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The Horizontal Electric Dipole in a Conducting Half-Space, II

Authors
Banos, Alfredo, Jr.
Wesley, James Paul

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THE HORIZONTAL ELECTRIC DIPOLE IN A CONDUCTING HALF-SPACE, II

Alfredo Baños, Jr. and James Paul Wesley

Physics Department and Institute of Geophysics, Los Angeles,

Scripps Institution of Oceanography, La Jolla

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ABSTRACT

This report, Part II, constitutes the culmination of a research study which was described initially in a paper of the same title, Part I, that appeared as SIO Reference 53-33, September 1953, and provides a further account of the mathematical theory involved in the determination of the electromagnetic field components generated by a horizontal electric dipole embedded in a conducting half-space separated from the non-conducting medium above by a horizontal plane. In particular, a detailed account is given of the computations involved for points of observation in the non-conducting medium when the depth of the source and the height of the point of observation are small in comparison with the horizontal range.

The first part of this report is concerned mainly with the general evaluation of the fundamental integrals for both media by the double saddle point method of integration developed earlier, and the salient feature of the present analysis is the fact that the new asymptotic expansions are term-wise differentiable to any order with respect to three essential parameters: horizontal range, depth (or height) of dipole source, and height (or depth) of the point of observation. It is shown that this important achievement is a consequence of applying the saddle point method of integration to a more judicious choice of exponent with the result that the asymptotic expansions presented here are much simpler than those reported in Part I.

The remainder of the report is concerned with the application of the new asymptotic expansions to the evaluation of the Cartesian components of the Hertzian vector and of the cylindrical components of the electromagnetic field vectors for points of observation in the non-conducting medium. Simplified approximations in which numerical substitutions can be readily made are presented for three distinct ranges corresponding to the asymptotic, the intermediate, and the near field; and, in each case, a detailed account is given of the power flow in the field. In addition, there is presented for the first time, for points of observation in the non-conducting medium, an approximation valid down to zero horizontal range, which is attained by equating to zero the propagation constant in the non-conducting medium. Numerical results are given in a manner similar to the numerical example presented in Part I.

CONTENTS TO PART II

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CORRECTIONS TO PART I

Apart from minor typographical errors that are for the most part self-evident, we wish to call the attention of the reader to a number of corrections to our earlier work, which in no wise affect the results given in Part I, but which should be presented here for the sake of accuracy. We list the various corrections by section number.

6.2a Transformation to the α_2-plane.– The specification of the poles, real and virtual, of the integrand for $V_1^{(2)}$ are incorrectly labelled in Fig. 7, although the significant poles which lie in the vicinity of the saddle point, $\alpha_2 = \frac{1}{2}\pi$, are correctly labelled. The proper identification is given in the next item.

6.2e Asymptotic expansion for $V_1^{(2)}$.– The specification of the poles given by Eq. (6.78) and subsequent discussion is incorrect. The correct labelling, in accordance with Eqs. (8.37), follows:

Putting $w_0 = \tan^{-1} n$ (principal value), the real poles P_1 occur in Fig. 7, with $m = 0, \pm 1, \pm 2, \ldots,$ at

$$\alpha_2 = \pm (\tfrac{1}{2}\pi + w_0) + 2m\pi \quad \text{for} \quad \lambda = \pm k_0,$$

and the virtual poles P_2 occur in Fig. 7 at

$$\alpha_2 = \pm (\tfrac{1}{2}\pi - w_0) + 2m\pi \quad \text{for} \quad \lambda = \pm k_0.$$

6.3a Extension of Watson's lemma to include the case at hand. - The discussion of this section as well as the interpretation of Fig. 8 should be completely recast in the light of our new findings as explained in detail in Sections 8.2c and 8.4d. Briefly, in accordance with the correct interpretation illustrated in Fig. 12, we must properly identify $\lambda^{\frac{1}{2}}$ and $\nu^{\frac{1}{2}}$, which are incorrectly given by Eqs. (6.92) and (6.93) respectively, as the $|x|$ and y intercepts of the boundary curves in Fig. 8. That is,

$$\lambda^{\frac{1}{2}} = |x_0| \approx (2k_2\rho)^{\frac{1}{2}} \quad \text{instead of Eq. (6.92);}$$

$$\nu^{\frac{1}{2}} = y_0 = (4k_2\rho)^{\frac{1}{2}} \quad \text{instead of Eq. (6.93),}$$

exactly as we found in the present correct analysis. It should be pointed out that the erroneous interpretation of Fig. 8 led to an estimate of the order of magnitude of the remainder of our asymptotic series which was grossly in error, but that otherwise the results given in Part I are correct as presented. For the sake of accuracy we list all the equations in which reference is made to the remainder as computed from the wrong values of λ and ν; namely, Eqs. (6.96), (6.102), (6.107), (6.119), (6.133), (6.142), (6.149), and (6.156), which remain correct as written so long as we give to λ and ν their correct values specified above.

6.3c Asymptotic expansion for the integrals $M_1^{(2)}$ and $\partial M_1^{(2)}/\partial z$.- The last paragraph in this section, pages 164 and 165, should be recast in the light of our findings concerning the behavior of the first few terms of our asymptotic series, Sections 8.4b and 8.5c. Whereas it remains true that the first few terms of our asymptotic series, old and new, behave as reciprocal powers of $ik_1\rho$, which in fact makes our results useful and practical, it is no longer true that higher order terms behave likewise. On the contrary, according to Watson's lemma, the higher order terms eventually behave as reciprocal powers of $ik_2\rho$.

7.1a Imposition of the condition $|k_1\rho| > 1$.- The first ten lines of the first paragraph in this section are correct as written. The remainder of this paragraph, beginning with the sentence: "Thus, we contend that all integrals ..., etc.," should be deleted because it is incorrect. Instead we should write:

Thus, we contend that all integrals of the type I_1, being exponentially attenuated, are negligible and certainly smaller than the order of magnitude of the error committed in the asymptotic evaluation of the integrals of the type I_2 which are computed over the path C_2 around the branch cut for γ_2 in the λ-plane. To see this, consider an integral of the form (5.10) in which $\phi(x)$ possesses a branch point at $x = x_1$ and is analytic for $|x| < \lambda^{\frac{1}{2}}$, where $\lambda^{\frac{1}{2}} < |x_1|$, and let the branch cut extend from $x = x_1$ to infinity within the sector $|\arg\{x\}| < \frac{1}{4}\pi$ as shown in Fig. 9. Then, according to Cauchy's theorem, the difference between the integral taken over path (1) and the integral taken over path (2) is clearly the contribution around the branch cut, but as already pointed out this contribution is exponentially attenuated and therefore negligible.

PREFACE

This report is the announced sequel to our original communication: "The Horizontal Electric Dipole in a Conducting Half-Space," which appeared as SIO Reference 53-33, September 1953, henceforth referred to as Part I. It provides a further account of the mathematical theory involved in the determination of the electromagnetic field components generated by a horizontal electric dipole embedded in a conducting half-space separated from the non-conducting medium above (Fig. 1) by a horizontal interface. In particular, we complete the results of Part I, which were confined to points of observation in the conducting medium, by giving a detailed account of the computations involved for points of observation in the non-conducting medium when the depth of the source and the height of the point of observation are small in comparison with the horizontal range. This report consists essentially of two additional Chapters which deal, respectively, with an improved method of obtaining the asymptotic expansions for the fundamental integrals, and with the application of the theory for points of observation in the non-conducting medium.

Thus, in Chapter VIII we give a detailed exposition of our new asymptotic expansions for the fundamental integrals which proved to be much simpler than those deduced in Part I. The new method of attack came about as a result of a careful re-appraisal of the methods already employed, for it soon became clear that, due to the greater complexity of the exponential behavior exhibited by the integrands of the fundamental integrals for the non-conducting medium, the formulas and results presented in Chapters VI and VII for the conducting medium, which in their most general form proved to be so cumbersome, would become even more unwieldy when applied to the non-conducting medium.

In addition to the complexity of our earlier expansions we had to contend with the unpleasant fact that our asymptotic series were not term-wise differentiable; that is, could not be differentiated term by term to yield correctly the asymptotic expansion of the corresponding derivative. As a consequence, we had to differentiate the original integral under the sign of integration and then apply anew the double saddle point method of integration to the new integrand to obtain the asymptotic expansion of the derivative, but only at the expense of considerable additional labor.

Furthermore, this critical analysis of our earlier methods disclosed the fact that the complexity of the resulting expansions and their lack of differentiability could be traced directly to the complexity itself of the exponent to which we were applying the double saddle point method of integration. And we soon realized that this complexity was merely a consequence of the fact that we had retained in the exponent the totality of the exponential behavior contained in the integrand, although we recognized that the saddle point method of integration does not require that the complete exponential behavior be retained for the purpose of determining the path of steepest descents, so long as the convergence of the integral is guaranteed beforehand.

Thus, to improve on our earlier expansions, we made a more judicious choice of exponent in applying the method of steepest descents, with the gratifying result that our new asymptotic expansions proved to be much simpler and, most important of all, term-wise differentiable to any order with respect to all pertinent parameters. Furthermore, partly as a consequence of the greater simplicity of the asymptotic series and partly because we approached more directly the problem of obtaining the coefficients of the double power series expansion of the amplitude function in our double integrals, we were able to compute in detail one more term of the asymptotic series than had been possible in the past; and, what is more important, we were able to estimate correctly the order of magnitude and the behavior of all higher order terms.

In this manner we established that higher order terms of our asymptotic series, old and new, behave like $(ik_2\rho)^{-n}$ rather than $(ik_1\rho)^{-n}$, as we had erroneously claimed in Part I, and we were led to re-examine the estimate of the error in our asymptotic series as deduced from Watson's lemma (see Corrections to Part I). This analysis, Sections 8.4b, 8.4d, and 8.4e, shows how to obtain the correct estimate of the error for sufficiently large number of terms, but gives no clue as to the behavior of the first few terms. However, we found that the first few terms of our new asymptotic expansions, like those in Part I, can be written as reciprocal powers of $ik_1\rho$ and, in fact, admit putting $k_2 = 0$ outright, all of which signifies that we have achieved results which prove useful in practice notwithstanding the apparent limitations imposed by Watson's lemma.

As a final remark on Chapter VIII we wish to point out that the present method of attack is quite general and yields at once the evaluation of the fundamental integrals over the contour C_2 around the branch cut for γ_2 (Fig. 10) for both media, thus affording a valuable check on the expansions of Part I. In addition, it is clear that, except for obvious modifications, the present method of attack can be applied equally as well to the evaluation of the contour integrals around the branch cut for γ_1; and, in fact, by a different choice of exponent and hence of branch cuts, it is evident that this method can be applied to other ranges of parameters besides points of observation close to the interface separating the two media. Thus we feel that, in Chapter VIII, we present general methods of wider applicability than is apparent from the applications discussed in this report.

In Chapter IX we consider the application of the general results of Chapter VIII to points in the non-conducting medium for which the height of the point of observation and the depth of the source are much smaller than the horizontal range. We compute the Cartesian components of the Hertzian vector, the cylindrical components of the electromagnetic field vectors E and H, and the components of the time average Poynting's vector, $\frac{1}{2}\mathrm{Re}\{E \times H^*\}$, which yields the power flow, for three distinct ranges of parameters.

First, we consider in Section 9.2 the asymptotic results for $\rho \rightarrow \infty$ or $|n^2k_2\rho| \gg 1$. This means that the horizontal range is so large that Sommerfeld's "numerical distance" is also large and, in fact, larger than unity. Although this range is of no practical interest in the present low frequency study, its examination shows that, for points of observation in the non-conducting medium close to the interface separating the two media (Fig. 1), we observe essentially a surface wave whose equiphase planes are tilted forward (towards increasing ρ) by a very small angle, while its equiamplitude planes are horizontal (parallel to the interface) with downward exponential attenuation. The results of this section are compared with the corresponding results for the conducting medium and the boundary conditions for the field components and for the power flow are verified.

In Section 9.3 we consider the intermediate range, $|n^2k_2\rho| \ll 1 \ll |k_2\rho|$, for which the horizontal range of the point of observation is large when measured in terms of the wavelength in the non-conducting medium, but for which Sommerfeld's numerical distance, approximately $|n^2k_2\rho|$, remains less than unity. This range again is of no practical importance in the low frequency case, but we consider it here in detail for completeness sake and for its historical importance, for it is in this range that the celebrated Sommerfeld attenuation formula applies which provoked in the past so many arguments concerning its validity and the existence of Sommerfeld's electromagnetic surface wave, Section 7.3. In contrast to the corresponding analysis for the conducting medium which was confined exclusively to the fundamental vectors, Section 7.1c, we have computed, for the non-conducting medium, the Hertzian vector, the field components, and the power flow. The salient feature of the present results is that the components of the power flow in the radial and vertical directions behave essentially as $1/\rho^2$ as would be the case for the radiation field of a dipole embedded in an unbounded non-dissipative medium. This result is to be contrasted with the asymptotic $(\rho \to \infty)$ power flow wherein all three components behave as $1/\rho^4$.

Next, we take up in Section 9.4 the so-called practical range, $k_2\rho \ll 1 \ll |k_1\rho|$, for which the horizontal range of the point of observation is large in terms of wavelengths in the conducting medium, but small in terms of wavelengths in the non-conducting medium. It is here where the usefulness of our asymptotic expansions is most clearly brought out: for in spite of the fact that now $k_2\rho \ll 1$, contrary to the limitations imposed by the magnitude of the remainder of an asymptotic series according to Watson's lemma (Section 8.4d), we still obtain useful and practical results for the components of the Hertzian vector, of the electromagnetic field vectors, and of the power flow. It should be emphasized at this juncture that it would have been impossible to obtain accurate results in this range if we had not retained as many as three terms in the asymptotic series for the fundamental integrals, which to our knowledge had never been obtained by our predecessors. Thus, we feel that our formulas for this range and for points of observation in the non-conducting medium constitute significant results which we believe are being presented here for the first time. Furthermore, our formulas for both media are eminently practical in the sense that numerical substitutions can be readily made.

In addition, there is presented for the first time in Section 9.5, for points of observation in the non-conducting medium, an approximation valid down to zero horizontal range, which is attained by equating to zero the propagation constant k_2 in the non-conducting medium. The results for this range, $0 \leqslant k_2\rho \lll 1$, constitute an extension of the Lien approximation which we reported in Section 7.4 for the conducting medium, and together with the results for the three distinct ranges reported here, give us a complete picture of the behavior of the field components as the horizontal range varies from zero to infinity.

Finally, we take up a numerical example with the same data used in Part I, Section 7.5, for the purpose of illustrating the application of our formulas to the computation of the field components in the non-conducting medium for the low frequency case. It is shown that, in the practical range, the horizontal field components vary inversely as the cube of the horizontal range of the point of observation, the z component of the electric field varies inversely as the square of the horizontal range, and the z component of magnetic intensity is negligibly small, varying inversely as the fourth power of the horizontal range; and, finally, all field components are exponentially attenuated with the depth of the source.

VIII. NEW ASYMPTOTIC EXPANSIONS FOR THE FUNDAMENTAL INTEGRALS

In Part I of this report we established asymptotic expansions for the fundamental integrals U_1 and V_1, Eqs. (2.88) and (2.89), by first resolving each integral into the sum of two terms in accordance with Eq. (2.107), and then treating each term, respectively, by the saddle point method for single integration (Section 5.1) and for double integration (Section 5.2). In applying the saddle point method of integration to U_1 and V_1 we chose to retain in the exponential factor of their respective integrands the totality of the exponential behavior therein contained, with the consequence that the resulting asymptotic expansions turned out to be not term-wise differentiable; that is, we established that, to obtain correctly the asymptotic expansion of a given derivative, we had to begin anew by differentiating under the sign of integration and then applying the saddle point method of integration to the new integrand, all of which proved to be a laborious procedure. Furthermore, especially in the case of the integrals over the contour C_2 around the cut for γ_2 (Figure 4), the resulting asymptotic expansions turned out to be much too cumbersome in their most general form, even though we were able to extract useful approximate expressions valid for restricted ranges of the pertinent parameters.

Thus, when we undertook the evaluation of the fundamental integrals U_2 and V_2, Eqs. (2.90) and (2.91), for points of observation in the non-conducting medium, we were led to re-appraise the whole approach with the aim in view of obtaining term-wise differentiable asymptotic series, and simplifying the resulting expansions to the point where numerical substitutions could be readily made. This program we have achieved and it is described in the present Chapter, except that for the moment we have confined our attention to the evaluation of the fundamental integrals over the contour C_2 around the cut for γ_2 (integrals of the type I_2, Section 2.5d), since we have already shown that, in the low frequency case, integrals of the type I_1 are exponentially attenuated and therefore negligible in the working range of parameters.

To achieve term-wise differentiable series it is sufficient to recognize that, in applying the saddle point method of integration, it is not necessary to retain in the exponent the complete exponential behavior of the integrand, which means that there exist several possible formulations of the problem, and we have merely selected the one that seems most appropriate for the present aims. In addition, we have endeavored to treat the fundamental integrals in a sufficiently general way to yield at once the integrals for both media, thus affording a check on our earlier work.

8.1 FUNDAMENTAL INTEGRALS

As explained in Chapter III, the Cartesian components of the Hertzian vectors and the cylindrical components of the electromagnetic field vectors are expressible in terms of four fundamental integrals: U_1 and V_1, Eqs. (2.88) and (2.89), for points of observation in the conducting medium, and U_2 and V_2, Eqs. (2.90) and (2.91), for points of observation in the non-conducting medium above (Figure 1). In this section we establish that all four fundamental integrals can be deduced from two new fundamental integrals, U and V defined below, which in turn are most readily evaluated by introducing two simpler auxiliary integrals that we have labelled J and K.

8.1a Differential equations.- All the integrals discussed here are of the general form

$$I(a,b,\rho) = \int_{-\infty}^{\infty} F(\lambda) e^{-\gamma_1 a - \gamma_2 b} H_0^1(\lambda\rho)\lambda d\lambda, \tag{8.1}$$

which is the generalization of the integral (2.92). In Eq. (8.1) a, b, and ρ are non-negative real parameters and γ_1 and γ_2, as defined by Eqs. (2.58), are given by

$$\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}} \quad \text{and} \quad \gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}} \tag{8.2}$$

in which λ is the (complex) variable of integration. The function $F(\lambda)$ is devoid of any exponential behavior.

It is readily seen, by inspection of the integrand, that all integrals of the type (8.1) satisfy the following partial differential equations:

$$\nabla_\rho^2 I + I_{aa} + k_1^2 I = 0; \tag{8.3}$$

$$\nabla_\rho^2 I + I_{bb} + k_2^2 I = 0; \tag{8.4}$$

$$I_{bb} - I_{aa} = (k_1^2 - k_2^2) I; \tag{8.5}$$

$$k_1^2 I_{bb} - k_2^2 I_{aa} = - (k_1^2 - k_2^2) \nabla_\rho^2 I; \tag{8.6}$$

$$k_1^4 I_{bb} - k_2^4 I_{aa} = - (k_1^4 - k_2^4)(\nabla_\rho^2 + k_o^2 I), \tag{8.7}$$

in which letter subscripts denote partial differentiation with respect to the corresponding parameters and where

$$\nabla_\rho^2 I = (1/\rho)(\partial/\partial\rho)(\rho \ \partial I/\partial\rho). \tag{8.8}$$

In Eq. (8.7) k_o^2 corresponds to the values of λ which yield the zeros of the Sommerfeld denominator, Eq. (2.94), and has the form, symmetric in k_1 and k_2,

$$k_o^2 = k_1^2 k_2^2/(k_1^2 + k_2^2) = k_2^2/(1+n^2), \quad n = k_2/k_1, \tag{8.9}$$

as given already in Eq. (2.95). These differential equations, (8.3) through (8.7), have proved extremely useful in the evaluation of the fundamental integrals.

<u>8.1b Fundamental forms U and V</u>.- We now adopt as fundamental integrals the forms

$$U(a,b,\rho) = \int_{-\infty}^{\infty} \frac{e^{-\gamma_1 a - \gamma_2 b}}{\gamma_1 + \gamma_2} H_o^1(\lambda\rho)\lambda d\lambda$$

$$= -\int_{-\infty}^{\infty} \frac{\gamma_1 - \gamma_2}{k_1^2 - k_2^2} e^{-\gamma_1 a - \gamma_2 b} H_o^1(\lambda\rho)\lambda d\lambda; \tag{8.10}$$

and

$$V(a,b;\rho) = \int_{-\infty}^{\infty} \frac{e^{-\gamma_1 a - \gamma_2 b}}{k_2^2\gamma_1 + k_1^2\gamma_2} H_o^1(\lambda\rho)\lambda d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{k_1^2\gamma_2 \quad k_2^2\gamma_1}{(k_1^4 - k_2^4)(\lambda^2 - k_o^2)} e^{-\gamma_1 a - \gamma_2 b} H_o^1(\lambda\rho)\lambda d\lambda, \tag{8.11}$$

in which the second forms are readily deduced upon making use of Eqs. (8.2) and (8.9), respectively, and in which it is necessary to assume that $a,b \geqslant 0$ in order to insure the convergence of the integrals. The forms U and V are not independent of each other; in fact, making use of Eqs. (8.3) through (8.7), we can exhibit U in terms of V as follows:

$$U = \frac{k_1^2 V_{bb} - k_2^2 V_{aa}}{k_1^2 - k_2^2} - V_{ab}; \tag{8.12}$$

$$U = - \nabla_\rho^2 V - V_{ab}; \tag{8.13}$$

$$U = k_1^2 V + (V_a - V_b)_a; \tag{8.14}$$

$$U = k_2^2 V - (V_a - V_b)_b, \tag{8.15}$$

in which only the last form, Eq. (8.15), had been previously pointed out by us in Eq. (2.71a).

Making use of the new fundamental forms (8.10) and (8.11), it is readily seen that our original fundamental integrals, as defined in Section 2.4d, may now be expressed, with $h \geqslant 0$, as

$$U_1 = U(h-z, 0, \rho), \qquad z \leqslant 0; \tag{8.16}$$

$$V_1 = k_1^2 V(h-z, 0, \rho), \qquad z \leqslant 0; \tag{8.17}$$

$$U_2 = U(h, z, \rho), \qquad z \geqslant 0; \tag{8.18}$$

$$V_2 = k_2^2 V(h, z, \rho), \qquad z \geqslant 0, \tag{8.19}$$

in which, it is recalled, U_1 and V_1 belong to points of observation in the conducting medium and have been amply treated in Part I, whereas U_2 and V_2 correspond to points of observation in the non-conducting medium and, hence, their evaluation constitutes the main purpose of the present report.

8.1c Auxiliary forms J and K.- To facilitate the asymptotic evaluation of the fundamental integrals U and V, as defined by Eqs. (8.10) and (8.11), it proves convenient to introduce the auxiliary integrals

$$J(a,b,\rho) = \int_{-\infty}^{\infty} e^{-\gamma_1 a-\gamma_2 b} H_0^1(\lambda\rho)\lambda d\lambda; \tag{8.20}$$

and

$$K(a,b,\rho) = \int_{-\infty}^{\infty} \frac{e^{-\gamma_1 a-\gamma_2 b}}{\lambda^2-k_0^2} H_0^1(\lambda\rho)\lambda d\lambda, \tag{8.21}$$

which are obviously related to each other as follows:

$$J = -(\nabla_\rho^2 K + k_0^2 K) = \frac{k_1^4 K_{bb} - k_2^4 K_{aa}}{k_1^4 - k_2^4}, \tag{8.22}$$

where the last equality follows immediately from Eq. (8.7).

In terms of the auxiliary integrals (8.20) and (8.21), we have at once for our fundamental integrals (8.10) and (8.11) the important working formulas

$$U = (J_a - J_b)/(k_1^2-k_2^2); \tag{8.23}$$

and

$$V = (k_2^2 K_a - k_1^2 K_b)/(k_1^4-k_2^4), \tag{8.24}$$

which are readily established by inspection.

8.1d Choice of cuts in the λ-plane.- In Eq. (8.1) the original path of integration is along the real axis in the λ-plane, $-\infty < \lambda < \infty$. Before discussing the present choice of cuts for γ_1 and γ_2 and the corresponding deformation of the path of integration it is well to review briefly the arguments that led to the choice of cuts described in detail in Section 2.5b and depicted graphically in Fig. 4. In Part I we were concerned mainly with the asymptotic evaluation of the fundamental integrals U_1 and V_1 for points of observation in the conducting medium. It is seen from Eqs. (8.16) and (8.17) that $b = 0$ in that case and therefore γ_2 does not enter into the exponential factor of the corresponding integrands. Thus, when $b = 0$, the choice of cuts for γ_2 is quite arbitrary so long as we comply with the requirement that $Re\{\gamma_2\} > 0$ on the original path of integration (assuming that k_2 has a positive imaginary past, no matter how small). As indicated in Section 2.5b we chose the cuts for γ_2 in such a way that $Im\{\gamma_2\} < 0$ everywhere on the cut λ-plane with $Re\{\gamma_2\} > 0$ along the real axis, and this was achieved by drawing the cuts for γ_2 as depicted in Fig. 4. When $b > 0$ no such freedom of choice is available to us.

As regards the cuts for γ_1 it was pointed out in Section 2.5b that, in order to guarantee the convergence of the integrals U_1 and V_1 as $z \longrightarrow -\infty$, we must choose the cuts for γ_1 in such a way that $Re\{\gamma_1\} > 0$ for all values of λ on the original path of integration and on the corresponding sheet of the Riemann surface; and this was achieved as described in Section 2.5b and depicted graphically in Fig. 4. More generally, the parameters a and b assume the values $a = h-z$ and $b = 0$ for points of observation in the conducting medium, or else $a = h$ and $b = z$ for points of observation in the non-conducting medium. Furthermore, in practice we find that the depth or height of the source as given by h and the depth or height of the point of observation given by z will in general be much smaller than the horizontal range ρ (see Fig. 1) and, in no practical case, would we be required to insure the convergence of the integrals (8.1) except for <u>finite</u> values of a and b. Therefore, in the present instance, we are no longer compelled to choose the cuts for γ_1 in the manner already described.

Thus, bearing in mind that the parameters a and b in our general integral (8.1) are to remain finite and smaller than ρ, we have chosen the cuts for γ_1 and γ_2, as shown in Fig. 10, by drawing half-lines in the upper half-plane parallel to the axis of imaginaries, starting at the respective branch points $\lambda = k_1$ and $\lambda = k_2$, and similarly for the lower half-plane; that is, the chosen cuts are specified by the conditions

$$\mathrm{Re}\left\{\lambda\right\} = \mathrm{Re}\left\{\pm k_1\right\} \quad \text{and} \quad \mathrm{Re}\left\{\lambda\right\} = \pm k_2 \tag{8.25}$$

(since k_2 is essentially real). It can be readily ascertained that this choice of cuts in no wise modifies the nature of the poles of the integrands that exhibit the Sommerfeld denominator

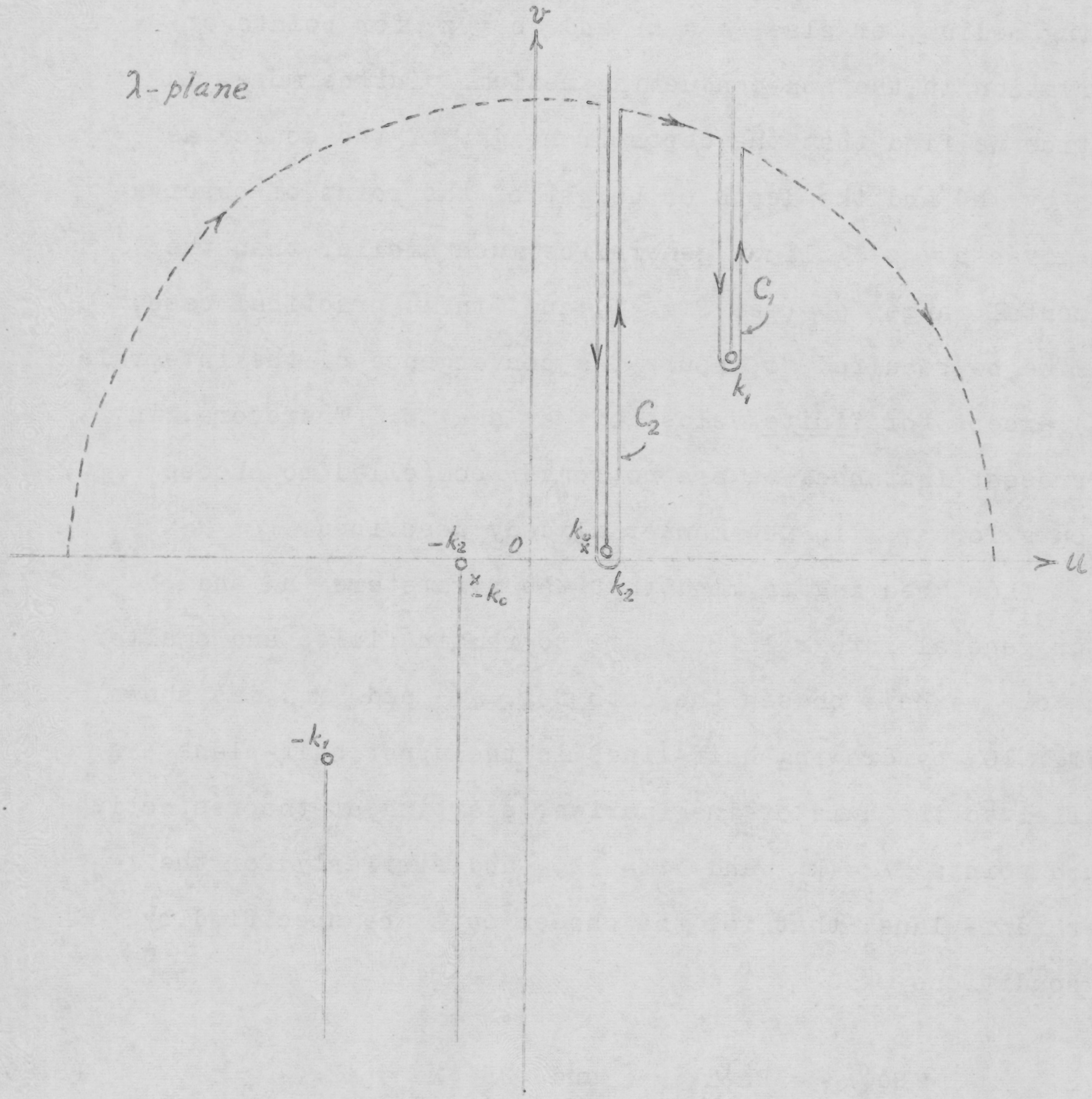

Fig. 10.- The λ-plane showing the new cuts for γ_1 and γ_2 and the deformed path of integration.

(2.94), as listed in the last column of Table II, page I-51, nor the structure of the Riemann surface of four sheets in the λ-plane shown schematically in Fig. 5. In this sense, much of the discussion in Section 2.5b and all of Section 2.5c are still applicable in the present instance.

Finally, following the arguments adduced in Section 2.5d we now deform the original path of integration as shown in Fig. 10. That is, starting on the real axis at $\lambda \longrightarrow -\infty$, the proposed path follows, first, the semi-circle at infinity in the second quadrant, next the contour C_2 completely around the upper branch cut for γ_2, then the contour C_1 completely around the corresponding branch cut for γ_1 and, finally, the semi-circle at infinity in the first quadrant terminating on the real axis at $\lambda \longrightarrow +\infty$. By Cauchy's theorem, the proposed path is completely equivalent to the original path along the real axis, for there are no singularities of the integrand between the two paths. Furthermore, it can be readily shown that the contribution over the semi-circle at infinity in the upper half-plane vanishes, with the result that we can express our original integral (8.1), in the manner of Eq. (2.107), as the sum of two integrals,

$$I = I_1 + I_2, \tag{8.26}$$

where I_1 and I_2 denote the integrals evaluated on the contours C_1 and C_2 respectively.

As shown later in Section 8.2a, the principal merit of the present choice of cuts is the fact that the contours C_1

and C_2 of Fig. 10 are <u>themselves</u> paths of steepest descents for the exponent $i\lambda\rho$ which is implicit in the asymptotic behavior of the Hankel function $H_0^1(\lambda\rho)$ in the integrand of (8.1), and it turns out that, if we apply the saddle point method of integration exclusively to the exponent $i\lambda\rho$, the resulting asymptotic expansions are term-wise differentiable to any order with respect to all three parameters a, b, and ρ. Furthermore, the resulting asymptotic expansions are considerably simpler than our earlier developments, with the result that we have been able to compute higher order terms, thus enabling us to ascertain more accurately the behavior of the first few terms of our new asymptotic expansions.

The present Chapter is concerned mainly with the evaluation of the fundamental integrals over the contour C_2, since we have already pointed out in Section 7.1a that the integrals over the contour C_1 are exponentially attenuated and therefore entirely negligible in the low frequency case. However, it should be pointed out here that the methods of evaluation described below for integrals over the contour C_2 are equally applicable, except for trivial modifications, to the evaluation of the integrals over the contour C_1.

8.2 EVALUATION OF $J^{(2)}$

The integral in question, $J(a,b,\rho)$, is the auxiliary form defined by Eq. (8.20) in which the path of integration is the real axis in the λ-plane. Deforming the path of integration in accordance with Fig. 10 and selecting for study and evaluation the integral over the path C_2 we have

$$J^{(2)}(a,b,\rho) = \int_{C_2} e^{-\gamma_1 a-\gamma_2 b} H_0^1(\lambda\rho)\lambda d\lambda, \tag{8.27}$$

where, it is recalled, C_2 denotes the contour around the upper branch cut for γ_2 which, in the λ-plane, is the half-line from $\lambda = k_2$ to $\lambda = k_2 + i\infty$ (parallel to the axis of imaginaries); i.e., C_2 is defined by the conditions

$$\mathrm{Re}\{\lambda\} = k_2; \quad \mathrm{Im}\{\lambda\} \geqslant 0 \quad \text{on} \quad C_2. \tag{8.28}$$

We now proceed to evaluate $J^{(2)}$ by the double saddle point method of integration following the essential prescriptions laid out in Section 5.2, except for a more enlightened approach to the problem of obtaining the power series expansions that are needed in the application of the method as described below.

8.2a The conformal transformation $\lambda = k_2\cos w$.- Following the analysis described in detail in Sections 6.2a and 6.2b, we now apply directly the conformal transformation

$$\lambda = k_2\cos w; \qquad d\lambda = -k_2\sin w\, dw, \tag{8.29}$$

according to which we have, from Eqs. (8.2),

$$\gamma_1 = (\lambda^2-k_1^2)^{\frac{1}{2}} = -ik_1(1-n^2\cos^2 w)^{\frac{1}{2}}; \tag{8.30}$$

and

$$\gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}} = ik_2 \sin w, \tag{8.31}$$

where $n = k_2/k_1$ as defined by Eq. (2.64). As shown in Fig. 11, the conformal transformation (8.29) maps the entire Sheet I of the Riemann surface in the λ-plane (Fig. 10) unto the curvilinear strip of width π whose boundaries represent, respectively, the mappings of the contour C_2 around the upper branch cut for γ_2 and of the corresponding contour about the lower branch cut.

In fact, the equation of the boundaries can be readily deduced from the conditions

$$\operatorname{Re}\{\cos w\} = \pm 1; \quad \operatorname{Im}\{\cos w\} \gtrless 0, \tag{8.32}$$

which follow at once from (8.28) and (8.29). Writing $\lambda = u + iv$ and imposing the first condition (8.32) we obtain

$$\cos u \cosh v = 1 \tag{8.33}$$

as the equation of the contour C_2, from which, solving for v, one readily deduces

$$v = \log \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}u). \tag{8.34}$$

It is readily seen from (8.34) that, as $v \longrightarrow \pm\infty$, $u \longrightarrow \mp\frac{1}{2}\pi$, respectively, and that the slope of the curve as it crosses the real axis at $u = 0$ is -1 as indicated

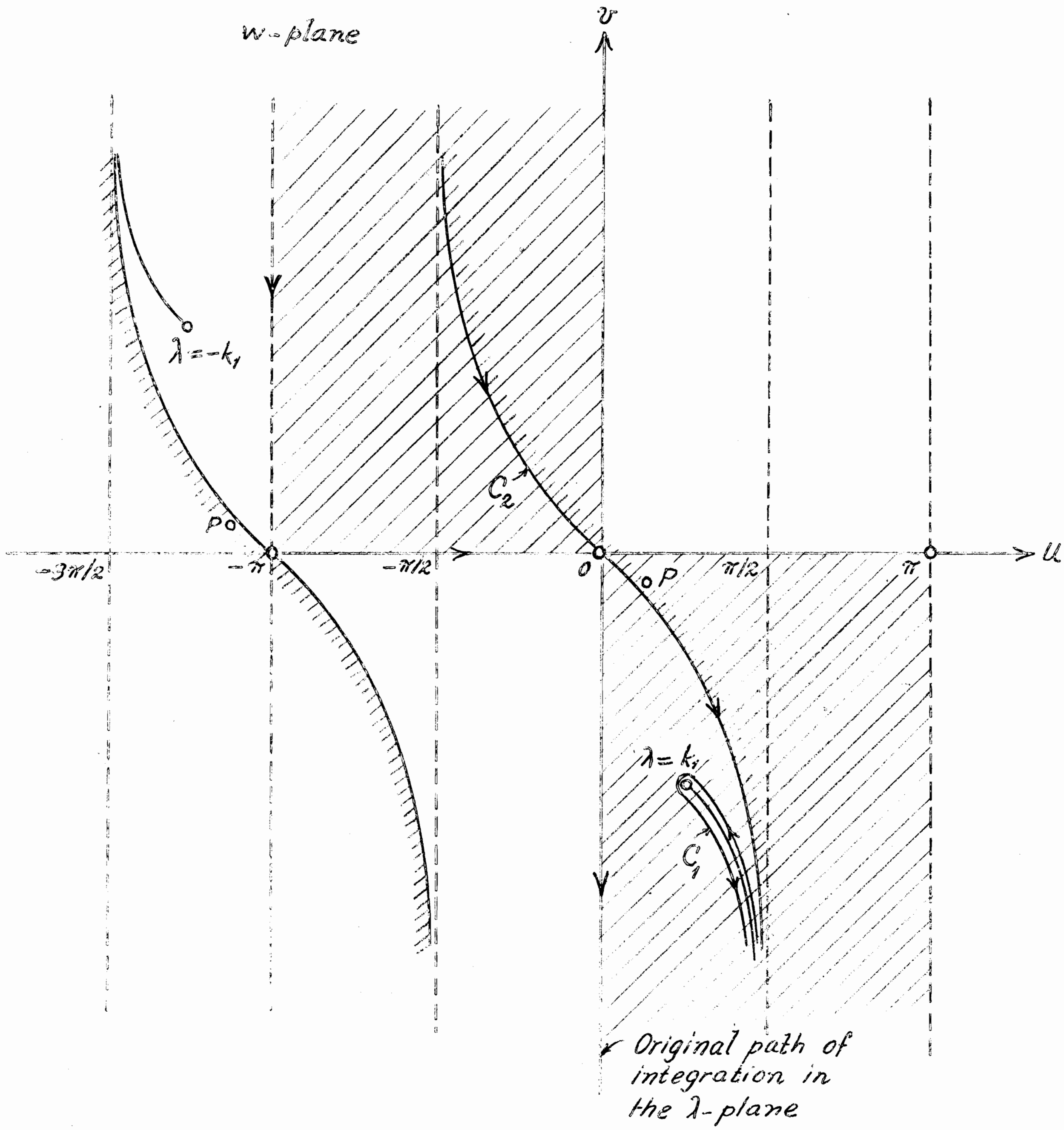

Fig. 11.- The w-plane illustrating the half-period strip corresponding to Sheet I in the λ-plane and the path of steepest descents C_2.

by the symmetric boundary labelled C_2 in Fig. 11. Similarly, one obtains for the contour around the lower branch cut for γ_2 in the λ-plane the equation

$$v = \log \tan(-\tfrac{1}{4}\pi - \tfrac{1}{2}u), \tag{8.35}$$

which corresponds to the symmetric left-hand boundary in Fig. 11 crossing the real axis at $u = -\pi$. It is clear from Section 2.5b and Fig. 5 that the half-period curvilinear strip shown in Fig. 11 corresponds to Sheet I of the Riemann surface in the λ-plane and that it is bordered by similar half-period strips corresponding to the (periodic) mapping of Sheet III unto the w-plane.

Next, we must display in Fig. 11 the location of the branch points of γ_1 and the mapping of the corresponding branch cuts. These branch points occur in the λ-plane at $\lambda = \pm k_1$ and, therefore, making use of Eq. (8.29), we readily ascertain that the branch points for γ_1 on Sheet I of the λ-plane occur, on the w-plane, at

$$w = w_1 = \cos^{-1}(1/n) \quad \text{for} \quad \lambda = k_1;$$

and (8.36)

$$w = -w_1 - \pi = -\cos^{-1}(1/n) - \pi \quad \text{for} \quad \lambda = -k_1,$$

where $n = k_2/k_1$ and $\cos^{-1}(1/n)$ signifies the principal value.

Finally, to complete the picture in the w-plane we need to indicate in Fig. 11 the position of the zeros of the

Sommerfeld denominator, Eq. (2.94), which constitute real poles of the corresponding integrands. In accordance with earlier discussion (Section 2.5b) the points $\lambda = \pm k_o$, where k_o is defined by Eq. (2.96), are zeros of the Sommerfeld denominator only on Sheets II and III. Therefore in Fig. 11, which displays Sheet I bordered by Sheet III, we find from Eq. (8.29) that the poles occur at

$$w = w_o = \tan^{-1} n \quad \text{for} \quad \lambda = k_o;$$

and (8.37)

$$w = - w_o - \pi = - \tan^{-1} n - \pi \quad \text{for} \quad \lambda = - k_o,$$

where $n = k_2/k_1$ and $\tan^{-1} n$ denotes the principal value. These poles, which are labelled P on Fig. 11, fall just <u>outside</u> of the principal half-period strip corresponding to Sheet I and, therefore, occur on Sheet III in accordance with <u>Table II</u>. It is recalled that the w-plane is in point of fact a Riemann surface of two sheets and that Fig. 11 exhibits only the one corresponding to Sheets I and III of the λ-plane. Thus, the poles that occur on Sheet II of the λ-plane are not displayed in Fig. 11.

Following the prescriptions set down in Section 6.2b, we now apply the conformal transformation (8.29) to the variable of integration in Eq. (8.27), obtaining the integral

$$J^{(2)}(a,b,\rho) = - k_2^2 \int_{C_2} F(w) e^{\phi(w)} dw, \qquad (8.38)$$

where C_2 is the path of integration shown in Fig. 11 and where the exponent $\phi(w)$ is given by

$$\phi(w) = ik_2\rho \cos w, \tag{8.39}$$

as a consequence of which we have, in Eq. (8.38),

$$F(w) = G(w) \cos w \sin w \left\{ H_0^1(k_2\rho \cos w) e^{-ik_2\rho \cos w} \right\}; \tag{8.40}$$

$$G(w) = \exp\left\{ ik_1 a(1-n^2\cos^2 w)^{\frac{1}{2}} - ik_2 b \sin w \right\}. \tag{8.41}$$

Thus, we have exhibited the integral $J^{(2)}(a,b,\rho)$ in the form (5.2) which we require in preparation for the application of the saddle point method of integration. And, it is clear from Eq. (8.39), that with the present choice of exponent the saddle point in which we are interested occurs at $w = 0$, for which the path of steepest descents is given by imposing the condition

$$\mathrm{Im}\left\{ ik_2\rho \cos w \right\} = \mathrm{Im}\left\{ ik_2\rho \right\} \tag{8.42}$$

in accordance with Eq. (6.62). But the condition (8.42) is seen to be precisely the condition (8.28) satisfied by the contour C_2 in both the λ and the w planes. Therefore, the chosen path of integration in Eq. (8.27) is already the path of steepest descents for the variable w and the exponent $\phi(w)$ given by Eq. (8.39).

8.2b Transformation to the x-plane.- In accordance with the prescriptions of Section 6.2c we now introduce the new variable of integration x defined by

$$\tfrac{1}{2}x^2 = \phi(0) - \phi(w) = ik_2\rho(1-\cos w), \tag{8.43}$$

from which we deduce, putting

$$\xi = cx; \qquad c = (ik_2\rho)^{-\frac{1}{2}}, \tag{8.44}$$

the following useful relations:

$$\begin{aligned} \cos w &= 1 - \tfrac{1}{2}\xi^2; \\ \sin w &= \xi(1-\tfrac{1}{4}\xi^2)^{\frac{1}{2}}; \\ \sin w(dw/dx) &= c\xi. \end{aligned} \tag{8.45}$$

It is recalled that, in accordance with Section 5.1, the conformal transformation (8.43), from the w-plane unto the x-plane, maps the path of steepest descents C_2 of Fig. 11 unto the real axis in the x-plane. Thus, transforming to the x variable of integration in Eq. (8.38) and bisecting the interval of integration in accordance with Eq. (5.10), we obtain

$$J^{(2)}(a,b,\rho) = - k_2^2 e^{ik2\rho} \int_0^{\infty} \left[F(w)+F(-w)\right](dw/dx)e^{-\frac{1}{2}x^2}dx, \tag{8.46}$$

in which $F(w)$ is given by Eqs. (8.40) and (8.41) with $w = w(x)$ in accordance with (8.43).

8.2c Representation as a double integral.- In preparation for the application of the double saddle point method of integration described in Section 5.2 we need to replace the bracket in Eq. (8.40) by the integral representation

$$H_0^1(k_2\rho \cos w)e^{-ik_2\rho \cos w} = \frac{4}{\pi}\int_0^\infty (4ik_2\rho \cos w - y^2)^{-\frac{1}{2}}e^{-\frac{1}{2}y^2}dy \qquad (8.47)$$

which follows from Eq. (5.50). Thus, introducing (8.47) into the integrand of (8.46) and, proceeding in accordance with Eq. (5.51), we obtain the double integral

$$J^{(2)}(a,b,\rho) = -2k_2^2e^{ik_2\rho}\cdot\frac{2}{\pi}\int_0^\infty\int_0^\infty \Phi(x,y)e^{-\frac{1}{2}(x^2+y^2)}dxdy, \qquad (8.48)$$

where the function $\Phi(x,y)$ is the product of two factors,

$$\Phi(x,y) = f(x)\ g(x,y), \qquad (8.49)$$

in which, with $w = w(x)$ from Eq. (8.43), we have

$$f(x) = \left[G(w)-G(-w)\right]\cos w \sin w\ (dw/dx); \qquad (8.50)$$

$$G(w) = \exp\left\{ik_1au(x) - ik_2bv(x)\right\}; \qquad (8.51)$$

$$u(x) = (1-n^2\cos^2 w)^{\frac{1}{2}} = (1-n^2)^{\frac{1}{2}}\left\{1 + \frac{n^2\xi^2}{2(1-n^2)} - \frac{n^2\xi^4}{8(1-n^2)^2} + \frac{n^4\xi^6}{16(1-n^2)^3} - \cdots\right\}, \quad |\xi|^2 < |2(1-n^2)/n|; \tag{8.52}$$

$$v(x) = \sin w = \xi(1-\tfrac{1}{4}\xi^2)^{\frac{1}{2}} = \xi\left\{1 - \frac{\xi^2}{8} - \frac{\xi^4}{128} - \frac{\xi^6}{1024} - \cdots\right\}, \quad |\xi|^2 < 4; \tag{8.53}$$

$$g(x,y) = (4ik_2\rho\ \cos w - y^2)^{-\frac{1}{2}} = \tfrac{1}{2}c\left\{1 - \tfrac{1}{4}c^2(2x^2+y^2)\right\}^{-\frac{1}{2}}, \tag{8.54}$$

in which, it is recalled, $c = (ik_2\rho)^{-\frac{1}{2}}$ and $\xi = cx$ in accordance with Eqs. (8.44).

<u>8.2d Expansion of $f(x)$</u>.- The application of the double saddle point method of integration requires the expansion of the function $\Phi(x,y)$, Eq. (8.49), into a double power series in x and y. To expand the factor $f(x)$ we first consider, from Eqs. (8.50) and (8.51), the function

$$\tfrac{1}{2}\left[G(w) - G(-w)\right] = \tfrac{1}{2}e^{ik_1au(x)}\left[e^{-ik_2bv(x)} - e^{ik_2bv(x)}\right] = -i\ \sin\left[k_2bv(x)\right]e^{ik_1au(x)}; \tag{8.55}$$

and, making use of the expansion (8.52), we obtain

$$e^{ik_1 au(x)} = A_o(1+A_2\xi^2+A_4\xi^4+A_6\xi^6+ \ldots), \quad |\xi|^2 < |2(1-n)/n|, \quad (8.56)$$

in which the expansion coefficients are

$$A_o = e^{ik_1 a(1-n^2)^{\frac{1}{2}}};$$

$$A_2 = \tfrac{1}{2} ik_1 an^2(1-n^2)^{-\frac{1}{2}};$$

$$A_4 = -\frac{1}{8}\, ik_1 an^2(1-n^2)^{-3/2} + \frac{1}{8}\,(ik_1 a)^2 n^4(1-n^2)^{-1}; \quad (8.57)$$

$$A_6 = \frac{1}{16}\, ik_1 an^4(1-n^2)^{-5/2} - \frac{1}{16}\,(ik_1 a)^2 n^4(1-n^2)^{-2}$$

$$+ \frac{1}{48}\,(ik_1 a)^3 n^6(1-n^2)^{-3/2}.$$

Proceeding similarly with the remaining factor in Eq. (8.55) we find, making use of Eq. (8.53), the expansion

$$- i \sin\left[k_2 bv(x)\right] = B_1\xi(1+B_2\xi^2+B_4\xi^4+B_6\xi^6+ \ldots), \quad |\xi|^2 < 4, \quad (8.58)$$

with the expansion coefficients

$$B_1 = - ik_2 b;$$

$$B_2 = -\frac{1}{3!}\left[(k_2 b)^2 + \frac{3}{4}\right];$$

$$(8.59)$$

$$B_4 = \frac{1}{5!}\left[(k_2 b)^4 + \frac{15}{2}\,(k_2 b)^2 - \frac{15}{16}\right];$$

$$B_6 = -\frac{1}{7!}\left[(k_2 b)^6 + \frac{105}{4}\,(k_2 b)^4 + \frac{315}{16}\,(k_2 b)^2 + \frac{315}{64}\right].$$

Multiplying out the expansions (8.56) and (8.58) in accordance with Eq. (8.55) we obtain the expansion

$$\tfrac{1}{2}\left[G(w)-G(-w)\right] = C_1\xi(1+C_2\xi^2+C_4\xi^4+C_6\xi^6+\ \ldots), \quad |\xi|^2 < 4, \qquad (8.60)$$

with the expansion coefficients

$$\begin{aligned} C_1 &= A_oB_1; \\ C_2 &= A_2 + B_2; \\ C_4 &= A_4 + A_2B_2 + B_4; \\ C_6 &= A_6 + A_4B_2 + A_2B_4 + B_6. \end{aligned} \qquad (8.61)$$

Finally, introducing Eqs. (8.45) and (8.60) into (2.50) we obtain the desired power series expansion for $f(x)$, namely

$$\begin{aligned} f(x) &= 2cC_1\xi^2(1-\tfrac{1}{2}\xi^2)(1+C_2\xi^2+C_4\xi^4+C_6\xi^6+\ \ldots) \\ &= D_2x^2 + D_4x^4 + D_6x^6 + D_8x^8 + \ \ldots, \quad |x|^2 < 4k_2\rho, \end{aligned} \qquad (8.62)$$

in which the expansion coefficients are

$$\begin{aligned} D_2 &= 2c^3C_1; & & \\ D_4 &= 2c^5C_1(C_2-\tfrac{1}{2}); & D_4/c^2D_2 &= C_2 - \tfrac{1}{2}; \\ D_6 &= 2c^7C_1(C_4-\tfrac{1}{2}C_2); & D_6/c^4D_2 &= C_4 - \tfrac{1}{2}C_2; \\ D_8 &= 2c^9C_1(C_6-\tfrac{1}{2}C_4); & D_8/c^6D_2 &= C_6 - \tfrac{1}{2}C_4. \end{aligned} \qquad (8.63)$$

The question of the radius of convergence of the power series expansion (8.62) deserves a word of explanation. First, the radius of convergence indicated for the expansion (8.56) is simply given by the branch points for γ_1 occurring in $u(x)$, Eq. (8.52); whereas, the radius of convergence of the expansion (8.58) is given by the branch points of the function $v(x)$, Eq. (8.53), which is seen to be much smaller than the radius of convergence for the expansion (8.56). In consequence, the power series expansion (8.62), which involves the product of the expansions (8.56) and (8.58), Eq. (8.60), is limited by the smaller of the two radii of convergence. In expressing the radius of convergence in (8.62) we have made use of Eqs. (8.44).

8.2e Expansion of $g(x,y)$.- According to Eq. (5.72), the function $g(x,y)$ defined by (8.54) possesses a double power series expansion of the form

$$g(x,y) = B_0^0 + (B_2^0x^2+B_0^2y^2) + (B_4^0x^4+B_2^2x^2y^2+B_0^4y^4) + \dots, \qquad (8.64)$$

in which the expansion coefficients, deduced directly from the second form of (8.54), are tabulated below:

$$\begin{array}{llll}
B_0^0 = \frac{1}{2}\, c & & & \\
B_2^0 = \frac{1}{8}\, c^3 & B_0^2 = \frac{1}{16}\, c^3 & & \\
B_4^0 = \frac{3}{64}\, c^5 & B_2^2 = \frac{3}{64}\, c^5 & B_0^4 = \frac{3}{256}\, c^5 & \\
B_6^0 = \frac{5}{256}\, c^7 & B_4^2 = \frac{15}{512}\, c^7 & B_2^4 = \frac{15}{1024}\, c^7 & B_0^6 = \frac{5}{2048}\, c^7
\end{array} \qquad (8.65)$$

where, it is recalled, $c = (ik_2\rho)^{-\frac{1}{2}}$.

The double power series expansion (8.64) is merely a formal development, using the binomial theorem, of the expression

$$g(x,y) = \tfrac{1}{2}c\left\{1 - \tfrac{1}{4}c^2(2x^2+y^2)\right\}^{-\frac{1}{2}} , \qquad (8.66)$$

and we must now determine for what domains of the complex x and y planes does the expansion (8.64) actually converge. To this end, we observe from (8.66) that the function $g(x,y)$ exhibits branch points for values of x and y satisfying the equation

$$2x^2 + y^2 = 4/c^2 = 4ik_2\rho. \qquad (8.67)$$

Thus, we note from Eq. (8.67) that, as x varies, $|y|$ must always remain less than $\left|4ik_2\rho - 2x^2\right|^{\frac{1}{2}}$; therefore, constructing the function

$$\begin{aligned} y = h(x) &= \left|4ik_2\rho - 2x^2\right|^{\frac{1}{2}}; \\ h(0) = (4k_2\rho)^{\frac{1}{2}}; &\quad h(x) \xrightarrow[x \to \infty]{} 2^{\frac{1}{2}}x, \end{aligned} \qquad (8.68)$$

which is depicted graphically in Fig. 12, we can assert that, for fixed real values of x $(0 \leqslant x < \infty)$, the function $g(x,y)$ possesses a convergent power series expansion in even powers of y provided $|y| < h(x)$. That is, provided that, for real values of y, this variable remains below the boundary

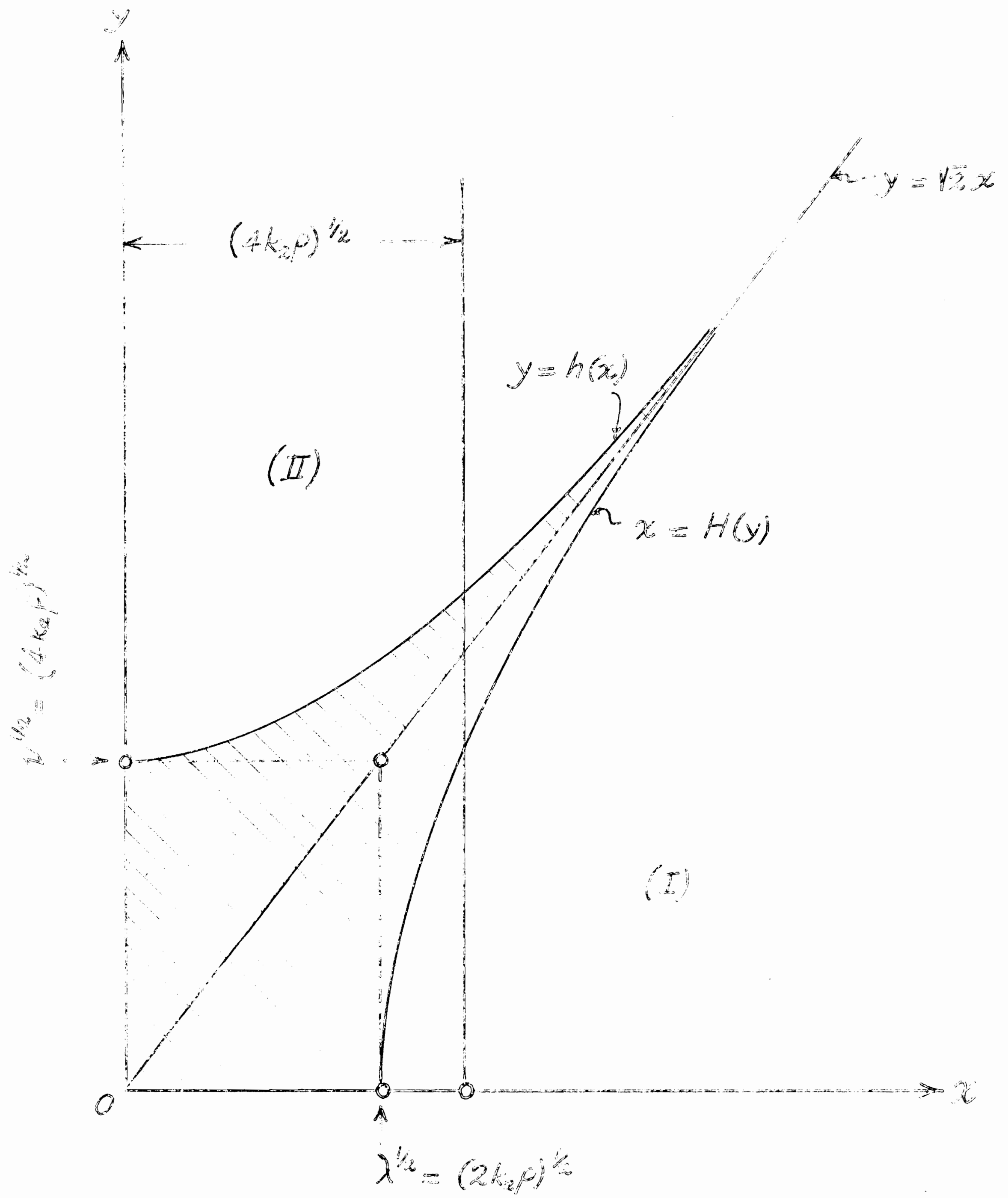

Fig. 12.- Domains of analiticity of the functions g(x,y), f(x), and $\Phi(x,y) = f(x)\, g(x,y)$ as explained in the text.

$h(x)$ and, hence, within the domain labelled region (I) in Fig. 12.

Similarly, we note from Eq. (8.67) that, as y varies, $|x|$ must always remain less than $|2ik_2\rho - \frac{1}{2}y^2|^{\frac{1}{2}}$. Hence, if we construct the function

$$x = H(y) = |2ik_2\rho - \tfrac{1}{2}y^2|^{\frac{1}{2}};$$
$$H(0) = (2k_2\rho)^{\frac{1}{2}}; \quad H(y) \xrightarrow[y \to \infty]{} 2^{-\frac{1}{2}}y, \tag{8.69}$$

which is likewise drawn in Fig. 12, we can assert that, for <u>fixed</u> real values of y $(0 \leqslant y \leqslant \infty)$, the function $g(x,y)$ possesses a convergent power series expansion in even powers of x provided $|x| \leqslant H(y)$. That is, provided that, for real values of x, this variable remains to the left of the boundary $H(y)$ and, hence, within the domain labelled region (II) in Fig. 12.

We note further from Eqs. (8.68) and (8.69) and from Fig. 12 that regions (I) and (II) overlap yielding a common region of analiticity (shaded region) within which the function $g(x,y)$ must be regarded as an analytic function of the two complex variables x and y. That is, we interpret this result by asserting that the double power series expansion (8.64) for $g(x,y)$ is valid and therefore convergent, provided $|x|$ and $|y|$ remain within the common region of analiticity bounded by the curves $h(x)$ and $H(y)$ and the coordinate axes in Fig. 12. In Section 8.4c we return to this matter indicating how these results govern the asymptotic behavior of our expansions and how do they enter into the estimate of the order of magnitude of the remainder. It will then be shown in what respect the similar analysis in Part I, Section 6.3a and Fig. 8, is partly erroneous and how must we correct our earlier estimates of the order of magnitude of the remainder (see Corrections to Part I).

8.2f Computation of the expansion terms.- Once in possession of the expansions (8.62) for $f(x)$ and (8.64) for $g(x,y)$ we can proceed to the calculation of the expansion terms in the asymptotic evaluation of the double integral (8.48) by merely following the prescriptions of Section 5.2b. Thus, making use of Eqs. (5.74) and (8.65) we obtain, in the present notation, the expansion terms:

$$\begin{aligned}
\Phi^{(0)} &= 0; \\
\Phi^{(1)} &= \tfrac{1}{2}cD_2; \\
\Phi^{(2)} &= \tfrac{1}{2}c^3D_2\left[\frac{7}{8} + \frac{3D_4}{c^2D_2}\right]; \\
\Phi^{(3)} &= \tfrac{1}{2}c^5D_2\left[\frac{225}{128} + \frac{33D_4}{8c^2D_2} + \frac{15D_6}{c^4D_2}\right]; \\
\Phi^{(4)} &= \tfrac{1}{2}c^7D_2\left[\frac{5445}{1024} + \frac{1467D_4}{128c^2D_2} + \frac{225D_6}{8c^4D_2} + \frac{105D_8}{c^6D_2}\right],
\end{aligned} \tag{8.70}$$

which must be here regarded as general formulas applicable to all the integrals discussed in this Chapter.

Next, substituting the coefficients (8.63) into (8.70) we obtain the expansion terms $\Phi^{(n)}$ corresponding to our integral $J^{(2)}(a,b,\rho)$, namely:

$$\Phi^{(o)} = 0;$$

$$\Phi^{(1)} = c^4 C_1;$$

$$\Phi^{(2)} = \tfrac{1}{2}c^6 C_1\left[6C_2 - \frac{5}{4}\right]; \tag{8.71}$$

$$\Phi^{(3)} = \frac{1}{8}c^8 C_1\left[120C_4 - 27C_2 - \frac{39}{16}\right];$$

$$\Phi^{(4)} = \frac{1}{48}c^{10} C_1\left[5040C_6 - 1170C_4 - \frac{999}{8}C_2 - \frac{1269}{64}\right],$$

where the C's are given by Eqs. (8.61) in terms of the A's and B's defined by Eqs. (8.57) and (8.59) respectively.

<u>8.2g Asymptotic expansion for $J^{(2)}$</u>.- Recalling from Eq. (5.67) that

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \Phi(x,y)e^{-\frac{1}{2}(x^2+y^2)}dx\,dy \sim \Phi^{(o)} + \Phi^{(1)} + \ldots, \tag{8.72}$$

and making use of this result in Eq. (8.48) we obtain, with $c = (ik_2\rho)^{-\frac{1}{2}}$, the asymptotic expansion

$$J^{(2)}(a,b,\rho) \sim -\frac{2k_2^2 C_1 e^{ik_2\rho}}{(ik_2\rho)^2}\left\{1 + \frac{6C_2 - 5/4}{2(ik_2\rho)} + \frac{120C_4 - 27C_2 - 39/16}{8(ik_2\rho)^2} + \ldots\right\}, \tag{8.73}$$

where the C's are given by Eqs. (8.61). Substituting the A's and B's from Eqs. (8.57) and (8.59) respectively, we obtain the asymptotic expansion

$$J^{(2)}(a,b,\rho) \sim \frac{2ik_2^3 b e^{ik_2\rho + ik_1 a(1-n^2)^{\frac{1}{2}}}}{(ik_2\rho)^2} \left\{ 1 + \frac{E_1(a,b)}{2(ik_2\rho)} + \frac{E_2(a,b)}{8(ik_2\rho)^2} + \frac{E_3(a,b)}{48(ik_2\rho)^3} + \cdots \right\}, \tag{8.74}$$

where the expansion coefficients E_n, expressed in terms of the C's, are

$$E_1(a,b) = 6C_2 - 5/4;$$

$$E_2(a,b) = 120C_4 - 27C_2 - 39/16; \tag{8.75}$$

$$E_3(a,b) = 5040C_6 - 1170C_4 - (999/8)C_2 - 1269/64,$$

whereas, expressed in terms of the A's and B's, Eqs. (8.61), they become

$$E_1(a,b) = 6(A_2+B_2) - 5/4;$$

$$E_2(a,b) = 120(A_4+A_2B_2+B_4) - 27(A_2+B_2) - 39/16; \tag{8.76}$$

$$E_3(a,b) = 5040(A_6+A_4B_2+A_2B_4+B_6) - 1170(A_4+A_2B_2+B_4) - (999/8)(A_2+B_2) - 1269/64.$$

Finally, making use of Eqs. (8.57) and (8.59) to express the A's and B's in terms of the convenient arguments

$$\alpha = ik_1 a; \qquad \beta = ik_1 b, \tag{8.77}$$

we obtain for the first two coefficients in the asymptotic expansion (8.74) the expressions

$$\begin{aligned} E_1(a,b) &= -2 + n^2\left[3\alpha(1-n^2)^{-\frac{1}{2}} + \beta^2\right]; \\ E_2(a,b) &= -3n^2\left[5\alpha(1-n^2)^{-3/2} + 7\alpha(1-n^2)^{-\frac{1}{2}} + 4\beta^2\right] \\ &\quad + n^4\left[15\alpha^2(1-n^2)^{-1} + 10\alpha\beta^2(1-n^2)^{-\frac{1}{2}} + \beta^4\right]. \end{aligned} \tag{8.78}$$

The coefficient $E_3(a,b)$ will not be used in further computations and thus is not displayed here.

Two points of interest concerning the asymptotic expansion (8.74) must be mentioned here. First, the question of the behavior of the first few terms, which is intimately related to the order of magnitude of the remainder, is not discussed now but we return to this matter in Sections 8.4b and 8.4c. The second question has to do with the behavior of the expansion coefficients E_n as $a,b \longrightarrow 0$. Thus, making use of Eqs. (8.57) and (8.59) and putting $a = 0$ and $b = 0$ we have, from Eqs. (8.76),

$$\begin{aligned} E_1(0,0) &= 6B_2(0) - 5/4 = -2; \\ E_2(0,0) &= 120B_4(0) - 27B_2(0) - 39/16 = 0; \\ E_3(0,0) &= 5040B_6(0) - 1170B_4(0) - (999/8)B_2(0) - 1269/64 = 0, \end{aligned} \tag{8.79}$$

in which the first two results also follow directly from Eqs. (8.78). These results strongly suggest, as will be proved later (Section 8.4e), that

$$E_m(0,0) = 0, \quad m \geqslant 2, \tag{8.80}$$

a fact which is extremely helpful later in predicting the behavior of higher order terms.

8.3 EVALUATION OF $K^{(2)}$

The integral in question, $K(a,b,\rho)$, is the auxiliary form defined by Eq. (8.21) in which the path of integration is the real axis in the λ-plane. In this section we are concerned with the evaluation of the integral

$$K^{(2)}(a,b,\rho) = \int_{C_2} \frac{e^{-\gamma_1 a-\gamma_2 b}}{\lambda^2 - k_o^2} H_o^1(\lambda\rho)\lambda d\lambda, \tag{8.81}$$

in which the path of integration C_2 is the contour around the upper branch cut for γ_2 in the λ-plane (Fig. 10).

8.3a Representation as a double integral.- To evaluate the integral (8.81) by the double saddle point method of integration, we proceed as in Section 8.2 by first applying the conformal transformation (8.29), which is illustrated in

Fig. 11, and then applying the conformal transformation (8.43) which maps the path of steepest descents C_2 into the real axis in the x-plane. In this manner we obtain for $K^{(2)}(a,b,\rho)$ an integral which, except for a constant factor, is formally identical to the integral (8.46). Finally, making use of the integral representation (8.47), we obtain the double integral

$$K^{(2)}(a,b,\rho) = -\,2e^{ik_2\rho} \cdot \frac{2}{\pi}\int_0^\infty\int_0^\infty \Phi(x,y)e^{-\frac{1}{2}(x^2+y^2)}dxdy, \qquad (8.82)$$

where the function $\Phi(x,y)$ is again the product of two factors, Eq. (8.49), in which $g(x,y)$ is still given by Eq. (8.54), but now we have for the first factor, with $w = w(x)$ from Eq. (8.43),

$$f(x) = \left[G(w) - G(-w)\right]\frac{\cos w\,\sin w\,(dw/dx)}{\cos^2 w - \cos^2 w_o}. \qquad (8.83)$$

In Eq. (8.83) the function $G(w)$ is still defined by Eqs. (8.51), (8.52), and (8.53) and, in accordance with the first of Eqs. (8.37), we have

$$\cos w_o = (1+n^2)^{-\frac{1}{2}}. \qquad (8.84)$$

<u>8.3b Expansion of $f(x)$</u>.- Comparing the factors (8.50) and (8.83) we notice that the only difference is the new denominator in (8.83), $\cos^2 w - \cos^2 w_o$, which we now proceed

to expand as follows. First we note, from Eqs. (8.45), that

$$\cos w = 1 - \tfrac{1}{2}\xi^2 \quad \text{and} \quad \cos w_o = 1 - \tfrac{1}{2}\xi_o^2, \tag{8.85}$$

where, from Eqs. (8.43) and (8.84),

$$\xi_o^2 = 2(1-\cos w_o) = 2\left[1-(1+n^2)^{-\frac{1}{2}}\right] = n^2(1+n^2)^{-\frac{1}{2}}Q \tag{8.86}$$

with

$$Q = 2n^{-2}\left[(1+n^2)^{\frac{1}{2}}-1\right] = 1 - \tfrac{1}{4}n^2 + \tfrac{1}{8}n^4 - \tfrac{5}{64}n^6 + \tfrac{7}{128}n^8 - \ldots . \tag{8.87}$$

Factoring out the denominator in (8.83) and making use of the expressions (8.85) we obtain the expansion

$$(\cos^2 w-\cos^2 w_o)^{-1} = \frac{1+n^2}{n^2}\left\{1 + \frac{1+n^2}{n^2}\,\xi^2 + \frac{4+7n^2+3n^4}{4n^4}\,\xi^4 + \ldots\right\}, \quad |\xi|^2 < |n|^2, \tag{8.88}$$

in which use has been made of the identity

$$\xi_o^2(4-\xi_o^2) = 4\sin^2 w_o = 4n^2/(1+n^2) \tag{8.89}$$

which follows readily from Eqs. (8.84) and (8.86). Finally, combining the expansion (8.88) with (8.45), in accordance with Eq. (8.83) we obtain the desired expansion for the factor

$$\frac{\cos w\,\sin w(dw/dx)}{\cos^2 w - \cos^2 w_o} = c\xi\,\frac{1+n^2}{n^2}\left\{1 + \frac{2+n^2}{2n^2}\,\xi^2 + \frac{4+5n^2+n^4}{4n^4}\,\xi^4 + \ldots\right\}, \quad |\xi|^2 \leqslant |n|^2. \tag{8.90}$$

The remaining factor in (8.83), $\left[G(w) - G(-w)\right]$, is identical to the corresponding factor in (8.50). Thus, combining the expansions (8.60) and (8.90) we now obtain for the new $f(x)$ the power series

$$f(x) = D_2x^2 + D_4x^4 + D_6x^6 + \dots , \quad |x|^2 < |n^2k_2\rho| , \qquad (8.91)$$

in which the expansion coefficients are

$$\begin{aligned} D_2 &= 2c^3C_1 \frac{1+n^2}{n^2} ; \\ D_4 &= 2c^5C_1 \frac{1+n^2}{n^2}\left[C_2 + \frac{2+n^2}{2n^2}\right] ; \\ D_6 &= 2c^7C_1 \frac{1+n^2}{n^2}\left[C_4 + \frac{2+n^2}{2n^2}\,C_2 + \frac{4+5n^2+n^4}{4n^4}\right] , \end{aligned} \qquad (8.92)$$

with the ratios

$$\begin{aligned} \frac{D_4}{c^2D_2} &= C_2 + \frac{2+n^2}{2n^2} ; \\ \frac{D_6}{c^4D_2} &= C_4 + \frac{2+n^2}{2n^2}\,C_2 + \frac{4+5n^2+n^4}{4n^4} . \end{aligned} \qquad (8.93)$$

The radius of convergence given for the expansion (8.91) has its origin in the presence of the pole which limits the expansion (8.88). The radius of convergence here is actually $|\xi| = |\xi_o|$, but if $|n|^2 \ll 1$ as is the case at low frequencies, we see from Eq. (8.86) that $|\xi_o|^2 \approx |n|^2$ as indicated.

8.3c Computation of the expansion terms.– Proceeding as in Section 8.2f and making use of the general formulas (8.70) we obtain at once, for $K^{(2)}(a,b,\rho)$, the expansion terms

$$\begin{aligned}
\Phi^{(0)} &= 0; \\
\Phi^{(1)} &= c^4 C_1 \frac{1+n^2}{n^2}; \\
\Phi^{(2)} &= \tfrac{1}{2} c^6 C_1 \frac{1+n^2}{n^2}\left[6\left(C_2 + \frac{2+n^2}{2n^2}\right) + \frac{7}{4}\right]; \\
\Phi^{(3)} &= \tfrac{1}{8} c^8 C_1 \frac{1+n^2}{n^2}\left[120\left(C_4 + \frac{2+n^2}{2n^2} C_2 + \frac{4+5n^2+n^4}{4n^4}\right)\right. \\
&\qquad \left. + 33\left(C_2 + \frac{2+n^2}{2n^2}\right) + \frac{225}{16}\right],
\end{aligned} \tag{8.94}$$

in which, as before, the C's are given by Eqs. (8.61) in terms of the A's and B's defined by Eqs. (8.57) and (8.59) respectively.

Substituting the expansion (8.72) for the double integral (8.82), making use of the coefficients (8.94), we have for our second auxiliary integral the three-term asymptotic expansion

$$K^{(2)}(a,b,\rho) \approx - \frac{2C_1 e^{ik_2\rho}}{(ik_2\rho)^2} \frac{1+n^2}{n^2}\left\{1 + \frac{E_1(a,b)}{2(ik_2\rho)} + \frac{E_2(a,b)}{8(ik_2\rho)^2} + \dots\right\} \tag{8.95}$$

where the new expansion coefficients are

$$E_1(a,b) = 6C_2 + 3\,\frac{2+n^2}{n^2} + \frac{7}{4}\ ;$$

$$E_2(a,b) = 120C_4 + \left[33 + 60\,\frac{2+n^2}{n^2}\right]C_2 + 30\,\frac{4+5n^2+n^4}{n^4} + 33\,\frac{2+n^2}{2n^2} + \frac{225}{16}\ . \tag{8.96}$$

8.3d Proof that the present asymptotic expansions are term-wise differentiable.- As a necessary step in the verification of the fact that our present asymptotic expansions admit the term by term partial derivatives, to any order, with respect to any of the three parameters a, b, and ρ, to yield the asymptotic expansions of the corresponding partial derivatives of the original integral, we need to verify by actual computation that Eqs. (8.22), for example, are indeed satisfied by the asymptotic expansions of the corresponding integrals over the path C_2. That is, we need to establish that

$$J^{(2)}(a,b,\rho) = -(\nabla_\rho^2 + k_0^2)K^{(2)}(a,b,\rho) \tag{8.97}$$

for the actual asymptotic expansions (8.73) and (8.95).

Thus, making use of the expansion (8.95), we obtain by term-wise differentiation with respect to ρ

$$-(\nabla_\rho^2 + k_0^2)K^{(2)}(a,b,\rho) \sim -\frac{2k_2^2C_1e^{ik_2\rho}}{(ik_2\rho)^2}\left\{1 + \frac{E_1 - 6(1+n^2)/n^2}{2(ik_2\rho)} + \frac{E_2 - (20E_1 - 32)(1+n^2)/n^2}{8(ik_2\rho)^2} + \dots\right\}, \tag{8.98}$$

where E_1 and E_2 are given by Eqs. (8.96). But, now, it is a matter of simple arithmetic to verify that the expansion coefficients in (8.98) reduce to

$$\begin{aligned} &E_1 - 6(1+n^2)/n^2 = 6C_2 - 5/4; \\ &E_2 - (20E_1-32)(1+n^2)/n^2 = 120C_4 - 27C_2 - 39/16 , \end{aligned} \tag{8.99}$$

which are seen to be identical to the coefficients in the expansion (8.73) for $J^{(2)}(a,b,\rho)$. Hence, we have established by actual computation that the asymptotic expansions (8.73) and (8.95) do satisfy Eq. (8.97), and we interpret this result as constituting proof of the fact that our present asymptotic expansions are term-wise differentiable to any order with respect to ρ. We reach this conclusion by noting that we obtain in the end the same asymptotic expansion (8.73) whether we apply the differential operator $-(\nabla_\rho^2+k_o^2)$ to the integrand of the definition integral (8.81) for $K^{(2)}(a,b,\rho)$ or term by term to its asymptotic expansion (8.95), which demonstrates the existence of the asymptotic expansions for the first and second order derivatives with respect to ρ and hence, by induction, for all higher order derivatives.

The proof of the term-wise differentiability of our asymptotic expansions with respect to a and b follows immediately by noting that the second equality in (8.22) must also be satisfied by our asymptotic expansion (8.95). However, there is no need to verify this fact by actual computation. It is sufficient to observe that, having established that our asymptotic expansions are indeed term-wise differentiable with respect to ρ, we need only refer to Eqs. (8.3) and (8.4), which are satisfied by all of our integrals, to establish that they are also term-wise differentiable with respect to a and b, to any order. This conclusion follows logically from the uniqueness of an asymptotic expansion and from the fact that an asymptotic series can always be integrated term by term.

Finally, we must remark that the term-wise differentiability of our present asymptotic expansions stems from the more enlightened choice of exponent in the application of the double saddle point method of integration. As a consequence of the new choice of exponent given by Eq. (8.39) we now obtain asymptotic series in which a and b appear only as polynomials in the numerators of successive terms, whereas ρ appears only as simple powers in the corresponding denominators.

8.4 EVALUATION OF $U^{(2)}$

According to Eq. (8.23), the integral in question, Eq. (8.10), when evaluated over the path C_2 can be computed as follows:

$$U^{(2)}(a,b,\rho) = \left[J_a^{(2)} - J_b^{(2)}\right]/(k_1^2-k_2^2). \tag{8.100}$$

Therefore, we need only compute the partial derivatives with respect to a and b of the asymptotic expansion (8.74) for

$J^{(2)}(a,b,\rho)$, combining the resulting expansions in accordance with (8.100).

8.4a Asymptotic expansion for $U^{(2)}$.- Carrying out the above prescriptions we obtain from Eq. (8.74), after some algebraic manipulations, the expansion

$$U^{(2)}(a,b,\rho) \sim -\frac{2ik_2 e^{ik_2\rho+ik_1a(1-n^2)^{\frac{1}{2}}}}{(1-n^2)(ik_1\rho)^2}\left\{F_o(a,b) + \frac{F_1(a,b)}{2(ik_2\rho)} + \frac{F_2(a,b)}{8(ik_2\rho)^2} + \frac{F_3(a,b)}{48(ik_2\rho)^3} + \cdots\right\}, \tag{8.101}$$

in which we have

$$\begin{aligned} F_o(a,b) &= 1 - ik_1b(1-n^2)^{\frac{1}{2}}; & F_o(0,0) &= 1; \\ F_1(a,b) &= F_oE_1 + b(E_{1b}-E_{1a}); & F_1(0,0) &= -2; \\ F_2(a,b) &= F_oE_2 + b(E_{2b}-E_{2a}); & F_2(0,0) &= 0; \\ F_3(a,b) &= F_oE_3 + b(E_{3b}-E_{3a}); & F_3(0,0) &= 0, \end{aligned} \tag{8.102}$$

where the coefficients E_m, $m = 1,2,3$, are given by Eqs. (8.75) or (8.76), and where the letter subscripts attached to E_m denote partial derivatives with respect to the corresponding parameter. It is noteworthy to point out that, in the present expansion (8.101), the coefficients satisfy the condition

$$F_m(0,0) = 0, \quad m \geqslant 2, \tag{8.103}$$

which follows at once from Eqs. (8.80) and (8.102). Furthermore, making use of Eqs. (8.76), (8.57), and (8.59), we deduce by induction from the structure of the terms and from the condition (8.80) that $E_m(a,b) = O(n^2)$, $m \geqslant 2$, and hence from Eqs. (8.102) that

$$F_m(a,b) = O(n^2), \quad m \geqslant 2. \tag{8.104}$$

<u>8.4b Behavior of the first few terms</u>.- It now becomes of importance to examine the relative order of magnitude of successive terms in the asymptotic expansion (8.101) for $U^{(2)}(a,b,\rho)$. It is seen by examining in detail the coefficients (8.102), in the light of the condition (8.104), that the bracket in the expansion (8.101) can be written to order of magnitude as

$$O(1) + \frac{O(1)}{ik_2\rho} + \frac{O(n^2)}{(ik_2\rho)^2} + \frac{O(n^2)}{(ik_2\rho)^3} + \frac{O(n^2)}{(ik_2\rho)^4} + \cdots . \tag{8.105}$$

Noting that the factor in front of the expansion (8.101) contains the term $k_2 = nk_1$, and incorporating n into the bracket we obtain

$$O(n) + \frac{O(n)}{ik_2\rho} + \frac{O(n^3)}{(ik_2\rho)^2} + \frac{O(n^3)}{(ik_2\rho)^3} + \frac{O(n^3)}{(ik_2\rho)^4} + \cdots , \tag{8.106}$$

which can be written in terms of reciprocal powers of $ik_1\rho$ as follows:

$$O(n) + \frac{O(1)}{ik_1\rho} + \frac{O(n)}{(ik_1\rho)^2} + \frac{O(1)}{(ik_1\rho)^3} + \frac{O(n^{-1})}{(ik_1\rho)^4} + \ldots \qquad (8.107)$$

This result suggests that, if $|n|^2 \ll 1$, the asymptotic series ought to be broken off retaining only the first three terms. This would appear to minimize the error in the asymptotic expansion of $J^{(2)}(a,b,\rho)$. It is also noted that, after the third term, all successive terms of the asymptotic series (8.107) behave like successive powers of $(ik_2\rho)^{-1}$ in accordance with the predictions based on Watson's lemma which are reviewed in the next section.

Accordingly, we propose for $U^{(2)}(a,b,\rho)$ the three-term asymptotic expansion

$$U^{(2)}(a,b,\rho) = -\frac{2ik_1 e^{ik_2\rho + ik_1 a(1-n^2)^{\frac{1}{2}}}}{(1-n^2)(ik_1\rho)^2}\left\{G_o(a,b) + \frac{G_1(a,b)}{2(ik_1\rho)} + \frac{G_2(a,b)}{8(ik_1\rho)^2} + O(ik_1\rho)^{-3}\right\}, \qquad (8.108)$$

in which the estimate of the error is possibly somewhat smaller than the actual error when $|n|^2 \ll 1$. The expansion coefficients in (8.108), displayed in full in terms of the parameters α and β introduced by Eqs. (8.77), become

$$G_0(a,b) = \quad nF_0(a,b) = n\left[1-\beta(1-n^2)^{\frac{1}{2}}\right];$$

$$G_1(a,b) = \quad F_1(a,b) = -2\left[1-\beta(1-n^2)^{\frac{1}{2}}\right] + 3n^2\left[(\alpha-\beta)(1-n^2)^{-\frac{1}{2}} - \alpha\beta + \beta^2 - \frac{1}{3}\beta^3(1-n^2)^{\frac{1}{2}}\right];$$

$$\begin{aligned} G_2(a,b) = n^{-1}F_2(a,b) = &-3n\left[5(\alpha-\beta)(1-n^2)^{-3/2} + 7(\alpha-\beta)(1-n^2)^{-\frac{1}{2}}\right. \\ &\left. - 5\alpha\beta(1-n^2)^{-3/2} - 7\alpha\beta + 12\beta^2 - 4\beta^3(1-n^2)^{\frac{1}{2}}\right] \\ &+ n^3\left[15\alpha(\alpha-2\beta)(1-n^2)^{-1} + 30\alpha\beta^2(1-n^2)^{-\frac{1}{2}} - 10\alpha\beta^3\right. \\ &\left. - 5\beta(3\alpha^2+2\beta^2)(1-n^2)^{-\frac{1}{2}} + 5\beta^4 - \beta^5(1-n^2)^{\frac{1}{2}}\right]; \end{aligned} \qquad (8.109)$$

and, the notable feature of this expansion is the fact that, in the limiting case in which we put $k_2 = 0$ or $n = 0$, none of the retained terms become infinite leading to the simple expression

$$U^{(2)}(a,b,\rho) = \frac{2ik_1 e^{ik_1a}}{(ik_1\rho)^2}\left\{\frac{1 - ik_1b}{ik_1\rho} + O(ik_1\rho)^{-3}\right\}, \quad (k_2=0). \quad (8.110)$$

8.4c Application of Watson's lemma to the present asymptotic expansions.- In Section 5.1a we considered a particular formulation of Watson's lemma as applied to the typical integral (5.10) in which the function $\Phi(x)$ exhibits no exponential behavior, Eqs. (5.11), (5.12), and (5.15). This time, however, with the aim in view of simplifying and improving our resulting asymptotic expansions, we have chosen to apply the saddle point method of integration to a different exponent, Eq. (8.39), with the consequence that the function $\phi(x)$, Eqs. (5.10) and (5.11), now exhibits the exponential behavior that arises from the factor $\exp\left\{-\gamma_1 a - \gamma_2 b\right\}$, Eq. (8.1), which is common to all integrals presently studied.

Therefore, we must reformulate Watson's lemma to take care of the new situation. Following Section 5.1a, we reword Watson's lemma as follows:

Lemma:- Let $\psi(u)$ be analytic within the unit circle $|u| < 1$; i.e., let $\psi(u)$ have the power series expansion

$$\psi(u) = \sum_{m=o}^{\infty} A_{2m}\lambda^m u^{2m}, \quad |u| < 1; \tag{8.111}$$

further, assume that

$$|\psi(u)| < Au^{2p}e^{\frac{1}{2}\eta\lambda u^2}, \tag{8.112}$$

where A is a positive number independent of u, p is a positive integer or zero, and $\eta = \eta(a,b)$ is a positive number, $0 \leqslant \eta < 1$, when u is real and $u \geqslant 1$. Then, the asymptotic expansion

$$e^{-\phi(o)}I \sim (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{m=o}^{\infty} \frac{(2m)!}{2^m m!} A_{2m} \tag{8.113}$$

is valid in the sense of Poincaré.

To establish the above result we note from Eqs. (8.111) and (8.112) that, if $M \geqslant p$ is a fixed integer, a constant B

can be found such that

$$\left| \Psi(u) - \sum_{m=o}^{M-1} A_{2m}\lambda^m u^{2m} \right| \leqslant Bu^{2M} e^{\frac{1}{2}\eta\lambda u^2} \tag{8.114}$$

whenever $u \geqslant 0$, whether $u \leqslant 1$ or $u \geqslant 1$; and therefore

$$e^{-\phi(o)} I = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left\{ \sum_{m=o}^{M-1} \frac{(2m)!}{2^m m!} A_{2m} + R_M \right\}, \tag{8.115}$$

where R_M, the remainder after M terms, is bounded as follows:

$$\begin{aligned} |R_M| &< \lambda^{\frac{1}{2}} B \int_o^{\infty} u^{2M} e^{-\frac{1}{2}\lambda(1-\eta)u^2} du \\ &= (\tfrac{1}{2}\pi)^{\frac{1}{2}} B \frac{(2M)!}{2^M M!} \left[\lambda(1-\eta)\right]^{-M} \\ &= O\left[\lambda(1-\eta)\right]^{-M} . \end{aligned} \tag{8.116}$$

Comparing this result with Eq. (5.19), we conclude that all of our earlier findings, in particular Eqs. (5.20a) and (5.20b) are still applicable if we merely replace λ in the form of the remainder by the smaller number $\lambda(1-\eta)$. Thus, it is seen that, unless $\eta \ll 1$, the magnitude of the remainder is increased due to the presence of exponential behavior in the factor $\Phi(x)$.

To apply these new findings to our present integrals we note that the exponential behavior in question arises from the factor

$$\tfrac{1}{2}\left[G(w) - G(-w)\right] = -i\,\sin\left[k_2 b v(x)\right] e^{ik_1 a u(x)}, \tag{8.117}$$

which first appeared in Eq. (8.50) and which is common to all our integrals. Thus, we must show that the absolute value of this factor, $\left|G(w) - G(-w)\right|$, remains bounded in accordance with condition (8.112) for $x \longrightarrow \infty$ on the path of steepest descents, $0 \leqslant x \leqslant \infty$, and indeed for all x within the allowable sector, $\left|\arg\{x\}\right| \leqslant \tfrac{1}{4}\pi$.

To this end, putting $w = u + iv$, we note from Eq. (8.43) and Fig. 11 that

$$\tfrac{1}{2}x^2 \xrightarrow[x \to \infty]{} \tfrac{1}{2}k_2 \rho e^{-v}, \tag{8.118}$$

and, from Eqs. (8.52) and (8.53), we deduce that

$$u(x) = (1-n^2\cos^2 w)^{\frac{1}{2}} \xrightarrow[x \to \infty]{} nx^2/(2k_2\rho),$$

and (8.119)

$$v(x) = \sin w \xrightarrow[x \to \infty]{} x^2/(2k_2\rho);$$

whence, substituting into (8.117), we have

$$\tfrac{1}{2}\left[G(w) - G(-w)\right] \xrightarrow[x \to \infty]{} -i\,\sin\left[\tfrac{1}{2}x^2(b/\rho)\right] e^{\frac{1}{2}ix^2(a/\rho)}, \tag{8.120}$$

whose absolute value obviously remains bounded for x real and $x \longrightarrow \infty$.

However, according to the theory of the saddle point method of integration, Section 5.1, the path of integration in Eq. (5.10) need not coincide with the positive half of the real axis in the x-plane, but may be taken as any contour joining the origin to the point at infinity and lying entirely within the sector $\left|\arg\{x\}\right| < \frac{1}{4}\pi$ as shown in Fig. 9. Therefore, in the present instance we deduce immediately from Eq. (8.120) that, within this sector, our factor remains bounded as

$$\tfrac{1}{2}\left|G(w) - G(-w)\right| < e^{\frac{1}{2}x^2(a+b)/\rho}, \tag{8.121}$$

which shows from Eqs. (5.12) that condition (8.112) is satisfied with $\eta(a,b) = (a+b)/\rho$; that is, according to Eq. (8.116), the remainder after M terms becomes in this case

$$\left|R_M\right| = O\left[\lambda(1-a/\rho-b/\rho)\right]^{-M}, \tag{8.122}$$

and thus it is clear that our proposed expansions remain valid only so long as $(a+b) < \rho$. This result merely reflects the fact that, with our present choice of cuts, the contour integrals, Eq. (8.26), converge only for finite values of a and b.

Furthermore, in Eq. (8.122), λ is the square of the radius of convergence of the power series expansion (5.11) for the function $\Phi(x)$; i.e., $\lambda^{\frac{1}{2}} = |x_0|$, where x_0 is the singularity nearest to the origin, pole or branch point, which the function $\Phi(x)$ exhibits in the x-plane. According to Eq. (8.62), the radius of convergence for the factor $f(x)$ defined by Eq. (8.50), which appears in the present integral $U^{(2)}(a,b,\rho)$, is given by $|x| = |x_2| = (4k_2\rho)^{\frac{1}{2}}$. This singularity arises from the branch points of the inverse of the transformation (8.43), which occur at all the other saddle points in the w-plane. In Fig. 11, which represents a portion of the w-plane, these additional saddle points occur at $w = \pm\pi$ and, in the x-plane they occur at $x = \pm x_2$, where $x_2 = (4ik_2\rho)^{\frac{1}{2}}$ as readily deduced from (8.43). This situation is illustrated in Fig. 13 which displays the conformal transformation from the w-plane, Fig. 11, into the x-plane in accordance with Eq. (8.43). As pointed out before, the path of steepest descents C_2 in the w-plane now becomes the real axis in the x-plane, and the shaded region in Fig. 11, which may be termed the valley of the exponent $-\frac{1}{2}x^2$ because $\mathrm{Re}\{\frac{1}{2}x^2\} \geqslant 0$ throughout this region, maps into the so-called allowable sector of the x-plane, $|\arg\{x\}| \leqslant \frac{1}{4}\pi$. In fact, the original path of integration which in the λ-plane, Fig. 10, is merely the real axis, and which in the w-plane, Fig. 11, becomes the path from $-\pi + i\,\infty$ to $-\pi$, from $-\pi$ to 0, and from 0 to $-i\,\infty$, now maps into the two half-lines in the x-plane, from $\infty\, e^{5i\pi/4}$ to 0, and from 0 to $\infty\, e^{-i\pi/4}$, as shown in Fig. 13.

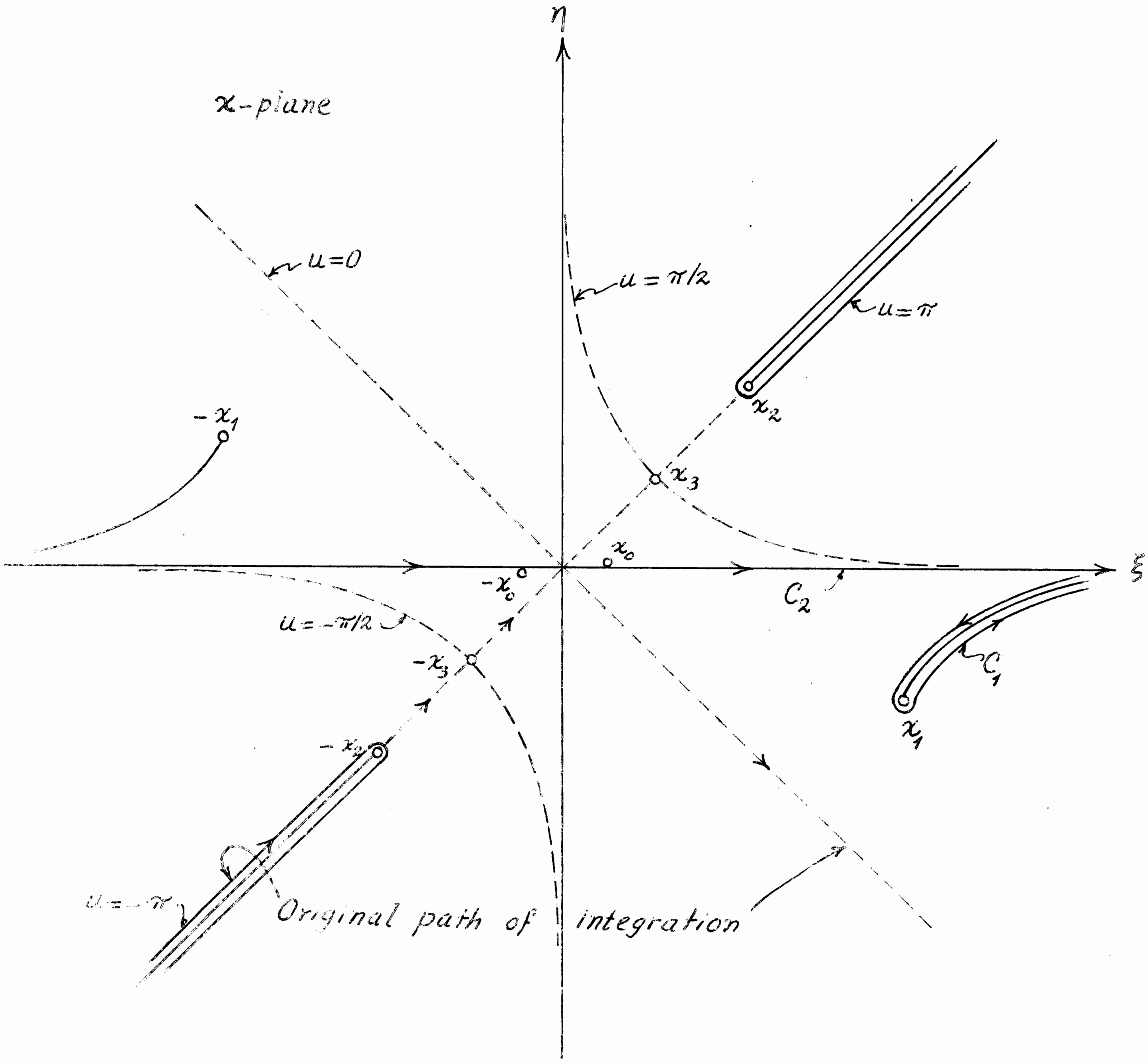

Fig. 13.- The x-plane displaying the full-period strip, $-\pi \leqslant u \leqslant \pi$, of the w-plane (Fig. 11) and the mapping of the path of steepest descents C_2 unto the real axis.

Finally, in accordance with the conditions imposed by Watson's lemma, it would appear from Eq. (8.62) that, for the expansions corresponding to $J^{(2)}(a,b,\rho)$ and $U^{(2)}(a,b,\rho)$, Eqs. (8.74) and (8.101) respectively, we should place $\lambda^{\frac{1}{2}} = (4k_2\rho)^{\frac{1}{2}}$ in the form of the remainder, Eq. (8.122). But, we must not overlook the fact that the bracket in Eq. (8.40) contains the Hankel function $H_0^1(k_2\rho \cos w)$ which exhibits branch points at $\lambda = 0$ in Fig. 10 or at $w = \pm \frac{1}{2}\pi$ in Fig. 11. And, therefore, the singularity <u>nearest</u> to the origin in the x-plane, Fig. 13, arises from the Hankel function in question and occurs at $x = \pm x_3$, where $x_3 = (2ik_2\rho)^{\frac{1}{2}}$. Accordingly, we must take $\lambda^{\frac{1}{2}} = (2\ k_2\rho)^{\frac{1}{2}}$ in Eq. (8.122) as properly representing the radius of convergence of the power series expansion (5.11). This result fully confirms the fact that, for M sufficiently large, the successive terms of the asymptotic series (8.74) and (8.101) behave as reciprocal powers of $ik_2\rho$.

<u>8.4d Extension of Watson's lemma to a double integral</u>.- In the present instance we have carried out the asymptotic expansions (8.74) and (8.101) by applying the double saddle point method of integration, Section 5.2, and we must now show in what respect our earlier extension of Watson's lemma to a double integral, Section 5.2a, is still applicable. To this end, we first call attention to the fact that, initially at least, the function $\mathfrak{F}(x,y)$ in the double integral (5.52) is assumed to be the product of two factors, $\mathfrak{F}(x,y) = f(x)g(y)$,

as indicated by Eqs. (5.54), (5.55), and (5.56). This assumption, it is recalled, leads to the form of the remainder after N grouped terms given by Eq. (5.64); that is,

$$|R_N| = O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^N , \qquad (8.123)$$

where $\lambda^{\frac{1}{2}}$ is the radius of convergence of the power series expansion (5.55) for $f(x)$, and $\nu^{\frac{1}{2}}$ is the radius of convergence for the corresponding power series expansion (5.56) for $g(y)$.

In the present instance the function $\Phi(x,y)$ which appears in the double integral (8.48) has the form given by Eq. (8.49) wherein the factor $f(x)$ possesses the power series expansion (8.62) which is valid for $|x|^2 \leqslant 4k_2\rho$, and thus we must now turn our attention to the factor $g(x,y)$ defined by Eq. (8.54). According to the discussion of Section 8.2e, the function $g(x,y)$ possesses the double power series expansion (8.64), with coefficients (8.65), which converges for all real values of x and y contained within the shaded strip of analiticity shown in Fig. 12. Therefore, we propose to identify the λ and ν in Eqs. (5.55), (5.56), and hence (8.123), with the x and y intercepts, respectively, of the curves $x = H(y)$ and $y = h(x)$ bounding the strip of analiticity; that is, if we put

$$\lambda^{\frac{1}{2}} = (2k_2\rho)^{\frac{1}{2}} \quad \text{and} \quad \nu^{\frac{1}{2}} = (4k_2\rho)^{\frac{1}{2}}, \qquad (8.124)$$

as shown in Fig. 12, we obtain a rectangle of analiticity bounded by the axes of coordinates and the lines $x = \lambda^{\frac{1}{2}}$ and $y = \nu^{\frac{1}{2}}$, within which the double power series expansion (8.64) certainly converges.

It is of interest to point out the origin of the singularities of the integrand $\Phi(x,y)$ and, in particular, of the function $g(x,y)$ which lead to the above identification of λ and ν. Referring to Fig. 13, we recognize $\lambda^{\frac{1}{2}} = (2k_2\rho)^{\frac{1}{2}}$ as the radius of convergence of $\Phi(x,y)$ regarded as an analytic function of x for fixed real y, $0 \leqslant y \leqslant \nu^{\frac{1}{2}}$, arising from the singularity nearest to the origin in the x-plane, which in this case is the branch point of the Hankel function for zero argument that still prevails in the factor $g(x,y)$; that is, as far as a power series expansion in x is concerned, we confirm the conclusion of the preceding section. Similarly, we recognize $\nu^{\frac{1}{2}} = (4k_2\rho)^{\frac{1}{2}}$ as the radius of convergence of $\Phi(x,y)$ regarded as an analytic function of y for fixed real x, $0 \leqslant x \leqslant \lambda^{\frac{1}{2}}$, arising from the singularity nearest to the origin in the y-plane, which in this case comes from the pair of branch points introduced into $g(x,y)$, Eqs. (8.54), by the inverse of the transformation (8.43).

Returning to the identification of λ and ν given by Eqs. (8.124), we note that the limitations imposed on the primitive expansions (5.55) and (5.56) are certainly satisfied and, hence, all of our earlier results, Eqs. (5.64) through (5.66), are still applicable with but one exception; namely, in accordance with the discussion of the preceding section, we now have, instead of (8.123), the form

$$|R_N| = O\left(\frac{1}{\lambda(1-\eta)} + \frac{1}{\nu}\right)^N , \tag{8.125}$$

where λ and ν are given by (8.124) and $\eta = (a+b)/\rho$ as in Eq. (8.122). However, we note from Eqs. (8.124) that in the present instance λ and ν are essentially of the same order of magnitude and, in fact, $O(k_2\rho)$. Therefore, if $\eta \ll 1$, which is the case of practical interest, we may write for the form of the remainder after N grouped terms

$$|R_N| = O(k_2\rho)^{-N} , \tag{8.126}$$

which is certainly in accord with the behavior of higher order terms in the asymptotic expansions (8.74) and (8.101), as already pointed out. This result, Eq. (8.126), is at variance with our earlier erroneous claim in Part I, Eqs. (6.92) and (6.93). See Corrections to Part I at the beginning of this report.

One final remark seems to be in order at this point. It is recalled that the form of the remainder given by Eq. (8.123) was deduced by assuming that $\Phi(x,y)$ has a valid double power series expansion for real values of x and y within the rectangle $0 \leq x < \lambda^{\frac{1}{2}}$ and $0 \leq y < \nu^{\frac{1}{2}}$. Actually, our present function $\Phi(x,y)$, Eq. (8.49), has a larger domain of analiticity which comprises the area in Fig. 12

bounded by the axes of coordinates, the boundary curves $x = H(y)$ and $y = h(x)$, and the vertical line $x = (4k_2\rho)^{\frac{1}{2}}$ corresponding to the radius of convergence of the expansion (8.62) for the factor $f(x)$. We suggest that it is this larger domain of analiticity that accounts for the observed behavior of the first few terms, as discussed in Section 8.4b, and which, in spite of the limitation implied by (8.126), furnishes us with useful results even though in practice $k_2\rho$ might be considerably less than unity. We would like to summarize the situation by observing that, after all, the form of the remainder given by Eq. (8.123) applies only for N sufficiently large, and, therefore, if we limit judiciously the number of terms in our expansions through careful examination of the behavior of the first few terms, we may obtain formulas that are useful beyond all expectations based on the form of the remainder.

8.4e <u>Verification of van der Pol's result</u>.- As pointed out in Section 7.2a, it was shown by van der Pol that the integral $U^{(2)}(0,0,\rho)$ can be integrated exactly with the result, Eqs. (8.10) and (8.26),

$$U^{(2)}(0,0,\rho) = \frac{1}{k_1^2-k_2^2}\int_{C_2}\gamma_2 H_0^1(\lambda\rho)\lambda d\lambda = \frac{2}{(k_1^2-k_2^2)\rho}\frac{\partial}{\partial\rho}\left(\frac{e^{ik_2\rho}}{\rho}\right), \quad (8.127)$$

which we have expressed in our notation. To verify the result

(8.127) we need only put $a = 0$ and $b = 0$ in the expansion (8.101) taking due account of the values assumed by the expansion coefficients in this special case, as listed in Eqs. (8.102). In this manner we obtain

$$U^{(2)}(0,0,\rho) = -\frac{2ik_2^3 e^{ik_2\rho}}{(k_1^2-k_2^2)(ik_2\rho)^2}\left(1 - \frac{1}{ik_2\rho}\right) - \frac{2}{(k_1^2-k_2^2)\rho}\,\frac{\partial}{\partial\rho}\left(\frac{e^{ik_2\rho}}{\rho}\right), \quad (8.128)$$

which agrees exactly with (8.127), thus furnishing an important and necessary check on our method of attack. See also Eq. (7.35) and subsequent discussion.

It is important to observe that the result (8.128) obtained from the expansion (8.101) is exact (i.e., not asymptotic) if we assume, as in Eq. (8.103), that all coefficients $F_m(0,0)$ vanish for $m \geqslant 2$. That we find agreement with van der Pol's result (8.127), which is exact, constitutes in our mind sufficient proof that the specified condition, Eq. (8.103), is indeed true for all $m \geqslant 2$, even though we were able to verify it only for $m = 2$ and $m = 3$.

<u>8.4f Verification of earlier evaluation of $U_1^{(2)}$</u>.– As a further check on our present work we find in Eq. (6.4) the fundamental integral

$$M_1^{(2)}(a,\rho) = \int_{C_2} \frac{\gamma_2 e^{-\gamma_1 a}}{k_1^2-k_2^2}\, H_0^1(\lambda\rho)\lambda d\lambda, \quad a = h-z \geqslant 0, \qquad (8.129)$$

which, according to Eqs. (8.16) and (8.23), may be written as

$$M_1^{(2)}(a,\rho) = U^{(2)}(a,0,\rho) = - J_b^{(2)}(a,0,\rho)/(k_1^2-k_2^2). \qquad (8.130)$$

Using the middle form we have at once, from Eq. (8.101), to two terms

$$M_1^{(2)}(a,\rho) \approx - \frac{2ik_2^3 e^{ik_2\rho+ik_1a(1-n^2)^{\frac{1}{2}}}}{(k_1^2-k_2^2)(ik_2\rho)^2}\left\{1 - \frac{2-3\ \alpha n^2(1-n^2)^{-\frac{1}{2}}}{2(ik_2\rho)} + \ldots\right\} \quad (8.131)$$

where we have made use of the first of Eqs. (8.78) to compute $E_1(a,0)$.

Now, Eq. (6.102), likewise to two terms reads

$$M_1^{(2)}(a,\rho) \approx - \frac{2ik_2^3(1+K)^{3/2}e^{ik_2\rho+ik_1a(1-n^2)^{\frac{1}{2}}}}{(k_1^2-k_2^2)(ik_2\rho)^2}$$

$$\times\left\{1 - \frac{8+24K+15K^2-n^2(8+9K)}{8(1-n^2)(ik_2\rho)} + \ldots\right\}, \qquad (8.132)$$

in which the factor K, according to Eq. (6.67), may be written as

$$K = \frac{na}{\rho(1-n^2)^{\frac{1}{2}}-na} = \frac{n^2\alpha}{(ik_2\rho)(1-n^2)^{\frac{1}{2}}-n^2\alpha}$$

$$= \frac{n^2\alpha}{(1-n^2)^{\frac{1}{2}}(ik_2\rho)}\left\{1 - \frac{n^2\alpha}{(1-n^2)^{\frac{1}{2}}(ik_2\rho)}\right\}^{-1}, \qquad (8.133)$$

where $\alpha = ik_1a$ in accordance with (8.77). Comparing the expansions (8.131) and (8.132) and suppressing the factor in front which is common to both, we obtain making use of (8.133) the expansion

$$(1+K)^{3/2}\left\{1 - \frac{8 + 24K + 15K^2 - n^2(8+9K)}{8(1-n^2)(ik_2\rho)}\right\}$$

$$= 1 - \frac{1 - (3/2)\alpha n^2(1-n^2)^{-\frac{1}{2}}}{ik_2\rho} + \ldots , \qquad (8.134)$$

in exact agreement with the bracket in Eq. (8.131). There is little doubt that the next term in Eq. (6.102) would also check with our present expansions. This additional check on our earlier and present expansions also serves to point out quite forcibly how much simpler our present developments are in comparison with the earlier asymptotic expansions and thus fully justifies the additional labor that has gone into developing the new asymptotic expansions.

8.5 EVALUATION OF $V^{(2)}$

According to Eq. (8.24) the integral in question, Eq. (8.11), when evaluated over the path C_2 can be computed as

$$V^{(2)}(a,b,\rho) = \frac{k_2^2K_a^{(2)} - k_1^2K_b^{(2)}}{k_1^4 - k_2^4} , \qquad (8.135)$$

in terms of the partial derivatives with respect to a and b of our second auxiliary integral $K^{(2)}(a,b,\rho)$ whose asymptotic expansion is given by Eq. (8.95), with expansion coefficients (8.96). Therefore, we need only compute the derivatives $K_a^{(2)}$ and $K_b^{(2)}$ and insert their asymptotic expansions into (8.135).

8.5a Asymptotic expansion for $V^{(2)}$.– First we rewrite the expansion (8.95) by inserting for $C_1 = A_0 B_1$ its explicit expression in terms of a and b in accordance with the first of Eqs. (8.57) and (8.59); thus, we have

$$K^{(2)}(a,b,\rho) \sim \frac{2ik_2 b e^{ik_2\rho + ik_1 a(1-n^2)^{\frac{1}{2}}}}{(ik_2\rho)^2} \frac{1+n^2}{n^2}$$

$$\times \left\{1 + \frac{E_1(a,b)}{2(ik_2\rho)} + \frac{E_2(a,b)}{8(ik_2\rho)^2} + \dots \right\}, \qquad (8.136)$$

where E_1 and E_2 are given by Eqs. (8.96). Next, we compute by term-wise differentiation of Eq. (8.136) the asymptotic expansions for the terms $k_2^2 K_a^{(2)}$ and $k_1^2 K_b^{(2)}$ and, before inserting them into (8.135), we exhibit both expansions with a common factor in front. Proceeding in this manner we obtain for $V^{(2)}(a,b,\rho)$ the asymptotic expansion

$$V^{(2)}(a,b,\rho) = - \frac{2ik_1^3 e^{ik_2\rho + ik_1 a(1-n^2)^{\frac{1}{2}}}}{(k_1^4 - k_2^4)(ik_2\rho)^2} \frac{1+n^2}{n^3} \left\{F_0(b) + \frac{F_1(a,b)}{2ik_2\rho}\right.$$

$$\left. + \frac{F_2(a,b)}{8(ik_2\rho)^2} + O(n^2 k_2 \rho)^{-3} \right\}. \qquad (8.137)$$

where the expansion coefficients are

$$F_o(b) = 1 - \beta n^2(1-n^2)^{\frac{1}{2}};$$

$$F_1(a,b) = F_oE_1 + b(E_{1b}-n^2E_{1a});$$

$$F_2(a,b) = F_oE_2 + b(E_{2b}-n^2E_{2a}),$$

where $E_1(a,b)$ and $E_2(a,b)$, originally reported in Eqs. (8.96), may be written, making use of Eqs. (8.57), (8.59), (8.61), and (8.77), in the various forms that follow:

$$\begin{aligned} E_1(a,b) &= 6C_2+3(2+n^2)/n^2+7/4 \\ &= n^{-2}\left\{6+4n^2+n^4\left[3\alpha(1-n^2)^{-\frac{1}{2}}+\beta^2\right]\right\} \\ &= n^{-2}\left\{6+4n^2+(3\alpha+\beta^2)n^4+(3/2)\alpha n^6+(9/8)\alpha n^8+0(n^{10})\right\}; \end{aligned} \tag{8.139}$$

$$\begin{aligned} E_2(a,b) &= 120C_4+\left[33+60(2+n^2)/n^2\right]C_2+30(4+5n^2+n^4)/n^4 \\ &\quad +33(2+n^2)/2n^2+225/16 \\ &= n^{-4}\left\{120+168n^2+n^4\left[48+60\alpha(1-n^2)^{-\frac{1}{2}}+20\beta^2\right]+n^6\left[39\alpha(1-n^2)^{-\frac{1}{2}}\right.\right. \\ &\quad \left.\left.-15\alpha(1-n^2)^{-3/2}+8\beta^2\right]+n^8\left[15\alpha^2(1-n^2)^{-1}+10\alpha\beta^2(1-n^2)^{-\frac{1}{2}}+\beta^4\right]\right\} \\ &= n^{-4}\left\{120+168n^2+(48+60\alpha+20\beta^2)n^4+(54\alpha+8\beta^2)n^6\right. \\ &\quad \left.+(39\alpha/2+15\alpha^2+10\alpha\beta^2+\beta^4)n^8+0(n^{10})\right\}. \end{aligned} \tag{8.140}$$

Finally, substituting the above expansions in powers of n^2 into Eqs. (8.138) we obtain the expansion coefficients in the form

$$
\begin{aligned}
F_0(b) &= 1-\beta n^2+\tfrac{1}{2}\beta n^4+(1/8)\beta n^6+(1/16)\beta n^8+O(n^{10});\\
F_1(a,b) &= n^{-2}\Big\{6+(4-6\beta)n^2+(3\alpha-\beta+3\beta^2)n^4+\big[(3/2)\alpha-\tfrac{1}{4}\beta\\
&\qquad -3\alpha\beta-\beta^3\big]n^6+\big[(9/8)\alpha-(5/8)\beta+\tfrac{1}{2}\beta^3\big]n^8+O(n^{10})\Big\};\\
F_2(a,b) &= n^{-4}\Big\{120+(168-120\beta)n^2+(48+60\alpha-108\beta+60\beta^2)n^4\\
&\qquad +(54\alpha-9\beta-60\alpha\beta+24\beta^2-20\beta^3)n^6+\tfrac{1}{2}(39\alpha-3\beta\\
&\qquad -48\alpha\beta+30\alpha^2+60\alpha\beta^2+4\beta^3+10\beta^4)n^8+O(n^{10})\Big\}.
\end{aligned}
\tag{8.141}
$$

We note from Eqs. (8.141) that successive coefficients of the asymptotic series in the bracket of (8.137) behave as follows:

$$F_0(b) = O(1), \quad F_1(a,b) = O(n^{-2}), \quad F_2(a,b) = O(n^{-4}),$$

from which we conclude that successive terms of the said asymptotic series behave as reciprocal powers of $n^2k_2\rho$ and that the remainder in (8.137) is properly estimated as $O(n^2k_2\rho)^{-3}$. This means that the series (8.137) is worthless for practical purposes and that only asymptotically, $\rho \rightarrow \infty$ or $|n^2k_2\rho| \gg 1$, do we derive any useful information from its leading term. The difficulty is readily traced to the presence of a pair of poles in the integrand of $K^{(2)}(a,b,\rho)$, Eq. (8.81), which is evident in the power series expansion (8.88) and which limits the radius of convergence for the function $f(x)$ as indicated in Eq. (8.91). These poles occur in the x-plane at $x = \pm x_0$, where x_0, from Eqs. (8.43) and

(8.86), may be written as

$$x_o = \left\{2ik_2\rho\left[1 - (1+n^2)^{-\frac{1}{2}}\right]\right\}^{\frac{1}{2}} = |n^2k_2\rho|^{\frac{1}{2}}\left[1 + (3i/8)|n|^2 + \ldots\right], \quad (8.142)$$

which shows that the pair of poles occur, for $|n|^2 \ll 1$, almost on the real axis and very close to the origin in the x-plane as shown in Fig. 13.

In accordance with our findings of Section 8.4d, we must now properly identify λ and ν in the form of the remainder (9.123) to conform with the present situation; thus, we place

$$\lambda^{\frac{1}{2}} \approx |n^2k_2\rho|^{\frac{1}{2}} \quad \text{and} \quad \nu^{\frac{1}{2}} = (4k_2\rho)^{\frac{1}{2}}, \quad (8.143)$$

which implies that, for $|n|^2 \ll 1$, the rectangle of analiticity for the function $\Phi(x,y)$ has become extremely narrow, Fig. 12. Further, since now $\lambda^{\frac{1}{2}} \ll \nu^{\frac{1}{2}}$, we may write simply for the remainder after N grouped terms, instead of (8.123), the form

$$|R_N| = O(\lambda^{-N}) = O|n^2k_2\rho|^{-N} \quad (8.144)$$

in complete accord with the conclusions of the preceding paragraph. Finally, we recall that the difficulty arising from the presence of a pair of poles very close to the saddle point can be remedied by the technique of the subtraction of the pole, Section 5.2c, which we take up in the next section.

<u>8.5b Evaluation of $V^{(p)}$</u>.- Following the method developed in Chapter V we resolve the integral $V^{(2)}(a,b,\rho)$ into the sum of two terms

$$V^{(2)} = V^{(s)} + V^{(p)}, \tag{8.145}$$

where $V^{(s)}$ denotes the evaluation of the integral $V^{(2)}$, as given by Eq. (8.135), after the removal of the pair of poles near the saddle point, and $V^{(p)}$ denotes the so-called contribution from the pole. In the present instance it proves convenient to evaluate first the contribution from the pole $K^{(p)}$ of the auxiliary integral $K^{(2)}$ and then, making use of Eq. (8.135), we compute

$$V^{(p)} = \frac{k_2^2 K_a^{(p)} - k_1^2 K_b^{(p)}}{k_1^4 - k_2^4} . \tag{8.146}$$

Thus, starting from Eq. (8.82) for $K^{(2)}(a,b,\rho)$, and following the prescriptions of Section 5.2c we have at once, from Eqs. (5.83) and (5.88),

$$K^{(p)} = -2x_o e^{ik_2\rho} \int_0^\infty \frac{e^{-\frac{1}{2}x^2}}{x^2 - x_o^2}\, dx \cdot \frac{4}{\pi} \int_0^\infty C(y) e^{-\frac{1}{2}y^2}\, dy$$

$$= - e^{ik_2\rho} i\pi C e^{-\frac{1}{2}x_o^2} \operatorname{erfc}(-ix_o/2^{\frac{1}{2}}), \tag{8.147}$$

where x_o is given by (8.142), and, from Eqs. (5.79) and (5.89),

$$C(y) = \lim_{x \to x_o} \left\{ \frac{x^2 - x_o^2}{2x_o} \Phi(x,y) \right\}, \tag{8.148}$$

and

$$C = \frac{4}{\pi} \int_0^\infty C(y) e^{-\frac{1}{2}y^2} \, dy \; . \tag{8.149}$$

Next, making use of Eqs. (8.49), (8.54), (8.83), and (8.148), we have

$$C(y) = - \tfrac{1}{2} \left[G(w_o) - G(-w_o) \right] g(x_o, y) , \tag{8.150}$$

from which, with the aid of Eqs. (8.55) and (8.47), we have from (8.149) for the factor $C(a,b,\rho)$ the expression

$$C(a,b,\rho) = - \tfrac{1}{2} \left[G(w_o) - G(-w_o) \right] \cdot \frac{4}{\pi} \int_0^\infty (4ik_2\rho \cos w_o - y^2)^{-\frac{1}{2}} e^{-\frac{1}{2}y^2} \, dy$$

$$= i \sin\left[nk_2 b(1+n^2)^{-\frac{1}{2}} \right] e^{ik_1 a(1+n^2)^{-\frac{1}{2}}} \cdot H_0^1(k_o\rho) e^{-k_o\rho} , \tag{8.151}$$

where $k_o = k_2(1+n^2)^{-\frac{1}{2}}$ in accordance with Eq. (8.9). Finally, in preparation for the application of formula (8.146) we compute, from (8.151), the difference

$$k_2^2 C_a - k_1^2 C_b = -ik_1 k_2^2 (1+n^2)^{-\frac{1}{2}} e^{ik_1 a(1+n^2)^{-\frac{1}{2}} - ink_2 b(1+n^2)^{-\frac{1}{2}}}$$

$$\times H_0^1(k_o\rho) e^{-ik_o\rho} \; . \tag{8.152}$$

Thus, making use of Eqs. (8.146), (8.147), and (8.152), we obtain the desired expression for the contribution from the pole,

$$V^{(p)}(a,b,\rho) = -\frac{\pi n^2(1+n^2)^{-\frac{1}{2}}}{k_1(1-n^4)} e^{ik_2\rho+ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times\left[H_0^1(k_0\rho)e^{ik_0\rho}\right] \cdot \left[e^{-\frac{1}{2}x_0^2}\ \mathrm{erfc}(-ix_0/2^{\frac{1}{2}})\right], \qquad (8.153)$$

which we now propose to expand asymptotically in order to compute $V^{(s)}(a,b,\rho)$ by means of the formula

$$V^{(s)} \sim V^{(2)} - V^{(p)}, \qquad (8.154)$$

which follows immediately from (8.145); that is, we obtain the asymptotic expansion of $V^{(s)}$ by subtracting from (8.137) the asymptotic expansion of (8.153).

8.5c Asymptotic expansion for $V^{(p)}$.— To this end, making use of well-known formulas, we expand the first bracket in (8.153) as follows:

$$H_0^1(k_0\rho)e^{-ik_0\rho} \sim \left[2/(i\pi k_0\rho)\right]^{\frac{1}{2}}\left\{1 + \frac{1}{4(2ik_0\rho)} + \frac{9}{32(2ik_0\rho)^2} + \frac{75}{128(2ik_0\rho)^3} + \cdots\right\}, \qquad (8.155)$$

which may be written, putting $c = (ik_2\rho)^{-\frac{1}{2}}$ and

$k_o = k_2(1+n^2)^{-\frac{1}{2}}$, in the form

$$H_0^1(k_o\rho)e^{-ik_o\rho} \sim \left[2\pi^{-1}c^2(1+n^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\left\{1 + \frac{1}{8}(1+n^2)^{\frac{1}{2}}c^2 + \frac{9}{128}(1+n^2)c^4 + \frac{75}{1024}(1+n^2)^{3/2}c^6 + \ldots\right\}. \tag{8.156}$$

The second bracket in (8.153) can be treated likewise by making use of the formula[1]

$$e^{z^2}\,\mathrm{erfc}(z) \sim (z\pi^{\frac{1}{2}})^{-1}\left\{1 - \frac{1}{2z^2} + \frac{3}{(2z^2)^2} - \frac{15}{(2z^2)^3} + \ldots\right\}, \tag{8.157}$$

which we now apply by putting $z^2 = -\frac{1}{2}x_o^2$ and $z = -ix_o/2^{\frac{1}{2}}$ obtaining

$$e^{-\frac{1}{2}x_o^2}\,\mathrm{erfc}(-ix_o/2^{\frac{1}{2}}) \sim \left[-ix_o(\pi/2)^{\frac{1}{2}}\right]^{-1}\left\{1+x_o^{-2}+3x_o^{-4}+15x_o^{-6} + \ldots\right\}$$

$$= \frac{ic(1+n^2)^{\frac{1}{4}}}{nQ^{\frac{1}{2}}(\pi/2)^{\frac{1}{2}}}\left\{1 + \frac{c^2}{\xi_o^2} + \frac{3c^4}{\xi_o^4} + \frac{15c^6}{\xi_o^6} + \ldots\right\}, \tag{8.158}$$

where we have made use of Eqs. (8.44), (8.86), and (8.87) to obtain the second form.

Substituting the expansions (8.156) and (8.158) into Eq. (8.153) we obtain the desired asymptotic expansion for the

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[1]W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics," (Chelsea Publishing Co., New York, 1949), p. 96.

contribution from the pole,

$$v^{(p)}(a,b,\rho) \sim -\frac{2iQ^{-\frac{1}{2}}e^{ik_2\rho}}{k_1(1-n^4)(ik_1\rho)}\, e^{ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times\left\{1+\frac{H_o}{ik_1\rho}+\frac{H_1}{2(ik_1\rho)^2}+\frac{H_2}{8(ik_1\rho)^3}+\dots\right\}, \qquad (8.159)$$

where the expansion coefficients are

$$H_o = n^{-3}\left[n^2\xi_o^{-2} + \frac{1}{8}n^2(1+n^2)^{\frac{1}{2}}\right];$$

$$H_1 = n^{-6}\left[6n^4\xi_o^{-4} + \frac{1}{4}n^4\xi_o^{-2}(1+n^2)^{\frac{1}{2}} + \frac{9}{64}n^4(1+n)^2\right]; \qquad (8.160)$$

$$H_2 = n^{-9}\left[120n^6\xi_o^{-6} + 3n^6\xi_o^{-4}(1+n^2)^{\frac{1}{2}} + \frac{9}{16}n^6\xi_o^{-2}(1+n^2) + \frac{75}{128}n^6(1+n^2)^{3/2}\right].$$

And, making use of the fact that $n^2\xi_o^{-2} = \frac{1}{2}(1+n^2)^{\frac{1}{2}}\left[1+(1+n^2)^{\frac{1}{2}}\right]$, which follows from (8.85), we obtain after expanding into powers of n^2 the forms

$$H_o = n^{-3}\left[1+\frac{7}{8}n^2+0+\frac{1}{64}n^6-\frac{3}{256}n^8+0(n^{10})\right];$$

$$H_1 = n^{-6}\left[6+\frac{37}{4}n^2+\frac{197}{64}n^4+0+\frac{1}{16}n^8+0(n^{10})\right]; \qquad (8.161)$$

$$H_2 = n^{-9}\left[120+273n^2+\frac{2985}{16}n^4+\frac{4209}{128}n^6+\frac{3}{64}n^8+0(n^{10})\right].$$

<u>8.5d Evaluation of $v^{(s)}$</u>.- According to Eq. (8.154) we now compute $v^{(s)}(a,b,\rho)$ as the difference between the asymptotic expansions (8.137) for $v^{(2)}(a,b,\rho)$ and (8.159)

for $V^{(p)}(a,b,\rho)$. In preparation for the indicated subtraction we first rewrite (8.137) in the same form as (8.159); that is, we exhibit successive terms as reciprocal powers of $(ik_1\rho)$ and we extract in front of the expansion the same factor that multiplies the bracket in (8.159). In this manner we obtain

$$V^{(2)}(a,b,\rho) \sim - \frac{2iQ^{-\frac{1}{2}}e^{ik_2\rho}}{k_1(1-n^4)(ik_1\rho)} e^{ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times F(a,b)\left\{\frac{n^{-3}F_o}{ik_1\rho} + \frac{n^{-4}F_1}{2(ik_1\rho)^2} + \frac{n^{-5}F_2}{8(ik_1\rho)^3} + \ldots\right\}, \qquad (8.162)$$

where the coefficients F_o, F_1, and F_2 are given by Eqs. (8.141) and where the factor $F(a,b)$ has the form

$$F(a,b) = (1+n^2)Q^{\frac{1}{2}}e^{ik_1a\left[(1-n^2)^{\frac{1}{2}}-(1+n^2)^{-\frac{1}{2}}\right]+ink_2b(1+n^2)^{-\frac{1}{2}}} . \qquad (8.163)$$

Combining the expansions (8.162) and (8.159) in accordance with Eq. (8.154) we obtain for the integral $V^{(s)}(a,b,\rho)$ the four-term asymptotic expansion

$$V^{(s)}(a,b,\rho) \sim \frac{2iQ^{-\frac{1}{2}}e^{ik_2\rho}}{k_1(1-n^4)(ik_1\rho)} e^{ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times\left\{1 + \frac{H_o-n^{-3}FF_o}{ik_1\rho} + \frac{H_1-n^{-4}FF_1}{2(ik_1\rho)^2} + \frac{H_2-n^{-5}FF_2}{8(ik_1\rho)^3} + \ldots\right\}, \qquad (8.164)$$

in which the only remaining problem is the evaluation of the indicated expansion coefficients which must be expanded in powers of n^2. Here, the coefficients F_o, F_1, and F_2 are given by Eqs. (8.141), whereas the coefficients H_o, H_1, and H_2 are similarly listed in (8.161). Thus, we need only the power series expansion of the factor $F(a,b)$, Eq. (8.163), which we carry out by successive steps yielding finally

$$F(a,b) = 1 + \left(\frac{7}{8} + \beta\right)n^2 + \left(-\frac{9}{128} - \frac{1}{2}\alpha + \frac{3}{8}\beta + \frac{1}{2}\beta^2\right)n^4 + \left(\frac{23}{1024} - \frac{3}{16}\alpha - \frac{1}{2}\alpha\beta - \frac{17}{128}\beta - \frac{1}{16}\beta^2 + \frac{1}{6}\beta^3\right)n^6 + O(n^8). \tag{8.165}$$

In further preparation for the evaluation of the expansion coefficients in (8.164) we compute, making use of Eqs. (8.141) and (8.165), the expansions

$$\begin{aligned} n^{-3}FF_o &= n^{-3}\left[1 + \frac{7}{8}n^2 - \left(\frac{9}{128} + \frac{1}{2}\alpha + \frac{1}{2}\beta\right)n^4 + \left(\frac{23}{1024} - \frac{3}{16}\alpha + \frac{1}{2}\beta + \frac{1}{16}\beta^2 - \frac{1}{3}\beta^3\right)n^6 + O(n^8)\right]; \\ n^{-4}FF_1 &= n^{-6}\left[6 + \frac{37}{4}n^2 + \frac{197}{64}n^4 - \left(\frac{75}{512} - \alpha - \beta^2\right)n^6 + O(n^8)\right]; \\ n^{-5}FF_2 &= n^{-9}\left[120 + 273\,n^2 + \frac{2985}{16}n^4 + \frac{4209}{128}n^6 + O(n^8)\right]. \end{aligned} \tag{8.166}$$

And, combining Eqs. (8.166) and (8.161) in accordance with (8.164) we obtain for the expansion coefficients the expressions

$$H_o - n^{-3}FF_o = \left(\frac{9}{128} + \frac{1}{2}\,\alpha + \frac{1}{2}\,\beta^2\right)n$$

$$-\left(\frac{7}{1024} - \frac{3}{16}\,\alpha + \frac{1}{2}\,\beta + \frac{1}{16}\,\beta^2 - \frac{1}{3}\,\beta^3\right)n^3 + O(n^5); \tag{8.167}$$

$$H_1 - n^{-4}FF_1 = \left(\frac{75}{512} - \alpha - \beta^2\right) + O(n^2);$$

$$H_2 - n^{-5}FF_2 = O(n^{-1}).$$

Accordingly, making use of Eqs. (8.167) in the expansion (8.164), we write for the contribution over the saddle point the asymptotic expansion

$$V^{(s)}(a,b,\rho) \sim \frac{2iQ^{-\frac{1}{2}}e^{ik_2\rho}}{k_1(1-n^4)(ik_1\rho)}\, e^{ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times\left\{1 + \frac{G_1(a,b)}{128(ik_1\rho)} + \frac{G_2(a,b)}{1024(ik_1\rho)^2} + \frac{O(n^{-1})}{(ik_1\rho)^3} + \ldots\right\}, \tag{8.168}$$

where the expansion coefficients are

$$G_1(a,b) = (9+64\alpha+64\beta^2)n - \left(\frac{7}{8} - 24\alpha+64\beta+8\beta^2 - \frac{128}{3}\,\beta^3\right)n^3 + O(n^5); \tag{8.169}$$

$$G_2(a,b) = (75 - 512\alpha - 512\beta^2) + O(n^2).$$

8.5e Behavior of the first few terms.- From an analysis of the coefficients (8.167) we can conclude that the general behavior of successive terms in the bracket of the expansion (8.168), as to order of magnitude, is as follows:

$$1 + \frac{O(n)}{ik_1\rho} + \frac{O(1)}{(ik_1\rho)^2} + \frac{O(n^{-1})}{(ik_1\rho)^3} + \frac{O(n^{-2})}{(ik_1\rho)^4} + \frac{O(n^{-3})}{(ik_1\rho)^5} + \dots,$$

which can be rewritten thus

$$1 + \frac{O(n)}{ik_1\rho} + \frac{O(1)}{(ik_1\rho)^2}\left\{1 + \frac{O(1)}{ik_2\rho} + \frac{O(1)}{(ik_2\rho)^2} + \frac{O(1)}{(ik_2\rho)^3} + \dots\right\},$$

which clearly indicates that the asymptotic series in the bracket of (8.168) eventually behaves like reciprocal powers of $(ik_2\rho)$ in full agreement with the predictions of Watson's lemma as presented in Section 8.4d and Eq. (8.126). It is seen that the first three terms of the bracket can be written in terms of reciprocal powers of $(ik_1\rho)$, exactly as we found in Part I for the corresponding expansions of $U^{(2)}(h-z,0,\rho)$ and $k_1^2V^{(s)}(h-z,0,\rho)$, but it is now clearly established that our series for $U^{(2)}(a,b,\rho)$ and $V^{(s)}(a,b,\rho)$ do not behave eventually as reciprocal powers of $(ik_1\rho)$ as had been erroneously claimed. See Corrections to Part I at the beginning of this report.

Thus, finally, if we follow the criterion that an asymptotic series ought to be stopped after the term which exhibits the smallest absolute value, then we should write for $V^{(s)}(a,b,\rho)$, at least for $|n|^2 \ll 1$, the two-term expansion formula

$$V^{(s)}(a,b,\rho) = \frac{2iQ^{-\frac{1}{2}}e^{ik_2\rho}}{k_1(1-n^4)(ik_1\rho)} e^{ik_1a(1+n^2)^{-\frac{1}{2}}-ink_2b(1+n^2)^{-\frac{1}{2}}}$$

$$\times\left\{1 + \frac{n(9+64\alpha+64\beta^2) + O(n^3)}{128(ik_1\rho)} + O(ik_1\rho)^{-2}\right\}, \qquad (8.170)$$

where the estimate of the remainder is possibly smaller than the actual error. Since the second term in the bracket of (8.170) is practically negligible in comparison with unity, we may find that in many cases the leading term of (8.170) gives an adequate representation for the integral $V^{(s)}(a,b,\rho)$.

IX. RESULTS FOR THE NON-CONDUCTING MEDIUM

The present Chapter parallels Chapter VII for the conducting medium except that we omit the solution for the case in which the point of observation and the source are both on the interface between the two media, $h = 0$ and $z = 0$, which has been discussed already in some detail in Section 7.2. The special case $z = 0$ for the point of observation on the interface, with the source at the depth h in the conducting medium, is discussed briefly in various sections when it is desired to verify the fact that the results for the non-conducting medium agree with the results for the conducting medium as determined by the boundary conditions.

The expansions for the fundamental integrals $U^{(2)}$ and $V^{(2)}$, obtained in Chapter VIII, are particularized for points of observation in the non-conducting medium by setting the parameters $a = h$ and $b = z$. As pointed out in Chapter VIII, the expansions for the fundamental integrals appropriate to the conducting medium are obtained by setting the parameters $a = h - z$ and $b = 0$; however, since we have already obtained in <u>Part I</u>, even though laboriously, adequate results for the conducting medium utilizing a different method of expansion, we do not here take advantage of the simpler developments to recalculate results for the conducting medium.

The method of evaluation of the fundamental integrals presented in Chapter VIII entails the following necessary restrictions on the parameters, namely:

$$|n| < 1, \qquad |k_1\rho| > 1; \qquad z/\rho < 1, \qquad h/\rho < 1. \tag{9.1}$$

It is important to note that we claim that our results are adequate for $|k_1\rho| > 1$, that is, for a horizontal range greater than a wavelength in the conducting medium, rather than for the more restrictive condition $k_2\rho > 1$, which means a horizontal range greater than a wavelength in the non-conducting medium, as might have been deduced from the conditions imposed by Watson's lemma. As pointed out in Chapter VIII, this apparent circumvention of Watson's lemma is a consequence of the resulting behavior of the first few terms of our asymptotic expansions, of which we have availed ourselves by judiciously limiting the number of terms retained, thereby minimizing the error of our asymptotic expansions and thus extending the range of applicability of our formulas.

If we impose, in addition to the necessary restrictions (9.1), the additional restrictions $k_2z \ll 1$ and $k_2h \ll 1$, then we achieve considerable simplification in our formulas. This means that the formulas developed in the present Chapter are all restricted to heights of the point of observation and depths of the source which are much less than a wavelength in the non-conducting medium (Fig. 1). However, at low frequencies, this does not constitute any serious practical restriction. Considering the terms in our series which are to be neglected, the present restriction means that we drop terms of $O(k_2 z)$ and $O(k_2 h)$ when compared with unity and that we treat $k_1 z$ and $k_1 h$ as being of $O(1)$. Thus, our present results apply to points of observation in the non-conducting medium for which the horizontal range ρ amounts to several conducting medium wavelengths, while the vertical height z above the interface is much less than one non-conducting medium wavelength, but might very well be several conducting medium wavelengths.

Accordingly, we can take the expansions of the fundamental integrals obtained in Chapter VIII, setting $a = h$ and $b = z$ and neglecting terms of $O(k_2 z)$ and $O(k_2 h)$ as compared with unity. Then, neglecting integrals of the type I_1, which are exponentially attenuated and therefore negligible, and substituting into the appropriate formulas of Chapter III we can obtain the Cartesian components of the Hertzian vector and the cylindrical components of the electromagnetic field vectors, from which one can deduce in turn the components of the time average Poynting's vector. However, the most general formulas obtained in this way are much too complicated to be of direct practical value; and it has been found advisable, as was done in Chapter VII, to develop formulas utilizing additional approximations valid for different horizontal ranges, in order to obtain simpler results in which numerical substitutions may be readily made. In particular, as in Chapter VII, we consider the three ranges: $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$, $|n^2 k_2 \rho| < 1 < k_2 \rho$, and $k_2 \rho < 1 < |k_1 \rho|$, where each range satisfies the condition $|k_1 \rho| > 1$. Actually, the simpler expressions given for the components of the electric and magnetic fields and for the Poynting's vector correspond to the more restricted ranges: $\rho \to \infty$ or $|n^2 k_2 \rho| \gg 1$, $|n^2 k_2 \rho| \ll 1 \ll k_2 \rho$, and $k_2 \rho \ll 1 \ll |k_1 \rho|$; which means that, for the transition ranges $k_2 \rho \approx 1$ and $|n^2 k_2 \rho| \approx 1$ one must re-examine the results without making the additional approximations which invalidate the simpler results of this Chapter in these transition regions.

In Section 9.5 is presented, we believe for the first time, an evaluation of the fundamental integral V_2, Eq. (8.19), in the form of a convergent infinite series which is valid for $k_2 \rho \ll 1$, such as was given by the Lien approximation for points of observation in the conducting medium, Section 7.4a. The asymptotic expansion for V_2 obtained by the saddle point method of integration, which is valid for $|k_1 \rho| > 1$, is shown to agree with this evaluation in the range of overlap, $k_2 \rho \ll 1 < |k_1 \rho|$; and, therefore, we are able to check our solution for the non-conducting medium with the same completeness that was possible for the solution in the conducting medium. Since this convergent power series evaluation of V_2 is valid for $k_2 \rho \ll 1$ and the saddle point method of evaluation is valid for $|k_1 \rho| > 1$, we are able to obtain results for all values of the horizontal range from $\rho = 0$ to $\rho \rightarrow \infty$.

In Section 9.6 we present a numerical example using the same values of the parameters given in Section 7.5 for the range $k_2\rho << 1 << |k_1\rho|$, the results being obtainable either from the converging series evaluation of V_2 or from the saddle point method of integration.

9.1 IMPOSITION OF THE CONDITION $|k_1\rho| > 1$

Considering the original resolution of the fundamental integrals into the sum of two integrals, Eq. (8.26), one about the branch cut C_1 designated by I_1 and the other about the branch cut C_2 designated by I_2 (Fig. 10), we may readily determine that all of the integrals of the type I_1 are negligible, being exponentially attenuated and of $O(e^{ik_1\rho})$, where $|k_1\rho| > 1$ and $k_1 = |k_1| e^{i\pi/4}$. This conclusion may be verified by actually evaluating the I_1 integrals by the saddle point method of integration, which we did in Part I and thus have not repeated here. Thus, in accordance with Eqs. (8.18) and (8.19), we obtain for $|k_1\rho| > 1$ the approximations of the fundamental integrals

$$\begin{aligned} U_2 &\approx U_2^{(2)} = U^{(2)}(h,z,\rho); \\ V_2 &\approx V_2^{(2)} = k_2^2 V^{(2)}(h,z,\rho), \end{aligned} \qquad (9.2)$$

which should be compared with the corresponding results for points of observation in the conducting medium, Eqs. (7.1).

9.1a Hertzian vector and field components for $|k_1\rho| > 1$.- Making use of Eqs. (3.6) and (3.9), which express the Cartesian components of the Hertzian vector in terms of the fundamental integrals U_2 and V_2, and employing the approximations (9.2) which also apply to the derivatives of the fundamental integrals, we obtain

$$\Pi_{x2} \approx \frac{ip}{4\pi k_2 \eta_2} U^{(2)};$$

$$\Pi_{z2} \approx -\frac{ip\cos\phi}{4\pi k_2 \eta_2} \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h}\right) V^{(2)}. \tag{9.3}$$

Similarly, from Eqs. (3.20), (3.21), and (3.22), the approximate expressions for the electric field components appropriate for $|k_1\rho| > 1$ become, making use of Eqs. (9.2),

$$E_{\rho 2} \approx \frac{i\omega\mu_o p}{4\pi} \cos\phi \left\{\frac{\partial^2 V^{(2)}}{\partial\rho^2} + U^{(2)}\right\};$$

$$E_{\phi 2} \approx -\frac{i\omega\mu_o p}{4\pi} \sin\phi \left\{\frac{1}{\rho}\frac{\partial V^{(2)}}{\partial\rho} + U^{(2)}\right\}; \tag{9.4}$$

$$E_{z2} \approx \frac{i\omega\mu_o p}{4\pi} \cos\phi \frac{\partial^2 V^{(2)}}{\partial h \partial\rho},$$

where the factor k_2/η_2 has been replaced by $\omega\mu_o$. And similarly, from Eqs. (3.27), (3.28), and (3.29), the appropriate approximate expressions for the magnetic field components become

$$H_{\rho 2} \approx \frac{p \sin\phi}{4\pi}\left\{\frac{\partial U^{(2)}}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h}\right)V^{(2)}\right\} ;$$

$$H_{\phi 2} \approx \frac{p \cos\phi}{4\pi}\left\{\frac{\partial U^{(2)}}{\partial z} + \frac{\partial^2}{\partial\rho^2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h}\right)V^{(2)}\right\} ; \qquad (9.5)$$

$$H_{z2} \approx -\frac{p \sin\phi}{4\pi}\frac{\partial U^{(2)}}{\partial\rho} .$$

It may be noted again, as pointed out in Sections 3.2b and 3.3b, that it is possible to express the Hertzian vector and the field components exclusively in terms of $V(h,z,\rho)$, which is achieved by eliminating $U(h,z,\rho)$ with the aid of any one of Eqs. (8.12) through (8.15), but no special advantage accrues from this procedure.

<u>9.1b Evaluation of $U^{(2)}$ for $|k_1\rho| > 1$</u>.- This evaluation has already been carried out in Section 8.4a, Eqs. (8.108) and (8.109); and it is now merely a matter of setting $a = h$ and $b = z$ and choosing the simplest form which will yield useful results. Thus, treating k_1h and k_1z as being of $O(1)$, which is equivalent to saying that we may neglect k_2h and k_2z as compared with unity, we find that each term of the asymptotic series for $U^{(2)}$ may be expanded in a power series in n^2. Since $|n|^2$ is a very small quantity (see <u>Table I</u>, page I-47), this expansion of the terms of the series for $U^{(2)}$ affords us a means of quickly determining what

simplifications may be made without impairing the validity of our results.

For the range $\rho \rightarrow \infty$ or $|n^2 k_2 \rho| > 1$ it will be found that it is possible to obtain simpler results without having to expand in powers of n^2; and since only the leading term survives in this range, we retain the leading term of $U^{(2)}$ without further approximations. The second term of the asymptotic series for $U^{(2)}$ is written neglecting terms of $O(n^4)$ as compared with unity, and the third term, the last term retained, is written neglecting terms of $O(n^2)$ as compared with unity. In this way we obtain from Eqs. (8.108) and (8.109),

$$U^{(2)} \sim \frac{2ie^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}}}{k_1(1-n^2)\rho^2} \Big\{ n\left[1-\beta(1-n^2)^{\frac{1}{2}}\right] - \left[1 - \beta - \tfrac{1}{2}n^2(3\alpha-4\beta-3\alpha\beta+3\beta^2-\beta^3) + O(n^4)\right](ik_1\rho)^{-1} - \tfrac{3}{2}n\left[3\alpha-3\beta-3\alpha\beta+3\beta^2-\beta^3+O(n^2)\right](ik_1\rho)^{-2}\Big\}, \qquad (9.6)$$

where according to Eqs. (8.77), setting $a = h$ and $b = z$,

$$\alpha = ik_1 h \quad \text{and} \quad \beta = ik_1 z. \qquad (9.7)$$

By inspecting Eq. (9.6) and assuming $k_1 h$ and $k_1 z$ are of $O(1)$ it may be seen that terms of $O(n^3/k_1\rho)$ and $O(n/k_1\rho)^2$ have been neglected as compared with unity, which may be verified as being proper by referring to <u>Table I</u>, page I-47, and recalling that $|k_1\rho| > 1$. As a final comment on our working formula for $U^{(2)}$, Eq. (9.6), we assume as in Eq. (8.108) that the remainder after the three terms of the asymptotic series presented is of $O(ik_1\rho)^{-3}$.

9.1c Evaluation of $V^{(2)}$ for $|k_1\rho| > 1$.- In order to achieve simpler formulas we include two results for $V^{(2)}(h,z,\rho)$, one of which is valid for $\rho \to \infty$ or $|n^2k_2\rho| > 1$ and is presented in the next section, while the other is valid for $|n^2k_2\rho| \approx 1 < |k_1\rho|$ and is presented here. Thus, collecting results from Section 8.5, Eqs. (8.145) and (8.153), setting $a = h$ and $b = z$, we have

$$V^{(2)}(h,z,\rho) = V^{(p)}(h,z,\rho) + V^{(s)}(h,z,\rho), \tag{9.8}$$

where $V^{(s)}$ may be obtained from Eqs. (8.168) and (8.169), while

$$V^{(p)} = -\frac{2Q^{-\frac{1}{2}}}{k_1^2(1-n^4)\rho}\, e^{ik_2\rho+in^{-1}k_0h-ink_0z} \times \left[(\tfrac{1}{2}\pi i k_0\rho)^{\frac{1}{2}}e^{-ik_0\rho}H_0^1(k_0\rho)\right]\left[-i(\pi/2)^{\frac{1}{2}}x_0e^{-\frac{1}{2}x_0^2}\operatorname{erfc}(-ix_0/2^{\frac{1}{2}})\right]; \tag{9.9}$$

where we have, from Eq. (8.9),

$$k_0 = k_2(1+n^2)^{-\frac{1}{2}}; \tag{9.10}$$

from Eq. (8.87),

$$Q = 2n^{-2}\left[(1+n^2)^{\frac{1}{2}} - 1\right] ; \tag{9.11}$$

and, from Eqs. (8.86) and (8.84),

$$x_o^2 = in^2k_2\rho(1+n^2)^{-\frac{1}{2}}Q = 2i(k_2-k_o)\rho \ . \tag{9.12}$$

In order to obtain the working formula for $V^{(s)}$ from Eqs. (8.168) and (8.169), setting $a = h$ and $b = z$, we proceed as in the previous section. Thus, treating k_1h and k_1z as being of $O(1)$, which is equivalent to neglecting k_2h and k_2z as compared with unity, we take the second and third terms of the three-term asymptotic series for $V^{(s)}$, Eqs. (8.169), whose coefficients are already expanded in powers of n^2. Neglecting terms of $O(n^3/k_1\rho)$ and $O(n/k_1\rho)^2$ as compared with unity, as was done to obtain our working formula for $U^{(2)}$, we obtain the working formula for $V^{(s)}$,

$$V^{(s)} \sim \frac{2Q^{-\frac{1}{2}}}{k_1^2(1-n^4)\rho}\, e^{ik_2\rho+in^{-1}k_oh-ink_oz}$$
$$\times\left\{1 + \tfrac{1}{2}n\left[\frac{9}{64} +\alpha+\beta^2+O(n^2)\right](ik_1\rho)^{-1}+\tfrac{1}{2}\left[\frac{75}{512} - \alpha-\beta^2+O(n^2)\right](ik_1\rho)^{-2}\right\}, \tag{9.13}$$

where α and β are given by Eqs. (9.7). For a discussion of the remainder in the above asymptotic series, see Section 8.5e.

The evaluation of $U^{(2)}$ as given by Eq. (9.6) plus the two evaluations of $V^{(2)}$, the one given by Eqs. (9.8), (9.9), and (9.13), and the other being given in the next section by

Eq. (9.15), are sufficient to yield by substitution into the formulas of Section 9.1a the Hertzian vector, the field components, and the time average Poynting's vector for the three principal ranges of the parameters which we consider for $|k_1\rho| > 1$. However, in order to simplify our results so that numerical substitutions may be readily made, we consider in the following sections further approximations appropriate to particular ranges of the parameters.

9.2 ASYMPTOTIC RESULTS FOR $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$

This range is properly characterized by the fact that Sommerfeld's numerical distance, Eq. (7.45), is large and consequently this range is primarily of academic interest, not only for the present low frequency case but also for radio frequencies. However, even though it is not possible to make actual physical measurements in this region, the character of the solution at infinity helps to establish a reasonable overall picture which may be of assistance in regions that are of practical interest.

The formulas presented below refer thus to the asymptotic range $\rho \to \infty$ and, consequently, are adequately represented by the leading terms of the corresponding asymptotic expansions. In particular, for $V^{(2)}$ we need only take the leading term of the expansion <u>before</u> the removal of the pole since now $|n^2 k_2 \rho| >> 1$. In any case, however, we are governed by the conditions (9.1) and, therefore, as $\rho \to \infty$ we must have

$z/\rho \to 0$ and $h/\rho \to 0$ since both z and h must remain finite.

9.2a Fundamental integrals for $\rho \to \infty$ or $|n^2 k_2 \rho| \gg 1$. — The expression of the fundamental integral $U^{(2)}$ appropriate for the present asymptotic range, $\rho \to \infty$, is obtained from the first term of Eq. (9.6), the second and third terms being negligibly small in this case. Thus,

$$U^{(2)} \sim \frac{2in\left[1-ik_1 z(1-n^2)^{\frac{1}{2}}\right]}{k_1(1-n^2)\rho^2} e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}} . \tag{9.14}$$

The corresponding expression of the fundamental integral $V^{(2)}$ is obtained directly from the results of Chapter VIII before the removal of the pole from the integrand. From Eqs. (8.137) and (8.138), preserving only the first term and setting $a = h$ and $b = z$, we obtain

$$V^{(2)} \sim \frac{2i\left[1-in^2k_1 z(1-n^2)^{\frac{1}{2}}\right]}{k_2^3(1-n^2)\rho^2} e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}} . \tag{9.15}$$

These results, Eqs. (9.14) and (9.15), may be checked directly with the corresponding asymptotic results for the conducting medium by comparing the respective integrals for $z = 0$. Thus, from Eqs. (2.67) and (2.69), we have

$$U_1 = U_2, \quad z = 0, \tag{9.16}$$

and from Eqs. (2.68) and (2.70), we have

$$n^2 V_1 = V_2, \quad z = 0; \tag{9.17}$$

and, in particular, evaluating the integrals over the path C_2 we obtain, for $z = 0$,

$$\begin{aligned} U_1^{(2)} &= M_1^{(2)} = U^{(2)}(h,0,\rho), \\ k_1^{-2} V_1^{(2)} &= V^{(2)}(h,0,\rho). \end{aligned} \tag{9.18}$$

Using Eqs. (9.18) we find that the asymptotic results for the non-conducting medium, Eqs. (9.14) and (9.15), agree with the corresponding results for the conducting medium, Eqs. (7.10) and (7.6). It is interesting to note that the asymptotic form of $U^{(2)}$, Eq. (9.14), may be obtained directly from the asymptotic form of $V^{(2)}$, Eq. (9.15), by applying any one of Eqs. (8.12) through (8.15).

9.2b Hertzian vector for $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$.- The form of the Cartesian components of the Hertzian vector suitable for the present range of parameters may be obtained by substituting $U^{(2)}$ as given by Eq. (9.14) and $V^{(2)}$ as given by Eq. (9.15) into Eqs. (9.3) and performing the differentiations. Thus, retaining only the leading terms, we have

$$\Pi_{x2} \sim \frac{ip\left[1-ik_1 z(1-n^2)^{\frac{1}{2}}\right]}{2\pi\sigma k_2(1-n^2)\rho^2}\, e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}},$$

$$\Pi_{z2} \sim -\frac{ip\cos\phi\left[1+n^2-in^2k_1 z(1-n^2)^{\frac{1}{2}}\right]}{2\pi\sigma k_2(1-n^2)^{\frac{1}{2}}n^3\rho^2}\, e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}}, \qquad (9.19)$$

where use has been made of the relation $k_2\eta_2 = i\sigma n^2$.

Apart from the $\cos\phi$ factor the relative order of magnitude of the two components of the Hertzian vector is

$$(\Pi_{z2}/\Pi_{x2}) = O(1/n^3), \qquad (9.20)$$

which indicates that in the low frequency case, as $\rho \rightarrow \infty$, the field may be described primarily in terms of the z components of the Hertzian vector as was found to be the case for the conducting medium, Eqs. (7.11). Since the z component, as shown in the static limit, Sections 4.4 and 4.5, is associated with a secondary source distributed over the boundary surface between the conducting and non-conducting media, the entire field in both the conducting and non-conducting media may be primarily ascribed to a surface phenomenon. Using the boundary conditions, the first of Eqs. (2.15) and (2.16), we find that the components of the Hertzian vector as given by Eqs. (9.19) for the non-conducting medium check with the corresponding components as given by Eqs. (7.11) for the conducting medium.

9.2c Electric field components for $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$.- The form of the electric field components suitable for the present asymptotic range, $\rho \to \infty$, may be obtained by substituting $U^{(2)}$ as given by Eq. (9.14) and $V^{(2)}$ as given by Eq. (9.15) into Eqs. (9.4), performing the differentiations; and preserving only the leading terms to yield

$$E_{\rho 2} \sim \frac{p \cos\phi}{2\pi \eta_2 \rho^2} e^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}} ;$$

$$E_{\phi 2} \sim \frac{p \sin\phi\, n^2 \left[1 - ik_1 z(1-n^2)^{\frac{1}{2}}\right]}{2\pi \eta_2 (1-n^2)\rho^2} e^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}} ; \qquad (9.21)$$

$$E_{z2} \sim \frac{p \cos\phi \left[1 - in^2 k_1 z(1-n^2)^{\frac{1}{2}}\right]}{2\pi \eta_2 n (1-n^2)^{\frac{1}{2}} \rho^2} e^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}} ,$$

where we have used the relation $\omega\mu_0/k_2 = 1/\eta_2$. These identical results, Eqs. (9.21), can also be obtained by substituting the Hertzian vector, Eqs. (9.19), into Eqs. (3.13) and preserving only the leading terms.

Apart from the $\sin\phi$ and $\cos\phi$ factors, it may be seen that for points near the interface between the two media the electric field components stand in the relative order of magnitude

$$(E_{\rho 2}/E_{\phi 2}/E_{z2}) = O(n/n^3/1), \qquad (9.22)$$

which may be compared with the results for the conducting medium, page I-190. That is, putting $E_{z2} = O(1)$ we have, in matrix notation, the order of magnitude comparison for the electric field components belonging to the two media,

$$\begin{pmatrix} E_{\rho 1} & E_{\phi 1} & E_{z1} \\ \\ E_{\rho 2} & E_{\phi 2} & E_{z2} \end{pmatrix} = O \begin{pmatrix} n & n^3 & n^2 \\ \\ n & n^3 & 1 \end{pmatrix} ; \tag{9.23}$$

from which we see that the corresponding horizontal components are of the same order of magnitude, whereas the vertical components stand in the ratio $(E_{z2}/E_{z1}) = O(1/n^2)$, all of which follows readily from the boundary conditions.

The electric field components, Eqs. (9.21), for the non-conducting medium are seen to agree with the corresponding electric field components, Eqs. (7.12), for the conducting medium upon using the following boundary conditions:

$$\begin{aligned} E_{\rho 1} = E_{\rho 2}, \quad E_{\phi 1} = E_{\phi 2}, \quad z = 0 \; ; \\ E_{z1} = n^2 E_{z2}, \quad z = 0 \; , \end{aligned} \tag{9.24}$$

where the first two conditions arise from the continuity of the horizontal components of the electric field and the last condition, which expresses the discontinuity of the z component of the electric field caused by a surface charge distribution, may be determined from the last of Eqs. (3.13) and the boundary conditions (2.15) and (2.16).

It is interesting to note that the nature of the electric field as given by Eqs. (9.21) changes drastically when we consider values of z of the order of, but necessarily less than, ρ. Thus, considering $|n^2k_2z| > 1$ we find that the ϕ and z components are much larger than their values near the interface, and that compared with the ϕ and z components the radial component is now negligibly small. The electric field is seen to take on more of the character of a free space radiating field for these large values of z. Since a discussion of the field on the hemisphere at infinity can be best carried out by using another asymptotic evaluation of the fundamental integrals in terms of $R = (\rho^2+z^2)^{\frac{1}{2}}$ instead of ρ, we will restrict ourselves here primarily to a discussion for points of observation near the interface, keeping in mind that the character of the solution will be quite different for points of observation sufficiently far removed from the interface.

<u>9.2d Magnetic field components for $\rho \to \infty$ or $|n^2k_2\rho| > 1$.</u>– The form of the magnetic field components suitable for the present range, $\rho \to \infty$, may be obtained by substituting $U^{(2)}$ as given by Eq. (9.14) and $V^{(2)}$ as given by Eq. (9.15) into Eqs. (9.5), performing the differentiations, and preserving only the leading terms to yield

$$H_{\rho 2} \sim \frac{np \sin\phi}{2\pi(1-n^2)^{\frac{1}{2}}\rho^2} e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}} ;$$

$$H_{\phi 2} \sim - \frac{p \cos\phi \left[1 - in^2 k_1 z (1-n^2)^{\frac{1}{2}}\right]}{2\pi (1-n^2)^{\frac{1}{2}} n \rho^2} e^{ik_2\rho + ik_1 h (1-n^2)^{\frac{1}{2}}} ; \qquad (9.25)$$

$$H_{z2} \sim \frac{n^2 p \sin\phi \left[1 - ik_1 z (1-n^2)^{\frac{1}{2}}\right]}{2\pi (1-n^2) \rho^2} e^{ik_2\rho + ik_1 h (1-n^2)^{\frac{1}{2}}} .$$

These same results can also be obtained by substituting the Hertzian vector, Eqs. (9.19), into Eqs. (3.23) and preserving only the leading terms.

Apart from the factors $\sin\phi$ and $\cos\phi$ and considering points of observation near the interface, it may be seen that magnetic field components stand in the relative order of magnitude

$$(H_{\rho 2}/H_{\phi 2}/H_{z2}) = 0(n^2/1/n^3), \qquad (9.26)$$

which is seen to be identical to the corresponding result for the magnetic field components in the conducting medium, page I-191; or, in matrix notation, letting $H_{\phi 2} = 0(1)$, we have

$$\begin{pmatrix} H_{\rho 1} & H_{\phi 1} & H_{z1} \\ H_{\rho 2} & H_{\phi 2} & H_{z2} \end{pmatrix} = 0 \begin{pmatrix} n^2 & 1 & n^3 \\ n^2 & 1 & n^3 \end{pmatrix} . \qquad (9.27)$$

The magnetic field components for the non-conducting medium, Eqs. (9.25), are seen to agree with the corresponding magnetic field components for the conducting medium, Eqs. (7.13), upon using the boundary condition that the magnetic field be continuous across the boundary, $z = 0$, between the conducting and non-conducting media.

As for the electric field components we find that the character of the magnetic field changes drastically when we let z be of the order of, but necessarily less than, ρ. The magnetic field for the present asymptotic range, $\rho \to \infty$, is seen to take on more of the character of a free space radiating field for these large values of z.

9.2e Power flow for $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$.- Confining our attention to points of observation near the interface, which means that we treat $k_1 z$ as being of $O(1)$, thus allowing us to neglect $k_2 z$ in comparison with unity, the time average Poynting's vector, $\langle S \rangle = \frac{1}{2}\mathrm{Re}\left\{ E \times H^* \right\}$, becomes upon substituting the asymptotic forms of the field components, Eqs. (9.21) and (9.25), and neglecting terms of $O(n^2)$ as compared with unity,

$$\langle S_{\rho 2} \rangle \sim \frac{2^{\frac{1}{2}} p^2 \cos^2\phi}{8\pi^2 \sigma \delta |n|^3 \rho^4} e^{-2h/\delta} ;$$

$$\langle S_{\phi 2} \rangle \sim \frac{2^{\frac{1}{2}} |n| p^2 \sin\phi \cos\phi}{8\pi^2 \sigma \delta \rho^4} \left[1 - 2z/\delta\right] e^{-2h/\delta} ; \qquad (9.28)$$

$$\langle S_{z2} \rangle \sim -\frac{p^2 \cos^2\phi}{8\pi^2 \sigma \delta |n|^2 \rho^4} e^{-2h/\delta} ,$$

where $\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}}$ is the skin depth and where the relation $2^{\frac{1}{2}}\eta_2 = \sigma\delta|n|$ has been used.

For points of observation near the interface the relative orders of magnitude of the net power flow in the three coordinate directions then become, apart from the trigonometric factors,

$$(\langle S_{\rho 2}\rangle / \langle S_{\phi 2}\rangle / \langle S_{z2}\rangle) = O(1/|n|^4/|n|), \qquad (9.29)$$

showing that most of the power flow is radial and parallel to the interface between the two media. It may be noted by examining the orders of magnitude of the field components, Eqs. (9.22) and (9.26), that the ϕ component of the time average Poynting's vector, $\langle S_{\phi 2}\rangle$, would have been expected to be of $O(|n|^2)$ as compared with the ρ component, $\langle S_{\rho 2}\rangle$; but it is found upon computation that the $|n|^2$ term is pure imaginary and represents only reactive power flow; therefore, the $|n|^4$ term is the largest term representing net power flow. A similar situation occurs in the conducting medium for $\langle S_{\rho 1}\rangle$, Eqs. (7.16), where the next larger term, not shown, is of $O(|n|^2)$ times the term retained and represents an actual power flow. Comparing the present asymptotic results for the non-conducting medium with the corresponding results for the conducting medium, page I-191, we have in matrix notation, letting $\langle S_{\rho 2}\rangle = O(1)$,

$$\begin{pmatrix} \langle S_{\rho 1}\rangle & \langle S_{\phi 1}\rangle & \langle S_{z1}\rangle \\ \langle S_{\rho 2}\rangle & \langle S_{\phi 2}\rangle & \langle S_{z2}\rangle \end{pmatrix} = O \begin{pmatrix} |n|^4 & |n|^4 & |n| \\ 1 & |n|^4 & |n| \end{pmatrix}, \tag{9.30}$$

where the trigonometric factors have not been included.

The z component of the net power flow, Eqs. (9.28), is negative and is seen to be identical to the z component of the net power flow in the conducting medium for $z = 0$, Eqs. (7.16). This result was to be anticipated from the continuity of the tangential field components across the interface. Thus, we conclude that the energy which supplies the conducting medium with a field for $\rho \rightarrow \infty$ comes essentially from the non-conducting medium directly above the point of observation in the conducting medium.

A symmetry between the ϕ and z components of the electric and magnetic fields exists, namely

$$e_\rho \times E_2 = \zeta_2 H_2, \tag{9.31}$$

where e_ρ is a unit vector directed in the outward radial direction and $\zeta_2 = 1/\eta_2 = k_2/\omega\epsilon_0$, as defined by Eqs. (2.3), which may be readily verified by inspecting Eqs. (9.21) and (9.25). This implies a cylindrical wave proceeding outward from the z axis and properly describes the major part of the power flow asymptotically, $\rho \rightarrow \infty$. This result may be contrasted with the description of the field for the conducting medium, Eq. (7.30) and subsequent discussion.

It is interesting to note that this cylindrical wave picture as provided by Eq. (9.31) is even more applicable for large values of z since the relative importance of the radial components of the electric and magnetic field decreases for very large values of z. Thus, for $|n^2 k_2 z| > 1$ but $z < \rho$, the electric field is given essentially by the ϕ and z components; for from Eqs. (9.21), neglecting unity as compared with $k_1 z$ or $n^2 k_1 z$, we have

$$E_{\phi 2} \sim -\frac{k_2^2 pz \sin\phi}{2\pi\sigma n(1-n^2)^{\frac{1}{2}}\rho^2}\, e^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}};$$

$$E_{z2} \sim -\frac{k_2^2 pz \cos\phi}{2\pi\sigma n^2\rho^2}\, e^{ik_2\rho + ik_1 h(1-n^2)^{\frac{1}{2}}}, \qquad (9.32)$$

where use of the relation $1/\eta_2 = -ik_2/n^2\sigma$ has been made. The magnetic field components for this asymptotic case $\rho \rightarrow \infty$ and $|n^2 k_2 z| > 1$, but $z < \rho$, may be obtained from Eqs. (9.32) by using the relation (9.31). The time average Poynting's vector as obtained from Eqs. (9.32) and (9.31) becomes, for points of observation far above the interface between the two media,

$$\langle S_{2\rho}\rangle = \frac{\omega^2\mu_0^2 z^2 p^2 \cos^2\phi}{8\pi^2\rho^4}\, e^{-2h/\delta}, \qquad (9.33)$$

where only the ρ component survives and n^2 has been dropped as compared with unity in the exponent. This result, Eq. (9.33), may be contrasted with the power flow near the interface between the two media, Eqs. (9.28).

From the electric and magnetic field components, Eqs. (9.21) and (9.25), and the relative order of magnitude of the x and z components of the Hertzian vector, Eqs. (9.20), we readily see that the field arises primarily from the Π_{z2} component of the Hertzian vector, the Π_{x2} component yielding a negligible field by comparison. Since Π_{z2} may be associated with a secondary source distributed over the boundary between the conducting and non-conducting media, we conclude that, for $\rho \longrightarrow \infty$, the best description of the field near the interface is that of a <u>true</u> surface wave. Indeed, apart from an amplitude factor which represents the source and the cylindrical nature of the field, we may refer to Stratton's discussion[2] of a surface wave. Considering only the single z component of E, Eqs. (9.21), and the single ϕ component of H, Eqs. (9.25), which according to Eqs. (9.22) and (9.26) essentially describe the field for $\rho \longrightarrow \infty$, we may make the approximate replacement

$$e^{-ink_2 z} \approx 1 - in^2 k_1 z(1-n^2)^{\frac{1}{2}} \tag{9.34}$$

for points of observation near the interface. Thus, fitting Stratton's description and including the time variation, we have

[2] J. A. Stratton, "Electromagnetic Theory," (McGraw-Hill Book Co., New York, 1941), Sections 9.13 and 9.14, pp. 516-524.

$$H_{\phi 2} \sim Ce^{ik_2\rho - ink_2 z - i\omega t} , \qquad (9.35)$$

where

$$C = - \frac{p \cos\phi \; e^{ik_1 h(1-n^2)^{\frac{1}{2}}}}{2\pi(1-n^2)^{\frac{1}{2}} n\rho^2} \qquad (9.36)$$

is a factor indicating the nature of the source and the cylindrical structure of the surface wave. The electric field which is of the same nature may be obtained from Eq. (9.35) by using Eq. (9.31).

Examining the exponential factor in Eq. (9.35) we see that it represents an inhomogeneous plane wave in which the equiphase planes are tilted forward from the vertical by the small angle $\tan^{-1}(|n|/2^{\frac{1}{2}})$ and the equiamplitude planes are parallel to the interface separating the two media. From Eqs. (9.28) it may be seen that the time average Poynting's vector makes the same small angle, $\tan^{-1}(|n|/2^{\frac{1}{2}})$, down with respect to the horizontal. In conclusion, the field for $\rho \rightarrow \infty$ or $|n^2 k_2 \rho| > 1$ may be described essentially as a true surface wave for points of observation near the interface separating the two media, but must be described as more of a free space radiating type field when the point of observation is far above the interface.

9.3 RESULTS FOR THE RANGE $|n^2 k_2 \rho| < 1 < k_2 \rho$

This range is for a point of observation farther than an air wavelength radially from the source while Sommerfeld's numerical distance, Eq. (7.44), is small in comparison with unity. Although this range is not of primary interest in the present low frequency case, it is included here for the sake of completeness and for the purpose of comparing our results with the solution on the surface, which includes in part the range in which the Sommerfeld formula, Eq. (7.46), applies. In order to establish an overall picture, the electric and magnetic field components and the time average Poynting's vector have been obtained for the mid-range $|n^2 k_2 \rho| \ll 1 \ll k_2 \rho$, which was not done in Part I for points of observation in the conducting medium.

It is important to note that the results for the present range of parameters will be strictly limited to points of observation near the interface. That is, we assume $k_1 h$ and $k_1 z$ are of order unity, or more precisely we may neglect $k_2 h$ and $k_2 z$ as compared with unity.

9.3a Fundamental integrals for $|n^2 k_2 \rho| < 1 < k_2 \rho$.- The expression for $U^{(2)}$ as given by Eq. (9.14) is also adequate for the present range of parameters, since the $U^{(2)}$ integral, Eq. (8.10), possesses no pole in the integrand, and no further approximation need be considered here. For the present range of parameters the pole occurring in the integrand of $V^{(2)}$ requires that we choose the evaluation of $V^{(2)}$ as presented by Eqs. (9.8), (9.9), and (9.13). In particular, we need only retain the first term of $V^{(s)}$, Eq. (9.13), as suggested at the end of Section 8.5e. Rather than retain $V^{(p)}$, Eq. (9.9), in its exact and complicated form, we find that we may simplify our result for $V^{(2)} = V^{(p)} + V^{(s)}$ by making further approximations of $V^{(p)}$. Thus, there are three possible ways in which $V^{(p)}$ may be approximated for the present range of parameters, leading to three approximations of $V^{(2)}$.

First, we may regard $k_2\rho$ as being much greater than unity but $|n^2k_2\rho|$ not much less than unity, which may be written $|n^2k_2\rho| < 1 << k_2\rho$. For this case the Hankel function appearing in Eq. (9.9) for $V^{(p)}$ may be expanded asymptotically, but the error function complement is retained unaltered. Thus, using the leading term of the asymptotic series for $V^{(s)}$, Eq. (9.13), and the leading term of the asymptotic series for the Hankel function, we obtain

$$V^{(2)} \sim \frac{2Q^{-\frac{1}{2}}}{k_1^2(1-n^4)\rho} e^{ik_2\rho+in^{-1}k_oh-ink_oz} \times\left\{1 + i(\pi/2)^{\frac{1}{2}}x_o e^{-\frac{1}{2}x_o^2} \operatorname{erfc}(-ix_o/2^{\frac{1}{2}})\right\}, \tag{9.37}$$

where k_o, Q, and x_o are defined by Eqs. (9.10), (9.11), and (9.12), respectively. This result, Eq. (9.37), may be checked by noting that it reduces to the Sommerfeld formula, Eq. (7.46), upon setting $z = 0$, $h = 0$, neglecting n^2 as compared with unity and multiplying by k_1^2 as prescribed by Eq. (8.17).

The second approximation of $V^{(p)}$ which is also appropriate to the present range of parameters may be obtained by considering $k_2\rho$ very much greater than unity and $|n^2k_2\rho|$ very much less than unity, which may be written $|n^2k_2\rho| \ll 1 \ll k_2\rho$. This range of parameters includes a considerable range of values for the radial distance ρ as may be ascertained from Table I, page I-47. In addition to expanding asymptotically the Hankel function appearing in Eq. (9.9) for $V^{(p)}$, we may now expand the error function complement about the zero of its argument. Thus, using the leading term of the asymptotic series for $V^{(s)}$, Eq. (9.13), and the leading terms of the asymptotic series for the Hankel function and of the power series for the error function complement appearing in Eq. (9.9) for $V^{(p)}$, we obtain

$$V^{(2)} \sim \frac{2Q^{-\frac{1}{2}}}{k_1^2(1-n^4)\rho}\, e^{ik_2\rho+in^{-1}k_0h-ink_0z} \times \left\{1 + i(\pi/2)^{\frac{1}{2}}x_0 e^{-\frac{1}{2}x_0^2}\right\}. \tag{9.38}$$

Finally, a third approximation of $V^{(p)}$ which is also appropriate to the present range of parameters may be obtained by considering $k_2\rho$ not much greater than unity and $|n^2k_2\rho|$

much less than unity, which may be written $|n^2 k_2 \rho| \ll 1 < k_2\rho$. For this case we retain the Hankel function appearing in Eq. (9.9) for $V^{(p)}$ while we do expand the error function complement about the zero of its argument. Thus, using the first term of $V^{(s)}$, Eq. (9.13), and the first term of the power series expansion of the error function complement, we obtain

$$V^{(2)} \sim \frac{2Q^{-\frac{1}{2}}}{k_1^2(1-n^4)\rho} e^{ik_2\rho + in^{-1}k_0 h - ink_0 z}$$
$$\times \left\{1 - \tfrac{1}{2}\pi n Q^{\frac{1}{2}} k_0 \rho e^{-ik_2\rho} H_0^1(k_0\rho)\right\} . \tag{9.39}$$

Using the second of Eqs. (9.18) we may obtain a valuable check on the results of the present section by comparing with the corresponding results for the conducting medium, Section 7.1c. Thus, setting $z = 0$ in Eqs. (7.17) and (9.37) and similarly in Eqs. (7.18) and (9.39) we obtain agreement according to the second of Eqs. (9.18). The apparent differences involve terms with the factor $K = O(nh/\rho)$ which have been dropped in the present ordering of terms for the non-conducting medium.

9.3b Hertzian vector for $|n^2 k_2 \rho| < 1 < k_2\rho$.- Since the form of $U^{(2)}$, Eq. (9.14), is the same for the present range of parameters as for the previous asymptotic case, Section 9.2a, and since the x component of the Hertzian vector, Π_{x2}, is given by the first of Eqs. (9.3), we need merely copy the first of Eqs. (9.19) to obtain the expression of Π_{x2} appropriate to the present range of parameters. The expression for the z component of the Hertzian vector appropriate for the present range of parameters may be obtained by substituting $V^{(2)}$ as given by Eqs. (9.8), (9.9), and (9.13) into the second of Eqs. (9.3) for Π_{z2}. In particular, we wish to limit ourselves to the mid-range $|n^2 k_2 \rho| \ll 1 \ll k_2\rho$, so that we are entitled to make the additional simplifying approximations which led to Eq. (9.38) for $V^{(2)}$. Thus, we obtain, for $|n^2 k_2\rho| \ll 1 \ll k_2\rho$,

$$\Pi_{x2} \sim \frac{ip\left[1-ik_1 z(1-n^2)^{\frac{1}{2}}\right]}{2\pi\sigma\kappa_2(1-n^2)\rho^2}\, e^{ik_2\rho+ik_1h(1-n^2)^{\frac{1}{2}}} ;$$

$$\Pi_{z2} \sim -\frac{p\cos\phi\, Q^{-\frac{1}{2}}}{2\pi\sigma(1-n^4)n\rho}\, e^{ik_2\rho+in^{-1}k_0h-ink_0z} \qquad (9.40)$$

$$\times\left\{1 + i(\pi/2)^{\frac{1}{2}}x_0 e^{-\frac{1}{2}x_0^2}\right\},$$

where k_0, Q and x_0 are defined by Eqs. (9.10), (9.11), and (9.12), respectively. If one were to compute Π_{z2} by substituting the approximate expression (9.38) for $V^{(2)}$ into the second of Eqs. (9.3) and performing the differentiation one would obtain a result which is in error by a term of $O(n^2)$. Thus, if terms of $O(n^2)$ are to be neglected as compared with unity the simpler procedure of using the approximate expression (9.38) for $V^{(2)}$ is legitimate; otherwise, it is necessary to use the expression of $V^{(2)} = V^{(p)} + V^{(s)}$ in which $V^{(p)}$, Eq. (9.9), is expressed exactly and then make the additional simplifying approximations after the differentiation indicated in the second of Eqs. (9.3) has been performed.

Comparing the orders of magnitude of Π_{x2} and Π_{z2}, as given by Eqs. (9.35), we obtain aside from the $\cos\phi$ factor,

$$(\Pi_{z2}/\Pi_{x2}) = O[k_1\rho/(1-ik_1z)] \quad , \tag{9.41}$$

which indicates that, for the present range of parameters and for points of observation near the interface, the field may again be described primarily in terms of the z component of the Hertzian vector. Since the Hertzian vector was not computed in this range for points of observation in the conducting medium, no direct check with the solution in the conducting medium is available for Eqs. (9.40).

9.3c Field components for $|n^2k_2\rho| < 1 < k_2\rho$.- In order to present simple results we limit ourselves to the important mid-range $|n^2k_2\rho| << 1 << k_2\rho$ and neglect n^2 as compared with unity. Thus, we may substitute $U^{(2)}$ as given by Eq. (9.14) and $V^{(2)}$ as given by the approximate expression (9.38) directly into Eqs. (9.4) and (9.5) and then neglect n^2 as compared with unity to obtain the electric and magnetic field components. Or else, substituting the exact expression for $V^{(2)}$, Eqs. (9.8), (9.9), and (9.13), considering only the leading term of $V^{(s)}$, into Eqs. (9.4) and (9.5) for the field components, performing the indicated differentiations, expanding the Hankel function asymptotically and the error function complement about the zero of its argument, and finally neglecting n^2 as compared with unity, leads to the same identical expressions for the electric and magnetic field components. Thus, we obtain for the electric field components appropriate to the present range of parameters, $|n^2k_2\rho| << 1 << k_2\rho$, the expressions

$$E_{\rho 2} \sim -\frac{k_2^2 p \cos\phi}{2\pi\sigma\rho}\left[1 + in(\tfrac{1}{2}\pi i k_2\rho)^{\frac{1}{2}}\right]e^{ik_2\rho+ik_1h} \; ;$$

$$E_{\phi 2} \sim \frac{k_2^2 p \sin\phi}{2\pi\sigma\rho}\,\frac{2}{ik_2\rho}\left[1 - \tfrac{1}{2}ik_1z + \tfrac{1}{2}in(\tfrac{1}{2}\pi i k_2\rho)^{\frac{1}{2}}\right]e^{ik_2\rho+ik_1h} \; ; \qquad (9.42)$$

$$E_{z2} \sim -\frac{k_2^2 p \cos\phi}{2\pi\sigma\rho}\,\frac{1}{n}\left[1 + in(\tfrac{1}{2}\pi i k_2\rho)^{\frac{1}{2}}\right]e^{ik_2\rho+ik_1h} \; .$$

Comparing orders of magnitude of the electric field components, neglecting $in(\frac{1}{2}\pi i k_2\rho)^{\frac{1}{2}}$ as compared with unity, we have, aside from the trigonometric factors,

$$(E_{\rho 2}/E_{\phi 2}/E_{z2}) = O\left[n/(2-ik_1z)(k_1\rho)^{-1}/1\right] \; . \qquad (9.43)$$

The ordering of the three components is the same as for the asymptotic range, $\rho \to \infty$ or $|n^2k_2\rho| > 1$, Eq. (9.22), but now the ϕ component is relatively larger. As was the case for the Hertzian vector, it is not possible to check Eqs. (9.42) for the electric field components directly with the solution for the conducting medium, since the electric field components in this range were not calculated in Part I.

Next, substituting $U^{(2)}$ as given by Eq. (9.14) and $V^{(2)}$ as given by Eq. (9.38) into Eqs. (9.5) and neglecting n^2 as compared with unity, we obtain the magnetic field components appropriate to the present range of parameters, $|n^2 k_2 \rho| << 1 << k_2 \rho$,

$$H_{\rho 2} \sim \frac{ik_2 np \sin\phi}{2\pi\rho} \frac{2}{ik_2\rho}\left[1 + \tfrac{1}{2} in(\tfrac{1}{2}\pi i k_2 \rho)^{\frac{1}{2}}\right] e^{ik_2\rho + ik_1 h} ;$$

$$H_{\phi 2} \sim \frac{ik_2 np \cos\phi}{2\pi\rho}\left[1 + in(\tfrac{1}{2}\pi i k_2 \rho)^{\frac{1}{2}}\right] e^{ik_2\rho + ik_1 h} ; \qquad (9.44)$$

$$H_{z2} \sim \frac{ik_2 np \sin\phi}{2\pi\rho} \frac{n}{ik_2\rho}\left[1 - ik_1 z\right] e^{ik_2\rho + ik_1 h}$$

Comparing orders of magnitude of the magnetic field components, neglecting $in(\frac{1}{2}\pi i k_2 \rho)^{\frac{1}{2}}$ as compared with unity, we have, not including the $\sin\phi$ and $\cos\phi$ factors,

$$(H_{\rho 2}/H_{\phi 2}/H_{z2}) = O\left[1/k_2\rho/n(1-ik_1 z)\right] . \qquad (9.45)$$

Again the ordering of the components does not differ from the asymptotic case, $\rho \rightarrow \infty$ of $|n^2 k_2 \rho| > 1$, Eq. (9.26), but now the ϕ component is relatively smaller. Since the magnetic field components were not calculated in Part I for the present range of parameters, we are again unable to make a direct check of the continuity of the magnetic field across the boundary.

9.3d Power flow for $|n^2k_2\rho| < 1 < k_2\rho$.- Again limiting our discussion to the important mid-range $|n^2k_2\rho| << 1 << k_2\rho$ and to points of observation near the interface; in particular, neglecting k_2h and k_2z as compared with unity, and neglecting n^2 as compared with unity, we may substitute the field components as given by Eqs. (9.42) and (9.44) into the time average Poynting's vector, $\langle S\rangle = \frac{1}{2}\mathrm{Re}\left\{E \times H^*\right\}$, to obtain the net power flow. Thus, we obtain

$$\langle S_{\rho 2}\rangle \sim \frac{k_2^3p^2 \cos^2\phi}{8\pi^2\sigma\rho^2} e^{-2h/\delta} ;$$

$$\langle S_{\phi 2}\rangle \sim \frac{3k_2^3p^2 \sin\phi \cos\phi}{8\pi^2\sigma\rho^2} \frac{|n|}{2^{\frac{1}{2}}} \left(\frac{\pi}{k_2\rho}\right)^{\frac{1}{2}} e^{-2h/\delta} ; \qquad (9.46)$$

$$\langle S_{z2}\rangle \sim -\frac{k_2^3p^2 \cos^2\phi}{8\pi^2\sigma\rho^2} \frac{|n|}{2^{\frac{1}{2}}} e^{-2h/\delta} ,$$

where $\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}}$ is the skin depth. By examining the relative orders of magnitude of the electric and magnetic field components, Eqs. (9.43) and (9.45), it might be concluded that the ϕ component of the time average Poynting's vector should be of $0\left[1/n(k_2\rho)^{\frac{1}{2}}\right]$ times greater than the value actually found above; this comes about from the fact that the leading term of the ϕ component of the Poynting's vector is pure imaginary and, when the real part is taken to obtain the time average Poynting's vector, this leading term vanishes leaving the next higher order term.

The relative orders of magnitude of the components of the net power flow per unit cross sectional area in the three coordinate directions become

$$(\langle S_{\rho 2} \rangle / \langle S_{\phi 2} \rangle / \langle S_{z2} \rangle) = 0\left[1/|n|(k_2\rho)^{-\frac{1}{2}}/|n|\right] , \qquad (9.47)$$

where the trigonometric factors have not been included. Comparing this result, Eq. (9.47), with the corresponding result, Eq. (9.29), for the asymptotic range, $\rho \rightarrow \infty$ or $|n^2k_2\rho| > 1$, we find that the ordering of the components has not changed, but now the ϕ component is relatively larger.

Considering only the largest of the magnetic field components, Eqs. (9.44), and the largest of the electric field components, Eqs. (9.42), namely $H_{\phi 2}$ and E_{z2}, neglecting $in(\frac{1}{2}\pi ik_2\rho)^{\frac{1}{2}}$ as compared with unity and reinstating explicitly the exponential factor e^{-ink_2z} which arises from the integral $V^{(s)}$, Eq. (9.38), it is seen that the field may again be essentially described as a true surface wave as in the asymptotic case $\rho \rightarrow \infty$ or $|n^2k_2\rho| > 1$, Section 9.2e. In particular, we may write including the time variation

$$H_{\phi 2} \sim C' e^{ik_2\rho - ink_2 z - i\omega t} \quad , \tag{9.48}$$

where

$$C' = \frac{ik_2 np \cos\phi \; e^{ik_1 h}}{2\pi\rho} \tag{9.49}$$

is a factor indicating the nature of the source and the cylindrical structure of the surface wave. The electric field, E_{z2}, has the same exponential behavior and may be obtained from Eq. (9.48) by using Eq. (9.31). Although the exponent in Eq. (9.48) describes an inhomogeneous plane wave as in the previous asymptotic case, the factor in front is no longer the same, as may be seen by comparing Eqs. (9.49) and (9.36). We thus conclude that, for points of observation near the interface, the field may be described primarily as a true surface wave for horizontal distances much greater than a wavelength in air.

9.4 RESULTS FOR THE RANGE $k_2\rho < 1 < |k_1\rho|$

This range is the one of primary interest to us in the present low frequency investigation and constitutes the practical range of parameters. This range implies that the horizontal distance of the point of observation, Fig. 1, amounts to several wavelengths in the conducting medium but is only a fraction of a wavelength in air as measured from the source. In addition, we consider the point of observation and the depth of the source as being near the interface between the two media; in particular, we treat $k_1 h$ and $k_1 z$ as being of $O(1)$ which, more precisely, means that we may neglect $k_2 h$ and $k_2 z$ as compared with unity, the depth of the source and the height of the point of observation being much less than a wavelength in air. Since the asymptotic evaluation of the fundamental integrals necessarily requires that z/ρ and h/ρ be less than unity, we need consider no additional restrictions on h and z when the horizontal range is very much less than a wavelength in air, or when $k_2\rho << 1$.

Since the leading terms of our asymptotic expansions for the fundamental integrals do not always turn out to be the largest terms for this range of parameters, we develop a direct power series in $ik_2\rho$ by expanding as well the exponential term $e^{ik_2\rho}$. This procedure is only possible by virtue of the fact that we have restricted our asymptotic series to three terms, stopping short of the terms which behave as reciprocal powers of $(ik_2\rho)$. Complete verification of the validity of this procedure is established in Section 9.5 where our asymptotic results for $k_2\rho << 1 < |k_1\rho|$ are found to agree with an independent evaluation valid for $0 \leqslant k_2\rho << 1$.

9.4a Fundamental integrals for $k_2\rho < 1 < |k_1\rho|$.- Considering $U^{(2)}$ as given by the three-term asymptotic expansion (9.6) we find that the leading term is no longer the largest for the present range of parameters; and, thus, to properly order terms we extract the factor $-1/[n(ik_1\rho)^2]$ from the bracket. Next, we expand the exponential factor $e^{ik_2\rho}$ since $k_2\rho < 1$; and, finally, we expand the resulting expression into a power series in n^2, retaining only the first power in n^2 and treating k_1h and k_1z as being of $O(1)$, to obtain

$$U^{(2)} \sim \frac{2in^2}{k_2^3\rho^4} e^{ik_1h} \Big\{\gamma(1-\tfrac{1}{2}\gamma^2)(1-\beta) + \tfrac{1}{2}n^2\Big[3(3\alpha-3\beta-3\alpha\beta+3\beta^2-\beta^3) + \gamma(2+5\alpha-7\beta-5\alpha\beta+6\beta^2-2\beta^3)\Big]\Big\} , \tag{9.50}$$

where α and β are defined by Eqs. (9.7) and where we have set $\gamma = ik_2\rho$; that is, collecting parameters

$$\alpha = ik_1h, \quad \beta = ik_1z, \quad \gamma = ik_2\rho . \tag{9.51}$$

In obtaining the result (9.50) we have neglected terms of order n^3, γ^4, and $n^2\gamma^2$ which are negligible in comparison with γ.

Considering $V^{(2)} = V^{(p)} + V^{(s)}$ we may obtain the expression of $V^{(p)}$ appropriate for the present range of parameters from Eq. (9.9) by expanding the Hankel function about the zero of its argument and the error function complement about the zero of its argument and neglecting n^2 as compared with unity. To obtain $V^{(s)}$ appropriate for the present range of parameters we proceed in a manner identical to that used to obtain the appropriate expression of $U^{(2)}$, Eq. (9.50). Thus, from Eq. (9.13) we first extract the factor $1/(ik_2\rho)^2$, in order to be able to better order the terms; we next expand the exponential factor $e^{ik_2\rho}$ into powers of $ik_2\rho$ since $k_2\rho < 1$; and, finally, we expand into a power series in n^2, preserving only the first power in n^2 and treating k_1h and k_1z as being of $O(1)$. In this way, combining $V^{(p)}$ and $V^{(s)}$ according to Eq. (9.8), we obtain

$$V^{(2)} \sim -\frac{2n^2}{k_2^4\rho^3}\, e^{ik_1h}\left\{\gamma^2(1+\gamma+\tfrac{1}{2}\gamma^2) + n\gamma^3 \log(2/\gamma^1 k_2\rho) + \tfrac{1}{2}n^2\left(\frac{75}{512} - \alpha - \beta^2 + \frac{39}{128}\gamma\right)\right\}, \tag{9.52}$$

where $\gamma^1 = 1.78107\ldots$ and where α, β, and γ are given by Eqs. (9.51). In obtaining this result, Eq. (9.52), we have neglected terms in γ^5, $n\gamma^4$, and $n^2\gamma^2$ as compared with terms in γ^2.

We may check our result for $U^{(2)}$, Eq. (9.50), by comparing with the results for the conducting medium using the first of Eqs. (9.18). The proper expression of $M_1^{(2)}$ appropriate for the present ordering of the parameters may be obtained from Eqs. (6.102) and (6.103) by extracting the factor $[n(1-n^2)^2(ik_1\rho)^4]^{-1}$, noting that $K \approx n^2\alpha/\gamma$ for $z = 0$, and expanding in powers of n^2 treating α or ik_1h as being of $O(1)$. In this way it is possible to demonstrate the complete agreement between $M_1^{(2)}$ and $U^{(2)}$ on the interface for the present range of parameters; the only reason that $M_1^{(2)}$ as given by Eq. (7.22) appears not to check with $U^{(2)}$ on the interface is a consequence of the fact that Eq. (7.22) was deduced assuming $\cot\theta_2$ of $O(1)$ and neglecting terms of $O(n^2)$. In a similar fashion, using the second of Eqs. (9.18) we find complete agreement between $V_1^{(2)}$ and $V^{(2)}$ for $z = 0$ when $V_1^{(2)} = W^{(p)} + W^{(s)}$, as given by Eqs. (6.121), (6.125), (6.133), and (6.134), is expanded in powers of n^2. The apparent discrepancy between Eq. (7.25) for $V_1^{(2)}$ and Eq. (9.52) for $V^{(2)}$ is again a consequence of having considered $\cot\theta_2$ as being of $O(1)$ and having neglected terms of $O(n^2)$ in order to derive Eq. (7.25).

If we neglect terms in n^2, $n\gamma$, and γ^2, the fundamental integrals, Eqs. (9.50) and (9.52), reduce to the simpler forms

$$U^{(2)} \sim -\frac{2}{k_1^2\rho^3}\left[1 - ik_1 z\right] e^{ik_1 h} \quad ; \qquad (9.53)$$

$$V^{(2)} \sim \frac{2}{k_1^2\rho}\left[1 + nik_2\rho \log(2/\gamma^1 k_2\rho)\right] e^{ik_1 h} \quad ,$$

where $\gamma^1 = 1.78107 \ldots.$

<u>9.4b Hertzian vector for $k_2\rho < 1 < |k_1\rho|$</u>.- Substituting Eq. (9.50) into the first of Eqs. (9.3) we obtain the expression of the x component of the Hertzian vector appropriate

to the present practical range,

$$\Pi_{x2} \sim \frac{ipe^{ik_1h}}{2\pi\sigma k_2^3\rho^4}\Big\{\gamma(1-\tfrac{1}{2}\gamma^2)(1-\beta) + \tfrac{1}{2}n^2\Big[3(3\alpha-3\beta-3\alpha\beta+3\beta^2-\beta^3) + \gamma(2+5\alpha-7\beta-5\alpha\beta+6\beta^2-2\beta^3)\Big]\Big\} \quad , \tag{9.54}$$

where α, β, and γ are defined by Eqs. (9.51). Considering the first of the boundary conditions (2.15), the first of Eqs. (7.2) for Π_{x1}, and the first of Eqs. (9.3) for Π_{x2}, and the fact that $M_1^{(2)} = U^{(2)}$ for $z = 0$ as indicated in the previous section, we see that Π_{x2} checks exactly with Π_{x1}. The reason that Eq. (7.26) for Π_{x1} does not apparently check with Π_{x2} for $z = 0$ is again a consequence of the fact that Eq. (7.26) was derived assuming $\cot\Theta_2 = O(1)$ and neglecting n^2 as compared with unity, whereas Eq. (9.54) has been derived preserving n^2 terms, treating k_1h and k_1z as of $O(1)$, or more precisely, neglecting k_2h and k_2z as compared with unity.

The z component of the Hertzian vector is obtained by substituting $V^{(2)} = V^{(p)} + V^{(s)}$, as given by Eqs. (9.9) and (9.13), into the second of Eqs. (9.3) and then making the same approximations that yielded Eq. (9.52) for $V^{(2)}$ in the previous section. Thus, neglecting terms in γ^4, $n\gamma^3$, $n^2\gamma^2$, and n^3 in comparison with γ, we obtain

$$\Pi_{z2} \sim -\frac{p\cos\phi\, e^{ik_1h}}{2\pi\sigma k_2^2 n\rho^3}\Big\{\gamma(1-\tfrac{1}{2}\gamma^2) + n\gamma^2$$

$$+ \tfrac{1}{2}n^2\Big[3\Big(\frac{535}{512} + \alpha - 2\beta + \beta^2\Big) + \gamma\Big(\frac{1671}{512} + \alpha - 6\beta + 2\beta^2\Big)\Big]\Big\} , \tag{9.55}$$

where α, β, and γ are given by Eqs. (9.51). As a check on this result we use the first of the boundary conditions (2.16) and compare with the z component of the Hertzian vector for the conducting medium, Π_{z1}, the second of Eqs. (7.26). We find complete agreement if we neglect terms in n^2, treating k_1h and k_1z as of $O(1)$; but we do not obtain complete agreement for the coefficient of n^2. A more detailed analysis of the results for the conducting medium still shows a lack of agreement to within $O(n^2)$ for the present method of ordering the parameters. Since two distinct types of asymptotic series have been developed for the two media, it is not surprising that such a small discrepancy should show up in the approximate coefficient of n^2. Indeed, although it would not be justified, we might alter the coefficient of n^2 appearing in Eq. (9.55) simply by including portions of higher order terms in the asymptotic series for Π_{z2}. Since the asymptotic series developed for the z component of the Hertzian vector in the conducting medium, Π_{z1}, involves the function K, Eq. (6.67), which includes ρ implicitly, some such rearrangement of terms must occur. It is, perhaps, significant that no such discrepancy for the fundamental integrals evaluated in the two media by the two different methods occurs as was verified in Section 9.4a.

If we now, in addition, neglect terms in n^2, γ^3, and $n\gamma^2$ as compared with γ, the components of the Hertzian vector from Eqs. (9.54) and (9.55), reduce to the simpler expressions

$$\Pi_{x2} \sim -\frac{p(1-ik_1 z)}{2\pi\sigma k_2^2 \rho^3} e^{ik_1 h} ;$$

$$\Pi_{z2} \sim -\frac{ip \cos\phi}{2\pi\sigma k_2 n \rho^2} e^{ik_1 h} , \qquad (9.56)$$

which upon using the boundary conditions (2.16) are seen to agree for $z = 0$ with the corresponding components of the Hertzian vector for the conducting medium as given by Eqs. (7.26a).

Comparing orders of magnitude of the two components of the Hertzian vector, Eqs. (9.56), we find, apart from the $\cos\phi$ factor,

$$(\Pi_{z2}/\Pi_{x2}) = O\left[k_1\rho/(1-ik_1 z)\right], \qquad (9.57)$$

which is the same result we found for the mid-range $|n^2 k_2 \rho| \ll 1 \ll k_2\rho$, Eq. (9.41). Again we see that the z component of the Hertzian vector is the most important component.

9.4c Field components for $k_2\rho < 1 < |k_1\rho|$.- The expressions for the electric field components appropriate to the present practical range of parameters may be obtained by substituting $U^{(2)}$ as given by Eq. (9.6) and $V^{(2)} = V^{(p)} + V^{(s)}$ as given by Eqs. (9.9) and (9.13) into Eqs. (9.4) and then making the approximations that led to Eq. (9.52) in Section 9.4a. Thus, neglecting terms in γ^4, $n^2\gamma^2$, n^3 and $n\gamma^3$, we obtain

$$E_{\rho 2} \sim \frac{p\cos\phi}{2\pi\sigma\rho^3}\, e^{ik_1h}\Big\{1 + \beta + \tfrac{1}{2}\gamma^2(1-\beta) + \tfrac{1}{3}\gamma^3(2-\beta) + n\gamma$$

$$- \tfrac{1}{2}n^2\Big[\frac{981}{512} + 3\alpha + 6\beta + 4\alpha\beta + 2\beta^2 + \beta^3$$

$$+ \gamma\Big(\frac{141}{512} - \alpha - 3\beta - 3\alpha\beta - \beta^2 - \beta^3\Big)\Big]\Big\} \quad ;$$

$$E_{\phi 2} \sim \frac{p\sin\phi}{2\pi\sigma\rho^2}\,\frac{1}{ik_2\rho}\, e^{ik_1h}\Big\{\gamma(1-\tfrac{1}{2}\gamma^2)(2-\beta) + n\gamma^2$$

$$+ \tfrac{1}{2}n^2\Big[3\Big(\frac{23}{512} + 4\alpha - 3\beta - 3\alpha\beta + 4\beta^2 - \beta^3\Big)$$

$$+ \gamma\Big(\frac{1149}{512} + 6\alpha - 9\beta - 5\alpha\beta + 8\beta^2 - 2\beta^3\Big)\Big]\Big\} \quad ; \qquad (9.58)$$

$$E_{z2} \sim -\frac{p\cos\phi}{2\pi\sigma\rho^3}\,\frac{1}{n}\, e^{ik_1h}\Big\{\gamma(1-\tfrac{1}{2}\gamma^2) + n\gamma^2$$

$$+ \tfrac{1}{2}n^2\Big[3\Big(\frac{535}{512} + \alpha + \beta^2\Big) + \gamma\Big(\frac{637}{512} + \alpha - 2\beta + 2\beta^2\Big)\Big]\Big\} \quad ,$$

where α, β, and γ are defined by Eqs. (9.51). Using the boundary conditions specified by Eqs. (9.24), we find agreement between Eqs. (9.58), for the electric field in the non-conducting medium, and the corresponding Eqs. (7.28), for the electric field in the conducting medium, to within terms of $O(n^2)$ for the present method of ordering parameters. A more detailed analysis of the results for the conducting medium still shows a small lack of agreement to within $O(n^2)$; and this must be attributed to the fact that two types of asymptotic series were used to obtain results in the two media as explained in the previous section.

The expressions for the magnetic field components appropriate for the present practical range of parameters may be obtained by substituting $U^{(2)}$ as given by Eq. (9.6) and $V^{(2)} = V^{(p)} + V^{(s)}$ as given by Eqs. (9.9) and (9.13) into Eqs. (9.5) and then making the approximations that led to Eq. (9.52) in Section 9.4a. Thus, neglecting terms in γ^4, $n^2\gamma^2$, n^3, and $n\gamma^3$, we obtain

$$H_{\rho 2} \sim \frac{p \sin\phi}{\pi k_1^2 n \rho^4} e^{ik_1 h} \Big\{ \gamma(1-\tfrac{1}{2}\gamma^2) + \tfrac{1}{2}n\gamma^2 + \tfrac{1}{2}n^2\Big[3\Big(\frac{2071}{1024} + 2\alpha - 4\beta + 2\beta^2\Big) + \gamma\Big(\frac{5245}{1024} + 3\alpha - 9\beta + 4\beta^2\Big)\Big]\Big\} \quad ;$$

$$H_{\phi 2} \sim -\frac{ip \cos\phi}{2\pi k_1 \rho^3} e^{ik_1 h} \Big\{ 1 + \tfrac{1}{2}\gamma^2 + \tfrac{2}{3}\gamma^3 + n\gamma - \tfrac{1}{2}n^2\Big[\frac{469}{512} + 3\alpha + 2\beta^2 + \gamma\Big(\frac{653}{512} + \alpha - 2\beta + \beta^2\Big)\Big]\Big\} \quad ; \qquad (9.59)$$

$$H_{z2} \sim -\frac{3p\ \sin\phi}{2\pi k_1^2\rho^4}\, e^{ik_1h}\left\{(1-\frac{1}{6}\gamma^2)(1-\beta) + \tfrac{1}{2}n^2\left[2 - 7\alpha + 5\beta\right.\right.$$

$$\left.\left. + 7\alpha\beta - 6\beta^2 + 2\beta^3 - 5\gamma(\alpha-b-\alpha\beta+\beta^2-\frac{1}{3}\beta^3)\right]\right\} \quad ,$$

where α, β, and γ are defined by Eqs. (9.51). Using the boundary condition that the magnetic field be continuous across the interface between the two media, we find agreement between Eqs. (9.59), for the magnetic field components in the non-conducting medium, and the corresponding magnetic field components, Eqs. (7.29), in the conducting medium to within terms of $O(n^2)$ for the present method of ordering the parameters. A more detailed analysis of the results for the conducting medium again shows, as in the case of the z component of the Hertzian vector and of the electric field components, a lack of agreement to within terms of $O(n^2)$ for the present method of ordering the parameters; and we again assign this small discrepancy to the difference between the two types of asymptotic series developed for the solution in the two media.

Neglecting terms in $n\gamma$, n^2, and γ^2 the electric field components as given by Eqs. (9.58) reduce to the very simple expressions

$$E_{\rho 2} \sim \frac{p\ \cos\phi(1+ik_1z)}{2\pi\sigma\rho^3}\, e^{ik_1h} \ ;$$

$$E_{\phi 2} \sim \frac{p \sin\phi(2-ik_1 z)}{2\pi\sigma\rho^3} e^{ik_1 h} \quad ; \qquad (9.60)$$

$$E_{z2} \sim -\frac{p \cos\phi}{2\pi\sigma\rho^3}(ik_1\rho)e^{ik_1 h} \quad ,$$

which upon comparing with Eqs. (7.28a) and setting $z = 0$ are seen to satisfy the boundary conditions, Eqs. (9.24). Comparing orders of magnitude of the electric field components, omitting the $\sin\phi$ and $\cos\phi$ factors, we obtain

$$(E_{\rho 2}/E_{\phi 2}/E_{z2}) = O\left[(1+ik_1 z)/(2-ik_1 z)/ik_1\rho\right] \; , \qquad (9.61)$$

which may be contrasted with the results for the asymptotic range, $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$, Eq. (9.22), and the results for the mid-range, $|n^2 k_2 \rho| << 1 << k_2\rho$, Eq. (9.43). Comparing Eqs. (9.60) for the electric field in the non-conducting medium with Eqs. (7.28a) for the electric field in the conducting medium, we find that the significant difference is that E_{z2} is of the $O(1/n^2)$ times larger than E_{z1}, as we would of course anticipate from the boundary conditions, Eqs. (9.24).

We may similarly reduce the expressions for the magnetic field components, Eqs. (9.59), to the following simple expressions by neglecting terms in $n\gamma$, n^2, and γ^2,

$$H_{\rho 2} \sim \frac{ip \sin\phi}{\pi k_1 \rho^3} e^{ik_1 h} \quad ;$$

$$H_{\phi 2} \sim - \frac{ip \cos\phi}{2\pi k_1 \rho^3} e^{ik_1 h} \quad ; \tag{9.62}$$

$$H_{z2} \sim - \frac{ip \sin\phi}{2\pi k_1 \rho^3} \frac{3(1-ik_1 z)}{ik_1 \rho} e^{ik_1 h} \quad ,$$

which may be seen to satisfy the boundary condition of continuity of the magnetic field across the boundary upon comparison with Eqs. (7.29a), setting $z = 0$. Comparing the orders of magnitude of the magnetic field components, Eqs. (9.62), omitting the $\sin\phi$ and $\cos\phi$ factors, we obtain

$$(H_{\rho 2}/H_{\phi 2}/H_{z2}) = 0\left[2/1/3(1-ik_1 z)(ik_1\rho)^{-1}\right], \tag{9.63}$$

which may be contrasted with the results for the asymptotic range, $\rho \rightarrow \infty$ or $|n^2 k_2 \rho| > 1$, Eq. (9.26), and the results for the mid-range $|n^2 k_2 \rho| << 1 << k_2\rho$, Eq. (9.45). This same ordering of the components of the magnetic field occurs for points of observation in the conducting medium, as may be seen by comparing Eq. (9.63) with Eqs. (7.29a) and as could have been anticipated from the boundary conditions.

<u>9.4d Power flow for $k_2\rho < 1 < |k_1\rho|$</u>.- Limiting our discussion to points of observation near the interface, that is, treating $k_1 h$ and $k_1 z$ as being of $O(1)$ or, more precisely, neglecting $k_2 h$ and $k_2 z$ as compared with unity, and neglecting terms of order $nk_2\rho$, n^2, and $(k_2\rho)^2$ as

compared with unity, the time average Poynting's vector, $\langle S \rangle = \frac{1}{2}\mathrm{Re}\left\{ E \times H^* \right\}$, appropriate for the present practical range of parameters, is obtained by using the electric field components as given by Eqs. (9.60) and the magnetic field components as given by Eqs. (9.62); thus,

$$\langle S_{\rho 2} \rangle \sim \frac{k_2 p^2 \cos^2\phi}{8\pi^2 \sigma \rho^4} (k_2\rho)^2 e^{-2h/\delta} ;$$

$$\langle S_{\phi 2} \rangle \sim \frac{k_2 p^2 \sin\phi \cos\phi}{8\pi^2 \sigma \rho^4} \frac{|n|}{2^{\frac{1}{2}}} e^{-2h/\delta} ; \qquad (9.64)$$

$$\langle S_{z2} \rangle \sim - \frac{k_2 p^2 (\cos^2\phi + 4\sin^2\phi)}{8\pi^2 \sigma \rho^4} \frac{|n|}{2^{\frac{1}{2}} (k_2\rho)^2} e^{-2h/\delta} .$$

Inspecting the ρ component of the power flow, the first of Eqs. (9.64), for the present practical range of parameters, we find that it is identical in form to the expression obtained for the mid-range, $|n^2 k_2\rho| << 1 << k_2\rho$, the first of Eqs. (9.46). The z component of the power flow may be seen to be identical to the z component of the power flow in the conducting medium, as derived from Eqs. (7.28a) and (7.29a); a result which could have been anticipated from the boundary conditions that the horizontal electric and magnetic field components be continuous across the interface between the two media.

The relative orders of magnitude of the net power flow per unit cross-sectional area in the three coordinate directions, omitting the trigonometric factors, become

$$\left(\langle S_{\rho 2} \rangle / \langle S_{\phi 2} \rangle / \langle S_{z2} \rangle \right) = O\left[(k_2\rho)^4/|n|(k_2\rho)^2/|n|\right] . \qquad (9.65)$$

It is seen, by inspecting Eq. (9.65), that the relative orders of magnitude of the components of the time average Poynting's vector are very much dependent upon the horizontal range ρ, within the present practical range of parameters. Thus, for $k_2\rho$ not much less than unity, that is, for a point of observation almost a wavelength in the non-conducting medium radially away from the source, the radial flow of energy predominates, the vertical and angular flow being both of $O(|n|)$ by comparison; whereas, in the case of $|k_1\rho|$ not much greater than unity, that is, for a horizontal range not much greater than a wavelength in the conducting medium, the vertical flow of energy predominates, the angular flow being of $O(|n|^2)$ and the radial flow being of $O(|n|^3)$ by comparison.

In conclusion, it may be noted from Eq. (9.57) and subsequent equations that the entire field for the present range of practical interest, $k_2\rho < 1 < |k_1\rho|$, may be primarily ascribed to the z component of the Hertzian vector and therefore is due primarily to a surface phenomenon; although it may be noted from Eq. (9.57) that, as the horizontal range decreases, the x component of the Hertzian vector plays an increasingly more important part until the point is reached when $|k_1\rho|$ is not much greater than unity, the x and z components of the Hertzian vector then playing almost equivalent roles. For the present range of parameters it is not possible to describe the field in terms of a true surface wave as was done for horizontal ranges greater than a wavelength in air, $k_2\rho > 1$, in Sections 9.2 and 9.3; but we may still describe the field as arising primarily from a secondary source distributed over the boundary surface. As we found for points of observation greater than a wavelength in air radially away from the source, $k_2\rho > 1$, we again find for the present practical range, $k_2\rho < 1 < |k_1\rho|$, that the energy supplied to the conducting medium comes essentially from the non-conducting medium directly above the point of observation in the conducting medium, as may be concluded from the continuity of the time average Poynting's vector across the interface.

9.5 RESULTS FOR THE RANGE $0 \leqslant k_2\rho \ll 1$

The corresponding discussion in Part I, Section 7.4, was concerned with the limiting case in which we put $k_2 = 0$; however, since the solution obtained by setting $k_2 = 0$ is equivalent to obtaining an approximate solution which is adequate for the range $0 \leqslant k_2\rho < 1$, that is for points of observation much less than a wavelength in the non-conducting medium radially away from the source, we feel that the title of the present section for the non-conducting medium describes the contents more accurately than the title of Section 7.4. As was adequately demonstrated for the conducting medium, Section 7.4, putting $k_2 = 0$ in our fundamental integral V_1

leads to the proper approximate solution, the Lien approximation, in the region $0 \leqslant k_2\rho \ll 1$. Thus, to generate an equivalent approximate solution for the non-conducting medium for the range $0 \leqslant k_2\rho << 1$, we again set $k_2 = 0$ in our fundamental integral V, Eq. (8.11), letting $a = h$ and $b = z$. The result of this evaluation is then used to obtain a valuable check on our asymptotic series expansion of $V^{(2)}(h,z,\rho)$ which is found to agree with the present approximation, obtained by setting $k_2 = 0$, in the region of overlap, $k_2\rho << 1 < |k_1\rho|$, in which both results are applicable. Thus, in exactly the same manner as was used to verify the range of applicability of our asymptotic solution in the conducting medium, we verify the important fact that our asymptotically derived results for the non-conducting medium are also valid for all points of observation greater than a few wavelengths in the conducting medium radially away from the source, $|k_1\rho| > 1$. As further checks on our approximate evaluation of $V(h,z,\rho)$, obtained by setting $k_2 = 0$, we note that it properly reduces to the static limit, Chapter IV, Eq. (4.4), for $|k_1\rho| << 1$ and that it agrees with the corresponding solution, the Lien approximation, for the conducting medium when we set $z = 0$ and use the last of Eqs. (9.18).

9.5a Evaluation of $V(h,z,\rho)$, letting $k_2 = 0$.- In order to obtain an approximate solution for the range of parameters $0 \leqslant k_2\rho << 1$, that is, for points of observation much less than a wavelength in air radially away from the source, we set $k_2 = 0$ in the expression for $V(h,z,\rho)$, obtained from Eq. (8.11) after putting $a = h$ and $b = z$. Thus, replacing the Hankel function $H_0^1(\lambda\rho)$ appearing in Eq. (8.11) by the Bessel function $J_0(\lambda\rho)$, according to the scheme indicated by Eq. (2.85) and accompanying discussion, we define

$$\Lambda_2 = 2 \int_0^\infty \lim_{k_2 \to 0} \left\{ \frac{e^{-\gamma_1 h - \gamma_2 z}}{k_1^2\gamma_2 + k_2^2\gamma_1} \right\} J_0(\lambda\rho)\lambda d\lambda$$

$$= 2k_1^{-2} \int_0^\infty e^{-\gamma_1 h - \lambda z} \, J_0(\lambda\rho) d\lambda \, , \tag{9.66}$$

which implies

$$\lim_{k_2 \to 0} \left\{ V(h,z,\rho) \right\} \approx \Lambda_2 \, , \tag{9.67}$$

as pointed out in the similar situation in Section 7.4, Eq. (7.59).

In order to reduce our integral (9.66) to known integrals, we expand $e^{-\lambda z}$ in powers of λz and interchange the process of summation and integration to obtain

$$\Lambda_2 = \frac{2}{k_1^2} \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!} \int_0^\infty \lambda^m e^{-\gamma_1 h} J_0(\lambda\rho) d\lambda \, . \tag{9.68}$$

The process of expanding in powers of λz is valid for all λ and z; and, therefore, the interchange of the processes of summation and integration is also valid. We next resolve the series into the sum of an odd and even series to obtain

$$\Lambda_2 = \frac{2}{k_1^2} \sum_{m=o}^{\infty} \frac{z^{2m}}{(2m)!} \int_0^{\infty} \lambda^{2m} e^{-\gamma_1 h} J_o(\lambda\rho) d\lambda$$

$$- \frac{2}{k_1^2} \sum_{m=o}^{\infty} \frac{z^{2m+1}}{(2m+1)!} \int_0^{\infty} \lambda^{2m} e^{-\gamma_1 h} J_o(\lambda\rho) \lambda d\lambda . \qquad (9.69)$$

The factor λ^{2m} may now be removed by the process of differentiation,

$$\left(\frac{\partial^2}{\partial h^2} + k_1^2\right)^m e^{-\gamma_1 h} = \lambda^{2m} e^{-\gamma_1 h} ; \qquad (9.70)$$

which yields, upon application to Eq. (9.69),

$$\Lambda_2 = \frac{2}{k_1^2} \sum_{m=o}^{\infty} \frac{z^{2m}}{(2m)!} \left(\frac{\partial^2}{\partial h^2} + k_1^2\right)^m \int_o^{\infty} e^{-\gamma_1 h} J_o(\lambda\rho) d\lambda$$

$$- \frac{2}{k_1^2} \sum_{m=o}^{\infty} \frac{z^{2m+1}}{(2m+1)!} \left(\frac{\partial^2}{\partial h^2} + k_1^2\right)^m \int_o^{\infty} e^{-\gamma_1 h} J_o(\lambda\rho)\lambda d\lambda \ , \tag{9.71}$$

where the remaining integrals to be evaluated have known expressions in closed form.

The second integral in Eq. (9.71) may be obtained from the Sommerfeld formula, Eq. (2.66), yielding

$$\int_o^{\infty} e^{-\gamma_1 h} J_o(\lambda\rho)\lambda d\lambda = - \frac{\partial}{\partial h} \frac{e^{ik_1 R}}{R} \ , \tag{9.72}$$

where we define

$$R = (\rho^2 + h^2)^{\frac{1}{2}} \ . \tag{9.73}$$

And the first integral in Eq. (9.72) may be obtained from the Lien formula, Eqs. (7.60) and (7.61), where we substitute $\gamma_1^2 = \lambda^2 - k_1^2$, yielding

$$\int_0^\infty e^{-\gamma_1 h} J_o(\lambda\rho)d\lambda = \tfrac{1}{2}\Lambda_1(\rho,h)$$

$$= -\tfrac{1}{2}i\pi \frac{\partial}{\partial h}\left\{J_o\left(\tfrac{1}{2}k_1\left[R-h\right]\right) H_o^1\left(\tfrac{1}{2}k_1\left[R+h\right]\right)\right\} , \tag{9.74}$$

where R is defined by Eq. (9.73). The Lien formula (9.74), which we have also expressed in Eqs. (7.60) and (7.61), was first used by Foster[3] and later by Lien[4] and is identical to a formula appearing without proof in Magnus and Oberhettinger.[5]

Dr. Oberhettinger was kind enough to indicate a proof to us personally; and, in order to dispel any question as to the validity of Eq. (9.74), we include his proof here. Considering the integral

$$\int_0^\infty \gamma_1^{-1} e^{-\gamma_1 h} J_o(\lambda\rho)d\lambda , \tag{9.75}$$

from which the integral in Eq. (9.74) may be readily obtained by differentiating with respect to h, we replace $\gamma_1^{-1}e^{-\gamma_1 h}$ by using the definition of $K_{\frac{1}{2}}(x)$ and its integral

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[3] R. M. Foster, Bell System Tech. J. 10, 408-419 (1931).

[4] R. H. Lien, Journal of App. Physics 24, 1-5 (1953).

[5] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics," (Chelsea Publishing Co., New York, 1949), p. 133.

representation appearing in Watson.[6] Thus,

$$\gamma_1^{-1}e^{-\gamma_1 h} = (2h/\pi\gamma_1)^{\frac{1}{2}}K_{\frac{1}{2}}(\gamma_1 h)$$

$$= (h/2\pi^{\frac{1}{2}})\int_0^{\infty} t^{-3/2}e^{-t-h^2(\lambda^2-k_1^2)/4t}dt \; . \qquad (9.76)$$

Substituting this result, Eq. (9.76), under the integral sign of Eq. (9.75), interchanging the order of integrations, integrating with respect to λ using a formula in Watson[7] which in our notation may be written as

$$\int_0^{\infty} J_o(\lambda\rho)e^{-h^2\lambda^2/4t}d\lambda = \pi^{\frac{1}{2}}t^{\frac{1}{2}}h^{-1}e^{-\rho^2t/2h^2}I_o(\rho^2t/2h^2), \qquad (9.77)$$

and finally making the substitution

$$t = k_1^2h^2/2v,$$

noting that $v \rightarrow i\infty$ for $t = 0$ since $k_1^2 = i|k_1|^2$, we may express our integral, Eq. (9.75), in the form

$$\tfrac{1}{2}\int_0^{i\infty} v^{-1}e^{\frac{1}{2}v-k_1^2(\rho^2+2h^2)/4v}I_o(k_1^2\rho^2/4v)dv \; . \qquad (9.78)$$

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[6]G. N. Watson, "A Treatise on the Theory of Bessel Functions," (The Macmillan Co., New York, 1944), 2nd edition, p. 183, Eq. (15).

[7]*Loc. cit.*, p. 394, Eq. (15).

And thus we finally obtain, upon using a formula in Watson,[8] and indicating the differentiation with respect to h, the result given by Eq. (9.74).

Substituting the values of the two integrals appearing in Eq. (9.71) as given by Eqs. (9.72) and (9.74), we obtain the convergent infinite series approximation of $V(h,z,\rho)$, according to Eq. (9.67),

$$\Lambda_2 = \frac{1}{k_1^2}\sum_{m=o}^{\infty}\left(\frac{\partial^2}{\partial h^2}+k_1^2\right)^m\left\{\frac{z^{2m}}{(2m)!}\Lambda_1(\rho,h)+\frac{2z^{2m+1}}{(2m+1)!}\frac{\partial}{\partial h}\frac{e^{ik_1R}}{R}\right\}, \tag{9.79}$$

where $\Lambda_1(\rho,h)$ is defined by Eq. (9.74) and R is defined by Eq. (9.73). This result may be checked in three different ways. First, we compare it with the static solution for points of observation much less than a wavelength in the conducting medium radially away from the source, $|k_1\rho| << 1$; in particular, we show that

$$\lim_{k_1 \to o}\left\{k_1^2\Lambda_2\right\} = \lim_{\omega \to o}\left\{k_1^2 V_2/k_2^2\right\} \tag{9.80}$$

as may be derived from Eqs. (9.67), (8.19), and (4.4). Second, making use of the fact that $V_1(h-z,\rho) = k_1^2 V(h-z,0,\rho)$, we show

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[8] Loc. cit., p. 439, Eq. (2), where we let $Z = \frac{1}{2}k_1(R+h)$ and $z = \frac{1}{2}k_1(R-h)$, $R^2 = \rho^2+h^2$.

that this result for Λ_2 satisfies

$$\Lambda_1 = k_1^2 \Lambda_2, \quad z = 0. \tag{9.81}$$

And, third, in the next section we compare this result with the asymptotic series for $V^{(2)}$ in the region $k_2\rho \ll 1 < |k_1\rho|$ in which both results are applicable.

Thus, to derive the limit of $k_1^2\Lambda_2$ as $k_1 \to 0$ we obtain from Eq. (9.79)

$$\lim_{k_1 \to 0} \left\{k_1^2 \Lambda_2\right\} = \sum_{m=0}^{\infty} \frac{\partial^{2m}}{\partial h^{2m}} \left\{\frac{z^{2m}}{(2m)!} \lim_{k_1 \to 0} \left[\Lambda_1(\rho,h)\right] + \frac{2z^{2m}}{(2m+1)!} \frac{\partial}{\partial h} \frac{1}{R}\right\}, \tag{9.82}$$

where, from Eq. (9.74),

$$\lim_{k_1 \to 0} \left\{\Lambda_1(\rho,h)\right\} = 2/R . \tag{9.83}$$

Substituting Eq. (9.83) into Eq. (9.82), we then obtain

$$\lim_{k_1 \to 0} \left\{k_1^2 \Lambda_2\right\} = 2 \sum_{m=0}^{\infty} \frac{z^m}{m!} \frac{\partial^m}{\partial h^m} \frac{1}{R} = 2 \sum_{m=0}^{\infty} \frac{z^m}{m!} \left\{\frac{\partial^m}{\partial z^m} \frac{1}{R_1}\right\}_{z=0} = \frac{2}{R_1}, \tag{9.84}$$

where $R_1 = \left[\rho^2 + (z+h)^2\right]^{\frac{1}{2}}$. This result, Eq. (9.84), also follows directly from the defining equation for $k_1^2 \Lambda_2$, Eq. (9.66), by setting $k_1 = 0$ and using a formula in Watson.[9] Thus comparing Eq. (9.84) with the static limit solution as obtained directly in Chapter IV, Eq. (4.4), and noting that $\lim_{\omega \to 0} \left\{k_1^2/k_2^2\right\} = (\sigma_1+\sigma_2)/\sigma_2$, we complete our check by noting that Eq. (9.80) is satisfied.

The second check is obtained by setting $z = 0$ in Eq. (9.79) which gives the $m = 0$ term as the only non-vanishing term, and Eq. (9.81) follows immediately. An interesting and particularly simple form for Λ_2 may be obtained for the important special case in which the source is at the interface between the two media. Thus, setting $h = 0$ in the defining equation for Λ_2, Eq. (9.66), and using a formula in Watson,[10] we obtain

$$\Lambda_2 = (2/k_1^2)(\rho^2+z^2)^{-\frac{1}{2}}. \tag{9.85}$$

As still another check on our series evaluation of Λ_2, Eq. (9.79), we set $h = 0$ after performing the differentiations to obtain for the first two terms

$$\Lambda_2 = \frac{2}{k_1^2}\left\{\frac{1}{\rho} - \frac{z^2}{2\rho^3} + \cdots\right\} , \tag{9.86}$$

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[9] *Loc. cit.*, p. 384, Eq. (1).

[10] *Loc. cit.*, p. 384, Eq. (1).

which agrees with Eq. (9.85) to within $O(z^4/\rho^5)$.

9.5b Comparison of Λ_2 with $V^{(2)}$ for $k_2\rho << 1 < |k_1\rho|$.- The present comparison in the practical range of interest, $k_2\rho < 1 < |k_1\rho|$, not only serves to check the evaluation of Λ_2, Eq. (9.79), but also yields an important verification of the range of validity that we have claimed for our asymptotic evaluation of the fundamental integral $V \approx V^{(2)}(h,z,\rho)$. The appropriate expression for the asymptotic evaluation of $V^{(2)}$ for the range of parameters $k_2\rho << 1 < |k_1\rho|$ may be obtained from Eqs. (9.8), (9.9), and (9.13) by setting $n = 0$; thus, noting that in this limit $V^{(2)} = V^{(s)}$, we obtain

$$V^{(2)} \sim \frac{2e^{ik_1h}}{k_1^2\rho}\left\{1 + \tfrac{1}{2}\left[\frac{75}{512} - \alpha - \beta^2\right](ik_1\rho)^{-2}\right\}, \tag{9.87}$$

where $\alpha = ik_1h$ and $\beta = ik_1z$. The expression of Λ_2 appropriate for the range $k_2\rho << 1 < |k_1\rho|$ may be obtained from Eq. (9.79) by making use of the asymptotic expansion of $\Lambda_1(\rho,h)$, Eqs. (7.62), (7.66), and (7.67) and neglecting all terms in $e^{ik_1\rho}$ since $|k_1\rho| > 1$; thus,

$$\Lambda_2 \sim \frac{1}{k_1^2}\sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!}\left(\frac{\partial^2}{\partial h^2} + k_1^2\right)^m \Lambda_1^{(2)}(\rho,h) . \tag{9.88}$$

Substituting the first two terms of the asymptotic expansion of $\Lambda_1^{(2)}(\rho,h)$ as given by Eq. (7.67), where $x_1 = R - h$ and $x_2 = R + h$, and $R = (\rho^2+h^2)^{\frac{1}{2}}$, as defined by Eq. (9.73), and neglecting terms of $O(ik_1\rho)^{-6}$, we obtain

$$\Lambda_2 \sim \frac{2e^{ik_1h}}{k_1^2\rho}\left\{1 - \tfrac{1}{2}\left[\alpha + \beta^2\right](ik_1\rho)^{-2} + \tfrac{3}{8}\left[3\alpha^2 + 3\beta^2 + 6\alpha\beta^2 + \beta^4\right](ik_1\rho)^{-4}\right\} . \tag{9.89}$$

Comparing our two results, Eqs. (9.87) and (9.89), it may be seen that they differ, to within the number of terms retained for $V^{(2)}$, by the amount

$$k_1^2(V^{(2)} - \Lambda_2) \sim \frac{75\ ik_1}{512\ (ik_1\rho)^3}\, e^{ik_1h} ; \tag{9.90}$$

which may be compared with the identical situation for the conducting medium, Eq. (7.71). It may be noted that Eqs. (7.69) and (7.70) may be rewritten by letting $\cot\Theta_2 = ik_1(h-z)/(ik_1\rho)$ and rearranging terms into even powers of $(ik_1\rho)^{-1}$, thus showing that the agreement obtained for the non-conducting medium is as good as the agreement obtained for the conducting medium. We have, therefore, established the fact that our asymptotic solution for $V \approx V^{(2)}(h,z,\rho)$ blends smoothly with a completely independent solution which is valid in a region of overlap $k_2\rho << 1 < |k_1\rho|$ and which, in particular, is valid for all horizontal ranges much less than a wavelength in the non-conducting medium, $0 \leqslant k_2\rho \ll 1$.

Since the asymptotic form of Λ_2, Eq. (9.89), may be used to compute the Hertzian vector and the electric and magnetic field components appropriate to the practical range, $k_2\rho \ll 1 < |k_1\rho|$, it is well to point out here two additional checks on this formula. First, setting $z = 0$ and using Eq. (9.81) we obtain agreement with the corresponding asymptotic expression for the conducting medium, Λ_1, as given by Eq. (7.69), setting $z = 0$. A second check on the validity of the asymptotic form of Λ_2, Eq. (9.89), may be obtained by setting $h = 0$ and noting that the result differs by the small amount

$$-9z^2/4k_1^4\rho^5 \qquad (9.91)$$

from the non-asymptotic result, Eq. (9.85), thus indicating that Eq. (9.89) is adequate for relatively small values of $k_1\rho$.

9.5c Hertzian vector and field components for $k_2 = 0$.- Besides setting $k_2 = 0$ or assuming the equivalent restriction $k_2\rho \ll 1$, we wish to restrict our attention to points of observation greater than a wavelength in the conducting medium radially away from the source; thus, we are concerned here with the practical range of interest $k_2\rho \ll 1 < |k_1\rho|$. The Hertzian vector appropriate for this range may be obtained from $\Lambda_2 \approx V^{(2)}(h,z,\rho)$ as given by Eq. (9.89), deriving an approximate expression for $U^{(2)}(h,z,\rho)$ by using any one of Eqs. (8.12) through (8.15), and substituting into Eqs. (9.3); thus, we have

$$k_2^2 \Pi_{x2} \sim -\frac{p(1-ik_1 z)}{2\pi\sigma\rho^3} e^{ik_1 h} ;$$
$$k_2^2 \Pi_{z2} \sim -\frac{ik_1 p \cos\phi}{2\pi\sigma\rho^2} e^{ik_1 h} . \tag{9.92}$$

The electric field components for the present practical range of interest may be obtained from Eqs. (9.58) by setting $k_2 = 0$, or else by using Λ_2, Eq. (9.89), as an approximation of $V^{(2)}$, substituting into any one of Eqs. (8.12) through (8.15) to obtain an approximation of $U^{(2)}$, and substituting these results into Eqs. (9.4); thus, we obtain

$$E_{\rho 2} \sim \frac{p \cos\phi}{2\pi\sigma\rho^3}(1+ik_1 z)e^{ik_1 h} ;$$
$$E_{\phi 2} \sim \frac{p \sin\phi}{2\pi\sigma\rho^3}(2-ik_1 z)e^{ik_1 h} ; \tag{9.93}$$
$$E_{z2} \sim -\frac{p \cos\phi}{2\pi\sigma\rho^3} ik_1\rho\, e^{ik_1 h} .$$

Similarly, by setting $k_2 = 0$ in Eqs. (9.59) or by using Λ_2 and Eqs. (9.5), we obtain for the magnetic field components

$$H_{\rho 2} \sim \frac{ip \sin\phi}{\pi k_1 \rho^3} e^{ik_1 h} \quad ;$$

$$H_{\phi 2} \sim - \frac{ip \cos\phi}{2\pi k_1 \rho^3} e^{ik_1 h} \quad ; \tag{9.94}$$

$$H_{z2} \sim - \frac{ip \sin\phi}{2\pi k_1 \rho^3} \left(\frac{3}{ik_1 \rho}\right) (1 - ik_1 z) e^{ik_1 h} \quad .$$

These results, Eqs. (9.92), (9.93), and (9.94), are seen to be identical to Eqs. (9.56), (9.60), and (9.62) respectively, which were derived using only the asymptotic evaluations of $V^{(2)}(h,z,\rho)$ and $U^{(2)}(h,z,\rho)$ and simplifying the results by making approximations appropriate to the practical range of interest, $k_2\rho < 1 < |k_1\rho|$, and in which no use was made of the device of setting $k_2 = 0$. Formulas (9.92), (9.93), and (9.94) for the Hertzian vector and for the field components appropriate for the practical range $k_2\rho << 1 < |k_1\rho|$ have actually been established by two completely independent methods, thus yielding a necessary and important check on our working formulas.

9.6 NUMERICAL EXAMPLE

The results of the present Chapter for the non-conducting medium are now illustrated for the important range of parameters, $k_2\rho << 1 << |k_1\rho|$, which is the range of greatest practical interest in the present low frequency case, by using a numerical example for which the parameters are chosen as in Section 7.5, Eqs. (7.78) and (7.79). Considering the present section and Section 7.5 together, our numerical example gives the field components in both media produced by a dipole source embedded in the conducting medium.

Apart from the exponential factor $e^{-ik_1 z}$, $z \leqslant 0$, the solution in the conducting medium is not particularly sensitive to changes in z; however, the present solution in the non-conducting medium requires that we delineate the region of applicability in z as well as in ρ. Considering that our results were obtained under the necessary restriction $z/\rho < 1$ and in addition we chose to treat $k_1 z$ and $k_1 h$ as being of $O(1)$, which is equivalent to neglecting $k_2 h$ and $k_2 z$ as compared with unity, we find that when the horizontal range is at the approximate minimum of 50 meters the value of z must be less than 50 meters and we find that, in order for $k_2 z \ll 1$ or for z to be much less than wavelength in the non-conducting medium, z must be less than 5000 meters. Thus, it is sufficient for this numerical example if $z < 50$ meters, quite independent of the value of ρ. However, if larger values of z need be considered then we have the less restricting condition $z < \rho$ which suffices for the present practical range of parameters by virtue of the fact that we have also restricted ρ to values much less than a wavelength in the non-conducting medium.

Following the discussion of Section 7.5, we find that to within an accuracy of 0.5% for $k_2\rho \leqslant 10^{-1} < 1 < 10 \leqslant |k_1\rho|$, the electric field components from Eqs. (9.60) or (9.93) become

$$E_{\rho 2} \sim \frac{p\ \cos\phi}{2\pi\,\sigma\,\rho^3}\,(1 - z/\delta + i\ z/\delta)\,e^{(i-1)h/\delta}\ ;$$

$$E_{\phi 2} \sim \frac{p\ \sin\phi}{2\pi\,\sigma\rho^3}\,(2 + z/\delta - i\ z/\delta)\,e^{(i-1)h/\delta}\ ; \qquad (9.95)$$

$$E_{z2} \quad \frac{p\ \cos\phi}{2\pi\,\sigma\,\rho^2}\,\frac{1 - i}{\delta}\ e^{(i-1)h/\delta}\ ,$$

where $\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}}$ is the skin depth and where $z < \rho$. Similarly, the magnetic field components from Eqs. (9.62) or (9.94) become

$$H_{\rho 2} \sim \frac{\delta p\ \sin\phi}{2\pi\rho^3}\,(1+i)\,e^{(i-1)h/\delta}\ ;$$

$$H_{\phi 2} \sim -\frac{\delta p\ \cos\phi}{4\pi\rho^3}\,(1+i)\,e^{(i-1)h/\delta}\ ; \qquad (9.96)$$

$$H_{z2} \sim \frac{3i\delta^2 p\ \sin\phi}{4\pi\rho^4}\,(1 + z/\delta\ -\ i\ z/\delta)\,e^{(i-1)h/\delta}\ ,$$

where z is again restricted to values less than ρ. The size of the error committed in these approximate results, Eqs. (9.95) and (9.96), may be obtained from the more accurate formulas given by Eqs. (9.58) and (9.59), respectively. Thus, we have neglected $|n|^2 = 10^{-9}$, $|n\gamma| \leq 10^{-5}$, and $|\gamma|^2 = (k_2\rho)^2 \leq 10^{-2}$ as compared with unity.

Introducing the function $f(\rho,h) = (p/\pi\,\rho^3)e^{-h/\delta}$ volts/meter which is defined in a manner similar to Eq. (7.84), we obtain for the magnitudes of the electric field components, from Eqs. (9.95),

$$|E_{\rho 2}| \sim \tfrac{1}{2}f(\rho,h)\left[1 - 2(z/\delta) + 2(z/\delta)^2\right]^{\frac{1}{2}} \cos\phi \;;$$

$$|E_{\phi 2}| \sim \tfrac{1}{2}f(\rho,h)\left[4 + 4(z/\delta) + 2(z/\delta)^2\right]^{\frac{1}{2}} \sin\phi \;; \qquad (9.97)$$

$$|E_{z2}| \sim 2^{-\frac{1}{2}}f(\rho,h)(\rho/\delta)\cos\phi \,.$$

Similarly, from Eqs. (9.96), the magnitudes of the magnetic field components become

$$|H_{\rho 2}| \sim 2^{-\frac{1}{2}}\sigma\delta\, f(\rho,z)\sin\phi \;;$$

$$|H_{\phi 2}| \sim (\sigma\delta/2^{3/2})f(\rho,z)\cos\phi \;; \qquad (9.98)$$

$$|H_{z2}| \sim (3\sigma\delta^2/4\rho)f(\rho,z)\left[1 + 2(z/\delta) + 2(z/\delta)^2\right]^{\frac{1}{2}}\sin\phi \,.$$

By examining Eqs. (9.95) and (9.96), we find that the horizontal field components for this present practical range, $k_2\rho \ll 1 \ll |k_1\rho|$, vary inversely as the cube of the horizontal range ρ, the z component of the electric field varies inversely as the square of the horizontal range, and the z component of the magnetic field is negligibly small, varying inversely as the fourth power of the horizontal range. The field components are all exponentially attenuated as a function of the depth of the source only. These results, Eqs. (9.97) and (9.98), may be compared and checked with the corresponding results for the conducting medium, Eqs. (7.85), by putting $z = 0$ and using the boundary conditions.

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