of-year problem solving assessment reflected the mathematics of the days observed; students were expected to make sense of and solve a variety of multiplication and division story problems involving multidigit numbers, using strategies of their own choosing. On the other hand, the assessment was administered as an independent activity. Students were expected to work individually, and were not provided with any additional support to make sense of the story problems, reflect on their strategies, or generate or compare their strategies with each other. In other words, while demonstrating competence within the context of the assessment *was* related to children's thinking about multiplication and division, competence *was not* related to providing explanations or engaging with others' ideas.

Students' responses on assessment items were coded according to the type of strategy used, as well as for the correctness of the final response. Table 4.1 displays the assessment items and the number and percentage of students who solved each item correctly. Overall, students successfully solved 74% of items (75% and 73% in Ms. J's and Ms. C's classrooms, respectively). Students in both classrooms were most successful in correctly solving item D (82%), and were most challenged by item A (62%). Students performed similarly (76% correct)

²⁰ The most closely related standards to those posed on the assessment are CCSSM 4.NBT.5 (for items A and D), 4.NBT.6 (item C), and 5.NBT.6 (item B). In terms of the previous 1997 California Mathematics Standards, the most closely related standards are 3.NS 2.4 (items A and D), 3.NS 2.5 (item C), and 5.NS 2.2 (item D).

in solving the two division problems, including the division problem involving a multidigit divisor (item B).

Table 4.1: End-of-Year Problem Solving Assessment Items and Results

Problem	Number (percentage) of students who correctly solved problem		
	Ms. J n = 23	Ms. C n=22	Total n=45
<i>A. Multiplication</i> 6 buses are taking students on a field trip. There are 29 students riding on each bus. How many students are riding on the buses?	15 (65%)	13 (59%)	28 (62%)
<i>B. Measurement division</i> There are 75 fish at the pet store. If the pet store owner wants to put 15 fish in each fish tank, how many fish tanks will she fill?	18 (78%)	16 (73%)	34 (76%)
<i>C. Partitive division</i> The candy store has 84 lollipops. They want to put the lollipops into 4 boxes so that there are the same number in each box. How many lollipops will go into each box?	17 (74%)	17 (77%)	34 (76%)
<i>D. Multiplication</i> Ms. Padilla is buying books at the bookstore. Each book costs \$16. How much will Ms. Padilla spend on 4 books?	19 (83%)	18 (86%)	37 (82%)
Total assessment items solved correctly	69 (75%)	64 (73%)	133 (74%)

While there was some slight variation in students' responses for particular items, students' overall performance on the problem solving assessment was quite similar across the two classrooms, with unadjusted mean scores of 3.00 ($\sigma = 1.35$) and 2.91 ($\sigma = 1.34$) for Ms. J's and Ms. C's classrooms respectively. An analysis was performed to examine relationships between students' end-of-year assessment scores and individuals' prior achievement, language status, and classroom teacher. The goal of the analysis was not to provide a exhaustive model for predicting individual student performance, but rather to investigate if variations in competence across classrooms, or possible language demands associated with providing explanation or engaging with others' ideas, impacted students' assessment outcomes. Because of the relatively small number of items on the assessment, and because students' scores did not follow a normal distribution (32 of 45 students scored 3 or 4), a Poisson loglinear regression model was used. The model provided a good fit to these data, $\chi^2_{(40)} = 17.607, p > .05$, and the results of the model were statistically significant $\chi^2_{(3)} = 8.586, p = .035$. Appendix F displays the model and adjusted means for each classroom and for students' language status (as identified by the state), controlling for students' second grade CST scores. CST data was not available for one student, who was excluded from this analysis. Model effects indicated that incoming grade 2 CST score was a significant predictor of the end-of-year assessment score ($p = .005$). Categorical predictors of classroom teacher and English learner classification were not statistically significant. In other words, when controlling for prior achievement, students' scores on the end-of-year problem solving assessment were similar across classrooms, and constructing competence around providing explanation and engaging with others' ideas did not result in differential outcomes for emergent bilingual students.

Comparing students' performance on the end-of-year problem solving assessment to what would be expected in other classrooms requires some extrapolation. As previously stated, the problem solving assessment was created by the research team; thus, large-scale performance data on the measure does not exist. And as no pre-test was administered, students' performance on this measure cannot be evaluated in terms of growth. However, the overall performance of other students on similar standardized testing items provides a rough proxy for evaluating students' performance on the problem solving assessment. Released test questions from the CST and NAEP mathematics assessments were reviewed to identify items most similar to those posed on the problem solving assessment (California Department of Education, 2009; National Center for Education Statistics, 2011). This review turned up only a handful of items that both reflected the specific mathematics content and also contained item-specific data on student achievement. Appendix G displays similar CST and NAEP multiplication and division items and corresponding correct response rates. Across these items third graders averaged 57% correct, and fourth graders 60% correct. Comparatively, recall that across the two study classrooms students correctly solved 74% of items on the end-of-year assessment. Moreover, unlike the open-ended problem solving assessment, these standardized test items were multiple choice, and a calculator was available for the NAEP item. Given the small sample of comparable items, comparisons between the end-of-year problems solving assessment and CST/NAEP items must be viewed with some caution. However, the available evidence suggests that students in Ms. J's and Ms. C's classrooms were quite successful in solving multidigit multiplication and division problems, and outperformed their peers across the state and nation.

Students in both classrooms used a wide range of strategies on the problem solving assessment. Figure 4.16 and 4.17 display the strategies used for each of the assessment items in

Figure 4.16: Students' strategies on the end-of-year assessment in Ms. J's classroom

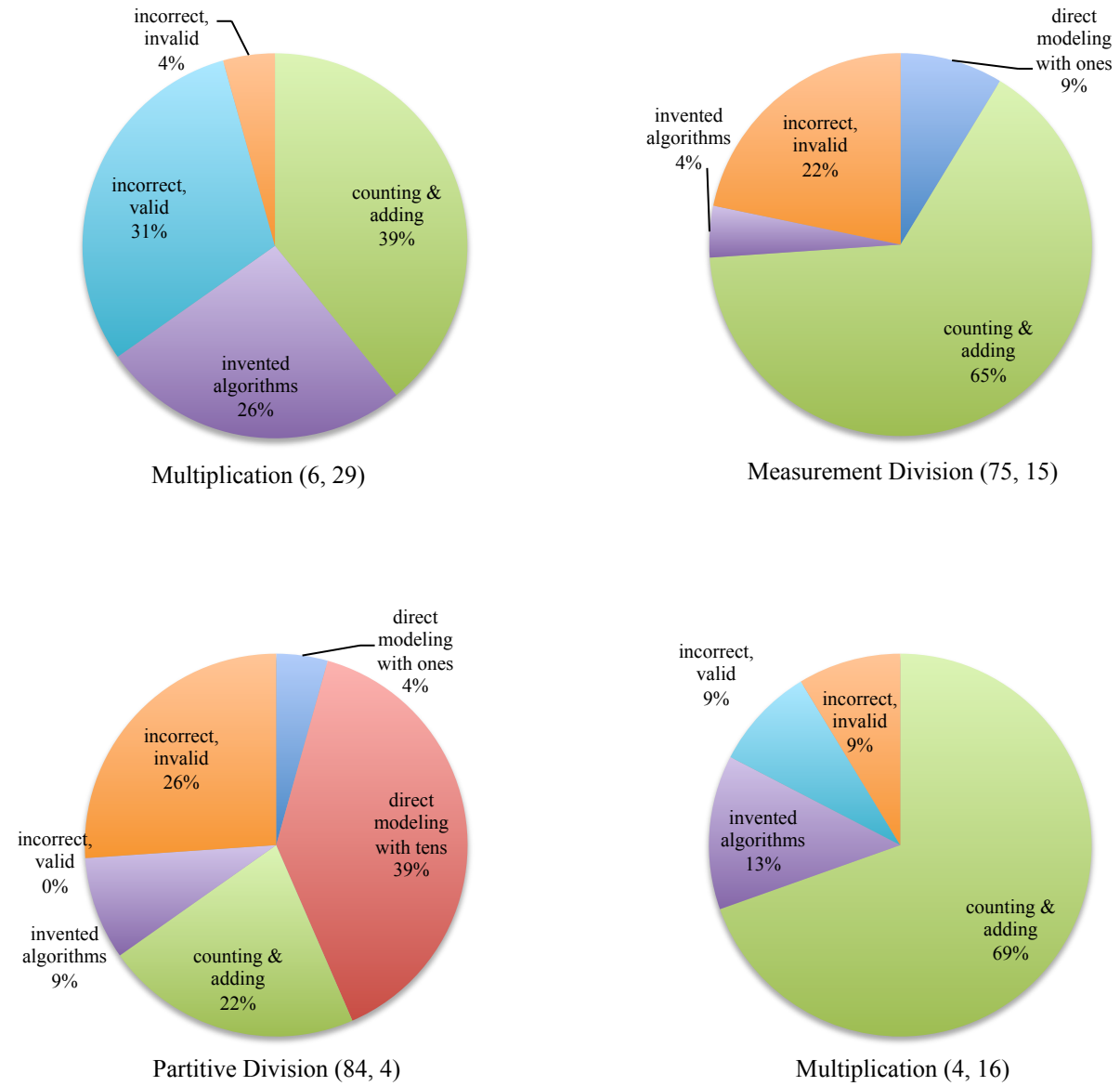
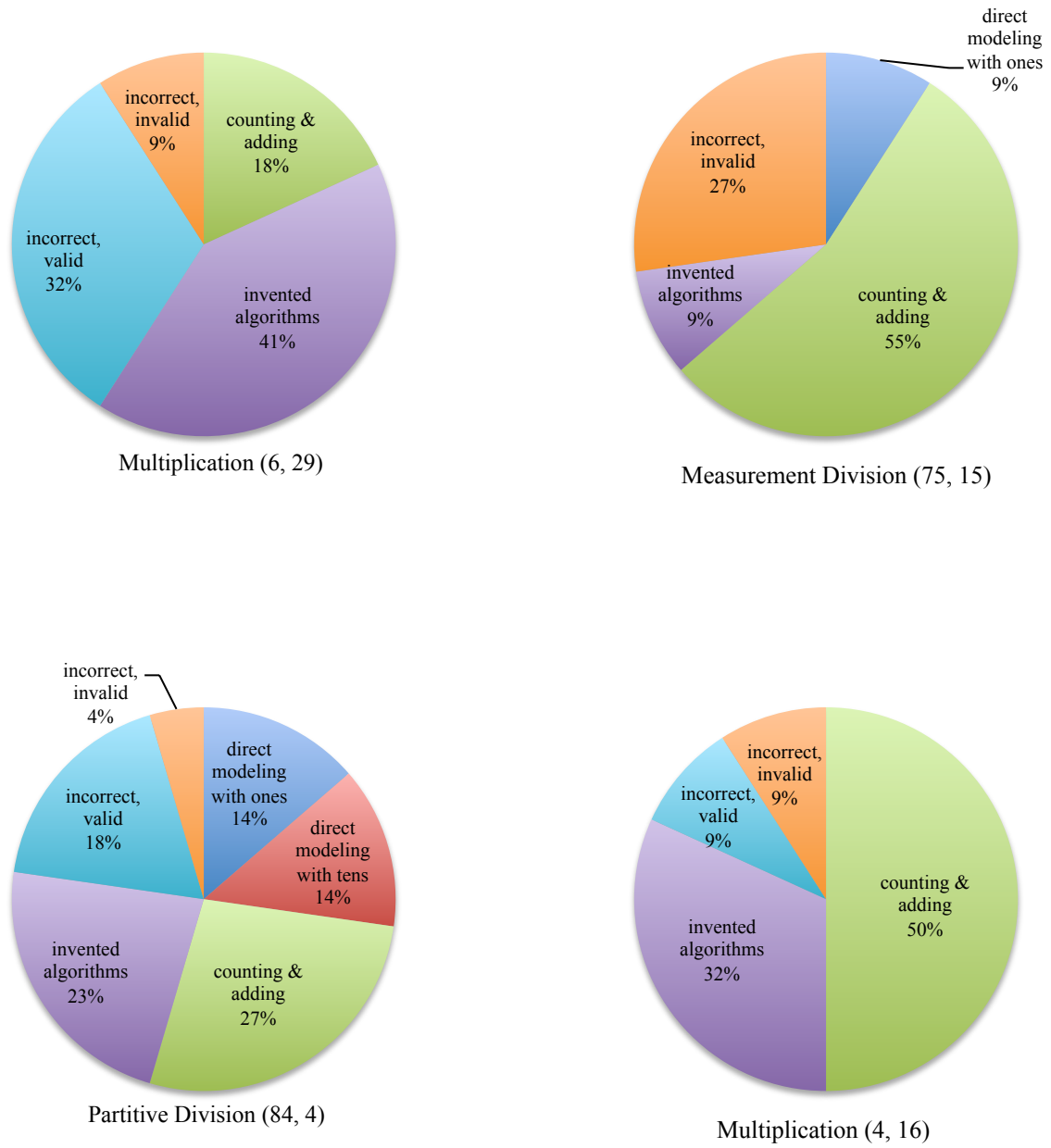


Figure 4.17: Students' strategies on the end-of-year assessment in Ms. C's classroom



each classroom. The percentages displayed for direct modeling, counting and adding, and invented algorithms are for strategies used on correct responses. For incorrect responses, students' strategies were coded as valid or invalid. For incorrect, valid strategies students correctly interpreted and set up the problem, but either made an error in counting or computing, or were unable to carry out the necessary computation. For incorrect, invalid strategies students either misinterpreted the problem (e.g., multiplying instead of dividing) or were unable to come up with an approach to solve the problem.

As previous research in children's thinking would suggest, students' strategies varied according to the structure of the problem and the numbers involved (Ambrose et al., 2003; Carpenter et al., 2015). For example, students in both classrooms were more likely to use counting and adding strategies for measurement division than partitive division, or when the amount in a group was easier to count by (counting or adding 16s as compared to 29s). Students were much more likely to use direct modeling strategies for the partitive division problem than for the other problems. Notably, invented algorithms were more common on the multiplication problem which involved a greater amount of groups, and a group size that could be easily adjusted to a decade number ($29 \rightarrow 30$), despite it being a more challenging problem.

Many of the strategies used by students in both classrooms would be considered quite sophisticated for third graders. Counting and adding or invented algorithm strategies required that students apply their understanding of place value and properties of operations (NGA & CCSSO, 2010). While students in both classrooms used a variety of sophisticated strategies, on average Ms. C's students used invented algorithms with greater frequency (26% of items, as compared with 13% of items in Ms. J's classroom). It should be noted that students were asked to solve the assessment items using any strategy they wanted, rather than their most sophisticated

or “efficient” strategy. It is certainly possible that some students who were able to solve problems using invented algorithms chose not to do so on this day for the assessment. In summary, like they did in solving problems with their partners on the days observed, students used a wide variety of strategies on the end-of-year problem solving assessment. Put another way, students took up notions of competence related to making sense of and solving story problems using a range of strategies within the alternate context of an independent problem solving assessment.

Collectively, the available data suggests that students in Ms. J and Ms. C’s classrooms performed quite well on the end-of-year problem solving assessment. Students correctly solved 74% of items, and used a range of sophisticated strategies in doing so. While students began the school year with comparable CST scores to those of their peers across the state, by the end of the year they solved multidigit multiplication and division story problems at a greater rate than that of other third and fourth grade students who solved similar items.

Summary: Constructing competence across two third-grade classrooms

The analyses presented in the previous sections illustrate the varied, overlapping ways in which competence was constructed in Ms. J’s and Ms. C’s classrooms. Students in both classrooms were consistently supported to solve problems in ways that made sense to them, to provide explanations that made the details of their thinking explicit, and to engage with each other’s mathematical ideas. Doing mathematics in these ways served to expand collective notions of competence, enabling students to be positioned competently for contributing in a wide range of ways.

In many ways, Ms. J's and Ms. C's classrooms were quite similar. Instruction in both classrooms focused on students' mathematical thinking, and provided consistent opportunities for students to explain their mathematical ideas and to engage with the ideas of others. In both classrooms explanation and engagement with others' ideas were leveraged to engage students in substantive mathematics, and in both classrooms students performed well on the end-of-year problem solving assessment. The shared features of Ms. J's and Ms. C's classrooms highlight the ways in which constructing competence around explanation and engagement with others' mathematical ideas served to expand competent ways of doing and participating in mathematics. Descriptions of these shared features are provided below:

- *Surfacing and framing partial understandings*: Across both classrooms students' errors or incomplete ideas were framed as productive, worthy of discussion with others, and as competent ways of participating in mathematics. These partial understandings were not positioned as misconceptions, nor were they ignored. Rather, they were given space and built upon, allowing for students' ideas to remain the focus of conversation while also advancing the mathematics. Positioning students' emergent ideas as reflecting understanding opened space for them to participate, and expanded collective notions of competence.
- *Exercising agency and authority*: In both classrooms students were consistently positioned as capable of making sense of mathematics and as authors of their own strategies. Students exercised agency collectively in determining whether or not particular solutions were correct and made sense, and individually by presenting their strategies to each other and responding to their peers' ideas. Students often interjected with questions or challenges as strategies were shared, enabling students whose

understanding of the mathematics under discussion was still emerging to assert their needs, take the floor, and contribute.

- *The teacher's role:* Teachers consistently centered students' ideas in interactions, and supported students to explain to and engage with each other through a variety of in-the-moment support moves. Teachers invited students to explain each other's ideas, and supported them to add on in novel ways, such as by inviting students to predict what a peer would do next, or by inviting students to make comparisons and connections across multiple strategies. The range of moves employed by both teachers created opportunities for students to contribute mathematically in both general and specific ways, and space for multiple ways of participating to be recognized as legitimate.

The nuances of these common features also revealed ways in which Ms. J's and Ms. C's classrooms were subtly different. While in both classrooms providing explanation and engaging with others' ideas were constructed as competent ways of participating, the two classrooms differed with respect to their relative emphases. Interactions in Ms. J's classroom tended to underscore the importance of providing explanations or working through struggle, while in Ms. C's classroom competence was more likely to be associated with interjecting to ask questions, or adding on in the service of more sophisticated mathematics. This influenced not so much the forms of agency that students exercised, but rather the manner in which they tended to do so. For example, consider the following two interactions in which students interjected as a peer began sharing a solution strategy.

Ms. J: Shonda, would you like to have a go?
Shonda: I think it's 165 minutes because I know 60 plus 60 equals 120, and then I added 10 four times.
Several students: What?

Nico: Where'd you get the 10 from?
 Shonda: (pointing to the board) From the—
 Giancarlo: How?
 Nico: But there's only 40, how did you get the 10 from the 40?
 Edwin: Oh she splitted it, she splitted it.
 Ms. J: Let her finish, let her finish.
 Shonda: So then I counted 120, 130, 140, 150, 160. And then I added 5 minutes to it so it's 165 minutes.
 [Ms. J, May 28, 10:20]

Ms. C: Melani do you want to start by telling us what you did to solve this?
 Melani: Yes. First what I did I drew, I put the 3 days.
 Ms. C: (quietly to class) Thumbs up if you did that too.
 Melani: Then I started, I put, I started counting by 5s until I got to 93.
 Several students: Why?/Why did you count by 5s?
 Karla: So that she can get to 93.
 Ricardo: But if you count by 5s you won't get to 93.
 Ms. C: Well, remember we're trying to understand her strategy. But do we understand *why* she's doing this? Because if we don't understand why she's dividing five, five, five then that's what we need to find out.
 Julisa: Melani, why are you putting 5s to go to 93?
 [Ms. C, January 24, 39:41]

Both interactions begin with invitations for the student to explain their strategy, and both explanations are momentarily interrupted when several students voice confusion about something said. These interjections are followed by other students' attempts to respond, at which point each teacher intervenes to provide some structure to the conversation by granting a single student the floor. However, the teachers' moves function differently in who is granted the floor, and thus what is communicated about the relationship between explaining and engaging with others' ideas. Ms. J's (gentle) directive to "let her finish, let her finish" returns the floor to Shonda to complete her explanation, suggesting that students should allow for an explanation to be completed before responding. On the other hand, Ms. C's normative statement about the goal of strategy sharing ("to understand her strategy") and subsequent prompt for question-asking ("if we don't understand why... then that's what we need to find out") function to award the floor to Julisa to ask a specific question. This suggests that in Ms. C's classroom interjecting with questions during another students' explanation is permissible and even helpful, as it can support

the student who is sharing to provide further explanation. It is important to note that in *both* classrooms students at times participated by interjecting into whole-class discussions. However, a general trend was that in Ms. J's classroom students often waited until the presenting student had finished sharing their idea, and then raised their hands to ask questions about the strategy. In Ms. C's classroom students tended to ask questions in-the-moment as strategies were being shared, with fewer questions occurring after a complete explanation had been provided.

This difference in the relative emphasis of providing explanation versus engaging with others' ideas was also evident in examining the contexts within which students' partial understandings emerged. Focusing on supporting students to provide explanations of their ideas allowed Ms. J to draw out students' productive ideas within incomplete or incorrect strategies. Ms. J would often position an aspect of the student's thinking as competent, and use this as a starting point to connect to the story context and to support the student to develop a valid strategy. Framing errors and unfinished strategies as reflecting partial understanding allowed students to be positioned as competent even as their ability to solve a given problem was still emerging.

In contrast, Ms. C often leveraged engagement with others' ideas as a way of supporting students to build from their partial understandings of more sophisticated strategies. Ms. C regularly framed students' existing strategies as containing the seeds of more sophisticated mathematics, and encouraged students to try out their emerging ideas to generate additional strategies and representations. In this way partial understandings were positioned as competent and leveraged to support students to try out more abstract and elegant mathematics.

To summarize, two predominant ways of participating were shared across both classrooms—providing explanation and engaging with others' mathematical ideas. Both teachers

routinely highlighted aspects of students' participation that were desirable and productive, but the two classrooms varied with respect to the relative emphasis placed on providing explanation versus engaging with each other's ideas. These emphases shaped the ways in which students exercised agency, and the contexts within which partial understandings were recognized.

CHAPTER 5: DISCUSSION

Competence as a Theoretical Lens for Understanding Learning and Teaching

This study examined the ways in which competence was constructed in two third-grade classrooms. Constructing competence around and in relation to students' explanations and engagement with each other's ideas broadened opportunities to participate. Analyses of interactions revealed the relational nature of competence; providing explanations and engaging with others' ideas were negotiated in relation to other ways of participating, exercising agency, the details of students' mathematical thinking, and teachers' in-the-moment support moves.

Both Ms. J's and Ms. C's classrooms created space for students to make sense of mathematics, and to be positioned competently for making a wide variety of contributions—including ways of contributing that were available to students when their understanding of the mathematics was still emerging. The two classrooms created space for and built upon students' partial understandings in different ways; teachers leveraged students' explanations and engagement with each other's ideas to advance instructional goals while supporting individuals to take up their space in class discussions. Analyses of the end-of-year problem solving assessment demonstrated that students developed a strong understanding of multiplication and division, both in relation to their own prior achievement as well as to that of their peers across the state. Thus, both classrooms expanded collective notions of competence, and were successful in supporting students to demonstrate their mathematical learning in the context of an independent assessment.

Leveraging the potential of explanation and engagement with others' ideas

Ms. J's and Ms. C's classrooms highlight the possibilities of instruction that supports students to explain their ideas and to engage with the ideas of others. Centering these ways of participating, in concert with a focus on the details of students' mathematical thinking, expanded what counted as competence and created opportunities for varied participation. In addition to being supported to solve problems in ways that made sense to them, students were provided opportunities to engage in mathematics from a variety of perspectives. They were able to see and hear mathematical ideas explained and represented in multiple ways, and to consider relationships between their own and others' solutions and perspectives. Listening to and communicating with others was at the core of mathematical activity in both classrooms; the heterogeneity of resources and perspectives was fundamental to learning (Rosebery, Ogonowski, DiSchino, & Warren, 2010).

We cannot conclude, however, that a focus on participating through explanation and engagement with others' ideas alone will necessarily broaden what counts as competence. The relational nature of competence illustrated in both classrooms in fact suggests the opposite—that inviting students to explain their ideas or to engage with each other's ideas were not enough. The variety of ways of participating available in each classroom, the ways that students' attempts to participate were framed positively, the ways that teachers' tailored their responses to individual students, and a consistent emphasis on the details of students' mathematical thinking—were essential in opening space for students to participate. Put another way, the extent to which explaining and engaging with each other's ideas can be successful in expanding competence was dependent upon teachers knowing their students as people, and upon creating ways for students to bring their whole selves to the learning of mathematics.

The range of ways that Ms. J and Ms. C responded to students' efforts to participate make it clear that the teachers did not expect every student to participate in the same ways. Rather, they created classroom environments where students could choose for themselves ways of participating that felt comfortable and reflected their own preferences. Within whole-group settings a student might explain their strategy to the class, interject with a question, make a prediction about another's idea, offer a new strategy, or simply listen and observe while privately sharing thoughts with their partner during turn and talk. During problem solving work partners might work together to develop strategies, solve in parallel, or appropriate strategies from each other through explaining and engaging with each other. Rather than assigning particular roles to individual students, students were free to choose from and take up a variety of roles across settings. To be clear, it is not that any and every way of participating was deemed acceptable; students were expected and supported to engage in substantive mathematical work together. But in doing so they were offered space to negotiate what it would mean for individuals to take up a variety of disciplinary practices in communicating their thinking, critiquing the reasoning of others, and making sense of mathematics (NGA & CCSSO, 2010). In this way Ms. J's and Ms. C's classrooms present an alternative conception of classroom norms. Rather than establishing norms with the goal of supporting all students to participate in similar ways, the norms in these two classrooms allowed for each student to find their own productive spaces for participating.

Many investigations of competent participation would investigate breaches in classroom norms as a way of understanding what counts as competence. However, such instances were exceedingly rare on the days observed. One might hypothesize that this was due in part to the time of year the classrooms were filmed, but in many classrooms challenges to normative ways of participating are readily observed throughout the school year. Ms. J's and Ms. C's classrooms

were successful in blurring the edges of what participation could look, sound, and feel like. What might be viewed as breaches in many classrooms—students’ incorrect or incomplete strategies, their interjections into classroom conversations, the tensions that emerged as they attempted to explain and engage with each other during pair work—were framed positively, and taken up by teachers as productive attempts to participate. Thus, rather than reading as norm violations, these ongoing negotiations instead took the form of teachers supporting and building on what students were already doing.

An expanded view of competence also played a part in shifting the distribution of power in the two classrooms. In crafting classrooms where students initiated engagement with each other, where individuals were consistently positioned as sense-makers and as authors of ideas, the teachers relinquished some forms of power and control typically wielded in classrooms, even in classrooms that attempt to adopt reform-oriented approaches to instruction. Students assumed responsibility for helping each other, exercised voice in determining whether or not conversations moved forward, and were given space to negotiate what it would mean to collaborate with each other as they worked. In supporting students to explain their ideas and to engage with each other, Ms. J and Ms. C were not just supporting students to learn mathematics, they were pushing back on traditional systems of authority and power.

Children’s thinking as an entry point into expanding competence

The ways in which both classrooms consistently framed students’ efforts to participate and solve problems as reflecting understanding was critical in creating opportunities for students to take up their space. Both teachers demonstrated an in-depth understanding of children’s mathematical thinking, but this extended beyond simply viewing children’s strategies in relation

to a developmental trajectory. Teachers' responses consistently positioned students' emerging strategies as reflecting mathematical understanding (not misconceptions), and students' varied ways of participating as valid attempts to contribute to the mathematical work of the classroom. Inquiring into students' thinking and participation was a way to, in Ms. C's words, "understand more about what (students) *can* do, and what they *do* know." This disposition—to view children's emergent strategies and efforts to participate as reflecting understanding—carries implications for teacher educators. As a common focus of teacher preparation and professional development, learning about student thinking can provide opportunities to move beyond procedural views of mathematics, and for alternative ways of solving problems to be viewed as "right." However, in focusing on identifying strategies, these programs may underemphasize the productive ideas within children's incorrect or incomplete strategies, or the underlying mathematical threads that connect children's early strategies to those that are more abstract and sophisticated. The cases of Ms. J and Ms. C suggest that supporting teachers to develop dispositions that attend to and build from students' "partial understandings" presents a promising focus for professional development, one that potentially integrates teachers' engagement in the learning and teaching of mathematics with a focus on fostering of asset-based perspectives of students (Battey & Franke, 2015).

Teachers' instructional practices also served to distribute authority among students, and to promote a view of mathematics learning that emphasized the importance of communicating one's ideas and of working toward understanding the ideas of others. Of note are three particular categories of moves: 1) inviting students to explain another student's strategy for the group, 2) inviting students to add on to another student's strategy by making a prediction, and 3) returning the floor to the original author of a strategy after it had been interpreted by peers. These moves

served to support students to engage with each other's ideas while also ensuring that individual student's ideas were represented faithfully. It can be challenging for teachers to orchestrate interactions in which multiple ideas are pursued without privileging certain students' contributions above others, or misrepresenting a student's idea once it has been shared with the group. These particular instructional practices offer a promising line of future inquiry for researchers and teacher educators concerned with understanding how distributions of intellectual authority relate to students' mathematics learning and their emergent mathematical identities (Langer-Osuna, 2017). And while the examples provided by Ms. J's and Ms. C's classroom illustrate the potential of these moves to support students' participation in collaborative mathematics, it would seem likely that many of the affordances of these practices would transcend disciplinary boundaries.

The social construction of competence

This study contributes to an emerging research base that views competence as a socially constructed phenomenon (Gresalfi et al., 2009; Hand, 2010; Jackson, 2009). Specifically, this study extends previous work by connecting examinations of competence with the details of teachers' instructional practices and the development of children's mathematical thinking. Rather than comparing notions of competence across "traditional" and "reform" classrooms, this study provides vivid illustrations of different dimensions of competence in two classrooms that took up current recommendations for ambitious mathematics teaching. The two classrooms highlight the affordances of explanation and engagement with each other's mathematical ideas for expanding collective notions of competence.

The cases of Ms. J's and Ms. C's classrooms illustrate the potential of competence as an analytical lens in understanding learning and teaching in classrooms. In many studies, analyses of classroom activity focus on particular features such as the use of cognitively demanding tasks, organizational structures for cooperative learning, or the use of specific "talk moves" in facilitating classroom discussions. This work has been invaluable in identifying critical components of learning environments, and provided researchers and practitioners with influential frameworks (e.g., Chapin et al., 2009; Cohen & Lotan, 2014; Stein et al., 2009). Generally speaking, however, in their specificity these analyses have tended to underemphasize the mutually constitutive nature of learning, teaching, content, and identity (Franke et al., 2007). Analyzing Ms. J's and Ms. C's classrooms from the perspective of only instructional tasks, organizational structure, or teaching moves would fail to capture what it was that made these two classrooms successful in supporting student learning. For example, an analysis of instructional tasks and their implementation would find that many of the problems solved by students would initially be classified as low-level, and that teachers often provided minimal support in the "launch" of these tasks (Jackson et al., 2013). Such an analysis would fail to capture that when enacted within the two classrooms' systems of competence, the cognitive demand of tasks was transformed; routine tasks were elevated to function as complex, challenging tasks (Johnson, in preparation). In contrast, the multidimensional analysis of competence presented in this study attends to its relational nature; the function of instructional tasks, organizational structures, student participation, teacher practice, and mathematics learning are understood in connection with each other.

The findings of this study also contribute to ongoing discussions in mathematics education of the nature of explicitness in teaching (e.g., Ball, Mann, & Shaughnessy, 2015;

Parks, 2010; Selling, 2016). The teachers in this study did not demonstrate procedures for solving problems to students, nor require that students master basic facts and skills before moving on to more complex mathematics. However, students were not left to flounder aimlessly when encountering struggle, wondering what they were expected to do or how they were expected to do it. Students were actively supported to make the details of their thinking explicit to the teacher and to each other. Ms. J and Ms. C helped students to know what it would mean to participate together in solving problems, to communicate their thinking, and to represent their ideas mathematically. While teachers sometimes provided explicit instructions for how students were expected to participate, clear directions and normative statements were not enough. Broadening competent ways of participating required that teachers vary their in-the-moment support as students took up opportunities to participate. This involved following up on what students said to support them to detail their thinking, scaffolding their engagement with others' ideas, and explicitly assigning competence to particular contributions and ways of participating. Making the rules of the game (in this case, formal "school" mathematics) explicit is but one aspect of supporting nondominant students' access to the culture of power (Delpit, 1988), but it is one that scholars in mathematics education have struggled to grapple with. The classrooms detailed in this study supported students to successfully take up a variety of mathematical practices, while at the same time creating space for students to adapt and make meaning of these practices in their own image.

A socially constructed, relational view of competence is consistent with recent research that argues that understanding students' opportunities to learn requires attending to the complex ways that students' emerging mathematical identities are intertwined with their opportunities to "take up their space" (Hand, 2012). This perspective, and the findings presented in this study,

suggests that as researchers and practitioners continue to work toward identifying and implementing equitable teaching practices (e.g., Bartell et al., 2017; McDuffie et al., 2014; NCTM, 2014), no practice in and of itself is inherently equitable—it is how these practices play out in relation to students’ uptake of opportunities to participate, and how these opportunities relate to who students are and who they want to become, that ultimately determine who is viewed as successful, and whose voices and ideas are given prominence in educational spaces. For example, in investigating the practice of assigning competence, researchers might inquire: Under what conditions is competence assigned? To whom and for what? In relation to what normative ways of knowing and participating? And by what outcomes is equity (or inequity) determined? Relationships between teachers, students, and researchers, and our understanding of each other’s stories and histories, filter the ways classroom activity is interpreted. Expanding what counts as mathematical competence is but one aspect of disrupting the systems and practices that structure inequities into and outside of schools.

Limitations and future directions

This study has focused on classrooms as the primary unit of analysis, rather than individual students. Such a study represents only a first step towards understanding how classrooms structure individual student’s opportunities to learn. Such a step is necessary, however, in making sense of the ways in which individual’s attempts to take up opportunities are situated in the immediate learning environment, and how varied conceptions of disciplinary practices may open or constrain particular ways of knowing, participating, or demonstrating understanding. Future work could examine connections between local constructions of

competence, individual student's efforts to participate, and students' own voices in reflecting on their experiences in doing mathematics across classrooms, and in and out of school.

In focusing on the construction of competence at a classroom level, the analyses presented have only touched on how notions of competence within Ms. J's and Ms. C's classroom might compare with those that students encountered elsewhere. Analyses of competence might be extended to examine these local negotiations of meaning in relation to those taking place within the larger sociopolitical structures of schooling and society. The classrooms studied here were fairly unique within their school; many of Ms. J and Ms. C's colleagues tended to favor traditional, didactic approaches to mathematics teaching. In interviews and in conversations with the research team, Ms. J and Ms. C shared concerns about whether their students' natural strategies and varied ways of participating would be cast aside in subsequent years. In this way competence was also being contested at a school level, and one can suppose, within the surrounding community as families and school leaders worked to implement current reforms related to standards and assessment (e.g., CCSSM). Future work that considers the social construction of competence across multiple levels of analysis could inform understanding of how classrooms, schools, policy, and broader society collectively function in shaping young people's opportunities to learn.

Finally, this study has not explicitly taken up investigations of the ways in which issues of race, class, or gender shape and are shaped by classroom interactions. This study does, however, illustrate possibilities for how classrooms might be reorganized to better take up the varied intellectual resources children bring with them to school—their understanding of everyday situations and their intuitive strategies for solving problems—and connect them with the doing of formal mathematics in school. Such illustrations are consistent with design

principles outlined by Nasir and colleagues for creating learning environments which 1) make the structure of the domain visible, 2) engage students in discourse through which they can identify themselves within the practices of the domain, and 3) help students to see relationships between their tacit, everyday knowledge and discourse, and academic knowledge and discourse (Nasir, Rosebery, Warren, & Lee, 2006). Future work could investigate relationships between students' informal ways of knowing and doing mathematics outside of school, and the ways that different classrooms draw from or neglect these resources.

CONCLUSION

Learning and teaching in classrooms is complex. Discussions of “good teaching” have often neglected the ways in which students’ uptake of opportunities to learn is dependent upon the ways of participating made available to them, and how these opportunities to “take up space” relate to who students are and who they hope to become. I have argued that examining competence within classrooms sheds light on the ways that participation, positionings, and student thinking mutually shape what it means to know and do mathematics, and in turn what it means to do school.

The benefits of creating classrooms in which students learn to explain their ideas, and to make sense of and engage with the ideas of others extend beyond content learning, and even beyond supporting individuals to see themselves as valued participants in classroom communities. Engagement with each other as essential component of doing school creates an avenue for children to share in responsibility for each other’s success, and to see themselves as participants in democratic processes. Reimagining classrooms as dialogic spaces—where multiple voices and a diversity of ideas are valued, and where difference, disagreement, and uncertainty are negotiated productively—offers a way for students and teachers to begin to understand the world through the eyes, ears, bodies, and histories of one another.

APPENDICES

APPENDIX A

Questions to guide semi-structured interviews

Round 1 (Spring, 2016)

1. Can you talk a little about what you think is important in supporting students to learn math?
2. What relationships do you see between your understanding of the development of children's thinking and the decisions you make as a teacher? (How does knowing about children's thinking help you?)
3. What does it mean for students to be good at math in your classroom? In what ways do you think this might be similar to or different from students' experiences in your colleagues' classrooms? Why? (Are there ways to be good at math in your classroom besides being able to solve problems? What kinds of things?)
4. What relationships do you see between your understanding of children's thinking and what it means for students to be viewed as successful in math?
5. One of the things I've noticed on the days that we filmed is that you sometimes praise or thank students in specific ways, but that these are not necessarily related to the mathematics that the students has done. Can you talk at all about the kinds of things you try to explicitly recognize students for, and why?

APPENDIX A (continued)

Round 2 (Fall, 2016)

1. One way the students participate in your classroom is through providing explanations. Can you talk about some of the ways you approach supporting students to explain their ideas?
 - Do you find that most students are able to participate in this way? If not, what do you do?
 - If you think about a student who was successful in explaining their ideas, how did you support them? What about somebody who struggled? What would you do to support them and why?
2. Can you talk about some of the ways you approach supporting students to engage with each other's ideas? Some of what you've tried and some of what learned to be effective?
3. One of the things that can be challenging for teachers is to negotiate multiple instructional goals, especially in classrooms where students use a wide range of strategies. I'm curious about how you think about your goals. Do you have different goals at different times of the year? Or different days? Are there things that stay the same?
 - Possible follow up: do you have different goals for different parts of a lesson? (e.g., warm-up vs. partner work vs. share out)
4. Another thing I'm curious about are times when errors or mistakes surface within the whole group? Are there particular ways that you would help people to think about that would be important in sharing errors/mistakes in the whole group?

APPENDIX B

Coding classroom interactions in Studiocode

<i>Dimension</i>	<i>Sub Code</i>	<i>Label</i>	<i>Description</i>
Multidimensional mathematics	Varied ways of participating	Providing explanation	Interactions in which students participate by providing explanations of their mathematical ideas or strategies
		Engaging in another's idea	Interactions in which students participate by engaging with another student's mathematical idea
		Drawing on context (to make sense of the mathematics)	Interactions in which students participate by working to understand the relationship between a story situation and a corresponding mathematical representation; or how their understanding of a story context can be used to generate a solution strategy
		Using varied representations/tools	Interactions in which students participate by using a variety of representations and/or tools to reflect mathematical ideas.
		Using varied strategies	Interactions in which students participate by using multiple or different strategies
		Helping others	Interactions in which students participate by taking responsibility for or helping another student to make sense of mathematics
		Trying out a new idea	Interactions in which students participate by extending their thinking in the service of a new or more sophisticated strategy or idea
		Voicing confusion	Interactions in which students participate by voicing confusion about the mathematics under discussion
		Working independently	Interactions in which independent work is constructed as a desirable way of participating
	Working together	Interactions in which solving problems together is constructed as a desirable way of participating	

Sociomathematical participation	Difference	Interactions in which what counts as mathematically different is negotiated
	Sophistication	Interactions in which what counts as sophisticated or efficient are negotiated
	Valid representation	Interactions in which what counts as an acceptable way of representing a mathematical idea is constructed/negotiated
	Valid justification	Episodes in which what counts as acceptable justification or proof are negotiated
	Correctness	Episodes in which what determines the correctness of a particular answer, strategy, or approach is negotiated

Taking up space

Supporting dialogic space	Student initiated bid for the floor	Instances in which a student calls out or otherwise initiates a bid to contribute or take the floor in whole-group conversation
	Return authority	Instances in which the teacher intentionally checks back with or returns the floor to the original author of an idea or strategy after it has been engaged with by others to make a determination about whether or not their idea has been interpreted as intended by the author
	One controls or dominates	Interactions in which one student's attempts to participate are limited or otherwise controlled by another
	Disciplinary agency	Interactions for which agency in deciding methods for solving problems is granted to an outside authority, such as the teacher or textbook
	Conceptual agency	Interactions for which agency for sense-making and decisions about how to solve problems is exercised by students (Pickering, as discussed by Greeno, 2011 and; Gresalfi et al., 2009)
Blurring distinctions	Informal language	Interactions in which students are encouraged to draw on informal language, registers, gestures, or their home language in order to communicate mathematical ideas
	Partial understanding	Interactions in which an incomplete, incorrect, or otherwise “in-progress” idea surfaces and is explored. Includes interactions in which the seeds of more sophisticated idea are recognized and built from
Systems of math education and schooling		Instances in which the mathematical work under discussion is connected with or referenced in relation to the systems or institutional structures of schooling (Hand, 2012)

Teacher support of
student
participation

Invitation moves	Explain	Invitational move for student(s) to explain their idea or to T invitation to explain could be to explain what you did or to explain what another did
	Add on	Invitational move for student(s) to add on to another's idea. Includes inviting students to make a prediction about what another student might do next in their strategy
	Agree/Disagree	Invitational move for student(s) to voice agreement or disagreement with an idea stated by a classmate
	Compare/Connect	Invitation to comparing the details of two or more strategies or representations, noting similarities, differences, or connections
Follow-up/support moves	Probe	Follow-up move to probe the details of a student's idea or to press for further explanation or justification (Kazemi & Stipek, 2001).
	Scaffold	Follow-up move to scaffold a student's explanation or sense-making, where the teacher takes over a portion of the mathematical work
	Position	A student's idea or way of participating is positioned positively by the teacher, a student's strategy or idea is positioned in relation to another strategy or idea (Franke et al., 2015)
	Revoice	The teacher restates or rephrases a student's idea to support other students to make sense of and engage with the idea, or to highlight or elaborate the mathematics within a student's idea (Chapin et al., 2009)
	Explicit assignment of competence	Instance in which the teacher explicitly praises, thanks, or otherwise deems a particular student's contribution or way of participating as productive or desirable
	Normative statement	Instance in which the teacher makes explicit statement about her expectations for normative practices in doing mathematics

Children's
mathematical
thinking

Direct modeling with ones	Strategies for solving multidigit multiplication and division problems that model the action or relationship within the story situation, and in which each quantity is represented in terms of individual units.
Direct modeling with tens	Strategies for solving multidigit multiplication and division problems that model the action or relationship within the story situation, and in which each quantity is represented using units of tens and ones.
Counting/adding	Strategies for solving multidigit multiplication and division problems in which each group is represented, but the amount in each group is abstracted (often through a numerical representation).
Invented algorithms	Strategies for solving multidigit multiplication and division problems in which both the number of group and the amount in each group is abstracted. These strategies often involve the strategic decomposition of numbers to perform calculations, or the creation of new units (e.g., groups of groups) which are then operated upon.
Non-valid	Strategies that do not represent the problem situation and cannot be used to accurately solve the problem
Other valid	Other valid strategies used to solve problems not encompassed by the above strategies (usually used during warm-up activities that did not involve multiplication or division)

Explanation and engagement with others' ideas (detailed)

Explanation	Complete and correct	Instances where a complete, fully detailed explanation is provided (can be by an individual student or collectively by several students across an interaction)
Engagement with others' ideas	Add on	Instances where students engage with another's idea by adding on to the idea under discussion
	Challenge	Instances where students engage with another's idea by disagreeing or challenging a detail of the other student's strategy or idea
	Explain another's idea	Instances where students engage with another's idea by explaining their strategy
	Ask questions	Instances where students engage with another's idea by asking a specific question about the details of another's strategy
	Compare/Connect strategies	Instances where students engage with another's idea by comparing the details of two or more strategies or representations, noting similarities, differences, or connections

APPENDIX C

Time and episodes across lesson phases

Ms. J's Classroom						
	<i>Warm-Up</i>	<i>Launch^a</i>	<i>Solving</i>	<i>Share out</i>	<i>Independent</i>	<i>Total</i>
January 22	18 min 7 episodes	5 min 0 episodes	17 min 5 episodes	13 min 2 episodes	0 min 0 episodes	58 min ^b 14 episodes
January 24	21 min 21 episodes	5 min 0 episodes	27 min 7 episodes	5 min 1 episode	0 min 0 episodes	59 min 29 episodes
March 26	19 min 6 episodes	2 min 0 episodes	23 min 7 episodes	14 min 2 episodes	0 min 0 episodes	60 min 15 episodes
May 28	26 min 11 episodes	1 min 0 episodes	36 min 8 episodes	15 min 2 episodes	12 min 3 episodes	92 min 24 episodes
Total	84 min 45 episodes	13 min 0 episodes	103 min 27 episodes	47 min 7 episodes	12 min 3 episodes	269 min 82 episodes
Mean	21 min 11.25 episodes	3 min 0 episodes	26 min 6.75 episodes	12 min 1.75 episodes	3 min .75 episodes	67 min 20.5 episodes

^a The lack of episodes during the launch is an artifact of coding episodes in terms of interactions involving students' strategies. Launching the problem involved supporting students to understand and engage with the problem, not the generating or reporting of strategies.

^b Times are rounded to the nearest minute. Total time is greater than sum of lesson phases due to the inclusion of transition times (for example, on January 22 transitions in Ms. J's classroom included 45 seconds between the warm up and launch, 1 minute between the launch and problem solving, and 2 minutes between problem solving and sharing strategies). Transition times were typically short.

APPENDIX C (continued)

Ms. C's Classroom						
	<i>Warm-Up</i>	<i>Launch</i>	<i>Solving</i>	<i>Share out</i>	<i>Independent</i>	<u>Total</u>
January 22	6 min 4 episodes	4 min 0 episodes	30 min ^c 10 episodes	18 min 2 episodes	9 min 5 episodes	71 min 21 episodes
January 24	9 min 11 episodes	1 min 0 episodes	28 min 4 episodes	14 min 2 episodes	6 min 1 episodes	58 min 18 episodes
March 26	18 min 4 episodes	4 min 0 episodes	21 min 5 episodes	14 min 3 episodes	6 min 0 episodes	63 min 12 episodes
May 28	18 min 5 episodes	1 min 0 episodes	18 min 6 episodes	16 min 3 episodes	1 min 0 episodes	56 min 14 episodes
Total	51 min 24 episodes	10 min 0 episodes	97 min 25 episodes	62 min 10 episodes	22 min 6 episodes	248 min 65 episodes
Mean	13 min 6 episodes	3 min 0 episodes	24 min 6.25 episodes	16 min 2.5 episodes	6 min 1.5 episodes	62 min 16.25 episodes

^c On January 22 problem solving was split between independent and collaborative time with partners. Students were first asked to attempt to solve the problem independently, and later directed to explain their strategies to their work partner and to continue to work together from that point forward. On all other days problem solving consisted entirely of collaborative work time in pairs.

APPENDIX D

Warm-Up Activities

	<u>Ms. J's Classroom</u>	<u>Ms. C's Classroom</u>
January 22	<ul style="list-style-type: none">• True/False Number Sentence $7 \times 10 = (4 \times 10) + (4 \times 10)$	<ul style="list-style-type: none">• Open Number Sentence $(10 \times 7) = (10 \times 2) + (10 \times \underline{\quad})$
January 24	<ul style="list-style-type: none">• How many ways can you decompose 15 or 150? (students' choice)	<ul style="list-style-type: none">• How many ways can you decompose 150?
March 26	<ul style="list-style-type: none">• Mental Math 6×21	<ul style="list-style-type: none">• Mental Math 6×21
May 28	<ul style="list-style-type: none">• Brief review of Choral Count from previous day and predict next numbers• Open Number Sentences<ol style="list-style-type: none">1. $75 \text{ min} = 1 \text{ hr } \underline{\quad} \text{ min}$2. $2 \text{ hr } 45 \text{ min} = \underline{\quad} \text{ min}$3. $5 \text{ hr } 30 \text{ min} = \underline{\quad} \text{ min}$	<ul style="list-style-type: none">• Open Number Sentences<ol style="list-style-type: none">1. $75 \text{ min} = 1 \text{ hr } \underline{\quad} \text{ min}$2. $2 \text{ hr } 45 \text{ min} = \underline{\quad} \text{ min}$3. $5 \text{ hr} = \underline{\quad} \text{ min}$

APPENDIX E

Story Problems and Number Sets

	<u>Ms. J's Classroom</u>	<u>Ms. C's Classroom</u>
January 22	<p>Katherine went on a trip to the beach. At the end of ____ days, she had collected ____ seashells. If she collected the same number of seashells each day, how many seashells did she collect each day? [Partitive Division]</p> <p>(2, 70) (3, 75) (3, 96)</p>	<p>Katherine went on a trip to the beach. At the end of ____ days, she had collected ____ seashells. If she collected the same number of seashells each day, how many seashells did she collect each day? [Partitive Division]</p> <p>(2, 70) (3, 75) (3, 96)</p>
January 24	<p>Neffy's family drove ____ miles over a week. If they divided the trip up equally over ____ days, how many miles did they drive each day? [Partitive Division]</p> <p>(93, 3) (105, 5) (150, 3)</p>	<p>Xitlaly's family drove ____ miles over the week. If they drove the same amount each day over ____ days, how many miles did they drive each day? [Partitive Division]</p> <p>(93, 3) (105, 5) (150, 3)</p>
March 26	<p>____ students in room 28 raised ____ dollars each for the jog-a-thon. How many dollars did those students raise in all? [Multiplication]</p> <p>(6, 15) (4, 32) (6, 55)</p>	<p>____ students in room 18 raised ____ dollars each for the jog-a-thon. How many dollars did those students raise in all? [Multiplication]</p> <p>(6, 15) (4, 32) (6, 55)</p>
May 28	<p>Leo is allowed to watch TV for ____ minutes a day. How many minutes of TV can he watch in ____ days? [Multiplication]</p> <p>(15, 8) (16, 6) (26, 4)</p>	<p>Gerardo is allowed to watch TV for ____ minutes a day. How many minutes of TV can he watch in ____ days? [Multiplication]</p> <p>(15, 8) (16, 6) (26, 4)</p>

APPENDIX F^d

Results from end-of-year problem solving assessment and previous year (grade 2) CST data

	End-of-year problem solving assessment				Grade 2 CST			
	<i>Mean (adjusted)</i>	<i>SD (adjusted)</i>	<i>Median</i>	<i>Mode</i>	<i>Mean</i>	<i>SD</i>	<i>Median</i>	<i>Mode</i>
Ms. J's classroom n=23	3.05 (2.74)	1.20 (.350)	4	4	365	56.3	345	345
Ms. C's classroom n=21	3.00 (3.15)	1.35 (.412)	4	4	394	57.9	399	339
Students classified as English learners n=25	3.24 (2.96)	1.01 (.356)	4	4	391	65.2	391	339
Overall n=44	3.02	1.27	4	4	380	58.3	369	339

^d The data presented in appendix 6 does not include one student from Ms. J's classroom, for whom CST data was not available.

APPENDIX F (continued)

Predictors of end-of-year problem solving assessment (Poisson loglinear regression)

Predictor	Coefficient (<i>B</i>)	Standard Error	Wald Chi-square (df = 1)	<i>p</i>
Grade 2 CST*	.004	.0015	7.965	.005
Classroom	.141	.185	.575	.448
Language status	.015	.191	.006	.936

*statistically significant for $p < .05$

APPENDIX G

Comparable CST and NAEP multidigit multiplication and division items

Problem	Source	Difficulty	Percent correct responses
<p>Third-grade students went to a concert in 8 buses. Each bus took 45 students. How many students went to the concert?</p> <p>A 320 B 360 C 380 D 3240</p>	Grade 3 CST (2009)	Advanced	66%
<p>Mr. Brown bought 6 towels. All the towels were the same price. The total cost was \$84. How much money did each towel cost?</p> <p>A \$11 B \$14 C \$78 D \$504</p>	Grade 3 CST (2009)	Proficient	53%
<p>A company has 6 big trucks. Each truck has 18 wheels. How many wheels is this in all?</p> <p>A 24 B 96 C 108 D 116</p>	Grade 3 CST (2009)	Proficient	53%
<p>Jeb paid \$72 for a magazine subscription. If he is paying \$4 for each issue of the magazine, how many issues of the magazine will he receive?</p> <p>A 18 B 20 C 22 D 24</p>	Grade 4 CST (2009)	Proficient	67%
<p>Patty expects that each tomato plant in her garden will bear 24 tomatoes. If there are 6 tomato plants in her garden, how many tomatoes does she expect? <i>(calculator available)</i></p> <p>A 4 B 18 C 30 D 144</p>	Grade 4 NAEP (2011)	Medium	53%

REFERENCES

- Ambrose, R., Baek, J. M., & Carpenter, T. P. (2003). Children's invention of multidigit multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructive adaptive expertise* (pp. 305–336). Mahwah, NJ: Lawrence Erlbaum Associates.
- Anyon, J. (2014). *Radical possibilities: Public policy, urban education, and a new social movement*. New York, NY: Routledge.
- Ball, D. L., Mann, L., & Shaughnessy, M. M. (2015). Explicitness in creating access for all students to complex mathematics. Presented at the Annual Meeting of the American Educational Research Association, Chicago, IL.
- Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, 48(1), 7–21.
<https://doi.org/10.5951/jresmetheduc.48.1.0007>
- Battey, D. (2013). “Good” mathematics teaching for students of color and those in poverty: The importance of relational interactions within instruction. *Educational Studies in Mathematics*, 82(1), 125–144. <https://doi.org/10.1007/s10649-012-9412-z>
- Battey, D., & Franke, M. L. (2015). Integrating professional development on mathematics and equity: Countering deficit views of students of color. *Education and Urban Society*, 13124513497788. <https://doi.org/10.1177/0013124513497788>
- Boaler, J. (2008). Promoting “relational equity” and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal*, 34(2), 167–194. <https://doi.org/10.1080/01411920701532145>
- Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171–200). Westport, CT: Ablex Publishing.
- Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, 110(3), 608–645.
- California Department of Education. (1997). Mathematics content standards for California public schools: Kindergarten through grade twelve. Retrieved from <http://www.cde.ca.gov/be/st/ss/documents/mathstandards.pdf>
- California Department of Education. (2009). STAR sample test questions. Retrieved June 8, 2017, from <http://starsamplequestions.org/starRTQ/search.jsp>
- California Department of Education. (2014). *California Standards Tests technical report: Spring 2013 administration*. Sacramento, CA: CDE Assessment Development and

- Administration Division. Retrieved from <http://www.cde.ca.gov/ta/tg/sr/documents/cst13techrpt.pdf>
- California Department of Education. (2016). CalEdFacts: Standardized testing and reporting. Retrieved June 19, 2017, from <http://www.cde.ca.gov/ta/tg/sr/cefstar.asp>
- Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17–40). Hillsdale, NJ: Erlbaum.
- Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, 24(5), 428–441. <https://doi.org/10.2307/749152>
- Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3–20. <https://doi.org/10.2307/1001789>
- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd edition). Portsmouth, NH: Heinemann.
- Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, 19(5), 385–401. <https://doi.org/10.2307/749173>
- Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loeff, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26(4), 499–531. <https://doi.org/10.3102/00028312026004499>
- Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 29(1), 3–20. <https://doi.org/10.2307/749715>
- Carpenter, T. P., Franke, M. L., Johnson, N. C., Turrou, A. C., & Wager, A. A. (2017). *Young children's mathematics: Cognitively guided instruction in early childhood education*. Portsmouth, NH: Heinemann.
- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.
- Carter, P. L., & Welner, K. G. (Eds.). (2013). *Closing the opportunity gap: What America must do to give every child an even chance*. New York, NY: Oxford University Press.
- Chapin, S. H., O'Connor, C., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn*. Sausalito, CA: Math Solutions.

- Civil, M. (2012). Opportunities to learn in mathematics education: Insights from research with “non-dominant” communities. In T. Y. Tso (Ed.), *36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 43–59). Taipei, Taiwan: PME.
- Civil, M., & Bernier, E. (2006). Exploring images of parental participation in mathematics education: Challenges and possibilities. *Mathematical Thinking and Learning*, 8(3), 309–330. https://doi.org/10.1207/s15327833mtl0803_6
- Cobb, P., & Hodge, L. L. (2011). Culture, identity, and equity in the mathematics classroom. In A. Sfard, K. Gravemeijer, & E. Yackel (Eds.), *A Journey in Mathematics Education Research* (pp. 179–195). Springer Netherlands.
- Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences*, 10(1–2), 113–163. https://doi.org/10.1207/S15327809JLS10-1-2_6
- Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (Third edition). New York, NY: Teachers College Press.
- Darling-Hammond, L. (2007). How America’s commitment to equity will determine our future. *Educational Researcher*, 36(6), 318–334. <https://doi.org/10.3102/0013189X07308253>
- Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people’s children. *Harvard Educational Review*, 58(3), 280–299. <https://doi.org/10.17763/haer.58.3.c43481778r528qw4>
- Diversity in Mathematics Education Center for Learning and Teaching (DiME). (2007). Culture, race, power, and mathematics education. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 405–433). Charlotte, NC: Information Age Publishing.
- Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, 17(3), 283–342.
- Empson, S. B. (2003). Low-performing students and teaching fractions for understanding: An interactional analysis. *Journal for Research in Mathematics Education*, 34(3), 305–343.
- Empson, S. B., & Levi, L. (2011). *Extending children’s mathematics: Fractions and decimals*. Portsmouth, NH: Heinemann.
- Enyedy, N., Rubel, L., Castellón, V., Mukhopadhyay, S., Esmonde, I., & Secada, W. (2008). Revoicing in a multilingual classroom. *Mathematical Thinking and Learning*, 10(2), 134–162.
- Erickson, F. (1992). Ethnographic microanalysis of interaction. In M. D. LeCompte, W. L. Millroy, & J. P. Goetz (Eds.), *The handbook of qualitative research in education* (pp. 201–225). Academic Press.

- Erickson, F. (1996). Going for the zone: The social and cognitive ecology of teacher-student interaction in classroom conversations. In D. Hicks (Ed.), *Discourse, learning, and schooling* (pp. 29–62). New York, NY: Cambridge University Press.
- Erickson, F. (2004). *Talk and social theory: Ecologies of speaking and listening in everyday life*. Cambridge, UK: Polity Press.
- Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 177–191). Mahwah, NJ: Lawrence Erlbaum Associates.
- Esmonde, I. (2009a). Ideas and identities: Supporting equity in cooperative mathematics learning. *Review of Educational Research, 79*(2), 1008–1043.
<https://doi.org/10.3102/0034654309332562>
- Esmonde, I. (2009b). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *Journal of the Learning Sciences, 18*(2), 247–284.
<https://doi.org/10.1080/10508400902797958>
- Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education, 44*(1), 288–315.
- Featherstone, H., Crespo, S., Jilk, L. M., Oslund, J. A., Parks, A. N., & Wood, M. B. (2011). *Smarter together! Collaboration and equity in the elementary math classroom*. Reston, VA: NCTM.
- Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403–434.
<https://doi.org/10.2307/749875>
- Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *High School Journal, 91*(1), 29–42.
- Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653–689.
<https://doi.org/10.3102/00028312038003653>
- Franke, M. L., Ing, M., Johnson, N. C., Turrou, A. C., & Webb, N. M. (in press). Dude, why did you start without me? Fostering engagement with others' mathematical ideas. In C. Chinn, E. Manalo, & Y. Uesaka (Eds.), *Promoting spontaneous use of learning and reasoning strategies: Theory, research, and practice*. New York, NY: Routledge.
- Franke, M. L., & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterizing generative growth. In T. Wood, Nelson, & J. Warfield (Eds.), *Beyond*

- Classical Pedagogy: Teaching Elementary School Mathematics* (pp. 47–74). Mahwah, N.J.: Erlbaum.
- Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 225–256). Charlotte, NC: Information Age Publishing.
- Franke, M. L., Turrou, A. C., Webb, N. M., Ing, M., Wong, J., Shin, N., & Fernandez, C. (2015). Student engagement with others' mathematical ideas: The role of teacher invitation and support moves. *The Elementary School Journal*, *116*(1), 126–148.
<https://doi.org/10.1086/683174>
- Greeno, J. G. (2011). A situative perspective on cognition and learning in interaction. In T. Koschmann (Ed.), *Theories of Learning and Studies of Instructional Practice* (pp. 41–71). New York, NY: Springer New York.
- Greeno, J. G., & MMAP. (1998). The situativity of knowing, learning, and research. *American Psychologist*, *53*(1), 5–26. <https://doi.org/10.1037/0003-066X.53.1.5>
- Gresalfi, M. S., Martin, T., Hand, V. M., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, *70*(1), 49–70.
- Gutiérrez, K. D., Baquedano-López, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind, Culture, and Activity*, *6*(4), 286–303.
- Gutiérrez, K. D., Morales, P. Z., & Martinez, D. C. (2009). Re-mediating Literacy: Culture, Difference, and Learning for Students From Nondominant Communities. *Review of Research in Education*, *33*(1), 212–245. <https://doi.org/10.3102/0091732X08328267>
- Gutiérrez, K. D., Rymes, B., & Larson, J. (1995). Script, counterscript, and underlife in the classroom: James Brown versus Brown v. Board of Education. *Harvard Educational Review*, *65*(3), 445–472.
- Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in Discourse for Mathematics Education* (pp. 17–33). Springer Netherlands.
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68.
- Hall, R., & Stevens, R. (2016). Interaction analysis approaches to knowledge in use. In A. A. DiSessa, M. Levin, & N. J. S. Brown (Eds.), *Knowledge and interaction: a synthetic agenda for the learning sciences* (pp. 72–108). New York, NY: Routledge.
- Hand, V. M. (2010). The co-construction of opposition in a low-track mathematics classroom. *American Educational Research Journal*, *47*(1), 97–132.
<https://doi.org/10.3102/0002831209344216>

- Hand, V. M. (2012). Seeing culture and power in mathematical learning: toward a model of equitable instruction. *Educational Studies in Mathematics*, 80(1–2), 233–247. <https://doi.org/10.1007/s10649-012-9387-9>
- Hand, V. M., Quindel, J., & Esmonde, I. (2010). Status and competence as entry points into discussions of equity in mathematics classrooms. In M. Q. Foote (Ed.), *Mathematics teaching & learning in K-12: Equity and professional development* (pp. 151–166). New York, NY: Palgrave Macmillan.
- Herrnstein, R. J., & Murray, C. (1994). *The bell curve: Intelligence and class structure in American life*. New York: Free Press.
- Howard, T. C. (2010). *Why race and culture matter in schools: Closing the achievement gap in America's classrooms*. New York, NY: Teachers College Press.
- Jackson, K. (2009). The social construction of youth and mathematics: The case of a fifth-grade classroom. In D. B. Martin (Ed.), *Mathematics teaching, learning, and liberation in the lives of Black children* (pp. 175–199). New York, NY: Routledge.
- Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, 44(4), 646–682.
- Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education*, 38(3), 258–288. <https://doi.org/10.2307/30034868>
- Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 169–202.
- Johnson, N. C. (in preparation). *The elevation of cognitive demand*. Unpublished manuscript, University of California, Los Angeles.
- Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *Journal of the Learning Sciences*, 4(1), 39–103. https://doi.org/10.1207/s15327809jls0401_2
- Kazemi, E., & Hintz, A. (2014). *Intentional talk: How to structure and lead productive mathematical discussions*. Portland, ME: Stenhouse.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal*, 102(1), 59–80.
- Ladson-Billings, G. (1997). It doesn't add up: African American students' mathematics achievement. *Journal for Research in Mathematics Education*, 28(6), 697. <https://doi.org/10.2307/749638>

- Ladson-Billings, G. (2006). From the achievement gap to the education debt: Understanding achievement in US schools. *Educational Researcher*, 35(7), 3–12.
- Ladson-Billings, G. (2013). Lack of achievement or loss of opportunity. In P. L. Carter & K. G. Welner (Eds.), *Closing the Opportunity Gap: What America Must Do to Give Every Child an Even Chance* (pp. 11–22). New York, NY: Oxford University Press.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29–63. <https://doi.org/10.3102/00028312027001029>
- Lampert, M. (2003). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.
- Lampert, M., Beasley, H., Ghouseini, H., Kazemi, E., & Franke, M. L. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional Explanations in the Disciplines* (pp. 129–141). Boston, MA: Springer US.
- Langer-Osuna, J. M. (2011). How brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education*, 11(3), 207–225. <https://doi.org/10.1080/14926156.2011.595881>
- Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education*, 48(3), 237–247. <https://doi.org/10.5951/jresmetheduc.48.3.0237>
- Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. New York, NY: Cambridge University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. New York, NY: Cambridge University Press.
- Martin, D. B. (2003). Hidden assumptions and unaddressed questions in mathematics for all rhetoric. *The Mathematics Educator*, 13(2), 7–21.
- Martin, D. B. (2009). Liberating the production of knowledge about African American children and mathematics. In D. B. Martin (Ed.), *Mathematics teaching, learning, and liberation in the lives of Black children* (pp. 3–36). New York, NY: Routledge.
- Martin, D. B. (2015). The collective Black and principles to actions. *Journal of Urban Mathematics Education*, 8(1), 17–23.
- Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, 3(2).

- McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, 32(3), 236–266. <https://doi.org/10.2307/749827>
- McDuffie, A. R., Foote, M. Q., Drake, C., Turner, E. E., Aguirre, J., Bartell, T. G., & Bolson, C. (2014). Use of video analysis to support prospective K-8 teachers' noticing of equitable practices. *Mathematics Teacher Educator*, 2(2), 108–140. <https://doi.org/10.5951/mathteceduc.2.2.0108>
- Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice*, 31(2), 132–141. <https://doi.org/10.1080/00405849209543534>
- Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, 4(2–3), 189–212. https://doi.org/10.1207/S15327833MTL04023_5
- Moses, R. P., & Cobb, C. E. (2001). *Radical equations: Civil rights from Mississippi to the algebra project*. Boston, MA: Beacon Press.
- Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between “cultural” and “domain” knowledge in the mathematics classroom and beyond. *Review of Research in Education*, 32(1), 187–240. <https://doi.org/10.3102/0091732X07308962>
- Nasir, N. S., Rosebery, A. S., Warren, B., & Lee, C. D. (2006). Learning as a cultural process: Achieving equity through diversity. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences*. New York, NY: Cambridge University Press. Retrieved from <http://dx.doi.org/10.1017/CBO9780511816833.030>
- National Center for Education Statistics. (2011). NAEP questions tool. Retrieved June 8, 2017, from <https://nces.ed.gov/nationsreportcard/nqt/>
- National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.
- National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington D.C. Retrieved from corestandards.org
- National Research Council. (2001). *Adding it up: Helping children learn mathematics*. (J. Kilpatrick, J. Swafford, & B. Findell, Eds.). Washington, DC: National Academy Press.
- National Research Council. (2009). *Mathematics learning in early childhood paths toward excellence and equity*. (C. T. Cross, T. A. Woods, & H. A. Schweingruber, Eds.). Washington, DC: National Academies Press.

- National Research Council. (2012). *Education for life and work: Developing transferable knowledge and skills in the 21st century*. Washington, D.C.: National Academies Press. Retrieved from <http://www.nap.edu/catalog/13398>
- Oakes, J. (2008). Keeping track: Structuring equality and inequality in an era of accountability. *Teachers College Record*, 110(3), 700–712.
- O'Connor, C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning, and schooling* (pp. 63–103). New York, NY: Cambridge University Press.
- O'Connor, C., & Michaels, S. (2007). When is dialogue “dialogic”? *Human Development*, 50(5), 275–285.
- Organisation for Economic Co-operation and Development. (2005). *The definition and selection of key competencies* (executive summary). Organisation for Economic Co-operation and Development.
- Parks, A. N. (2010). Explicit versus implicit questioning: Inviting all children to think mathematically. *Teachers College Record*, 112(7), 1871–1896.
- Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teacher’s pedagogical content beliefs in mathematics. *Cognition and Instruction*, 6(1), 1–40.
- Resnick, L. B., Michaels, S., & O'Connor, C. (2010). How (well-structured) talk builds the mind. In D. D. Preiss & R. J. Sternberg (Eds.), *Innovations in educational psychology: Perspectives on learning, teaching and human development* (pp. 163–194). New York, NY: Springer.
- Rosebery, A. S., Ogonowski, M., DiSchino, M., & Warren, B. (2010). “The coat traps all your body heat”: Heterogeneity as fundamental to learning. *Journal of the Learning Sciences*, 19(3), 322–357. <https://doi.org/10.1080/10508406.2010.491752>
- Saldaña, J. (2013). *The Coding Manual for Qualitative Researchers* (Second Edition). Los Angeles, CA: SAGE.
- Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education*, 47(5), 505–551. <https://doi.org/10.5951/jresematheduc.47.5.0505>
- Sfard, A. (1998). On Two Metaphors for Learning and the Dangers of Choosing Just One. *Educational Researcher*, 27(2), 4–13. <https://doi.org/10.3102/0013189X027002004>
- Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). *Implementing standards-based mathematics instruction: A casebook for professional development (Second edition)* (2nd Edition). New York, NY: Teachers College Press.

- Stinson, D. W. (2004). Mathematics as “gate-keeper” (?): Three theoretical perspectives that aim toward empowering all children with a key to the gate. *He Mathematics Educator*, 14(1), 8–18.
- Studiocode Group. (2015). Studiocode (Version 10). Sydney, Australia. Retrieved from <http://www.studiocodegroup.com/products/>
- Taylor, E. V. (2009). The purchasing practice of low-income students: The relationship to mathematical development. *Journal of the Learning Sciences*, 18(3), 370–415.
- Turner, E. E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English learners’ participation in mathematical discussion: Shifting positionings and dynamic identities. *Journal for Research in Mathematics Education*, 44(1), 199–234.
- Turner, E. E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children’s multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, 15(1), 67–82.
- Van de Walle, J. A. (2007). *Elementary and middle school mathematics: Teaching developmentally* (Sixth Edition). Boston, MA: Pearson.
- Wager, A. A. (2014). Noticing children’s participation: Insights into teacher positionality toward equitable mathematics pedagogy. *Journal for Research in Mathematics Education*, 45(3), 312–350.
- Wager, A. A., & Whyte, K. (2013). Young children’s mathematics: Whose home practices are privileged? *Journal of Urban Mathematics Education*, 6(1), 81–95.
- Webb, N. M., Franke, M. L., De, T., Chan, A. G., Freund, D., Shein, P., & Melkonian, D. K. (2009). “Explain to your partner”: teachers’ instructional practices and students’ dialogue in small groups. *Cambridge Journal of Education*, 39(1), 49–70. <https://doi.org/10.1080/03057640802701986>
- Webb, N. M., Franke, M. L., Ing, M., Turrou, A. C., & Johnson, N. C. (2015). Student participation, teacher instructional practices, and the development of mathematical understanding in the elementary classroom. In R. M. Gillies (Ed.), *Collaborative learning: Developments in research and practice* (pp. 47–68). New York, NY: Nova Press.
- Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, N., & Turrou, A. C. (2014). Engaging with others’ mathematical ideas: Interrelationships among student participation, teachers’ instructional practices, and learning. *International Journal of Educational Research*, 63, 79–93. <https://doi.org/10.1016/j.ijer.2013.02.001>
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458–477. <https://doi.org/10.2307/749877>