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Representational Change and Numerical Estimation: Effect of Progressive Alignment on the Breadth of Transfer

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Abstract

The ability to carry out effortless structural alignment is a hallmark of human cognitive processing. We tested whether this mechanism might explain how humans learn to represent the ratio characteristics of the decimal system. Specifically, children were asked to align the familiar ratio structure of object sets (e.g., how 10□ is more similar to 1□ than to 100□) with the ratio structure of the decimal system (e.g., how 10 is more similar to 1 than to 100) in an effort to change how children represent the magnitudes of larger numbers, such as those between 0-1,000, 0-10,000, and 0-100,000. Results indicated that progressive alignment of high and low scales led children to generalize linear representations of numerical magnitude to the highest numbers tested, which otherwise elicited the logarithmic pattern of estimates typical of younger children, infants, and non-human animals.

Keywords: representational change, progressive alignment, numerical magnitudes, estimation, analogy

Analogy and Representational Change

Analogy provides a potentially powerful mechanism for representational change, allowing broad and rapid generalization of novel information across multiple contexts. This general perspective on representational change, drawn from computational models of cognition and cognitive development (Doumas, Hummel, & Sandhofer, 2008; Gentner, 1983; Hummel & Holyoak, 2003), artificial grammar learning in infants (Marcus, Vijiyan, Bandi Rao, & Vishton, 1999), and historical changes in scientific concepts (Gentner et al., 1997; Holyoak & Thagard, 1995), immediately suggests analogy as a candidate mechanism for development of numerical representations. To test whether analogy could serve this developmental function, we examined whether progressive alignment—a means of fostering analogies in young children (Kotovsky & Gentner, 1996)—would lead numeric representations used at low numerical ranges (0-100) to be spontaneously generalized to progressively higher numerical ranges (0-1,000, 0-10,000, and 0-100,000).

Progressive alignment is a procedure that allows young children to make similarity comparisons over concrete, perceptual similarities in order to notice higher-order relational commonalities (Kotovsky & Gentner, 1996). For example, when shown stimulus arrays like oOo and xXx, children might initially describe them as having 'two little ones on either side of a big one'. After children practice making such comparisons, however, children's ability to notice higher-order relational commonalities is facilitated. For example, arrays like oOo and xXx might be described with the relational terms, "baby, daddy, baby". In this way, carrying out a concrete similarity comparison highlights the relational structure common between the stimuli and makes this relational structure more salient. When the relational structure becomes more salient, the structure can be abstracted and transferred to other related problems.

These findings on the effect of progressive alignment on transfer suggests a mechanism for generally improving children's understanding of how numerical magnitudes are related to one another. Like other stimuli, symbolic numbers have both surface and relational similarities that children can compare. For example, the symbols "1" and "7" look more similar than do "1" and "3", but within the decimal system of numerical magnitudes, 1 and 3 are more similar than 1 and 7. In our experiments, we sought to capitalize on the surface similarities of numbers to draw attention to the relational structure of the formal decimal system, and thereby elicit application of this structure for novel numeric ranges.

Development of Numerical Representations

Across a wide range of tasks, children normally improve their expectations about the magnitudes denoted by symbolic numerals. For example, on a number line estimation task, children are presented with a series of lines flanked by a number (e.g., 0 and 1000), a third number above the line (e.g., 230), and no other markings. When asked to estimate the position of this third number, children's estimates of the positions of the numbers would ideally increase linearly with the actual value of the third number, thereby reflecting representation of the ratio characteristics of the formal decimal system. In fact, however, children's estimates do not increase linearly—at least not initially. On 0-1,000 number lines, sixth graders' estimates increase linearly, but second graders' estimates increase logarithmically (Siegler & Opfer, 2003); on 0-100 number lines, second graders' estimates increase linearly (Geary, Hoard, Nugent, & Byrd-Craven, in press; Geary, et al., 2007; Siegler & Booth, 2004; Siegler & Opfer, 2003), whereas kindergartners' estimates increase logarithmically

(Siegler & Booth, 2004); and on 0-10 problems, kindergartners' estimates increase linearly, whereas preschoolers' estimates increase logarithmically (Opfer & Thompson, 2006). Finally, evidence for logarithmic-to-linear shifts is not unique to number lines. Parallel changes have been found in estimation of real objects, money, answers to arithmetic problems, measurements of novel units, and the categorization and comparison of symbolic numbers (Booth & Siegler, 2006; Laski & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, in press).

Theoretically, these initial expectations that numerical magnitudes increase logarithmically are interesting because they are consistent with Fechner's Law, in which the apparent magnitude of a quantity increases logarithmically with actual value. This logarithmic representation of numerical magnitude is apparently widespread across species and age groups and is consistent with the quantitative performance of time-pressured adults, young preschoolers, human infants, and such non-human animals as pigeons, rats, and monkeys (Brannon, 2005; Banks & Hill, 1974; Feigenson, Dehaene, & Spelke, 2004; Moyer & Landauer, 1967; Gallistel & Gelman, 1992; Roberts, 2005; Xu & Spelke, 2000). In these groups, too, the difference between 1 and 10 seems larger (or is more quickly detected) than the difference between 101 and 110, much as these numbers would be spaced on a logarithmic ruler. Thus, it appears that the natural mental number line is logarithmically scaled, whereas the decimal system that children must eventually learn in school is linearly scaled (Dehaene, Dehaene-Lambert, & Cohen, 1998).

How does the logarithmic-to-linear shift in numeric representations take place with increasing age or experience? An important mechanism implicated in previous work on number line estimation (Opfer & Siegler, 2007; Opfer & Thompson, 2008) is children's use of analogy to structure their generalization of "log discrepant" information. According to this view, children normally encounter information that does not match their logarithmic representation of numerical magnitudes (e.g., hearing 150 referring to a relatively small part of 1,000 items). If children already apply linear representations in some numerical contexts (e.g., for small numeric ranges), such experiences of log discrepancy may lead them to draw analogies between the two contexts and to extend the linear representation to numerical ranges where they previously used logarithmic representations. For example, if a second grader is shown that her estimate of the position of 150 on a 0-1,000 number line is too high, and also is shown the correct position of 150 within that range, she may draw the analogy "150 is to the 0-1,000 range as 15 is to the 0-100 range." This analogy may lead her to rely on a linear representation for the 0-1,000 range on subsequent estimation problems. Consistent with this account of an analogy being drawn at the level of the entire representation (as opposed to being restricted to numbers near the point of feedback), several studies (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, in press) have found that log discrepant information yields broad and rapid representational change often after only a single trial of feedback.

The Present Study

The present study addressed two novel questions about such broad and rapid changes: (1) How widely can children generalize linear representations of numerical magnitudes? and (2) Is analogy a mechanism of representational change that is sufficient to produce generalization to larger numerical contexts? These two questions are important for identifying the mechanism of logarithmic-to-linear shifts, and previous work has not addressed these two questions directly.

To directly test whether analogy could serve as a mechanism for representational change in the numeric domain, we brought number lines for high scales (0-1,000, 0-10,000, and 0-100,000) into progressive alignment with scales that children already represent linearly (0-100). The alignment procedure was "progressive" in two senses. First, all children's comparisons, regardless of the experimental condition to which they were assigned, were supported by increasing the perceptual similarity of low and high scales. Specifically, by having the color of units appearing on 0-100 scales (pears, cherries, and carrots) matching the color of zeros denoting order of magnitude on 0-1,000, 0-10,000, and 0-100,000 scales, we used color information to highlight the similarity between 0-100 and 0-1,000 scales, 0-100 and 0-10,000 scales, and 0-100 and 0-100,000 scales (see Figure 1). Second, in the alignment condition, children's comparisons of low to high scales were supported by allowing children to directly compare 0-100 problems with 0-1,000, 0-10,000, and 0-100,000 problems; in contrast, children in the no alignment condition were not given this opportunity. Thus, all participants received the benefit of overlapping perceptual similarity across the low and high scales, but the underlying structural relation between the low and high scales was made more salient in the alignment condition than the no alignment condition by allowing children to compare scales directly. Finally, to test for representational change, we examined numerical estimates on a 0-1,000 post-test, where children were given no feedback, no perceptual support, and no opportunity to compare problems to lower scales.

Method

Participants

Participants were 30 second graders (*mean age* = 7.75, *SD* = .46; 13 girls, 17 boys) enrolled in public elementary schools in the midwestern United States.

Design and Procedure

Children estimated the placement of numbers on number lines over three phases of the experiment: training, generalization and posttest (Figure 1). The number line task presented children with a line flanked by two hatch marks, the left hatch mark labeled "0," and the right hatch mark labeled with either "100," "1000," "10000," or "100000." On each trial, children were asked to estimate the position of a number (one per number line) by making a hatch mark through the line. The numbers to be estimated (2, 5, 8, 11, 15, 25, 49, 61, 73, 94, and multiples thereof) were chosen to reduce the influence of specific knowledge (e.g., that 50 is half of 100) and to over-sample at the low end of the range to maximize the discriminability of the logarithmic and linear functions.

To test the effect of progressive alignment on transfer from training to generalization and posttest problems, participants were randomly assigned to one of two experimental conditions in this between-subjects design: no alignment (N = 14) and alignment (N = 16) (cf. Figure 1a and 1b). In the no alignment condition, participants received training and generalization problems one at a time; thus, participants could not directly compare new problems to old problems. In the alignment condition, participants received identical training problems, but generalization problems were presented alongside previously solved training problems, thereby allowing children in the alignment group to compare generalization problems to training problems (i.e., to compare units to orders of magnitude). Thus, in both conditions, children were told that generalization problems were "just like" the training problems, but only the alignment condition allowed perceptual comparison of training and generalization problems.

In the training phase (Figure 1, left column), participants were given 40 0-100 number line problems, and they received feedback on their estimates (for feedback procedure, see Opfer & Siegler, 2007). All training problems specified units (i.e., pears, cherries, and carrots), and after completing these problems, children were shown that their estimates did not differ much over different units. In the generalization phase (Figure 1, middle column), we highlighted similarity of generalization and training problems (color of units matched color of zeros denoting order of magnitude) by asking participants "to try some more problems just like the ones you just finished." Then, participants were asked to make estimates for 40 new number lines (0-100, 0-1,000, 0-10,000) without feedback. In the posttest phase (Figure 1, right column), participants were asked to estimate the position of 10 numbers on 0-1,000 number lines without receiving feedback and without perceptual support.

Results

The performance of children in the alignment and no alignment groups was compared over all three experimental phases: training, generalization, and posttest.

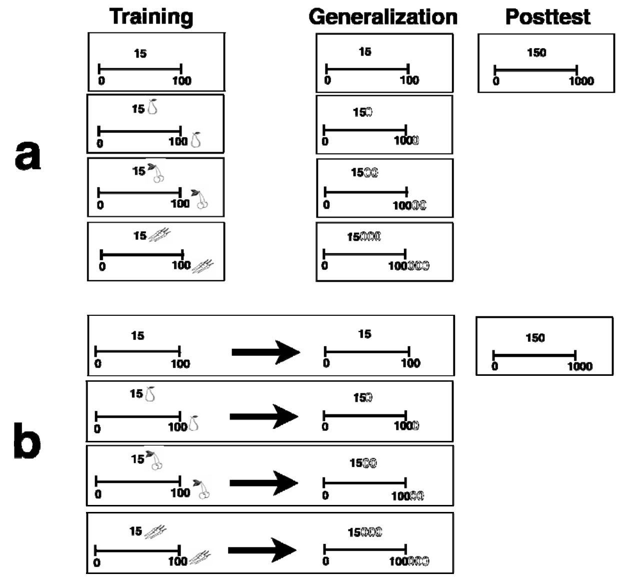

Figure 1: Illustration of stimuli: a) No Alignment condition, and b) Alignment condition. Pears = green; cherries = red; carrots = orange.

Training

Based on previous studies (Booth & Siegler, 2006; Geary, et al., 2007; Laski & Siegler, 2007; Siegler & Booth, 2004; Siegler & Opfer, 2003), we had expected that second graders' estimates in the 0-100 range would be highly accurate, due to their use of a linear representation for familiar numbers, regardless of either the specific units tested or experimental condition. To test this hypothesis, we first examined accuracy of numerical estimates across the training phase.

Accuracy was measured by first converting the magnitude estimate for each number (the child's hatch mark) to a numeric value (the linear distance from the "0" mark to the child's hatch mark), then dividing the result by the total length of the line. The magnitude of each child's error was calculated by taking the mean absolute difference between each of the child's estimated values and the actual values.

As expected, accuracy did not vary over units or experimental condition (Fs < 1), with participants' accuracy hovering at ceiling levels for the no alignment (M_{Blank} = 91%, M_{Pear} = 93%, M_{Cherry} = 93%, M_{Carrot} = 92%) and alignment condition (M_{Blank} = 88%, M_{Pear} = 95%, M_{Cherry} = 94%, M_{Carrot} = 94%). To examine whether this high accuracy was the result of children relying on linear representations of numerical magnitude, estimates were next regressed against actual values using the logarithmic and linear regression functions. Again, participants in the alignment condition produced more linear than logarithmic series of estimates (Blank R^2 = .87; Pear R^2 = .998; Cherry R^2 = .996; Carrot R^2 = .996) as did participants in the no alignment condition (Blank R^2 = .99; Pear R^2 = .993;

Cherry R^2 = .996; Carrot R^2 = .995). Thus, no differences among groups existed prior to the generalization phase.

Generalization

We next examined the effect of alignment on generalization. To do so, we first compared accuracy of estimates generated by the alignment group to the no alignment group across the four orders of magnitude (see Figure 2) by conducting a 2 (condition: alignment, no alignment) x 4 (scale: 0-100, 0-1,000, 0-10,000, 0-100,000) repeated measures ANOVA on accuracy scores. There was a main effect of scale, $F(3, 26) = 21.06$, $p < .0001$, and condition, $F(1, 28) = 13.65$, $p < .0001$, and these main effects were qualified by a significant condition x scale interaction, $F(3, 26) = 5.15$, $p < .01$.

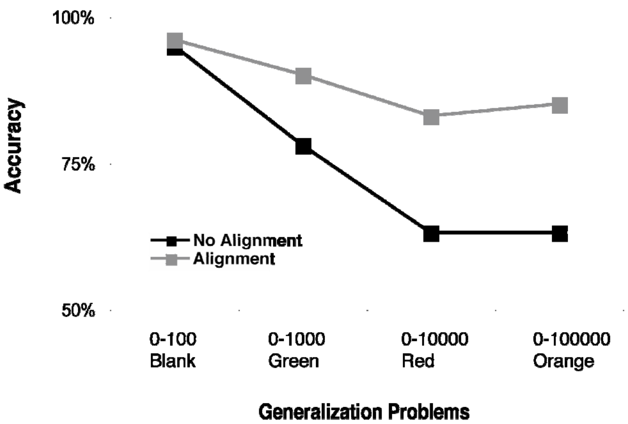

Figure 2: Accuracy across generalization problems.

Post-hoc t-tests indicated that the effect of alignment was tied to the magnitude of the scale, with larger effects for larger numbers. Thus, there was no effect of alignment for 0-100 number line problems, $t(28) = .83$, $p > .05$, *ns*, presumably because participants already possessed a linear representation on these numbers. In contrast, alignment significantly improved accuracy for 0-1,000 problems (no alignment, M = 78%, alignment; M = 90%), $t(28) = 3.47$, $p < .01$; for 0-10,000 problems (no alignment, M = 63%; alignment, M = 83%), $t(28) = 2.61$, $p = .01$; and 0-100,000 problems (alignment, M = 63%; no alignment, M = 85%), $t(28) = 4.06$, $p < .001$.

To test whether these improvements in accuracy came from children substituting a linear representation of number for a logarithmic one, we compared the fit of the best fitting linear and logarithmic functions to the median numerical estimates across both experimental conditions in the generalization phase (see Figure 3).

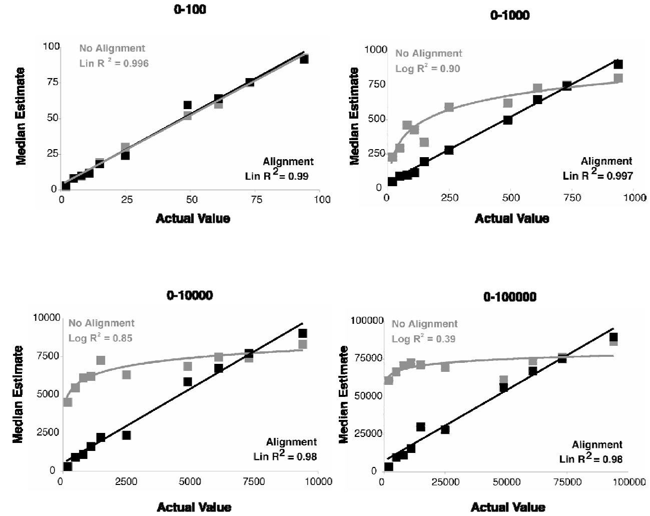

Figure 3: Linear and logarithmic model fits on generalization problems.

As previously, the linear function was the best-fitting function for both experimental conditions on the 0-100 generalization problems (alignment, R^2 = .99; no alignment, R^2 = .996). The experimental groups differed, however, in whether the best-fitting function for their median estimates across the other generalization problems (0-1,000, 0-10,000, and 0-100,000) was the logarithmic or the linear function. For the no alignment group, the best fitting function was the logarithmic one across the 0-1,000 and 0-10,000 generalization problems (log R^2 = .90 and .85, respectively). On the 0-100,000 generalization problems, both the linear (R^2 = .46) and logarithmic function (R^2 = .39) provided a poor fit of the children's estimates in the no alignment condition. Across the 0-1,000, 0-10,000, 0-100,000 generalization problems, the alignment group generated estimated best fit by the linear function (lin R^2 = .997, .98, and .97, respectively). Thus, results indicated that the alignment group successfully generalized the learned linear representation to larger orders of magnitude, whereas the no alignment group failed to generalize their learning.

In summary, these findings indicate that alignment caused children to generalize linear representations broadly, across 0-1,000, 0-10,000, and 0-100,000 problems, whereas simply providing children feedback in the 0-100 range and highlighting the similarity between units and orders of magnitude was insufficient for children to "scale up" their learned linear representation to successively larger numerical magnitudes.

Posttest

To examine whether alignment caused the representational change implied by performance on the generalization problems, we tested numerical estimation on a post-test where we eliminated perceptual support (i.e., no fruit/vegetable icons or colored zeros). If children grasped the analogy implied by the progressive alignment, we reasoned, their estimates on the number line should continue to increase linearly, even when perceptual support was lacking.

As illustrated in Figure 4, participants' estimates in the alignment group were indeed better fit by the linear function (R^2 = .95) than by the logarithmic function (R^2 = .76), whereas the estimates of the no alignment group were better fit by the logarithmic function (R^2 = .91) than by the linear (R^2 = .66). Thus, not only were the alignment group's estimates best fit by the linear function across all generalization problems, this trend continued at posttest, indicating that progressive alignment provided the most supportive condition for participants to "scale up" the linear representation to larger numerical contexts.

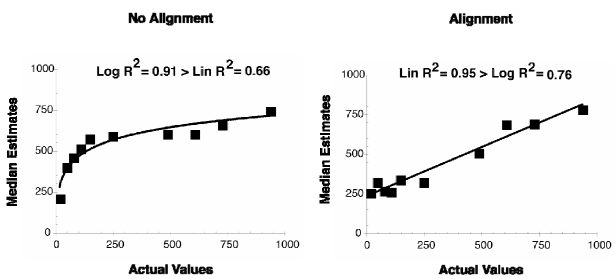

Figure 4: Linear and logarithmic model fits on posttest problems.

General Discussion

The ability to carry out effortless structural alignment is a hallmark of human cognitive processing (Gentner & Markman, 1997). In this experiment, children were asked to align the familiar ratio structure of units (e.g., how 10□ is more similar to 1□ than to 100□) with the ratio structure of the decimal system (e.g., how 10 is more similar to 1 than to 100) in an effort to change how children represent the magnitudes of larger numbers, such as those between 0-1,000, 0-10,000, and 0-100,000. Results supported the assumption that progressive alignment was necessary to generalize a linear representation of 0-100 to higher numbers. Specifically, although we found that feedback on 0-100 problems caused all children to generate linear estimates on 0-100 "training problems," only children in the alignment condition generalized their linear representation to 0-1,000, 0-10,000, and 0-100,000 "generalization problems," and only children in the alignment group continued to generate linear estimates on problems without perceptual support, alignment, and feedback ("post-test").

These results are important for three reasons. First, these results provide the first behavioral evidence that a known cognitive mechanism—analogy—allows a pre-existing numerical representation to be extended to represent the magnitudes of a potentially infinite range of new numbers. Second, results are important developmentally because they indicate both the conditions under which broad and abrupt cognitive changes take place, as well as the conditions under which cognitive development is slow and gradual. Currently, these two types of cognitive changes figure strongly in competing theories of cognitive development, but the theory of representational change depicted here points to a way of reconciling them within an architecture that is capable of relational reasoning (e.g., Doumas, Hummel, & Sandhofer, 2008; Gentner, 1983; Hummel & Holyoak, 2003). Finally, by showing that progressive alignment leads young children to adopt a linear representation of number and abandon a logarithmic one, it seems that the methodology used in this experiment could be a simple classroom intervention. Given previous evidence of causal and correlational links between numerical estimation and other mathematical skills (Booth & Siegler, 2006, in press), we believe this intervention is likely to be effective too.

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