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Multiplicative Lattice Versions of Some Results From Noetherian Commutative Rings

Author
Majcherek, Daniel Joseph

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Multiplicative Lattice Versions of Some Results From Noetherian Commutative Rings

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by

Daniel Joseph Majcherek

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Dissertation Committee:

Professor David E. Rush, Chairperson
Professor Jacob Greenstein
Professor Mei-Chu Chang

The Dissertation of Daniel Joseph Majcherek is approved:

Committee Chairperson

University of California, Riverside

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"A little sleep, a little slumber, a little folding of the hands to rest, and poverty will come upon you like a robber, and want like an armed man."

Proverbs 6:10-11, ESV

ABSTRACT OF THE DISSERTATION

Multiplicative Lattice Versions of Some Results
From Noetherian Commutative Rings

by

Daniel Joseph Majcherek

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2014
Professor David E. Rush, Chairperson

The goal of this paper is to continue building lattice theory by generalizing known results from commutative rings. Our focus will be results concerning strong Mori lattices as Mori lattices have been rather developed already in [7], [22], and others sources.

Our first main objective will be to obtain more results concerning quotient field lattices, which have been significantly developed in [6], [7], and [22]. Most notably, the quotient field lattice version of the Nagata Theorem will be proven, following the approach of Chang Hwan Park and Mi Hee Park in [23]. This result had not yet been generalized to quotient field lattices. Also, some results concerning composition series will receive lattice versions, generalizing results from [23] as well as [15].

Some concepts from commutative ring theory that have not yet been applied to lattices will be introduced, such as certain star operations, particularly the w-operation. This will allow us to develop strong Mori lattices and begin to

characterize them, working toward achieving lattice versions of results from [27] and [28]. We will also introduce a lattice version of w-invertibility and obtain lattice versions of basic results. This will give a brief characterization of strong Mori lattices and lay the groundwork for further, more detailed characterizations as discussed in the final section.

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Preface

Consider the ideals of a commutative ring R with identity. It is well known that this set of ideals forms what is known as a complete lattice monoid. Many familiar algebraic structures are also complete lattice monoids, such as certain sets of submodules, graded submodules, and graded ideals.

Although the development of lattice theory as an abstract version of the ideal theory of commutative rings began with the 1924 paper [16] of W. Krull, this development received little notice until M. Ward and D.P. Dilworth gave lattice analogues of several of Noether's theorems on primary decomposition in Noetherian commutative rings in [29], [30], and [31]. However, the definitions given at that time were not adequate to prove lattice versions of more advanced results in commutative ring theory, such as Krull's principal ideal theorem. The main difficulty was the need to find a more suitable lattice analogue of a principal ideal. In [8], Dilworth gave such a definition and defined a **Noether lattice** $\mathcal{L}$ to be a complete modular lattice satisfying ACC with a commutative, associative multiplication that distributes over arbitrary joins, with the largest element $I \in \mathcal{L}$ as the multiplicative identity and such that each element of $\mathcal{L}$ is a join of principal elements. By, proving important results such as the principal element theorem (the lattice analogue of Krull's principal theorem), he paved the way for further development in complete lattice monoids (Among the early contributors to the development of multiplicative lattices were the authors of the UCR desertations [6, 7, 11, 14, 18, 19, 32] written under the direction of Louis J. Rattliff Jr.).

In recent years, there has been a surge of interest in lattice theory and new lattice structures have been defined and developed, including lattices of fractions of a multiplicative lattice. In 2005, Rush, Okon, and Wallace further developed what are known as quotient field lattices, obtaining the Krull Akizuki Theorem for quotient field lattices as well as the Mori-Nagata Theorem. This development had begun with W.P. Brithinee in [7] several decades earlier around 1971 as well as D.D. Anderson, a student of Irving Kaplansky, in his thesis in 1974 (see [1]).

My goal is to continue this extension of results in commutative algebra to lattice theory. Of course there is a rapidly expanding supply of theorems in commutative algebra that might seem like good candidates to extend from lattices of ideals of commutative rings to multiplicative lattices. So the question arises as to how to choose such theorems in commutative algebra. The definition of multiplicative lattices is broad enough to contain four different classes of examples: the set of ideals of a commutative ring, the set of homogeneous ideals of a graded ring, the set of ideals of a commutative multiplicative monoid with a zero element and certain multiplicative lattices which arise from having a star operation as we will show in Chapter 4. There is a growing trend of researchers trying to find graded ring analogues of known theorems on commutative rings. This is generally not a routine exercise because many such theorems do not carry over. We focus our attention on the recent papers [23], [24] which extend several important results from rings to graded rings, and on [27], [28] for lattices which arise from star operations.

In the introduction, we define core concepts, the structures of interest of this paper, and give some simple examples. We will also give the definition of a quotient field lattice as well as some examples, which will help us prove some deeper results.

In the second section we will start generalizing results from [23]. For this we will state results from [22] that are relevant. This section mainly concerns quotient field lattices. We will introduce lattice versions of familiar concepts such as valuation rings, Krull rings, and integral closure, to name a few. The reader is referred to [6] and [7] for additional results concerning many of these topics.

In section three we turn our attention to structures known as lattice modules and use them to extend Nagata's Theorem on two-dimensional Noetherian domains to quotient field lattices. Many theorems from [24] are also generalized in this section. We restate the Principal Element Theorem of Dilworth and prove a general form of the Eakin-Nagata Theorem. These results rely on the version of the Krull Akizuki Theorem proven in [22] and the author strongly recommends referring to this source.

Section four is the bulk of our work. We will investigate strong Mori lattices and special types of lattice modules in an attempt to generalize results concerning strong Mori domains and strong Mori modules to lattices and lattice modules. Star operations and w-elements are defined and developed for lattices. This section spans many works, including [24], [27], [28], and others.

The final section investigates basic results considering w-invertibility in preparation for future endeavors. It would be valuable to obtain lattice versions of both

[27, Theorem 5.4] and [28, Theorem 2.8], which would be a significant characterization of strong Mori lattices. However, to complete these results, we need to obtain lattice versions of results from [21].

Chapter 1

Preliminaries

We begin by defining a cl-monoid, a basic form of our structures of interest, and giving classic examples of cl-monoids, relating them to familiar structures from commutative algebra. After defining basic terms, we will introduce quotient field lattices, along with examples. The section finishes with the introduction of localization in quotient field lattices.

Definition 1.0.1. A **complete lattice monoid** (or simply a **cl-monoid**) is a complete lattice-ordered (multiplicative) monoid $\mathcal{L}$ that satisfies the following:

1. If 0 is the smallest element of $\mathcal{L}$, then $0A = 0$ for every $A \in \mathcal{L}$.

2. For any family $\{B_\lambda \in \mathcal{L} \mid \lambda \in \Lambda\}$ and $A \in \mathcal{L}$, $A(\bigvee B_\lambda) = (\bigvee AB_\lambda)$ (the multiplication distributes over arbitrary joins).

Denote by R the multiplicative identity of the cl-monoid $\mathcal{L}$. A cl-monoid $\mathcal{L}$ is said to be **integral** if R is the largest element in $\mathcal{L}$. Note that $\{A \in \mathcal{L} \mid A \leq R\}$ is a sub-cl-monoid of $\mathcal{L}$, which we shall denote $\mathcal{I}$.

Definition 1.0.2. *Let $\mathcal{L}$ be a cl-monoid. An element $A \in \mathcal{L}$ is said to be a* ***ring element*** *if $R \leq A$ and $AA \leq A$.*

Example 1.0.3. [22, Example 2.1]

The following are examples of cl-monoids the reader should keep in mind.

1. *Let R be a subring of a commutative ring $R^{'}$. The set of R-submodules of $R^{'}$ is a cl-monoid.*

2. *Let M be a cancellative torsion-free abelian monoid and R be a graded subring of the M-graded commutative ring $R^{'}$. The set of graded R-submodules of $R^{'}$ is a cl-monoid.*

3. *Let M be a multiplicative cancellative monoid with quotient group G. With the symbol 0, define a multiplication on $G \cup \{0\} = G_0$ as follows: $g0 = 0g = 0$ for every $g \in G$. Note that $M \cup \{0\} = M_0$ is a subsemigroup of G_0. Now, if we let $\mathcal{L}(M_0, G_0) = \{A \subseteq G_0 \mid M_0 A \subseteq A\}$ and order $\mathcal{L}(M_0, G_0)$ by inclusion with a multiplication defined by $AB = \{ab \mid a \in A, b \in B\}$, then $\mathcal{L}(M_0, G_0)$ is a cl-monoid.*

These examples demonstrate that many familiar structures in commutative ring theory are cl-monoids. More sources of examples will be developed in Chapter 4.

Some familiar concepts and definitions also appear in this context, which we now introduce. The next definitions will be used very frequently.

Definition 1.0.4. Let $\mathcal{L}$ be a cl-monoid. **Residuation** is defined for A, $B \in \mathcal{L}$

as $(A : B) = \bigvee\{C \in \mathcal{L} \mid CB \leq A\}$. When needed, we will add a subscript such as $(A :_{\mathcal{I}} B)$ to represent $\bigvee\{C \in \mathcal{I} \mid CB \leq A\}$.

Definition 1.0.5. Let $\mathcal{L}$ be a cl-monoid. An element $M \in \mathcal{L}$ is **meet principal** (respectively, **join principal**) if $(B \wedge (A : M))M \geq A \wedge MB$ (respectively, $(AM \vee B) : M \leq A \vee (B : M)$) for all $A, B \in \mathcal{L}$ (note the reverse inequalities always hold). If M is both meet principal and join principal, it is a **principal** element of $\mathcal{L}$ and if every element of $\mathcal{L}$ is a join of principal elements, $\mathcal{L}$ is **principally generated**.

The term "principal" is indeed a reference to the concept of a principal ideal of a ring, as a principal ideal in a ring will satisfy the conditions of the preceding definition when viewed in the context of a cl-monoid. The following is a weaker version of principality.

Definition 1.0.6. Let $\mathcal{L}$ be a cl-monoid. An element $M \in \mathcal{L}$ is **weak meet principal** (respectively, **weak join principal**) if for all $A \in \mathcal{L}$, $A \wedge M = M(A : M)$ (respectively, $A \vee (0 : M) = (AM : M)$). If M is both weak meet principal and weak join principal, it is a **weak principal** element of $\mathcal{L}$.

We define a few more basic terms that we will use frequently.

Definition 1.0.7. Let $\mathcal{L}$ be a cl-monoid. An element $A \in \mathcal{L}$ is **invertible** if $AB = R$ for some $B \in \mathcal{L}$. If this is the case, $B = (R : A)$, denoted $B = A^{-1}$.

Definition 1.0.8. Let $\mathcal{L}$ be a cl-monoid. An element $M \in \mathcal{L}$ is **compact** if $A \leq \bigvee_{i \in I} A_i$, $A_i \in \mathcal{L}$ implies that $A \leq \bigvee_{i \in I'} A_i$, for some finite subset I' of I. If every element of $\mathcal{L}$ is a join of compact elements, $\mathcal{L}$ is **compactly generated**.

Definition 1.0.9. Let $\mathcal{L}$ be a cl-monoid and let A, B, and $C \in \mathcal{L}$. We say that $\mathcal{L}$ is **modular** if $A \leq B$ implies that $B \wedge (A \vee C) \leq A \vee (B \wedge C)$.

The above examples of a cl-monoid demonstrate that any results we prove for cl-monoids can be applied to the cases given in Example 1.6 and more to obtain results without having to prove them in each case. Also, the proofs are often more eloquent when dealing with cl-monoids.

Our goal is to generalize results from commutative rings to cl-monoids. Since many results we will be dealing with concern quotient fields, localizations, and many other concepts already familiar to the reader, we will need cl-monoid versions corresponding to these ideas. We begin by defining a quotient field lattice.

Definition 1.0.10. Let $\mathcal{L}$ be a cl-monoid. $\mathcal{L}$ is a **quotient field lattice** (abbreviated q.f. lattice) if the following hold:

1. $AK = K$ for every nonzero $A \in \mathcal{L}$, where $K = \vee\mathcal{L}$ is the largest element of $\mathcal{L}$.

2. $\mathcal{L}$ is principally generated.

3. There exists a compact, invertible element in $\mathcal{L}$.

4. For every $A \in \mathcal{L} \setminus \{0\}$, $A \wedge R \neq 0$.

To see some examples, we only need make some slight modifications to the examples of cl-monoids.

Example 1.0.11. [22, Example 2.4]

1. *Let R be an integral domain and $R^{'} = K$ the quotient field of R. The set of R-submodules of $R^{'} = K$ is a q. f. lattice.*

2. *With M as in the previous example, let $R = \oplus_{m \in M} R_m$ be an M-graded integral domain and let $R^{'} = R_S$ where S is the set of nonzero homogeneous elements of R. The set of graded R-submodules of $R^{'} = R_S$ is a q. f. lattice.*

3. *Number* 3 *from the previous example is also a q. f. lattice.*

In [7, Proposition 1.21] it was shown that principality, invertibility, and meet principality of an element are equivalent in a q. f. lattice (invertibility implies principality in any cl-monoid by [7, Corollary 1.20]).

Definition 1.0.12. [22, Definition 3] Let $\mathcal{L}$ be a q. f. lattice and let $\mathcal{S} \subseteq \mathcal{I}$. If each $s \in \mathcal{S}$ is $\mathcal{L}$-principal, $R \in \mathcal{S}$ and $ss^{'} \in \mathcal{S}$ for each $s, s^{'} \in \mathcal{S}$, then $\mathcal{S}$ is said to be a **multiplicative subset for** R. If P is a prime element of $\mathcal{I}$, denote $\mathcal{S}(P) = \{s \in \mathcal{I} \mid s \text{ is } \mathcal{L}\text{-principal and } s \not\leq P\}$.

Definition 1.0.13. [22, Definition 4] Let $\mathcal{L}$ be a cl-monoid and let $\mathcal{S}$ be a multiplicative subset for R. If $A \in \mathcal{L}$, the **localization of** A **at** $\mathcal{S}$ is $A_{\mathcal{S}} = \vee\{A : s \mid s \in \mathcal{S}\}$. If $\mathcal{S}$ is $\mathcal{S}(P)$ for a prime element P of $\mathcal{I}$, we denote $A_{\mathcal{S}(P)}$ by A_P.

Localization of lattice elements satisfies properties similar to those of localization in rings and modules. See [22].

Chapter 2

Some Results on Quotient Field Lattices

In this chapter we will recall important results already proven for quotient field lattices, such as a version of the Krull Akizuki Theorem and the Mori Nagata Theorem. First, we will state results from [7] regarding integral and almost integral elements in a quotient field lattice and Noetherian lattices, which are simply lattices that satisfy the ascending chain condition on elements of the lattice.

2.1 Basic results and integral elements

We begin this section by stating some definitions and results from [7] and [22], including b-dependent and f-dependent elements, the lattice counterparts of being integral or almost integral over a ring. We will need them to obtain further results generalizing [23]. First, we prove a few basic lemmas we will use later as well as state essential definitions.

Definition 2.1.1. Let A be an element of $\mathcal{L}$. We define the **radical of** A to be $\sqrt{A} = \bigvee\{X \in \mathcal{L} \mid X \text{ is principal and } X^n \leq A \text{ for some positive integer } n\}$. If $A = \sqrt{A}$, A is a **radical element of** $\mathcal{L}$.

Definition 2.1.2. Let $\mathcal{L}$ be a cl-monoid. An element $A \in \mathcal{I}$ is **maximal** in $\mathcal{I}$ if for any $B \in \mathcal{I}$ such that $A \leq B$, it follows that either $B = R$ or $B = A$.

The following is a generalization of [15, Theorem 87] and the proof given there has been adapted to obtain the lattice version.

Lemma 2.1.3. *Let $\mathcal{L}$ be a cl-monoid such that $\mathcal{I}$ satisfies the ACC on radical elements. Then any radical element of $\mathcal{I}$ is a finite meet of prime elements of $\mathcal{I}$.*

Proof. Assume the assertion fails. Let $\mathcal{A}$ be the set of radical elements of $\mathcal{I}$ that are not a finite meet of prime elements of $\mathcal{I}$. Since $\mathcal{I}$ satisfies ACC on radical elements, $\mathcal{A}$ contains a maximal element, call it I, which cannot be a prime element. Let $A, B \in \mathcal{I}$ be principal such that $AB \leq I$ and $A \nleq I$, $B \nleq I$. Define $J = \sqrt{(I \vee A)}$ and $K = \sqrt{(I \vee B)}$. Since I is maximal, J and K can be written as a finite meet of prime elements of $\mathcal{I}$.

Now, let $X \leq J \wedge K$ be principal. Then $X^m \leq (I \vee A)$ and $X^n \leq (I \vee B)$ for positive integers m and n and $X^{m+n} \leq (I \vee A)(I \vee B) \leq I$ so that $X \in \sqrt{I} = I$. We have just shown that $J \wedge K \leq I$ and it is clear that $I \leq J \wedge K$, so $I = J \wedge K$, contradicting our hypothesis. $\square$

Now we can obtain the following corollary which generalizes both [15, Theorem 88] and [23, Corollary 1.2] from commutative rings to cl-monoids.

Corollary 2.1.4. *Let $\mathcal{L}$ be a cl-monoid such that $\mathcal{I}$ satisfies the ACC on radical elements. For any $I \in \mathcal{I}$, there are only a finite number of prime ideals minimal over I.*

Proof. By [7, Lemma 3.5], if $A \in \mathcal{L}$ such that $A \leq P$ for a prime element $P \in \mathcal{L}$, then there is a minimal prime element $Q \in \mathcal{L}$ such that $A \leq Q \leq P$. The previous lemma can then be restricted to such minimal prime elements. Since a prime element contains A iff it contains $\mathrm{Rad}(A)$, the result follows. □

The following lemma generalizes both [15, Theorem 53] and [23, Lemma 2.1] from integral domains to q. f. lattices.

Lemma 2.1.5. *Let $\mathcal{L}$ be a q. f. lattice with identity element R and let $\mathcal{S}$ be a multiplicative subset for R. Then $R = \bigwedge\{R_P \mid P$ is a maximal prime of a principle element in $\mathcal{I}\}$.*

Proof. It is known that R_P is a ring element of $\mathcal{L}$, hence we have $R \leq R_P$ for each prime element P of $\mathcal{I}$. So $R \leq \bigwedge\{R_P \mid P$ is a maximal prime of a principal element in $\mathcal{I}\}$.

To show that $\bigwedge\{R_P \mid P$ is a maximal prime of a principal element in $\mathcal{I}\} \leq R$, let $U \in \mathcal{L}$ be principal with $U \leq \bigwedge\{R_P \mid P$ is a maximal prime of a principal element in $\mathcal{I}\}$. Then, $U = (A :_{\mathcal{I}} B) = AB^{-1}$ for some $\mathcal{L}$-principal elements A,$B \in \mathcal{I}$, i.e $UB = A \leq R$ [22, Lemma 2.5(7)].

Let $I = (B :_{\mathcal{I}} A)$. If $I = R$, then $A \leq B$ in $\mathcal{I}$. So $BU = A \leq B$, and thus $U \leq (B :_{\mathcal{I}} B) = R$. If $I < R$, then $I \leq P$ for some maximal element $P \in \mathcal{I}$. Then $U \leq R_P$ implies that $SU \leq R$ some $\mathcal{L}$-principal $S \in \mathcal{I}$ with $S \not\leq P$.

So $(SU)B \leq B$ implies $SA \leq B$. So $S \leq (B :_{\mathcal{I}} A) \leq P$, a contradiction, meaning the proof is complete. $\square$

In order to proceed, we require the definitions of b-dependent and f-dependent elements, as well as some basic facts and characterizations.

Definition 2.1.6. Let $\mathcal{L}$ be a q. f. lattice. An element $A \in \mathcal{L}$ is **b-dependent on** R if and only if there exists a finite join B of $\mathcal{L}$-principal elements and a positive integer n such that $(R \vee A)^n \leq B$ and $AB \leq B$.

Definition 2.1.7. Let $\mathcal{L}$ be a q. f. lattice. The element of $\mathcal{L}$, $R_b = \vee\{B_i \mid B_i$ is b-dependent on R is called the **b-closure of** R **in** $\mathcal{L}$. Brithinee [6, Definition 3.38] defined an element $A \in \mathcal{L}$ to be **integral over** R if it is a join of elements in $\mathcal{L}$ that are b-dependent on R. Hence R_b is also called the **integral closure of** R **in** $\mathcal{L}$. $\mathcal{I}$ is **integrally closed** if $R = R_b$.

Definition 2.1.8. Let $\mathcal{L}$ be a q. f. lattice. An element $A \in \mathcal{L}$ is **fractionary** if there is a nonzero $D \in \mathcal{I}$ such that $DA \in \mathcal{I}$.

We have seen the q. f. lattice version of integral closure. The following theorem introduces the q. f. lattice version of complete integral closure.

Theorem 2.1.9. [7, Lemma 4.6] *The following conditions are equivalent:*

1. $R[A]$ *is fractionary, where* $R[A] = R \vee A \vee A^2 \vee A^3 \cdots$.

2. *There exists a non-zero* $D \in \mathcal{I}$ *such that* $DA \leq D$.

3. *There exists a compact element* $F \in L$ *such that* $R[A] \leq F$.

Definition 2.1.10. Let $\mathcal{L}$ be a q. f. lattice. An element $A \in \mathcal{L}$ is **f-dependent on** R if it satisfies one of the above equivalent conditions.

Definition 2.1.11. Let $\mathcal{L}$ be a q. f. lattice. The element of $\mathcal{L}$, $R_f = \vee\{B_i \mid B_i$ is f-dependent on $R\}$ is called the **complete integral closure of** R **in** $\mathcal{L}$. We say that $\mathcal{I}$ is **completely integrally closed** if $R = R_f$.

The following is an extremely useful characterization of b-dependent elements.

Proposition 2.1.12. [6, Proposition 3.26] *Let $\mathcal{L}$ be a q. f. lattice with an element $A \in \mathcal{L}$. Then A is b-dependent on R if and only if there exists a compact element $B \in \mathcal{L}$ such that $B \geq R$ and $AB \leq B$. Also, if A is b-dependent on R, then A is f-dependent on R.*

Proof. ($\Rightarrow$) This is clear from the definition of b-dependence. For the converse, assume there is a compact element $B \in \mathcal{L}$ with $B \geq R$ and $AB \leq B$. Then $A \leq AB \leq B$ and $A^k \leq B$ for any positive integer k by mathematical induction, which implies that $R[A] \leq B$. Of course, this means that $(R \vee A)^n \leq B$ for any positive integer n and $AB \leq B$, hence A is b-dependent on R. The inequality $R[A] \leq B$ implies that A is f-dependent on R. $\square$

In [23, Lemma 2.2], it is shown that if R is a graded domain and S is the set of nonzero homogeneous elements of R, then the complete integral closure R^* of R in the quotient field of R is a subring of R_S. Since the set of graded submodules of R_S is a q. f. lattice $\mathcal{L}$ by [22, Example 2.4], this follows from [7, Lemma 4.8] which states that R_f is a ring element of $\mathcal{L}$.

Note that we have just shown that $R_b \leq R_f$ in the last proposition. In the case that $\mathcal{L}$ is as in Example 1.0.11, the following gives the counterpart for *b*-dependence. We include the proof for the convenience of the reader.

Lemma 2.1.13. [6, Proposition 3.33] *R_b is a ring element.*

Proof. Since R satisfies the definition of b-dependence (take $B = R$ in the definition), we have $R \leq R_b$ and we need only to show that R_b is subidempotent. Consider, then, arbitrary elements A and B that are b-dependent on R. Using our characterization above, there exist C and D compact with $C \geq R$, $AC \leq C$ and $D \geq R$, $BD \leq D$. Consider the element $E = CD$. We have $R = R^2 \leq CD = E$, showing that $E \geq R$ and since products of compact elements are compact in q. f. lattices, E is compact as well. Since $ABE = ABCD = ACBD \leq CD = E$, we have $ABE \leq E$ so that AB is b-dependent on R. We have just shown an element in R_b^2 is in R_b, so $R_b^2 \leq R_b$ and the proof is complete. $\square$

The next corollary, proven in [6, 3.36], follows easily. It generalizes [23, Lemma 2.3] to q. f. lattices.

Corollary 2.1.14. *If $\mathcal{I}$ satisfies ACC, then $R_b = R_f$.*

Proof. It remains to show that $R_f \leq R_b$. Assume that $\mathcal{I}$ satisfies ACC. Then, each fractionary element of $\mathcal{L}$ is compact. Hence for an element $A \in \mathcal{L}$ with A f-dependent on R we have $R[A]$ is fractionary, and hence compact. Since $R[A] \geq R$, $AR[A] \leq R[A]$ and $R[A]$ is compact, it follows that A is b-dependent on R. $\square$

2.2 The global transform and Krull lattices

Now we will define more advanced lattice structures, such as a valuation lattice and Krull lattice. We will introduce the lattice version of a global transform as well, so that we may list important results in Noetherian lattices, as well as add a few more to the roster as we continue to work through [23]. We will end the section by listing the version of the Krull-Akizuki Theorem and the Mori Nagata Theorem proven in [22].

We will soon begin obtaining q. f. lattice versions of results involving strong Mori domains. We will need to use results concerning Mori lattices to reach that point, as well as a lattice version of the v-operation, which we define first.

Definition 2.2.1. Let $\mathcal{L}$ be a cl-monoid and $\mathcal{F}$ the set of fractionary elements of $\mathcal{L}$. For an $I \in \mathcal{F}$, denote $(R : I)$ by I^{-1} and $(I^{-1})^{-1}$ by I_v. If $I = I_v$, then I is called a **divisorial** element of $\mathcal{L}$ or a v**-element** of $\mathcal{L}$.

A q.f. lattice that satisfies ACC on integral divisorial elements is a **Mori Lattice**.

The map $v : \mathcal{F} \to \mathcal{F}$ defined by $I \mapsto I_v$ is called the v**-operation**.

The map $t : \mathcal{F} \to \mathcal{F}$ defined by $I \mapsto I_t = \bigvee\{J_v \mid J \leq I$ and J is finitely generated$\}$ is the t**-operation** and if $I = I_t$, then I is called a t**-element of** $\mathcal{L}$.

The following lemma contains the result [23, Lemma 2.4].

Lemma 2.2.2. *Let $\mathcal{L}$ be a q. f. lattice such that $\mathcal{I}$ satisfies ACC with multiplicative identity R and let R_b be the integral closure of R in $\mathcal{L}$. Let $A_1, A_2, \cdots, A_r \in \mathcal{I}(R_b)$ be principal in $\mathcal{L}$. Then $(R : (R : \vee_{i=1}^{r} A_i)) \leq (R_b : (R_b : \vee_{i=1}^{r} R_b A_i))$.*

Proof. Recall that since $\mathcal{I}$ is Noetherian, each element of $\mathcal{I}$ is compact, hence finitely generated since $\mathcal{L}$ is a q. f. lattice. Write $(R : (R : \vee_{i=1}^r A_i)$ as $((\vee_{i=1}^r A_i)^{-1})^{-1} = (\vee_{i=1}^r A_i)_v$.

Since $A_i \in \mathcal{I}(R_b)$, there is an element $B \in \mathcal{I}$ such that $A_iB \leq B$ for all $i = 1, \ldots, r$. So, $(\vee_{i=1}^r A_i)B \leq B$. Applying the v-operation, we have $(\vee_i A_i)_v B_v \leq ((\vee_i A_i)B)_v \leq B_v$. So $(\vee_i A_i)_v \leq (B_v : B_v)$. We know $B_v \in \mathcal{I}$ and hence is finitely generated so that $(B_v : B_v) \leq R_b$. Hence $(\vee_{i=1}^r A_i)_v \leq R_b$.

Let $X \leq (R_b : \vee_i R_b A_i)$ be principal, so that $XA_i \leq R_b$. If we replace A_i with XA_i in the previous paragraph, we have $(\vee_i XA_i)_v = X(\vee_i A_i)_v \leq R_b$. Therefore $(R_b : \vee_i R_b A_i)(\vee_i A_i)_v \leq R_b$. Thus $(R : (R : \vee_i A_i) = (\vee A_i)_v \leq (R_b : (R_b : \vee R_b A_i))$, completing the proof. □

We will need the following generalization of the global transform of a ring R.

Definition 2.2.3. If A is a ring element of a q. f. lattice $\mathcal{L}$, **the global transform of** A is $T(A) = \vee\{B \in \mathcal{L} \mid B$ is $\mathcal{L}$-principal and $(A : B)$ is greater than some product of maximal members of $\mathcal{I}(A)\}$.

The following theorems are proved in [22]. We will need them for the next generalization. First, we define the dimension of a lattice.

Definition 2.2.4. The **dimension** of a multiplicative lattice $\mathcal{L}$, denoted $\dim(\mathcal{L})$, is the supremum of the lengths of chains of prime elements of $\mathcal{L}$. The **height** of a prime element P of $\mathcal{L}$ is the supremum of the lengths of all chains of prime elements $\leq P$.

Theorem 2.2.5. [22, Theorem 7.4] *If $\mathcal{L}$ is a q. f. lattice and A is a ring element of $\mathcal{L}$ with $\mathcal{I}(A)$ Noetherian, then*

(1) *For each maximal element M of $\mathcal{I}(A)$, $T(A) : M = T(A)$;*

(2) *For each ring element B of $\mathcal{L}(A)$ with $B \leq T(A)$, $\mathcal{I}(B)$ is Noetherian; and*

(3) *If $\dim(\mathcal{I}) = 1$, then $T(A) = K$, where $K = \vee\mathcal{L}$ as above.*

Corollary 2.2.6. [22, Corollary 7.5] (The Krull-Akizuki Theorem for q.f. lattices) *Let $\mathcal{L}$ be a q. f. lattice with $\mathcal{I}$ Noetherian. If $\dim(\mathcal{I}) = 1$, then for each ring element A of $\mathcal{L}$, $\mathcal{I}(A)$ is Noetherian.*

Lemma 2.2.7. [22, Lemma 8.1] *Let $\mathcal{L}$ be a q. f. lattice with global transform $T = T(R)$ and assume $\mathcal{I}$ is Noetherian. Let $B = T \wedge R_b$. If P is a maximal element of $\mathcal{I}(B)$ and $P = [b : a]_{\mathcal{I}(B)}$ for some principal elements a, $b \in \mathcal{I}(B)$, then P is an invertible element of $\mathcal{L}(B)$.*

Lemma 2.2.8. *Let $P \in \mathcal{L}$ be an associated prime of a principal element of R. Then P is a t-element of $\mathcal{L}$.*

Proof. We wish to show that $P = \bigvee\{(J^{-1})^{-1} \mid J \leq P$ and J is finitely generated $\}$. The $\leq$ portion is clear, since $C \leq (R : (R : C)) = (R : C^{-1}) = (C^{-1})^{-1}$ for all $C \in \mathcal{L}$.

To show the $\geq$ direction, let C be finitely generated with $C \leq \bigvee\{(J^{-1})^{-1} \mid J \leq P$ and J is finitely generated $\}$. So, $C \leq (J^{-1})^{-1}$ for some $J \leq P = (A : B)$ for $A, B \in \mathcal{L}$ with A principal since P is a prime associated with a principal element of $\mathcal{L}$. Hence $C(R : J) \leq R$ and $C(R : (A : B)) \leq R$. Since $BA^{-1} \leq (R : (A : B))$, we have $C(BA^{-1}) \leq R$, meaning $CB \leq A$ and $C \leq (A : B)$, i.e. $C \leq P$. $\square$

The following is a generalization of [23, Lemma 2.6].

Lemma 2.2.9. *Let $\mathcal{L}$ be a q. f. lattice with $\mathcal{I}$ Noetherian. Let R_b be the integral closure of R in $\mathcal{L}$. Then the set of associated prime elements of nonzero principal elements of $\mathcal{I}(R_b)$ is equal to the set of height-one prime elements of $\mathcal{I}(R_b)$.*

Proof. Obviously any height-one prime element is an associated prime of a principal element. (See for example [26, Theorems 4.11 and 4.12].)

For the converse, let $Q = (A : C)$ be an associated prime of a principal element $A \in \mathcal{I}(R_b)$, let $P = Q \wedge R$, let $D = T(R_P) \wedge (R_b)_{R\backslash P}$ and let $M = Q(R_b)_{R\backslash P} \wedge D$.

Since Q is an associated prime in $\mathcal{I}(R_b)$ of a principal ideal a, Q is a t-element of $\mathcal{I}(R_b)$. Since $R_P \leq D \leq (R_b)_{R\backslash P}$ and $\mathcal{I}(D)$ is Noetherian by Theorem 2.2.5(2), then M is finitely generated.

We seek to show M is maximal in $\mathcal{I}(D)$. We know that $Q(R_b)_{R\backslash P}$ is an associated prime of a principal element of $\mathcal{I}((R_b)_{R\backslash P}$. So $Q(R_b)_{R\backslash P}$ is a t-element of $\mathcal{I}((R_b)_{R\backslash P})$.

By Lemma 2.2.8, $M = Q(R_P)_b \wedge D$ is a t-element of $\mathcal{I}(D)$ (since $D_b = (R_P)_b)$). It is also compact since $\mathcal{I}(D)$ is Noetherian, and it is finitely generated since it is in a q. f. lattice. Because D is integral over R_P and PR_P is a maximal element of $\mathcal{I}(R_P)$, M is a maximal element of $\mathcal{I}(D)$. By Lemma 2.2.7, M is invertible.

Assume that Q is not a minimal prime element of $\mathcal{I}(R_b)$ and let Q_0 be a prime element of $\mathcal{I}(R_b)$ such that $0 < Q_0 < Q$. So, $Q_0(R_b)_{R\backslash P} \wedge D < Q(R_b)_{R\backslash P} \wedge D = M$.

We also have $Q_0(R_b)_{R\setminus P} \wedge D < M^n$ for all $n \geq 1$ because M is invertible. Let X be a nonzero element with $X \leq Q_0(R_b)_{R\setminus P} \wedge D$. Then we have the infinite strictly increasing chain $XM^{-1} < XM^{-2} < \cdots < B$, contradicting the fact that $\mathcal{I}(D)$ is Noetherian. Therefore Q has height one. $\square$

We state some more definitions before we can state the generalization of the Mori-Nagata Theorem for q. f. lattices.

Definition 2.2.10. [7, Definition 4.1] Let $\mathcal{L}$ be a q. f. lattice. If, for every pair of $\mathcal{L}$-principal elements A, $B \in \mathcal{I}$, either $A \leq B$ or $B \leq A$, then $\mathcal{I}$ is called a **valuation lattice**.

Definition 2.2.11. Let $\mathcal{L}$ be a q. f. lattice. If $\mathcal{I}$ is a Noetherian valuation lattice then $\mathcal{I}$ is called a **discrete valuation lattice**.

Definition 2.2.12. Let $\mathcal{L}$ be a q.f. lattice. If for a family $\{C_\lambda \mid \lambda \in \Lambda\}$ of ring elements of $\mathcal{L}$ and each $X \in \mathcal{L}$ principal, $XC_\lambda = C_\lambda$ for all but finitely many $\lambda \in \Lambda$, then the family is **locally finite**.

Definition 2.2.13. [22, Definition 15] Let $\mathcal{L}$ be a q.f. lattice and let $\mathcal{P}$ denote the set of prime elements of $\mathcal{I}$ of height one. $\mathcal{I}$ is a **Krull lattice** if the following hold:

1. If $P \in \mathcal{P}$, then $\mathcal{I}(R_P)$ is a Noetherian valuation lattice.

2. If $M \in \mathcal{I}$ is $\mathcal{L}$-principal, then there are only finitely many $P \in \mathcal{P}$ such that $M \leq P$.

3. $R = \bigwedge\{R_P \mid P \in \mathcal{P}\}$.

The following result contains the graded Mori-Nagata theorem. See [22, Theorems 9.1, 9.2 and Corollary 9.3], [23, Theorem 2.10], and [24, Theorem 5.3].

Theorem 2.2.14. [22, Theorem 8.4] (Mori-Nagata Theorem for q.f. lattices) *If $\mathcal{L}$ is a q.f. lattice with $\mathcal{I}$ Noetherian, then $\mathcal{I}(R_b)$ is a Krull Lattice.*

Chapter 3

Nagata's Theorem on two-dimensional Noetherian domains

A well-known result of Nagata states that the integral closure of a two-dimensional Noetherian domain is Noetherian [25, Theorem 33.12]. In this section, we give a generalization of this theorem to multiplicative lattics. This result also contains the generalization on Nagata's theorem given in [24, Theorem 6.8] for graded domains.

3.1 Lattice modules and composition series

In order to generalize results about modules over a commutative ring, lattice modules were defined. This section shall introduce the definition and basic properties of lattice modules as well as define a composition series of a lattice module to generalize results in [24] to lattice modules.

Definition 3.1.1. Let $\mathcal{L}$ be a multiplicative lattice. A complete lattice $\mathcal{M}$ equipped with a multiplication $\mathcal{L} \times \mathcal{M} \to \mathcal{M}$ is a **left lattice module over $\mathcal{L}$** (or simply an **$\mathcal{L}$-module**) if for $a, b \in \mathcal{L}$, $\{a_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{L}$, $A \in \mathcal{M}$, $\{B_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{M}$:

1. $(ab)A = a(bA)$;
2. $(\vee_\lambda a_\lambda)(\vee_\gamma B_\gamma) = \vee_{\lambda,\gamma} a_\lambda B_\gamma$;
3. $RA = A$; and
4. $0_\mathcal{L} A = 0_\mathcal{M}$, where $0_\mathcal{M}$ denotes the smallest element of $\mathcal{M}$.

We will denote M to be the greatest element of $\mathcal{M}$ ($M = \vee\mathcal{M}$).

We will adopt the following convention. If, for example, M is finitely generated, we often say that $\mathcal{M}$ is a finitely generated $\mathcal{L}$-module.

Many definitions and notations carry over to lattice modules, such as residuation, compact elements, etc.

Let $\mathcal{M}$ be an $\mathcal{L}$-module with $a, b \in \mathcal{L}$ and $A, B \in \mathcal{M}$. Denote $a : b$ (respectively $A : b$, $A : B$) to be the largest element $c \in \mathcal{L}$ (respectively $C \in \mathcal{M}$, $c \in \mathcal{L}$) such that $bc \leq a$ (respectively $bC \leq A$, $cB \leq A$).

Definition 3.1.2. An element A of an $\mathcal{L}$-module $\mathcal{M}$ is **compact** if whenever $A \leq \vee\{B_\lambda \mid \lambda \in \Lambda\}$ for some family $\{B_\lambda \mid \lambda \in \Lambda\}$ of elements of $\mathcal{M}$, there is a finite subset $\Gamma \subseteq \Lambda$ such that $A \leq \vee\{B_\lambda \mid \lambda \in \Gamma\}$. We say that $\mathcal{M}$ is **compactly**

generated (or a **CG-module**) if each element of $\mathcal{M}$ is a join of compact elements of $\mathcal{M}$.

Definition 3.1.3. With $\mathcal{M}$ as above, an element $A \in \mathcal{M}$ is **meet principal** (respectively **join principal**) if for all $b \in \mathcal{L}$ and $B \in \mathcal{M}$, we have

$$(b \wedge (B : A))A \geq bA \wedge B \quad \text{(respectively} \quad (bA \vee B) : A \leq b \vee (B : A)\ ).$$

The reverse inequalities always hold. If A is meet principal and join principal, it is **principal** and $\mathcal{M}$ is **principally generated** if every element of $\mathcal{M}$ is a join of principal elements of $\mathcal{M}$.

We will now generalize the results of Park and Park [24] concerning the composition series of an R-module, M to lattice modules. They used their results to prove the Krull-Akizuki Theorem for graded Noetherian integral domains [24, Theorem 4.2]. Rush, Okon, and Wallace have previously generalized the weaker version of the Krull-Akizuki Theorem [15, Theorem 93] to the case of q.f. lattices [22, Corollary 7.5], as has already been pointed out in the previous chapter.

Definition 3.1.4. Let $\mathcal{M}$ be an $\mathcal{L}$-module. We define $\mathcal{M}$ to be **irreducible** if $A \leq M$ implies $A = 0_{\mathcal{M}}$ or $A = M$.

Definition 3.1.5. Let $\mathcal{M}$ be an $\mathcal{L}$-module. A chain $M = M_0 > M_1 > \cdots > M_{r-1} > M_r = (0)$ of elements of $\mathcal{M}$ is called a **composition series of $\mathcal{M}$** if each $[M_{i+1}, M_i]$ is irreducible. Then r is the length of the composition series, which we denote as $\mathrm{le}(\mathcal{M}) = r$.

The reader should note that notation will be slightly abused for convenience. When we say that $\mathrm{le}(M) = r$, we mean $\mathrm{le}([0, M]) = r$, but of course $[0, M]$ is just $\mathcal{M}$. As an example, for $A \in \mathcal{M}$, we will write $\mathrm{le}(A) = r$, but we mean $\mathrm{le}([0, A]) = r$. Likewise, by a composition series of M, we mean a composition series of $[0, M] = \mathcal{M}$, and so forth.

The following generalizes [24, Theorem 3.1] from graded modules to lattice modules.

Theorem 3.1.6. *Let $\mathcal{M}$ be a lattice module with a composition series of length r. Then any composition series of $\mathcal{M}$ has length r and any chain of elements of $\mathcal{M}$ can be refined to a composition series.*

Proof. We will use mathematical induction on r, the length of the composition series of $M \in \mathcal{M}$. The theorem is obviously true for $r = 1$. So, assume $r > 1$ and that the theorem holds for all lattice modules with a composition series of length less than r.

So, let $M = M_0 > M_1 > \cdots > M_{r-1} > M_r = (0)$ be a composition series of length r. By the induction hypothesis M has no composition series of length less than r. We must show that any chain of elements of M has length less than or equal to r.

Let $M = N_0 > N_1 > \cdots > N_s = (0)$ be a distinct chain of length s of elements of M. We proceed by cases. Assume $N_1 = M_1$. Then, M_1 has a composition series $M_1 > \cdots > M_r = (0)$ of length $r - 1$. From the induction hypothesis, we have $s - 1 \leq r - 1$ and $s \leq r$.

Now, assume $N_1 < M_1$. We can form a chain $M_1 > N_1 > N_2 > \cdots > N_s = (0)$ of length s. By the induction hypothesis, we have $s \leq r - 1$ so that $s \leq r$.

Lastly, assume that $N_1 \not\leq M_1$. Then, as lattice elements of $\mathcal{M}$, we have $M = M_1 \vee N_1$. Applying the second isomorphism theorem gives us the isomorphism $[M_1, M] = [M_1, M_1 \vee N_1] \cong [M_1 \wedge N_1, N_1]$, which is also irreducible.

Note that M_1 has a composition series of length $r-1$. Also, $M_1 \wedge N_1 < M_1$, so $M_1 \wedge N_1$ has a composition series of length $\leq r - 2$ by the induction hypothesis. From our chains, there are no elements $\leq M$ between N_1 and $M_1 \wedge N_1$ so that N_1

has a composition series of length $\leq r - 1$. So, we have $s - 1 \leq r - 1$ and $s \leq r$, again from the induction hypothesis. $\square$

The following generalizes [24, Lemma 3.2] from graded integral domains to q. f. lattices.

Lemma 3.1.7. *Let $\mathcal{L}$ be a q.f. lattice with* $\dim \mathcal{I} = 0$. *Then, for a* **finitely generated** *$\mathcal{I}$-module $\mathcal{M}$ and for any $A \in \mathcal{M}$, A has a finite length composition series.*

Proof. Since $\dim \mathcal{I} = 0$, R is irreducible in $\mathcal{L}$ which is a $\mathcal{L}$-module. Let $A \in \mathcal{M}$ be finitely generated and write $A = X_1 \vee \cdots \vee X_n$ where each X_i is a principal element of A. We proceed using induction on n. For $n = 1$, we have $A = X_1 \cong [(0 : X_1), R]$. However, $(0 : X_1) = 0$ since $\mathcal{L}$ is irreducible, meaning A is as well, so the $\mathrm{le}(A) = 1$. Now assume $n > 1$ and define $B \in \mathcal{M}$ as $B = X_2 \vee \cdots \vee X_n$. Note that $0 \leq B \leq A$ and that $[B, A] \cong X_1$, either it is 0 or it is irreducible. Thus $\mathrm{le}(A) = \mathrm{le}([B, A]) + \mathrm{le}(B) \leq 1 + \mathrm{le}(B) < \infty$. □

The following lemma generalizes [24, Lemma 3.3] from graded rings to multiplicative lattices.

Lemma 3.1.8. *Let $\mathcal{L}$ be a multiplicative lattice with $\mathcal{I}$ Noetherian such that* $\dim \mathcal{I} = 0$. *Then, for an $\mathcal{L}$-module $\mathcal{M}$ such that M is finitely generated, $\mathcal{M}$ has finite length.*

Proof. Note that $\mathrm{Min}(\mathcal{I}) = \mathrm{Spec}(\mathcal{I}) = \mathrm{Max}(\mathcal{I})$ because $\dim \mathcal{I} = 0$. By Corollary 2.1.4 there are a finite number of minimal prime elements of $\mathcal{I}$, call them $P_1, \cdots, P_n$. Note that $P_1 \cdots P_n \leq P_1 \wedge \cdots \wedge P_n = \mathrm{Rad}(0)$, which is finitely generated since $\mathrm{Rad}(0) \in \mathcal{I}$. Since this implies $(\mathrm{Rad}(0))^k = 0$ for some k, we have

$(P_1 \cdots P_n)^k = 0$. Let $A \in \mathcal{M}$ be finitely generated and consider the following chain of elements of $\mathcal{M}$:

$$M \geq P_1 M \geq P_1^2 M \geq \cdots \geq P_1^k M \geq P_1^k P_2 M \geq P_1^k P_2^2 M \geq$$

$$\cdots \geq P_1^k P_2^k M \geq P_1^k P_2^k P_3 M \geq \cdots \geq P_1^k P_2^k \cdots P_n^k M = 0.$$

Now let B and $P_i B$ be any two consecutive elements in this chain. Since each P_i is finitely generated and M is finitely generated, $[P_i B, B]$ is a finitely generated element of $\mathcal{M}'$, where $\mathcal{M}'$ is a $\mathcal{I}/P_i$-module. Since $\mathcal{I}/P_i$ is a q.f. lattice with $\dim(\mathcal{I}/P_i) = 0$, $\text{le}([P_i B, B]) \leq \infty$ from the previous lemma. Since $\text{le}(M)$ is the sum of the lengths of all these chains, which have finite length, $\text{le}(M)$ is finite also. □

Park and Park proved a generalization of the Principal Ideal Theorem for the graded Noetherian case [24, Theorem 3.5]. An even more general theorem had already been proven by R.P. Dilworth, however [8, Theorem 6.4]. We state it here; the reader may refer to the source for the proof, which resembles Kaplansky's proof of the Principal Ideal Theorem but makes use of congruence classes of prime elements of a Noether lattice. For reference, we also state the definition of an r-lattice from [2], which is a generalization of a Noether lattice.

Definition 3.1.9. A multiplicative lattice $\mathcal{L}$ is an r**-lattice** if it satisfies the following:

1. $\mathcal{L}$ is modular
2. $\mathcal{L}$ is principally generated

3. $\mathcal{L}$ is compactly generated

4. R is compact

If, in addition, $\mathcal{L}$ satisfies the ascending chain condition, $\mathcal{L}$ is a **Noether lattice**.

Theorem 3.1.10 (Principal Element Theorem)**.** *Let $\mathcal{L}$ be a Noether lattice and let $M \in \mathcal{L}$ be principal. If $P \in \mathcal{L}$ is minimal among the prime elements of $\mathcal{L}$ with $M \leq P$, then* $\mathrm{ht}(P) \leq 1$.

We state a few more definitions in order to proceed. For the rest of this chapter, $\mathcal{L}$ represents a multiplicative lattice unless otherwise stated.

Definition 3.1.11. Let $\mathcal{M}$ be an $\mathcal{L}$-module with maximal element M. Then $\mathcal{M}$ (or M) is **torsion-free** if $(0 :_{\mathcal{M}} X) = 0$ for every non-zero $X \in \mathcal{L}$.

Definition 3.1.12. Let $\mathcal{M}$ be a lattice module over a multiplicative lattice $\mathcal{L}$ and let $M \in \mathcal{M}$ be torsion-free. We define the **rank** of M to be the cardinality of any subset of principal elements $A_i \leq M$ of $\mathcal{M}$ that is maximally linearly independent over $\mathcal{I}$. We assume that any two such subsets have the same cardinality (examples where this holds include the set of R-submodules of an R-module or if R and M are graded, the set of graded submodules of M).

The following is a lattice module version of [20, Lemma, p. 84] and [24, Lemma 4.1].

Lemma 3.1.13. *Let $\mathcal{L}$ be a multiplicative lattice with $\mathcal{I}$ Noetherian and* $\dim \mathcal{I} = 1$. *Let $\mathcal{M}$ be an $\mathcal{L}$-module such that M is torsion-free and has finite rank r. Then for a principal element $A \in \mathcal{L}$ we have* $\mathrm{le}([AM, M]) \leq r \cdot \mathrm{le}([A, R])$.

Proof. First we will prove the case when M is finitely generated. Select principal elements $X_1, \cdots, X_n \in \mathcal{M}$ for $i - 1, .., n$ with each X_i linearly independent over $\mathcal{I}$. Let $B = X_1 \vee \cdots \vee X_n$. For any principal element $X \leq M$, there is a principal element $D \in \mathcal{I}$ such that $DX \leq B$. Let $C = M/B$, which is a finitely generated element of $\mathcal{M}$ with $DC = 0$. We know that $\mathcal{I}/D$ is Noetherian with $\dim \mathcal{I}/D = 0$ and that C is a finitely generated element of the lattice module over $\mathcal{I}/D$. By Lemma 3.1.8, $\mathrm{le}(C) < \infty$. Let $A \in \mathcal{I}$ be principal and consider the exact sequence of homomorphisms of the following $\mathcal{L}$-modules:

$$[A^n B, B] \to [A^n M, M] \to [A^n C, C].$$

From this we know that $\mathrm{le}([A^n M, M]) \leq \mathrm{le}([A^n B, B]) + \mathrm{le}([A^n C, C]) \leq \mathrm{le}([A^n B, B]) + \mathrm{le}(C)$ for all $n > 0$.

Since B and M are both torsion-free elements of $\mathcal{M}$, we have $[A^{i+1}E, A^i E] \cong [AE, E]$ as $\mathcal{L}$-modules, likewise $[A^{i+1}M, A^i M] \cong [AM, M]$. Using this fact with the above inequality, it follows that $n \cdot \mathrm{le}([AM, M]) \leq n \cdot \mathrm{le}([AB, B]) + \mathrm{le}(C)$ for all $n > 0$. Once again invoking Lemma 3.1.8 we know that $\mathrm{le}([AM, M])$ and $\mathrm{le}([AB, B])$ are finite and this combined with the previous inequality implies that $\mathrm{le}([AM, M]) \leq \mathrm{le}([AB, B])$.

Recall that $B = X_1 \vee \cdots \vee X_r$ with $X_1, \cdots, X_r$ principal elements of $\mathcal{M}$ that are linearly independent over $\mathcal{I}$, which gives us $\mathrm{le}([AB, B]) = r \cdot \mathrm{le}([AR, R]) = r \cdot \mathrm{le}([A, R])$, implying that $\mathrm{le}([AM, M]) \leq r \cdot \mathrm{le}([A, R])$. The proof is complete for M finitely generated.

For the case when M is not finitely generated, consider the $\mathcal{L}$-submodule

$[AM, M]$ and let $Y_1^{'}, \cdots, Y_s^{'}$ be principal elements of $[AM, M]$. Let $N = Y_1^{'} \vee \cdots \vee Y_s^{'}$ be a finitely generated element of $[AM, M]$. For each $Y_i^{'}$, select a an inverse image $Y_i \in \mathcal{M}$ and define $M_1 = Y_1 \vee \cdots \vee Y_n$. Since we know that $N \cong [AM, AM \vee M_1] \cong [AM \wedge M_1, M_1]$ as elements of $\mathcal{M}$, it follows that

$$\mathrm{le}(N) = \mathrm{le}([AM \wedge M_1, M_1]) \leq \mathrm{le}([AM_1, M_1]) \leq r^{'} \cdot \mathrm{le}([A, R]) \leq r \cdot \mathrm{le}([A, R])$$

where $r^{'}$ is the rank of M_1. Note that this inequality depends on M_1, not N, so that $\mathrm{le}([AM, M]) \leq r \cdot \mathrm{le}([A, R])$ as desired. $\square$

3.2 Lattice modules and the ascending chain condition

This section contains many results concerning Noetherian lattice modules as well as lattice modules over Noetherian lattices. These results will be instrumental in obtaining the Nagata Theorem for quotient field lattices.

The following lemma generalizes [24, Lemma 6.1] from graded modules to lattice modules.

Lemma 3.2.1. *Let $\mathcal{M}$ be an $\mathcal{L}$-module such that M is finitely generated. If $\mathcal{I}$ is Noetherian, then $\mathcal{M}$ is Noetherian.*

Proof. We proceed by induction on n, the number of generators of M. Obviously, the assertion is true for $n = 0$, so let $M = X_1 \vee \cdots \vee X_n$ where each X_i is a principal element of $\mathcal{M}$. Let $M^{'} = X_2 \vee \cdots \vee X_n$. Note that $[0_{\mathcal{M}}, M^{'}]$ is Noetherian by the induction hypothesis. Let $N \in \mathcal{M}$ with $N \leq M$. We will show

N is finitely generated. Let I be the set of all principal elements of $\mathcal{I}$ such that $I = (N \vee M' :_{\mathcal{L}} X_1)$. Since $I \in \mathcal{I}$ which is Noetherian, I is finitely generated, so let $I = A_1 \vee \cdots \vee A_m$. For each $i = 1, \cdots, m$, let $Y_i \leq N$ such that $Y_i \in M' \vee A_i X_1$. If $N' = Y_1 \vee \cdots \vee Y_m \leq N$, we have $N = N' \vee (N \wedge M')$. Since $[0_{\mathcal{M}}, M']$ is Noetherian and $N \wedge M' \leq M'$, $N \wedge M'$ is finitely generated so that N is also finitely generated, which implies that $\mathcal{M}$ is Noetherian. $\square$

The following lemma is a version of [24, Lemma 6.2] for q. f. lattices.

Lemma 3.2.2. *Let $\mathcal{L}$ be a q. f. lattice such that $A \in \mathcal{L}$ is a ring element with A integral over R. Then* $\dim \mathcal{I} = \dim \mathcal{I}(A)$.

Proof. Note that lattice versions of the "going up" and "incomparable" conditions were proven in [6, Proposition 3.5.1 and Theorem 3.59] and we will use them in this proof. For a chain $P_1 < P_2 < \cdots < P_n$ of prime elements of $\mathcal{I}$, there exists a chain of prime elements $Q_1 < Q_2 < \cdots < Q_n$ of $\mathcal{I}(A)$ such that $Q_i \wedge R = P_i$ by the GU condition. So, $\dim \mathcal{I} \leq \dim \mathcal{I}(A)$. For the converse, let $Q_1 < Q_2 < \cdots < Q_n$ be a chain of distinct prime elements of $\mathcal{I}(A)$. By the INC condition, we can construct a chain $Q_1 \wedge R < Q_2 \wedge R < \cdots < Q_n \wedge R$ of distinct prime elements of R, each an element of $\mathcal{I}$. Hence $\dim \mathcal{I}(A) \leq \dim \mathcal{I}$. $\square$

The following lemma generalizes [24, Lemma 6.3] from graded rings to multiplicative lattices.

Lemma 3.2.3. *Let $\mathcal{L}$ be a multiplicative lattice and let $I_1, \cdots, I_n$ be elements of $\mathcal{I}$ such that $\bigwedge_{i=1}^{n} I_i = 0$ and each $\mathcal{I}/I_i = [I_i, R]$ is Noetherian. Then, $\mathcal{I}$ is Noetherian.*

Proof. It is sufficient to prove the lemma for $n = 2$. Since $(I_1 \vee I_2) \in \mathcal{I}/I_2$, a Noetherian multiplicative lattice, there are a finite number of principal elements, $A_1, \cdots A_r$ with each $A_1 \leq I_1$ such that $I_1 \vee I_2 = A_1 \vee \cdots \vee A_r \vee I_2$. Since $I_1 \wedge I_2 = 0$, $I_1 = A_1 \vee \cdots \vee A_n$ and I_1 is finitely generated. A similar argument shows that I_2 is finitely generated. Let P be a prime element of $\mathcal{I}$. We have $I_1 \wedge I_2 = 0 \leq P$ and so $I_1 \leq P$ or $I_2 \leq P$. Assume that $I_1 \leq P$. We have I_1 and $P \vee I_1$ (as an element of $\mathcal{I}/I_1$) are finitely generated, so P is finitely generated as well and $\mathcal{I}$ is Noetherian. □

Before we continue, we need the following definition.

Definition 3.2.4. Let $\mathcal{M}$ be an $\mathcal{L}$-module with maximal element M. Then $\mathcal{M}$ (or M) is **faithful** if $(0 :_{\mathcal{L}} M) = 0$.

The following generalizes [24, Lemma 6.4] from graded modules to lattice modules.

Lemma 3.2.5. *Let $\mathcal{L}$ be a multiplicative lattice and $\mathcal{M}$ an $\mathcal{L}$-module. If $M \in \mathcal{M}$ is faithful and $\mathcal{M}$ is Noetherian, then $\mathcal{I}$ is Noetherian.*

Proof. By assumption, M is finitely generated, so write $M = X_1 \vee \cdots \vee X_n$ for principal elements $X_i \leq M$ for $i = 1, \cdots, n$. Hence each $[0, X_i]$ is Noetherian and since $X_i \cong \mathcal{I}/(0 :_{\mathcal{L}} X_i) = [(0 :_{\mathcal{L}} X_i), R]$ as elements of $\mathcal{M}$, $[(0 :_{\mathcal{L}} X_i), R]$ is Noetherian. We have $\bigwedge_{i=1}^{n}(0 :_{\mathcal{L}} X_i) = (0 :_{\mathcal{L}} M) = 0$ from the hypothesis that M is faithful. The result now follows from the previous lemma. □

The following lemma extends [24, Lemma 6.5] to lattice modules.

Lemma 3.2.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-module such that M is finitely generated and faithful over $\mathcal{I}$. If the subset of elements of $\mathcal{M}$ of the form IM for $I \in \mathcal{I}$ satisfies ACC, then $\mathcal{I}$ is Noetherian.*

Proof. If we can show that $\mathcal{M}$ is Noetherian then we obtain the result from the previous lemma. Suppose to the contrary that $\mathcal{M}$ is not Noetherian. Let $\mathcal{S}$ be the set of elements of $\mathcal{M}$ of the form IM for $I \in \mathcal{I}$ such that $[IM, M]$ is not Noetherian. We know that $\mathcal{S}$ is not empty since it contains $0_{\mathcal{M}}$ and that it must contain a maximal element since it satisfies ACC. We will simply call the maximal element IM. If we pass from M to $[IM, M]$ and from $\mathcal{I}$ to $[(0 : [IM, M])$, we can further assume that even though $\mathcal{M}$ is not Noetherian, $[IM, M]$ is Noetherian for any $I \in \mathcal{I}$.

Now we define $\mathcal{T}$ to be the set of elements N of $\mathcal{M}$ such that any element of $[N, M]$ is faithful over $\mathcal{I}$. Again, $0_{\mathcal{M}} \in \mathcal{T}$. Give $\mathcal{T}$ the partial order of $\mathcal{M}$. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a chain in $\mathcal{T}$ such that $N' = \bigvee_{\lambda\in\Lambda} N_\lambda$. We will show that $[N', M]$ is faithful over $\mathcal{I}$ implying that $N' \in \mathcal{T}$. Suppose it is not, which means there is a nonzero principal element $A \in \mathcal{I}$ such that $AM \leq N'$. Since we assumed M is finitely generated, it follows that $AM \leq N_\lambda$ for some $\lambda \in \Lambda$, but this contradicts $[N_\lambda, M]$ being faithful over $\mathcal{I}$.

Using Zorn's Lemma, let N_0 be the maximal element of $\mathcal{T}$ so that $[N_0, M]$ is faithful. If it is Noetherian, it follows from Lemma 3.2.5 that $\mathcal{I}$ is Noetherian and then $\mathcal{M}$ is Noetherian from Lemma 3.2.1, so assume that $[N_0, M]$ is not Noetherian.

Now, if we consider $[N_0, M]$ instead of $\mathcal{M}$, we can further assume that $[N, M]$

is not faithful for any $N \in \mathcal{M}$, i.e. $(0 :_{\mathcal{I}} [N, M]) \neq 0$. Hence we have a principal $B \in \mathcal{I}$ with $BM \leq N$. Also, since $[BM, M]$ is Noetherian, every element of $[BM, N]$ is finitely generated. Since M is finitely generated, so is AM, hence N as well. $\square$

The next theorem generalizes the Eakin-Nagata Theorem and [24, Lemma 6.6] to multiplicative lattices.

Theorem 3.2.7. (Eakin-Nagata Theorem for Multiplicative Lattices) *Let $\mathcal{L}$ be a q.f. lattice and assume $\mathcal{I}(A)$ is Noetherian for some ring element $A \in \mathcal{L}$ such that A is finite over $\mathcal{I}$. Then $\mathcal{I}$ is also Noetherian.*

Proof. Since A is a ring element of $\mathcal{L}$, it is a faithful $\mathcal{I}$-module. Therefore the result follows by Lemma 3.2.6. $\square$

We use the following result which generalizes a well-known result of I. S. Cohen.

Lemma 3.2.8. [2, Theorem 2.5] *If $\mathcal{I}$ is an integral multiplicative lattice and $I \in \mathcal{I}$ is maximal among elements $A < R$ which are not finitely generated, then I is prime.*

The following definition is the lattice version of a primary ideal.

Definition 3.2.9. [7, Definition 3.8] Let $\mathcal{L}$ be a c.l. monoid. An element $Q \in \mathcal{I}$ is **primary** if for every A and B in $\mathcal{I}$, $AB \leq Q$ implies that $A \leq Q$ or $B^k \leq Q$ for some positive integer k. If Q is primary, then let $P(Q) = \vee\{C \in \mathcal{I} \mid C^k \leq Q\}$, which is the unique minimal prime greater than or equal to Q and is called the **prime associated with** Q. We then say that Q is $P(Q)$**-primary**.

The next theorem generalizes the Approximation Theorem for Krull rings [20, Theorem 12.6] which is needed to generalize the Nagata Theorem.

Theorem 3.2.10. [20, Theorem 12.6] *Let $\mathcal{L}$ be a q. f. lattice such that $\mathcal{I}$ is a Krull Lattice. Let $\mathcal{P}$ be the set of prime elements of $\mathcal{I}$ of height one. For any $P_1, ..., P_r \in \mathcal{P}$ and $n_1, ..., n_r \in \mathbb{Z}$, there exists an $X \in \mathcal{L}$ such that:*

1. *$v_i(X) = n_i$ for $1 \leq i \leq r$*

2. *$v_P(X) \geq 0$ for all $P \in \mathcal{P} \setminus \{P_1, ..., P_r\}$*

where v_i and v_P are the additive valuations of $\mathcal{L}$ corresponding to P_i and P.

Proof. The proof is readily adapted from Matsumura's proof. Let $Y_1 \in \mathcal{I}$ be such that $Y_1 \leq P_1$ but $Y_2 \not\leq P_1^{(2)} \vee P_2 \vee \cdots \vee P_r$. Then, $v_i(Y_1) = \delta_{1i}$ for $1 \leq i \leq r$. Likewise, let principal elements $Y_2, \cdots, Y_r \in \mathcal{I}$ be such that $v_i(Y_j) = \delta_{ij}$. Now, set

$$Y = \prod_{i=1}^{r} Y_i^{e_i}$$

and let $P_1^{'}, \cdots, P_s^{'}$ be all the prime elements of $\mathcal{P} \setminus \{P_1, \cdots, P_r\}$ such that $v_p(Y) < 0$. For each $j = 1, \cdots, s$ choose an element $T_j \leq P_j^{'}$ with $T_j \not\leq P_1 \vee \cdots \vee P_r$. If k is sufficiently large, the element $X = Y(T_1 \cdots T_s)^k$ is the element we seek. $\square$

The next theorem generalizes a classical result [20, Theorem 12.7], and [24, Lemma 6.7] to multiplicative lattices.

Theorem 3.2.11. *Let $\mathcal{L}$ be a q. f. lattice such that $\mathcal{I}$ is a Krull lattice and let $\mathcal{P}$ denote the set of prime elements of $\mathcal{I}$ of height one. If $\mathcal{I}/P$ is Noetherian for each $P \in \mathcal{P}$, then $\mathcal{I}$ is Noetherian.*

Proof. By 3.2.8, it suffices to show that each nonzero prime element Q is finitely generated. Let $a \leq Q$ be a nonzero principal element of $\mathcal{I}$. By [7, Lemma 4.22.2], $a = P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)}$ for some $P_1, \ldots, P_n \in \mathcal{P}$, where the P_i and r_i are uniquely determined up to order. If each $R/P_i^{(r_i)}$ is Noetherian by Lemma 3.2.3, so that Q/aR is is finitely generated. But then Q is finitely generated. Therefore it suffices to show that $R/P^{(n)}$ is Noetherian for each $P \in \mathcal{P}$ and positive integer n.

Choose a principal element $a \leq P$ with $a \not\leq P^{(2)}$. Then $a = P \cap P_2^{(n_2)} \cap \cdots \cap P_r^{(n_r)}$ for distinct $P_i \in \mathcal{P}$ and $n_2, \ldots, n_r \in \mathbb{N}$. For $i = 2, \ldots, n$, choose a principal element $b_i \leq P_i$ with $b_i \not\leq P$ for $i = 2, \ldots, r$.

If $a = P$, let $x = a$. Otherwise let $x = ab_2^{-n_2} \cdots b_r^{-n_r}$. Then $x \in R_{\mathcal{S}}$, where $\mathcal{S}$ is the set of nonzero principal elements of R. By [7, Lemma 4.25.2], $R_Q \in \mathcal{L}$ is a Noetherian valuation ring element for each $Q \in \mathcal{P}$. Let v_Q be the normalized additive valuation determined by R_Q.

Since $b_i \leq P_i$ with $b_i \not\leq P$, then $v_P(b_i) = 0$ for $i = 2, \ldots, r$. Thus since $v_P(a)$ $= 1$, then $v_P(x) = 1$.

If $Q = P_i$, $i \in \{2, \ldots, r\}$, then $v_{P_i}(x) = v_{P_i}(a) - v_{P_i}(b_2^{n_2} \cdots b_r^{n_r}) \leq 0$ since $v_{P_i}(a)$ $= n_i \leq v_{P_i}(b_i^{n_i})$ and $v_{P_i}(b_j^{n_j}) \geq 0$ if $j \in \{2, \ldots, r\} \setminus \{i\}$. If $Q \in \mathcal{P} \setminus \{P, P_2, \ldots, P_r\}$, then $v_Q(x) = v_Q(a) - v_Q(b_2^{n_2} \cdots b_r^{n_r}) \leq 0$ since $v_Q(a) = 0$ in this case. Thus $v_P(x)$ $= 1$ and $v_Q(x) \leq 0$ for all $Q \in \mathcal{P} \setminus \{P\}$. Let $A = R[x] = R \vee x \vee x^2 \vee \cdots$. **Claim:** $A/xA \cong R/P$.

Since $v_P(x) = 1$, $xA \leq xR_P = PR_P$ and thus $(xA \wedge R) \leq (PR_P \wedge R) = P$. For the opposite inclusion, let $y \leq P$ be a principal element. Then $v_P(y) \geq v_P(x)$ $= 1$. So $v_P(yx^{-1}) = v_P(y) - v_P(x) \geq 0$. If $Q \in \mathcal{P} \setminus \{P\}$, then $v_Q(x) = 0$. So again $v_Q(yx^{-1}) = v_Q(y) - v_Q(x) \geq 0$. Thus $yx^{-1} \leq \wedge\{R_Q \mid Q \in \mathcal{P}\} = R$. So for any principal $y \leq P$, we have $y \leq xR \leq xA$. Thus $P \leq xA \wedge R$. Therefore $P = xA \wedge R$. Further since $A = R + xA$, then $A/xA = (R + xA)/xA \cong R/(xA \wedge R)$ $= R/P$.

Now since any prime element containing $x^n A$ contains xA, it follows by Cohen's theorem that $A/x^n A$ is Noetherian for any $n \geq 1$. Since $A/x^n A \cong [x^n A, A]$ is a lattice module over $\mathcal{I}$ with $(x^n A \wedge R)) \leq (0 :_{\mathcal{I}} A/x^n A)$, then $A/x^n A$ is a lattice module over $R/(x^n A \wedge A) = [x^n A \wedge R, R]$. Further since $A/x^n A$ is generated over $\mathcal{I}$ by $1, x, \ldots, x^{n-1}$, it is also generated over $R/(x^n A \wedge A)$ by $1, x, \ldots, x^{n-1}$. Thus it follows from Eakin's theorem that that $R/(x^n A \wedge R)$ is Noetherian. Further since $x^n A \wedge R \leq x^n R_P \wedge R = P^n R_P \wedge R = P^{(n)}$, then $R/P^{(n)}$ is Noetherian. $\square$

At last, we are ready to prove the Nagata Theorem for q. f. lattices.

Theorem 3.2.12. (Nagata Theorem for q. f. Lattices) Let $\mathcal{L}$ be a q.f. lattice such that $\mathcal{I}$ is Noetherian and $\dim \mathcal{I} \leq 2$. Then $\mathcal{I}(R_b)$ is also Noetherian with $\dim \mathcal{I}(R_b) \leq 2$.

Proof. By the Krull-Akizuki Theorem for q. f. lattices, Lemma 2.1.13, and Corollary 2.1.14, the theorem holds if $\dim \mathcal{I} \leq 1$ and so we may assume that $\dim \mathcal{I} = 2$. Hence $\mathcal{I}(R_b)$ is a Krull lattice and $\dim \mathcal{I}(R_b) = 2$ by the Mori Nagata Theorem for q. f. lattices and Lemma 3.2.2. According to the previous theorem, it is sufficient to show that $\mathcal{I}(R_b)/P^{'} \cong [P^{'}, R_b]$ is Noetherian for every $P^{'} \in \mathcal{P}^{'}$, where $\mathcal{P}^{'}$ is the set of all prime elements of $\mathcal{I}(R_b)$ of height one.

We know that $[P', R_b]$ is trivially Noetherian if $P^{'}$ is a maximal element of $\mathcal{I}(R_b)$ because $[P^{'}, R_b]$ would have no nontrivial elements. So, we assume that $P^{'}$ is not a maximal element of $\mathcal{I}(R_b)$. Let $P = P^{'} \wedge R$, which is not a maximal element of $\mathcal{I}$ because R_b is integral over R. We know that the height of $P = 1$ because $\dim(\mathcal{I}) = 2$, and each prime element of $\mathcal{I}(R_b)$ lying over P has height 1.

Since $\mathcal{I}(R_b)$ is a Krull lattice, there are only a finite number of prime elements of $\mathcal{I}(R_b)$ lying over P; call them $P^{'}, P_2^{'}, \cdots, P_r^{'}$. Now choose a principal element $X_i \leq P^{'} \setminus P_i^{'}$ for each $i = 2, \cdots, r$. Note that $\mathcal{I}(R[X_2, \cdots, X_r]) = \mathcal{I}(R \vee X_2 \vee X_2^2 \vee \cdots \vee X_i \vee X_i^2 \vee \cdots \vee X_r \vee X_r^2 \vee \cdots)$ is a sublattice of $\mathcal{I}(R_b)$ which is finite over $\mathcal{I}$, $\mathcal{I}(R[X_2, \cdots, X_r])$ is Noetherian with dimension 2 by Lemma 3.2.1 and Lemma 3.2.2.

By using $\mathcal{I}(R[X_2, \cdots, X_r])$ instead of $\mathcal{I}$, it is safe to assume that $P^{'}$ is the unique prime element of $\mathcal{I}(R_b)$ lying over P.

Since $\mathcal{I}(R_P)$ is Noetherian with dimension 1, The Krull-Akizuki Theorem for q.f. lattices states that $\mathcal{I}((R_P)_b)/A(R_P)_b = \mathcal{I}((R_b)_P)/A(R_b)_P$ is a finite $\mathcal{I}(R_P)$-module for every nonzero principal element $A \in \mathcal{I}(R_P)$. More specifically, $\mathcal{I}((R_P)_b)/P'(R_b)_P$ is a finite $\mathcal{I}(R_P)/PR_P$-module, and since $\mathcal{I}/P$ is Noetherian with dimension 1, $\mathcal{I}(R_b)/P'$ is Noetherian as well by the Krull-Akizuki Theorem for q.f. lattices and we are done. $\square$

Chapter 4

Strong Mori lattices and lattice modules

In this section we will generalize results in [27] and [28] concerning strong Mori domains and results in [23] concerning graded strong Mori domains to lattices as well as obtain results concerning lattice modules. Of particular interest will be w-elements of lattices and lattice modules, which are the lattice versions of w-ideals and w-submodules.

4.1 The w-operation on a q. f. lattice

To give lattice versions of some of these results, we first define the lattice version of a star operation and the w-operation. Star operations were first defined and studied by Krull in [16] and are now ubiquitous in commutative ideal theory. It is interesting that star operations also furnish a good class of models for the abstract ideal theory that he first advocated in [17] as we shall see.

The following definition is a lattice version of a Glaz-Vasconcelos ideal.

Definition 4.1.1. Let $\mathcal{L}$ be a q. f. lattice and let $J \in \mathcal{I}$. We will call J a **Glaz-Vasconcelos element** (denoted $J \in \mathrm{GV}(\mathcal{L})$) if J is finitely generated and $J_v = R$ (so that $J^{-1} = R$).

It is well-known that a product of Glaz-Vasconcelos ideals is also a Glaz-Vasconcelos ideal [27, Lemma 1.1]. This is also true for Glaz-Vasconcelos elements, as we quickly prove.

Lemma 4.1.2. *If $J_1, J_2 \in \mathrm{GV}(\mathcal{L})$, then $J_1 J_2 \in \mathrm{GV}(\mathcal{L})$.*

Proof. It is obvious that $J_1 J_2$ is finitely generated and we need only show that $(J_1 J_2)^{-1} \leq R$. Consider a principal element $X \in \mathcal{L}$ such that $X \leq (J_1 J_2)^{-1}$. So $X(J_1 J_2) \leq R$, implying that $X J_1 \leq J_2^{-1} = R$. Hence $X \leq J_1^{-1} = R$. $\square$

Now we will introduce the lattice version of a star-operation.

Definition 4.1.3. Let $\mathcal{L}$ be a q. f. lattice and let $\mathcal{F}$ denote the set of nonzero fractional elements of $\mathcal{L}$. A map $* : \mathcal{F} \to \mathcal{F}$ is a **star operation** if for all $A, B \in \mathcal{F}$ and all nonzero principal elements $a \in \mathcal{L}$, the following hold:

1. $a^* = a$ and $(aA)^* = aA^*$,

2. $A \leq A^*$ and $A \leq B \Rightarrow A^* \leq B^*$, and

3. $(A^*)^* = A^*$.

The v-operation mentioned in Definition 2.2.1 is a classic example of a star operation in ring theory.

Definition 4.1.4. Let $\mathcal{L}$ be a q. f. lattice and define a map $w : \mathcal{F} \to \mathcal{F}$ by $I \mapsto I_w$ where $I_w = \bigvee\{X \in \mathcal{L} \mid X$ is $\mathcal{L}$-principal and $JX \leq I$ for some $J \in \mathrm{GV}(\mathcal{L})\}$. This map is called the w-**operation**. If $I = I_w$, I is called a w-**element** of $\mathcal{L}$.

The following lemma generalizes [23, Lemma 3.3].

Lemma 4.1.5. *Let $\mathcal{L}$ be a q. f. lattice. If $I \in \mathcal{L}$ is fractionary, then I_w is also fractionary.*

Proof. Note that $J \in \mathrm{GV}(\mathcal{L})$ implies J is integral since $J \leq J_v \leq R$. So, since I is fractionary, there is a $D \in \mathcal{I} \setminus \{0\}$ such that $DI \in \mathcal{I}$. Hence if $A \leq I_w$, that is, $JA \leq I$ for some $J \in \mathrm{GV}(\mathcal{L})$, then we have $DJA \leq DI \leq R \in \mathcal{I}$. So $DA \leq (DI)_w \leq R_w = R$ for each $A \in I_w$. So $DI_w \leq R$, and thus I_w is fractionary. □

The w-operation is one of the most important examples of a star operation on fractional ideals of an integral domain. We will show it also gives a star operation on fractional elements of a q. f. lattice.

Proposition 4.1.6. *The map w defined above is a star operation on $\mathcal{F}$.*

Proof. First, note that for any $I \in \mathcal{F}$, $I \leq I_w$ by letting $J = R$. Now let a be a nonzero principal element of $\mathcal{L}$ and let $X \leq a_w$. Then there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX \leq a$. Since a is principal, a is invertible by [7, Proposition 1.21] or [22, Proposition 2.3]. So $JXa^{-1} \leq R$, and thus $Xa^{-1} \in J^{-1} = R$. So $X \leq aR = a$. Hence $a_w = a$.

To show that $(aA)_w = aA_w$, first we show that $(aA)_w \leq aA_w$. Suppose $X \in \mathcal{L}$ is principal and $X \leq (aA)_w$. Then for some $J \in \mathrm{GV}(\mathcal{L})$, $JX \leq aA$. So

$J(Xa^{-1}) \leq A$. So $(Xa^{-1}) \leq A_w$ and thus $X \leq aA_w$. So $(aA)_w \leq aA_w$. The opposite inequality is similar. Thus property (1) of star-operations holds.

For property (2), assume $A \leq B$ for $A, B \in \mathcal{F}$ and that $X \leq A_w$. Then there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX \leq A \leq B$, implying that $X \leq B_w$. So $A_w \leq B_w$.

Finally, let $X \leq (A_w)_w$ for $A \in \mathcal{F}$. Then there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX \leq A_w$ and there is a $J_1 \in \mathrm{GV}(\mathcal{L})$ such that $J_1 JX \leq A$. Since $J_1 J \in \mathrm{GV}(\mathcal{L})$ it follows that $X \leq A_w$. □

With this, we may define a strong Mori lattice, the lattice version of a strong Mori domain.

Definition 4.1.7. A q.f. lattice $\mathcal{L}$ is a **strong Mori lattice** if it has ACC on integral w-elements.

4.2 The r-lattice $\mathcal{L}_w$

It is known that the set $\mathcal{L}_w(R)$ of w-ideals of a commutative ring R is an r-lattice [3, Theorem 3.5], and thus $\mathcal{L}_w(R)$ is a Noether lattice if and only if R is a strong Mori domain. Hence many of the known results on strong Mori domains follow from results on Noether lattices. For example, if $\mathcal{L}$ is a Noether lattice, then the principal element theorem of Krull and many other results on Noetherian commutative rings carry over to $\mathcal{L}_w(R)$ [8].

In this subsection, we will show that if $\mathcal{L}$ is a strong Mori lattice, then the sublattice $\mathcal{L}_w$ of w-elements of $\mathcal{L}$ is a Noether lattice. Thus we will automatically have the Principal Element Theorem for the sublattice $\mathcal{L}_w$ of w-elements of $\mathcal{L}$ and many other results from Noetherain commutative ring theory will carry over to $\mathcal{L}_w$. To see this we follow the approach used in [3, Section 3].

Theorem 4.2.1. (cf. [3, Theorem 3.1].) *Let $* : \mathcal{F} \to \mathcal{F}$ be a star operation on the*

q. f. lattice $\mathcal{L}$ with $\mathcal{I}$ modular. and let $\mathcal{L}_*$ denote the set of integral $*$-elements of $\mathcal{F}$ together with $\{0\}$. We partial order $\mathcal{L}_*$ by the partial order on $\mathcal{L}$ and define the meet of a family $\{A_\lambda \mid \lambda \in \Lambda\}$ to be the meet in $\mathcal{L}$ and define the join of the family $\{A_\lambda \mid \lambda \in \Lambda\}$ to be $(\vee\{A_\lambda \mid \lambda \in \Lambda\})^*$. We define multiplication on $\mathcal{L}_*$ by $A \circ B = (AB)^*$. Then $\mathcal{L}_*$ is a multiplicative lattice.

Proof. It is immediate that $\mathcal{L}_*$ is a complete lattice. To see that multiplication distributes over joins, observe that $A \circ (\vee\{B_\lambda \mid \lambda \in \Lambda\})^* = (A(\vee\{B_\lambda \mid \lambda \in \Lambda\})^*)^*$ $= (A(\vee\{B_\lambda \mid \lambda \in \Lambda\}))^* = (\vee\{AB_\lambda \mid \lambda \in \Lambda\})^*$. $\square$

Before we continue, we must define a useful property of a star operation.

Definition 4.2.2. A star operation $* : \mathcal{F} \to \mathcal{F}$, where $\mathcal{F}$ is the set of fractional elements of a q.f. lattice $\mathcal{L}$ is said to **distribute over (finite) meets** if $(A \wedge B)^* = A^* \wedge B^*$ for all $A, B \in \mathcal{F}$.

The following generalizes [27, Proposition 2.5] to q. f. lattices. The proof works in the lattice module case as well and we will investigate w-elements of a lattice module soon enough.

Proposition 4.2.3. *Let $\mathcal{L}$ be a q. f. lattice such that $A, B \in \mathcal{L}$. Then $(A \wedge B)_w = A_w \wedge B_w$, i.e. w distributes over (finite) meets.*

Proof. It is clear that $(A \wedge B)_w \leq A_w \wedge B_w$. Let $X \in A_w \wedge B_w$ be principal. Then there exist $J_1, J_2 \in \mathrm{GV}(\mathcal{L})$ such that $J_1 X \leq A$ and $J_2 X \leq B$. Hence $J_1 J_2 X \leq A \wedge B$. We know that $J_1 J_2 \in \mathrm{GV}(\mathcal{L})$ and so $X \in (A \wedge B)_w$. $\square$

Now we are ready to continue working with $\mathcal{L}_w$. We proceed to prove [3, Theorem 3.2] for our case.

Theorem 4.2.4. (cf. [3, Theorem 3.2].) *If $* : \mathcal{F} \to \mathcal{F}$ be a star operation on $\mathcal{F}$ that distributes over (finite) meets. Then $\mathcal{L}_*$ is a modular lattice.*

Proof. Let A, B and $C \in \mathcal{L}_*$ and suppose $A \leq B$. Recall that the join of A and C in $\mathcal{L}_*$ is defined as $(A \vee C)^*$. We have $B \wedge (A \vee C)^* = B^* \wedge (A \vee C)^* = (B \wedge (A \vee C))^* = (A \vee (B \wedge C))^*$, which is the join of A and $(B \wedge C)$ in $\mathcal{L}_*$. $\square$

The following generalizes [3, Theorem 3.3] to $\mathcal{L}_w$.

Theorem 4.2.5. (cf. [3, Theorem 3.3].) *If $* : \mathcal{F} \to \mathcal{F}$ be a star operation on $\mathcal{F}$ that distributes over (finite) meets. Then each principal $A \in \mathcal{L}$ is principal in $\mathcal{L}_*$. Thus $\mathcal{L}_*$ is principally generated.*

Proof. Since $\mathcal{L}_*$ is modular, it suffices by [2, Proposition 1.1] to show that each principal element $A \in \mathcal{L}$ is weak principal in $\mathcal{L}_*$. To see that A is weak meet principal, it suffices by [4, Lemma 1(a)] to show that if $B \in \mathcal{L}_*$ and $B \leq A$, then $B = AQ$ for some $Q \in \mathcal{L}_w$. But if we take $Q = (B : A)$, then $(B : A) = BA^{-1} \wedge R \in \mathcal{L}_*$.

To show each each principal element $A \in \mathcal{L}$ is weak join principal in $\mathcal{L}_*$, it suffices by [4, Lemma 1(d)] to show that $BA = CA$ implies that $B \leq C \vee (0 : A)$. But this is clear since $\mathcal{L}$ is a q. f. lattice, and thus has no nonzero zero-divisors and each principal $A \in \mathcal{L}$ is invertible. □

An important characteristic of a star operation is being of finite character, which we proceed to define for lattices.

Definition 4.2.6. *Let $\mathcal{L}$ be a q. f. lattice and $* : \mathcal{F} \to \mathcal{F}$ be a star operation on $\mathcal{F}$. We say that $*$ has **finite character** if for every $A \in \mathcal{F}$, $A^* = \bigvee\{B^* \mid B \in \mathcal{F}$ is finitely generated and $B \leq A\}$.*

We will show that w has finite character for elements in $\mathcal{F}$, which is a special case of the lattice version of [3, Theorem 2.7].

Theorem 4.2.7. *Let $\mathcal{L}$ be a q. f. lattice. Then w has finite character on $\mathcal{F}$.*

Proof. Let $I \in \mathcal{F}$ and suppose there is a principal element $X \leq I_w$. Then there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX \leq I$. Write $XJ = (XJ_1 \vee \cdots \vee XJ_n)$ since J is finitely generated, thus so is XJ. Now $X \leq (XJ_1 \vee \cdots \vee XJ_n)_w$ so that $X \leq \bigvee\{B_w \mid B \in \mathcal{F}$ is finitely generated and $B \leq I$. The reverse containment always holds and so w has finite character. □

We are now fully equipped to complete our objective for $\mathcal{L}_w$. We proceed to obtain part of [3, Theorem 3.4] for $\mathcal{L}_w$.

Theorem 4.2.8. *The following hold for $\mathcal{L}_w$:*

1. *For each nonzero, principal $X \in \mathcal{L}$, X is compact in $\mathcal{L}_w$.*

2. *$\mathcal{L}_w$ is compactly generated with R compact.*

Proof. Suppose that there is a principal $X \in \mathcal{L}$ such that $X \leq \bigvee A$ for some subset $A = \{A_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{L}_w$. Then $X \leq (\bigvee A_\lambda)_w \leq \bigvee\{B_w \mid B \in \mathcal{F}$ is finitely generated and $B \leq \bigvee A_\lambda\}$ since w has finite character. This means that $X \leq B_w$ for some $B \in \mathcal{F}$. Since B is finitely generated, we may write $B \leq A_{\lambda_1} \vee \cdots \vee A_{\lambda_n}$ for some $\lambda_1, \cdots, \lambda_n \in \Lambda$. Hence $X \leq B_w \leq A_{\lambda_1} \vee \cdots \vee A_{\lambda_n}$ so that X is compact. Finally, since $\mathcal{L}_w$ is principally generated and R is principal, the second portion follows. □

The next theorem follows from our previous results. See [3, Theorem 3.5 and Theorem 3.6].

Theorem 4.2.9. *$\mathcal{L}_w$ is an r-lattice. Further, $\mathcal{L}$ is strong Mori if and only if $\mathcal{L}_w$ is a Noether lattice.*

Proof. Since by Lemma 4.2.7, w has finite character, then by Theorem 4.2.8, each principal element of $\mathcal{L}_w$ is compact in $\mathcal{L}_w$. Thus $\mathcal{L}_w$ is an r-lattice. $\square$

4.3 Strong Mori lattices

Now that we have sufficiently established the w-elements we are ready to begin our investigation of strong Mori lattices.

Let $w\text{Max}(\mathcal{I}) = \{M \in \mathcal{I} \mid M \text{ is a maximal proper } w\text{-element of } \mathcal{I}\}$.

The following lemma generalizes [23, Theorem 3.5].

Lemma 4.3.1. *Let $\mathcal{L}$ be a q. f. lattice. Then $R = \bigwedge\{R_P \mid P \in w\text{Max}(\mathcal{I})\}$ and $I_w = \bigwedge\{IR_P \mid P \in w\text{Max}(\mathcal{I})\}$ for each $I \in \mathcal{L}$.*

Proof. According to Lemma 2.1.5, $R = \bigwedge\{R_P \mid P$ is a maximal prime of a principal element in $\mathcal{I}\}$. Again, we know that any P that is an associated prime of a principal element of $\mathcal{L}$ is a t-prime element, hence a w-element as well. Also, $P \leq M$ for some $M \in w\text{Max}(\mathcal{I})$, which means that $R_M \leq R_P$, proving the first part of the lemma.

To prove the second part, let $I \in \mathcal{I}$ and let $X \leq I_w$ be principal. Then $XJ \leq I$ for some $J \in \text{GV}(\mathcal{L})$. Since $J_w = R$, $J \not\leq P$ for any $P \in w\text{Max}(\mathcal{I})$ so that $X \leq IR_P$ for all $P \in w\text{Max}(\mathcal{I})$.

Conversely, let $X \leq \bigwedge\{IR_P \mid P \in w\text{Max}(\mathcal{I})\}$ be principal. It follows that $(I : X) \not\leq A$ for any $A \in w\text{Max}(\mathcal{I})$ so that $(I : X)_w = R$. Hence there exists a $J \in \text{GV}(\mathcal{L})$ such that $J \leq (I : X)$, meaning $XJ \leq I$ and $X \leq I_w$. $\square$

We will need the following definition to proceed. We define finite type for lattice modules now for later use.

Definition 4.3.2. *Let $\mathcal{M}$ be an $\mathcal{L}$-module such that A is a w-element. We say that A* ***is of finite type*** *if $A = B_w$ for some finitely generated element $B \leq A$. If M is of finite type, then $\mathcal{M}$ is a finite type lattice module.*

The following Theorem generalizes [28, Theorem 1.9] from integral domains to q.f. lattices.

Theorem 4.3.3. *Let $\mathcal{L}$ be a q. f. lattice. Then $\mathcal{L}$ is a strong Mori lattice if and only if $\mathcal{I}(R_P)$ is Noetherian for all $P \in w\mathrm{Max}(\mathcal{I})$ and $\{R_P \mid P \in w\mathrm{Max}(\mathcal{I})\}$ is locally finite.*

Proof. ($\Rightarrow$) Let $A \in \mathcal{I}(R_P)$, $P \in w\mathrm{Max}(\mathcal{I})$. We know $A = IR_P$ for some nonzero $I \in \mathcal{I}$. From the previous lemma, we have $I_w R_P = (\bigwedge\{IR_P \mid P \in w\mathrm{Max}(\mathcal{I})\})R_P = IR_P$. Since $\mathcal{L}$ is a strong Mori lattice, there is a finitely generated $J \leq \mathcal{I}$ such that $I_w = J_w$. Hence $A = IR_P = I_w R_P = J_w R_P = JR_P$, which is finitely generated, so every element of $\mathcal{I}(R_P)$ is finitely generated so that $\mathcal{I}(R_P)$ is Noetherian.

Claim: Every $P \in w\mathrm{Max}(\mathcal{I})$ is divisorial. Assume to the contrary so that $P \neq P_v$. Then $P \leq P_v$ and since P_v is also a w-element, $P \in w\mathrm{Max}(\mathcal{I})$ implies that $P_v = R$. Again, $\mathcal{L}$ is a strong Mori lattice so that $P = J_w$ for a finitely generated $J \in \mathcal{I}$. Hence we have $R = P_v = (J_w)_v = J_v$, meaning $J \in \mathrm{GV}(\mathcal{L})$. Then $P = J_w = R$, a contradiction.

Let X be a principal element of $\mathcal{L}$ and assume that $XR_P \neq R_P$ for infinitely many prime elements $P_1, P_2, \cdots \in w\mathrm{Max}(\mathcal{I})$, meaning $X \leq P$ for those primes. We can construct an increasing chain of v-elements of $\mathcal{L}$: $XP_1^{-1} \leq X(P_1 \wedge P_2)^{-1} \leq X(P_1 \wedge P_2 \wedge P_3)^{-1} \leq \cdots$ We have that $\mathcal{L}$ is a strong Mori lattice and so the chain must stabilize for some positive integer n, i.e. $X(P_1 \wedge \cdots \wedge P_n)^{-1} = X(P_1 \wedge \cdots \wedge P_m)^{-1}$ for all $m \geq n$. So, $P_1 \wedge \cdots \wedge P_m = (P_1 \wedge \cdots \wedge P_m)_v = (P_1 \wedge \cdots \wedge P_n)_v = (P_1 \wedge \cdots \wedge P_n)$, a contradiction. Hence $\{R_P \mid P \in w\mathrm{Max}(\mathcal{I})\}$ is locally finite.

($\Leftarrow$) Let $I \in \mathcal{I}$ be a nonzero w-ideal and let $X \leq I$ be nonzero and principal. Let $P_1, \cdots, P_n$ be the elements of $w\mathrm{Max}(\mathcal{I})$ such that $X \leq P_i$ for $i = 1, \cdots, n$. By assumption, $\mathcal{I}(R_{P_i})$ is Noetherian, so IR_{P_i} is finitely generated. Let $IR_{P_i} = \{X_1, \cdots, X_m\}$ with each X_j principal and $X_j \leq I$ for $j = 1, \cdots, m$. Now, let J be the element of $\mathcal{I}$ generated by X and the X_j's. If $P \in w\mathrm{Max}(\mathcal{I})$, we have $JR_P = IR_P$ and hence $J_w = I_w = I$. So, every w-element is of finite type and $\mathcal{L}$ is a strong Mori lattice. $\square$

The following Theorem generalizes [28, Theorem 3.5] from integral domains to q. f. lattices.

Theorem 4.3.4. *Let $\mathcal{L}$ be a strong Mori lattice. Then $\mathcal{I}(R_f)$ is a Krull lattice.*

Proof. From the previous lemma, we know that $\{R_P \mid P \in w\text{Max}(\mathcal{I})\}$ is locally finite and each $\mathcal{I}(R_P)$ is Noetherian. Thus $R_f = \bigwedge\{(R_P)_f \mid P \in w\text{Max}(\mathcal{I})\}$ by [22, Lemma 6.14] with $\{(R_P)_f \mid P \in w\text{Max}(\mathcal{I})\}$ locally finite. The Mori Nagata Theorem for lattices, Theorem 4.3.3, says that each $(R_P)_b$ is a Krull lattice. But since each $\mathcal{I}(R_P)$ is Noetherian, $(R_P)_b = (R_P)_f$ for each $P \in w\text{Max}(R)$. $\square$

By [23, Example 3.8], the above theorem does not hold for rings if R_f is replaced by R_b.

We state this useful result from [7].

Theorem 4.3.5. [7, Theorem 4.27] *Let $\mathcal{L}$ be a q. f. lattice. Then $\mathcal{I}$ is a Krull lattice if and only if $\mathcal{I}$ is completely integrally closed and $\mathcal{L}$ is a Mori lattice.*

The following corollary generalizes part of [28, Theorem 2.8] from integral domains to q. f. lattices.

Corollary 4.3.6. *Let $\mathcal{L}$ be a q. f. lattice. Then $\mathcal{I}$ is a Krull lattice if and only if $\mathcal{I}$ is integrally closed and $\mathcal{L}$ is a strong Mori lattice.*

Proof. ($\Rightarrow$) Since $\mathcal{I}$ is a Krull lattice, it is completely integrally closed and $\mathcal{L}$ is Mori, i.e. it satisfies the ACC on integral v-elements. Hence we only need show that every integral w-element is also a v-element. Let I be a w-element of $\mathcal{I}$.

Claim: $(II^{-1})_v = R$. It is sufficient to show that $(II^{-1})^{-1} = R$. Obviously, $(II^{-1})(II^{-1})^{-1} \leq R$ and $II^{-1} \leq R$. Since $\mathcal{I}$ is completely integrally closed, $R = R_f$, $(II^{-1})(II^{-1})^{-1} \leq R_f$, and $II^{-1} \leq R_f$. Hence from [22, Proposition 4.1] we have $DII^{-1} \leq D$ for some nonzero $D \in \mathcal{I}$. Since $D(II^{-1})(II^{-1})^{-1} \leq DR = D$ and $D(II^{-1}) \in \mathcal{I}$, we have shown that $(II^{-1})^{-1}$ is f-dependent on R, i.e. $(II^{-1})^{-1} \leq R_f = R$.

We have shown that II^{-1} is a v-element of the Krull lattice $\mathcal{I}$ so that there exists a finitely generated element $J \in \mathcal{I}$ such that $J \leq II^{-1}$ and $J_v = (II^{-1})_v$, so $J \in \mathrm{GV}(\mathcal{L})$. Since $JI_v \leq (II^{-1})I_v = I(I^{-1}I_v) \leq IR = I$, $I_v \leq I_w = I$.

$(\Leftarrow)$ Once we show R is completely integrally closed, we are done. Let $\mathcal{I}$ be integrally closed. Write $R = \bigwedge\{R_P \mid P \in \mathcal{M}\}$. By Theorem 4.3.3, $\mathcal{I}(R_P)$ is Noetherian for each $P \in \mathcal{M}$ and hence $(R_P)_f = (R_P)_b = R_P$ with the last equality following from the fact that $\mathcal{I}$ is integrally closed. Hence each R_P is completely integrally closed, as well as their meet. $\square$

4.4 w-elements of a lattice module

We now return to our discussion of lattice modules. The main goal of this portion is to generalize results from [27] and [28].

For convenience, we restate the definition of a w-element H where H is an element of a lattice module $\mathcal{M}$ rather than an element of a q. f. lattice. As above, $\mathcal{S}$ denotes the multiplicative subset of nonzero principal elements of $\mathcal{L}$ and M is the largest element of $\mathcal{M}$.

Definition 4.4.1. Let $\mathcal{M}$ be an $\mathcal{L}$-module and let H be a torsion-free element of $\mathcal{M}$. Let $H_w = \bigvee\{X \leq M_{\mathcal{S}} \mid X \text{ is } \mathcal{L}\text{-principal and } JX \leq H \text{ for some } J \in \mathrm{GV}(\mathcal{L})\}$. If $H = H_w$, H is a w**-element** of $\mathcal{M}$. If M is a w-element, we say that $\mathcal{M}$ is a w**-$\mathcal{L}$-module**.

Note that it is easy to see from the definition that for a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of w-elements of $\mathcal{M}$, $\bigwedge_{\lambda\in\Lambda} A_\lambda$ is also a w-element of $\mathcal{M}$. It follows as in Proposition 4.2.3 that the w-operation distributes over finite meets of elements in $\mathcal{M}$.

In [27], H_w was defined as the **w-envelope of H**. The next theorem generalizes [27, Theorem 2.1] to lattice modules.

Theorem 4.4.2. *Let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module. Then*

1. *M_w is a w-element of $\mathcal{M}$*

2. *If F is a w-element of $\mathcal{M}$ with $M \leq F$, then $M_w \leq F$.*

Proof. (1) Assume that $JX \leq M_w$ for some $J \in \mathrm{GV}(\mathcal{L})$ and principal $X \leq (M_w)_\mathcal{S}$. Write $J = I_1 \vee \cdots \vee I_n$ for principal elements $I_1, \cdots, I_n \in \mathcal{I}$ and so $I_i X \leq M_w$ for $i = 1, \cdots, n$. Hence there exist $J_1, \cdots, J_n \in \mathrm{GV}(\mathcal{L})$ such that $J_i I_i X \leq M$ for $i = 1, \cdots, n$. It follows that $(J_1 \cdots J_n J)X \leq M$, but $(J_1 \cdots J_n J) \in \mathrm{GV}(\mathcal{L})$ so that $X \in M_w$.

(2) If there is a principal $X \leq M_w$, then $X \leq M_\mathcal{S} \leq F_\mathcal{S}$ and $JX \leq M \leq F$ for some $J \in \mathrm{GV}(\mathcal{L})$. Since F is a w-element, $X \leq F$. $\square$

The following generalizes [27, Lemma 1.2] to lattice modules.

Lemma 4.4.3. *Let A be an element of a torsion-free $\mathcal{L}$-module $\mathcal{M}$. If A is a w-element, then $I = (A : M)$ is a w-element of $\mathcal{I}$.*

Proof. Let $JX \leq I$ for some $J \in \mathrm{GV}(\mathcal{L})$ and a principal element $X \in \mathcal{L}$. Then $JXM \leq A$, meaning that $XM \leq (XM)_\mathcal{S} = (JXM)_\mathcal{S} \leq A_\mathcal{S}$, and hence $XM \leq A$. Furthermore, $JX \leq I \leq R$, so $X \leq J^{-1} = R$, so that $X \leq I$. $\square$

The following generalizes [27, Lemma 1.3] to lattice modules.

Theorem 4.4.4. *Let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module and let $A \in \mathcal{M}$. If A is a w-element, then $J \not\leq (A : M)$ for any $J \in \mathrm{GV}(\mathcal{L})$.*

Proof. If there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $J \leq (A : M)$, then $JM \leq A$, implying that $M \leq A$ so that $M = A$ contrary to hypothesis. □

The following generalizes [27, Proposition 1.5] to lattice modules.

Proposition 4.4.5. *Let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module. If M_P is a w-element of the $\mathcal{I}(R_P)$-module $\mathcal{I}(M_P)$ for all maximal elements $P \in \mathcal{I}$, then M is a w-element.*

Proof. Suppose $JX \leq M$ for some $J \in \mathrm{GV}(\mathcal{L})$ and a principal element $X \leq M_{\mathcal{S}}$. Let $I = (M :_{\mathcal{I}} X)$. Assume that $I < R$. Then $I \leq P$ for some P maximal in $\mathcal{I}$, so that $I_P < R_P$ in $\mathcal{I}(R_P)$. It follows that $X \not\leq M_P$ in $\mathcal{I}(M_P)$. However, we know that $J_P \leq \mathrm{GV}(\mathcal{I}(R_P))$ and $J_P X \leq M_P$ in $\mathcal{I}(R_P)$, a contradiction. So, $I \not< R$ and hence $X \leq M$. □

The following generalizes [27, Lemma 1.6] to lattice modules.

Proposition 4.4.6. *For a q. f. $\mathcal{L}$, let I be a w-element of $\mathcal{I}$. Then $\sqrt{I}$ is also a w-element of $\mathcal{I}$.*

Proof. Suppose that $JX \leq \sqrt{I}$ for some $J \in \mathrm{GV}(\mathcal{L})$ and a principal element $X \in \mathcal{L}$. Hence there is a positive integer n such that $(JX)^n \leq I$ and $X \in \mathcal{I}$ as in the proof of 4.4.3. Since a product of elements in $\mathrm{GV}(\mathcal{L})$ is also in $\mathrm{GV}(\mathcal{L})$ and $(JX)^n = J^n X^n \leq I$, we have $X^n \leq I$ from the assumption that I is a w-element, hence $X \in \sqrt{I}$. □

The following generalizes [27, Proposition 2.3] from integral domains to q. f. lattices.

Proposition 4.4.7. *Let $\mathcal{L}$ be a q. f. and let $I \in \mathcal{I}$. Then $I_w = R$ iff there exists a $J \in \mathrm{GV}(\mathcal{L})$ such that $J \leq I$.*

Proof. If $I_w = R$, then there is an $J \in \mathrm{GV}(\mathcal{L})$ such that $JR \leq I$, meaning $J \leq I$. For the converse, if $J \leq I$, then $JR \leq I$ so that $R \leq I$. Hence $I = R$ and $I_w = R$. □

The following generalizes [27, Proposition 2.4] from integral domains to q. f. lattices.

Proposition 4.4.8. *Let $\mathcal{L}$ be a q. f. lattice with $I \in \mathcal{I}$. Then $(\sqrt{I})_w = \sqrt{I_w}$*

Proof. Since $I \leq I_w$, $\sqrt{I} \leq \sqrt{I_w}$ so that $(\sqrt{I})_w \leq (\sqrt{I_w})_w = \sqrt{I_w}$ because $\sqrt{I_w}$ is a w-element by the previous proposition. For the reverse inclusion, let $X \in \sqrt{I_w}$ be principal. Hence there is a positive integer n and a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX^n \leq I$. As in the previous proposition, $(JX)^n = J^n X^n \leq I$, meaning $X \in (\sqrt{I})_w$. □

The following generalizes [27, Proposition 2.7] to lattice modules.

Proposition 4.4.9. *Let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module and let $J \in \mathrm{GV}(\mathcal{L})$. Then $(JM)_w = M_w$.*

Proof. Let $X \leq (JM)_w$ be principal. Then there is a $J_1 \in \mathrm{GV}(\mathcal{L})$ such that $J_1 X \leq JM \leq M$, meaning $X \in M_w$.

Conversely, if $X \in M_w$, then $J_1 X \leq M$ for some $J_1 \in \mathrm{GV}(\mathcal{L})$. Hence $JJ_1X \leq JM$, of course implying that $X \leq (JM)_w$. □

The following generalizes [27, Proposition 2.8] to lattice modules.

Proposition 4.4.10. *Let $I \in \mathcal{I}$ and let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module. Then $(IM)_w = (I_w M_w)_w$.*

Proof. The $\leq$ direction is clear. Let $X \leq (I_w M_w)_w$. Then there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX \leq I_w M_w \leq M_w$. It follows that there is a $J_1 \in \mathrm{GV}(\mathcal{L})$ such that $J_1 J X \leq M$. Similarly, there is a $J_2 \in \mathrm{GV}(\mathcal{L})$ such that $J_2 J X \leq I$. Hence $J_2 J_1 J^2 X^2 \leq J_2 J_1 J^2 X \leq IM$. Since a product of elements in $\mathrm{GV}(\mathcal{L})$ is in $\mathrm{GV}(\mathcal{L})$, $X \leq (IM)_w$. □

4.5 Prime and primary w-elements

We turn our attention once again to primary elements, but first we define a primary element of lattice module, which is of course similar the the definition of a primary element of a cl-monoid.

Definition 4.5.1. Let $\mathcal{M}$ be an $\mathcal{L}$-module. An element $Q \in \mathcal{M}$ with $Q < M$ is **primary** if for any elements $A \in \mathcal{I}$, $B \in \mathcal{M}$, $AB \leq Q$ implies that $B \leq Q$ or $A^k M \leq Q$ for some positive integer k. In this case, $P(Q) = \vee\{C \in \mathcal{M} \mid C^k M \leq Q\}$ is the **prime associated with** Q and Q is $P(Q)$-primary.

The following generalizes [27, Theorem 3.1] to lattice modules.

Theorem 4.5.2. *Let $\mathcal{M}$ be a w-$\mathcal{L}$-module and let P be a prime w-element of $\mathcal{I}$ such that $A \in \mathcal{M}$ is P-primary. Then A is a w-element of $\mathcal{M}$.*

Proof. Let $JX \leq A$ for some $J \in \mathrm{GV}(\mathcal{L})$ and a principal element $X \leq A_{\mathcal{S}}$ in $\mathcal{I}(M_{\mathcal{S}})$. Since M is a w-element and $A \leq M$, we know that $X \leq M$. By Theorem 4.4.4, $J \not\leq P$ since P is a w-element, meaning $X \leq A$. □

The following generalizes [27, Proposition 3.2] to lattice modules.

Proposition 4.5.3. *Let $\mathcal{M}$ be a w-$\mathcal{L}$-module such that $A \in \mathcal{M}$ is P-primary for a prime element $P \in \mathcal{I}$. If A_P is a w-element of the $\mathcal{I}(R_P)$-module $\mathcal{I}(M_P)$, then A is a w-element.*

Proof. Let $JX \leq A$ for some $J \in \mathrm{GV}(\mathcal{L})$ and assume without loss of generality that X is a principal element of $\mathcal{M}$. We have $J_P X = (JX)_P \leq A_P$. thus $X \leq A_P \wedge M = A$. □

The next proposition is a variation of 4.3.1.

Proposition 4.5.4. *Let $\mathcal{M}$ be a w-$\mathcal{L}$-module. Then $M = \bigwedge\{M_P \mid P \in w\mathrm{Max}(\mathcal{I})\}$.*

Proof. The $\leq$ direction is clear. Let $X \in \bigwedge\{M_P \mid P \in w\mathrm{Max}(\mathcal{I})\}$ be principal. Note that $I = \vee\{B \in \mathcal{I} \mid BX \in \mathcal{M}\}$ is a w-element of $\mathcal{I}$. Since I is not contained in a maximal w-element of $\mathcal{I}$, we have $I = R$ which means $X \in \mathcal{M}$. □

It is clear from the definition of a w-element that if $\mathcal{M}$ is a torsion-free $\mathcal{L}$-module and X is a principal element of $\mathcal{I}$, then if M and $N \leq M$ are w-elements, then $(N :_{\mathcal{M}} X)$ is also a w-element.

The next lemma is a generalization of a simple, well-known fact and is listed for convenience [27, Lemma 3.5].

Lemma 4.5.5. 1. *Let $P \leq R$ in $\mathcal{I}$. Then P is prime iff $(P :_{\mathcal{I}} X) = P$ for every principal $X \in \mathcal{I}$, $X \nleq P$.*

2. *Let $\mathcal{M}$ be an $\mathcal{L}$-module with $N \leq M$ and $P = (N : M)$. Then N is prime iff $(N :_{\mathcal{M}} X) = N$ for every principal $X \in \mathcal{I}$, $X \nleq P$.*

Proof. The "if" portion of (1) is obvious. For the converse, let $A, B \in \mathcal{I}$ such that $AB \leq P$. Assume that $B \nleq P$. Then $AB = P$ by hypothesis so that $B \leq P$. The proof of 2 is similar. □

The following generalizes [27, Proposition 3.6] to lattice modules.

Proposition 4.5.6. *Let $\mathcal{M}$ be a w-$\mathcal{L}$-module and let P be a prime element of $\mathcal{I}$. If N is a w-element of $\mathcal{M}$ which is maximal among the w-elements B of $\mathcal{M}$ with $(B : M) = P$, then N is prime.*

Proof. For any $A \in \mathcal{I}$ with $A \nleq P$, we have $((N :_{\mathcal{M}} A) : M) = P$ since P is prime. Now, $N \leq (N :_{\mathcal{M}} A)$, the maximality of N gives us $N = (N :_{\mathcal{M}} A)$. Applying the previous lemma completes the proof. $\square$

The following is a variation of Lemma 4.4.4.

Theorem 4.5.7. *Let $\mathcal{M}$ be a w-$\mathcal{L}$-module with N a primary element of $\mathcal{M}$. Then N is a w-element iff $J \nleq (N : M)$ for any $J \in \mathrm{GV}(\mathcal{L})$.*

Proof. Suppose that $JX \leq N$ for some $J \in \mathrm{GV}(\mathcal{L})$ and a principal element $X \in \mathcal{M}$. If we assume $J \leq \sqrt{(N : M)}$ then there is a positive integer k such that $J^k \leq (N : M)$ because J is finitely generated, but this contradicts our hypothesis. Then, we have $X \leq N$ because N is primary. $\square$

We will generalize [27, Theorem 4.2] to lattice modules by making use of the following lemma.

Lemma 4.5.8. *Let $\mathcal{L}$ be a multiplicative lattice and $\mathcal{M}$ an $\mathcal{L}$-module. If for some $N \in \mathcal{M}$ and a principal element $I \in \mathcal{I}$, both $N \vee IM$ and $(N :_{\mathcal{M}} I)$ are finitely generated, then N is finitely generated.*

Proof. Note that $I(N :_{\mathcal{M}} I) = N \wedge IM$. Hence if $(N :_{\mathcal{M}} I)$ is finitely generated, so is $N \wedge IM$.

Since $[IM, N \vee IM] \cong [N \wedge IM, N]$ and $N \vee IM$ is finitely generated, then $[N \wedge IM, N]$ is finitely generated with $N \wedge IM$ also finitely generated. Hence N is finitely generated. $\square$

We will prove a lattice module version of Cohen's Theorem using the above lemma.

Theorem 4.5.9 (Cohen's Theorem for Lattice Modules)**.** *Let $\mathcal{L}$ be a multiplicative lattice and $\mathcal{M}$ an $\mathcal{L}$-module and let $N \in \mathcal{M}$, $N < M$ be maximal among all elements of $\mathcal{M}$ which are not finitely generated. Then N is a prime element of $\mathcal{M}$.*

Proof. Suppose that N is not prime. Then there exits principal elements $I \in \mathcal{I}$ and $A \in \mathcal{M}$ such that $IA \leq N$ but $A \not\leq N$ and $IM \not\leq N$. Then $N \not\leq (N :_{\mathcal{M}} I)$ and $N \not\leq (N \vee IM)$. By the choice of N, $(N :_{\mathcal{M}} I)$ and $(N \vee IM)$ are finitely generated. Thus N is finitely generated by the above lemma, contrary to hypothesis. □

The lattice version of [27, Theorem 4.2] now follows as a special case from the previous theorem and the fact that $\mathcal{L}_w$ is an r-lattice.

Theorem 4.5.10. *Let $\mathcal{L}$ be a multiplicative lattice. If $P \in \mathcal{I}$ such that P is maximal in the set of all non-finite type w-elements of $\mathcal{L}$, then P is prime.*

The following generalizes [27, Theorem 4.3] from integral domains to q. f. lattices.

Theorem 4.5.11. *For a q. f. lattice $\mathcal{L}$, TFAE.*

1. *$\mathcal{L}$ is a strong Mori lattice.*
2. *Each w-element of $\mathcal{L}$ is of finite type.*
3. *Each prime w-element of $\mathcal{L}$ is of finite type.*

Proof. (1) $\Rightarrow$ (2) Let $A \in \mathcal{L}$ be a w-element and define $\mathcal{A}$ be the set of all w-elements of $\mathcal{L}$ that are w-finite with $A_i \leq A$ for every $A_i \in \mathcal{A}$. Since $\mathcal{A}$ contains

0, it is non-empty and hence contains a maximal element B since $\mathcal{L}$ satisfies ACC on w-elements. Hence $B = C_w$ for a finitely generated $C = C_1 \vee \cdots \vee C_n \in \mathcal{L}$ where each C_i for $i = 1, \cdots, n$ is principal and $C \leq B = C_w \leq A$. For each principal element $a \leq A$, define $D_a = a \vee C_1 \vee \cdots \vee C_n$. Note that $(D_a)_w \in \mathcal{A}$ with $C_w \leq (D_a)_w$. Since $B = C_w$ is maximal in $\mathcal{A}$, $C_w = (D_a)_w$ for every principal $a \leq A$ so that $a \leq D_a \leq (D_a)_w = C_w$ and $A \leq C_w$ (since $\mathcal{L}$ is principally generated). This means that $A = C_w$.

(2) $\Rightarrow$ (1) Let $A_1 \leq A_2 \leq \cdots$ be a chain of w-elments in $\mathcal{L}$, which are of finite-type by hypothesis. We know $\bigvee_{i \geq a} A_i \in \mathcal{L}$ is also a w-element, hence of finite-type. Let $\bigvee_{i \geq a} A_i = B_w$ where $B = B_1 \vee \cdots \vee B_n \leq \bigvee_{i \geq a} A_i$ where each B_j for $j = 1, \cdots, n$ is a principal element of $\mathcal{L}$. Each B_j must be an element of some A_k and so there is an index m such that $B_j \leq A_m$ for $i = 1, \cdots, n$. Hence $B_w \leq \bigvee_{i \geq a} A_i \leq A_m$ so that $A_i = A_m$ for $i \geq m$.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) If the collection of all non-finite type w-elements of $\mathcal{L}$ is non-empty, it contains a maximal element by Zorn's Lemma. This maximal element must be prime by Theorem 4.5.10 which is of finite type by our hypothesis, a contradiction.

□

The following generalizes [27, Proposition 4.7] from integral domains to q. f. lattices.

Proposition 4.5.12. *For a strong Mori lattice $\mathcal{L}$, $\mathcal{I}(R_{\mathcal{S}})$ is a strong Mori lattice for every localization $R_{\mathcal{S}}$.*

Proof. By [7, Proposition 2.13], we have $A \wedge R = \vee\{(A : S) \wedge R \mid S \in \mathcal{S}\}$ and $(A \wedge R)_{\mathcal{S}} = A$ for any $A \in \mathcal{I}(R_{\mathcal{S}})$ and any $S \in \mathcal{S}$. Note that $(A \wedge R)_w = A_w \wedge R_w = A_w \wedge R$ by Proposition 4.2.3. Hence, the above equalities become $A_w \wedge R = \vee\{(A_w : S) \wedge R \mid S \in \mathcal{S}\}$ and $(A_w \wedge R)_{\mathcal{S}} = A_w$ for any $A \in \mathcal{I}(R_{\mathcal{S}})$ and $S \in \mathcal{S}$. So, if A is a w-element, by taking its meet with R, we may contract it to a w-element of $\mathcal{I}$. Hence any strictly increasing chain of w-elements of $\mathcal{I}(R_{\mathcal{S}})$ contracts to a strictly increasing chain of w-elements of $\mathcal{I}$ and this chain must stabilize, so much the chain in $\mathcal{I}(R_{\mathcal{S}})$ so that $\mathcal{I}(R_{\mathcal{S}})$ is a strong Mori lattice. $\square$

Next we generalize [27, Theorem 4.8] and [27, Theorem 4.9] from strong Mori domains to strong Mori lattices. For this, let $\mathcal{L}$ be a q. f. lattice with $I \in \mathcal{I}$. We define a lattice version of Kaplansky's $\mathscr{Z}(I)$, adopting his notation [15]. Let $\mathscr{Z}(I)$ denote the set of all principal elements $A \in \mathcal{I}$ such that $AB \leq I$ for some principal element $B \in \mathcal{I}$ with $B \not\leq I$. We will let $\mathrm{Max}_{\mathcal{I}}\mathscr{Z}(I)$ represent the set of elements of $\mathcal{I}$ that are maximal among elements contained in $\mathscr{Z}(I)$.

To obtain these two results, we use the following lattice module version of Herstein's result ([15, Theorem 6]).

Lemma 4.5.13. *Let $\mathcal{L}$ be a multiplicative lattice and $\mathcal{M}$ an $\mathcal{L}$-module with maximal element M. Let $I \in \mathcal{I}$ be maximal among elements in $\mathcal{I}$ of the form $(0 :_{\mathcal{L}} X)$ for nonzero principal elements $X \leq A$. Then I is prime.*

Proof. Assume that $CD \leq I$ and that $C \not\leq I$. Then $CX \neq 0$. Then $D \leq (0 :_{\mathcal{L}} CX) \geq (0 :_{\mathcal{L}} X)$. By hypothesis, $(0 :_{\mathcal{L}} CX) = (0 :_{\mathcal{L}} X)$. Thus $D \leq I = (0 :_{\mathcal{L}} X)$. So I is prime. $\square$

The following theorem generalizes [27, Theorem 4.8] from strong Mori domains to strong Mori lattices. For this we use the following definition from [13, page 106].

Definition 4.5.14. A multiplicative lattice $\mathcal{L}$ is said to **satisfy the union condition on primes** if given prime elements $P_1, \ldots, P_k \in \mathcal{L}$ and an element $A \in \mathcal{L}$ such that for each $i = 1, \ldots, k$, $A \nleq P_i$, then there exists a principal element $E \leq A$ such that $E \nleq P_i$ for $i = 1, \ldots, k$.

It is known that the lattice of ideals in a semigroup and the lattice of graded ideals of a graded ring generally do not satisfy the union condition on primes. See [5, pp. 140-141] and [24, Section 7] respectively.

Theorem 4.5.15. *Let $\mathcal{L}$ be a strong Mori lattice which satisfies the union condition on primes and let $I \in \mathcal{I}$ be a w-element. Then every element of* $\mathrm{Max}_{\mathcal{I}}\mathcal{Z}(I)$ *is of the form $(I :_{\mathcal{I}} X)$ for a principal element $X \in \mathcal{I}$, $X \nleq I$.*

Proof. Let $\Gamma = \{(I :_{\mathcal{I}} X) \mid X \in \mathcal{I}, X \nleq I\}$. Since $\mathcal{L}$ is a strong Mori lattice, every member of Γ is contained in a maximal element of Γ. Index these maximal elements by the set Λ and let $P_\lambda = (I :_{\mathcal{I}} X_\lambda)$ for each $\lambda \in \Lambda$. Then each P_λ is a prime w-element of $\mathcal{L}$.

Let $A = \bigvee\{X_\lambda \mid \lambda \in \Lambda\} \in \mathcal{I}$. Since $\mathcal{L}$ is a strong Mori lattice, there is a finite subset $\Lambda' = \{\lambda_1, \ldots, \lambda_n\}$ of Λ such that $A_w = B_w$, where $B = X_{\lambda_1} \vee \cdots \vee X_{\lambda_n}$. Let $\alpha \in \Lambda \setminus \Lambda'$.

We claim that $P_{\lambda_1} \wedge \ldots \wedge P_{\lambda_n} \leq P_\alpha$. That is $(I : X_{\lambda_1}) \wedge \ldots \wedge (I : X_{\lambda_n}) \leq (I : X_\alpha)$.

Let $h \in (I : X_{\lambda_1}) \wedge \ldots \wedge (I : X_{\lambda_n})$. We have $X_\alpha \leq A_w = B_w = (X_{\lambda_1} \vee \cdots \vee X_{\lambda_n})_w$. So there is a $J \in \mathrm{GV}(\mathcal{L})$ such that $JX_\alpha \leq B$.

Also, $JhX_{\lambda_i} \leq I$ for $i = 1, \ldots, k$. But I is a w-element. So $hX_{\lambda_i} \leq I$ for $i = 1, \ldots, n$. So $h \in (I : X_\alpha) = P_\alpha$, proving the claim. Thus $P_{\lambda_i} \leq P_\alpha$ for some $i = 1, \ldots, n$. But these primes $P_\lambda = (I : X_\lambda)$ are all maximal among ideals of that form. So $P_\alpha = P_{\lambda_i}$ for some i.

It remains to see that if $H \in \mathrm{Max}_{\mathcal{I}}\mathcal{Z}(I)$, then $H \leq P_{\lambda_i}$ for some $i = 1, \ldots, n$. But if $H \nleq P_i$ for each $i = 1, \ldots, n$, then by the union condition on primes, there exists a principal $E \leq H$ with $E \nleq P_i$ for each $i = 1, \ldots, n$. But this is a contradiction since we have shown above that for each principal zero-divisor A on $[I, R]$, we have $A \leq P_{\lambda_i}$ for some $i = 1, \ldots, n$. $\square$

We will need the lattice version of [9, Proposition 1.1(5)] in order to prove [27, Theorem 4.9]. We prove it for the case of the w-operation, though it certainly holds for star operations in general satisfying the required characteristics.

Lemma 4.5.16. *Let $\mathcal{L}$ be a q. f. lattice. Then if P is a prime element minimal over a w-element $I \in \mathcal{I}$ of finite type, then P is a w-element.*

Proof. Let J be a finitely generated element of $\mathcal{I}$ such that $J \leq P$. We only need to show that $J_w \leq P$. Since P is minimal over I, $PR_P = \text{Rad}(IR_P)$ in $\mathcal{I}(R_P)$ which means there is a positive integer n such that $J^n R_P \leq IR_P$. Thus there is a principal element $S \in \mathcal{I}$, $S \not\leq P$ such that $SJ^n \leq I$. It follows from Proposition 4.4.10 that

$$S(J_w)^n \leq S((J_w)^n)_w = S(J^n)_w = (SJ^n)_w \leq I_w = I \leq P.$$

Since $S \not\leq P$, we have $J_w \leq P$. □

The following generalizes [27, Theorem 4.9] from strong Mori domains to strong Mori lattices.

Theorem 4.5.17. *Let $\mathcal{L}$ be a strong Mori lattice and let $I \in \mathcal{I}$ be a w-element. Then there are only finitely many prime elements in $\mathcal{I}$ minimal to I and every minimal prime element $C \leq I$ is of the form $(I :_{\mathcal{I}} X)$ for some principal element $X \in \mathcal{I}$, $X \not\leq I$.*

Proof. Let P be a prime element of $\mathcal{I}$ minimal over I. Then P_P is the unique prime element of $\mathcal{I}(R_P)$ that contains I_P (see [7]) and P is a w-element of $\mathcal{I}$ by the previous lemma. Hence P is of finite type and there is a finitely generated element

$N \in \mathcal{I}$ such that $N \leq P$ and $P = N_w$. Select a $Q \in w\text{Max}(\mathcal{I})$ with $P \leq Q$. Then, $\mathcal{I}(R_Q)$ is Noetherian by Theorem 4.3.3 as well as $\mathcal{I}(R_P)$. Now $P_P = (I_P :_{\mathcal{I}(R_P)} X)$ for some principal element $X \in \mathcal{I}(P_P)$, $X \nleq I_P$ so that $XP_P \leq I_P$ and since N is finitely generated, there exists a principal element $S \in \mathcal{I}$, $S \nleq P$ such that $SX \in \mathcal{I}$ and $SXN \leq I$. Hence $SXP = SXN_w = (SXN)_w \leq I_w = I$, meaning $P = (I :_{\mathcal{I}} SX)$ with $SX \in \mathcal{I}$, $SX \nleq I$.

If there are infinitely many minimal prime elements to I, let $\mathcal{S}$ be the set of w-elements of $\mathcal{I}$ that have infinitely minimal primes and $I \leq S'$ for each $S' \in \mathcal{S}$. Since $\mathcal{S}$ is nonempty and $\mathcal{L}$ is a strong Mori lattice, we may choose a maximal element of $\mathcal{S}$, call it T. We know that T is not prime, but $T = \sqrt{T}$ by Proposition 4.4.6. There exist principal elements $A, B \in \mathcal{I}$, $A, B \nleq T$ such that $AB \in T$. Then $(T \vee A)_w$ and $(T \vee B)_w$ have only finitely minimal primes. We will show $T = \sqrt{(T \vee A)_w} \wedge \sqrt{(T \vee B)_w}$ to reach a contradiction. We only need to verify "$\geq$."

Let $X \leq \sqrt{(T \vee A)_w} \wedge \sqrt{(T \vee B)_w}$ be principal. Then there exists a positive integer k such that $X^k \leq (T \vee A)_w \wedge (T \vee B)_w$. Thus we have $JX^k \leq (T \vee A) \wedge (T \vee B)$ for some $J \in \text{GV}(\mathcal{I})$. For each principal $D \leq J$, write $DX^k = U \vee I_1 A = V \vee I_2 B$ for principal elements $U, V \leq T$ and principal elements $I_1, I_2 \in \mathcal{I}$. Then $VB \vee I_2 B^2 = UB \vee I_1 AB \leq T$, so that $I_2 B^2 \leq T$. Since T is a radical element, this shows that $I_2 B \leq T$, whence it follows that $JX^k \leq T$. Again, since T is a radical element, as well as a w-element, it follows that $X \leq T$ and the proof is complete. $\square$

Chapter 5

Future work: w-invertibility and strong Mori lattice characterization

The rest of section 4 in [27] deals with primary decomposition. Since primary decomposition for elements of a multiplicative lattice or lattice module has already been studied in substantial detail in [8], [26], and elsewhere, we proceed on to consider w-invertibility of lattice elements.

5.1 Basic results

We will begin by proving basic properties of w-invertibility for quotient field lattices. The ultimate goal of this chapter is to obtain a portion of the lattice version of [28, Theorem 2.8], which we already have a small piece of.

The following is a lattice version of w-invertibility. See [27, Definition 6].

Definition 5.1.1. Let $\mathcal{L}$ a q. f. lattice with $A \in \mathcal{F}$. We say that A is **w-invertible** if $(AB)_w = R$ for some $B \in \mathcal{F}$.

The following generalizes [27, Proposition 5.1] from integral domains to q. f. lattices.

Proposition 5.1.2. *For a q. f. lattice $\mathcal{L}$ and an element $A \in \mathcal{F}$, TFAE:*

1. *A is w-invertible.*

2. *$(AA^{-1})_w = R$*

3. *$J \leq AA^{-1}$ for some $J \in \mathrm{GV}(\mathcal{L})$.*

4. *A_w is w-invertible.*

Proof. (1) $\Rightarrow$ (2) If A is w-invertible, there is a $B \in \mathcal{F}$ such that $AB \leq (AB)_w = R$, implying that $B \leq A^{-1}$ so that $R = (AB)_w \leq (AA^{-1})_w \leq R_w = R$.

(2) $\Rightarrow$ (3) If $(AA^{-1})_w = R$, then $JR \leq AA^{-1}$ for some $J \in \mathrm{GV}(\mathcal{L})$ and $J \leq AA^{-1}$.

(3) $\Rightarrow$ (4) If $J \leq AA^{-1} \leq A_wA^{-1} \leq R$ for some $J \in \mathrm{GV}(\mathcal{L})$, then $(A_wA^{-1})_w = R$ by Proposition 4.4.7 and hence A_w is w-invertible.

(4) $\Rightarrow$ (1) If A_w is w-invertible, there is a $B \in \mathcal{F}$ such that $R = (A_wB)_w = (A_wB_w)_w$ from Proposition 4.4.10. $\square$

If we define v-invertibility in a similar manner to Definition 5.1.1, the proof of Proposition 5.1.2 works by replacing "w" with "v." Combining this with the definition of the v-operation, it can be reduced to the following:

Definition 5.1.3. [7, Definition 4.1] Let $\mathcal{L}$ be a q. f. lattice such that $A \in \mathcal{F}$ with $A \neq 0$. We define A to be v**-invertible** if $(AA^{-1})^{-1} = R$.

It is known that $I \subseteq I^* \subseteq I_v$ for a fractional ideal I in an integral domain, R and a star operation on the fractional ideals of R. We also have $I \leq I^* \leq I_v$ in a q. f. lattice for any $I \in \mathcal{F}$ and a star operation on $\mathcal{F}$, which implies that any w-invertible element is v-invertible. We will use this fact later.

The following generalizes [27, Proposition 5.2] from integral domains to q. f. lattices.

Proposition 5.1.4. *Let $\mathcal{L}$ be a q. f. lattice with $A, B \in \mathcal{F}$. Then A and B are w-invertible iff AB is.*

Proof. ($\Rightarrow$) By hypothesis, there are $C, D \in \mathcal{F}$ such that $(AC)_w = R$ and $(BD)_w = R$. Obviously, $CD \in \mathcal{F}$ and $(ABCD)_w = (ACBD)_w = ((AC)_w(BD)_w)_w = R_w = R$ by Proposition 4.4.10.

($\Leftarrow$) By hypothesis, there is an $E \in \mathcal{F}$ such that $(ABE)_w = R$. Then $R = (A(BE))_w = (B(AE))_w$. So A and B are w-invertible. $\square$

If $*$ is a star operation and $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family in $\mathcal{F}$ such that $\bigwedge_{\lambda\in\Lambda}(A_\lambda)^* \neq 0$, then it is easy to see that $\bigwedge_{\lambda\in\Lambda}(A_\lambda)^* = (\bigwedge((A_\lambda)^*)^*$. Now the v-operation is defined by $A \mapsto A_v = \bigwedge\{a \mid a \text{ is } \mathcal{L}\text{-principal and } A \leq a\}$, and if $A \leq a$ where a is principal, then $A^* \leq a^* = a$, and thus $A^* \leq \bigwedge\{a \mid a \text{ is } \mathcal{L}\text{-principal and } A \leq a\} = A_v$. That is, the v-operation is the largest star operation on $\mathcal{F}$, where we define $*_1 \leq *_2$ for star operations $*_1$, $*_2$ to mean $A^{*_1} \leq A^{*_2}$ for each $A \in \mathcal{F}$. Also since $a \in (R :_{\mathcal{L}} A) = A^{-1}$ if and only if $A \leq a^{-1}$, it follows that $A^{-1} = (A^{-1})_v = (A^*)^{-1}$.

Now suppose $R = (AB)^*$. Then multiplying by A^{-1} and applying the star operation gives $(A^{-1}R)^* = (A^{-1}(AB))^*$. So $A^{-1} = (A^{-1})^* = ((AB)^*A^{-1})^* = ((AB)A^{-1})^* = (AA^{-1}B)^* \leq B^*$. Also, from $AB^* \leq (AB^*)^* = (AB)^* = R$, we get opposite inequality: $B^* \leq A^{-1}$. So, we get that if $(AB)^* = R$, then $B^* = A^{-1}$. Since if A is $*$-invertible, so is A^{-1}, then $A^* = A_v$. Since $R = (AA^{-1})^* \leq (AA^{-1})_v \leq R$, we get $*$-invertible implies v-invertible. In fact if $\star_1 \leq \star_2$ are star operations and if A is $*_1$-invertible, then A is $*_2$-invertible.

The following generalizes [27, Proposition 5.3] from integral domains to q. f. lattices, preserving a useful property of w-invertible elements of q. f. lattices.

Proposition 5.1.5. *Let $\mathcal{L}$ be a q. f. lattice with $A \in \mathcal{F}$ w-invertible. Then A_w is of finite type.*

Proof. Let $A \in \mathcal{F}$ be w-invertible. Since w is of finite character by Theorem 4.2.7, $R \leq (AA^{-1})_w = \vee\{F_w \mid \text{where } F \text{ is finitely generated and } F \subseteq AA^{-1}\}$. That is $R \leq F_w$ for some element F_w of finite type with $F \leq AA^{-1}$. So there are finitely generated elements $G \leq A$ and $H \leq A^{-1}$ such that $G_w = A_w$ and $H_w = A^{-1}$. In particular, $A_w = G_w$ is of finite type. □

5.2 Future objectives

With basic results concerning w-invertibility in hand, we have obtained a crucial set of information needed to obtain lattice versions of [27, Theorem 5.4] and [28, Theorem 2.8]. Unfortunately, the other required part has not yet been obtained by the author. The purpose of this section is to inform the reader of immediate goals concerning further research.

First we obtain a few more results concerning w-elements of a lattice module, beginning by generalizing [28, Lemma 1.5] from integral domains to q. f. lattices.

Lemma 5.2.1. *Let $\mathcal{M}$ be a torsion-free $\mathcal{L}$-module and let $A, B \in \mathcal{M}$. Then $(A \vee B)_w = (A_w \vee B_w)_w$.*

Proof. The "$\leq$" direction is clear. It is sufficient to show that $A_w \vee B_w \leq (A \vee B)_w$. Let $X \leq A_w \vee B_w$ be principal so that $X = U \vee V$ for principal elements $U \leq A_w$ and $V \leq B_w$. Hence there exist $J_1, J_2 \in \mathrm{GV}(\mathcal{L})$ such that $J_1 U \leq A$ and $J_2 V \leq B$. thus $J_1 J_2 X \leq A \vee B$ and $X \leq (A \vee B)_w$. □

The following generalizes part of [28, Lemma 2.5] from integral domains to q. f. lattices.

Lemma 5.2.2. *Let $\mathcal{L}$ be a q. f. lattice and let P be a w-invertible prime w-element of $\mathcal{I}$. Then P is a maximal w-element and P is also a v-element.*

Proof. Suppose Q is an element of $\mathcal{I}$ with $P < Q$. Since P is w-invertible, P is of finite type so that $P = B_w$ for a finitely generated element $B \in \mathcal{I}$. Let $J = B \vee C$, where C is principal such that $C \leq Q$ but $C \not\leq P$ and let $X \leq J^{-1}$ be principal. We have $XCB \leq B \leq P$, implying that $XB \leq P$. thus $XP = XB_w = (XB)_w \leq P_w = P$, which means $X(PP^{-1}) \leq PP^{-1} \leq R$. Consequently, $X \in (PP^{-1})^{-1} = ((PP^{-1})_w)^{-1} = R^{-1} = R$. We have shown that $J^{-1} = R$, i.e. $J \in \mathrm{GV}(\mathcal{L})$. Since $J \leq Q$, it follows that $Q_w = R$ by Proposition 4.4.7, completing the proof. □

The above lemma yields the following corollary, which generalizes [28, Corollary 2.6] from integral domains to q. f. lattices.

Corollary 5.2.3. *Let P be a w-invertible prime element of $\mathcal{I}$. Then either $P_w = R$ or P is a maximal w-element.*

Proof. If $P_w \neq R$ then P is a w-element and the result follows from the previous lemma. $\square$

The following theorem is a lattice version of a portion of [28, Theorem 2.8], which is a significant characterization of Krull lattices. Although most of this theorem has been obtained, a few crucial implications are missing which requires lattice versions of the results of [21].

Theorem 5.2.4. *TFAE for a q. f. lattice $\mathcal{L}$:*

1. *$\mathcal{I}$ is a Krull lattice.*

2. *Every non-zero element of $\mathcal{I}$ is invertible.*

3. *Every non-zero prime w-element of $\mathcal{I}$ is w-invertible.*

4. *Every non-zero element $I \in \mathcal{I}$ with $I_w \neq R$ can be written as a unique w-product of a finite number of prime w-elements of $\mathcal{I}$.*

5. *$\mathcal{I}$ is an integrally closed and $\mathcal{L}$ is a strong Mori lattice.*

6. *$\mathcal{L}$ is a strong Mori lattice and $\mathcal{I}(R_P)$ is a discrete valuation lattice for each $P \in w\mathrm{Max}R$.*

7. *Every maximal w-element of $\mathcal{I}$ is w-invertible and has height one.*

8. $\mathcal{L}$ is a strong Mori lattice and every maximal w-element of $\mathcal{I}$ is w-invertible.

Some of these implications are trivial and others follow from some of our previous theorems, but there are several that are incomplete. Generalizing [21] would complete this significant characterization of strong Mori lattices, as well as giving a lattice version of [27, Theorem 5.4]. Also, many results concerning star operations in commutative rings have yet to be generalized to lattices. Furthermore, one may also consider the lattice formed by a certain semistar operation and the properties it would have, which up to this point has not yet been investigated.

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