Title
Boundary Asymptotics for Convex and Strongly Pseudoconvex Domains

Author
Martin, Alec

Publication Date
2021

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Boundary Asymptotics for Convex and Strongly Pseudoconvex Domains

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Alexander Henri Martin

June 2021

Dissertation Committee:

Dr. Bun Wong, Chairperson
Dr. Yat Sun Poon
Dr. Frederick Wilhelm

The Dissertation of Alexander Henri Martin is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

I would like to thank Rachel for her enduring support through this entire endeavor. I would also like to especially thank my advisor, Dr. Bun Wong, for his incredible insight and patience.

ABSTRACT OF THE DISSERTATION

Boundary Asymptotics for Convex and Strongly Pseudoconvex Domains

by

Alexander Henri Martin

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2021
Dr. Bun Wong, Chairperson

We present two results. The first is a converse to a theorem first proved by Wong which says the ratio of intrinsic measures approaches 1 near the boundary of a strongly pseudoconvex domain; we show that for a particular type of domain the boundary is strongly pseudoconvex if the ratio of intrinsic measures approaches 1 near the boundary. The argument is primarily one from Zimmer using the scaling method. What we did is show that the ratio of intrinsic measures is a function which respects this scaling process. Our second contribution was done in an attempt to use one particular step of Huang and Xiao's proof of the S.-Y. Cheng conjecture to settle the Ramadanov conjecture. While unsuccessful in this regard, we were able to make this step more direct and we ultimately show that if the Bergman metric is asymptotically Kähler-Einstein enough near the boundary of a C^∞ strongly pseudoconvex domain Ω then the boundary $\partial\Omega$ must be spherical. This result is of interest on its own but it also provides a more direct proof of the S.-Y. Cheng conjecture and may be used in further work on the Ramadanov conjecture.

Contents

Chapter 1

Overview

There are two main contributions in this work: Theorem 4.6 and Theorem 5.13. Theorem 4.6 is the subject of chapter 4 and Theorem 5.13 is the subject of chapter 5. In this overview we give the meaning of these results relative to the rest of the theory, as well as cover the organizational structure of the current work.

Before talking about Theorem 4.6 we should recall the well-known theorem of Bun Wong [56]:

Theorem 1.1. *Let Ω be a bounded strongly pseudoconvex domain with noncompact automorphism group. Then Ω is biholomorphic to the unit ball.*

Theorem 1.1 has had a significant impact on the field of several complex variables, see [43] for an overview of the many generalizations and different paths stemming from it. Part of Wong's original proof of Theorem 1.1 was Theorem 4.1, stated here in chapter 4. Without getting into too many details, Theorem 4.1 says that a particular intrinsic function approaches 1 near a particular type of boundary point – a point of strong pseudoconvexity.

By intrinsic function we mean a function whose value derives from the complex structure of the domain, i.e. which is invariant under biholomorphisms. As it turns out there are quite a few intrinsic functions in the field of several complex variables.

The intrinsic function which Wong used – the ratio of the intrinsic measures (Definition 3.17) – is closely related to another intrinsic function called the squeezing function which has gathered attention in recent years. In particular Theorem 4.1 has an analogue for the squeezing function. It is worth knowing (see the proof of Example 4.2) that the squeezing function tending to 1 implies the ratio of intrinsic measures tends to 1. There is further discussion of how the twointrinsic functions compare in the beginning of chapter 4.

Zimmer [60] proved a partial converse to the anlogue of Theorem 4.1 with respect to the squeezing function, and he also noted that his proof applies to a wide class of intrinsic functions. We show that the ratio of intrinsic measures is one such function for which Zimmer's proof applies and use it to prove Theorem 4.6, which is Zimmer's converse but stated for the ratio of intrinsic measures instead of the squeezing function. Because of the relationship between squeezing function and ratio of intrinsic measures, Zimmer's result follows from Theorem 4.6.

Although Theorem 4.6 presented here is original, it seems that the same result was arrived at in a very similar manner by Borah and Kar [7] and in particular their referee, stated there as Theorem 1.3. We conclude this portion by mentioning that the converse is only partial because it requires Ω to be convex, which is quite a restrictive assumption.

The other main contribution of this work is Theorem 5.13, which was proven in an attempt at taking on the Ramadanov conjecture (section 5.1). There are actually two related conjectures described in that section – the S.-Y. Cheng conjecture and the Ramadanov conjecture. The work of Fu and Wong [27] shows that the S.-Y. Cheng conjecture would follow from the Ramadanov conjecture.

Both were open until quite recently, but in [31] Huang and Xiao confirmed the S.-Y. Cheng conjecture. Part of their proof involved the work of Fu and Wong. In the present work we investigate their proof of the S.-Y. Cheng conjecture in an attempt to tackle the Ramadanov conjecture. As it turns out, we were unable to settle the Ramadanov conjecture. Instead we obtained a more direct proof of the S.-Y. Cheng conjecture which does not use the result of Fu-Wong connecting the two conjectures, so in a sense we got the opposite of what we set out for. However we still obtained a more direct route to proving the S.-Y. Cheng conjecture and another possible route of settling the Ramadanov conjecture, and this is presented as Theorem 5.13.

In terms of the organization of the current work we start with an overview of the field of several complex variables in chapter 2, introducing the various concepts which permeate the subject. Chapter 3 introduces the more specific features of the subject involving intrinsic metrics and measures and certain properties enjoyed by them. No new material is presented in either chapter 2 or 3.

Chapter 4 introduces the background and proof of Theorem 4.6. As mentioned Theorem 4.6 is a partial converse to Theorem 4.1 of Wong, and an example (due to Fornaess and Wold) is given to show that a full converse is not possible. Our proof, as we will see, pulls heavily from that of Zimmer [60]; it is essentially Zimmer's proof but applied in our setting. Of particular interest is Proposition 4.13 and Theorem 4.6.

Chapter 5 introduces the background behind the S.-Y. Cheng and Ramadanov conjectures, as well as Huang and Xiao's proof of the S.-Y. Cheng conjecture. Along the way is a discussion of biholomorphic invariants used in the proof. That chapter concludes with our proof of Theorem 5.13 and how it provides a more direct proof of the S.-Y. Cheng conjecture.

Chapter 2

Background

The unit interval and the real line are diffeomorphic; this can be accomplished with a scaled version of the tangent function. In higher dimensions, and with a little more work, any convex open subset of $\mathbb{R}^n$ can be shown to be diffeomorphic to $\mathbb{R}^n$ (the proof of this is apparently mathematical folklore – a discussion and proof can be found at [46]). Taking domain to mean nonempty connected open subset of $\mathbb{C}^n$, it is not particularly interesting to study convex domains if we only look at the smooth structure.

The situation is much different in complex analysis. There are holomorphic maps from the unit disc $\mathbb{D}$ to the plane which fill the plane, but any holomorphic map from $\mathbb{C}$ to $\mathbb{D}$ is necessarily bounded and therefore must be constant by Liouville's theorem. Thus the plane and the disc are two distinct convex domains in $\mathbb{C}$, as there is no biholomorphism between them.

This begs the question, a primary motivator for the present work, of how many holomorphically distinct convex domains there are in $\mathbb{C}^n$. In the single variable case we have a complete answer to this question. Recall the Riemann mapping theorem:

Theorem 2.1 (Riemann Mapping Theorem [16], [41])**.** *If* $\Omega \subseteq \mathbb{C}$ *is open, simply connected, and* $\Omega \neq \mathbb{C}$*, then* Ω *is biholomorphic to the disc* $\mathbb{D}$*.*

This remarkable theorem completely answers the question in dimension 1, as any convex set is either the plane itself or is biholomorphic to the disc.

The proof of the Riemann mapping theorem, at least the version found in [41], can be considered geometric in nature and actually provides motivation for two intrinsic metrics discussed in section 3. First we should look at the disc through geometer's eyes.

The disc should be recognized as fundamental to any geometer because it is the disc, under the Poincaré metric ds, which serves as a model for hyperbolic geometry. Explicitly, the infinitesimal metric ds is defined at a point $z \in \mathbb{D}$ as

$$ds_z^2 = \frac{dz d\overline{z}}{(1 - z\overline{z})^2}.$$

The Poincaré distance and metric are invariant under rotations and Möbius transformations of the disc. All automorphisms of the disc are compositions of Möbius transformations and rotations [16], so the Poincaré metric is invariant under automorphisms of the disc. Geometrically, the disc under the Poincaré metric is highly symmetric for this reason. In particular, the curvature is constant.

Because of the Rieman mapping theorem there is not much to investigate in one complex dimension, at least not in the direction we are going. The plane is the plane and any other simply connected domain is equivalent to the disc.

The Riemann mapping theorem fails catastrophically in higher dimensions. Even the two canonical ways to extend the notion of the disc – the ball and the polydisc – are biholomorphically distinct in any dimension higher than 1. This classical result dates back to Poincaré [51] and essentially comes from an algebraic study of the automorphism groups of the two domains, in particular the automorphisms which fix the identity. The ball is detectably more homogeneous than the polydisc.

A full classification of simply connected domains in $\mathbb{C}^n$ as powerful as the Riemann mapping theorem would necessarily be very complicated, to the point that it is considered hopeless. We proceed with the task regardless, seeking any way of classifying particular domains in $\mathbb{C}^n$ up to biholomorphism in the effort to better understand them.

2.1 Notations

We write $B_\epsilon(z)$ to mean

$$B_\epsilon(z) = \{p \in \mathbb{C}^n : |z - p| < \epsilon\} .$$

We use $A \Subset B$ to mean A is compactly contained in B.

If $\Omega \subseteq \mathbb{C}^n$ is a domain, $1 \leq \alpha \leq n$, and $r : \Omega \to \mathbb{C}$ is a C^1 function then we write

$$r_\alpha = \frac{\partial r}{\partial z_\alpha}.$$

Likewise we write

$$r_{\overline{\alpha}} = \frac{\partial r}{\partial \overline{z}_\alpha}.$$

If $D \subseteq \mathbb{C}^m$ is another domain and $f : \Omega \to D$ is a C^1 function then for each $p \in \Omega$ we write

$$df_p$$

to mean the differential map at p, where we are considering $\mathbb{C}^n = \mathbb{R}^{2n}$ and $\mathbb{C}^m = \mathbb{R}^{2m}$.

We say O is $C^{2+\epsilon}$ if there is a defining function ρ (see Definition 2.8) for the domain Ω such that ρ is $C^{2+\epsilon}$ smooth. That is, ρ is C^2 and the double derivatives of ρ are ϵ-Hölder continuous, i.e.

$$|f(x) - f(y)| \leq C\,|x - y|^{\epsilon}$$

where C is a constant and f is any of the second partial derivatives of ρ.

We will be using big O and little o notation, particularly in chapter 5. For completion we define them here.

Definition 2.2. *Let f, g be real valued functions on some domain $\Omega \subseteq \mathbb{C}^n$ and let $p \in \overline{\Omega}$. We say $f = O(g)$ if there is some constant $C > 0$ and some $\epsilon > 0$ so that*

$$|f(z)| \leq Cg(z)$$

for all $z \in \Omega \cap B_{\epsilon}(p)$.

We don't need to define $f = O(g^k)$ for any $k \in \mathbb{N}$ since the above definition applies directly, but we say $f = O(g^{\infty})$ to mean $f = O(g^k)$ for all $k \in \mathbb{N}$.

Definition 2.3. *Let f, g be real valued functions on some domain $\Omega \subseteq \mathbb{C}^n$ and let $p \in \overline{\Omega}$. We say $f = o(g)$ if for every $C > 0$ there is an $\epsilon > 0$ so that*

$$|f(z)| \leq Cg(z)$$

for all $z \in \Omega \cap B_{\epsilon}(p)$.

Finally we want to define what it means for a function f to be $C^k(\overline{\Omega})$ where Ω is a domain and $1 \leq k \leq \infty$. We take it to mean simply that there is a larger domain Ω' such that $\Omega \Subset \Omega'$ and such that f extends to a C^k function on Ω'. We don't need a more technical definition which localizes the boundary because we only use it for domains where the boundary is a smoothly embedded submanifold of $\mathbb{C}^n$.

2.2 Domain of Holomorphy

We start with some examples of domains in $\mathbb{C}^2$ with profoundly different behavior in terms of holomorphic functions.

Example 2.4. *Let $\mathbb{B}_2$ be the unit ball in $\mathbb{C}^2$. Then there is a holomorphic function $f : \mathbb{B}_2 \to \mathbb{C}$ which does not extend beyond the boundary of the ball.*

Proof. For each point $p \in \mathbb{C}^2$ with $\|p\| = 1$ let

$$f_p(z) = \langle z - p, p \rangle.$$

Then $f_p(p) = 0$ and moreover $f_p^{-1}(0)$ is the plane tangent to the sphere at p. Taking $\frac{1}{f_p}$ gives a function holomorphic in the ball but which cannot be defined continuously at p.

Take a sequence of points $\{p_n\}$ which is dense in the sphere. Define the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n f_{p_n}(z)}.$$

Then f is holomorphic in the ball but f cannot be defined continuously at any point in the sphere, so f does not extend beyond the boundary of the ball. $\square$

This example is not particularly new to $\mathbb{C}^2$ as opposed to $\mathbb{C}$: the same trick could be used on any domain in $\mathbb{C}$ to obtain a holomorphic function which does not extend beyond its boundary. The next example, however, has no analogue in $\mathbb{C}$:

Example 2.5. *Let $\Delta \subset \mathbb{C}^2$ be the polydisc $\mathbb{D} \times \mathbb{D}$. Denote by $\frac{1}{2}\Delta$ the scaling of Δ by $\frac{1}{2}$. Let Ω be $\Delta \smallsetminus \overline{\frac{1}{2}\Delta}$. This domain, a kind of higher-dimensional analogue of an annulus, is called Hartogs' domain.*

Let $f : \Omega \to \mathbb{C}$ be any holomorphic function. Then there is an extension $F : \Delta \to \mathbb{C}$ such that F is holomorphic and $F|_\Omega = f$. This is called Hartogs' phenomenon.

Proof. The original proof is due to Hartogs [30], but a proof in English can be found in [40]. □

The fact that some, but not all, domains Ω can be enlarged to a larger domain Ω' in such a way that every holomorphic function on Ω must extend holomorphically to Ω' is a strange and fascinating phenomenon in the theory of several complex variables.

These examples motivate the definition of domain of holomorphy. Essentially a domain of holomorphy is a domain which admits a holomorphic function which does not extend beyond the boundary. What precisely we mean by extend beyond the boundary is a bit technical, mostly to ensure that topological obstructions do not interfere with the ability to extend a function.

Definition 2.6. *We say a domain Ω is a domain of holomorphy if there is no domain $D \subset \mathbb{C}^n$ where $D \smallsetminus \Omega \neq \varnothing$ which admits an open set $U \subseteq \Omega \cap D$ with the property that any holomorphic function $f : \Omega \to \mathbb{C}$ admits an $F : D \to \mathbb{C}$ which is holomorphic and which has $F|_U = f|_U$.*

Historically, the study of several complex variables has been more or less the endeavour to understand domains of holomorphy. The question is about extending holomorphic functions beyond the boundary, so naturally the boundary of a domain is very important to the study of domains of holomorphy.

2.3 Pseudoconvexity

Convex domains have nice properties; we will get into the particulars later. However, convexity is not a notion which respects complex structure. It possible for two domains to be biholomorphic where one is convex and the other is not. This is in particular a consequence of the Riemann mapping theorem. Luckily there is a complex analytic version of convexity which does respect complex structure: pseudoconvexity.

Definition 2.7. *We say a domain $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex, sometimes specified as Hartogs pseudoconvex, if there is a continuous plurisubharmonic exhaustion function $\phi : \Omega \to \mathbb{R}$.*

We do not delve deeper because we will not need this version of pseudoconvexity; details can be found in Krantz [40]. Suffice it to say that Hartogs pseudoconvexity is a biholomorphic invariant. It is not too difficult to show that a domain of holomorphy is Hartogs pseudoconvex. It turns out the converse holds as well, though this is much more difficult. See the discussion after Theorem 2.10.

There is another form of pseudoconvexity, Levi pseudoconvexity, which applies to C^2 domains and for which the inspiration from convexity is more clear. Details can be found in Krantz [42] and [40] as to how Levi pseudoconvexity derives from convexity. Levi pseudoconvexity requires the concept of a defining function:

Definition 2.8. *Suppose Ω is a C^1 domain and let ρ be a defining function for Ω. That is, ρ is a C^1 function from $\mathbb{C}^n$ to $\mathbb{R}$ with*

- $\rho(\Omega) > 0$

- $\rho\left(\overline{\Omega}^c\right) < 0$

- $\forall x \in \partial\Omega, \nabla\rho_x \neq 0.$

Note that nonvanishing of the gradient of ρ at $\partial\Omega$ means that $\partial\Omega$ is a C^1 embedded submanifold of $\mathbb{C}^n$. If ρ can be chosen to be C^k smooth for some $1 \leq k \leq \infty$ then we say Ω is a C^k domain.

For a C^2 domain Ω and a point $p \in \partial\Omega$ we define whether p is a point of (Levi) pseudoconvexity or not as follows: let w be a holomorphic tangent vector to $\partial\Omega$ at p. That is, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$ and

$$\sum_{k=1}^{n} \left(\frac{\partial\rho}{\partial z_k}\right)_p w_k = 0.$$

Definition 2.9. *We say p is a point of (Levi) pseudoconvexity if for all holomorphic tangent vectors w we have*

$$\sum_{j,k=1}^{n} -\left(\frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}\right)_p w_j \overline{w_k} \geq 0.$$

If all $p \in \partial\Omega$ are points of pseudoconvexity then we say Ω is pseudoconvex. Levi pseudoconvexity and Hartogs pseudoconvexity are equivalent for C^2 domains. There are in fact several concepts which turn out to be equivalent for a domain Ω to be pseudoconvex, these are given later in Theorem 2.10.

2.4 Other Statements for Pseudoconvexity

We mention some more holomorphic interpretations of geometric convexity in this section and then state Theorem 2.10 which says all these concepts are equivalent.

Suppose $\Omega \subseteq \mathbb{C}^n$ is a domain. Let $K \subseteq \Omega$. Denote $\hat{K}$ to be all those $x \in \Omega$ such that for all holomorphic $f : \Omega \to \mathbb{C}$ we have

$$|f(x)| \leq \sup_{z \in K} |f(z)|.$$

If $\hat{K}$ is compactly contained in Ω whenever K is compactly contained in Ω then we say Ω is holomorphically convex. This is a holomorphic variant of the real case where a domain is convex if and only if it is convex with respect to the family of real-valued affine linear functions.

Another way we can obtain a holomorphic version of the definition of convexity is by looking at the analytic version of line segments on the unit ball. Suppose we have a nonconstant holomorphic map $\mathfrak{d} : \mathbb{D} \to \mathbb{C}^n$. We call $\mathfrak{d}$ an analytic disc and we use $\mathfrak{d}$ to refer to both the map $\mathfrak{d}$ and the image $\mathfrak{d}(\mathbb{D})$. If $\mathfrak{d} : \mathbb{D} \to \mathbb{C}^n$ extends continuously to $\overline{\mathbb{D}}$ then we say $\mathfrak{d}$ is a closed analytic disc and we call $\mathfrak{d}(\partial\mathbb{D})$ the boundary of $\mathfrak{d}$.

Suppose Ω is a domain and we have a family $\{\mathfrak{d}_\alpha\}_{\alpha \in A}$ of closed analytic discs in $\mathbb{C}^n$, indexed by some set A. Suppose $\cup_{\alpha \in A} \partial \mathfrak{d}_a$ is compactly contained in Ω. If $\cup_{\alpha \in A} \mathfrak{d}_\alpha$ is also contained in Ω, we call Ω closed with respect to $\{\delta_a\}$. The form of pseudoconvexity associated to this phenomenon is called Kontinuitätssatz and it means that Ω is closed with respect to all families of analytic discs with the aforementioned boundary containment condition. This property is in one sense the most immediate extension of the real version of convexity since it involves the complex analogue of the unit interval.

Now that we have a few notions of pseudoconvexity, we should see why they are important.

Theorem 2.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then the following are equivalent:*

- *Ω is a domain of holomorphy*

- *Ω is holomorphically convex*

- *The Kontinuitätssatz is satisfied for Ω.*

- *Ω is Hartogs pseudoconvex*

If Ω happens to have C^2 boundary then we can add one more item to the list of equivalences:

- *Ω is Levi pseudoconvex*

By far the most difficult part of Theorem 2.10 to prove is that a pseudoconvex domain is a domain of holomorphy, whatever notion of pseudoconvexity we mean. This is known as the Levi problem, posed in 1911 by E. E. Levi. It was proven in 1954 by Oka [49]. Pathways to proving it involve sheaf cohomology or L^2 estimates of the $\overline{\partial}$ operator, neither of which we need to get into in the current work. A full proof of Theorem 2.10 can be found in Krantz [40], where a major portion of that book is dedicated to proving the Levi problem.

2.5 Strong Pseudoconvexity

Definition 2.11. *Suppose Ω is a C^2 pseudoconvex domain defined by ρ and suppose $p \in \partial\Omega$. Suppose we have strict inequality in the pseudoconvexity condition, i.e. for all nonzero holomorphic tangent vectors w we have*

$$\sum_{j,k=1}^{n} -\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)_p w_j \overline{w_k} > 0.$$

Then we call p a point of strong pseudoconvexity. If Ω is bounded and if all points in $\partial\Omega$ are points of strong pseudoconvexity then we say Ω is strongly pseudoconvex.

The notions of tangent vector, pseudoconvex point, and strongly pseudoconvex point are independent of the choice of defining function.

A natural question is whether strong pseudoconvexity is a biholomorphic invariant. This turns out to be more difficult than the equivalent question for pseudoconvexity. The question was answered in the affirmative by Bell [5] for smoothly bounded domains:

Theorem 2.12. *Suppose $D \subset \mathbb{C}^n$ is a bounded, smoothly bounded, strongly pseudoconvex domain, that $\Omega \subset \mathbb{C}^n$ is a bounded, smoothly bounded domain, and that D and Ω are biholomorphic. Then Ω is must be strongly pseudoconvex.*

The equivalent question for convex domains was answered by Zimmer in [60], and we will give much more details and a slightly alternate proof in chapter 4.

Theorem 2.13. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded C^2 strongly pseudoconvex domain and suppose $D \subset \mathbb{C}^n$ is a $C^{2+\epsilon}$ convex domain for some $\epsilon > 0$. Suppose Ω and D are biholomorphic. Then every point in ∂D is a point of strong pseudoconvexity.*

Note that Theorem 2.13 does not require D to be bounded. The difference between Zimmer's proof and ours is minor – Zimmer based his argument on one particular intrinsic function and we show that the same proof holds for a different intrinsic function.

If $\Omega \subset \mathbb{C}^n$ is a Levi pseudoconvex domain and $p \in \partial\Omega$, we have a test for if p is a point of strong pseudoconvexity given in Krantz [40]:

Theorem 2.14. *Suppose $\Omega \subset \mathbb{C}^n$ $(n > 1)$ is a Levi pseudoconvex domain. Suppose $p \in \partial\Omega$ is a point of strong pseudoconvexity. Then there is no analytic disc $\mathfrak{d}$ in $\mathbb{C}^n$ centered at p such that*

$$\lim_{z \to 0} \frac{\operatorname{dist}(\mathfrak{d}(z), \partial\Omega)}{\|\mathfrak{d}(z) - p\|^2} = 0.$$

2.6 Convexity

Let Ω be a convex domain in $\mathbb{C}^n$. By convex we mean geometrically convex, i.e. for all $x, y \in \Omega$ the (real) line segment connecting x to y is completely contained in Ω.

We will be studying convex domains in detail, specifically bounded convex domains. Convex domains are simpler than general domains for several reasons. We list below some tools available while studying convex domains which do not apply for general nonconvex domains.

One technical trick which we use repeatedly in chapter 4 involves a particular way of writing Ω as an increasing union of compactly-contained subdomains, each of which is biholomorphic to Ω. Given any convex domain Ω, we can translate so that $0 \in \Omega$. Then for any number r with $0 < r < 1$ we can rescale Ω by r to obtain $r\Omega$. Given such an r the domain $r\Omega$ is compactly contained in Ω and $z \mapsto rz$ is a biholomorphism between Ω and $r\Omega$. Moreover, the family $\{r\Omega\}_{0<r<1}$ is an increasing family which exhausts Ω. This trick is foundational to many of our intermediate steps along the way of proving Theorem 4.6.

Another useful property of convex domains involves a projection process for a point in the boundary:

Proposition 2.15. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain which is not the whole of $\mathbb{C}^n$ and let*

$p \in \partial\Omega$. Then there is a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}$ such that $f(p) = 0$ and $f(\Omega) \subseteq \mathcal{H}$ where $\mathcal{H}$ is the right half plane $\{z : \mathrm{Re}(z) > 0\}$. This map f is called the affine projection at b.

Proof. Through translation $z \mapsto z - b$ we can assume $b = 0$.

By virtue of being convex, there is a real affine hypersurface L passing through 0 which does not intersect Ω. Take $x_1, \cdots, x_{n-1} \in \mathbb{C}^n$ such that the complex span of $\{x_i\}$ lies in L. Then there is an $x_n \in \Omega$ such that $\langle x_i, x_n \rangle = 0$ for all $1 \leq i \leq n-1$.

Let $f(z) = \langle z, x_n \rangle$. Then $f(\Omega)$ lies in the right half plane and $f(0) = 0$ by construction. □

The half plane is biholomorphic to the disc, so a corollary of Proposition 2.15 is that any convex domain except the whole of $\mathbb{C}^n$ admits a nonconstant bounded holomorphic function.

A third tool involves distance to the boundary. Let $\Omega \subset \mathbb{C}_n$ be a convex domain. For any $z \in \Omega$ we define the distance to the boundary as

$$\delta_\Omega(z) = \inf\{\|z - w\| : w \in \partial\Omega\}.$$

Additionally given a $v \in \mathbb{C}^n$ with $v \neq 0$ we define the directional distance to the boundary as

$$\delta_\Omega(z, v) = \inf\{\|w - z\| : w \in \partial\Omega \cap (z + \mathbb{C}v)\}.$$

These distance functions give useful analytic tools to study convex domains, and the convexity gives these functions strength to classify the domain in ways not so easily found in the nonconvex case. For example, we can use them to detect strong pseudoconvexity:

Proposition 2.16. *Let $\Omega \subset \mathbb{C}^n$ be a C^2 domain and let $p \in \partial\Omega$. Then p is not a point of strong pseudoconvexity if and only if there is a $v \in T_p\Omega$ so that*

$$\lim_{r \to 0} \frac{r}{\left(\delta_\Omega(p + r\mathbf{n}_p; v)\right)^2} = 0.$$

Here $\mathbf{n}$ is the inward normal to $\partial\Omega$ at p.

If Ω is slightly smoother than C^2 then there is a stronger result available, stated in Zimmer [60]:

Proposition 2.17. *Let Ω be a $C^{2+\epsilon}$ domain and let $p \in \partial\Omega$. Then p is not a point of strong pseudoconvexity if and only if there is a $C, \delta > 0$ and a unit vector $v \in T_p\Omega$ so that*

$$\delta_\Omega(p + r\mathbf{n}_p; v) \geq C r^{\frac{1}{2+\epsilon}}$$

for every $0 < r \leq \delta$. By $\mathbf{n}_p$ we mean the unit inward normal vector at p.

Proof. Choose a unit vector $v \in T_p\Omega$ such that

$$\lim_{r \to 0} \frac{r}{\left(\delta_\Omega(p + r\mathbf{n}_p; v)\right)^2} = 0. \tag{2.1}$$

Such a v exists because Ω is convex and weakly pseudoconvex at p. The $C^{2+\epsilon}$ boundary condition combines with (2.1), which is essentially the second derivative of the defining function at p in the direction of v, to give that for all $r > 0$ small enough

$$\frac{r}{\left(\delta_\Omega(p + r\mathbf{n}_p; v)\right)^2} \leq C\left(\delta_\Omega(p + r\mathbf{n}_p; v)\right)^\epsilon.$$

But this is exactly the condition we are looking for. $\square$

We conclude this section with Narasimhan's lemma (as given in Krantz [40]), connecting strong pseudoconvexity to convexity:

Lemma 2.18. *Let Ω be a bounded C^2 domain and let $p \in \partial\Omega$ be a point of strong pseudoconvexity. Then there is a neighborhood U of p and a biholomorphic mapping ϕ on U so that $\phi(U \cap \Omega)$ is strongly convex at p.*

In fact, Narashimhan's lemma can be refined to state that a strongly pseudoconvex point p is one where local coordinates can be chosen so that $\partial\Omega$ near p agrees with the sphere up to order 2. Fefferman showed this can be improved to order 4 agreement. Those points for which this agreement is not merely up to some order but is in fact holomorphic are special, hence:

Definition 2.19. *Let M be a real hypersurface in $\mathbb{C}^n$, i.e. the boundary of a domain. We say M is spherical if at every point $p \in M$ there is a choice of holomorphic coordinates $z_1, \cdots, z_n$ so that near p M is locally described by*

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1.$$

Spherical boundary means that the boundary can be holomorphically locally represented by pieces of the ball. This is a local property, though: it does not mean Ω is biholomorphic to the ball or that $\partial\Omega$ is holomorphically equivalent to the sphere in a global sense.

We state the following theorem from Burns and Schnider [10] to illustrate that there are domains Ω such that $\partial\Omega$ is spherical but Ω is not biholomorphic to the ball. Moreover, not only do such domains exist but also they are not particularly rare.

Theorem 2.20. *There exist continuous families of domains $\Omega \subset \mathbb{C}^n$ with spherical boundaries each such that the fundamental group $\pi_1(\partial\Omega)$ is the free group on m generators, where m is arbitrary.*

2.7 Normal Families

Suppose $\Omega \subseteq \mathbb{C}^n, D \subseteq \mathbb{C}^m$ are domains, meaning connected open subsets. Let $D(\Omega)$ denote the set of holomorphic functions from Ω to D. The set $D(\Omega)$ is a topological space under the compact open topology. That is, holomorphic functions $f_k \in D(\Omega)$ converge to some holomorphic $f \in D(\Omega)$ if $f_k \to f$ uniformly on compact subsets of Ω. It may be worth noting that if $f : \Omega \to D$ is merely continuous and $f_k \in D(\Omega)$ with $f_k \to f$ uniformly on compact sets then f must be holomorphic.

Let f_k be a sequence of functions in $D(\Omega)$. We say f_k is compactly divergent if for all compact $K \subset \Omega$ and $K' \subset D$ there is an $N \in \mathbb{N}$ such that for all $k > N$ we have $f_k(K) \cap K' = \varnothing$.

We only really need to know one thing about compact divergence: if f_k has a fixed point – a $z \in \Omega$ and $p \in D$ with $f_k(z) = p$ for all k – then f_k is not compactly divergent. The same holds even if z is not a fixed point but an almost fixed point in the sense that $f_k(z) \to p$ as $k \to \infty$.

Definition 2.21. *Suppose $\mathcal{F}$ is a family of holomorphic functions from Ω to D, i.e. $\mathcal{F} \subseteq D(\Omega)$. We say $\mathcal{F}$ is a normal family if for every sequence $f_k \in \mathcal{F}$ either f_k is compactly divergent or there is some $f \in D(\Omega)$ and a subsequence j of k such that $f_j \to f$.*

A major result involving normal families is Montel's theorem. In preparation for its statement, we must define what it means for a family $F \subseteq \mathbb{C}^m(\Omega)$ to be locally uniformly bounded. This means that for all $z \in \Omega$ there is a neighborhood $U \subseteq \Omega$ with $z \in U$ and some constant $M_U > 0$ such that for all $f \in \mathcal{F}$ and all $w \in U$ we have $\|f(w)\| \leq M_U$.

Theorem 2.22 (Montel)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\mathcal{F}$ be a family in $\mathbb{C}^m(\Omega)$. Suppose $\mathcal{F}$ is locally uniformly bounded. Then $\mathcal{F}$ is a normal family.*

In particular, Montel's theorem gives us the phenomenon that if $D \subset \mathbb{C}^m$ is a bounded domain and $\mathcal{F} \subset \mathbb{C}^m(\Omega)$ consists only of holomorphic functions from Ω to D then $\mathcal{F}$ is a normal family, and any sequence cannot be compactly divergent because $\overline{D}$ is compact. Thus there are always limiting functions for sequences in F (after taking a subsequence), though the limiting functions are only guaranteed to map into $\mathbb{C}^m$ and not necessarily D.

Corollary 2.23. *Suppose $\Omega \subseteq \mathbb{C}^n$ and $D \subset \mathbb{C}^m$ where D is a bounded domain. Suppose for each k we have a holomorphic $f_k : \Omega \to D$. Then there is a holomorphic function $f : \Omega \to \overline{D}$ such that some subsequence of f_k converges to f uniformly on compact sets.*

It would be nice if Corollary 2.23 could simply state that $D(\Omega)$ is a normal family whenever D is bounded, but this is not always the case. First we should acknowledge that compact divergence is relative to the codomain, for Corollary 2.23 never results in a compactly divergent sequence with respect to the codomain $\mathbb{C}^m$ but does not mention the same with respect to the codomain D:

Example 2.24. *For each $k \in \mathbb{N}$ let $f_k \in \mathbb{D}(\mathbb{D})$ be the transformation*

$$f_k(z) = 1 + \frac{1}{k}(z - 1).$$

Corollary 2.23 *guarantees that the sequence f_k is normal when considered as functions in $\mathbb{C}(\mathbb{D})$, and clearly f_k approaches the constant function* 1*. However, although each*

f_k maps into $\mathbb{D}$ the limit function does not because $1 \notin \mathbb{D}$. In fact $\{f_k\}$ is a compactly divergent family in $\mathbb{D}(\mathbb{D})$.

Perhaps more interestingly we may have a sequence which is not compactly divergent but which still requires the closure condition in corollary Corollary 2.23:

Example 2.25. *Let $\Omega \subset \mathbb{C}^2$ be the set*

$$\{(z_1, z_2) : |z_1| < 1, |z_2| < 2, z_1 \notin [0, 1)\}.$$

For each k let $f_k : \mathbb{D} \to \Omega$ be the function

$$f_k(z) = \left(-\frac{1}{k}, z\right).$$

Again Corollary 2.23 applies but the limiting function is $f(z) = (0, z)$ and $f(0) \notin \Omega$.

The main idea behind Example 2.25 is that the domain is not pseudoconvex. In fact, this example shows the Kontinuitätssatz fails in this particular domain.

Under a mild boundary smoothness conditions we get the following result due to Kerzman [32] involving maps from the disc into a domain:

Theorem 2.26. *Suppose $\Omega \subset \mathbb{C}^n$ is pseudoconvex with C^1 boundary. Then $\Omega(\mathbb{D})$ is a normal family.*

We can show a similar result involving convex domains which makes no boundary regularity assumptions:

Theorem 2.27. *Suppose $\Omega \subset \mathbb{C}^n$ is a domain and suppose $D \subset \mathbb{C}^m$ is a bounded convex domain. Then $D(\Omega)$ is a normal family.*

Proof. Let $\{f_k\}$ be a sequence of holomorphic functions from Ω to D. Suppose that $\{f_k\}$ is not compactly divergent. By Corollary 2.23 we know that, after passing to a subsequence k' of k, there is a holomorphic function $f : \Omega \to \overline{D}$ which is the limit of $\{f_{k'}\}$. We need to show that either $f(\Omega) \subseteq \partial D$ or $f(\Omega) \subseteq D$.

Suppose for the sake of contradiction that $f(\Omega)$ is only partially contained in the boundary of D. Take an $\epsilon > 0$ and points $p, a \in \Omega$ such that $|p-a| = \frac{\epsilon}{2}$, $B_\epsilon(p) \subseteq \Omega$, $f(p) \in \partial\Omega$, and $f(a) \in \Omega$. Let $g : \mathbb{D} \to \Omega$ be the map

$$g(z) = p + 2z(a-p).$$

Choose the affine projection π from Proposition 2.15 relative to p. The map $\pi \circ g : \mathbb{D} \to \overline{\mathcal{H}}$ is holomorphic and $h(g(0)) = 0$. By the open mapping theorem $\pi \circ g$ must be constant. But $\pi\left(g\left(\frac{1}{2}\right)\right) \in \mathcal{H}$, so $\pi \circ g$ is not constant. Thus we have our contradiction. $\square$

Chapter 3

Intrinsic Metrics and Measures

The theory of several complex variables has the interesting phenomenon that the complex structure of a manifold induces several metrics called the intrinsic metrics. These intrinsic metrics derive from the complex structure of a domain or manifold and as such are invariant under biholomorphisms, linking geometry to the subject in a more natural way than is possible in the real setting.

There are two intrinsic metrics which we are interested in: the Bergman metric and the Kobayashi metric. The Bergman metric arises from functional analysis and the Kobayashi metric can be considered a construction which arises as a key step in the standard proof of the Riemann mapping theorem. In essence the Kobayashi metric measures how large a disc can holomorphically fit inside a domain.

3.1 Bergman Metric

We describe the construction of the Bergman metric on bounded domains in $\mathbb{C}^n$, although the construction is more general. It just happens that there are fewer difficulties in the case when the domain is bounded and this is the case we are focusing on anyway. We are following Kobayashi [36] and Krantz [40] for this construction.

Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain. Let $\mathcal{H}^2$ denote all L^2 holomorphic functions on Ω, i.e.

$$\mathcal{H}^2 = \left\{ f : \Omega \to \mathbb{C} : f \text{ is holomorphic and } \int_\Omega |f|^2 d\mu < \infty \right\}.$$

The integral here is taken with respect to the Lebesgue measure. Note that, because we have restricted Ω to be bounded, all the polynomials are in $\mathcal{H}^2$ so $\mathcal{H}^2$ is an infinite dimensional vector space. We have the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}^2$ given by

$$\langle f, g \rangle = \int_\Omega f\overline{g} d\mu.$$

Then $\mathcal{H}^2$ is an infinite dimensional (separable) Hilbert space. As a consequence of Cauchy's integral formula the evaluation map $f \mapsto f(z)$ is a bounded linear functional for any $z \in \Omega$, so by the Riesz representation theorem there is an element $R_z \in \mathcal{H}^2$ such that for all $f \in \mathcal{H}^2$,

$$f(z) = \langle f, R_z \rangle = \int_\Omega f\overline{R_z} d\mu.$$

Definition 3.1. *Given $z, w \in \Omega$ define the Bergman kernel K to be*

$$K(z,w) = \overline{\langle R_z, R_w \rangle} = \overline{R_z(w)}.$$

Then K is holomorphic in z, antiholomorphic in w, and for all $f \in \mathcal{H}$,

$$f(z) = \int_\Omega f(w) K(z,w) d\mu_w.$$

That is, the Bergman kernel is a reproducing kernel for $\mathcal{H}^2$. We can compute the Bergman kernel if we are given an orthonormal basis of $\mathcal{H}^2$. Suppose e_k is an orthonormal basis of $\mathcal{H}^2$. Then

$$K(z,w) = \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)}.$$

This series in independent of choice of orthonormal basis. The Bergman kernel respects a particular transformation law: if $\phi : \Omega \to \Omega'$ is a biholomorphism and $z, w \in \Omega$ then

$$K_\Omega(z,w) = K_{\Omega'}(\phi(z),\phi(w)) \det d\phi(z)\overline{\det d\phi(w)}. \tag{3.1}$$

Note the diagonal $K(z,z)$ is strictly positive for all $z \in \Omega$.

Definition 3.2. *We obtain the Bergman metric B from the Bergman kernel K by setting*

$$B = \partial\overline{\partial} \ln K(z,z).$$

That is, the $(\alpha\beta)^{th}$ component of the Bergman metric, denoted $B_{\alpha\overline{\beta}}$, is given by

$$B_{\alpha\overline{\beta}} = (\log K)_{\alpha\overline{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log K(z,z).$$

For $\xi \in \mathbb{C}^n$ we denote by $|\xi|_{B,z}$ the norm of ξ under the Bergman metric at z,

$$|\xi|_{B,z} = \sum_{i,j=1}^{n} B_{\alpha\overline{\beta}}(z)\xi_i\overline{\xi_j}.$$

The Bergman metric is a Kähler metric which is invariant under biholomorphisms. Specifically, if $\phi : \Omega \to \Omega'$ is a biholomorphism and $z \in \Omega, \xi \in \mathbb{C}^n$ then

$$|\xi|_{B,z} = |d\phi(\xi)|_{B,z}.$$

Every bounded domain admits the Bergman metric, i.e. the metric is nondegenerate. The first question involving the Bergman metric is whether it is complete for a bounded domain Ω. Here complete means that Cauchy sequences (with respect to the distance induced by B) necessarily converge to a point in Ω.

Bremmerman [9] showed that a domain which is complete with respect to the Bergman metric is necessarily a domain of holomorphy, but that the converse is not true. That is, there are domains of holomorphy which are not complete with respect to the Bergman metric. However, Kerzman [33] and Greene and Wu [29] showed that a C^2 strongly pseudoconvex domain is complete with respect to the Bergman metric.

As an example of the utility of the Bergman metric, observe the following theorem of Lu [45]:

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with complete Bergman metric and suppose the Bergman metric has holomorphic sectional curvature constantly equal to $-c^2$ where $c > 0$. Then Ω is biholomorphic to the ball $\mathbb{B}_n$ and*

$$c^2 = \frac{2}{n+1}.$$

It is worth mentioning that Ω in Theorem 3.3 does not need to be assumed to be simply connected.

We conclude this section with an example; the unit ball in $\mathbb{C}^n$. As a starting point we take the formula for the Bergman kernel on the ball, calculated in full detail in [40]:

Proposition 3.4. *The Bergman kernel on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ is given by*

$$K_{\mathbb{B}}(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \overline{w})^{n+1}}.$$

On the one hand we can compute the Bergman metric directly from this formula:

Proposition 3.5. *The Bergman metric B on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ is given by*

$$B_{\alpha\overline{\beta}}(z) = (n+1)\frac{\delta_{\alpha\beta}(1-|z|^2) + z_\beta\overline{z_\alpha}}{(1-|z|^2)^2}.$$

Proof. This is a routine calculation. The diagonal Bergman kernel on the ball is

$$K(z,z) = \frac{n!}{\pi^n}\frac{1}{(1-z\cdot\overline{z})^{n+1}}$$

so

$$\begin{aligned}\log K(z,z) &= \log\left(\frac{n!}{\pi^n}\right) - (n+1)\log(1-z\cdot\overline{z})\\ &= \log\left(\frac{n!}{\pi^n}\right) - (n+1)\log\left(1-\sum_{i=1}^{n} z_i\overline{z_i}\right).\end{aligned}$$

Taking the derivatives,

$$\begin{aligned}(\log K(z,z))_\alpha &= (n+1)\frac{\overline{z_\alpha}}{1-\sum_{i=1}^n z_i\overline{z_i}}\\ (\log K(z,z))_{\alpha\overline{\beta}} &= (n+1)\frac{\delta_{\alpha\beta}(1-z\cdot\overline{z}) + z_\beta\overline{z_\alpha}}{(1-z\cdot\overline{z})^2}.\end{aligned}$$

□

On the other hand, it will help us later to rewrite these formulas in terms of the canonical defining function for the ball:

$$\rho(z) = 1-|z|^2.$$

Clearly ρ is a C^∞ (indeed real analytic) function which defines the unit ball in $\mathbb{C}^n$. The diagonal Bergman kernel and metric can be written quite succinctly in terms of ρ:

Proposition 3.6. *The diagonal Bergman kernel on the ball with defining function $\rho(z) = 1-|z|^2$ is given by*

$$K(z,z) = \frac{n!}{\pi^n}\rho^{-(n+1)}(z)$$

and the Bergman metric by

$$B_{\alpha\overline{\beta}}(z) = (n+1)\frac{z_\beta\overline{z_\alpha} + \delta_{\alpha\beta}\rho(z)}{\rho^2(z)}.$$

3.2 Kobayashi Metric

The constructions of the Kobayashi metric and its properties can be found in Kobayashi [38] and Krantz [41]. We describe the construction here as well.

Suppose $\Omega \subseteq \mathbb{C}^n$ is a domain. The differential Kobayashi pseudometric dK_Ω evaluated at $(x,v) \in \Omega$ is defined as

$$\begin{aligned} dK_\Omega(x,v) &= \inf\left\{|t| : t \in \mathbb{C} \text{ and } \exists f : \mathbb{D} \to \Omega \,\text{s.t.}\, f(0) = x, df_0\left(t\frac{\partial}{\partial z}\right) = v\right\} \\ &= \inf\left\{\alpha > 0 : \exists f \in \Omega(\mathbb{D}) \,\text{s.t.}\, df_0\left(\frac{\partial}{\partial z}\right) = \frac{1}{\alpha}v\right\}. \end{aligned}$$

Proposition 3.7. *Suppose $\Omega_1 \subseteq \mathbb{C}^n, \Omega_2 \subseteq \mathbb{C}^m$ are domains and we have a holomorphic map $f : \Omega_1 \to \Omega_2$. Then for all $z \in \Omega_1$ and $v \in \mathbb{C}^n \setminus \{0\}$ we have*

$$dK_{\Omega_2}(f(z), df_z(v)) \leq dK_{\Omega_1}(z,v).$$

Proof. Any holomorphic function on Ω_2 pulls back to one on Ω_1, so this is a consequence of the definition. □

We call this the distance decreasing property of the Kobayashi metric. One may wonder whether the Kobayashi metric can be infinite or zero at a certain point in some domain. One the one hand we have:

Proposition 3.8. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a domain, $z \in \Omega$, and $v \in \mathbb{C}^n \setminus \{0\}$. Then*

$$dK_\Omega(z,v) \leq \frac{\|v\|}{\delta_\Omega(z,v)}.$$

Proof. If $\delta_\Omega(z,v) < \infty$ then there is an affine map $a : \mathbb{D} \to \Omega$ with $a(0) = z$ and

$$da_0\left(\frac{\partial}{\partial z}\right) = \frac{\delta_\Omega(z,v)}{\|v\|}v.$$

If $\delta_\Omega(z,v) = \infty$ then for any $r > 0$ we can find an affine $b : \mathbb{D} \to \Omega$ with $b(0) = z$ and

$$db_0\left(\frac{\partial}{\partial z}\right) = rv$$

and hence $dK_\Omega(z,v) = 0$. □

This in particular shows the Kobayashi metric is always finite. Moreover the Kobayashi metric can vanish in some domains. In particular the Kobayashi metric constantly vanishes on the domain $\mathbb{C}$, and it vanishes in the z_2 direction in the product domain $\mathbb{D} \times \mathbb{C} \subset \mathbb{C}^2$.

Definition 3.9. *We say the Kobayashi metric is nondegenerate if for all $x \in \Omega, v \in \mathbb{C}^n \setminus \{0\}$ we have that $dK_\Omega(x,v) > 0$. We say the domain Ω is hyperbolic if the Kobayashi metric is nondegenerate, and complete hyperbolic if Ω is hyperbolic and complete with respect to the Kobayashi metric.*

Kobayashi [39] showed that any domain which is complete hyperbolic is a domain of holomorphy. Among the many other results found in [39] is enough to prove the following in particular:

Theorem 3.10. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Then Ω is hyperbolic.*

If additionally Ω is convex, or if Ω has C^2 boundary and is strongly pseudoconvex, then Ω is complete hyperbolic.

It would be nice if the Kobayashi metric value were actually attained by an analytic disc. Clearly this cannot happen for any domain – particularly any time the Kobayashi metric vanishes – but we can provide a situation where it must be attained:

Proposition 3.11. *Suppose $\Omega \subset \mathbb{C}^n$ is convex and bounded. Let $x \in \Omega$ and $v \in \mathbb{C}^n \setminus \{0\}$. Then there is a holomorphic map $f : \mathbb{D} \to \Omega$ with $f(0) = x$ and $df_0\left(\frac{\partial}{\partial z}\right) = \frac{1}{dK_\Omega(x,v)}v$.*

Proof. Let $\alpha = dK_\Omega(p, v)$. Then for every $k \in \mathbb{N}$ there is an $f_k : \mathbb{D} \to \Omega$ with $f_k(0) = x$ and

$$(df_k)_0\left(\frac{\partial}{\partial z}\right) = \frac{1}{t_k}v$$

where $\alpha \le t_k \le \alpha + \frac{1}{k}$. Then

$$\frac{1}{\alpha + \frac{1}{k}}\|v\| \le \left\|(df_k)_0\left(\frac{\partial}{\partial z}\right)\right\| \le \frac{1}{\alpha}\|v\|.$$

Because Ω is convex and $f_k(0) = x$ for all k Theorem 2.27 states that, after passing to a subsequence, there is a holomorphic function $f : \mathbb{D} \to \Omega$ with $f_k \to f$. Then $\left\|df_0\left(\frac{\partial}{\partial z}\right)\right\| = \frac{1}{\alpha}\|v\|$. Precomposing by a rotation in the disc if necessary, we can assume that $df_0\left(\frac{\partial}{\partial z}\right) = \frac{1}{\alpha}v$, so we have obtained the function we seek. □

We can drop the convexity assumption above if we impose a mild boundary smoothness condition:

Proposition 3.12. *Suppose Ω is a bounded C^1 pseudoconvex domain in $\mathbb{C}^n$. Then for all $z \in \Omega$ and $v \in \mathbb{C}^n \setminus \{0\}$ there is a holomorphic $f : \mathbb{D} \to \Omega$ with $f(0) = z$ and*

$$df_0\left(\frac{\partial}{\partial z}\right) = \frac{1}{dK_\Omega(z,v)}v.$$

Proof. The proof is identical to that of Proposition 3.11 except we use Theorem 2.26 in place of Theorem 2.27. ☐

The Kobayashi distance K_Ω between two points $x, y \in \Omega$ is defined over chains of holomorphic maps from the disc which link x to y. Specifically, suppose we have finitely many points $z_i \in \Omega$ for $0 \le i \le k$ such that $x = z_0$ and $y = z_k$. Suppose also we have $f_i \in \mathcal{F}_K(\Omega)$ and $a_i, b_i \in \mathbb{D}$ for $1 \le i \le k$ with $f_i(a_i) = z_{i-1}, f_i(b_i) = z_i$. We call this a chain connecting x to y, and we define its Kobayashi length as

$$\sum_{i=1}^{k} s(a_i, b_i).$$

Recall s is the Poincaré distance on the disc $\mathbb{D}$. The Kobayashi distance between x and y is then defined to be the infemum of the Kobayashi lengths of all chains which connect x to y.

Royden [54] showed that the Kobayashi distance is the integrated form of the differential Kobayashi metric. Royden's result combines with Proposition 3.7 to show

Proposition 3.13. *Suppose $\Omega_1 \subset \mathbb{C}^n, \Omega_2 \subset \mathbb{C}^m$ are domains and suppose $f : \Omega_1 \to \Omega_2$ is holomorphic. Then for all $x, y \in \Omega_1$*

$$K_{\Omega_2}(f(x), f(y)) \le K_{\Omega_1}(x, y).$$

We call this the distance decreasing property of the Kobayashi distance. Much of the utility of the Kobayashi metric and distance come from their distance decreasing properties, particularly with respect to the set inclusion map with respect to subdomains. In particular the Kobayashi metric is a biholomorphic invariant:

Proposition 3.14. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains and suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphism. Then for all $x, y \in \Omega_1$ and $v \in \mathbb{C}^n \setminus \{0\}$*

$$dK_{\Omega_1}(x, v) = dK_{\Omega_2}(f(x), df_x(v))$$

and

$$K_{\Omega_1}(x, y) = K_{\Omega_2}(f(x), f(y)).$$

Proof. The distance decreasing properties applied to f give that

$$dK_{\Omega_1}(x, v) \geq dK_{\Omega_2}(f(x), df_x(v))$$

and

$$K_{\Omega_1}(x, y) \geq K_{\Omega_2}(f(x), f(y)).$$

The same properties applied to f^{-1} give

$$dK_{\Omega_1}(x, v) \leq dK_{\Omega_2}(f(x), df_x(v))$$

and

$$K_{\Omega_1}(x, y) \leq K_{\Omega_2}(f(x), f(y)).$$

□

Proposition 3.15. *The Kobayashi distance on the disc is equal to the Poincaré distance, i.e. for all $x, y \in \mathbb{D}$,*

$$K_{\mathbb{D}}(x, y) = s(x, y).$$

Proof. Let $x, y \in \mathbb{D}$ and take any $f : \mathbb{D} \to \mathbb{D}$ holomorphic. By the Schwarz-Pick lemma (which says s is distance decreasing on $\mathbb{D}$),

$$s(f(x), f(y)) \leq s(x, y).$$

Choosing the identity function and the single-link chain connecting x to y gives $d_{K,\mathbb{D}}(x,y) \leq s(x,y)$. Now suppose we have any chain in $\mathbb{D}$ connecting x to y. That is, suppose we have $a_i, b_i \in \mathbb{D}$ and holomorphic $f_i : \mathbb{D} \to \mathbb{D}$ such that $f_i(b_i) = f_{i+1}(a_{i+1})$, $f_1(a_1) = x$, and $f_k(b_k) = y$. The triangle inequality of s provides that

$$s(x,y) \leq \sum_{i=1}^{k} s(f_i(a_i), f_i(b_i)).$$

Because $K_{\mathbb{D}}(x,y)$ is the infemum over all such sums, $s(x,y) \leq K_{\mathbb{D}}(x,y)$ proving our result. □

3.3 Intrinsic Measures

The intrinsic metrics have related measures. The Kobayashi-Eisenman measure M_{Ω}^{E} was introduced by Eisenman in [20] and it measures how large a ball can be holomorphically fit inside a domain. The Carathéodory measure M_{Ω}^{C} measures how small a ball in which Ω can be holomorphically fit inside. We are following Wong [57] for the notations.

The two instrinsic measures are given in terms of the coordinates $(z_1, \cdots, z_n)$ centered at a fixed point x. They are given by

$$M_{\Omega}^{E}(z) = \left| M_{\Omega}^{E}(z) \right| dV$$

$$M_{\Omega}^{C}(z) = \left| M_{\Omega}^{C}(z) \right| dV.$$

Here dV is the euclidean volume and

$$\left| M_{\Omega}^{E}(x) \right| = \inf \left\{ \frac{1}{R^{2n}} : \exists f : B_R(0) \to \Omega, f(0) = x, \det[df_0] = 1 \right\}$$

$$= \inf \left\{ (\det[df_0])^{-1} : f : \mathbb{B}_n \to \Omega, f(0) = x, \det[df_0] > 0 \right\},$$

$$\left|M_\Omega^C(x)\right| = \sup\left\{\frac{1}{R^{2n}} : \exists f : \Omega \to B_R(0), f(x) = 0, \det[df_x] = 1\right\}$$

$$= \sup\left\{\det[df_x] : f : \Omega \to \mathbb{B}_n, f(x) = 0, \det[df_x] > 0\right\}.$$

If Ω is bounded then Ω can be fit into the ball by a combined translation/scaling and hence $\left|M_\Omega^C(x)\right| > 0$.

The Carathéodory measure bounds the Kobayashi measure from below, i.e.

$$\left|M_\Omega^C(z)\right| \le \left|M_\Omega^E(z)\right|.$$

The Carathéodory and Kobayashi-Eisenman measures enjoy the same distance decreasing property under holomorphic maps (in the same dimension) as the Kobayashi metric and hence are both biholomorphic invariants.

Proposition 3.16. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $x \in \Omega$. If $D \subset \mathbb{C}^n$ is a domain and $f : \Omega \to D$ is holomorphic then*

$$\left|M_D^E(f(x))\right| |\det[df_x]| \le \left|M_\Omega^E(x)\right|$$

and

$$\left|M_D^C(f(x))\right| |\det[df_x]| \le \left|M_\Omega^C(x)\right|.$$

In particular, if f is a biholomorphism then

$$\left|M_D^E(f(x))\right| |\det[df_x]| = \left|M_\Omega^E(x)\right|$$

and

$$\left|M_D^C(f(x))\right| |\det[df_x]| = \left|M_\Omega^C(x)\right|.$$

Moreover, in the case that f is a biholomorphism and if the Carathéodory measure does not vanish at x,

$$\frac{\left|M_D^E(f(x))\right|}{\left|M_D^C(f(x))\right|} = \frac{\left|M_\Omega^E(x)\right|}{\left|M_\Omega^C(x)\right|}.$$

This last function is the one Wong used in [57] (stated here as Theorem 4.1), and we name it:

Definition 3.17. *For a domain Ω, the function*

$$f(z) = \frac{\left|M_\Omega^E(z)\right|}{\left|M_\Omega^C(z)\right|}$$

is called the ratio of intrinsic measures.

We can gather some information about the Kobayashi-Eisenman measure from the Kobayashi metric, namely:

Proposition 3.18. *Let Ω be a hyperbolic domain, meaning the Kobayashi metric is nondegenerate. Then the Kobayashi-Eisenman measure coefficient does not vanish. That is, for all $x \in \Omega$ we have $\left|M_\Omega^E(x)\right| > 0$.*

Proof. Kobayashi [38], Chapter IX (Miscellany), Theorem 1.10, page 118. □

We can obtain representative functions for the measure as in the case with the metrics.

Proposition 3.19. *Suppose the domain $\Omega \subset \mathbb{C}^n$ is convex and bounded. Let $x \in \Omega$. Then there is a holomorphic map $f : \mathbb{B}_n \to \Omega$ with $f(0) = x$ and*

$$(\det[df_0)])^{-1} = \left|M_\Omega^E(x)\right|.$$

There is also a holomorphic map $g : \Omega \to \mathbb{B}_n$ with $g(x) = 0$ and

$$\det[dg_x] = \left|M_\Omega^C(x)\right|.$$

Proof. By Theorem 3.10 Ω is hyperbolic. Then Proposition 3.18 forces $\left|M_\Omega^E(x)\right| > 0$ for all $x \in \Omega$. Let $\alpha = \left|M_\Omega^E(x)\right|^{-1}$. Then for every $k \in \mathbb{N}$ there is an $f_k : \mathbb{B}_n \to \Omega$ with $f_k(0) = x$ and

$$(\det[(df_k)_0])^{-1} = t_k$$

where $\alpha^{-1} \le t_k \le \alpha^{-1} + \frac{1}{k}$. Then

$$\alpha \ge \det[(df_k)_0] \ge \frac{k\alpha}{k+\alpha}.$$

Now Montel's theorem gives a holomorphic $f : \mathbb{B}_n \to \overline{\Omega}$ with, after passing to a subsequence, $f_k \to f$. Because Ω is convex, Theorem 2.27 states that f is actually a map from $\mathbb{B}_n$ to Ω because $f(0) = x \in \Omega$. Then $\det[df_0] = \alpha$.

Let $\beta = \left|M_\Omega^C(x)\right|$. For each k there is a $g_k : \Omega \to \mathbb{B}_n$ with $g_k(x) = 0$ and $\det[(dg_k)_x] = s_k$ where $\beta - \frac{1}{k} \le s_k \le \beta$. By Theorem 2.27, there is a map $g : \Omega \to \mathbb{B}^n$ such that some subsequence of g_k converges to g. Then $g(x) = 0$ and $\det[dg_x] = \beta$. □

3.4 Intrinsic Metrics and Measures on a Fixed Domain

In this section we collect some behavior about how the intrinsic metrics and measures behave as we vary the base point in a fixed domain.

Proposition 3.20. *Suppose $\Omega \subset \mathbb{C}^n$ is a convex domain which is not $\mathbb{C}^n$ and suppose for each k we have a subdomain $U_k \subseteq \Omega$. Assume that $U_k \subseteq U_{k+1}$ and that U_k exhaust Ω. Then for each $x \in \Omega$,*

$$\left|M_\Omega^E(x)\right| = \lim_{k\to\infty} \left|M_{U_k}^E(x)\right|$$

and

$$\left|M_{\Omega}^{C}(x)\right| = \lim_{k\to\infty}\left|M_{U_k}^{C}(x)\right|.$$

Moreover if we have $v \in \mathbb{C}^n \setminus \{0\}$ *then*

$$dK_{\Omega}(x,v) = \lim_{k\to\infty} dK_{U_k}(x,v).$$

Proof. We follow the proof of a similar statement found in Royden, P. M. Wong, Krantz [55]. In particular, the statement about the Kobayashi metric is contained in [55] and we adapt the argument to the measure case here.

Let $x \in \Omega$. Reindex by skipping the first few terms if necessary so that $x \in U_1$. Also translate if necessary so that $x = 0$. This loses no generality.

Let $\alpha_k = \left|M_{U_k}^{E}(0)\right|$ and $\alpha = \left|M_{\Omega}^{E}(0)\right|$. By monotonicity of the measure with respect to set inclusion, $\alpha_k \geq \alpha_{k+1} \geq \alpha$. Then α_k is a bounded monotone sequence, so there is some $c \geq \alpha$ with $\alpha_k \to c$.

If we have strict inequality where $c > \alpha$ then there is some holomorphic $g : \mathbb{B}_n \to \Omega$ with $g(0) = 0$ and $\det[dg_0] = \lambda > 0$ where $\lambda^{-1} < c$. Then there is some $\epsilon > 0$ so that $\lambda^{-1} < (1-\epsilon)^{2n}c$. Define $h : \mathbb{B}_n \to \Omega$ as

$$h(z) = g((1-\epsilon)z).$$

Then $h(0) = 0$ and $\det[dh_0] = (1-\epsilon)^{2n}\lambda > c^{-1}$, so $\det[dh_0]^{-1} < c$. However, the $(1-\epsilon)$ ball is compactly contained in $\mathbb{B}_n$, so its image under g (which is the image of the whole ball under h) is compactly contained in Ω. Then there is some k_0 such that $h(\mathbb{B}_n) \subseteq U_{k_0}$. Now by definition of α_{k_0} we have

$$\alpha_{k_0} \leq \det[dh_0]^{-1} < c.$$

Recall that α_k is a decreasing sequence which converges to c, so this is a contradiction. Thus $c = \alpha$ and we have proven the claim for the Kobayashi-Eisenman measure.

The proof for the Carathéodory measure is similar but we provide it for completeness, particularly to showcase why Ω is assumed to be convex. Define β_k to be $\left|M^C_{U_k}(0)\right|$ and let $\beta = \left|M^C_\Omega(0)\right|$. The Carathéodory measure is also decreasing under inclusions, so we have that β_k approaches some d with $d \geq \beta$. Note $\beta_k \geq d$.

If $d > \beta$ then we can find some $\epsilon > 0$ so that

$$(1-\epsilon)^{2n} d > \beta.$$

Note that $(1-\epsilon)\Omega$ is compactly contained in Ω because Ω is convex. Hence there is some k_0 large enough that $(1-\epsilon)\Omega \subseteq U_{k_0}$. Choose a holomorphic function $g : U_{k_0} \to \mathbb{B}_n$ with $g(0) = 0$ and $\det[dg_0] = \beta_{k_0}$. Define the function $h : \Omega \to \mathbb{B}_n$ as

$$h(z) = g((1-\epsilon)z).$$

Then $h(0) = 0$ and

$$\det[dh_0] = (1-\epsilon)^{2n}\beta_{k_0} \geq (1-\epsilon)^{2n} d > \beta.$$

This contradicts the definition of β as supremum. Thus $\beta_k \to \beta$ as claimed. $\square$

Proposition 3.21. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain. If $x_k \in \Omega$ and $x_k \to x \in \Omega$ then*

$$\left|M^E_\Omega(x_k)\right| \to \left|M^E_\Omega(x)\right|$$

and

$$\left|M^C_\Omega(x_k)\right| \to \left|M^C_\Omega(x)\right|.$$

Proof. Let $\alpha_k = \left|M_\Omega^E(x_k)\right|, \alpha = \left|M_\Omega^E(x)\right|$, and $\beta_k = \left|M_\Omega^C(x_k)\right|, \beta = \left|M_\Omega^C(x)\right|$.

First we show $\alpha \le \liminf \alpha_k$. Take a subsequence j of k so that $\liminf \alpha_k = \lim \alpha_j$. Using Proposition 3.19 let $f_j : \mathbb{B}_n \to \Omega$ such that

$$f_j(0) = x_j$$

$$\det[(df_j)_0] = \alpha_j^{-1}.$$

By Theorem 2.27 there is a holomorphic $f : \mathbb{B}_n \to \Omega$ which is the limit of a subsequence of f_j. Then $f(0) = x$ and $\det[df_0] = \lim \alpha_j^{-1}$.

By definition of α we therefore have $\alpha \le (\det[df_0])^{-1}$, so $\alpha \le \liminf \alpha_k$.

Now we want to show $\alpha \ge \limsup \alpha_k$. Let $f : \mathbb{B}_n \to \Omega$ be holomorphic with $f(0) = x$ and $\det[df_0] = \alpha^{-1}$, using Proposition 3.19 again. Let r be a number with $0 < r < 1$. Let $s_r : \mathbb{C}^n \to \mathbb{C}^n$ be the scaling

$$s_r(z) = x + r(z - x).$$

Let $\Omega_r = s_r(\Omega)$. Note that $\Omega_r \Subset \Omega$ because Ω is convex. Define the function g_k as

$$g_k(z) = s_r(f(z)) + x_k - x.$$

For k large enough, the image of g_k will lie in Ω. Observe $g_k(0) = x_k$ and $\det[(dg_k)_0] = r^{2n}\det[df_0] = r^{2n}\alpha^{-1}$. Therefore

$$\alpha_k \le (\det[(dg_k)_0])^{-1} = r^{-2n}\alpha.$$

Then $\limsup \alpha_k \le r^{-2n}\alpha$. This holds as $r \to 1$, so $\limsup \alpha_k \le \alpha$. Thus we have shown the Kobayashi-Eisenman measure to be continuous.

Now we show the Carathéodory measure is continuous for a fixed domain. We begin by showing $\beta \ge \limsup \beta_k$. Choose a subsequence j of k so that $\limsup \beta_k = \lim \beta_j$.

For each j let $f_j : \Omega \to \mathbb{B}_n$ be holomorphic such that $f_j(x_j) = 0$ and $\det[(df_j)_{x_j}] = \beta_j$. This uses Proposition 3.19.

Passing to a subsequence if necessary, let $f : \Omega \to \mathbb{B}_n$ such that $f_j \to f$, this is possible by Theorem 2.27. Then $f(x) = 0$ and $\det[df_x] = \lim \det[(df_j)x_j] = \lim \beta_j$. Then by definition of β we have

$$\beta \geq \lim \beta_j = \limsup \beta_k.$$

Now we show $\beta \leq \liminf \beta_k$. Suppose $\epsilon > 0$ is small. Define $r_\epsilon : \mathbb{C}^n \to \mathbb{C}^n$ as

$$r_\epsilon(z) = x + (1-\epsilon)(z-x).$$

Let $\Omega_\epsilon = r_\epsilon(\Omega)$. Note that because Ω is convex, $\Omega_\epsilon \Subset \Omega$. For each ϵ there is some $K_\epsilon > 0$ so that the translation $\Omega_\epsilon - x_k + x \Subset \Omega$ for $k \geq K_\epsilon$, this is because $x_k \to x$. Let U_ϵ be defined as

$$U_\epsilon = \bigcup_{k \geq K_\epsilon} \Omega_\epsilon - x_k + x.$$

Note that K_ϵ can be chosen so that $U_\epsilon \Subset \Omega$ as well. By the decreasing property of the Carathéodory measure

$$\left|M^C_{U_\epsilon}(x)\right| \leq \left|M^C_{\Omega_\epsilon - x_k + x}(x)\right| = \left|M^C_{\Omega_\epsilon}(x_k)\right|.$$

Both families $\{\Omega_\epsilon\}$ and $\{U_\epsilon\}$ are increasing as $\epsilon \to 0$ and both exhaust Ω.

Again by the decreasing property,

$$\beta = \left|M^C_{\Omega}(x)\right| \leq \left|M^C_{U_\epsilon}(x)\right|.$$

Fixing $\epsilon > 0$ for now we obtain a K_ϵ so that for all $k \geq K_\epsilon$,

$$\left|M^C_{U_\epsilon}(x)\right| \leq \left|M^C_{\Omega_\epsilon}(x_k)\right|.$$

Therefore

$$\beta \leq \liminf_{k\to\infty} \left| M^C_{\Omega_\epsilon}(x_k) \right|.$$

Letting $\epsilon \to 0$ in this estimate gives, again by Proposition 3.20,

$$\beta \leq \liminf \beta_k.$$

This shows the Carathéodory measure is continuous as well. $\square$

Chapter 4

Asymptotically Equivalent Measures

We start by recalling the following theorem of Bun Wong [57]:

Theorem 4.1. *Let Ω be a C^2 bounded strongly pseudoconvex domain. Then $\left|M^E_\Omega(z)\right|$ and $\left|M^C_\Omega(z)\right|$ are asymptotically equal, meaning for each $\epsilon > 0$ there is a $\delta > 0$ such that if $z \in \Omega$ with $\delta_\Omega(z) < \delta$ then*

$$\left|\frac{\left|M^E_\Omega(z)\right|}{\left|M^C_\Omega(z)\right|} - 1\right| < \epsilon.$$

Technically [57] assumes that Ω is smoothly bounded but the C^2 case is very similar.

Naturally we are interested in studying the converse. In that direction, note the following example due to Fornæss and Wold [24]:

Example 4.2. *There is a C^2 weakly pseudoconvex domain with asymptotically equivalent intrinsic measures.*

Proof. In the aptly titled paper [24] Fornæss and Wold construct a bounded convex C^2 domain Ω which is weakly pseudoconvex and which has the property that the squeezing function s_Ω tends to 1 on the boundary. We will not use the definition of the squeezing function directly, but we will give it below (after the conclusion) for completeness.

Deng, Guan, and Zhang in [17] showed the following estimate:

$$s_\Omega^{2n}(z) \leq \frac{\left|M_\Omega^E(z)\right|}{\left|M_\Omega^C(z)\right|} \leq s_\Omega^{-2n}(z).$$

Clearly if $s_\Omega \to 1$ near the boundary then the measures are asymptotically equivalent. Therefore Ω is a bounded C^2 convex domain with asymptotically equivalent intrinsic measures but which is not strongly pseudoconvex. $\square$

Definition 4.3. *The squeezing function is defined over the family $f_z(\Omega)$ of holomorphic function from Ω into $\mathbb{B}_n$ taking z to 0:*

$$s_\Omega(z) = \sup_{f_z}\{S_{\Omega,f_z}(z)\}$$

where

$$S_{\Omega,f_z}(z) = \sup\{r > 0 : B_r(0) \subseteq f_z(\Omega)\}.$$

In essence the squeezing function measures how big a ball can fit inside Ω while simultaneously forcing Ω to fit inside the unit ball. It is similar to, but different from, the ratio of intrinsic functions.

There are other similarities between the squeezing function and the ratio of intrinsic measures. For example, both the squeezing function and the intrinsic measures are defined over families of holomorphic functions to/from the ball. For both, attaining 1 at some point in the domain means Ω is biholomorphic to the ball. And both approach 1 near strongly pseudoconvex boundary points.

In [60] Zimmer proved the following:

Theorem 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a $C^{2+\epsilon}$ convex domain for some $\epsilon > 0$. Suppose $p \in \partial\Omega$ and suppose*

$$\lim_{\Omega \ni z \to p} s_\Omega(z) = 1.$$

Then p is a point of strong pseudoconvexity.

He actually proved a gap theorem related to Theorem 4.4:

Theorem 4.5. *For any $n \geq 2$ and $\epsilon > 0$ there is some δ, depending only on n and ϵ, so that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with $C^{2+\epsilon}$ boundary and such that*

$$s_\Omega(z) \geq 1 - \delta$$

outside a compact subset of Ω then Ω is strongly pseudoconvex.

Although stated for the squeezing function, he also mentioned that the same proof holds for a wide class of functions found in the theory of several complex variables. In the present work we will have shown that the function of Theorem 4.1, the ratio of the intrinsic measures, is one such function. In the current chapter we will give the necessary details for applying the argument and then present Zimmer's proof as it applies to the following:

Theorem 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a $C^{2+\epsilon}$ convex domain for some $\epsilon > 0$. Suppose $p \in \partial\Omega$ and suppose*

$$\limsup_{\Omega \ni z \to p} \frac{\left|M^E_\Omega(z)\right|}{\left|M^C_\Omega(z)\right|} = 1.$$

Then p is a point of strong pseudoconvexity.

Note that, by nature of the relationship between the squeezing function and the ratio of intrinsic measures, Theorem 4.4 follows from Theorem 4.6.

Zimmer's proof actually implies the stronger gap version of Theorem 4.6 (e.g. Theorem 4.5) as well. It may be of interest to see if there is a comparison between the gap parameter δ from Theorem 4.5 and the corresponding parameter for the gap version of Theorem 4.6, and such a correspondence could be considered another comparison between the squeezing function and the ratio of intrinsic measures. It would also be of interest to see more explicitly how this gap parameter depends on the boundary parameter ϵ, in the hopes of somehow taking ϵ to 0, but we have no results in this direction to present at the moment.

An immediate consequence of Theorem 4.6, also noted by Zimmer in [60], is that Theorem 2.13 is a corollary of Theorem 4.6. Technically Zimmer showed that Theorem 2.13 is a corollary of Theorem 4.4, but the same argument holds:

Corollary 4.7. *Theorem* 2.13 *follows from Theorem* 4.6*.*

Recall Theorem 2.13 states: Suppose $\Omega \subset \mathbb{C}^n$ is a bounded C^2 strongly pseudoconvex domain and suppose $D \subset \mathbb{C}^n$ is a $C^{2+\epsilon}$ convex domain for some $\epsilon > 0$. Suppose Ω and D are biholomorphic. Then every point in ∂D is a point of strong pseudoconvexity.

Proof. Let $\Omega \subset \mathbb{C}^n$ be a bounded C^2 strongly pseudoconvex domain and let $D \subset \mathbb{C}^n$ be a $C^{2+\epsilon}$ domain which is biholomorphic to Ω under the biholomorphism ϕ. Pick a point $p \in \partial D$.

The function $\frac{|M_\Omega^E(z)|}{|M_\Omega^C(z)|}$ is a biholomorphic invariant, so

$$\limsup_{D \ni z \to p} \frac{\left|M_D^E(z)\right|}{\left|M_D^C(z)\right|} = \limsup_{D \ni z \to p} \frac{\left|M_\Omega^E\left(\phi^{-1}(z)\right)\right|}{\left|M_\Omega^C\left(\phi^{-1}(z)\right)\right|}.$$

But Theorem 4.1 says this limit is 1 because $\phi^{-1}(z) \to \partial\Omega$. Thus by Theorem 4.6 p is a point of strong pseudoconvexity. $\square$

The proof of Theorem 4.6 relies on the scaling method. Originally introduced by Frankel [26] and Pinchuk [50], the scaling method has proven to be a powerful tool in the field of several complex variables. The scaling method requires a notion of convergence of sets which is simplest when the sets are convex – hence the convexity assumptions – but it does not rely much on boundary regularity. In particular C^∞ boundaries are often no more useful in the scaling process than those of particular finite-order regularity.

The scaling method consists of a sequence of biholomorphisms f_k from Ω to domains Ω_k. In a successful application there is a domain Ω_∞ which is the limit of the domains Ω_k; these details will be given shortly. Whereas Frankel and Pinchuk both constructed the maps f_k to that they converge to a biholomorphism $f_\infty : \Omega \to \Omega_\infty$, Zimmer's scaling sequence does not result in biholomorphism. Instead Ω_∞ will be constructed so as to preserve a certain boundary condition at a particular point.

Before discussing the scaling method in detail we first need to describe the topology on the domains we are using.

4.1 Convergence of Convex Domains

The scaling process always results in unbounded domains, so we begin by studying a nice class of unbounded convex domains. Specifically we are interested in hyperbolic (not necessarily bounded) convex domains. There is a simple classification to distinguish convex domains which are hyperbolic due to Barth [1]:

Theorem 4.8. *A convex domain $\Omega \subset \mathbb{C}^n$ is hyperbolic if and only if Ω contains no complex affine lines.*

That is, the Kobayashi metric K_Ω is a complete proper metric if and only if for all $z \in \Omega$ and $0 \neq v \in \mathbb{C}^n$,

$$\{w \in \mathbb{C} : z + wv \in \Omega\} \neq \mathbb{C}.$$

Consider the space $\mathbb{X}_n$ of all hyperbolic convex domains, which is equivalently all convex domains which do not contain any affine complex lines. We describe a topology on $\mathbb{X}_n$ via the Hausdorff distance between domains.

Define the distance from a point to a set as, for $x \in \mathbb{C}^n$ and $A \subseteq \mathbb{C}^n$,

$$d(x, A) = \inf_{a \in A} d(x, a).$$

For any $\epsilon > 0$ and $A \subseteq \mathbb{C}^n$, define the neighborhood $N_\epsilon(A)$ to be

$$N_\epsilon(A) = \{x \in \mathbb{C}^n : d(x, A) < \epsilon\}.$$

Given two sets $A, B \subset \mathbb{C}^n$, recall that the Hausdorff distance between A and B, denoted $d(A, B)$, is defined as

$$d(A, B) = \inf\{\epsilon > 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}.$$

This distance is a complete and proper metric on the space of nonempty compact subsets of $\mathbb{C}^n$ [52], and we are interested in compact subsets because any bounded convex set is uniquely determined by its closure (and that closure is compact). If we take the metric topology induced by the Hausdorff distance, this gives us a notion of convergence of compact subsets of $\mathbb{C}^n$. That is, now we know what it means for A_k to converge to A where A_k, A are all bounded convex domains in $\mathbb{C}^n$.

Given some $A \in \mathbb{X}_n$ (possibly unbounded), for any $R > 0$ define the compact set $\overline{A^R}$ as

$$\overline{A^R} = \overline{A \cap B_R(0)}.$$

Here $B_R(0)$ is the euclidean ball of radius R centered at 0.

This defines a topology on the space $\mathbb{X}_d$ of hyperbolic convex domains by declaring what is meant by the statement $A_k \to A$ for $A_k, A \in \mathbb{X}_d$. We say $A_k \to A$ if for all $R > 0$ which are large enough (i.e. $\exists R_0 > 0$ s.t. $\forall R > R_0$),

$$\overline{A_k^R} \to \overline{A^R}$$

where this convergence is with respect to the Hausdorff distance on compact sets. We call this topology on $\mathbb{X}_n$ the local Hausdorff topology.

We are going to end up proving results about how the intrinsic metrics and measures interact with this topology. We begin with a lemma which greatly aids us in that endeavor:

Lemma 4.9. *Suppose $\Omega_k \to \Omega$ for $\Omega_k, \Omega \in \mathbb{X}_n$. Then for every compact set $K \subset \Omega$ eventually $K \subset \Omega_k$, i.e. there is an $N > 0$ such that for all $k > N$, $K \subset \Omega_k$.*

Proof. Uniformly translate if necessary so that Ω contains the origin. To be clear, we are translating each domain the same amount.

Let $K \subset \Omega$ be compact. Then there is some $R > 0$ so that $K \subset B_R(0)$. Increasing R if necessary, we get $\Omega_k^R \to \Omega^R$ using the notation above. Thus we can assume that Ω and each Ω_k are contained in $B_R(0)$, the unbounded case follows from the simple fact that $\Omega_k^R \subseteq \Omega_k$.

Given a number r with $0 < r < 1$ define $r\Omega$ to be the scaling of Ω by r, i.e.

$$r\Omega = \{rz : z \in \Omega\}.$$

Then $r\Omega \subset \Omega$ because Ω is a convex set which contains the origin. Moreover, the family $\{r\Omega\}_{0<r<1}$ covers Ω so in particular it covers K. Then, because K is compact, there is a finite subfamily which also covers K. This is an increasing family so that means we can fix a single value for r between 0 and 1 for which $K \subset r\Omega$.

Now the Hausdorff distance from $r\Omega$ to $\partial\Omega$ is strictly positive because r is strictly less than 1. Because $\Omega_k \to \Omega$ and each domain is convex, it must be that Ω_k eventually contains $r\Omega$ and therefore K as well. □

The following normal family lemma will be helpful later on:

Lemma 4.10. *Suppose $D \subset \mathbb{C}^m$ is a domain and fix some $x \in D$.*

Suppose Ω_k, Ω, are uniformly bounded convex domains with $\Omega_k \to \Omega$. That is, there is some $R > 0$ such that Ω_k, Ω are all contained in $B_R(0)$. Suppose we also have holomorphic functions $f_k : D \to \Omega_k$ and an $x \in D$ such that $f_k(x) \to p$ where $p \in \Omega$.

Then there is a holomorphic $f : D \to \Omega$ such that some subsequence of f_k converges to f.

Proof. By uniform translation we can assume that $p = 0$. Then $0 \in \Omega_k$ for all k large enough by Lemma 4.9.

Our uniform bounded assumption means that, by Montel's theorem, there is a holomorphic $f : D \to \overline{\Omega}$ such that $f_k \to f$ after passing to a subsequence. We take this subsequence, so $f_k \to f$ uniformly on compact sets where the codomain is $\mathbb{C}^n$ and $f : D \to \overline{\Omega}$ is holomorphic. We want to show $f(D) \subseteq \Omega$, missing the boundary.

Let $0 < r < 1$ and look at the scaled domain $r\Omega$. It is compactly contained in Ω, so for k large enough $\overline{r\Omega} \subset \Omega_k$ by Lemma 4.9. Moreover, $r\Omega_k \to r\Omega$ in the Hausdorff topology so for k large enough $\overline{r\Omega_k} \subset \Omega$.

Then the functions $rf_k : D \to r\Omega_k$ map into Ω and $rf_k \to rf$. Moreover $rf_k(x) \to 0$. By Theorem 2.27 rf is a map into Ω, so $rf(D) \subseteq \Omega$. This holds for all $0 < r < 1$, and $rf \to f$ as $r \to 1$, so by Theorem 2.27 again $f : D \to \Omega$ as required. □

4.2 Continuity of the Intrinsic Metrics and Measures

There are several topologies at play here and we want to investigate how the intrinsic metrics and measures respect these topologies. We have seen in Proposition 3.21 that the Kobayashi and Carathéodory measures are continuous if the domain is fixed. We investigate now what happens as the domain varies. These proofs were motivated by Bracci, Gaussier, and Zimmer [8]; in particular they presented a proof of the statements for the Kobayashi metric and we adapt the argument for the intrinsic measures. We state the following without proof, since the proof is contained in [8]:

Proposition 4.11. *Suppose Ω_k, Ω are domains in $\mathbb{X}_n$ with $\Omega_k \to \Omega$ and $x_k \in \Omega_k$, $x \in \Omega$, and $x_k \to x$ in $\mathbb{C}^n$. Moreover suppose $v_k, v \in \mathbb{C}^n$ with $v_k \to v$. Then*

$$K_{\Omega_k}(x_k, v_k) \to K_\Omega(x, v).$$

We will have to modify slightly the argument of [8] to show the equivalent statement for the intrinsic measures. First we address the Kobayashi-Eisenman measure:

Proposition 4.12. *Let Ω_k, Ω be in $\mathbb{X}_n$ and suppose $\Omega_k \to \Omega$. Let $x_k \in \Omega_k, x \in \Omega$, and suppose $x_k \to x$ in $\mathbb{C}^n$. Then $\left|M^E_{\Omega_k}(x_k)\right| \to \left|M^E_{\Omega}(x)\right|$.*

Proof. Let $\alpha_k = \left|M^E_{\Omega_k}(x_k)\right|$ and $\alpha = \left|M^E_{\Omega}(x)\right|$. First we show $\alpha \le \liminf \alpha_k$.

Fix some $R > 0$. Define Ω^R_k, Ω^R to be Ω_k, Ω intersected with $B_R(0)$. Suppose R is large enough so the domains contain their corresponding points and then define $\alpha^R_k = \left|M^E_{\Omega^R_k}(x_k)\right|$ and $\alpha^R = \left|M^E_{\Omega^R}(x)\right|$. Note $\Omega^R_k \to \Omega^R$ in the Hausdorff topology by definition of $\Omega_k \to \Omega$. Also, by Proposition 3.20,

$$\lim_{R\to\infty} \alpha^R_k = \alpha_k$$

and

$$\lim_{R\to\infty} \alpha^R = \alpha.$$

Choose a subsequence j of k so that $\lim \alpha^R_j = \liminf \alpha^R_k$. For each j let f_j be a holomorphic map from $\mathbb{B}_n$ to Ω^R_j so that $f_j(0) = x_j$ and $(\det[(df_j)_0])^{-1} = \alpha^R_j$. By Lemma 4.10 we can pass to a subsequence and obtain a holomorphic map $f : \mathbb{B}_n \to \Omega$ with $f_j \to f$. Then $f(0) = x$ and $(\det[df_0])^{-1} = \lim \alpha^R_j$. Thus

$$\alpha^R \le \lim \alpha^R_j = \liminf \alpha^R_k$$

Letting $R \to \infty$ gives, by Proposition 3.20, that $\alpha \le \liminf \alpha_k$.

Now we show $\alpha \ge \limsup \alpha_k$. For each $0 < r < 1$ define K_r to be

$$K_r = \{f(z) : f : \mathbb{B}_n \to \Omega, \|z\| < r\}.$$

Because Ω is hyperbolic K_r is compactly contained in Ω. Skip the first few terms if necessary so that K_r is contained in each Ω_k, this is possible by Lemma 4.9. Also skip some terms if necessary so that each x_k is in K_r.

For each k let $f_k : \mathbb{B}_n \to \Omega$ be so that $f_k(0) = x_k$ and $(\det[(df_k)_0])^{-1} = \left|M_\Omega^E(x_k)\right|$. Define the function $f_{k,r}$ as

$$f_{k,r}(z) = f_k(rz).$$

Then the image of $f_{k,r}$ lies in K_r, so in particular $f_{k,r}$ is a holomorphic map from $\mathbb{B}_n$ to Ω_k. Also $f_{k,r}(0) = x_k$ and

$$(\det[(df_{k,r})_0])^{-1} = r^{-2n}\left|M_\Omega^E(x_k)\right|.$$

Therefore $\alpha_k \leq r^{-2n}\left|M_\Omega^E(x_k)\right|$. Taking the lim sup of both sides gives, by Proposition 3.21, that

$$\limsup \alpha_k \leq r^{-2n}\alpha.$$

Letting $r \to 1$ gives that $\limsup \alpha_k \leq \alpha$, making $\alpha_k \to \alpha$. □

Working with the Carathéodory measure is a little different because instead of changing the codomain we are changing the domain.

Proposition 4.13. *Suppose $\Omega_k, \Omega \in \mathbb{X}_n$, that $x_k \in \Omega_k, x \in \Omega$, and $x_k \to x$. Then*

$$\left|M_\Omega^C(x)\right| = \lim\left|M_{\Omega_k}^C(x_k)\right|.$$

Proof. Let $\beta_k = \left|M_{\Omega_k}^C(x_k)\right|$ and $\beta = \left|M_\Omega^C(x)\right|$. First we show $\beta \geq \limsup \beta_k$.

Let $R > 0$ be big and set $\Omega_k^R = \Omega \cap B_R(0)$, $\Omega^R = \Omega \cap B_R(0)$. Skip the first few k if necessary so that $x_k \in \Omega_k^R$ for all k. Let $\beta_k^R = \left|M_{\Omega_k^R}^C(x_k)\right|$ and $\beta^R = \left|M_{\Omega^R}^C(x)\right|$. By Proposition 3.20 $\beta_k^R \to \beta_k$ and $\beta^R \to \beta$ as $R \to \infty$.

Take a subsequence j of k so that $\lim \beta_j = \limsup \beta_k$. For each j let $f_j : \Omega_j^R \to \mathbb{B}_n$ be holomorphic so that $f_j(x_j) = 0$ and $\det[(df_j)_{x_j}] = \beta_j^R$, possible via Proposition 3.19.

Let $0 < r < 1$. Then $r\Omega^R \Subset \Omega_k^R$ for all k large enough so each f_j restricts to $r\Omega_R$. Moreover $\beta^R = r^{2n} \left| M_{r\Omega^R}^C(0) \right|$ and

$$\left| M_{r\Omega_R}^C(0) \right| \geq \beta_j^R.$$

Therefore

$$\beta^R \geq r^{2n} \beta_j^R.$$

Take $j \to \infty$, $R \to \infty$, and $r \to 1$ and we have

$$\beta \geq \lim \beta_j = \limsup \beta_k.$$

For the other direction, to show $\beta \leq \liminf \beta_k$, take $r, R, \Omega_k^R, \Omega^R, \beta_k^R$, and β^R as before. Uniformly translate so that $x = 0$.

As k gets large enough, $r\Omega_k^R \Subset \Omega^R$. Let the domain S_k be Ω_k translated so that x_k becomes $0 \in S_k$ and take $S_k^R = S_k \cap B_R(0)$. Then $rS_k^R \Subset \Omega^R$ for k large enough as well, since $x_k \to 0$. Let $f : \Omega^R \to \mathbb{B}_n$ be such that $f(0) = 0$ and $\det[df_0] = \beta^R$, possible via Proposition 3.19. Define the function $f_{r,k} : S_k^R \to \mathbb{B}_n$ by

$$f_{r,k}(z) = f(rz).$$

Note the value for $f_{r,k}$ does not depend on k, but the domain does. Clearly $f_{r,k}(0) = 0$ and

$$\det\left[\left(df_{r,k}\right)_0\right] = r^{2n} \beta^R.$$

By definition of β_k^R, because $f_{r,k}$ is defined on S_k^R (which is just a translation of Ω_k^R) we have

$$\beta_k^R \geq r^{2n} \beta^R$$

for all k large enough. Take $R \to \infty$, $\liminf$ over k, and $r \to 1$ to get

$$\liminf \beta_k \geq \beta.$$

Thus $\lim \beta_k = \beta$. □

This concludes the continuity properties of the intrinsic metrics and measures which we will utilize. Next we will see how to obtain convergent sequences of domains.

4.3 What is Blowing Up?

Blowing up a domain Ω consists of obtaining a sequence of affine transformations A_k such that the domains $A_k(\Omega)$ converge to some domain Ω_∞ and a sequence of points $x_k \in \Omega$ so that $A_k(x_k) \to x_\infty$ where $x_\infty \in \Omega_\infty$. The domain Ω_∞ is in general simpler than Ω in some way, and the points x_k typically approach the boundary $\partial\Omega$ so that only the boundary asymptotic behavior in Ω affects Ω_∞. Before getting into how the blowup maps are chosen for the proof of Theorem 4.6, we should address how we deal with convergence of domains.

The nature of the local Hausdorff topology on $\mathbb{X}_n$ means that it is actually quite common for a sequence of domains to have a convergent subsequence. We state this precisely in Theorem 4.15, but first recall the Blaschke selection theorem, a proof of which can be found in Price [52]:

Theorem 4.14 (Blaschke Selection Theorem)**.** *The family of nonempty compact subsets of $\overline{B_R(0)} \subset \mathbb{C}^n$ is compact with respect to the Hausdorff topology, i.e. every sequence of* *compact subsets of $\overline{B_R(0)}$ has a subsequence which converges to some compact subset of $\overline{B_R(0)}$.*

With the Blaschke selection theorem we can show that a quite simple condition suffices for some sequence of domains to contain a convergent subsequence:

Theorem 4.15. *Suppose $p \in \mathbb{C}^n$ and fix a bounded neighborhood U of p. Suppose Ω_k is a sequence of convex domains in $\mathbb{C}^n$, each of which contains U. Then there is a domain Ω and a subsequence k' of k so that $\Omega_{k'} \to \Omega$ in the local Hausdorff topology.*

Proof. Let Ω_k be given. For each $m \in \mathbb{N}$ define

$$\Omega_k^m = \Omega_k \cap B_m(0).$$

Recall $B_m(0)$ is the ball of (euclidean) radius m centered at 0. By the Blaschke selection theorem, the sequence $\overline{\Omega_k^1}$ has a limit point $\overline{\Omega^1}$. Let $k_{1,j}$ be a subsequence of k so that

$$\overline{\Omega_{k_{1,j}}^1} \xrightarrow{j} \overline{\Omega^1}.$$

Assume that for $m \geq 1$ we have constructed domains $\overline{\Omega^i} \subseteq \overline{B_i(0)}$ for each $1 \leq i \leq m$ and subsequences $k_{i,j}$ such that for all $1 \leq i \leq m$ we have

$$\overline{\Omega_{k_{i,j}}^i} \xrightarrow{j} \overline{\Omega^i}.$$

Suppose also that for $1 \leq i < m$, the sequence $k_{i+1,j}$ is a subsequence of $k_{i,j}$. Then look at the sequence

$$\overline{\Omega_{k_{m,j}}^{m+1}}.$$

By the Blaschke selection theorem again, this sequence has a limit point $\overline{\Omega^{m+1}} \subseteq \overline{B_{m+1}(0)}$. Define $k_{m+1,j}$ to be a subsequence of $k_{m,j}$ so that

$$\overline{\Omega^{m+1}_{k_{m+1,j}}} \to \overline{\Omega^{m+1}}.$$

Repeat for all $m \in \mathbb{N}$.

Now we have a convenient increasing property, and that is that $\overline{\Omega^m} \subseteq \overline{\Omega^{m+1}}$. To see this, start with the convergence

$$\overline{\Omega^{m+1}_{k_{m+1,j}}} \to \overline{\Omega^{m+1}}.$$

Intersecting all with $\overline{B_m(0)}$ gives

$$\overline{\Omega^{m+1}_{k_{m+1,j}}} \cap \overline{B_m(0)} = \overline{\Omega^{m}_{k_{m+1,j}}} \to \overline{\Omega^{m+1}} \cap \overline{B_m(0)}.$$

However, $k_{m+1,j}$ is a subsequence of $k_{m,j}$ and we know

$$\overline{\Omega^{m}_{k_{m,j}}} \to \overline{\Omega^{m}}.$$

Therefore $\overline{\Omega^m} = \overline{\Omega^{m+1}} \cap \overline{B_m(0)}$ because they are both limits of the same sequence $\overline{\Omega^{m}_{k_{m+1,j}}}$, showing $\overline{\Omega^m} \subseteq \overline{\Omega^{m+1}}$.

Define $\overline{\Omega}$ to be the union of all $\overline{\Omega^m}$. Then the diagonal subsequence $k_{j,j}$ gives us convergence

$$\overline{\Omega_{k_{j,j}}} \to \overline{\Omega}$$

in the local Hausdorff topology. As the limit of convex sets, $\overline{\Omega}$ is convex. For m large enough each $\overline{\Omega^m}$ contains the neighborhood U, so $U \subset \overline{\Omega}$. That is, the interior Ω of $\overline{\Omega}$ is nonempty because it contains at least U. Also

$$\Omega_{k_{j,j}} \to \Omega$$

because convex domains are uniquely determined by their closures. □

For the actual blowup process we will utilize affine transformations. To be explicit, an affine transformation is an affine linear map on $\mathbb{C}^n$ which is bijective. Affine transformations are biholomorphisms and $\mathbb{X}_n$ is closed under affine transformations; we can check this with the condition in Theorem 4.8.

4.4 Convex Domains Biholomorphic to the Ball

We would like to obtain a condition which particular convex domains biholomorphic to the ball must share. These domains are special because they arise as the limits of the blowup process. This condition is presented by Zimmer in [60]. Suppose $\Omega \subset \mathbb{C}^n$ is convex. We say $\Omega \in \mathcal{H}_n$ if the following two conditions are satisfied:

- $\Omega \cap \mathrm{Span}\{e_2, \cdots, e_n\} = \varnothing$
- $\Omega \cap (\mathbb{C} \cdot e_1) = \{z_1 e_1 : z_1 \in \mathcal{H}\}$

Here $\mathcal{H}$ is the right half plane in $\mathbb{C}$, $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{C}^n$.

In [60] is a proof of the following Lyapunov exponent result for the unit ball $\mathbb{B}^n$:

Lemma 4.16. *Suppose $\gamma_1, \gamma_2 : [0, \infty) \to \mathbb{B}_n$ are two geodesic rays under the Kobayashi metric. If*

$$\liminf_{s,t \geq 0} K_{\mathbb{B}_n}(\gamma_1(s), \gamma_2(t)) < \infty$$

then there is a $T \in \mathbb{R}$ such that

$$\lim_{t \to \infty} K_{\mathbb{B}_n}(\gamma_1(t), \gamma_2(t+T)) = 0.$$

Moreover, if γ_1 and γ_2 lie in the same complex geodesic then

$$\lim_{t\to\infty} \frac{1}{t} \log K_{\mathbb{B}_n}(\gamma_1(t), \gamma_2(t+T)) = -2.$$

Otherwise

$$\lim_{t\to\infty} \frac{1}{t} \log K_{\mathbb{B}_n}(\gamma_1(t), \gamma_2(t+T)) = -1.$$

This result, Lemma 4.16, is used in Zimmer [60] to prove the following asymptotic characteristic of domains in $\mathcal{H}_n$ which are biholomorphic to the ball:

Theorem 4.17. *Suppose $\Omega \in \mathcal{H}_n$ is biholomorphic to the ball $\mathbb{B}_n$. Let $V = Span\{e_2, \cdots, e_n\} \setminus \{0\}$. Then for any $v \in V$*

$$\lim_{r\to\infty} \frac{1}{r} \log \delta_\Omega(e^r e_1, v) = \frac{1}{2}.$$

4.5 Blowing Up to Prove Theorem 4.6

We begin by recalling the statement: Let Ω be a $C^{2+\epsilon}$ convex domain in $\mathbb{C}^n$. Suppose $p \in \partial\Omega$ and

$$\limsup_{\Omega \ni z \to p} \frac{|M_\Omega^E(z)|}{|M_\Omega^C(z)|} = 1.$$

Then p is a point of strong pseudoconvexity. We present the proof next, again mentioning that this is essentially the proof from Zimmer [60] applied to our situation.

Proof of Theorem 4.6. We begin by saying that Ω cannot contain an affine complex line, for if it did then the Carathéodory measure would identically vanish and hence the assumption of asymptotically equivalent measures could not hold. Thus our assumption only applies to hyperbolic convex domains and hence $\Omega \in \mathbb{X}_n$.

Through an affine isometry we can assume that $p = 0$ and that $\mathbf{n}$, the inward normal to $\partial\Omega$ at 0, is $e_1 = (1, 0, \cdots, 0)$.

Suppose for the sake of contradiction that 0 is not a point of strong pseudoconvexity. Recall that by Proposition 2.17 we can then find a $C, \delta > 0$ and a unit vector $v \in T_p\Omega$ so that $\delta e_1 \in \Omega$ and

$$\delta_\Omega(re_1; v) \geq Cr^{\frac{1}{2+\epsilon}} \tag{4.1}$$

for every $0 < r \leq \delta$. With an affine isometry we can further assume that $v = e_2$.

We will later want an explicit cone in Ω with vertex at the origin to aid in the blowup process. The boundary is C^2 and the point is not strongly pseudoconvex, so if we focus more to the origin by shrinking δ if necessary we can ensure

$$re_1 + r\mathbb{D}e_1 \subset \Omega \tag{4.2}$$

for all $0 < r \leq \delta$.

Now to obtain a sequence which forms the basis of our blowup sequence. By (4.1), for all $\alpha > 0$

$$\lim_{r\to 0} \frac{r^{\frac{1}{2+\epsilon}+\alpha}}{\delta_\Omega(re_1; e_2)} = 0.$$

Pick α_k, r_k both going to 0 so that

$$\lim_{k\to\infty} \frac{r_k^{\frac{1}{2+\epsilon}+\alpha_k}}{\delta_\Omega(r_k e_1; e_2)} = 0.$$

Let C_k be so that

$$\delta_\Omega(r_k e_1; e_2) = C_k r_k^{\frac{1}{2+\epsilon}+\alpha_k}.$$

Note $C_k \to \infty$. We want to compare

$$\delta_\Omega(re_1; e_2), C_k r^{\frac{1}{2+\epsilon}+\alpha_k}$$

for $r_k \leq r \leq \delta$. We know the two are equal when $r = r_k$, and we can extrapolate what happens as r increases because $\partial\Omega$ is convex. In particular the boundary term $\delta_\Omega(re_1, e_2)$ can only grow more slowly as r gets bigger, for if its growth accelerated too much then the boundary would have a region of concavity. That is, if we increase r_k a little then we can ensure

$$\delta_\Omega(re_1; e_2) \leq C_k r^{\frac{1}{2+\epsilon}+\alpha_k}$$

for all $r_k \leq r \leq \delta$ while still preserving equality for $r = r_k$.

After increasing r_k, we want to be sure that r_k still goes to 0 as $k \to \infty$. If the $\{r_k\}$ no longer went to 0 then, because $C_k \to \infty$, the boundary distances $\delta_\Omega(re_1; e_2)$ would be unbounded and hence Ω would contain an affine complex line. This cannot happen because Ω is hyperbolic, so $\{r_k\}$ still goes to 0.

By construction $\delta_\Omega(r_k e_1; e_2) = C_k r_k^{\frac{1}{2+\epsilon}+\alpha_k}$, so let $\lambda_k \in \mathbb{C}$ be such that $r_k e_1 + \lambda_k e_2 \in \partial\Omega$ and

$$|\lambda_k| = C_k r_k^{\frac{1}{2+\epsilon}+\alpha_k}.$$

We will use λ_k as a correctional rotation. Now to construct the affine maps. Let $A_k(z) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as

$$A_k(z_1, z_2, \cdots, z_n) = \left(\frac{z_1}{r_k}, \frac{z_2}{\lambda_k}, z_3, \cdots, z_n\right).$$

With $\Omega_k = A_k(\Omega)$, we have by (4.2) that $e_1 + \mathbb{D}e_1 \subset \Omega_k$. Moreover,

$$\left(i\mathbb{R} \times \mathbb{C}^{n-1}\right) \cap \Omega_k = \varnothing$$

$$e_1 + e_2 \in \partial\Omega_k$$

$$e_1 + \mathbb{D}e_2 \subset \Omega_k.$$

Thus, by Theorem 4.15, $\Omega_j \to \Omega_\infty$ for some domain Ω_∞ and some subsequence j of k.

We claim that $\Omega_\infty \cap (\mathbb{C} \cdot e_1)$ consists of the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cdot e_1$. To see this, choose $0 < r \leq \infty$ and $\eta > 0$ and let $S(r, \eta)$ be the truncated cone

$$S(r, \eta) = \{z \in \mathbb{C} : 0 < |z| < r \text{ and } |\operatorname{Im}(z)| < \eta \operatorname{Re}(z)\}.$$

For any η there is an $r > 0$ so that $S(r, \eta) \cdot e_1 \subset \Omega$. Then for any $\eta > 0$ we have

$$S\left(\frac{r}{r_j}, \eta\right) \cdot e_1 \subset \Omega_j$$

and hence the whole cone $S(\infty, \eta) \subset \Omega_\infty$. This holds for all $\eta > 0$ so our claim that

$$\Omega_\infty \cap (\mathbb{C} \cdot e_1) = \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cdot e_1$$

holds.

Now we are ready to state the boundary behavior so desired of Ω_∞: we claim that

$$\delta_{\Omega_\infty}(re_1; e_2) \leq r^{\frac{1}{2+\epsilon}} \tag{4.3}$$

for $r < 1$. This boundary behavior will be half of the contradiction driving our proof. To see why (4.3) holds, observe for $0 < r < 1$

$$\delta_{\Omega_j}(re_1; e_2) = \frac{1}{|\lambda_j|} \delta_\Omega(r_j r e_1; e_2) \leq \frac{1}{|\lambda_j|} C_j (r_j r)^{\frac{1}{2+\epsilon} + \alpha_j} = r^{\frac{1}{2+\epsilon} + \alpha_j}.$$

Taking $j \to \infty$ gives (4.3).

For the other half of the contradiction, we begin with the sequence $q_j = r_j e_1 \in \Omega$ such that $A_j(q_j) = e_1$. Moreover we know $q_j \to 0 \in \partial\Omega$. Thus by assumption

$$\limsup_{j \to \infty} \frac{|M_\Omega^E(q_j)|}{|M_\Omega^C(q_j)|} = 1.$$

The biholomorphic transformation properties of the intrinsic metrics means that

$$\frac{\left|M^E_{\Omega_j}(e_1)\right|}{\left|M^C_{\Omega_j}(e_1)\right|} = \frac{\left|M^E_{\Omega}(q_j)\right|}{\left|M^C_{\Omega}(q_j)\right|}$$

and hence

$$\limsup_{j\to\infty} \frac{\left|M^E_{\Omega_j}(e_1)\right|}{\left|M^C_{\Omega_j}(e_1)\right|} = 1.$$

We take a subsequence l of j so that

$$\limsup_{j\to\infty} \frac{\left|M^E_{\Omega_j}(e_1)\right|}{\left|M^C_{\Omega_j}(e_1)\right|} = \lim_{l\to\infty} \frac{\left|M^E_{\Omega_l}(e_1)\right|}{\left|M^C_{\Omega_l}(e_1)\right|} = 1.$$

This does not change the convergence, i.e. $\Omega_l \to \Omega_\infty$ as well.

By Proposition 4.12 and Proposition 4.13, these measures respect the convergence $\Omega_l \to \Omega_\infty$. That is,

$$\frac{\left|M^E_{\Omega_\infty}(e_1)\right|}{\left|M^C_{\Omega_\infty}(e_1)\right|} = 1.$$

By Wong's Theorem 4.18, Ω_∞ is therefore biholomorphic to the ball. But then Zimmer's Theorem 4.17 states that

$$\lim_{r\to\infty} \frac{1}{r} \log \delta_\Omega(e^r e_1; e_2) = \frac{1}{2}. \tag{4.4}$$

We claim (4.4) provides the other half of the contradiction, being incompatible with (4.3). For by (4.3)

$$\log \delta_{\Omega_\infty}(e^r e_1; e_2) \leq \log\left((e^r)^{\frac{1}{2+\epsilon}}\right) = \frac{r}{2+\epsilon}.$$

Hence

$$\limsup_{r\to\infty} \frac{1}{r} \log \delta_{\Omega_\infty}(e^r e_1; e_2) \leq \frac{1}{2+\epsilon} < \frac{1}{2}.$$

This is the contradiction from which we can conclude that p must be a point of strong pseudoconvexity. $\square$

For reference here is Wong's theorem about intrinsic measures [56]:

Theorem 4.18. *Let Ω be a bounded domain and suppose there is some point $z \in \Omega$ where*

$$\left|M_\Omega^E(z)\right| = \left|M_\Omega^C(z)\right|.$$

Then Ω is biholomorphic to the ball.

We do not give the proof here, instead referring the reader to either Wong [57] or Krantz [40]. The proof is fairly classical but is not nearly as immediate as that of the corresponding statement for the squeezing function.

Chapter 5

The Bergman Kernel on Strongly Pseudoconvex Domains

This chapter is devoted to proving Theorem 5.13, which was done in the hopes of settling the Ramadanov conjecture. Theorem 5.13 essentially says that if the Bergman metric of Ω is asymptotically Kähler-Einstein enough then $\partial\Omega$ is spherical. This is interesting on its own and it can also be used to make one of the key steps of Huang and Xiao's proof [31] of the S.-Y. Cheng conjecture more direct. We start by stating the S.-Y. Cheng and Ramadanov conjectures, then study the ball since the conclusions of both conjectures involve the ball. We then cover the required concepts and finally state and prove Theorem 5.13.

5.1 Conjectures

The ball is the canonical strongly pseudoconvex domain. In some sense, strong pseudoconvexity is a measure of how much a domain is like the ball near the boundary.

That is not to say that all strongly pseudoconvex domains are biholomorphic to the ball – recall Lemma 2.18 and the discussion immediately following. Generic strongly pseudoconvex domains only agree with the ball up to some order. On the other hand, there are many holomorphic invariants associated to any smooth strongly pseudoconvex domain; this is in fact one route by which to show that two such domains are generically holomorphically distinct. It is natural to question whether a domain which admits the same invariants as the ball must be equivalent to the ball, and we mention two particular such conjectures below.

As we will see in section 5.3, the Bergman metric on the ball is Kähler-Einstein. Cheng [11] and Yau [58] conjectured that any smooth strongly pseudoconvex domain where the Bergman metric is Kähler-Einstein must be biholomorphic to the ball; we will refer to this conjecture as the S.-Y. Cheng conjecture in the present work. Burns and Graham [28] confirmed the S.-Y. Cheng conjecture in $\mathbb{C}^2$ circa 1987. Huang and Xiao [31] confirmed the S.-Y. Cheng conjecture in all dimensions higher than 2 circa 2020. For reference we present their proof in section 5.7.

There is a related conjecture posed by Ramadanov in [53]. On a C^∞ bounded strongly pseudoconvex domain Ω the two functions ϕ and ψ in Fefferman's expansion of the Bergman kernel (see section 5.4) are invariantly (partially) defined; in particular ψ is invariantly defined up to $O(\rho^\infty)$ where ρ is a defining function for Ω. Ramadanov posed:

Conjecture 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a C^∞ bounded strongly pseudoconvex domain which has that ψ vanishes to infinite order at the boundary, i.e. $\psi = O(\rho^\infty)$, in Fefferman's expansion of the Bergman kernel. Then Ω has spherical boundary.*

The Ramadanov conjecture has been confirmed for $n = 2$ for quite some time – see Theorem 3.2 in Graham [28] where he attributes the solution to personal communication with Burns. In fact ψ only needs to vanish to $O(\rho^2)$ for this conclusion. However, the Ramadanov conjecture is still open for higher dimensions.

Fu and Wong [27], in particular Proposition 5.9, showed that if the Bergman metric is Kähler-Einstein then $\psi = O(\rho^\infty)$. Nemirovski and Shafikov [48] explicitly recognized that because of Fu-Wong's result the S.-Y. Cheng conjecture would follow from the Ramadanov conjecture. As it turns out, the S.-Y. Cheng conjecture was settled first.

The motivation for the present chapter in the present work was to take Huang and Xiao's proof of the S.-Y. Cheng conjecture and see how far it can go in terms of prove the Ramadanov conjecture. Perhaps unsurprisingly their proof of the S.-Y. Cheng conjecture does not also prove the Ramadanov conjecture. However we were able to weaken the assumption of the S.-Y. Cheng conjecture and still obtain the spherical boundary conclusion, see Theorem 5.13.

In particular it is not necessary to assume that the Bergman metric is Kähler-Einstein on the whole domain, or even that ψ vanishes at all, but only that the Bergman metric becomes asymptotically Kähler-Einstein enough near the boundary. This is only if we want to conclude that the boundary is spherical – we are not claiming that Ω is biholomorphic to the ball under the weaker assumption. That conclusion still requires the full assumption that the Bergman metric is Kähler-Einstein.

5.2 Motivation: The Unit Ball

The following Monge-Ampère operator was introduced by Fefferman in [23]:

$$J(u) = (-1)^n \det \begin{bmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{bmatrix}$$

We will discuss its utility shortly but for now we simply compute for on the canonical defining function on the unit ball because this calculation motivates our calculation in Theorem 5.13.

Proposition 5.2. *The canonical defining function for the ball,*

$$\rho(z) = 1 - \sum_{i=1}^{n} z_i \overline{z_i},$$

yields that $J(\rho)$ is constantly 1.

Proof. We begin by computing the derivatives of ρ:

$$\rho_\alpha(z) = -\overline{z_\alpha}$$

$$\rho_{\overline{\beta}}(z) = -z_\beta$$

$$\rho_{\alpha\overline{\beta}}(z) = -\delta_{ij}.$$

$$J(\rho) = (-1)^n \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}} \end{bmatrix} = (-1)^n \det \begin{bmatrix} 1 - |z|^2 & -z_\beta \\ -\overline{z_\alpha} & -I_n \end{bmatrix}$$

To compute the determinant we do some row operations. For each $1 \le k \le n$ take the $(k+1)^{th}$ row, scale it by $-z_k$, and add it to the first row. None of these operations changes the determinant:

$$J(\rho) = (-1)^n \begin{bmatrix} 1 & 0 \\ -\overline{z_\alpha} & -I_n \end{bmatrix}$$

We conclude by expanding along the top row, obtaining $J(\rho) = 1$. □

5.3 The Kähler-Einstein Condition

Recall that the Bergman metric is given by $(\log K)_{\alpha\overline{\beta}}$ where K is the diagonal of the Bergman kernel. We have seen that any bounded stongly pseudoconvex domain is complete with respect to the Bergman metric. We now look at its to curvature, following the particular approach of Fu-Wong [27].

Let $G(z) = \det[(\log K)_{\alpha\overline{\beta}}]$. The Ricci tensor is given by the components

$$R_{\alpha\overline{\beta}} = -(\log G)_{\alpha\overline{\beta}} = -\Big(\log\det\Big[(\log K)_{\gamma\overline{\delta}}\Big]\Big)_{\alpha\overline{\beta}}.$$

The Bergman metric is Kähler-Einstein if

$$R_{\alpha\overline{\beta}} = c(\log K)_{\alpha\overline{\beta}}$$

for some constant c. The Bergman invariant function M is defined as

$$M(z) = \frac{G(z)}{K(z,z)} = \frac{\det[(\log K)_{\alpha\overline{\beta}}]}{K}.$$

In [23] Fefferman introduced a Monge-Ampère operator which detects some invariants of the boundary of a strongly pseudoconvex domain.

For any positive C^2 function u defined on the strongly pseudoconvex domain Ω, we can form the $(n+1)\times(n+1)$ square matrix of partial derivatives

$$\begin{bmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{bmatrix}.$$

Define the operator J as

$$J(u) = (-1)^n \det \begin{bmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{bmatrix}.$$

As shown in Fu-Wong [27], Fefferman's Monge-Ampère operator J is related to checking whether the Bergman metric is Kähler-Einstein, see Proposition 5.4 below. First we prove a well-known formula involving J:

Proposition 5.3. *For any positive function u, we have*

$$J(u) = u^{n+1} \det \left[(-\log u)_{\alpha\overline{\beta}} \right].$$

Note this matrix has dimensions $n \times n$.

Proof. We start by evaluating the derivatives of $\log u$:

$$(\log u)_\alpha = \frac{u_\alpha}{u}$$

$$(\log u)_{\alpha\overline{\beta}} = \frac{u u_{\alpha\overline{\beta}} - u_\alpha u_{\overline{\beta}}}{u^2} = u^{-1} u_{\alpha\overline{\beta}} - u^{-2} u_\alpha u_{\overline{\beta}}.$$

Now we do some algebra starting from $J(u)$.

$$J(u) = (-1)^n \det \begin{bmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{bmatrix}.$$

For each $1 \le k \le n$ scale the first row by $-u^{-1} u_k$ and add to the $(k+1)^{th}$ row:

$$J(u) = (-1)^n \det \begin{bmatrix} u & u_{\overline{\beta}} \\ 0 & u_{\alpha\overline{\beta}} - u^{-1} u_\alpha u_{\overline{\beta}} \end{bmatrix}.$$

Pull a factor of u out of each row:

$$J(u) = (-1)^n u^{n+1} \det \begin{bmatrix} 1 & u^{-1}u_{\overline{\beta}} \\ & \\ 0 & u^{-1}u_{\alpha\overline{\beta}} - u^{-2}u_\alpha u_{\overline{\beta}} \end{bmatrix}.$$

Bringing a factor of -1 into each of the bottom n rows and expanding along the first column gives our desired result. □

The constant $\frac{n!}{\pi^n}$ occurs naturally while studying the Bergman kernel, see in Proposition 3.4 that it is part of the formula for the Bergman kernel on the ball. Set $C_n = \frac{\pi^n}{n!}$ to cancel this coefficient in later computations.

We will later use a clever trick presented in Fu-Wong [27] giving alternative descriptions of how the Bergman metric can be Kähler-Einstein.

Proposition 5.4. *The following are equivalent:*

$$M = (n+1)^n C_n \iff |J(K)| = (n+1)^n C_n K^{n+2} \iff J\left((C_n K)^{-\frac{1}{n+1}}\right) = 1.$$

Moreover, these conditions are all equivalent to the Bergman metric being Kähler-Einstein.

Proof. For the equivalence of the first two,

$$M = \frac{\det\left[(\log K)_{\alpha\overline{\beta}}\right]}{K} = (-1)^n \frac{J(K)}{K^{n+2}}$$

by the identity in Proposition 5.3. Their equivalence follows immediately.

Now we compute $J\left((C_n K)^{\frac{-1}{n+1}}\right)$ using the same identity.

$$J\left((C_n K)^{\frac{-1}{n+1}}\right) = (C_n K)^{-1} \det\left[\left(-\log(C_n K)^{\frac{-1}{n+1}}\right)_{\alpha\overline{\beta}}\right]$$

$$= (C_n K)^{-1} \det\left[\frac{1}{n+1}\left(\log C_n + \log K\right)_{\alpha\overline{\beta}}\right]$$

$$= \frac{\det\left[(\log K)_{\alpha\overline{\beta}}\right]}{(n+1)^n C_n K} = \frac{M}{(n+1)^n C_n}.$$

From this we conclude that the third is equivalent to the first and hence all three are equivalent.

Now to show the equivalence of these conditions to the Bergman metric being Kähler-Einstein. If M is constant then log of it is constant as well, so its derivatives are

$$\log\left(\frac{\det[(\log K)_{\gamma\overline{\delta}}]}{K}\right)_{\alpha\overline{\beta}} = 0.$$

But then

$$-\left(\log\det[(\log K)_{\gamma\overline{\delta}}]\right)_{\alpha\overline{\beta}} = -(\log K)_{\alpha\overline{\beta}},$$

so the Bergman metric is Kähler-Einstein with coefficient -1.

The reverse direction is less elementary. If the Bergman metric is Kähler-Einstein then we know

$$-\left(\log\det[(\log K)_{\gamma\overline{\delta}}]\right)_{\alpha\overline{\beta}} = c(\log K)_{\alpha\overline{\beta}}$$

for some constant c. The Ricci curvature of the Bergman metric on a smooth storngly pseudoconvex domain approaches -1 as z approaches the boundary; this has been proven in several contexts, see Cheng Yau [12], Klembeck [35], and even Theorem 5.13 below. Therefore c is -1. Hence

$$\left(\log\det[(\log K)_{\gamma\overline{\delta}}]\right)_{\alpha\overline{\beta}} = (\log K)_{\alpha\overline{\beta}}.$$

Then

$$0 = \left(\log\frac{\det[(\log K)_{\gamma\overline{\delta}}]}{K}\right)_{\alpha\overline{\beta}} = (\log M)_{\alpha\overline{\beta}}.$$

By Theorem 2 of Diedrich [18] this means M approaches $(n+1)^n C_n$ as z approaches the boundary. But $\log M$ is pluriharmonic, so by the maximum principle M is constantly $(n+1)^n C_n$. □

We can now show quite painlessly that the Bergman metric on the ball is Kähler-Einstein. In fact, this is an immediate consequence of Propositions 3.6, 5.2, and 5.4 because the function inside J on the third equivalent statement, $(C_n K)^{\frac{-1}{n+1}}$, turns out to be precisely ρ on the ball.

It is worth mentioning the following theorem of Cheng and Yau [12]:

Theorem 5.5. *Let Ω be a C^∞ bounded strongly pseudoconvex domain. Then Ω admits a unique complete Kähler-Einstein metric.*

We will discuss more about this shortly, but first we need the concept of a Fefferman defining function.

5.4 Fefferman's Expansion of the Bergman Kernel

This section focuses on the Bergman kernel $K(z,w)$. Of particular interest is the diagonal K of the Bergman kernel,

$$K(z) = K(z,z)$$

For one, the diagonal function is the potential for the Bergman metric:

$$B_{\alpha\overline{\beta}} = -(\log K)_{\alpha\overline{\beta}}$$

Another reason to focus on the diagonal is that the kernel $K(z,w)$ can be expressed asymptotically in terms of the diagonal up to infinite order [2]:

$$K(z,w) = \sum_{\alpha} \frac{1}{\alpha!} \left(\overline{\partial}^{\alpha} K(z,z)\right)(z)\overline{(w-z)^{\alpha}} + O((w-z)^{\infty})$$

Our knowledge of the diagonal of the Bergman kernel on strongly pseudoconvex domains comes in large part from the work of Fefferman [2], [21], [23]. He showed that if two strongly pseudoconvex domains Ω_1, Ω_2 share a piece of their boundary near a common point $p \in \partial\Omega_1 \cap \partial\Omega_2$ then the two Bergman kernel functions differ by a smooth function near p. Using this fact and the 4^{th} order contact of strongly pseudoconvex domains with the sphere, he was able to estimate the kernel near the boundary of a strongly pseudoconvex domain. Fefferman was able to express the kernel as

$$K(z) = \frac{\phi(z)}{\rho(z)^{n+1}} + \psi(z)\log(\rho(z)).$$

Here ρ is a defining function for Ω, and ϕ and ψ are smooth functions on $\overline{\Omega}$. Moreover, choosing a different defining function for the same domain will yield a (trivially) biholomorphic domain and hence induce a smooth change in the Bergman kernel. That is, ϕ is determined up to order ρ^n and ψ is determined up to order ρ^{∞} regardless of choice of ρ. By $O(\rho^{\infty})$ we mean $O(\rho^k)$ for all k.

As an example of the power of this asymptotic expansion, Klembeck used it in [35] to show that the curvature tensor of the Bergman metric on a smooth strongly pseudoconvex domain approaches the curvature tensor of the metric with constant holomorphic sectional curvature $-4/(n+1)$. He then went on to show that if ρ is the defining function which is signed distance to the boundary, then the curvature tensor of the Kähler metric with components $(-\log \rho)_{\alpha\overline{\beta}}$ approaches the curvature tensor of the metric with constant holomorphic sectional curvature -4.

5.5 Fefferman Defining Function

When Fefferman introduced the Monge-Ampère operator J on a smooth strongly pseudoconvex domain Ω, he also introduced the corresponding Dirichlet problem of finding a positive C^2 function ρ satisfying the following requirements:

$$\begin{aligned} J(\rho) &= 1 \quad \text{in } \Omega \\ \rho &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

Specifically we require that ρ be defined and C^2 on a neighborhood of $\overline{\Omega}$ and that $\rho > 0$ in Ω. Such a ρ is necessarily a defining function of Ω:

Proposition 5.6. *Any solution to the Dirichlet problem (5.1) is a defining function for Ω.*

Proof. Suppose ρ is a solution of (5.1) for the strongly pseudoconvex domain Ω. Then, by assumption, $\rho > 0$ in Ω and $\rho = 0$ on $\partial\Omega$. We merely need to show that $\nabla\rho$ does not vanish on $\partial\Omega$.

For the sake of contradiction suppsose that $\nabla\rho$ does vanish at a point $p \in \partial\Omega$. Then

$$\left(\det \begin{bmatrix} \rho & 0 \\ 0 & \rho_{\alpha\overline{\beta}} \end{bmatrix}\right)_p = 0$$

On the other hand,

$$\lim_{\Omega \ni z \to p} \left(\rho \det \left[\rho_{\alpha\overline{\beta}}\right]\right)_z = 1.$$

Now ρ vanishes in this limit by assumption because $\rho(p) = 0$. This is a contradiction because $\rho_{\alpha\overline{\beta}}$ are all bounded functions on $\overline{\Omega}$. □

As we calculated in Proposition 5.2, the canonical defining function $1 - |z|^2$ on the ball is a Fefferman defining function. In fact it is an exact solution to the Dirichlet problem (5.1), not only an asymptotic partial solution.

Starting with any smooth strongly pseudoconvex domain Ω, Fefferman described a process in [23] for modifying a defining function ρ to make it more like a solution to the Dirichlet problem (5.1). That is, suppose Ω has a C^∞ defining function ρ. Fefferman's process constructs a sequence of C^∞ defining functions $\rho_0, \cdots, \rho_{n+1}$ such that

$$\begin{aligned}
\rho_0 &= \rho & & \\
\rho_1 &= \rho_0 + O(\rho), & J(\rho_1) &= 1 + O(\rho) \\
\rho_2 &= \rho_1 + O(\rho^2), & J(\rho_2) &= 1 + O(\rho^2) \\
&\vdots & &\vdots \\
\rho_{n+1} &= \rho_n + O(\rho^n), & J(\rho_{n+1}) &= 1 + O(\rho^{n+1})
\end{aligned}$$

Fefferman's original process halts here due to the requirement that ρ_k remain C^∞. The specific process halts because it encounters division by 0 in this step. He conjectured that closer approximate solutions could be determined if the steps ρ_k were allowed to contain logarithmic terms as well.

Cheng and Yau [12] showed that any C^∞ strongly pseudoconvex domain admits a unique Kähler-Einstein metric given by a function u, i.e.

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i d\overline{z}_j,$$

where u is a real-valued function satisfying

$$\det\left[\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}\right] = e^{(n+1)u}$$

and such that u blows up to infinity on $\partial\Omega$.

Let v be the positive function $v = e^{-u}$. Then v is a solution to the Dirichlet problem (5.1). Cheng and Yau, recognizing that v cannot be C^{n+2} on $\overline{\Omega}$ because of the logarithmic terms in Fefferman's process, were able to show that v is $C^{n+\frac{3}{2}-\delta}$ on $\overline{\Omega}$ for all $\delta > 0$.

Thus we have two partial solutions to the Dirichlet problem (5.1): If we want a C^∞ solution ρ then we can do so if we weaken the requirement that $J(\rho)$ be constantly 1 to the requirement that $J(\rho) = 1 + O(\rho^{n+1})$. These are called Fefferman defining functions and any C^∞ smooth bounded strongly pseudoconvex domain admits a Fefferman defining function. Moreover the Fefferman defining function is uniquely determined in the sense that if ρ and ρ' are two such defining functions then $\rho - \rho' = O(\rho^{n+1})$.

If instead we want a solution ρ such that $J(\rho) \equiv 1$ then we can do so but we have to weaken the regularity so that ρ is only required to be $C^{n+\frac{3}{2}-\delta}$ on $\overline{\Omega}$ for all $\delta > 0$. This approach gives a Kähler-Einstein metric on Ω and the Kähler-Einstein metric is unique.

5.6 Chern Moser Invariants

Chern-Moser invariants are biholomorphic invariants attached to a domain at a given boundary point. They were introduced by Chern and Moser in [13] for analytic hypersurfaces, which apply when the defining function is analytic, but the invariants remain well-defined for merely C^∞ surfaces as well. The material is summarized in [28] and [22] as well. We start with the analytic case.

Suppose $\Omega \subset \mathbb{C}^n$ is a real analytic strongly pseudoconvex domain, meaning there is a defining function ρ for Ω which is real analytic and such that Ω is strongly pseudoconvex.

To consider ρ as a real function we write it as

$$\rho(z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n})$$

where $z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n}$ are real coefficients.

Suppose that $0 \in \partial\Omega$. Normal form requires singling out one coordinate, so we set $z^n = x + iy$ where x, y are real. Then Ω is said to be in normal form if $\partial\Omega$ is given by the equation

$$2x = |z_1|^2 + \cdots + |z_{n-1}|^2 + \sum_{|\alpha|,|\beta|\geq 2, l\geq 0} A^l_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta y^l.$$

Here α, β are multiindexes of length $(n-1)$ and $A^l_{\alpha\overline{\beta}}$ are complex numbers. There are some restrictions on these numbers given below. The numbers $A^l_{\alpha\overline{\beta}}$ are called the Chern-Moser coefficients.

The restrictions on the Chern-Moser coefficients are that:

- $A^l_{\alpha\overline{\beta}} = \overline{A^l_{\beta\overline{\alpha}}}$
- $\mathrm{Tr}(A^l_{\alpha\overline{\beta}})_{|\alpha|=|\beta|=2} = 0$, i.e. $\sum_{p=1}^{n-1} A^l_{pa\overline{pb}} = 0$ for all l, a, b
- $\mathrm{Tr}^2(A^l_{\alpha\overline{\beta}})_{|\alpha|=3,|\beta|=2} = 0$, i.e. $\sum_{p,q=1}^{n-1} A^l_{pqa\overline{pq}} = 0$ for all l, a
- $\mathrm{Tr}^2(A^l_{\alpha\overline{\beta}})_{|\alpha|=2,|\beta|=3} = 0$, i.e. $\sum_{p,q=1}^{n-1} A^l_{pq\overline{pqb}} = 0$ for all l, b
- $\mathrm{Tr}^3(A^l_{\alpha\overline{\beta}})_{|\alpha|=|\beta|=3} = 0$, i.e. $\sum_{p,q,r=1}^{n-1} A^l_{pqr\overline{pqr}} = 0$ for all l

If Ω is analytic and $p \in \partial\Omega$ then there is a normal form domain Ω' and a biholomorphism from a neighborhood of p to a neighborhood of the origin which takes Ω to Ω'. That is, any analytic strongly pseudoconvex domain can be locally biholomorphically turned into normal form at any point on the boundary.

The specific normal form for a given Ω and $p \in \partial\Omega$ is calculated by computing the Taylor expansion of ρ at p. This process is well-defined for C^∞ domains, giving rise to the same set of Chern-Moser invariants as for the analytic case.

Chern Moser invariants are not uniquely determined, which we can see from the symmetry in the normal form. For one, rescaling $z_1, \cdots, z_{n-1}$ by ϵ and z_n by ϵ^2 preserves the normal form. There are other symmetries too, see Graham [28] for a thorough treatment. In short we call H the group of automorphisms of a domain in normal form which extend to a neighborhood of the origin and which fix the origin.

The normal form is a biholomorphic invariant, but it must be interpreted in a way which is well-defined. Hence:

Definition 5.7. *Let Ω be a C^∞ bounded strongly pseudoconvex domain and let $p \in \Omega$. Let $\{A^l_{\alpha\overline{\beta}}\}$ be the normal form coefficients of $\partial\Omega$ at p and let N be the hypersurface described by these coefficients. Let $w \geq 0$. An invariant of weight w is a polynomial P in the normal form coefficients such that for all $h \in H$*

$$P(hN) = |\det[dh_0]|^{\frac{2w}{n+1}} P(N).$$

Graham [28] showed that the only weight 0 invariants are the constants, that there are no nonzero weight 1 invariants, and that there are no nonzero weight 2 invariants in $\mathbb{C}^2$. In $\mathbb{C}^n$ for $n \geq 3$ the space of weight 2 invariants is one dimensional and is spanned by $\|A^0_{2\overline{2}}\|^2$, defined as

$$\|A^0_{2\overline{2}}\|^2 = \sum_{|\alpha|=|\beta|=2} \left|A^0_{\alpha\overline{\beta}}\right|^2.$$

This particular invariant $\|A^0_{2\overline{2}}\|^2$ vanishes precisely when p is a CR umbilic point. A neighborhood in the boundary which consists entirely of CR umbilic points is a spherical piece of the boundary.

The Bergman kernel transforms according to (3.1) in chapter 3, and hence is related to Chern Moser invariants. In particular we have the following formula from Graham [28], which was also shown independently by Christoffers [15] using direct calculation without the theory of Chern-Moser invariants:

Theorem 5.8. *Let Ω be a C^∞ bounded strongly pseudoconvex domain and let K be the diagonal Bergman kernel. Let ρ be a Fefferman defining function for Ω and take ϕ, ψ to be $C^\infty(\overline{\Omega})$ such that*

$$K = \frac{\phi}{\rho^{n+1}} + \psi \log \rho.$$

Then

$$\phi = \frac{n!}{\pi^n} + O(\rho^2).$$

Moreover, on the boundary

$$\frac{\phi - \frac{n!}{\pi^n}}{\rho^2} = c_n \left\| A^0_{2\overline{2}} \right\|^2$$

where c_n is a universal constant depending only on the dimension and $c_n \neq 0$ when $n \geq 3$.

From Theorem 5.8 we can conclude that $\partial\Omega$ is spherical if and only if

$$\phi = \frac{n!}{\pi^n} + O(\rho^3).$$

5.7 Proof of the Cheng Conjecture

We first present an argument from Fu-Wong [27] that the Kähler-Einstein condition forces a particular smoothness of the Bergman kernel.

Proposition 5.9. *Suppose Ω is a smooth strongly pseudoconvex domain such that the Bergman metric is Kähler-Einstein. If ρ is a C^∞ defining function for Ω and the diagonal Bergman kernel has asymptotic expansion*

$$K = \frac{\phi}{\rho^{n+1}} + \psi \log \rho$$

then ψ is $O(\rho^\infty)$.

Proof. We first introduce some notation to simplify calculations. For $k \geq 0$ let $P(k)$ mean a function of the form

$$f_0 + f_1 \log \rho + f_2 (\log \rho)^2 + \cdots + f_k (\log \rho)^k$$

where each f_i is $C^\infty(\overline{\Omega})$. Then $P(k)$ is a vector space (in fact a $C^\infty(\overline{\Omega})$ module) for each k and if $f \in P(a)$ and $g \in P(b)$ then $fg \in P(a+b)$.

We begin by showing a claim on derivatives. Take the function $f \in P(k)$ as above. We claim $\rho f_\alpha, \rho f_{\overline{\beta}} \in P(k)$. In fact,

$$f_\alpha = (f_0)_\alpha + (f_1)_\alpha \log \rho + \rho^{-1} \rho_\alpha f_1 \log \rho + \cdots + (f_k)_\alpha (\log \rho)^k + k \rho^{-1} \rho_\alpha f_k (\log \rho)^{k-1}.$$

Then $\rho f_\alpha \in P(k)$. In an identitcal manner $\rho f_{\overline{\beta}} \in P(k)$ as well.

We will abuse notation for $P(k)$ in a manner analogous to big O notation; i.e. we will write $f = P(k)$ to mean $f \in P(k)$. In this notation, the above conclusion can be stated that $(P(k))_\alpha = \rho^{-1} P(k)$. Moreover, for any j, k we have $(\rho^j P(k))_\alpha = \rho^{j-1} P(k)$ because

$$\left(\rho^j P(k)\right)_\alpha = \left(j \rho^{j-1} \rho_\alpha\right) P(k) + \rho^j \left(\rho^{-1} P(k)\right)$$

$$= \rho^{j-1} \left(j \rho_\alpha P(k) + P(k)\right) = \rho^{j-1} P(k).$$

The equivalent statement holds for $(\rho^j P(k))_{\overline{\beta}}$ identically.

Back to the task at hand. Recall by Proposition 5.4 that the Bergman metric being Kähler-Einstein is equivalent to the following identity:

$$|J(K)| = (n+1)^n C_n K^{n+2}$$

We rewrite K as $\rho^{-(n+1)}(\phi + \rho^{n+1}\psi \log \rho) = \rho^{-(n+1)}P(1)$ for convenience. The right hand side is fairly straightforward to compute,

$$(n+1)^n C_n K^{n+2} = (n+1)^n C_n \rho^{-(n+1)(n+2)}(\phi + \rho^{n+1}\psi \log \rho)^{n+2}$$

$$= (n+1)^n C_n \psi^{n+1} (\log \rho)^{n+2} + \rho^{-(n+1)(n+2)} P(n+1).$$

For the left hand side we first compute the derivatives of K, utilizing our shorthand notation P:

$$K_\alpha = \rho^{-(n+2)} P(1)$$

$$K_{\overline{\beta}} = \rho^{-(n+2)} P(1)$$

$$K_{\alpha\overline{\beta}} = \rho^{-(n+3)} P(1).$$

Looking at $|J(K)|$, it is

$$\det \begin{bmatrix} K & K_{\overline{\beta}} \\ K_\alpha & K_{\alpha\overline{\beta}} \end{bmatrix} = \det \begin{bmatrix} \rho^{-(n+1)}P(1) & \rho^{-(n+2)}P(1) \\ \rho^{-(n+2)}P(1) & \rho^{-(n+3)}P(1) \end{bmatrix}.$$

We can factor $\rho^{-(n+1)}$ out of the first row, $\rho^{-(n+2)}$ out of the other n rows, and ρ^{-1} out of each column but the first (so n of them) and get

$$|J(K)| = \rho^{-(n+1)}\rho^{-n(n+2)}\rho^{-n} \det \begin{bmatrix} P(1) & P(1) \\ P(1) & P(1) \end{bmatrix} = \rho^{-(n^2+4n+1)} \det \begin{bmatrix} P(1) & P(1) \\ P(1) & P(1) \end{bmatrix}.$$

This is an $(n+1)\times(n+1)$ matrix, so by the product property of $P(1)$ we have

$$|J(K)| = \rho^{-(n^2+4n+1)}P(n+1).$$

That is, the Kähler-Einstein assumption on the Bergman metric forces

$$\rho^{-(n^2+4n+1)}P(n+1) = (n+1)^n C_n \psi^{n+1}(\log\rho)^{n+2} + \rho^{-(n^2+3n+2)}P(n+1).$$

Multiplying by a sufficient power of ρ (in particular n^2+4n+2) and doing some simple algebraic manipulations, we have

$$\rho^{n^2+4n+2}\psi^{n+1}(\log\rho)^{n+2} = P(n+1).$$

Let $f_{n+1} = \rho^{n^2+4n+2}\psi^{n+1}$ and let $f_0, \cdots, f_n \in C^\infty(\overline{\Omega})$ be such that

$$f_{n+1}(\log\rho)^{n+1} = -f_0 - f_1\log\rho - \cdots - f_n(\log\rho)^n.$$

Then

$$f_0 + f_1\log\rho + \cdots + f_{n+1}(\log\rho)^{n+1} = 0.$$

It then follows from Lemma 2.2 of [27] (Proposition 5.10 below) that each f_k is $O(\rho^\infty)$, so in particular $f_k = O(\rho^\infty)$. This is only possible if $\psi = O(\rho^\infty)$. $\square$

The argument above and the Proposition below are from Fu-Wong [27].

Proposition 5.10. *Suppose $f_0, \cdots, f_k$ are C^∞ on $(-\epsilon, \epsilon)$ and suppose*

$$f_0(t) + f_1(t)\log t + \cdots + f_k(t)(\log t)^k = 0.$$

Then each f_i is $O(t^\infty)$, i.e. each vanishes to infinite order at $t = 0$.

Proof. Suppose to the contrary that some of the f_i has a finite order. Let m be the smallest such order, and choose $0 \le j \le k$ to be the largest such that f_j has order m. Then $f_j = a_m t^k + O(t^{m+1})$ where a_m is a nonzero real constant. By multiplying through with $t^{-m}(\log t)^{-j}$ we have

$$0 = \sum_{i=0}^{k} t^{-m} f_i(t)(\log t)^{k-j}$$
$$= \sum_{i=0}^{j-1} t^{-m} f_i(t)(\log t)^{i-j} + (a_m + O(t)) + \sum_{i=j+1}^{k} t^{-m} f_i(t)(\log t)^{i-j}$$

All the functions $t^{-m} f_i(t)$ are smooth to the boundary, and $t^{-m} f_i(t) \to 0$ for $i > j$. Thus letting $t \to 0$ gives us $a_m = 0$, which contradicts the choice of a_m. □

It was shown in Graham [28], who attributed it to a personal communication with Burns, that if $\Omega \subset \mathbb{C}^2$ is a C^∞ strongly pseudoconvex domain such that $\psi = O(\rho^2)$ on Ω then Ω has spherical boundary. This confirms the Cheng conjecture for $\mathbb{C}^2$, as was pointed out by Fu and Wong [27].

We were able to refine Proposition 5.10 to the version presented below. This was done while investigating how much the asymptotically Kähler-Einstein assumption of Theorem 5.13 forces ψ to vanish. We didn't obtain a clean statement in that direction so we merely present the refined version and move on.

Proposition 5.11. *Let $n \in \mathbb{N}, \epsilon > 0$ and suppose $f_0, \cdots, f_k$ are $C^{n+\epsilon}$ on $(-\epsilon, \epsilon)$. Assume*

$$f_0(t) + f_1(t)\log t + \cdots + f_k(t)(\log t)^k = o(t^n).$$

Then each f_i is $o(t^n)$.

Proof. Suppose to the contrary that some of the f_i have

$$\lim_{t\to 0} t^{-l} f_i(t) \neq 0 \tag{5.2}$$

where $l \le n$. Let m be the smallest such l and choose $0 \le j \le k$ to be the largest such that (5.2) applies to f_j with $l = m$. Then $f_j = a_m t^m + o(t^m)$ where a_m is a nonzero constant. By multiplying through with $t^{-m}(\log t)^{-j}$ we have

$$o(t^{n-m}) = \sum_{i=0}^{k} t^{-m} f_i(t)(\log t)^{k-j}$$

$$= \sum_{i=0}^{j-1} t^{-m} f_i(t)(\log t)^{i-j} + (a_m + o(1)) + \sum_{i=j+1}^{k} t^{-m} f_i(t)(\log t)^{i-j} \tag{5.3}$$

All the functions $t^{-m} f_i(t)$ are continuous at 0. For the first sum, where $i < j$, $(\log t)^{i-j} \to 0$ as $t \to 0$. For the second sum $i > j$ so, by choice of m, $t^{-m} f_i(t) \to 0$. The smoothness assumption on f_i means that $t^{-m-\epsilon} f_i(t)$ converges as $t \to 0$. Then

$$\lim_{t \to 0} t^{-m-\frac{\epsilon}{2}} f_i(t)(\log t)^{i-j} = 0.$$

This forces $t^{-m} f_i(t)(\log t)^{i-j} \to 0$ as well. Thus letting $t \to 0$ in (5.3) means $a_m = 0$, which contradicts our choice of a_m. Thus each f_i is $o(t^n)$. □

We are now ready to present Huang and Xiao's proof of the S.-Y. Cheng conjecture for dimensions higher than 2:

Theorem 5.12. *The S.-Y. Cheng conjecture holds in $\mathbb{C}^n$ for all $n \ge 3$.*

Proof. Suppose $n \ge 3$ and $\Omega \subset \mathbb{C}^n$ is a C^∞ bounded strongly pseudoconvex domain such that the Bergman metric is Kähler-Einstein. Let ρ be a Fefferman defining function for Ω and let K be the diagonal Bergman kernel with Fefferman expansion

$$K = \frac{\phi}{\rho^{n+1}} + \psi \log \rho.$$

Take the function u defined as

$$u = \left(\frac{\pi^n}{n!} K\right)^{\frac{-1}{n+1}}.$$

By Proposition 5.4 $J(u) = 1$ because the Bergman metric is Kähler-Einstein. Moreover, by Proposition 5.9 $\psi = O(\rho^\infty)$. Thus u extends to a smooth function on $\overline{\Omega}$. This makes u a Fefferman defining function itself. On the one hand, take the Fefferman expansion with respect to u:

$$K = \frac{\phi'}{u^{n+1}} + \psi \log u.$$

On the other hand, note that by construction

$$K = \frac{\frac{n!}{\pi^n}}{u^{n+1}}.$$

Because ϕ and ϕ' are invariantly defined up to $O(\rho^{n+1})$, we have shown $\phi = \frac{n!}{\pi^n} + O(\rho^{n+1})$. Therefore for any $p \in \partial\Omega$

$$\lim_{z \to p} \rho^{-2}(z) \left(\phi(z) - \frac{n!}{\pi^n} \right) = 0.$$

By Theorem 5.8 p is a CR umbilic point in $\partial\Omega$. This holds for all $p \in \partial\Omega$ so $\partial\Omega$ is spherical. Because every boundary point is spherical it follows from Nemirovskii and Shafikov [47] that Ω is holomorphically covered by the ball. Descending the Bergman metric of the ball onto Ω gives a Kähler-Einstein metric with constant holomorphic sectional curvature. The Kähler-Einstein metric on Ω is unique by Theorem 5.5, and the Bergman metric is Kähler-Einstein, so the Bergman metric has constant holomorphic sectional curvature. Then Ω must be biholomorphic to the ball by Theorem 3.3. □

5.8 Spherical Boundary Asymptotic Condition

We took one key step Huang and Xiao's proof of the S.-Y. Cheng conjecture (Theorem 5.12) and tried applying it to the Ramadanov conjecture. As it turns out, we were unsuccessful in settling the Ramadanov conjecture. We were, however, able to weaken the assumption from the Bergman metric being Kähler-Einstein to a particular statement which essentially says the Bergman metric is asymptotically Kähler-Einstein enough near the boundary.

This result (Theorem 5.13) is interesting on its own, as it provides a way of classifying spherical boundaries in terms of asymptotic boundary behavior of the Bergman kernel, but it also provides a more direct proof of the S.-Y. Cheng conjecture than Huang and Xiao's original approach particularly because it does not rely on Fu-Wong 5.10 or require any vanishing order of ψ.

Recall Proposition 5.4 which says the Bergman metric is Kähler-Einstein if and only if the Bergman invariant function M is constant, particularly is the constant $\frac{(n+1)^n\pi^n}{n!}$.

Theorem 5.13. *Let $\Omega \subset \mathbb{C}^n$ be a bounded C^∞ strongly pseudoconvex domain where $n \geq 3$ with Fefferman defining function ρ. Let M be the Bergman invariant function of Ω and let $p \in \partial\Omega$. Then p is a CR umbilic point in $\partial\Omega$ if and only if*

$$\lim_{z\to p} \rho^{-2}(z)\left(M(z) - \frac{(n+1)^n\pi^n}{n!}\right) = 0. \tag{5.4}$$

Proof. Let K be the diagonal Bergman kernel and ϕ, ψ be $C^\infty(\overline{\Omega})$ such that K has asymptotic expansion

$$K = \frac{\phi}{\rho^{n+1}} + \psi \log\rho = \frac{\phi + \psi\rho^{n+1}\log\rho}{\rho^{n+1}}.$$

We know that $\phi = \frac{n!}{\pi^n}$ on $\partial\Omega$. We cannot ascertain much about ψ on $\partial\Omega$ as there are smooth strongly pseudoconvex domains for which $\psi = 0$ on $\partial\Omega$ and those for which $\psi \neq 0$ on $\partial\Omega$, but we do not actually need to know much about ψ for our calculations.

Let $\Phi = \phi + \psi\rho^{n+1}\log\rho$ so that

$$K = \frac{\Phi}{\rho^{n+1}}.$$

Then Φ is $C^n(\overline{\Omega})$. Moreover, $\Phi = \phi + o(\rho^n)$. Because $n \geq 3$, $\Phi = \phi + o(\rho^3)$.

Let a, b be smooth functions on $\overline{\Omega}$ so that

$$\phi = \frac{n!}{\pi^n}\left(1 + a\rho^2 + b\rho^3 + O(\rho^4)\right).$$

This is possible, specifically the part about the vanishing of the ρ^1 term, by Theorem 5.8. Moreover $a(p) = 0$ if and only if p is a CR umbilic point of $\partial\Omega$.

Because $\Phi = \phi + o(\rho^3)$ we have

$$\Phi = \frac{n!}{\pi^n}\left(1 + a\rho^2 + b\rho^3\right) + o(\rho^3).$$

Let P be $\frac{\pi^n}{n!}\Phi$, so $P = 1 + a\rho^2 + b\rho^3 + o(\rho^3)$, and let the function u be defined as

$$u = \left(\frac{\pi^n}{n!}K\right)^{\frac{-1}{n+1}}$$

$$= \left(\frac{\pi^n}{n!}\Phi\rho^{-(n+1)}\right)^{\frac{-1}{n+1}} = P^{\frac{-1}{n+1}}\rho.$$

The quantity $J(u)$, which takes the form

$$J(u) = (-1)^n \det\begin{bmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{bmatrix},$$

can be considered a measure of how Kähler-Einstein the Bergman metric is – recall Proposition 5.4. Hence we are interested in computing $J(u)$.

We start by computing the derivatives of u. First the α derivative:

$$u_\alpha = \frac{-1}{n+1}P^{\frac{-(n+2)}{n+1}}P_\alpha\rho + P^{\frac{-1}{n+1}}\rho_\alpha$$

$$= P^{\frac{-1}{n+1}}\left(\rho_\alpha - \frac{1}{n+1}P^{-1}P_\alpha\rho\right)$$

$$= u\left(\rho^{-1}\rho_\alpha - \frac{1}{n+1}P^{-1}P_\alpha\right).$$

The $\overline{\beta}$ derivative is similar:

$$u_{\overline{\beta}} = P^{\frac{-1}{n+1}}\left(\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\overline{\beta}}\rho\right)$$

$$= u\left(\rho^{-1}\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\overline{\beta}}\right).$$

Now for the $\alpha\overline{\beta}$ derivative:

$$u_{\alpha\overline{\beta}} = (u_{\overline{\beta}})_\alpha = u_\alpha\left(\rho^{-1}\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\overline{\beta}}\right)$$

$$+u\left(-\rho^{-2}\rho_\alpha\rho_{\overline{\beta}} + \rho^{-1}\rho_{\alpha\overline{\beta}} + \frac{1}{n+1}P^{-2}P_\alpha P_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\alpha\overline{\beta}}\right)$$

$$= u\left(\rho^{-1}\rho_\alpha - \frac{1}{n+1}P^{-1}P_\alpha\right)\left(\rho^{-1}\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\overline{\beta}}\right)$$

$$+u\left(-\rho^{-2}\rho_\alpha\rho_{\overline{\beta}} + \rho^{-1}\rho_{\alpha\overline{\beta}} + \frac{1}{n+1}P^{-2}P_\alpha P_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\alpha\overline{\beta}}\right)$$

$$= P^{\frac{-1}{n+1}}\rho\left(\rho^{-2}\rho_\alpha\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}\rho^{-1}\left(P_\alpha\rho_{\overline{\beta}} + P_{\overline{\beta}}\rho_\alpha\right) + \frac{1}{(n+1)^2}P^{-2}P_\alpha P_{\overline{\beta}}\right)$$

$$+P^{\frac{-1}{n+1}}\rho\left(-\rho^{-2}\rho_\alpha\rho_{\overline{\beta}} + \rho^{-1}\rho_{\alpha\overline{\beta}} + \frac{1}{n+1}P^{-2}P_\alpha P_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\alpha\overline{\beta}}\right)$$

$$= P^{\frac{-1}{n+1}}\rho\left(-\frac{1}{n+1}P^{-1}\rho^{-1}\left(P_\alpha\rho_{\overline{\beta}} + P_{\overline{\beta}}\rho_\alpha\right) + \frac{n+2}{(n+1)^2}P^{-2}P_\alpha P_{\overline{\beta}}\right)$$

$$+P^{\frac{-1}{n+1}}\rho\left(\rho^{-1}\rho_{\alpha\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\alpha\overline{\beta}}\right)$$

$$= P^{\frac{-1}{n+1}}\left(-\frac{1}{n+1}P^{-1}\left(P_\alpha\rho_{\overline{\beta}} + P_{\overline{\beta}}\rho_\alpha\right) + \frac{n+2}{(n+1)^2}P^{-2}P_\alpha P_{\overline{\beta}}\rho + \rho_{\alpha\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\alpha\overline{\beta}}\rho\right).$$

Back in $J(u)$ we now know

$$J(u) = (-1)^n \det\begin{bmatrix} P^{\frac{-1}{n+1}}\rho & P^{\frac{-1}{n+1}}\left(\rho_{\overline{\beta}} - \frac{1}{n+1}P^{-1}P_{\overline{\beta}}\rho\right) \\ P^{\frac{-1}{n+1}}\left(\rho_\alpha - \frac{1}{n+1}P^{-1}P_\alpha\rho\right) & u_{\alpha\overline{\beta}} \end{bmatrix}.$$

We utilize some determinant tricks to simplify this calculation. Factor $P^{\frac{-1}{n+1}}$ out of each of the $n+1$ rows and we have

$$J(u) = \frac{(-1)^n}{P} \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} - \frac{1}{n+1} P^{-1} P_{\overline{\beta}} \rho \\ \rho_\alpha - \frac{1}{n+1} P^{-1} P_\alpha \rho & X_{\alpha\overline{\beta}} \end{bmatrix}$$

where

$$X_{\alpha\overline{\beta}} = -\frac{1}{n+1} P^{-1} \left(P_\alpha \rho_{\overline{\beta}} + P_{\overline{\beta}} \rho_\alpha \right) + \frac{n+2}{(n+1)^2} P^{-2} P_\alpha P_{\overline{\beta}} \rho + \rho_{\alpha\overline{\beta}} - \frac{1}{n+1} P^{-1} P_{\alpha\overline{\beta}} \rho.$$

Add $\frac{1}{n+1} P^{-1} P_{\overline{\beta}}$ times the first column to column $\beta+1$, which does not change the determinant, to get

$$J(u) = \frac{(-1)^n}{P} \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha - \frac{1}{n+1} P^{-1} P_\alpha \rho & Y_{\alpha\overline{\beta}} \end{bmatrix}$$

where

$$Y_{\alpha\overline{\beta}} = -\frac{1}{n+1} P^{-1} P_\alpha \rho_{\overline{\beta}} + \frac{1}{n+1} P^{-2} P_\alpha P_{\overline{\beta}} \rho + \rho_{\alpha\overline{\beta}} - \frac{1}{n+1} P^{-1} P_{\alpha\overline{\beta}} \rho.$$

Add $\frac{1}{n+1} P^{-1} P_\alpha$ times the first row to row $\alpha+1$ to get

$$J(u) = \frac{(-1)^n}{P} \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \frac{1}{n+1} P^{-2} P_\alpha P_{\overline{\beta}} \rho + \rho_{\alpha\overline{\beta}} - \frac{1}{n+1} P^{-1} P_{\alpha\overline{\beta}} \rho \end{bmatrix}$$

$$= \frac{(-1)^n}{P} \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}} + \frac{1}{n+1} \rho \left(P^{-2} P_\alpha P_{\overline{\beta}} - P^{-1} P_{\alpha\overline{\beta}} \right) \end{bmatrix}$$

$$= \frac{(-1)^n}{P} \det \begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}} - \frac{1}{n+1} \rho \frac{P P_{\alpha\overline{\beta}} - P_\alpha P_{\overline{\beta}}}{P^2} \end{bmatrix}.$$

Recall

$$P = 1 + a\rho^2 + b\rho^3 + o(\rho^3).$$

We now approximate to order $o(\rho^2)$. First we use the fact that

$$\frac{1}{1+x} = 1 - x + O(x^2).$$

Let $x = a\rho^2 + o(\rho^2)$ and we have

$$P^{-1} = 1 - a\rho^2 + o(\rho^2).$$

Next we notice that

$$-(\log P)_{\alpha\overline{\beta}} = -\left(\frac{P_\alpha}{P}\right)_{\overline{\beta}} = \frac{P_\alpha P_{\overline{\beta}} - P_{\alpha\overline{\beta}}P}{P^2}.$$

The approximation

$$\log(1+x) = x + O(x^2)$$

helps us here. Let $x = a\rho^2 + b\rho^3 + o(\rho^3)$ and we get

$$\log P = a\rho^2 + b\rho^3 + o(\rho^3).$$

We need the derivatives of this term, so

$$(\log P)_\alpha = a_\alpha\rho^2 + 2a\rho\rho_\alpha + 3b\rho^2\rho_\alpha + o(\rho^2)$$

$$(\log P)_{\alpha\overline{\beta}} = 2a_\alpha\rho\rho_{\overline{\beta}} + 2a_{\overline{\beta}}\rho\rho_\alpha + 2a\rho_\alpha\rho_{\overline{\beta}} + 2a\rho\rho_{\alpha\overline{\beta}} + 6b\rho\rho_\alpha\rho_{\overline{\beta}} + o(\rho)$$

$$= 2a\rho_\alpha\rho_{\overline{\beta}} + 2(a_\alpha\rho_{\overline{\beta}} + a_{\overline{\beta}}\rho_\alpha + a\rho_{\alpha\overline{\beta}} + 3b\rho_\alpha\rho_{\overline{\beta}})\rho + o(\rho).$$

Therefore

$$J(u) = (-1)^n\left(1 - a\rho^2 + o(\rho^2)\right)\times$$

$$\det\begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}} - \frac{2}{n+1}\left(a\rho_\alpha\rho_{\overline{\beta}}\rho + (a_\alpha\rho_{\overline{\beta}} + a_{\overline{\beta}}\rho_\alpha + a\rho_{\alpha\overline{\beta}} + 3b\rho_\alpha\rho_{\overline{\beta}})\rho^2\right) + o(\rho^2) \end{bmatrix}.$$

Add $\frac{2}{n+1}\left(a\rho_\alpha\rho + (a_\alpha + 3b\rho_\alpha)\rho^2\right)$ times the first row to row $\alpha+1$ to give

$$J(u) = (-1)^n\left(1 - a\rho^2 + o(\rho^2)\right)\times$$

$$\det\begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha\left(1 + \frac{2}{n+1}a\rho^2\right) + o(\rho^2) & \rho_{\alpha\overline{\beta}} - \frac{2}{n+1}\left(a_{\overline{\beta}}\rho_\alpha + a\rho_{\alpha\overline{\beta}}\right)\rho^2 + o(\rho^2)\end{bmatrix}.$$

Add $\frac{2}{n+1}a_{\overline{\beta}}\rho^2$ times the first column to column $\beta+1$ to get

$$J(u) = (-1)^n\left(1 - a\rho^2 + o(\rho^2)\right)\det\begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha\left(1 + \frac{2}{n+1}a\rho^2\right) + o(\rho^2) & \rho_{\alpha\overline{\beta}} - \frac{2}{n+1}a\rho_{\alpha\overline{\beta}}\rho^2 + o(\rho^2)\end{bmatrix}$$

$$= (-1)^n\left(1 - a\rho^2\right)\det\begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha\left(1 + \frac{2}{n+1}a\rho^2\right) & \rho_{\alpha\overline{\beta}}\left(1 - \frac{2}{n+1}a\rho^2\right)\end{bmatrix} + o(\rho^2).$$

Pull a factor of $\left(1 + \frac{2}{n+1}a\rho^2\right)$ out from the first column and pull a factor of $\left(1 - \frac{2}{n+1}a\rho^2\right)$ out from the n other columns, giving

$$J(u) = (-1)^n\left(1 - a\rho^2\right)\left(1 + \frac{2}{n+1}a\rho^2\right)\left(1 - \frac{2}{n+1}a\rho^2\right)^n\times$$

$$\det\begin{bmatrix} \rho\left(1 + \frac{2}{n+1}a\rho^2\right)^{-1} & \rho_{\overline{\beta}}\left(1 - \frac{2}{n+1}a\rho^2\right)^{-1} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}}\end{bmatrix} + o(\rho^2).$$

Pull a factor of $\left(1 - \frac{2}{n+1}a\rho^2\right)^{-1}$ out of the first row and we have

$$J(u) = (-1)^n\left(1 - a\rho^2\right)\left(1 + \frac{2}{n+1}a\rho^2\right)\left(1 - \frac{2}{n+1}a\rho^2\right)^{n-1}\times$$

$$\det\begin{bmatrix} \rho\left(1 - \frac{4}{(n+1)^2}a^2\rho^4\right)^{-1} & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}}\end{bmatrix} + o(\rho^2)$$

$$= (-1)^n\left(1 - a\rho^2\right)\left(1 + \frac{2}{n+1}a\rho^2\right)\left(1 - \frac{2}{n+1}a\rho^2\right)^{n-1}\det\begin{bmatrix} \rho & \rho_{\overline{\beta}} \\ \rho_\alpha & \rho_{\alpha\overline{\beta}}\end{bmatrix} + o(\rho^2)$$

$$= \left(1 - a\rho^2\right)\left(1 + \frac{2}{n+1}a\rho^2\right)\left(1 - \frac{2}{n+1}a\rho^2\right)^{n-1} J(\rho) + o(\rho^2).$$

Recall ρ is a Fefferman deifnign function, meaning $J(\rho) = 1 + O(\rho^{n+1})$. Thus

$$J(u) = \left(1 - a\rho^2\right)\left(1 + \frac{2}{n+1}a\rho^2\right)\left(1 - \frac{2}{n+1}a\rho^2\right)^{n-1} + o(\rho^2).$$

Time for one last estimation. We use the fact that

$$(1+x)^{n-1} = 1 + (n-1)x + O(x^2)$$

to get that

$$\begin{aligned} J(u) &= 1 + \left(-1 + \frac{2}{n+1} - 2\frac{n-1}{n+1}\right) a\rho^2 + o(\rho^2) \\ &= 1 - 3\frac{n-1}{n+1} a\rho^2 + o(\rho^2). \end{aligned} \tag{5.5}$$

We want to rewrite (5.5) in a way which more directly involves the Bergman metric. To do this we rewrite $J(u)$ using Proposition 5.3:

$$\begin{aligned} J(u) = J\left(\frac{\pi^n}{n!}K\right)^{\frac{-1}{n+1}} &= \frac{\det\left[-\left(\log\left(\left(\frac{\pi^n}{n!}K\right)^{\frac{-1}{n+1}}\right)\right)_{\alpha\overline{\beta}}\right]}{\frac{\pi^n}{n!}K} \\ &= \frac{n!\det\left[\frac{1}{n+1}\left(\log\frac{\pi^n}{n!} + \log K\right)_{\alpha\overline{\beta}}\right]}{\pi^n K} \\ &= \frac{n!\det\left[\left(\log K\right)_{\alpha\overline{\beta}}\right]}{(n+1)^n \pi^n K}. \end{aligned}$$

Therefore

$$\frac{n!}{(n+1)^n\pi^n} M = 1 - 3\frac{n-1}{n+1}a\rho^2 + o(\rho^2),$$

or

$$M - \frac{(n+1)^n\pi^n}{n!} = -3\frac{(n-1)(n+1)^{n-1}\pi^n}{n!}a\rho^2 + o(\rho).$$

Therefore

$$\lim_{z\to p}\rho^{-2}(z)\left(M(z)-\frac{(n+1)^n\pi^n}{n!}\right)=-3\frac{(n-1)(n+1)^{n-1}\pi^n}{n!}a(p).$$

Because $a(p)=0$ if and only if p is a CR umbilic point in $\partial\Omega$, we have shown our claim. $\square$

Chapter 6

Conclusions

We have presented two main results: Theorem 4.6 and Theorem 5.13.

The first of these is a partial converse to Wong's Theorem 4.1, and we were able to provide Example 4.2, coming from Fornaess and Wold, to show that a full converse is not possible. It should not go without mention that the same example was given in Zimmer [60] for the analogous counterexample with respect to the squeezing function.

Our proof of Theorem 4.6 essentially followed that of Zimmer's similar statement for the squeezing function, with the blowup process borrowed directly from Zimmer. Any discrepencies between our proof of Theorem 4.6 and Zimmer's proof in [60] are not so much significant modifications to the argument as much as they are merely artifacts of restating it; we are applying his argument directly. Our contribution comes from showing that the ratio of intrinsic measures is one such intrinsic function for which Zimmer's argument applies.

As it turns out, Theorem 4.6 was independently proved by Borah and Kar in [7] and was published before our presentation of the current work. Both here and in [7] are results about the continuity of the intrinsic measures for convergent domains, since this is what is necessary to apply Zimmer's blowup process.

The relevant continuity proofs (Proposition 4.12 and Proposition 4.13) provided in the current work are original, being influenced primarily by Zimmer [61] and Royden, Wong P.M., and Krantz [55]. Of particular interest is our Proposition 4.13, which was apparently desired in Borah-Kar [7] but which was not given there. Instead a weaker version was stated and proven there; see their Proposition 3.3.

It would be interesting to drop the convexity assumption in Theorem 4.6. The proof is by the blowup process, however, and that process currently only applies to convex domains. An attempt to drop the convexity assumption was made and it led to an attempt to execute a blowup process on potentially nonconvex domains, but it was ultimately fruitless and hence not presented here.

Theorem 5.13 is a different story. It came from an attempt to utilize Huang and Xiao's proof of the S.-Y. Cheng conjecture to prove the Ramadanov conjecture. Specifically ψ was traced through a key step of their argument, made under the Kähler-Einstein assumption, to see what still holds under the weaker $\psi = O(\rho^\infty)$ vanishing assumption.

As it turns out, ψ ultimately played very little role in the argument. This is perhaps disappointing for one striving to settle the Ramadanov conjecture, but it did lead to a refinement of the proof of the S.-Y. Cheng conjecture. Instead of assuming the Bergman metric is Kähler-Einstein and obtaining that the boundary is spherical, we can weaken the assumption to a particular way of saying how much the Bergman metric is asymptotically Kähler-Einstein as z approaches the boundary. This does appear to be previosuly unknown and may end up finding utility elsewhere. It does not refine the S.-Y. Cheng conjecture itself but merely the proof of the conjecture, since the full Kähler-Einstein assumption is still required in a later step to obtain the conclusion of being biholomorphic to the ball.

Future work on the Ramadanov conjecture can potentially take a few different directions, each in the hopes of using Theorem 5.13 to obtain the spherical conclusion. For one, the result of Cheng and Yau that every smooth strongly pseudoconvex domain admits a unique complete Kähler-Einstein metric is certainly of interest, and it could be hoped that some connection between the Bergman metric and the Kähler-Einstein metric could be made under the assumption that $\psi = O(\rho^\infty)$. But see Proposition 1.9 of Graham [28] which says there are real analytic strictly pseudoconvex domains Ω such that $\partial\Omega$ is not spherical but for which the solution u to Fefferman's Dirichlet problem $J(u) = 1$ is solved to infinite order for a function u which is $C^\infty(\overline{\Omega})$.

Another route could be to study why precisely ψ does not vanish on some domains, specifically how the vanishing of ψ and that of the higher derivatives of ϕ are related. A relationship there along the lines of ϕ having nonvanishing second ρ-derivative meaning ψ has some finite order would certainly confirm the Ramadanov conjecture, but it does not seem apparent that our calculation in Theorem 5.13 is in the right direction for such a result.

A third potential direction for progress on the Ramadanov conjecture could involve a blowup process. Recall Lemma 2.18 which says strongly pseudoconvex domains are locally convex on the boundary. It seems feasible to structure an argument for a contradiction if the hypothesis of Theorem 5.13 and the nonvanishing assumption $\psi \neq O(\rho^\infty)$ are made simultaneously, using the vanishing order of ψ to obtain the affine maps used in the blowup in a way inspired by how Zimmer used the vanishing of the second derivative of ρ in his blowup process in [60]. This direction was the most appealing to the author while performing the work presented here. An attempt at this type of blowup process was made, but it also wound up fruitless and so was not presented.

Bibliography

[1] Theodore J. Barth. Convex domains and Kobayashi hyperbolicity. *Proc. Amer. Math. Soc.*, 79(4):556–558, 1980.

[2] Michael Beals, Charles Fefferman, and Robert Grossman. Strictly pseudoconvex domains in $\mathbf{C}^n$. *Bull. Amer. Math. Soc. (N.S.)*, 8(2):125–322, 1983.

[3] E. Bedford and S. I. Pinchuk. Convex domains with noncompact groups of automorphisms. *Mat. Sb.*, 185(5):3–26, 1994.

[4] Eric Bedford, John P. D'Angelo, Robert E. Greene, and Steven G. Krantz, editors. *Several complex variables and complex geometry. Part 3*, volume 52 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 1991.

[5] Steven R. Bell. Biholomorphic mappings and the $\bar{\partial}$-problem. *Ann. of Math. (2)*, 114(1):103–113, 1981.

[6] Stefan Bergman. *The kernel function and conformal mapping*. American Mathematical Society, Providence, R.I., revised edition, 1970. Mathematical Surveys, No. V.

[7] Diganta Borah and Debaprasanna Kar. Boundary behavior of the Carathéodory and Kobayashi–Eisenman volume elements. *Illinois Journal of Mathematics*, 64(2):151 – 168, 2020.

[8] Filippo Bracci, Hervé Gaussier, and Andrew Zimmer. The geometry of domains with negatively pinched kähler metrics, 2018.

[9] H. J. Bremermann. Holomorphic continuation of the kernel function and the Bergman metric in several complex variables. pages 349–383, 1955.

[10] D. Burns, Jr. and S. Shnider. Spherical hypersurfaces in complex manifolds. *Invent. Math.*, 33(3):223–246, 1976.

[11] S.-Y. Cheng. Open problems. *Conference on Nonlinear Problems in Geometry held in Katata*, 1979.

[12] Shiu Yuen Cheng and Shing Tung Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[13] S. S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta Math.*, 133:219–271, 1974.

[14] Wing-Sum Cheung and Bun Wong. Remarks on two theorems of Qi-Keng Lu. *Sci. China Ser. A*, 51(4):773–776, 2008.

[15] Henry Joseph Christoffers. *The Bergman Kernel and Monge-Ampere Approximations.* ProQuest LLC, Ann Arbor, MI, 1981. Thesis (Ph.D.)–The University of Chicago.

[16] John B. Conway. *Functions of one complex variable.* Springer-Verlag, New York-Heidelberg, 1973. Graduate Texts in Mathematics, 11.

[17] Fusheng Deng, Qi'an Guan, and Liyou Zhang. Properties of squeezing functions and global transformations of bounded domains. *Trans. Amer. Math. Soc.*, 368(4):2679–2696, 2016.

[18] Klas Diederich. Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudo-konvexen Gebieten. *Math. Ann.*, 187:9–36, 1970.

[19] Simon Donaldson. *Riemann surfaces*, volume 22 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2011.

[20] Donald Alfred Eisenman. *Intrinsic Measures on Complex Manifolds and Holomorphic Mappings.* ProQuest LLC, Ann Arbor, MI, 1969. Thesis (Ph.D.)–University of California, Berkeley.

[21] Charles Fefferman. The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.*, 26:1–65, 1974.

[22] Charles Fefferman. Parabolic invariant theory in complex analysis. *Adv. in Math.*, 31(2):131–262, 1979.

[23] Charles L. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.

[24] John Erik Fornæ ss and Erlend Fornæ ss Wold. A non-strictly pseudoconvex domain for which the squeezing function tends to 1 towards the boundary. *Pacific J. Math.*, 297(1):79–86, 2018.

[25] Sidney Frankel. Complex geometry of convex domains that cover varieties. *Acta Math.*, 163(1-2):109–149, 1989.

[26] Sidney Frankel. Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 183–208. Amer. Math. Soc., Providence, RI, 1991.

[27] Siqi Fu and Bun Wong. On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics. *Math. Res. Lett.*, 4(5):697–703, 1997.

[28] C. Robin Graham. Scalar boundary invariants and the Bergman kernel. In *Complex analysis, II (College Park, Md., 1985–86)*, volume 1276 of *Lecture Notes in Math.*, pages 108–135. Springer, Berlin, 1987.

[29] R. E. Greene and H. Wu. Curvature and complex analysis. *Bull. Amer. Math. Soc.*, 77:1045–1049, 1971.

[30] Fritz Hartogs. Einige folgerungen aus der cauchyschen integralformel bei funktionen mehrerer veränderlichen. 1906.

[31] Xiaojun Huang and Ming Xiao. Bergman-Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries. *J. Reine Angew. Math.*, 770:183–203, 2021.

[32] N Kerzman. Taut manifolds and domains of holomorphy in $\mathbb{C}^n$. 16(4):675, 1969.

[33] Norberto Kerzman. The Bergman kernel function. Differentiability at the boundary. *Math. Ann.*, 195:149–158, 1972.

[34] Kang-Tae Kim and Steven G. Krantz. Complex scaling and geometric analysis of several variables, 2008.

[35] Paul F. Klembeck. Kähler metrics of negative curvature, the Bergmann metric near the boundary, and the Kobayashi metric on smooth bounded strictly pseudoconvex sets. *Indiana Univ. Math. J.*, 27(2):275–282, 1978.

[36] Shoshichi Kobayashi. Geometry of bounded domains. *Trans. Amer. Math. Soc.*, 92:267–290, 1959.

[37] Shoshichi Kobayashi. On complete Bergman metrics. *Proc. Amer. Math. Soc.*, 13:511–513, 1962.

[38] Shoshichi Kobayashi. *Hyperbolic manifolds and holomorphic mappings*, volume 2 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1970.

[39] Shoshichi Kobayashi. Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.*, 82(3):357–416, 1976.

[40] Steven G. Krantz. *Function theory of several complex variables*. AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1992 edition.

[41] Steven G. Krantz. The Carathéodory and Kobayashi metrics and applications in complex analysis. *Amer. Math. Monthly*, 115(4):304–329, 2008.

[42] Steven G. Krantz. Pseudoconvexity, analytic discs and invariant metrics. *Bull. Allahabad Math. Soc.*, 23(2):245–262, 2008.

[43] Steven G. Krantz. The impact of the theorem of Bun Wong and Rosay. *Complex Var. Elliptic Equ.*, 59(7):966–985, 2014.

[44] L. Lempert. Holomorphic retracts and intrinsic metrics in convex domains. *Anal. Math.*, 8(4):257–261, 1982.

[45] Qi-Ken Lu. On Kaehler manifolds with constant curvature. *Chinese Math.–Acta*, 8:283–298, 1966.

[46] What are the open subsets of $\mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$, 2021. From Math Overflow, https://mathoverflow.net/questions/4468.

[47] S. Yu. Nemirovskiĭ and R. G. Shafikov. Uniformization of strictly pseudoconvex domains. I. *Izv. Ross. Akad. Nauk Ser. Mat.*, 69(6):115–130, 2005.

[48] S. Yu. Nemirovskiĭ and R. G. Shafikov. Conjectures of Cheng and Ramadanov. *Uspekhi Mat. Nauk*, 61(4(370)):193–194, 2006.

[49] Kiyosi Oka. Sur les fonctions analytiques de plusieurs variables. IV. Domaines d'holomorphe et domaines rationellement convexes. *Jpn. J. Math.*, 17:517–521, 1941.

[50] Sergey Pinchuk. The scaling method and holomorphic mappings. In *Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 151–161. Amer. Math. Soc., Providence, RI, 1991.

[51] M Henri Poincaré. Les fonctions analytiques de deux variables et la représentation conforme. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 23(1):185–220, 1907.

[52] G. Baley Price. On the completeness of a certain metric space with an application to Blaschke's selection theorem. *Bull. Amer. Math. Soc.*, 46:278–280, 1940.

[53] I. P. Ramadanov. A characterization of the balls in $\mathbf{C}^n$ by means of the Bergman kernel. *C. R. Acad. Bulgare Sci.*, 34(7):927–929, 1981.

[54] H. L. Royden. Remarks on the Kobayashi metric. In *Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970)*, pages 125–137. Lecture Notes in Math., Vol. 185, 1971.

[55] Halsey Royden, Pit-Mann Wong, and Steven G. Krantz. The Carathéodory and Kobayashi/Royden metrics by way of dual extremal problems. *Complex Var. Elliptic Equ.*, 58(9):1283–1298, 2013.

[56] B. Wong. Characterization of the unit ball in $\mathbf{C}^n$ by its automorphism group. *Invent. Math.*, 41(3):253–257, 1977.

[57] Bun Wong. Biholomorphic characterization of the ball in $\mathbb{C}^n$ by its automorphism group. University of Toronto, unpublished.

[58] S. T. Yau. Problem section, seminar on differential geomtry (s.-t. yau, eds.). *Ann. of Math. Studies*, 102:669–706, 1982.

[59] Andrew Zimmer. A gap theorem for the complex geometry of convex domains. *Trans. Amer. Math. Soc.*, 370(10):7489–7509, 2018.

[60] Andrew Zimmer. Characterizing strong pseudoconvexity, obstructions to biholomorphisms, and Lyapunov exponents. *Math. Ann.*, 374(3-4):1811–1844, 2019.

[61] Andrew M. Zimmer. Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type. *Math. Ann.*, 365(3-4):1425–1498, 2016.