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Mullineux involution and crystal isomorphisms

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Abstract. We develop a new approach for the computation of the Mullineux involution for the symmetric group and its Hecke algebra using the notion of crystal isomorphism and the Iwahori–Matsumoto involution for the affine Hecke algebra of type A. As a consequence, we obtain several new elementary combinatorial algorithms for its computation, one of which is equivalent to Xu algorithm (and thus Mullineux original algorithm). We thus obtain a simple interpretation of these algorithms and a new elementary proof that they indeed compute the Mullineux involution.

Keywords. Symmetric group, Mullineux involution, crystal graph

Mathematics Subject Classifications. 20C08, 05E10

1. Introduction

The Mullineux problem is a long standing problem in the representation theory of the symmetric groups which has been studied by various authors since the end of the 70's. Let $\mathfrak{S}_n$ be the symmetric group on n letters with $n > 1$. It is known that the irreducible representations of $\mathfrak{S}_n$ over the field of complex numbers are naturally labeled by the partitions of n (the sequences of non increasing positive integers of total sum n)

$$\mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n) = \{\rho_\lambda \mid \lambda \text{ partition of } n\}.$$

The characters and the dimensions of these representations may also been easily computed thanks to the combinatorics of partitions. There are exactly two non isomorphic representations of $\mathfrak{S}_n$ with dimension 1: the trivial representation which is labeled by the partition (n) and the sign representation ε, labeled by the partition $(\underbrace{1.\dots.1}_{n \text{ times}})$. As a consequence, if λ is a partition of n, there exists another partition μ such that $\rho_\mu \simeq \varepsilon \otimes \rho_\lambda$. It is natural to ask how one can compute μ from λ. The result is that μ is the conjugate partition of λ, that is the partition defined by interchanging rows and columns in the Young diagram of λ (the Young diagram of λ is the finite collection of boxes arranged in left-justified rows, with λ_k boxes in the k-th row for all $k \geqslant 1$).

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Of course, all the above questions and problems arise when we replace $\mathbb{C}$ by an arbitrary field k and in particular by a field of characteristic $p > 0$. In this case, the irreducible representations have first been constructed in [Jam76]. They are labeled by a subset of partitions called the set of p-regular partitions (the partitions of n where the non zero parts are not repeated p or more times)

$$\mathrm{Irr}_k(\mathfrak{S}_n) = \{\widetilde{\rho}_\lambda \mid \lambda \ p\text{-regular partition of } n\}.$$

We also have two one-dimensional representations: the trivial representation and the sign representation ε and they are non isomorphic if and only if $p \neq 2$. By contrast, we still not even know how to compute the dimensions of these representations in general. The other mentioned problem still makes sense in this context. Namely, if λ is a p-regular partition then there exists a unique p-regular partition μ such that

$$\widetilde{\rho}_\mu \simeq \varepsilon \otimes \widetilde{\rho}_\lambda.$$

If we set $m_p(\lambda) := \mu$, we thus obtain an involution m_p on the set of p-regular partitions.

If $p = 2$ then it is clear that $m_p = \mathrm{Id}$ (because then ε is nothing but the trivial representation) but in general, it is difficult to describe m_p. In fact, this map may even be defined in the context of Hecke algebras of type A at a p-root of unity. In this case, p does not need to be a prime but just a positive integer (greater than 2). The associated involution that we obtain coincides with m_p if p is prime. A natural problem is thus to find an explicit description of this involution m_e on the set of e-regular partitions for all $e \in \mathbb{N}_{>1}$. This is the main subject of the present paper.

In [Mul79], Mullineux first gave a conjectural algorithm for computing this involution (which will be called the Mullineux involution in the sequel). Later, another equivalent algorithm has been obtained by Xu [Xu99, Xu97]. In [Kle95], Kleshchev gave another combinatorial recursive algorithm for computing the Mullineux involution but it was not clear at that time why this algorithm would be equivalent to the Mullineux (and the Xu) algorithm. Ford and Kleshchev gave a proof of this fact later in [FK97]. Another proof was presented in [BO98] by Bessenrodt and Olsson. In [BK03], Brundan and Kujawa gave another proof using works by Serganova on the general linear supergroup. We also note that recently, Fayers [Fay22] has provided another way for computing the involution.

The aim of this paper is to present several elementary combinatorial (and recursive) algorithms for the computation of the involution using the Kleshchev result. These algorithms are based on the results of [JL10, JL13] and on the following points:

1. Each simple module for the Hecke algebra of type A labeled by an e-regular partition of rank n can be seen as a simple module for the affine Hecke algebra of type A.

2. The Mullineux map at the level of Hecke algebra coincide with the so called Iwahori–Matsumoto involution for the affine Hecke algebra of type A.

3. The Iwahori–Matsumoto involution may be computed using an analogue involution at the level of Ariki–Koike algebras associated to a multicharge $\mathbf{s} \in \mathbb{Z}^l$.

4. This later involution may be computed using the Mullineux involution for Hecke algebras of type A on e-regular partitions with rank (strictly) less than n.

As a consequence, to compute the image of an e-regular partition of rank n under the Mullineux involution, we are reduced to compute several images of e-regular partitions of rank strictly less than n under the Mullineux involution. This thus gives a recursive algorithm to solve our problem. In fact, depending on the *multicharge*, we choose for our Ariki–Koike algebras, we obtain several different algorithms. It turns out that for a particular choice of multicharge, our algorithm is equivalent to Xu algorithm. This yields a new elementary proof for the fact that the Mullineux and the Xu algorithm give an answer for the Mullineux problem. This also gives a new interpretation of these algorithms (another interpretation is also given in [BK03]). We note that the paper [DY19] also studies a link between the crystal isomorphisms (called wall crossing) and the Mullineux involution.

The paper will be organized as follows. In section 2, we recall some basic facts on the representation theory of affine Hecke algebras of type A and of Ariki–Koike algebras. We also recall several results coming from [JL10, JL09a] concerning the labelling of the simple modules for these algebras and the relations between them. Section 3 introduces the Mullineux and the Iwahori–Matsumoto involutions and shows how these two maps are related. In section 4, we study combinatorial properties of partitions and multipartitions which will be used in the following sections. Section 5 gives the algorithms we get for computing the Mullineux involution. The last section shows that Xu algorithm can be interpreted as one of our algorithms.

2. Hecke algebras

In this first section, we recall the definitions of the affine Hecke algebra of type A and of the Ariki–Koike algebras. We then give a brief overview of their representation theories. Finally, we explain the relations between the known parametrizations of the simple modules for these algebras. The main references for these parts are [Ari02] and [GJ11].

2.1. Affine Hecke algebra of type A

Let $n \in \mathbb{Z}_{>0}$. Let $q \in \mathbb{C}^*$ be a primitive root of unity of order $e > 1$. The *Iwahori–Hecke algebra* $H_n(q)$ of type A is the unital associative $\mathbb{C}$-algebra generated by T_0, T_1, …, T_{n-1} and subject to the relations:

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}\ (i = 1, \dots, n-2), \\ T_iT_j &= T_jT_i\ (|i-j| > 1), \\ (T_i - q)(T_i + 1) &= 0\ (i = 1, \dots, n-1). \end{aligned}$$

The *affine Hecke algebra* $\mathcal{H}_n(q)$ is the unital associative $\mathbb{C}$-algebra which is isomorphic to

$$H_n(q) \otimes_{\mathbb{C}} \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

as a $\mathbb{C}$-vector space and such that $H_n(q)$ and $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ are both subalgebras of $\mathcal{H}_n(q)$ with the following additional relations:

$$T_iX_iT_i = qX_{i+1},\ T_iX_j = X_iT_j,$$

for all $(i, j) \in \{1, \dots, n-1\}^2$ with $i \neq j$.

We denote by Mod_n the category of finite dimensional $H_n(q)$-modules such that for all $j = 1, \dots, n$, the eigenvalues of the X_j are power of q. The simple objects $\mathrm{Irr}(H_n(q))$ in Mod_n can be naturally labeled by the set of aperiodic multisegments that we now define:

Definition 2.1. Let $l \in \mathbb{N}_{>0}$ and let $i \in \mathbb{Z}/e\mathbb{Z}$. The *segment* of length l and head i is the sequence of consecutive *residues* (i.e elements of $\mathbb{Z}/e\mathbb{Z}$, identified with $\{0, 1, \dots, e-1\}$) $[i, i+1, \dots, i+l-1]$ in $\mathbb{Z}/e\mathbb{Z}$. The residue $i \in \mathbb{Z}/e\mathbb{Z}$ is then called the *head* of the segment and the residue $i+l-1$ the *tail* of the segment. A *multisegment* is a formal sum of segments. A multisegment is said to be *aperiodic* if for every $l \in \mathbb{Z}_{>0}$, there exists $i \in \mathbb{Z}/e\mathbb{Z}$ such that there is no segment with length l and tail i appearing in the multisegment. We denote by $\mathfrak{M}_e$ the set of aperiodic multisegments. The length of a multisegment is the sum of the lengths of the the segments appearing in it and is denoted by $|\psi|$. We denote by $\mathfrak{M}_e(n)$ the set of aperiodic multisegments of length n.

Example 2.2. For $e = 3$, the multisegment:

$$[0, 1, 2, 0] + [0] + [1] + [1, 2] + [2, 0],$$

is an aperiodic multisegment of length 10 whereas

$$[0, 1, 2, 0] + [0] + [0, 1] + [1, 2] + [2, 0],$$

is a multisegment of length 10 which is not aperiodic.

By the geometric realization of $H_n(q)$ by Chriss and Ginzburg [CG97], we know that one may naturally label the simple modules in Mod_n by the set $\mathfrak{M}^e(n)$ of aperiodic multisegments of length n. We thus have:

$$\mathrm{Irr}(H_n(q)) = \{L_\psi \mid \psi \in \mathfrak{M}^e(n)\}.$$

We note that the parametrization of irreducible affine Hecke algebras modules using these multisegments and the connections with the representation theory of Ariki–Koike algebras and crystal graph in the case $e = \infty$ were first studied in [Vaz02].

2.2. Ariki–Koike algebras

Let $l \in \mathbb{N}_{>0}$[1] As above, we fix a primitive root of unity $q \in \mathbb{C}^*$ of order $e > 1$. Let $P_l := \mathbb{Z}^l$ and let $\{z_i \mid i = 1, \dots, l\}$ be the canonical basis of P_l. Let $\mathfrak{S}_l$ be the symmetric group generated by the transpositions $\sigma_i := (i, i+1)$ for $i = 1, \dots, l-1$. The extended affine symmetric group $\widehat{\mathfrak{S}}_l$ is the semidirect product $P_l \rtimes \mathfrak{S}_l$ with the relations given by $\sigma_i z_j = z_j \sigma_i$ for $j \neq i, i+1$ and $\sigma_i z_i \sigma_i = z_{i+1}$ for $i = 1, \dots, l-1$ and $j = 1, \dots, l$. This group is generated by the σ_i for $i = 1, \dots, l-1$ and by $\tau := z_l \sigma_{l-1} \dots \sigma_1$ (see [JL10, §5.1]).

[1] Our main result developed in Section 5 only uses the case $l = 2$ but part of our results is settled in the general case $l \in \mathbb{N}_{>0}$, see §3.3. We have thus chosen to stick to the general case in this section to be self-contained.

It acts faithfully on $\mathbb{Z}^l$ as follows: for any $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$:

$$\begin{array}{rcll} \sigma_c.\mathbf{s} & = & (s_1, \dots, s_{c-1}, s_{c+1}, s_c, s_{c+2}, \dots, s_l) & \text{for } c = 1, \dots, l-1 \text{ and} \\ z_i.\mathbf{s} & = & (s_1, s_2, \dots, s_i + e, \dots, s_l) & \text{for } i = 1, \dots, l. \end{array}$$

and we have

$$\tau.\mathbf{s} = (s_2, \dots, s_l, s_1 + e).$$

Let $\mathfrak{s}$ be an orbit with respect to the above action and let $\mathbf{s} := (s_1, \dots, s_l) \in \mathbb{Z}^l$ be an element in this orbit. The Ariki–Koike algebra $\mathcal{H}_n^{\mathfrak{s}}(q)$ is the quotient $H_n(q)/I_{\mathfrak{s}}$ where $I_{\mathfrak{s}} := \langle \prod_{1 \leqslant j \leqslant l} (X_1 - q^{s_j}) \rangle$. If $l = 1$, this is a Hecke algebra of type A (of finite type), and if $l = 2$ a Hecke algebra of type B (of finite type). One can see that the above algebra is well defined and depends only on the orbit of $\mathbf{s}$ modulo the action of $\widehat{\mathfrak{S}}_l$ (and on q).

The representation theory of this algebra has been intensively studied in a many works. We refer to [Ari02, GJ11] and the references theirein. We will only recall what is needed for the results of the present paper. The analogues of the multisegments in the context of Ariki–Koike algebras are the multipartitions that we now define. For this, let us give some additional combinatorial definitions.

A *partition* is a nonincreasing sequence $\lambda = (\lambda_1, \cdots, \lambda_m)$ of nonnegative integers. One can assume this sequence is infinite by adding parts equal to zero. The *rank* of the partition is by definition the number $|\lambda| = \sum_{1 \leqslant i \leqslant m} \lambda_i$. We say that λ is a partition of n, where $n = |\lambda|$. By convention, the unique partition of 0 is the empty partition $\varnothing$.

More generally, for $l \in \mathbb{Z}_{>0}$, an *l-partition* $\boldsymbol{\lambda}$ of n is a sequence of l partitions $(\lambda^1, \dots, \lambda^l)$ such that the sum of the ranks of the λ^j is n. The number n is then called the *rank* of $\boldsymbol{\lambda}$ and it is denoted by $|\boldsymbol{\lambda}|$. The set of l-partitions is denoted by Π^l and the set of l-partitions of rank n is denoted by $\Pi^l(n)$. Let $\boldsymbol{\lambda}$ be an l-partition. The *nodes* or the *boxes* of $\boldsymbol{\lambda}$ are by definition the elements of the Young diagram of $\boldsymbol{\lambda}$:

$$[\boldsymbol{\lambda}] := \{(a, b, c) \mid a \geqslant 1,\ c \in \{1, \dots, l\},\ 1 \leqslant b \leqslant \lambda_a^c\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\}.$$

The *content* of a node $\gamma = (a, b, c)$ of $\boldsymbol{\lambda}$ is the element $b - a + s_c$ of $\mathbb{Z}$ and the residue is the content modulo $e\mathbb{Z}$. If $l = 1$ (that is when we consider a partition instead of a multipartition), then the Young diagram is identified with a subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ in an obvious way.

Since the works of Ariki and Lascoux–Leclerc–Thibon, it is known that the representation theory of these algebras is closely related to the representation theory of quantum groups. In particular, one can naturally label the simple modules by the crystal basis of a certain integrable representation for the quantum group of affine type $A_{e-1}^{(1)}$. We will not give the details of all the consequences of this fact but we summarize this below. Again, we refer to [GJ11] for a complete study. For all choices of $\mathbf{s} \in \mathfrak{s}$, we can define a certain subset of l-partitions which are called Uglov l-partitions. This subset of multipartitions is denoted by $\Phi_{e,\mathbf{s}}(n)$. These classes of multipartitions, which strongly depends on the choice of $\mathbf{s}$, can all be seen as non trivial generalizations of the set of e-regular partitions:

- For all $s \in \mathbb{Z}$, we define:

$$\mathcal{A}_e^l[s] := \{(s_1, \dots, s_l) \in \mathbb{Z}^l \mid s_1 = s \leqslant s_2 \leqslant \dots s_l < s + e\}.$$

This is a fundamental domain for the action of $\widehat{\mathfrak{S}}_l$ on $\mathbb{Z}^l$. If $\mathrm{s} \in \mathcal{A}_e^l[s]$, then the l-partitions in $\Phi_{e,\mathbf{s}}(n)$ are known as FLOTW l-partitions and they have a non recursive definition: we have $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^l) \in \Phi_{\mathbf{s},e}(n)$ if and only if:

1. For all $j = 1, \ldots, l-1$ and $i \in \mathbb{Z}_{>0}$, we have:
$$\lambda_i^j \geqslant \lambda_{i+s_{j+1}-s_j}^{j+1}.$$
2. For all $i \in \mathbb{Z}_{>0}$, we have:
$$\lambda_i^l \geqslant \lambda_{i+e+s_1-s_l}^1.$$
3. For all $k \in \mathbb{Z}_{>0}$, the set
$$\{\lambda_i^j - i + s_j + e\mathbb{Z} \mid i \in \mathbb{Z}_{>0},\ \lambda_i^j = k, j = 1, \ldots, l\},$$
is a proper subset of $\mathbb{Z}/e\mathbb{Z}$.

- If $\mathrm{s} = (s_1, \ldots, s_l)$ satisfies for all $i = 1, \ldots, l-1$, $s_{i+1} - s_i > n-1$ (we say that s is *very dominant*, it is also sometimes referred as the "asymptotic case" in the literature) then the set $\Phi_{e,\mathbf{s}}(n)$ is known as the set Kleshchev l-partitions. If s'' satisfy the same property, then the associated set $\Phi_{e,\mathbf{s}''}(n)$ is the same.
- If $l = 1$, the set $\Phi_{e,(s)}(n)$ is simply the set of e-regular partitions

It turns out that each set $\Phi_{e,\mathbf{s}}(n)$ with $\mathrm{s} \in \mathfrak{s}$ gives a natural labelling for the irreducible representations of the Ariki–Koike algebra $\mathcal{H}_n^{\mathfrak{s}}(q)$ (see [GJ11, §5]). As a consequence, there are several natural possibilities for the labelling of the simple modules of $\mathcal{H}_n^{\mathfrak{s}}(q)$, one for each choice of an element in the orbit $\mathfrak{s}$. For more details on these parametrizations, we refer to [GJ11]. Thus, one can write:
$$\mathrm{Irr}(\mathcal{H}_n^{\mathfrak{s}}(q)) = \{D_{\mathbf{s}}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_{e,\mathbf{s}}(n)\},$$
with $\mathrm{s} \in \mathfrak{s}$. By [Bow22], each of these labellings has an interpretation in terms of a cellular structure. Lastly, clearly, if s and s′ are in the same orbit, there is a map:
$$\Psi_e^{\mathbf{s}\to\mathbf{s}'} : \sqcup_{n\geqslant 0}\Phi_{(e,\mathbf{s})}(n) \to \sqcup_{n\geqslant 0}\Phi_{(e,\mathbf{s}')}(n),$$
which is the unique bijection satisfying the following property. For all $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$ then:
$$D_{\mathbf{s}}^{\boldsymbol{\lambda}} \simeq D_{\mathbf{s}'}^{\Psi_e^{\mathbf{s}\to\mathbf{s}'}(\boldsymbol{\lambda})}.$$
Note that for n fixed, we have $\Psi_e^{\mathbf{s}\to\mathbf{s}'}(\Phi_{(e,\mathbf{s})}(n)) = \Phi_{(e,\mathbf{s}')}(n)$. This bijection has been explicitly described in [JL10] in a combinatorial way using crystal isomorphisms (the coincidence of the crystal isomorphisms with these bijections is proved in [Jac17, Prop. 3.7]). We recall this description in the next subsection in the case $l = 2$ (a program in GAP3 is available for computing it in all cases [Jac12]). In the next sections, the following particular case: $\mathrm{s} = (s_1, s_2)$ and $\mathrm{s}' = (s_1, s_2 + e)$ will be of particular interest.

Remark 2.3. If s′ and s″ are both very dominant multicharges in the same orbit then $\Psi_e^{\mathbf{s}\to\mathbf{s}'}$ is the identity.

Example 2.4. Assume that $e = 3$. Take $\mathfrak{s} = (0 + 3\mathbb{Z}, 1 + 3\mathbb{Z})$. Take $n = 3$, then, we have

$$\Phi_{3,(0,1)}(3) = \{(\varnothing,(3)),((1),(1,1)),((1),(2)),((2),(1)),((2,1),\varnothing),((3),\varnothing)\}$$
$$\Phi_{3,(0,4)}(3) = \{(\varnothing,(3)),((1),(1,1)),((1),(2)),((2),(1)),(\varnothing,(2,1)),((1,1),(1))\}$$
$$\Phi_{3,(1,0)}(3) = \{((3),\varnothing),((1),(1,1)),((1),(2)),((1,1),(1)),((2,1),\varnothing),((2),(1))\} = \Phi_{3,(4,0)}(3)$$

So that :

$$\begin{aligned}\mathrm{Irr}(\mathcal{H}_n^{\mathfrak{s}}(q)) &= \{D_{(0,1)}^{(\varnothing,(3))}, D_{(0,1)}^{((1),(1,1))}, D_{(0,1)}^{((1),(2))}, D_{(0,1)}^{((2),(1))}, D_{(0,1)}^{((2,1),\varnothing)}, D_{(0,1)}^{((3),\varnothing)}\}\\ &= \{D_{(0,4)}^{(\varnothing,(3))}, D_{(0,4)}^{((1),(1,1))}, D_{(0,4)}^{((1),(2))}, D_{(0,4)}^{((2),(1))}, D_{(0,4)}^{(\varnothing,(2,1))}, D_{(0,4)}^{((1),(1,1))}\}\\ &= \{D_{(1,0)}^{((3),\varnothing)}, D_{(1,0)}^{((1),(1,1))}, D_{(1,0)}^{((1),(2))}, D_{(1,0)}^{((1,1),(1))}, D_{(1,0)}^{((2,1),\varnothing)}, D_{(1,0)}^{((2),(1))}\}\end{aligned}$$

2.3. Description of the crystal isomorphisms

First let us assume that $l = 2$ and that $(s_1, s_2) \in \mathbb{Z}^2$. Let $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$. We follow the presentation in [JL10].

We define the minimal integer $d \geqslant |s_1 - s_2|$ such that $\lambda^1_{d+1+s_1-s_2} = \lambda^2_{d+1} = 0$ if $s_2 \geqslant s_1$, and otherwise the minimal integer $d \geqslant |s_1 - s_2|$ such that $\lambda^2_{d+1+s_2-s_1} = \lambda^1_{d+1} = 0$. To (λ^1, λ^2), we associate its s-symbol of length d. This is the following two-rows array.

- If $s_1 \leqslant s_2$ then:

$$S(\lambda^1,\lambda^2) = \begin{pmatrix} s_2 - d + \lambda^2_d & \dots & \dots & & s_2 - 2 + \lambda^2_2 & s_2 + \lambda^2_1 - 1 \\ s_2 - d + \lambda^1_{d+s_1-s_2} & \dots & s_1 + \lambda^1_1 - 1. & & & \end{pmatrix}$$

- if $s_1 > s_2$ then:

$$S(\lambda^1,\lambda^2) = \begin{pmatrix} s_1 - d + \lambda^2_{d+s_2-s_1} & \dots & s_2 + \lambda^2_1 - 1 & & & \\ s_1 - d + \lambda^1_d & \dots & \dots & & s_1 - 2 + \lambda^1_2 & s_1 + \lambda^1_1 - 1. \end{pmatrix}$$

We will write $S(\lambda^1,\lambda^2) = \binom{L_2}{L_1}$ where the top row (resp. the bottom row) corresponds to λ^2 (resp. λ^1). Of course, it is easy to recover the 2-partition from the datum of its symbol. From this symbol, we define a new symbol $\binom{\widetilde{L}_2}{\widetilde{L}_1}$ as follows.

- Suppose first $s_2 \geqslant s_1$. Consider $x_1 = \min\{t \in L_1\}$. We associate to x_1 the integer $y_1 \in L_2$ such that

$$y_1 = \begin{cases} \max\{z \in L_2 \mid z \leqslant x_1\} \text{ if } \min\{z \in L_2\} \leqslant x_1, \\ \max\{z \in L_2\} \text{ otherwise.} \end{cases} \tag{2.1}$$

  We repeat the same procedure to the lines $L_2 - \{y_1\}$ and $L_1 - \{x_1\}$. By induction this yields a sequence $\{y_1, \dots, y_{d+s_1-s_2}\} \subset L_2$. Then we define $\widetilde{L}_2$ as the line obtained by reordering the integers of $\{y_1, \dots, y_{d+s_2-s_1}\}$ and $\widetilde{L}_1$ as the line obtained by reordering the integers of $L_2 - \{y_1, \dots, y_{d+s_1-s_2}\} + L_1$ (i.e. by reordering the set obtained by replacing in L_2 the entries $y_1, \dots, y_{d+s_1-s_2}$ by those of L_1). We obtain a "symbol" $\binom{\widetilde{L}_2}{\widetilde{L}_1}$.

- Now, suppose $s_2 < s_1$. Consider $x_1 = \min\{t \in L_2\}$. We associate to x_1 the integer $y_1 \in L_1$ such that

$$y_1 = \begin{cases} \min\{z \in L_1 \mid x_1 \leqslant z\} \text{ if } \max\{z \in L_1\} \geqslant x_1, \\ \min\{z \in L_1\} \text{ otherwise.} \end{cases} \tag{2.2}$$

  We repeat the same procedure to the lines $L_1 - \{y_1\}$ and $L_2 - \{x_1\}$ and obtain a sequence $\{y_1, \ldots, y_{d+s_1-s_2}\} \subset L_1$. Then we define $\widetilde{L}_1$ as the line obtained by reordering the integers of $\{y_1, \ldots, y_{d+s_2-s_1}\}$ and $\widetilde{L}_2$ as the line obtained by reordering the integers of $L_1 - \{y_1, \ldots, y_{d+s_2-s_1}\} + L_2$. We obtain a "symbol" $\binom{\widetilde{L}_2}{\widetilde{L}_1}$.

The new symbol $\binom{\widetilde{L}_2}{\widetilde{L}_1}$ that we obtain is canonically associated to a bipartition $(\overline{\lambda}^1, \overline{\lambda}^2)$ and the multicharge (s_2, s_1). Applying again this transformation gives the bipartition (λ^1, λ^2). Thus the above transformation is an involution.

The crystal isomorphisms in the case $l = 2$ are thus entirely determined from the following results proved in [JL10]:

1. We have $\Psi_e^{(s_1,s_2)\to\sigma_1(s_1,s_2)}(\lambda^1, \lambda^2) = (\overline{\lambda}^1, \overline{\lambda}^2)$.
2. We have $\Psi_e^{(s_1,s_2)\to\tau.(s_1,s_2)}(\lambda^1, \lambda^2) = (\lambda^2, \lambda^1)$.
3. For all $\sigma = x_1. \ldots .x_m \in \widehat{\mathfrak{S}}_2$ with $x_i \in \{\sigma_1, \tau\}$ for all $i = 1, \ldots, m$, we have:
$$\Psi_e^{(s_1,s_2)\to\sigma.(s_1,s_2)} = \Psi_e^{x_2.\ldots.x_m.(s_1,s_2)\to\sigma.(s_1,s_2)} \circ \ldots \circ \Psi_e^{(s_1,s_2)\to x_m.(s_1,s_2)}.$$

In the general case $l \in \mathbb{N}_{>0}$ and $\mathbf{s} \in \mathbb{Z}^l$, now:

1. For all $c = 1, \ldots, l-1$, we have $\Psi_e^{(s_1,s_2)\to\sigma_c(s_1,s_2)}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$, where $\mu^j = \lambda^j$ for all $j \neq c, c+1$, $\mu^c = \overline{\lambda}^c$ and $\mu^{c+1} = \overline{\lambda}^{c+1}$.
2. We have $\Psi_e^{\mathbf{s}\to\tau.\mathbf{s}}(\boldsymbol{\lambda}) = (\lambda^2, \ldots, \lambda^l, \lambda^1)$.
3. For all $\sigma = x_1. \ldots .x_m \in \widehat{\mathfrak{S}}_2$ with $x_i \in \{\sigma_1, \ldots, \sigma_{l-1}, \tau\}$ for all $i = 1, \ldots, m$, we have:
$$\Psi_e^{\mathbf{s}\to\sigma.\mathbf{s}} = \Psi_e^{x_2.\ldots.x_m.\mathbf{s}\to\sigma.\mathbf{s}} \circ \ldots \circ \Psi_e^{\mathbf{s}\to x_m.\mathbf{s}}.$$

Example 2.5. Assume that $(s_1, s_2) \in \mathbb{Z}^2$ with $s_1 \leqslant s_2$. In the next sections, we will be particularly interested in the computation of $\Psi_e^{(s_1,s_2)\to(s_1,s_2+e)}$. Let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \in \Phi_{(e,\mathbf{s})}(n)$, we then write its symbol:

$$S(\lambda^1, \lambda^2) = \begin{pmatrix} s_2 - d + \lambda^2_d & \ldots & \ldots & & s_2 - 2 + \lambda^2_2 & s_2 + \lambda^2_1 - 1 \\ s_2 - d + \lambda^1_{d+s_1-s_2} & \ldots & s_1 + \lambda^1_1 - 1 & & & \end{pmatrix}.$$

We then perform the above algorithm to obtain a new symbol $\binom{\widetilde{L}_2}{\widetilde{L}_1}$ which must be of the form :

$$\begin{pmatrix} y_{d+s_1-s_2} & \ldots & y_1 & & \\ x_d & \ldots & \ldots & x_2 & x_1. \end{pmatrix}$$

We then consider the following symbol:

$$\begin{pmatrix} 0 & \dots & e-1 & x_d+e & \dots & \dots & x_2+e & x_1+e \\ y_{d+s_1-s_2} & \dots & y_1. \end{pmatrix}$$

By the discussion above, this is the (s_1, s_2+e)-symbol of the bipartition

$$\Psi_e^{(s_1,s_2)\to(s_1,s_2+e)}(\lambda^1,\lambda^2) = (\overline{\lambda}^2, \overline{\lambda}^1)$$

(more details and examples can be found in [Jac07])

Example 2.6. We keep the example 2.4, one can check that the map $\Psi_e^{(0,1)\to(0,4)}$ is given as follows

$$\begin{array}{cccc} \Psi_e^{(0,1)\to(0,4)}: & \Phi_{3,(0,1)} & \to & \Phi_{3,(0,4)} \\ & (\varnothing,(3)) & \mapsto & (\varnothing,(3)) \\ & ((1),(1,1)) & \mapsto & ((1),(1,1)) \\ & ((1),(2)) & \mapsto & (\varnothing,(2,1)) \\ & ((2),(1)) & \mapsto & ((2),(1)) \\ & ((2,1),\varnothing) & \mapsto & ((1,1),(1)) \\ & ((3),\varnothing) & \mapsto & ((1),(2)) \end{array}$$

More examples can be found in [JL10].

2.4. Aperiodic multisegments and multipartitions

Let $\mathfrak{s}$ be an orbit of $\mathbb{Z}^l$ with respect to the action of the affine symmetric group (recall the definition of the action in §2.2). If V is a simple module for the Ariki–Koike algebra then it is also a simple $H_n(q)$-module in the category Mod_n. Hence there exists a unique aperiodic multisegment ψ such that $V \simeq L_\psi$ (as a $H_n(q)$-module). As a consequence, for any $\mathbf{s} \in \mathfrak{s}$ we have a well defined map:

$$\chi^n_{e,\mathbf{s}} : \Phi_{(e,\mathbf{s})}(n) \to \mathfrak{M}_e(n),$$

which is defined as follows. Let $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$, then we have a unique $\chi^n_{(e,\mathbf{s})}(\boldsymbol{\lambda}) \in \mathfrak{M}_e(n)$ such that:

$$D^{\boldsymbol{\lambda}}_{\mathbf{s}} \simeq L_{\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda})}.$$

By [AJL11], this map may be described as follows:

- Assume first that $\mathbf{s} \in \mathcal{A}^l_e[s]$. For all non zero part λ^c_i of $\boldsymbol{\lambda}$, we associate the segment

  $$[(1-i+s_c)+e\mathbb{Z},\dots,\lambda^c_i - i + s_c + e\mathbb{Z}].$$

  By [AJL11], The multisegment $\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is just the formal sum of all the segments associated to the non zero part of $\boldsymbol{\lambda}$.

- As a consequence, in general, if $\mathbf{s}' \in \mathfrak{s}$ and $\mathbf{s} \in \mathcal{A}^l_e[s] \cap \mathfrak{s}$, then

  $$\chi^n_{e,\mathbf{s}'}(\boldsymbol{\lambda}) = \chi^n_{e,\mathbf{s}}(\Psi_e^{\mathbf{s}'\to\mathbf{s}}(\boldsymbol{\lambda})).$$

Given an aperiodic multisegment ψ, It is now natural to try to find the multicharges $\mathbf{s}$ such that $\{\psi\}$ has a non empty preimage for the map $\chi^n_{e,\mathbf{s}}$. This question has been completely solved in [JL09a]. There always exist such multicharges (they are not unique in general) which are called *admissible multicharges*. By [AJL11], $\chi^n_{e,\mathbf{s}}$ is injective so that if $\mathbf{s}$ is admissible for ψ there exists a unique $\boldsymbol{\lambda}$ such that $\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \psi$. This l-partition will be called *admissible* (with respect to ψ). By definition, we have the proposition below where we use the following notation. For $\mathbf{s}$ and $\mathbf{t}$ two multicharges, we denote $\mathbf{s} \subset \mathbf{t}$ if and only if, for all $j \in \mathbb{Z}/e\mathbb{Z}$, the number of integers congruent to j in $\mathbf{s}$ is less or equal to the number of integers congruent to j in $\mathbf{t}$.

Proposition 2.4.1. *Assume that* $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$ *and that we have two multicharges such that* $\mathbf{s} \subset \mathbf{t}$ *then* $\mathbf{t}$ *is admissible for the multisegment* $\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda})$.

Proof. Set $\mathbf{s} = (s_1, \ldots, s_l)$ and $\mathbf{t} = (t_1, \ldots, t_m)$. Assume that $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$ then as a $H_n(q)$-module, we have that $\prod_{1\leqslant j\leqslant l}(X_1 - q^{s_j})$ acts as 0 on $D^{\boldsymbol{\lambda}}_{\mathbf{s}} \simeq L_{\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda})}$. As a consequence, as $\mathbf{s} \subset \mathbf{t}$, we have that $\prod_{1\leqslant j\leqslant m}(X_1 - q^{t_j})$ acts as 0 on $L_{\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda})}$. This implies that it is a well-defined $\mathcal{H}^{\mathbf{t}}_n(q)$-module and the result follows. □

Remark 2.7. One can also prove the above proposition combinatorially using the descriptions of the admissible multicharges.

3. The Mullineux and the Iwahori–Matsumoto involutions

The aim of this section is to introduce the Mullineux involution for the symmetric group and its analogues in the context of Ariki–Koike algebras and affine Hecke algebras.

3.1. Iwahori–Matsumoto involution for affine Hecke algebras of type A

We have an involution $\sharp$ on $H_n(q)$ which has been defined by Iwahori and Mastumoto in [IM65]:

$$T_i^\sharp = -qT_i^{-1},\ X_j^\sharp = X_j^{-1}$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. The Iwahori–Matsumoto involution naturally induces an involution on the set of aperiodic multisegments. We have an involution:

$$\sharp : \mathfrak{M}^e(n) \to \mathfrak{M}^e(n),$$

defined for all $\psi \in \mathfrak{M}^e(n)$ by

$$L_\psi^\sharp = L_{\psi^\sharp}.$$

Remark 3.1. We have in fact two others well defined involutions on $H_n(q)$ which are defined as follows:

- The Zelevinsky involution τ defined in [MgW86] :

  $$T_i^\tau = -qT_{n-i}^{-1},\ X_j^\tau = X_{n+1-j}^{-1},$$

  for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$.

- The involution ∇ :
$$T_i^{\nabla} = -qT_{n-i},\ X_j^{\nabla} = X_{n+1-j},$$
for $i = 1, \dots, n-1$ and $j = 1, \dots, n$.

We have for all $x \in H_n(q)$:
$$x^{\tau} = (x^{\nabla})^{\sharp} = (x^{\sharp})^{\nabla}.$$
These two involutions thus also induce involutions on the set $\mathfrak{M}^e(n)$ and they have been studied in [JL09a].

3.2. Mullineux involution for Ariki–Koike algebras

Assume that $\mathbf{s} \in \mathbb{Z}^l$. Then we have a well-defined algebra isomorphism:
$$\gamma : \mathcal{H}_n^{\mathfrak{s}}(q) \to \mathcal{H}_n^{\mathfrak{s}}(q^{-1}),$$
which is defined on the generators as follows:
$$T_0 \mapsto T_0,\ T_i \mapsto -qT_i.$$
This map naturally induces bijections on the indexing sets of the simple modules of Ariki–Koike algebras. Let $\mathfrak{s}^{\sharp}$ be the orbit of $(-s_1, \dots, -s_l)$ modulo the action of the affine symmetric group. Let $\mathbf{v} \in \mathfrak{s}^{\sharp}$ then we have a map:
$$m_e^{\mathbf{s}\to\mathbf{v}} : \Phi_{(e,\mathbf{s})}(n) \to \Phi_{(e,\mathbf{v})}(n),$$
defined as follows. Let $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$, then there exists a unique $\boldsymbol{\mu} \in \Phi_{(e,\mathbf{v})}(n)$ such that
$$(D_{\mathbf{s}}^{\boldsymbol{\lambda}})^{\gamma} \simeq D_{\mathbf{v}}^{\boldsymbol{\mu}},$$
and we set
$$m_e^{\mathbf{s}\to\mathbf{v}}(\boldsymbol{\lambda}) = \boldsymbol{\mu}.$$
This map has been described in [JL10]. If $l = 1$ and e is prime then it coincides with the usual Mullineux involution of the symmetric group that we have defined in the introduction. If $l = 1$, then it corresponds to the Mullineux involution of the Hecke algebra of type A described in [Bru98] which will simply be denoted by m_e (it does not depend on $\mathbf{s}$). In this paper, we will give an algorithm for computing m_e.

Remark 3.2. If λ is a partition and γ a node of its Young diagram (English convention), the γ-hook of λ id by definition the set of all the nodes at the right and at the bottom of γ (including γ). The length of the hook is the number of nodes it contains. We say that λ is an e-core if all the hooks have length strictly less than e. If λ is an e-core then $m_e(\lambda)$ can be easily described: it is just the conjugation of λ (as in the semisimple case), see [Mul79] (when e is a prime but the results generalizes easily if e is an integer).

More generally, it is a natural question to ask how one can describe all the maps $m_e^{\mathbf{s}\to\mathbf{v}}$ for any $\mathbf{s}$, $\mathbf{v}$. It turns out that by [JL09b, Prop. 4.2], knowing the map m_e, one can describe it quite easily in a particular case:

Proposition 3.2.1. *Assume that* $\mathbf{s}$ *is very dominant. Let* $\mathbf{s}^\sharp := (-s'_1, \dots, -s'_l)$ *be a very dominant multicharge such that* $s'_i \equiv s_i + e\mathbb{Z}$ *for all* $i = 1, \dots, l$*. Then for all* $\boldsymbol{\lambda} \in \Phi_{(e,\mathfrak{s})}(n)$*, we have:*

$$m_e^{\mathbf{s}\to\mathbf{s}^\sharp}(\boldsymbol{\lambda}) = (m_e(\lambda^1), \dots, m_e(\lambda^l)).$$

As a consequence, this result, combining with the fact that we know how to compute the natural bijection between the various parametrizations of the simple modules of Ariki–Koike algebras permits to describe all the Mullineux involutions (assuming that we know m_e). Indeed, let $\mathbf{v}_1 \in \mathfrak{s}$ and let $\mathbf{v}_2 \in \mathfrak{s}^\sharp$. Let $\mathbf{s}_1 \in \mathfrak{s}$ be a very dominant multicharge. Then we have:

$$m_e^{\mathbf{v}_1\to\mathbf{v}_2} = \Psi_e^{\mathbf{s}_1^\sharp\to\mathbf{v}_2} \circ m^{\mathbf{s}_1\to\mathbf{s}_1^\sharp} \circ \Psi_e^{\mathbf{v}_1\to\mathbf{s}_1}$$

where $\mathbf{s}_1^\sharp$ is as in the above proposition.

Example 3.3. We keep the setting of example 2.4. For $n = 3$, the multicharge $(0,4)$ is very dominant, so the above result applies in this case. One can take $\mathbf{s}^\sharp = (0,5)$ which is also very dominant. Using the fact that $m_3(3) = (2,1)$, $m_3(1.1) = (2)$, we obtain

$$\begin{array}{cccc} m_e^{(0,4)\to(0,5)} & \Phi_{3,(0,1)}(3) & \to & \Phi_{3,(0,5)}(3) \\ & (\varnothing,(3)) & \mapsto & (\varnothing,(2,1)) \\ & ((1),(1,1)) & \mapsto & ((1),(2)) \\ & (\varnothing,(2,1)) & \mapsto & (\varnothing,(3)) \\ & ((2),(1)) & \mapsto & ((1,1),(1)) \\ & ((1,1),(1)) & \mapsto & ((2),(1)) \\ & ((1),(2)) & \mapsto & ((1),(1,1)) \end{array}$$

Now combining with our cristal isomorphism in Example 2.6, we for example obtain

$$\begin{array}{cccc} m_e^{(0,1)\to(0,5)}: & \Phi_{3,(0,1)} & \to & \Phi_{3,(0,5)} \\ & (\varnothing,(3)) & \mapsto & (\varnothing,(2,1)) \\ & ((1),(1,1)) & \mapsto & ((1),(2)) \\ & ((1),(2)) & \mapsto & (\varnothing,(3)) \\ & ((2),(1)) & \mapsto & ((1,1),(1)) \\ & ((1,1),(1)) & \mapsto & ((2),(1)) \\ & ((3),\varnothing) & \mapsto & ((1),(1,1)) \end{array}$$

3.3. Relations between the involutions

Now we put all the above results together to deduce relations between the various involutions we have defined. The following result is proved in [JL09a].

Theorem 3.3.1. *Let* ψ *be an aperiodic multisegment and let* $\mathbf{s} \in \mathcal{A}_e^l[s]$ *be an admissible multicharge for* ψ*. Set* $\mathbf{s}^t = (-s_l, \dots, -s_1) \in \mathcal{A}_e^l[-s_l]$ *then we have:*

$$\psi^\sharp = \chi_{e,\mathbf{s}^t}^n \circ m_e^{\mathbf{s}\to\mathbf{s}^t} \circ (\chi_{e,\mathbf{s}}^n)^{-1}(\psi)$$

As a consequence, the Iwahori–Matsumoto involution may be computed as follows. Take an aperiodic multisegment ψ.

- Choose an admissible multicharge s for ψ and compute $\boldsymbol{\lambda} := (\chi^n_{e,\mathbf{s}})^{-1}(\psi)$ using §2.4 and the algorithm described in [JL09a].
- Compute $\nu := m_e^{\mathbf{s}\to\mathbf{s}^t}(\boldsymbol{\lambda})$ using the discussion in the last section.
- Compute $\psi^\sharp := \chi^n_{e,\mathbf{s}^t}(\nu)$ using the algorithm described in [JL09a].

Example 3.4. Take $e = 3$ and the multisegment $[0] + [0, 1, 2] + [1, 2, 0]$. One can see that $(0, 1)$ is admissible for this multisegment and we have $(\chi^7_{3,(0,1)})^{-1}(\psi) = ((3), (3, 1))$.

We need to compute $m_e^{\mathbf{s}\to\mathbf{s}^t}((3), (3, 1))$. To do this, we first compute $\Psi_e^{(0,1)\to(0,7)}((3), (3, 1))$ as $(0, 7)$ is very dominant. We obtain the bipartition $((1), (3, 3))$. Now by Proposition 3.2.1:

$$m_e^{(0,7)\to(0,8)}((1), (3, 3)) = (m_3(1), m_3(3, 3)) = ((1), (6)).$$

Again, we compute $\Psi_e^{(0,8)\to(-1,0)}((1), (6)) = ((6), (1))$ and thus we get

$$\psi^\sharp := [0] + [2, 0, 1, 2, 0, 1].$$

Now, let us explain how one can deduce an algorithm for computing the Mullineux involution for e-regular partitions. This is based on the following elementary remark. Let $\lambda \in \Phi_{e,(0)}$ be an e-regular partition and consider the aperiodic multisegment $\psi := \chi^n_{e,(0)}(\lambda)$ (recall that this is nothing but the formal sum of the segments given by the rows of the Young diagram of λ). The above theorem shows that:

$$m_e(\lambda) = (\chi^n_{e,(0)})^{-1}(\psi^\sharp).$$

So now we are reduced to compute $(\chi^n_{e,(0)})^{-1}(\psi^\sharp)$. Take $\mathbf{s} \in \mathcal{A}^l_e[0]$ such that $l > 1$ then by Proposition 2.4.1, this is an admissible multicharge. We have:

$$\psi^\sharp = \chi^n_{e,\mathbf{s}^t} \circ m_e^{\mathbf{s}\to\mathbf{s}^t} \circ (\chi^n_{e,\mathbf{s}})^{-1}(\psi)$$

Now $\boldsymbol{\mu} := (\chi^n_{e,\mathbf{s}})^{-1}(\psi)$ is the admissible l-partition (associated to s) and the main problem is thus to compute $m_e^{\mathbf{s}\to\mathbf{s}^t}(\boldsymbol{\mu})$. We have already seen that this can be done in three steps:

1. Compute the crystal isomorphism $\Psi_e^{\mathbf{s}\to\mathbf{v}}(\boldsymbol{\mu}) = (\nu^1, \dots, \nu^l)$ where $\mathbf{v}$ is very dominant (recall that this means that $\mathbf{v} = (s_1, s_2 + k_2 e, \dots, s_l + k_l e)$ with $(k_{j+1} - k_j)e > n - 1$ for $j > 1$). This can be done recursively by applying the algorithm described in §2.3 k times.
2. By Proposition 3.2.1, $m_e^{\mathbf{v}\to\mathbf{v}^\sharp}(\boldsymbol{\nu})$ can be computed by applying the Mullineux map component by component. As $|\nu| = |\lambda|$, if we assume that at least two components of the l-partition $(\nu^1, \dots, \nu^l)$ are non empty, all of the components are of rank $< n$ and we know how to compute the Mullineux involution by induction.
3. Apply again a crystal isomorphism $\Psi_e^{\mathbf{v}^\sharp\to\mathbf{s}^t}$.

In the next section, we will apply the above algorithm in the case where $l = 2$ and in particular show that the condition for applying our induction in step 2 is always satisfied (except in the case where $\mathbf{s} = (s_1, s_2)$ and $s_1 = s_2$). It is unclear which hypothesis we need to satisfy this condition for $l > 2$. We suggest that the multicharges $(s_1, s_2, \dots, s_l)$ such that $s_1 \neq s_2$ are however good candidates.

4. Combinatorial properties

In this section, we will try to find simple combinatorial ways to compute several objects that we have already defined: this concerns the admissible multicharges and multipartitions and the crystal isomorphisms.

4.1. On admissible multipartitions

If λ and μ are two partitions, we denote by $\lambda \sqcup \nu$ the partition obtained by concatenation (and reordering the parts if necessary).

Assume that we have an e-regular partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ (that is $\lambda \in \Phi_{e,(s)}(n)$ for any $s \in \mathbb{Z}$). Let $\mathbf{s} \in \mathcal{A}_e^l[s]$. By Proposition 2.4.1, $\mathbf{s}$ is an admissible multicharge. The aim of this subsection is to show that one can easily construct the associated admissible l-partition $\boldsymbol{\lambda} \in \Phi_{e,\mathbf{s}}(n)$ such that $\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \chi^n_{e,(s)}(\lambda)$ (recall that $\chi^n_{e,\mathbf{s}}$ is always injective) without considering the notion of multisegments. To do this, one can use the algorithm developed in [JL09a] from the datum of the multisegment $\chi^n_{e,(s)}(\lambda)$ or we can argue as follows. Let $l' \in \{1, \ldots, l\}$ be minimal such that $s_{l'} = s_l$. We construct $\boldsymbol{\lambda}$ by induction on the rank of $\boldsymbol{\lambda}$ as follows.

If $\lambda = \varnothing$ then $\boldsymbol{\lambda} := \varnothing$ and we are done. Otherwise, set

$$\mathbf{s}' := \begin{cases} (\underbrace{s_l, \ldots, s_l}_{l-l'+2}, s_2 + e, \ldots, s_{l'-1} + e) & \text{if } l' \neq 1 \\ \mathbf{s} & \text{if } l' = 1 \end{cases}$$

Note that we have $\mathbf{s}' \in \mathcal{A}_e^l[s_l]$. We denote $m := \lambda_1 + \ldots + \lambda_{e+s-s_l}$.

By induction, we have constructed the l-partition $\boldsymbol{\nu} \in \Phi_{(e,\mathbf{s}')}(n-m)$ such that we have

$$\chi^{n-m}_{e,\mathbf{s}'}(\boldsymbol{\nu}) = \chi^{n-m}_{e,(s_l)}(\lambda_{e+s-s_l+1}, \lambda_{e+s-s_l+2}, \ldots, \lambda_r).$$

We then define $\boldsymbol{\lambda}$ as follows

- If we have $l' = 1$ then $\lambda^1 = (\lambda_1, \ldots, \lambda_e) \sqcup \nu^l$ and $\lambda^j = \nu^{j-1}$ if $j \neq 1$.
- Otherwise, $\lambda^1 = (\lambda_1, \ldots, \lambda_{e+s-s_l}) \sqcup \nu^{2+l-l'}$ and $\lambda^j = \nu^{j+1-l'}$ for $j > 1$ where the indices are understood modulo l.

Proposition 4.1.1. *With this construction, we have $\boldsymbol{\lambda} \in \Phi_{e,\mathbf{s}}(n)$ and $\chi^n_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \chi^n_{e,(s)}(\lambda)$.*

Proof. We prove the proposition by induction. The result is trivial when $n = 0$. Keeping the above notations, one can assume that $\boldsymbol{\nu} \in \Phi_{(e,\mathbf{s}')}(n-m)$. First one can perform exactly the same procedure as in §2.4 for the description of the map $\chi^n_{e,\mathbf{s}}$ to associate to $\boldsymbol{\lambda}$ a multisegment (even if we have - not already - proved that $\boldsymbol{\lambda}$ is in $\Phi_{e,\mathbf{s}}(n)$). By construction, this multisegment is nothing but $\chi^n_{e,(s)}(\lambda)$. It is thus an aperiodic multisegment. This proves condition 3 of FLOTW l-partition for $\boldsymbol{\lambda}$ (see the definition in §2.2). Hence, we just need to show that the l-partition satisfies the two first points.

- If $l' = 1$, by induction, we have $\nu^j \geqslant \nu^{j+1}$ for all $j = 1, \ldots, l-1$. This implies that $\lambda^j_i \geqslant \lambda^{j+1}_i$ for all $j = 2, \ldots, l-1$ and that $\lambda^l_i \geqslant \lambda^1_{i+e}$ for all $i \geqslant 1$ and we get that $\lambda^1_i \geqslant \lambda^2_i$ because $(\lambda_1, \ldots, \lambda_e)$ are the greatest parts of λ and because $\nu^l_i \geqslant \nu^1_{i+e}$ for all $i > 0$.

- If $l' \neq 1$, by the property of FLOTW l-partitions, we have that
$$\boldsymbol{\mu} := (\nu^{l-l'+3}, \dots, \nu^{l}, \nu^{1}, \dots, \nu^{l-l'+1}, \nu^{l-l'+2})$$
is in $\Phi_{e,\mathbf{v}}(n-m)$ for $\mathbf{v} = (s_2, \dots, s_{l'-1}, s_l, \dots, s_l, s_l)$ and we can thus conclude using the fact that $\lambda^1_j = \lambda_j$ if $j = 1, \dots, e+s-s_l$ and $\lambda^1_j = \mu^l_{j-(e+s-s_l)}$ otherwise. □

In the case where $l = 2$ (which is the case that we will mostly studied in the forthcoming sections), the multipartition $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ is easy to obtain. One can assume that $s_1 = 0$, then we have
$$\lambda^1 = (\lambda_1, \dots, \lambda_{e-s_2}, \lambda_{2e-s_2+1}, \dots, \lambda_{3e-s_2}, \dots, \lambda_{2ke-s_2+1}, \dots, \lambda_{3ke-s_2}, \dots),$$
and
$$\lambda^2 = (\lambda_{e-s_2+1} \dots, \lambda_{2e-s_2}, \lambda_{3e-s_2+1}, \dots, \lambda_{4e-s_2}, \dots\ \lambda_{3ke-s_2+1}, \dots, \lambda_{4ke-s_2}, \dots).$$
In other words, the first $e - s_2$ parts of λ goes to λ^1 then the next e parts to λ^2 then the next e parts to λ^1 and so on.

Example 4.1. Let us take $e = 4$, $\lambda = (8, 8, 6, 6, 4, 3, 3, 2, 1, 1)$, then the associated Young tableau (with the residues of each node marked in the associated box) is:

0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2
2	3	0	1	2	3		
1	2	3	0	1	2		
0	1	2	3				
3	0	1					
2	3	0					
1	2						
0							
1							

Take $\mathrm{s} = (0, 2, 2)$. Following the algorithm, we first have $l' = 2$. Then $\mathrm{s}' = (2, 2, 2)$. We have $m = \lambda_1 + \lambda_2$ and we need to compute $\boldsymbol{\nu}$ such that
$$\chi^{n-m}_{4,(2,2,2)}(\boldsymbol{\nu}) = \chi^{n-m}_{4,(2)}(6, 6, 4, 3, 3, 2, 1, 1)$$
We obtain $\boldsymbol{\nu} = ((6, 6, 4, 3), (3, 2, 1, 1), \varnothing)$ and we have $\boldsymbol{\lambda} = ((8, 8), (6, 6, 4, 3), (3, 2, 1, 1))$.

In the case where $l = 2$, we have:

- If $\mathrm{s} = (0, 0)$, we have $\boldsymbol{\lambda} = ((8, 8, 6, 6, 1, 1), (4, 3, 3, 2))$.
- If $\mathrm{s} = (0, 1)$, we have $\boldsymbol{\lambda} = ((8, 8, 6, 2, 1, 1), (6, 4, 3, 3))$.
- If $\mathrm{s} = (0, 2)$, we have $\boldsymbol{\lambda} = ((8, 8, 3, 2, 1, 1), (6, 6, 4, 3))$.
- If $\mathrm{s} = (0, 3)$, we have $\boldsymbol{\lambda} = ((8, 3, 3, 2, 1), (8, 6, 6, 4, 1))$.

Using this, we have thus constructed a map
$$\theta^n_{e,\mathbf{s}} : \Phi_{e,(0)}(n) \to \Phi_{e,\mathbf{s}}(n)$$
which associates to λ the l-partition $\boldsymbol{\lambda}$ constructed above (we will sometime omit the subscript n).

4.2. Crystal isomorphisms

In this second subsection, we study in details the crystal isomorphisms restricted to the multipartitions in the image of $\theta_{e,\mathbf{s}}$ in the case where $l = 2$. The first aim is to simplify the procedure to compute it, the second is to show certain crucial properties which will help us to prove the correctness of our algorithm.

Let λ be an e-regular partition and assume that $\mathbf{s} = (0, s)$. We also assume that λ is non empty and that r is maximal such that $\lambda_r \neq 0$. Let $(\lambda^1, \lambda^2) := \theta_{(e,(0,s))}(\lambda)$ and consider the associated symbol with length te with t sufficiently large. It is thus of the following form :

$$\begin{pmatrix} \alpha_{te} & \dots & \alpha_{(t-1)e+1} & \dots & \alpha_{2e} & \dots & \alpha_{e+1} & \alpha_e & \dots & \alpha_{s+1} & \dots & \alpha_1 \\ \beta_{te-s} & \dots & \beta_{(t-1)e-s+1} & \dots & \beta_{2e-s} & \dots & \beta_{e-s+1} & \beta_{e-s} & \dots & \beta_1 & & \end{pmatrix}$$

By definition of the symbol, we here have $\alpha_j := \lambda^2_j - j + s$ for $j = 1, \dots, ke$ and $\beta_j := \lambda^1_j - j$ for $j = 1, \dots, ke - s$. We denote $(\mu^1, \mu^2) := \Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1, \lambda^2)$ (so that, as usual, $ke > n - 1$ and thus so that the multicharge $(0, s + ke)$ is very dominant). We have:

$$\Psi_e^{(0,s)\to(0,s+k.e)} = \Psi_e^{(0,s+(k-1)e)\to(0,s+ke)} \circ \Psi_e^{(0,s+(k-2)e)\to(0,s+(k-1)e)} \circ \dots \circ \Psi_e^{(0,s)\to(0,s+e)}$$

and we will see how this map can be more easily described in the cases we are interested in (see the second paragraph below)

Assume that $\lambda \neq \emptyset$ and that $\mu^2 = \emptyset$ then the algorithm for the computation of $\Psi_e^{(0,s)\to(0,s+k.e)}$ easily shows that that this can happen if and only if $\Psi_e^{(0,s)\to(0,s+k.e)}$ is the identity. Indeed if $(\mu^1, \mu^2) := \Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1, \lambda^2)$ then $|\mu^2| \geqslant |\lambda^2|$ so $\mu^2 = \lambda^2 = \emptyset$ and $\mu^1 = \lambda^1$. The fact that $\Psi_e^{(0,s)\to(0,s+k.e)}$ is the identity implies that:

$$\{\beta_i \mid i = 1, \dots, ke - s\} \subset \{\alpha_i \mid i = 1, \dots, ke\}.$$

In this case, we also need to have $r \leqslant e - s$. Now we have for all $i = 1, \dots, ke$, $\alpha_i = -i + s$ and also $\beta_j \leqslant \alpha_j$ for all $j = 1, \dots, ke - s$. As a consequence, we have

$$\lambda^2_1 - 1 \leqslant -1 + s.$$

and thus $\lambda^2_2 \leqslant s$. We conclude

Proposition 4.2.1. *Under the above notations, assume that $\mu^2 = \emptyset$ then $\lambda = \lambda^1$ is an e-core.*

Proof. The above discussion shows that λ has at most $e - s$ non empty rows and at most s columns. This implies that the hooks of λ has at most length $e - 1$ and thus that λ is an e-core. □

Now let us see what we can say if $\mu^1 = \varnothing$. Before this, we show below that the image of $\boldsymbol{\lambda}$ under a crystal isomorphism can be quite easily computed in the case where $\boldsymbol{\lambda}$ is in the image of $\theta_{e,\mathbf{s}}$ which is the case we are interested in here.

Keeping, the above notations, for all $i = 1, \dots, k-1$, we have $\alpha_{ie} = \lambda_{2ie-s} - ie + s$ and $\beta_{ie-s+1} = \lambda_{2ie-s+1} - (ie - s + 1)$. So we have $\alpha_{ie} + ie - s \geqslant \beta_{ie-s+1} + ie - s + 1$ and thus $\alpha_{ie} > \beta_{ie-s+1}$.

In addition $\alpha_{ie+1} = \lambda_{2ie+1-s} - (ie+1) + s$ and $\beta_{(i+1)e-s} = \lambda_{2ie-s} - ((i+1)e - s)$. So we have $\beta_{(i+1)e-s} + ((i+1)e - s) \geqslant \alpha_{ie+1} + (ie+1) - s$. So $\beta_{(i+1)e-s} + e > \alpha_{ie+1}$.

These calculations show that one can perform our crystal isomorphism step by steps in the "blocks" of the symbol separated by vertical lines below. First recall in Example 2.5 how the crystal isomorphisms $\Psi_e^{(0,s')\to(0,s'+e)}$ can be described.

$$\left(\begin{array}{ccc|c|ccc|ccccc} \alpha_{ke} & \dots & \alpha_{(k-1)e+1} & \dots & \alpha_{2e} & \dots & \alpha_{e+1} & \alpha_e & \dots & \alpha_{s+1} & \dots & \alpha_1 \\ \beta_{ke-s} & \dots & \beta_{(k-1)e-s+1} & \dots & \beta_{2e-s} & \dots & \beta_{e-s+1} & \beta_{e-s} & \dots & \beta_1 & & \end{array}\right)$$

We see that all the calculations in the blocks are trivial except in the rightmost. After one step of the crystal isomorphism we get

$$\left(\begin{array}{ccc|c|ccc|ccccccc} 0 & \dots & e-1 & \dots & \beta_{3e-s}+e & \dots & \beta_{2e-s+1}+e & \beta_{2e-s}+e & \dots & \beta_{e+1} & \dots & \beta_{e-s+1}+e & \dots & \alpha'_1 \\ \alpha_{ke} & \dots & \alpha_{(k-1)e+1} & \dots & \alpha_{2e} & \dots & \alpha_{e+1} & \beta'_{e-s} & \dots & \beta'_1 & & & & \end{array}\right)$$

and we see that the properties above are always satisfied. In particular, with the notation above, we have.

$$\beta'_{e-s} + e > \beta_{e-s+1} + e.$$

Now, take the right end of our first symbol:

$$\left(\begin{array}{cccccc} \alpha_e & \dots & \alpha_{s+1} & \alpha_s & \dots & \alpha_1 \\ \beta_{e-s} & \dots & \beta_1 & & & \end{array}\right)$$

We already know that $\beta_{e-s} + e > \alpha_1$. Assume that we have $\lambda^1_j \neq 0$ so that $\beta_j > -j$. Then we claim that this implies that we have $\beta_j \geqslant \alpha_{s+j-1}$. To do this, note that we have:

$$\beta_j \geqslant \beta_{j-1} + 1 \geqslant \dots \geqslant \beta_{e-s} + (e - s - j) > \alpha_1 - s - j.$$

Now we have $\alpha_1 \geqslant \alpha_2 + 1 \geqslant \dots \geqslant \alpha_{s+j-1} + (s + j - 2)$. So

$$\beta_j > \alpha_{s+j-1} - 2.$$

The only problem may appear if $\beta_j = \alpha_{s+j-1} - 1$ and this implies that all the inequalities above are in fact equalities. We thus have:

$$\beta_j = \beta_{j-1} + 1 = \dots = \beta_{e-s} + (e - s - j),$$

and

$$\alpha_1 = \alpha_2 + 1 \geqslant \dots = \alpha_{s+j-1} + (s+j-2) = \beta_j + s - j - 1 = \beta_{j-1} + s - j = \dots = \beta_{e-s} + e - 1.$$

This case implies that we have an e-period in the sense of [JL13, Def. 2.2]. Such property is impossible for Uglov l-partitions by [JL13, Prop. 5.1].

This discussion implies that, under the notation above, if we have $\beta_j > -j$ then we must have $\beta'_j > -j$ so that the associated part of the partition is also non zero. By a direct induction, we thus deduce:

Proposition 4.2.2. *Let $0 < s < e$ and let λ be an e-regular partition and $(\lambda^1, \lambda^2) := \theta_{(e,(0,s))}(\lambda)$. Assume that $(\mu^1, \mu^2) := \Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1, \lambda^2)$ for $k >> 0$ (so that $(0, s+k.e)$ is very dominant, see §2.2). Then $|\mu^1| \neq 0$.*

Remark 4.2. In the case where $s = 0$, the above discussion also shows that if $(\lambda^1, \lambda^2) := \theta_{(e,(0,0))}(\lambda)$ then $\Psi_e^{(0,s)\to(0,k.e)}(\lambda^1, \lambda^2) = (\varnothing, \lambda)$ for $k >> 0$. As a consequence, this choice of multicharge cannot be used to get our recursive algorithm to compute the Mullineux involution because then it would require the computation of $m_e(\lambda)$ ... to compute $m_e(\lambda)$.

Example 4.3. Let us take $e = 3$ and the partition $\lambda = (12, 10, 8, 8, 6, 5, 4, 4, 3, 2, 2, 1)$. With $\mathbf{s} = (0, 1)$, we have $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) = ((8, 8, 6, 3, 2, 2), (12, 10, 5, 4, 4, 1))$. With the above notations and $k = 2$, we thus obtain the following symbol for $\boldsymbol{\lambda}$.

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 5 & 6 & 8 & 12 & 15 & 16 \\ 0 & 1 & 3 & 7 & 8 & 10 & 16 & 19 & \end{array}\right)$$

We perform our algorithm of §2.3 to get:

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 3 & 7 & 8 & 10 & 12 & 16 & 19 \\ 0 & 1 & 2 & 5 & 6 & 8 & 15 & 16 & \end{array}\right)$$

The symbol for $\Psi_3^{(0,1)\to(0,4)}(\boldsymbol{\lambda})$ is

$$\left(\begin{array}{ccc|ccc|cccccc} 0 & 1 & 2 & 3 & 4 & 6 & 10 & 11 & 13 & 15 & 19 & 22 \\ 0 & 1 & 2 & 5 & 6 & 8 & 15 & 16 & & & & \end{array}\right)$$

We can apply our simplified procedure to get:

$$\left(\begin{array}{ccc|ccc|cccccc} 0 & 1 & 2 & 5 & 6 & 8 & 10 & 11 & 15 & 16 & 19 & 22 \\ 0 & 1 & 2 & 3 & 4 & 6 & 13 & 15 & & & & \end{array}\right)$$

and the symbol for $\Psi_3^{(0,1)\to(0,7)}(\boldsymbol{\lambda})$ is

$$\left(\begin{array}{ccc|ccc|ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 8 & 9 & 11 & 13 & 14 & 18 & 19 & 22 & 25 \\ 0 & 1 & 2 & 3 & 4 & 6 & 13 & 15 & & & & & & & \end{array}\right)$$

and so on.

5. The algorithm

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be an e-regular partition of rank n. We can now present a recursive algorithm for computing $m_e(\lambda)$. First by Remark 3.2, one can assume that λ is not an e-core. The algorithm now consists in the following steps:

1. Choose $0 < s < e$ and consider the bipartition $(\lambda^1, \lambda^2) := \theta_{(e,(0,s))}(\boldsymbol{\lambda})$.

2. Compute $(\mu^1, \mu^2) := \Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1, \lambda^2)$ for $k >> 0$. By Propositions 4.2.1 and 4.2.2, we now that $|\mu^1| < n$ and $|\mu^2| < n$.

3. By induction, we know $m_e(\mu^1)$ and $m_e(\mu^2)$ and we can thus compute:
$$(\kappa^1, \kappa^2) := \Psi_e^{(0,-s+ke)\to(0,e-s)}(m_e(\mu^1), m_e(\mu^2)).$$

4. We have $m_e(\lambda) = \theta^{-1}_{(e,(0,e-s))}(\kappa^1, \kappa^2)$.

Note that in principle, one can choose an arbitrary multicharge s instead of $(0, s)$ (as soon as the second point at the end of subsection 3.3 is satisfied) but the complexity of the algorithm for the computation of the crystal isomorphism from s to a very dominant multicharge increases. However, one can expect that some particular multicharge can lead to interesting fast new algorithms. We now show how one can perform the above steps.

5.1. Steps 1 and 2

It follows from Section 4.2 that the first two steps can be both implemented by the process below. This is just a translation of the process described in this section in the language of Young tableaux. Let $0 < s < e$ and set $\mathbf{s} = (0, s)$. We set $\boldsymbol{\lambda}[1] = (\lambda_1, \dots, \lambda_{e-s})$ and $\boldsymbol{\lambda}[2] = (\lambda_{e-s+1}, \dots, \lambda_r)$, we write the Young tableau of $\boldsymbol{\lambda}[1]$ with the associated contents and just below, the Young tableau of $\boldsymbol{\lambda}[2]$ with the associated contents with respect to the multicharge $(0, s)$.

$\boldsymbol{\lambda}[1]$						
0	1	2	3	$\dots$	$\dots$	$\lambda_1 - 1$
-1	0	1	2	$\dots$	$\lambda_2 - 2$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(e-s-1)$	$\dots$	$\dots$	$\lambda_{e-s} - (e-s)$			
$\boldsymbol{\lambda}[2]$						
s	$s+1$	$\dots$	$\lambda_{e-s+1} - 1 + s$			
$s-1$	$\dots$	$\lambda_{e-s+2} - 2 + s$				
$\dots$	$\dots$					

For example, take $\lambda = (10, 8, 7, 5, 4, 4, 3, 2, 1, 1)$. Take $e = 4$ and $s = 1$

$\boldsymbol{\lambda}[1]$									
0	1	2	3	4	5	$\mathbf{6}$	$\mathbf{7}$	$\mathbf{8}$	$\mathbf{9}$
-1	0	1	2	3	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$		
-2	-1	0	1	2	$\mathbf{3}$	$\mathbf{4}$			
$\boldsymbol{\lambda}[2]$									
1	2	3	4	5					
0	1	2	3						
-1	0	1	2						
-2	-1	0							
-3	-1								
-4									
-5									

Now, starting with the first part of $\boldsymbol{\lambda}[1]$, consider the content of the rightmost box, say c. In $\boldsymbol{\lambda}[2]$, we consider the rightmost boxes and we take the one with the greatest content which is less than c, say c' (if it does not exist we switch to the second part of $\boldsymbol{\lambda}[1]$). Then we remove the boxes of the first part of $\boldsymbol{\lambda}[1]$ with content greater than c' into this part in $\boldsymbol{\lambda}[2]$ (in other words, we move the "truncated first row" containing the boxes grater than c to the row in $\boldsymbol{\lambda}[2]$).

It is clear that we still have a partition. Then, we do the same for the second part of $\boldsymbol{\lambda}[1]$ and so on until we reach the last part of $\boldsymbol{\lambda}[1]$. Wwe continue this process until we reach the last part of $\boldsymbol{\lambda}[1]$.

In our example, we must remove the boxes in bold in the first partition above, and add the boxes in bold in the second partition below.

$\boldsymbol{\lambda}[1]$									
0	1	2	3	4	5				
-1	0	1	2	3					
-2	-1	0	1	2					
$\boldsymbol{\lambda}[2]$									
1	2	3	4	5	$\mathbf{6}$	$\mathbf{7}$	$\mathbf{8}$	$\mathbf{9}$	
0	1	2	3	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$			
-1	0	1	2	$\mathbf{3}$	$\mathbf{4}$				
-2	-1	0							
-3	-1								
-4									
-5									

We then collect all the parts of $\boldsymbol{\lambda}[2]$ that are above the smallest part we have modified, in a partition μ. So here $\mu = (9, 7, 6)$. The new partition $\boldsymbol{\lambda}[2]$ is given by the remaining parts and we add e to the contents of all the boxes in it. Thus in the example, the first row of $\boldsymbol{\lambda}[2]$ starts with a box of content $-2+4=2$. We then move to the step above and continue the process until we cannot do anything. The remaining parts of $\boldsymbol{\lambda}[2]$ are then added to μ. Finally the partition $\boldsymbol{\lambda}[1]$ is the first component of $\Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1,\lambda^2)$ and μ is the second.

$\boldsymbol{\lambda}[1]$					
0	1	2	3	4	$\mathbf{5}$
-1	0	1	2	$\mathbf{3}$	
-2	-1	0	$\mathbf{1}$	$\mathbf{2}$	

$\boldsymbol{\lambda}[2]$		
2	3	4
1	2	
0		
-1		

It becomes :

$\boldsymbol{\lambda}[1]$				
0	1	2	3	4
-1	0	1	2	
-2	-1	0		

$\boldsymbol{\lambda}[2]$			
2	3	4	$\mathbf{5}$
1	2	$\mathbf{3}$	
0	$\mathbf{1}$	$\mathbf{2}$	
-1			

We have now $\mu=(9,7,6,4,3,3)$, and we the above process:

$\boldsymbol{\lambda}[1]$				
0	1	2	3	$\mathbf{4}$
-1	0	1	$\mathbf{2}$	
-2	-1	0		

$\boldsymbol{\lambda}[2]$
3

gives :

$\boldsymbol{\lambda}[1]$			
0	1	2	3
-1	0	1	
-2	-1	0	
$\boldsymbol{\lambda}[2]$			
3	$\mathbf{4}$		
$\mathbf{2}$			

and then $\mu = (9,7,6,4,3,3,2,1)$. There is nothing we can do now. The bipartition we are searching for is $((4,3,3),((9,7,6,4,3,3,2,1))$

5.2. Step 3 and 4

At this stage, we have computed $(\mu^1,\mu^2) := \Psi_e^{(0,s)\to(0,s+k.e)}(\lambda^1,\lambda^2)$. By induction, we thus know $(\nu^1,\nu^2) := (m_e(\mu^1), m_e(\mu^2))$ and we must reverse the process considered in §5.1. to get our bipartition:

$$\Psi_e^{(0,-s+ke)\to((0,e-s)}(\nu^1,\nu^2).$$

This is done as follows.

We write the Young tableau of ν^1 with the associated contents for each box, and just below, the Young tableau of ν^2 with the associated contents charged by $ke-s$ where k is sufficiently large (that is, the content of the box (a,b) is $b-a+(ke-s)$). Keeping the above example, we have by induction $m_4(4,3,3) = (10)$ and $m_4(9,7,6,4,3,3,2,1) = (14,7,7,3,3,1)$. So we consider the bipartition $((10),(14,7,7,3,3,1)$ and the multicharge is $(0,3)$.

ν^1													
0	1	2	3	4	5	6	7	8	9				
ν^2													
19	20	21	22	23	24	25	26	27	28	29	30	31	32
18	19	20	21	22	23	24							
17	18	19	20	21	22	23							
16	17	18											
15	16	17											
14													

At each step, starting from the bottom of ν^2, we see if one can remove boxes from ν^2 to add it to ν^1 as in the subsection above (except that we remove the box from the other partition). Note that the number of rows in ν^1 is fixed so we can only add boxes in the first $e-s$ rows of ν^1. Then we remove e from all the contents of the boxes of ν^2. In the example, we have nothing to do so we remove e from all the contents of the second partitions and again one more time.

ν^1													
0	1	2	3	4	5	6	7	8	9				
ν^2													
15	16	17	18	19	20	21	22	23	24	25	26	27	28
14	15	16	17	18	19	20							
13	14	15	16	17	18	19							
12	13	14											
11	12	13											
10													

Then we can add a box of content 10 and we subsract e from all the contents. We then successively obtain the following bipartitions.

ν^1													
0	1	2	3	4	5	6	7	8	9	**10**			
ν^2													
15	16	17	18	19	20	21	22	23	24	25	26	27	28
14	15	16	17	18	19	20							
13	14	15	16	17	18	19							
12	13	14											
11	12	13											

and then:

ν^1													
0	1	2	3	4	5	6	7	8	9	**10**	**11**		
ν^2													
3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	5	6	7	8							
1	2	3	4	5	6								
0	1	2											
-1	0	1											

and then:

ν^1																
0	1	2	3	4	5	6	7	8	9	10	11	**12**	**13**	**14**	**15**	**16**
ν^2																
3	4	5	6	7	8	9	10	11								
2	3	4	5	6	7	8										
1	2	3	4	5	6											
0	1	2														
$\overline{1}$	0	1														

At the end, the concatenation (with reordering of the parts if necessary) of the two partitions we get must be $m_e(\lambda)$. In our example, we obtain $((17),(9,7,6,3,3))$ so that $m_e(\lambda)=(17,9,7,6,3,3)$.

5.3. Example

Let us keep our running example $\lambda=(10,8,7,5,4,4,3,2,1,1)$, $l=2$ and $e=4$ but this time, we take $s=2$. The first two steps will give:

$\boldsymbol{\lambda}[1]$									
0	1	2	3	4	5	6	7	8	$\mathbf{9}$
-1	0	1	2	3	4	5	$\mathbf{6}$		
$\boldsymbol{\lambda}[2]$									
2	3	4	5	6	7	8			
1	2	3	4	5					
0	1	2	3						
-1	0	1	2						
-2	-1	0							
-3	-1								
-4									
-5									

$\rightarrow$

$\boldsymbol{\lambda}[1]$								
0	1	2	3	4	5	6	7	$\mathbf{8}$
-1	0	1	2	3	4	$\mathbf{5}$		
$\boldsymbol{\lambda}[2]$								
4	5	6	7					
3	4	5	6					
2	3	4						
1	2							
0								
-1								

$\rightarrow$

$\boldsymbol{\lambda}[1]$							
0	1	2	3	4	5	6	$\mathbf{7}$
-1	0	1	2	3	4		
$\boldsymbol{\lambda}[2]$							
5	6						
4							
3							

$\rightarrow$

$\boldsymbol{\lambda}[1]$						
0	1	2	3	4	5	$\mathbf{6}$
-1	0	1	2	3		
$\boldsymbol{\lambda}[2]$						
8						
7						

and thus, we obtain the bipartition $((6,6),(8,6,5,4,4,3,1,1,1)$ which is thus the bipartition

$$\Psi_4^{(0,2)\to(0,2+4k)}((10,8,3,2,1,1),(7,5,4,4)).$$

Now, by induction, we know $m_4(6,6) = (6,6)$ and $m_4(8,6,5,4,4,3,1,1,1) = (15,7,5,4,1,1)$. We now perform Steps 3 and 4 for $((6,6),(15,7,5,3,1,1))$.

ν^1														
0	1	2	3	4	5									
−1	0	1	2	3	4									
ν^2														
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
9	10	11	12	13	14	15								
8	9	10	11	12										
7	8	9	10											
6														
5														

and then:

ν^1														
0	1	2	3	4	5	6								
−1	0	1	2	3	4	5								
ν^2														
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	6	7	8	9	10	11								
4	5	6	**7**	**8**										
3	4	5	**6**											

and then:

ν^1													
0	1	2	3	4	5	6	7	8					
−1	0	1	2	3	4	5	6						
ν^2													
2	3	4	5	6	7	8	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	2	3	4	5	6								

and then:

ν^1															
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
−1	0	1	2	3	4	5	6	7							
ν^2															

We obtain the bipartition $((17,9),(7,6,3,3))$ and we conclude that $m_e(\lambda) = (17,9,7,6,3,3)$ as in the last section.

6. Xu algorithm

In [Xu99, Xu97], Xu has given an algorithm for the computation of the Mullineux involution which is derived from the original Mullineux's algorithm. We here recall this algorithm and then show that it can be seen as a particular case of ours. This will in particular give a new proof that the algorithm computes the Mullineux involution.

6.1. The algorithm

To describe Xu algorithm, we will need some additional combinatorial definitions. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be an e-regular partition with $\lambda_r \neq 0$. The *rim* of λ is the subset of the Young diagram of $\boldsymbol{\lambda}$ consisting in the (i, j) such that $(i+1, j+1)$ is not in $[\lambda]$. The e-*rim* is now the subset $\{(a_1, b_1), \dots, (a_m, b_m)\}$ of the rim of λ which is obtained by following the rim of λ from right to left and top to bottom, and moving down one row every time the number of nodes we have is dividible by e.

Example 6.1. Let $e = 3$ and $\lambda = (7.4.2.2)$. The e-rim is given by the nodes marked by a star.

					★	★	★
		★	★	★			
		★					
	★	★					

Assume that the cardinality of the e-rim of λ is m. The *truncated* e-*rim* of λ is by definition the set of nodes (i, j) in the e-rim of λ such that $(i, j-1)$ is also in the e-rim of λ. If e does not divide m, we add also the node (r, x) in the e-rim of λ such that $(r, x-1)$ is not in the e-rim (recall that r is the length of the partition). We now define $\widetilde{\lambda}$ to be the partition obtained by removing the truncated e-rim from λ. It is easy to see that this partition is e-regular with rank strictly less than the rank of λ.

Example 6.2. Let $e = 3$ and $\lambda = (8, 5, 3, 3)$. The truncated e-rim is given by the nodes marked by a star.

						★	★
			★	★			
		★					

So the partition $\widetilde{\lambda}$ is $(6, 3, 3, 2)$.

Now we define a map

$$X_e : \Phi_{(e,(0))} \to \Phi_{(e,(0))}$$

recursively as follows. We define $X_e(\varnothing) = \varnothing$ and if $\lambda \in \Phi_{(e,(0))}(n)$ with $n \neq 0$ then $X_e(\lambda)$ is obtained by adding a column of length $n - |\widetilde{\lambda}|$ to $X_e(\widetilde{\lambda})$.

Theorem 6.1.1 (Xu). *We have* $X_e = m_e$.

We here give a new proof of this Theorem using the crystal isomorphisms.

Example 6.3. We keep the above example. We can compute $X_3(6,3,3,2) = (8,2,2,1,1)$, now we have exactly 5 nodes in the truncated 3-rim of λ so $X_e(7,4,2,2)$ is obtained by adding a column of length 5 to $(8,2,2,1,1)$ and we get $X_3(8,5,3,3) = (9,3,3,2,2)$.

6.2. Relation with crystal isomorphisms

We will see in this subsection that Xu algorithm is equivalent to ours in the case where we choose $s = e-1$. For λ an e-regular partition, we denote by $\widetilde{\lambda}$ the partition obtained by removing the truncated p-rim as in Xu algorithm. We denote by r the number of boxes in the truncated p-rim.

Proposition 6.2.1. *We have* $\Psi_e^{(0,e-1)\to(0,e-1+ke)} \circ \theta_{e,(0,e-1)}(\lambda) = (r, \widetilde{\lambda})$ *(*$k >> 0$*)*

Proof. We denote $\boldsymbol{\lambda}[2] = (\lambda_2, \ldots, \lambda_e)$. We begin with the two first steps of our algorithm which are described in §5.1. Assume first that one cannot add any "truncated row" of λ_1 in $\boldsymbol{\lambda}[2]$. This means that there exists $k > 0$ such that $\lambda_{k+1} - k + e - 1 = \lambda_1 - 1$ and we have the following partitions:

$\boldsymbol{\lambda}[1]$										
0	1	2	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	x	$\ldots$	λ_1-1
$\boldsymbol{\lambda}[2]$										
$e-1$	e	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$x+e-1$		
$e-2$	$e-1$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	λ_3+e-3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$e-s+1$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	λ_s-s+e				
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\vdots$						
$e-k$	$\ldots$	$\ldots$	$\ldots$	λ_1-1						
$e-k-1$	$\ldots$	$\ldots$	$\ldots$	λ_1-2						

(with $x = \lambda_2 - 1$)

Then the partition $(\lambda_2, \ldots, \lambda_{k+1})$ corresponds to the partition $(\lambda_1, \ldots, \lambda_k)$ with the very first truncated e-rim removed. If $\lambda_{k+1} = 0$ then we are done and λ_1 is the number of nodes in the truncated p-rim minus 1. In this case the number of elements in the associated e-rim is not e. Otherwise we get e boxes in the associated rim and we must go to the second step of our algorithm.

Assume that one can add a truncated row of length r. Assume that the row is added in the part λ_{k+1}. Then the partition $(\lambda_2, \ldots, \lambda_{k+1})$ corresponds to the partition $(\lambda_1, \ldots, \lambda_k)$ with a truncated e-rim removed.

$\boldsymbol{\lambda}[1]$										
0	1	2	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	x	$\dots$	λ_1-1
$\boldsymbol{\lambda}[2]$										
$e-1$	e	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$x+e-1$		
$e-2$	$e-1$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	λ_3+e-3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$e-k+1$	$\dots$	$\dots$	$\dots$	λ_1	$\dots$	λ_k-k+e				
$e-k$	$\dots$	x	$\dots$	λ_1-1						

Assume that λ_{k+1} is non zero. Note that the length of the truncated p-rim is $e-k$. By induction, the first $e-k$ nodes of the partition $\boldsymbol{\lambda}[1]$ will not moved in our algorithm. We can thus just argue by induction by replacing $\boldsymbol{\lambda}[1]$ with the partition $\boldsymbol{\lambda}[1]-(e-k)$ to find $\boldsymbol{\lambda}[2]$ and take into account that we must add $e-k$ (the length of the truncated p-rim) to the partition we obtain at the end of our algorithm. Note that the content of the leftmost node in our first partition will be now $e-k$ and the contet of the leftmost node of the second partition $(e-k)-e-1$ so the induction can be done. □

On the other hand, we now have the following result:

Proposition 6.2.2. *We have* $\Psi_e^{(0,1)\to(0,1+ke)}\circ\theta_{e,(0,1)}(\lambda)=(m_e((t)),\lambda-1)$ *where* t *is the length of the first column of* λ *(*$k>>0$*). Here* (t) *denotes the partition with one part equal to* t *and* $\lambda-1$ *is the partition obtained from* λ *by decreasing each non zero part by* 1*.*

Proof. We use the algorithm described in Subsection 5.2, using these notation, we are in the following configuration:

$\boldsymbol{\lambda}[1]$						
0	1	2	3	$\dots$	$\dots$	λ_1-1
-1	0	1	2	$\dots$	λ_2-2	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(e-2)$	$\dots$	$\dots$	$\lambda_{e-1}-(e-1)$			
$\boldsymbol{\lambda}[2]$						
1	2	$\dots$	$\dots$	λ_e		
0	$\dots$	$\dots$	$\lambda_{e+1}-1$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$-(e-3)$	$\dots$	$\dots$	$\lambda_{2e-2}-(e-2)$			
$-(e-2)$	$\dots$	$\lambda_{2e-1}-(e-1)$				
$\vdots$	$\vdots$	$\vdots$				

The first step of our algorithm thus gives:

$\boldsymbol{\lambda}[1]$						
0	1	2	3	$\dots$	$\dots$	λ_e
-1	0	1	2	$\dots$	$\lambda_{e+1}-1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(e-2)$	$\dots$	$\dots$	$\lambda_{2e-2}-(e-2)$			

$\boldsymbol{\lambda}[2]$				
1	2	$\dots$	$\dots$	λ_1-1
0	$\dots$	$\dots$	λ_2-2	
$\vdots$	$\vdots$	$\vdots$		
$-(e-3)$	$\dots$	$\lambda_{e-1}-(e-1)$		
$-(e-2)$	$\dots$	$\lambda_{2e-1}-(e-1)$		
$\vdots$	$\vdots$	$\vdots$		

and now, we have to perform the algorithm for the following configuration of partitions:

$\boldsymbol{\lambda}[1]$						
0	1	2	$\dots$	$\dots$	$\dots$	λ_e
-1	0	1	$\dots$	$\dots$	$\lambda_{e+1}-1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(e-2)$	$\dots$	$\dots$	$\lambda_{2e-2}-(e-2)$			

$\boldsymbol{\lambda}[2]'$				
2		$\dots$	$\dots$	$\lambda_{2e-1}+1$
1	$\dots$	$\dots$	λ_{2e}	
$\vdots$	$\vdots$	$\vdots$		
$-(e-2)$	$\dots$	$\lambda_{3e-3}-e+3$		
$\vdots$	$\vdots$	$\vdots$		

which thus leads to

$\boldsymbol{\lambda}[1]$						
0	1	2	3	$\dots$	$\dots$	$\lambda_{2e-1}+1$
-1	0	1	2	$\dots$	λ_{2e}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(e-2)$	$\dots$	$\dots$	$\lambda_{3e-3}-e+3$			
$\boldsymbol{\lambda}[2]'$						
2		$\dots$	$\dots$	λ_e		
1	$\dots$	$\dots$	$\lambda_{e+1}-1$			
$\vdots$	$\vdots$	$\vdots$				
$-(e-2)$	$\dots$	$\lambda_{2e-2}-(e-2)$				
$-(e-3)$	$\dots$	$\lambda_{3e-2}-(e-2)$				
$\vdots$	$\vdots$	$\vdots$				

Now, we come to the last step, assume that s is maximal such that $\lambda_{ke-k+s} \neq 0$ (so that $ke-k+s$ is the length of the first column of λ). Then, we are in the following configuration where we have an addable $k+1$-node in the second partition.

$\boldsymbol{\lambda}[1]$					
0	1	$\dots$	$\dots$	$\dots$	$\lambda_{ke-k+1}+k-1$
1	0	$\dots$	$\dots$	$\lambda_{ke-k+2}+k-2$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$-(s-1)$	$\dots$	$\dots$	$\lambda_{ke-k+s}+k-s$		
$-s$	$\dots$	$k-s-1$			
$\vdots$	$\vdots$	$\vdots$			
$-(e-2)$	$\dots$	$k-(e-1)$			
$\boldsymbol{\lambda}[2]'$					

and we obtain for $\boldsymbol{\lambda}[1]$:

0	$\dots$	$\dots$	k
-1	0	$\dots$	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-(s-1)$	$\dots$	$k-s$	$k+1-s$
$-s$	$\dots$	$k-s-1$	
$\vdots$	$\vdots$	$\vdots$	
$-(e-2)$	$\dots$	$k-(e-1)$	

The first partition is the image of the partition $((k+1)(e-1)-(-s-2+e+1)) = (ke-k+s)$ under the Mullineux involution. The second partition we get in the algorithm is $\lambda - 1$ which is exactly what we wanted. □

Let us now explain in which way our two algorithms are equivalent in the case where we choose $\mathrm{s} = (0, e-1)$. Let λ be an e-regular partition and recall the 4 steps of our algorithm at the beginning of §5.

1. By Proposition 6.2.1, after the two first steps of our algorithm, we obtain $(r, \widetilde{\lambda})$ where r is the number of boxes in the truncated rim.

2. By induction, we know $m_e(\widetilde{\lambda})$ and the third step of our algorithm consists in the computation of the image of $(m_e(r), m_e(\widetilde{\lambda}))$ with respect to $\Psi_e^{(0,1+ke)\to(0,1)}$ (for $k >> 0$).

3. By Proposition 6.2.2 that we apply to $\mu = m_e(\lambda)$, we have $\Psi_e^{(0,1)\to(0,1+ke)} \circ \theta_{e,(0,1)}(\mu) = (m_e(t), \mu - 1)$ (where t is the length of the first column of μ) so $m_e(\lambda)$ is the partition obtained by adding a row of length r to $m_e(\widetilde{\lambda})$ as in Xu algorithm.

The above result thus shows that Xu algorithm indeed computes the Mullineux involution.

Remark 6.4. In [BK03], Brundan and Kujawa gave another interpretation of the Xu algorithm using the representation theory of the supergroup $GL(n|n)$. It would be interesting to understand the connection of this work with ours.

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