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Abstract

Both seasonal unit roots and seasonal heterogeneity are common in seasonal data. When testing seasonal unit roots under seasonal heterogeneity, it is unclear if we can apply tests designed for seasonal homogeneous settings, for example the non-periodic HEGY test (Hylleberg, Engle, Granger, and Yoo, 1990). In this paper, the validity of both augmented HEGY test and unaugmented HEGY test is analyzed. The asymptotic null distributions of the statistics testing the single roots at 1 or -1 turn out standard and pivotal, but the asymptotic null distributions of the statistics testing any coexistence of roots at 1, -1, i, or $-i$ are non-standard, non-pivotal, and not directly pivotable. Therefore, the HEGY tests are not directly applicable to the joint tests for the concurrence of the roots. As a remedy, we bootstrap augmented HEGY with seasonal independent and identically distributed (iid) bootstrap, and unaugmented HEGY with seasonal block bootstrap. The consistency of both bootstrap procedures is established. Simulations indicate that for roots at 1 and -1 seasonal iid bootstrap augmented HEGY test prevails, but for roots at $\pm i$ seasonal block bootstrap unaugmented HEGY test enjoys better performance.

Keywords: Seasonality, Unit root, AR sieve bootstrap, Block bootstrap, Functional central limit theorem.

1 Introduction

Seasonal unit roots and seasonal heterogeneity often coexist in seasonal data, hence the importance to design seasonal unit root tests that allow for seasonal heterogeneity. In particular, given the following heterogeneous quarterly data $\{Y_{4t+s} : t = 1, ..., T,\ s = -3, ..., 0\}$ (see also Ghysels and Osborn, 2001, and Franses and Paap, 2004), generated by

$$\alpha_s(L)Y_{4t+s} = V_{4t+s}. \tag{1.1}$$

Suppose $\boldsymbol{V}_t = (V_{4t-3}, ..., V_{4t})'$ is a weakly stationary vector-valued process. Suppose for all $s = -3, ..., 0$, the roots of $\alpha_s(L)$ are on or outside the unit circle. If for some s, the roots of $\alpha_s(L)$ are all outside the unit circle, suppose the data are a stretch of a process $\{Y_{4t+s}, t = 1, 2, ..., s = -3, ..., 0\}$; otherwise, suppose $Y_{-3} = Y_{-2} = Y_{-1} = Y_0 = 0$, all $\alpha_s(L)$ share the same set of roots on the unit circle, and this set of roots on the unit circle is a subset of $\{1, -1, \pm i\}$. We aim to test if all $\alpha_s(L)$ share roots at 1, -1, or $\pm i$. To address this task, Franses (1994) and Boswijk, Franses, and Haldrup (1997) limit their scope to finite order seasonal AutoRegressive (AR) data, and apply Johansen's method (1988) to seasonal unit root tests in seasonal heterogeneous setting. However,

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their approaches cannot directly test the existence of a certain root without first checking the number of seasonal unit roots. Also working on finite order seasonal AR data, Ghysels, Hall, and Lee (1996) enable the direct test for the existence of a certain root. However, their simulations show that their procedure is less powerful than the non-periodic augmented HEGY test.

Does non-periodic HEGY test work in the seasonally heterogeneous setting? To the best of our knowledge, no literature has offered a satisfactory answer. Burridge and Taylor (2001a) analyze the behavior of augmented HEGY test when only seasonal heteroscadasticity exists; Castro and Osborn (2008) put augmented HEGY test in the periodic integrated model, a model related but mutually exclusive with model (1.1). No literature has ever touched the behavior of unaugmented HEGY test proposed by Breitung and Franses (1998), the important semi-parametric version of HEGY test. Since unaugmented HEGY test does not assume the noise having an AR structure, it may suit our non-parametric noise in (1.1) better.

To check the legitimacy of non-periodic HEGY tests in the seasonally heterogeneous setting (1.1), this paper derives the asymptotic null distributions of the unaugmented HEGY test and the asymptotic null distributions of the augmented HEGY test whose order of lags goes to infinity. It turns out that, the asymptotic null distributions of the statistics testing the single roots at 1 or -1 are standard. More specifically, for each single root at 1 or -1, the asymptotic null distributions of the augmented HEGY statistics are identical to that of Augmented Dickey-Fuller (ADF) test (Dickey and Fuller, 1979), and the asymptotic null distributions of the unaugmented HEGY statistics are identical to those of Phillips-Perron test (Phillips and Perron, 1988). However, the asymptotic null distributions of the statistics testing any combination of roots at 1, -1, i, or $-i$ depend on the seasonal heterogeneity parameters, and are non-standard, non-pivotal, and not directly pivotable. Therefore, when seasonal heterogeneity exists, both augmented HEGY and unaugmented HEGY tests can be straightforwardly used to test single roots at 1 or -1, but cannot be directly applied to the joint tests for the coexistence of any roots.

As a remedy, this paper proposes the application of bootstrap. In general, bootstrap's advantages are two fold. Firstly, bootstrap helps when the asymptotic distributions of the statistics of interest cannot be found or simulated. Secondly, even when the asymptotic distributions can be found and simulated, bootstrap method may enjoy second order efficiency. For the aforementioned problem, bootstrap therefore serves as an appealing solution. Firstly, it is hard to estimate the seasonal heterogeneity parameters in the asymptotic null distribution, and to simulate the asymptotic null distribution. Secondly, bootstrap seasonal unit root test may inherit the good second order efficiency (Park, 2003) of bootstrap non-seasonal unit root test.

The only methodological literature we find on bootstrapping HEGY test is Burridge and Taylor (2004). Their paper centers on seasonal heteroscadasticity, designs a bootstrap-aided augmented HEGY test, reports its simulation result, but does not give theoretical justification for their test. Indeed, from the discussion of the present paper, it will be seen (Remark 3.8) that the bootstrap approach of Burridge and Taylor (2004) is valid when nothing but seasonal variance varies, but is invalid under the general seasonal heterogeneous setting (1.1).

To cater to the general heterogeneous setting (1.1), this paper designs new bootstrap tests, namely 1) seasonal iid bootstrap augmented HEGY test, and 2) seasonal block bootstrap unaugmented HEGY test. To generate bootstrap replicates, the first test get residuals from season-by-season augmented HEGY regressions, and then applies seasonal iid bootstrap to the whitened regression errors. On the other hand, the second test starts with season-by-season unaugmented HEGY regressions, and then handle the correlated errors with seasonal block bootstrap proposed by Dudek, Lekow, Paparoditis, and Politis (2014). Our paper establishes the Functional Central Limit Theory (FCLT) for both bootstrap tests. Based on the FCLT, the validity for both bootstrap approaches is proven. To the best of our knowledge, this result gives the first justification for bootstrapping HEGY tests under (1.1).

This paper proceeds as follows. Section 2 formalizes the settings, presents the assumptions, and states the hypotheses. Section 3 gives the asymptotic null distributions of the augmented HEGY test statistics, details the algorithm of seasonal iid bootstrap augmented HEGY test, and establishes its consistency. Section 4 presents the asymptotic null distributions of the unaugmented HEGY test statistics, specifies the algorithm of seasonal block bootstrap unaugmented HEGY test, and proves its consistency. Section 5 compares the simulation performance of the two aforementioned tests. Appendix includes the proofs of all theorems.

2 Settings

Recall the quarterly data $\{Y_{4t+s} : t = 1, ..., T,\ s = -3, ..., 0\}$ generated by the seasonal AR model,

$$\alpha_s(L)Y_{4t+s} = V_{4t+s}, \tag{2.1}$$

where $LY_{4t+s} = Y_{4t+s-1}$, $\alpha_s(L) = 1 - \sum_{j=1}^{4} \alpha_{j,s} L^j$. Let V_{4t+s} and $\alpha_{j,s}$ be the regression errors and regression coefficients of (2.1), respectively. More specifically, V_{4t+s} is the distance between Y_{4t+s} and the vector space generated by Y_{4t+s-j}, $j = 1, ..., 4$, and $\alpha_{j,s}$ is the coefficient of the projection of V_{4t+s} on the aforementioned vector space. Let $\boldsymbol{\epsilon}_t = (\epsilon_{4t-3}, ..., \epsilon_{4t})'$, $B\boldsymbol{\epsilon}_t = \boldsymbol{\epsilon}_{t-1}$. Denote by AR($p$) a AutoRegressive process with order p, by VMA(∞) a Vector Moving Average process with infinite moving average order, and by VARMA(p, q) a Vector AutoRegressive Moving Average process with autoregressive order p and moving average order q. Let $Re(z)$ be the real part of complex number z. Let $\lfloor x \rfloor$ be the largest integer smaller or equal to real number x, and $\lceil x \rceil$ be the smallest integer larger or equal to x.

Assumption 1.A. *Assume*

$$\boldsymbol{V}_t = \boldsymbol{\Theta}(B)\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\Theta}(B) = \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i B^i$; the (j, k) entry of $\boldsymbol{\Theta}_i$, denoted by $\boldsymbol{\Theta}_i^{(j,k)}$, satisfies $\sum_{i=1}^{\infty} i|\boldsymbol{\Theta}_i|^{(j,k)} < \infty$ for all j and k; the determinant of $\boldsymbol{\Theta}(z)$, as a function of z, has all roots outside the unit circle; $\boldsymbol{\Theta}_0$ is a lower diagonal matrix whose diagonal entries equal 1; $\boldsymbol{\epsilon}_t$ is a vector-valued white noise process with mean zero and covariance matrix $\boldsymbol{\Omega}$; and $\boldsymbol{\Omega}$ is diagonal.

Assumption 1.A assumes that $\{\boldsymbol{V}_t\}$ is VMA(∞) with respect to white noise innovation. This is equivalent to the assumption that $\{\boldsymbol{V}_t\}$ is a weakly stationary process with no deterministic part in the multivariate Wold decomposition. The assumptions on $\boldsymbol{\Theta}_0$ and the determinant of $\boldsymbol{\Theta}(z)$ ensure the causality and the invertibility of $\{\boldsymbol{V}_t\}$ and the identifiability of $\boldsymbol{\Omega}$.

Assumption 1.B. *Assume*

$$\boldsymbol{V}_t = \boldsymbol{\Psi}(B)^{-1}\boldsymbol{\Lambda}(B) \equiv \boldsymbol{\Theta}(B)\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\Psi}(B) = \sum_{i=0}^{p} \boldsymbol{\Psi}_i B^i$; $\boldsymbol{\Lambda}(B) = \sum_{i=0}^{q} \boldsymbol{\Lambda}_i B^i$; determinants of $\boldsymbol{\Psi}(z)$ and $\boldsymbol{\Lambda}(z)$ have all roots outside the unit circle; $\boldsymbol{\Psi}_0$ is the identity matrix; $\boldsymbol{\Lambda}_0$ is a lower diagonal matrix whose diagonal entries equal 1; $\boldsymbol{\epsilon}_t$ is a vector-valued white noise process with mean zero and covariance matrix $\boldsymbol{\Omega}$; and $\boldsymbol{\Omega}$ is diagonal.

Assumption 1.B restricts $\{\boldsymbol{V}_t\}$ to be VARMA(p, q) with respect to white noise innovation. Compared to the VMA(∞) model in Assumption 1.A, VARMA(p, q)'s main restraint is its exponentially decaying autocovariance. Again, the assumptions on $\boldsymbol{\Psi}_0$, $\boldsymbol{\Lambda}_0$ and the determinant of $\boldsymbol{\Psi}(z)$ and $\boldsymbol{\Lambda}(z)$ in Assumption 1.B ensure the causality and the invertibility of $\{\boldsymbol{V}_t\}$ and the identifiablity of $\boldsymbol{\Omega}$.

At this stage $\{\boldsymbol{\epsilon}_t\}$ is only assumed to be a white noise sequence of random vectors. In fact, $\{\boldsymbol{\epsilon}_t\}$ also need to be weakly dependent.

Assumption 2.A. *(i)* $\{\boldsymbol{\epsilon}_t\}$ *is a fourth-order stationary martingale difference sequence with finite* $4+\delta$ *moment for some* $\delta > 0$. *(ii)* $\exists K > 0$, $\forall$ i, j, k, *and* l, $\sum_{h=-\infty}^{\infty} |Cov(\epsilon_i\epsilon_j, \epsilon_{k-h}\epsilon_{l-h})| < K$.

Assumption 2.B. *(i)* $\{\boldsymbol{\epsilon}_t\}$ *is a strictly stationary strong mixing sequence with finite* $4+\delta$ *moment for some* $\delta > 0$. *(ii)* $\{\boldsymbol{\epsilon}_t\}$*'s strong mixing coefficient* $a(k)$ *satisfies* $\sum_{k=1}^{\infty} k(a(k))^{\delta/(4+\delta)} < \infty$.

Notice the higher moment $\{\boldsymbol{\epsilon}_t\}$ has, the weaker assumption we require on the strong mixing coefficient of $\{\boldsymbol{\epsilon}_t\}$ in Assumption 2.B. The strong mixing condition in Assumption 2.B actually guarantees (ii) of Assumption 2.A (see Lemma 4).

Hypotheses. *We tackle the following set of null hypotheses. The alternative hypotheses are the complement of the null hypotheses.*

$$
\begin{aligned}
&H_0^1: && \alpha_s(1) = 0,\ \forall s = -3, ..., 0.\\
&H_0^2: && \alpha_s(-1) = 0,\ \forall s = -3, ..., 0.\\
&H_0^{1,2}: && \alpha_s(1) = \alpha_s(-1) = 0,\ \forall s = -3, ..., 0.\\
&H_0^{3,4}: && \alpha_s(i) = \alpha_s(-i) = 0,\ \forall s = -3, ..., 0.\\
&H_0^{1,3,4}: && \alpha_s(1) = \alpha_s(i) = \alpha_s(-i) = 0,\ \forall s = -3, ..., 0.\\
&H_0^{2,3,4}: && \alpha_s(-1) = \alpha_s(i) = \alpha_s(-i) = 0,\ \forall s = -3, ..., 0.\\
&H_0^{1,2,3,4}: && \alpha_s(1) = \alpha_s(-1) = \alpha_s(i) = \alpha_s(-i) = 0,\ \forall s = -3, ..., 0.
\end{aligned}
$$

Indeed, the alternative hypotheses can be written as one-sided. Recall we suppose that for all $s = -3, ..., 0$, the roots of $\alpha_s(L)$ are either on or outside the unit circle. Since $\alpha_s(0) = 1$, by the intermediate value theorem, $\alpha_s(1) \neq 0$ implies $\alpha_s(1) > 0$, $\alpha_s(-1) \neq 0$ implies $\alpha_s(-1) > 0$, and $\alpha_s(i) \neq 0$ implies $Re(\alpha_s(i)) > 0$. To further analyze the roots of $\alpha_s(L)$, HEGY (Hylleberg, Engle, Granger, and Yoo, 1990) propose the partial fraction decomposition

$$\frac{\alpha_s(L)}{1-L^4} = \lambda_{0,s} + \frac{\lambda_{1,s}}{1-L} + \frac{\lambda_{2,s}}{1+L} + \frac{\lambda_{3,s}L + \lambda_{4,s}}{1+L^2};$$

thus

$$
\begin{aligned}
\alpha_s(L) = {} & \lambda_{0,s}(1-L^4)\\
& + \lambda_{1,s}(1+L)(1+L^2) + \lambda_{2,s}(1-L)(1+L^2)\\
& + \lambda_{3,s}(1-L)(1+L)L + \lambda_{4,s}(1-L)(1+L).
\end{aligned} \tag{2.2}
$$

Substituting (2.2) into (2.1), we get

$$(1-L^4)Y_{4t+s} = \sum_{j=1}^{4} \pi_{j,s} Y_{j,4t+s-1} + V_{4t+s}, \tag{2.3}$$

where

$$
\begin{aligned}
Y_{1,4t+s} &= (1+L)(1+L^2)Y_{4t+s}, & Y_{2,4t+s} &= -(1-L)(1+L^2)Y_{4t+s},\\
Y_{3,4t+s} &= -L(1-L^2)Y_{4t+s}, & Y_{4,4t+s} &= -(1-L^2)Y_{4t+s},\\
\pi_{1,s} &= -\lambda_{1,s}, & \pi_{2,s} &= -\lambda_{2,s},\\
\pi_{3,s} &= -\lambda_{4,s}, & \pi_{4,s} &= \lambda_{3,s},
\end{aligned} \tag{2.4}
$$

Indeed, $\pi_{j,s}$ relates to the root of $\alpha_s(z)$, for example, $\alpha_s(1) = 4\lambda_{1,s}$; hence the proposition below.

Proposition 2.1 (HEGY, 1990)**.**

$$\begin{array}{ll} \alpha_s(1) = 0 \iff \pi_{1,s} = 0, & \alpha_s(1) \neq 0 \iff \pi_{1,s} < 0, \\ \alpha_s(-1) = 0 \iff \pi_{2,s} = 0, & \alpha_s(-1) \neq 0 \iff \pi_{2,s} < 0, \\ \alpha_s(i) = 0 \iff \alpha_s(-i) = 0 \iff \pi_{3,s} = \pi_{4,s} = 0, & \alpha_s(i) \neq 0 \iff \alpha_s(-i) \neq 0 \iff \pi_{3,s} < 0. \end{array}$$

By Proposition 2.1, the test for the null hypotheses can be carried on by checking the corresponding $\pi_{j,s}$. Further, $\pi_{j,s}$ can be estimated by Ordinary Least Squares (OLS). Unfortunately, the OLS cannot be readily applied season by season on the regression (2.3) (see Ghysels and Osborn, 2001, p. 158). On the other hand, it is unsure if the OLS can be implemented well on a non-periodic regression equation, for example, (3.2) and (4.1).

When we run non-periodic regressions in succeeding sections, the seasonally heterogeneous sequence $\{V_{4t+s}\}$ is fitted in seasonal homogeneous AR models. Consider fitting $\{V_{4t+s}\}$ in the AR(1) model $V_t = \phi V_{t-1} + e_t$, where ϕ is defined such that the OLS estimator $\hat{\phi}$ converges in probability to ϕ. Given this definition, it can be shown that $\phi = \tilde{\gamma}(1)/\tilde{\gamma}(0)$, where

$$\tilde{\gamma}(h) = \frac{1}{4} \sum_{s=-3}^{0} E[V_{4t+s} V_{4t+s-h}]. \tag{2.5}$$

Now let $\{\tilde{V}_t\}$ be a weakly stationary sequence with mean zero and autocovariance function $\tilde{\gamma}$. Such $\{\tilde{V}_t\}$ exists because $\tilde{\gamma}$ is a positive semi-definite function. Fitting $\{\tilde{V}_t\}$ in the AR(1) model $\tilde{V}_t = \tilde{\phi}\tilde{V}_{t-1} + e_t$, we get $\tilde{\phi} = \tilde{\gamma}(1)/\tilde{\gamma}(0) = \phi$. This indicates that when fitting the seasonally heterogeneous sequence $\{V_{4t+s}\}$ in AR models, the AR coefficients defined as the limits of the OLS estimators are identical to the AR coefficients of the seasonally homogeneous sequence $\{\tilde{V}_t\}$. We call $\{\tilde{V}_t\}$ a misspecified constant parameter representation (see also Osborn, 1991) of $\{V_{4t+s}\}$, and will refer to this concept in later sections.

3 Seasonal iid bootstrap Augmented HEGY Test

3.1 Augmented HEGY test

In seasonally homogeneous setting

$$\alpha(L)Y_t = V_t, \quad t = 1 + k, ..., 4T, \tag{3.1}$$

where $\alpha(L) = \sum_{i=0}^{4} \alpha_i L^i$, the augmented HEGY test detailed below copes with the roots of $\alpha(L)$ at 1, -1, and $\pm i$. By calculations similar to (2.2), HEGY (1990) get

$$(1 - L^4)Y_t = \sum_{j=1}^{4} \pi_j Y_{j,t-1} + \sum_{i=1}^{k} \phi_i (1 - L^4) Y_{t-i} + e_t, \tag{3.2}$$

where augmentations $(1 - L^4)Y_{t-i}$, $i = 1, 2, ..., k$, pre-whiten the time series $(1 - L^4)Y_t$ up to an order of k. As the sample size $T \to \infty$, let $k \to \infty$, so that the residual $\{e_t\}$ is asymptotically uncorrelated. Let $\hat{\pi}_i$ be the OLS estimator of π_i, t_i be the t-statistics corresponding to $\hat{\pi}_i$, and F_{34} be the F-statistic corresponding to $\hat{\pi}_3$ and $\hat{\pi}_4$. Other F-statistics F_{12}, F_{124}, F_{134}, and F_{1234} can be defined similarly. In seasonally homogeneous configuration, HEGY (1990) proposes to reject H_0^1 if $\hat{\pi}_1$ is too small, reject H_0^2 if $\hat{\pi}_2$ is too small, reject $H_0^{3,4}$ if F_{34} is too large, and reject other composite hypotheses if their corresponding F-statistics are too large. (In proposition 3.1 we will show this rejection rule works for seasonally heterogeneous data $\{Y_{4t+s}\}$.)

3.2 Real World Asymptotics

Now we apply the augmented HEGY test to seasonally heterogeneous processes. Namely, we run regression equation (3.2) with $\{Y_{4t+s}\}$ generated by (2.1). Our results show that when testing roots at 1 or -1 individually, the t-statistics t_1, t_2, and the F-statistics have standard and pivotal asymptotic distributions. On the other hand, when testing joint roots at 1 and -1, and when testing hypotheses that involve roots at $\pm i$, the asymptotic distributions of the t-statistics and the F-statistics are non-standard, non-pivotal, and not directly pivotable.

Theorem 3.1. *Assume that Assumption 1.B and one of the Assumption 2.A or 2.B hold. Further, assume $T \to \infty$, $k = k_T \to \infty$, $k = o(T^{1/3})$, and $ck > T^{1/\alpha}$ for some $c > 0$, $\alpha > 0$. Then under $H_0^{1,2,3,4}$, the asymptotic distributions of $\hat{\pi}_i$, t_i, $i = 1, 2$, and F-statistics are given by*

$$
\begin{aligned}
t_j &\Rightarrow \frac{\int_0^1 W_j(r)dW_j(r)}{\sqrt{\int_0^1 W_j^2(r)dr}} \equiv \xi_j, \ \text{j=1,2}, \\
F_{12} &\Rightarrow \frac{1}{2}(\xi_1^2 + \xi_2^2), \quad F_{34} \Rightarrow \frac{1}{2}(\xi_3^2 + \xi_4^2), \\
F_{134} &\Rightarrow \frac{1}{3}(\xi_1^2 + \xi_3^2 + \xi_4^2), \quad F_{234} \Rightarrow \frac{1}{3}(\xi_2^2 + \xi_3^2 + \xi_4^2), \\
F_{1234} &\Rightarrow \frac{1}{4}(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2), \quad \text{with} \\
\xi_3 &= \frac{\lambda_3^2 \int_0^1 W_3(r)dW_3(r) + \lambda_4^2 \int_0^1 W_4(r)dW_4(r)}{\sqrt{(\lambda_3^2 + \lambda_4^2)(\frac{1}{2}\lambda_3^2 \int_0^1 W_3^2(r)dr + \frac{1}{2}\lambda_4^2 \int_0^1 W_4^2(r)dr)}}, \\
\xi_4 &= \frac{\lambda_3\lambda_4(\int_0^1 W_3(r)dW_4(r) - \int_0^1 W_4(r)dW_3(r))}{\sqrt{(\lambda_3^2 + \lambda_4^2)(\frac{1}{2}\lambda_3^2 \int_0^1 W_3^2(r)dr + \frac{1}{2}\lambda_4^2 \int_0^1 W_4^2(r)dr)}},
\end{aligned}
$$

where $\boldsymbol{c}_1 = (1, 1, 1, 1)'$, $\boldsymbol{c}_2 = (1, -1, 1, -1)'$, $\boldsymbol{c}_3 = (0, -1, 0, 1)'$, and $\boldsymbol{c}_4 = (-1, 0, 1, 0)'$, $\lambda_i = \sqrt{\boldsymbol{c}_i'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_i/4}$, $W_i = \boldsymbol{c}_i'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\boldsymbol{W}/2\lambda_i$, $\boldsymbol{W}(t) = (W_1(t), W_2(t), W_3(t), W_4(t))'$ is a four-dimensional standard Brownian motion.

Remark 3.1. Notice the Theorem 3.1 allows the simultaneous increase of T and k when deriving the asymptotic distributions of double-indexed sequences of random variables. On the other hand, Galbraith and Zinde-Walsh (1999) and Castro and Osborn (2008, 2011) firstly fix k and let $T \to \infty$, and then let $k \to \infty$. Indeed, their limiting results based the aforementioned two-step convergences of k and n cannot imply the limiting results when $T \to \infty$ and $k \to \infty$ simultaneously. (For example, see Billingsley (1999), Theorem 3.2).

Remark 3.2. The asymptotic distributions presented in the Theorem 3.1 degenerate to the distributions in Burridge and Taylor (2001b) and Castro, Osborn and Taylor (2012) when $\{V_{4t+s}\}$ is a seasonally homogeneous sequence with homoscedastic noise, and to the distributions in Burridge and Taylor (2001a) when $\{V_{4t+s}\}$ is a seasonally homogeneous finite-order AR sequence with heteroscedastic noise.

Remark 3.3. Notice W_i's are standard Brownian motions. When $\{V_{4t+s}\}$ is seasonally homogeneous (Burridge and Taylor, 2001b, Castro et al., 2012), W_i's are independent, so are the asymptotic distributions of t_1 and t_2. On the other hand, when $\{V_{4t+s}\}$ has seasonal heterogeneity, W_i's are in general independent, so t_1 and t_2 are in general dependent, even asymptotically. Hence, when testing $H_0^{1,2}$, it is problematic to test H_0^1 and H_0^2 separately and calculate the level of the test with the independence of t_1 and t_2 in mind. Instead, the test of $H_0^{1,2}$ should be handled with F_{12}. Further, because of the dependence of t_1 and t_2, the asymptotic distribution of F_{12} under heterogeneity is different from its counterpart when $\{V_{4t+s}\}$ is seasonally homogeneous. Hence, the non-periodic augmented HEGY test cannot be directly applied to test $H_0^{1,2}$.

Remark 3.4. When $\{V_{4t+s}\}$ is only seasonally heteroscedastic (Burridge and Taylor, 2001a), $\mathbf{\Theta}(1)$ does not occur in the asymptotic distributions of the F-statistics. On the other hand, when $\{V_{4t+s}\}$ has generic seasonal heterogeneity, $\mathbf{\Theta}(1)$ impacts firstly the correlation between Brownian motions W_3 and W_4, and secondly the weights λ_3 and λ_4.

Remark 3.5. As Burridge and Taylor (2001a) point out, the dependence of the asymptotic distributions on weights λ_3 and λ_4 can be expected. Indeed, $Y_{3,4t+s} = Y_{4,4t+s-1}$ is the partial sum of $\{-V_{4t+s-1}, V_{4t+s-3}, ...\}$, while $Y_{3,4t+s} = Y_{4,4t+s-1}$ is the partial sum of $\{-V_{4t+s}, V_{4t+s-2}, ...\}$. Since these two partial sums differ in their variances, both $\sum Y_{3,4t+s}$ and $\sum Y_{4,4t+s}$ involve two different weights λ_3 and λ_4.

Remark 3.6. Theorem 3.1 presents the asymptotics when $\{Y_t\}$ has all roots at 1, -1, and $\pm i$. When $\{Y_{4t+s}\}$ has some but not all roots at 1, -1, and $\pm i$, we let $U_t = (1 - L^4)Y_t$, $\boldsymbol{U} = (U_{4t-3}, U_{4t-2}, U_{4t-1}, U_{4t})'$, and calculate $\boldsymbol{H}(z)$ such that $\boldsymbol{U}_t = \boldsymbol{H}(B)\boldsymbol{\epsilon}_t$. The asymptotic distributions can be expressed with respective to on $\boldsymbol{H}(z)$ and end up identical with those given in Theorem 3.1, where $\{Y_{4t+s}\}$ has all roots.

While the preceding results give the asymptotic behaviors of the testing statistics under the null hypotheses, the proposition below describes the asymptotics under the alternative hypotheses, and justifies the rejection rules in section 3.1 when applying HEGY test to seasonally heterogeneous processes.

Proposition 3.1. *When $\{Y_{4t+s}\}$ does not have roots at 1, -1, or $\pm i$, the OLS estimates $\hat{\pi}_j$ in* (3.2)*, $j = 1, 2, 3$, respectively, converge in probability to negative values.*

Proof. We take the root at 1 as an example. Suppose $\{Y_{4t+s}\}$ has no root at 1, namely $\pi_{1,s} < 0$ for some s. If $\{Y_{4t+s}\}$ has other nuisance root (for example, -1) then by the asymptotic orthogonality of regression equation (3.2), the corresponding predictor (for example, $Y_{2,t-1}$) of the nuisance root (for example, -1) can be excluded from (3.2) without changing the asymptotics of $\hat{\pi}_1$. All remaining predictors in (3.2) contain a filter (for example, $1 + L$) that filter out the nuisance root (for example, -1). Hence we can assume without loss of generality that $\{Y_{4t+s}\}$ has no root at 1, -1, or $\pm i$. Notice that (1.1) can be written as $\boldsymbol{A}(B)\boldsymbol{Y}_t = \boldsymbol{V}_t$, or $|\boldsymbol{A}(B)|\boldsymbol{Y}_t = \boldsymbol{A}^*\boldsymbol{V}_t \equiv \dot{\boldsymbol{V}}_t$, or $|\boldsymbol{A}(B)|Y_{4t+s} = \dot{V}_{4t+s}$, where $\boldsymbol{A}^*(B)$ is the adjugate matrix of $\boldsymbol{A}(B)$. Notice $\{Y_{4t+s}\}$ in $|\boldsymbol{A}(B)|Y_{4t+s} = \dot{V}_{4t+s}$ has seasonally homogeneous AR coefficients. If $\{Y_{4t+s}\}$ has no root at 1, -1, or $\pm i$, then $|\boldsymbol{A}(z)|$ has its roots outside the unit circle. Then $\hat{\pi}_1$ converges (Berk, 1974) to a negative value (see Proposition 2.1). □

3.3 Seasonal iid bootstrap algorithm

To accommodate the non-standard, non-pivotal asymptotic null distributions of the augmented HEGY test statistics, we propose the application of bootstrap. In particular, the bootstrap replications are created as follows. Firstly, we pre-whiten the data season by season to obtain uncorrelated noises. Although these noises are uncorrelated, they are not white due to seasonally heteroscadasticity. Hence secondly we resample season by season in order to generate bootstrapped noise. Finally, we post-color the bootstrapped noise. The detailed algorithm of this seasonal iid bootstrap augmented HEGY test is given below.

Algorithm 3.1. *Step 1: calculate the t-statistics t_1, t_2, and the F-statistics F from the non-periodic augmented HEGY test regression*

$$(1-L^4)Y_t = \sum_{j=1}^{4} \hat{\pi}_j Y_{j,t-1} + \sum_{i=1}^{k} \hat{\phi}_i (1-L^4) Y_{t-i} + e_t;$$

Step 2: record OLS estimators $\hat{\pi}_{j,s}$, $\hat{\phi}_{i,s}$ and residuals $\hat{\epsilon}_{4t+s}$ from the season-by-season regression

$$(1-L^4)Y_{4t+s} = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y_{j,4t+s-1} + \sum_{i=1}^{k} \hat{\phi}_{i,s} (1-L^4) Y_{4t+s-i} + \hat{\epsilon}_{4t+s};$$

Step 3: let $\check{\epsilon}_{4t+s} = \hat{\epsilon}_{4t+s} - \frac{1}{T}\sum_{t=\lfloor k/4 \rfloor+1}^{T} \hat{\epsilon}_{4t+s}$. Store demeaned residuals $\{\check{\epsilon}_{4t+s}\}$ of the four seasons separately, then independently draw four iid samples from each of their empirical distributions, and then combine these four samples into the vector $\{\epsilon^{\star}_{4t+s}\}$, with their seasonal orders preserved;

Step 4: set all $\hat{\pi}_{j,s}$ corresponding to the null hypothesis to be zero. For example, set $\pi_{3,s} = \pi_{4,s} = 0$ for all s when testing roots at $\pm i$. Let $\{Y^{\star}_t\}$ be generated by

$$(1-L^4)Y^{\star}_{4t+s} = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y^{\star}_{j,4t+s-1} + \sum_{i=1}^{k} \hat{\phi}_{i,s} (1-L^4) Y^{\star}_{4t+s-i} + \epsilon^{\star}_{4t+s};$$

Step 5: get t-statistics $t^{\star}_1$, $t^{\star}_2$, and the F-statistics F^{*} from the regression

$$(1-L^4)Y^{\star}_t = \sum_{j=1}^{4} \hat{\pi}^{\star}_j Y_{j,t-1} + \sum_{i=1}^{k} \hat{\phi}^{\star}_i (1-L^4) Y^{\star}_{t-i} + e^{\star}_t;$$

Step 6: run step 3, 4, and 5 for B times to get B sets of statistics $t^{\star}_1$, $t^{\star}_2$, and the bootstrapped F-statistics $F^{\star}$. Count separately the numbers of $t^{\star}_1$, $t^{\star}_2$ and $F^{\star}$ than which t_1, t_2, and the F-statistics F are more extreme. If these numbers are higher than $B(1-level)$, then we consider t_1, t_2, and the F-statistics F extreme, and reject the corresponding hypotheses.

Remark 3.7. It seems also reasonable to keep steps 1, 2, 3, 5, and 6 of the Algorithm 3.1, but change the generation of $\{Y^{\star}_t\}$ in step 4 to

$$(1-L^4)Y^{\star}_{4t+s} = \sum_{i=1}^{k} \hat{\phi}_{i,s} (1-L^4) Y^{\star}_{4t+s-i} + \epsilon^{\star}_{4t+s}. \tag{3.3}$$

This new algorithm is in fact theoretically invalid for the tests of any coexistence of roots (see Remark 3.3, 3.4, and 3.6), but it is valid for individual tests of roots at 1 or -1, due to the pivotal asymptotic distributions of t_1 and t_2 in Theorem 3.1.

Remark 3.8. If we keep steps 1, 3, 5, and 6 of Algorithm 3.1, but run regression equations with seasonally homogeneous coefficients $\hat{\pi}_j$ and $\hat{\phi}_i$ in steps 2 and 4, then this algorithm is identical with Burridge and Taylor (2004). However, this algorithm cannot in step 2 fully pre-whiten the time series, and it leaves the regression error $\{e_t\}$ serially correlated. When $\{e_t\}$ is bootstrapped by seasonal iid bootstrap, this serial correlation structure is ruined. As a result, $(1-L^4)Y^{\star}_t$ differs from $(1-L^4)Y_t$ in its correlation structure, in particular $\boldsymbol{\Theta}(1)$. Hence, the conditional distributions of the bootstrapped F-statistics, for example $F^{\star}_{34}$, differ from the distribution of the original F-statistics, for example F_{34} (see Remark 3.3 and 3.4).

3.4 Seasonal iid bootstrap asymptotics

Now we walk toward the justification of the seasonal iid bootstrap augmented HEGY test (Algorithm 3.1). Since the derivation of the real-world asymptotic distributions in Theorem 3.1 calls on FCLT (see Lemma 1), the justification of bootstrap approach also requires FCLT in the bootstrap world. From now on, let P°, E°, Var°, Std°, Cov° be the probability, expectation, variance, standard deviation, and covariance, respectively, conditional on our data $\{Y_{4t+s}\}$.

Theorem 3.2. *Suppose the assumptions in Theorem 3.1 hold. Let* $S_T^\star(u_1, u_2, u_3, u_4)$

$$= \frac{1}{\sqrt{4T}}(\sum_{t=1}^{\lfloor 4Tu_1 \rfloor} \epsilon_t^\star/\sigma_1^\star, \sum_{t=1}^{\lfloor 4Tu_2 \rfloor} (-1)^t \epsilon_t^\star/\sigma_2^\star, \sum_{t=1}^{\lfloor 4Tu_3 \rfloor} \sqrt{2}\sin(\frac{\pi t}{2})\epsilon_t^\star/\sigma_3^\star, \sum_{t=1}^{\lfloor 4Tu_4 \rfloor} \sqrt{2}\cos(\frac{\pi t}{2})\epsilon_t^\star/\sigma_4^\star),$$

where

$$\sigma_1^\star = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}\epsilon_t^\star], \qquad \sigma_2^\star = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}(-1)^t\epsilon_t^\star],$$

$$\sigma_3^\star = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}\sqrt{2}\sin(\frac{\pi t}{2})\epsilon_t^\star], \qquad \sigma_4^\star = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}\sqrt{2}\cos(\frac{\pi t}{2})\epsilon_t^\star].$$

Then, no matter which hypothesis is true, $S_T^\star \Rightarrow \boldsymbol{W}$ *in probability as* $T \to \infty$*, where* $\boldsymbol{W}(t) = (W_1(t), W_2(t), W_3(t), W_4(t))'$ *is a four-dimensional standard Brownian motion.*

By the FCLT given by Theorem 3.2 and the proof of Theorem 3.1, in probability the conditional distributions of $t_i^\star$, $i = 1, 2$, and $F^\star$ converge to the limiting distributions of t_i, $i = 1, 2$, and F, respectively. Notice that conditional on $\{Y_{4t+s}\}$, $\{Y_{4t+s}^\star\}$ is a finite-order seasonal AR process, so the derivation of the conditional distributions of $t_i^\star$, $i = 1, 2$, and $F^\star$ turns out easier than the that of Theorem 3.1, and in particular does not involve the fourth moments of $\{Y_{4t+s}^\star\}$. Hence the justification of the seasonal iid bootstrap augmented HEGY test.

Corollary 3.1. *Suppose the assumptions in Theorem 3.1 hold. Then,*

$$\sup_x |P^\circ(t_i^\star \leq x) - P(t_i \leq x)| \xrightarrow{p} 0, \ i = 1, 2,$$

$$\sup_x |P^\circ(F^\star \leq x) - P(F \leq x)| \xrightarrow{p} 0.$$

4 Seasonal block bootstrap unaugmented HEGY test

4.1 Unaugmented HEGY test

In the proceeding section our analysis focuses on the augmented HEGY test, a extension of the ADF test to the seasonal unit root setting. An important alternative of the ADF test is the Phillips-Perron test (Phillips and Perron, 1988). While the ADF test assumes an AR structure over the noise and thus becomes parametric, its semi-parametric counterpart, Phillips-Perron test, allows a wide class of weakly dependent noises. Unaugmented HEGY test (Breitung and Franses, 1998), as the extension of Phillips-Perron test to the seasonal unit root, inherits the semi-parametric nature and does not assume the noise to be AR. Given seasonal heterogeneity, it will be shown in Theorem 4.1 that the unaugmented HEGY test estimates seasonal unit root consistently under the very general

VMA(∞) class of noise (Assumption 1.A), instead of the more restrictive VARMA(p,q) class of noise (Assumption 1.B), which is needed for the augmented HEGY test.

Now we specify the unaugmented HEGY test. Consider regression

$$(1-L^4)Y_t = \sum_{j=1}^{4} \hat{\pi}_j Y_{j,t-1} + V_t. \tag{4.1}$$

Let $\hat{\pi}_j$ be the OLS estimator of π_j, t_j be the t-statistic corresponding to $\hat{\pi}_j$, and F_{34} be the F-statistic corresponding to $\hat{\pi}_3$ and $\hat{\pi}_4$. Other F-statistics F_{12}, F_{124}, F_{134}, and F_{1234}. Similar to the Phillips-Perron test (Phillips and Perron, 1988), the unaugmented HEGY test allows the use of both $\hat{\pi}_j$ and t_j in testing roots at 1 or -1. As in the augmented HEGY test, we reject H_0^1 if $\hat{\pi}_1$ (or t_1) is too small, reject H_0^2 if $\hat{\pi}_2$ (or t_2) is too small, and reject the joint hypotheses if the corresponding F-statistics are too large. The following results give the asymptotic null distributions of $\hat{\pi}_j$, t_j, and the F-statistics.

4.2 Real world asymptotics

Theorem 4.1. *Assume that Assumption 1.A and one of Assumption 2.A or Assumption 2.B hold. Then under $H_0^{1,2,3,4}$, as $T \to \infty$,*

$$\begin{aligned}
(4T)\hat{\pi}_i &\Rightarrow \frac{\lambda_i^2 \int_0^1 W_i(r)dW_i(r) + \Gamma^{(i)}}{\lambda_i^2 \int_0^1 W_i^2(r)dr}, \ \textit{for } i=1,2,\\
(4T)\hat{\pi}_3 &\Rightarrow \frac{\lambda_3^2 \int_0^1 W_3(r)dW_3(r) + \lambda_4^2 \int_0^1 W_4(r)dW_4(r) + \Gamma^{(3)}}{\frac{1}{2}(\lambda_3^2 \int_0^1 W_3^2(r)dr + \lambda_4^2 \int_0^1 W_4^2(r)dr)},\\
(4T)\hat{\pi}_4 &\Rightarrow \frac{\lambda_3\lambda_4(\int_0^1 W_3(r)dW_4(r) - \int_0^1 W_4(r)dW_3(r)) + \Gamma^{(4)}}{\frac{1}{2}(\lambda_3^2 \int_0^1 W_3^2(r)dr + \lambda_4^2 \int_0^1 W_4^2(r)dr)},\\
t_i &\Rightarrow \frac{\lambda_i^2 \int_0^1 W_i(r)dW_i(r) + \Gamma^{(i)}}{\sqrt{\tilde{\gamma}(0)\lambda_i^2 \int_0^1 W_i^2(r)dr}} \equiv \mathscr{D}_i, \ \textit{for } i=1,2,\\
t_3 &\Rightarrow \frac{\lambda_3^2 \int_0^1 W_3(r)dW_3(r) + \lambda_4^2 \int_0^1 W_4(r)dW_4(r) + \Gamma^{(3)}}{\sqrt{\tilde{\gamma}(0)\frac{1}{2}(\lambda_3^2 \int_0^1 W_3^2(r)dr + \lambda_4^2 \int_0^1 W_4^2(r)dr)}} \equiv \mathscr{D}_3\\
t_4 &\Rightarrow \frac{\lambda_3\lambda_4(\int_0^1 W_3(r)dW_4(r) - \int_0^1 W_4(r)dW_3(r)) + \Gamma^{(4)}}{\sqrt{\tilde{\gamma}(0)\frac{1}{2}(\lambda_3^2 \int_0^1 W_3^2(r)dr + \lambda_4^2 \int_0^1 W_4^2(r)dr)}} \equiv \mathscr{D}_4\\
F_{12} &\Rightarrow \frac{1}{2}(\mathscr{D}_1^2 + \mathscr{D}_2^2), \quad F_{34} \Rightarrow \frac{1}{2}(\mathscr{D}_3^2 + \mathscr{D}_4^2),\\
F_{134} &\Rightarrow \frac{1}{3}(\mathscr{D}_1^2 + \mathscr{D}_3^2 + \mathscr{D}_4^2), \quad F_{234} \Rightarrow \frac{1}{3}(\mathscr{D}_2^2 + \mathscr{D}_3^2 + \mathscr{D}_4^2),\\
F_{1234} &\Rightarrow \frac{1}{4}(\mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2 + \mathscr{D}_4^2),
\end{aligned}$$

where $\boldsymbol{c}_1 = (1,1,1,1)'$, $\boldsymbol{c}_2 = (1,-1,1,-1)'$, $\boldsymbol{c}_3 = (0,-1,0,1)'$, $\boldsymbol{c}_4 = (-1,0,1,0)'$, $\lambda_i = \sqrt{\boldsymbol{c}_i'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_i/4}$, $W_i = \boldsymbol{c}_i'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\boldsymbol{W}/2\lambda_i$, $\boldsymbol{W}(t) = (W_1(t), W_2(t), W_3(t), W_4(t))'$ is a four-dimensional standard Brownian motion, $\tilde{\gamma}(j)$ are defined in (2.5), $\Gamma^{(1)} = \sum_{j=1}^{\infty}\tilde{\gamma}(j)$, $\Gamma^{(2)} = \sum_{j=1}^{\infty}(-1)^j\tilde{\gamma}(j)$, $\Gamma^{(3)} = \sum_{j=1}^{\infty}\cos(\pi j/2)\tilde{\gamma}(j)$, and $\Gamma^{(4)} = -\sum_{j=1}^{\infty}\sin(\pi j/2)\tilde{\gamma}(j)$.

Remark 4.1. The results in Theorem 4.1 degenerate to the asymptotics in Burridge and Taylor (2001ab) when $\{V_{4t+s}\}$ is uncorrelated, and degenerate to the asymptotics in Breitung and Franses (1998) when $\{V_{4t+s}\}$ is seasonally homogeneous .

Remark 4.2. When $\{V_{4t+s}\}$ is seasonally homogeneous (Breitung and Franses, 1998), the asymptotic distributions of $(\hat{\pi}_1, t_1)$ and $(\hat{\pi}_2, t_2)$ are independent. On the other hand, when $\{V_{4t+s}\}$ has seasonal heterogeneity, $(\hat{\pi}_1, t_1)$ and $(\hat{\pi}_2, t_2)$ are dependent, as what we have seen for augmented HEGY test (Remark 3.3). Hence, when testing $H_0^{1,2}$, it is problematic to test H_0^1 and H_0^2 separately and calculate the level of the test with the independence of $(\hat{\pi}_1, t_1)$ and $(\hat{\pi}_2, t_2)$ in mind. Instead, the test of $H_0^{1,2}$ should be handled with F_{12}.

Remark 4.3. The parameters λ_i have the same definition as in Theorem 3.2. Since $\lambda_1^2 = \sum_{j=-\infty}^{\infty} \tilde{\gamma}(j)$, and $\lambda_2^2 = \sum_{j=-\infty}^{\infty} (-1)^j \tilde{\gamma}(j)$, the asymptotic distributions of $\hat{\pi}_i$ and t_i, $i = 1, 2$, only depends on the autocorrelation function of $\{\tilde{V}_t\}$, the misspecified constant parameter representation of $\{V_{4t+s}\}$. Since $\{\tilde{V}_t\}$ can be considered as a seasonally homogeneous version of $\{V_{4t+s}\}$, we can conclude that the asymptotic behaviors of the tests for single roots at 1 or -1 are not affected by the seasonal heterogeneity in $\{V_{4t+s}\}$. On the other side, the asymptotic distributions of the F-statistics does not solely depend on $\{\tilde{V}_t\}$. Hence, the test for the concurrence of roots at 1 and -1 and the tests involving roots at $\pm i$ are affected by the seasonal heterogeneity.

Remark 4.4. To remove the nuisance parameters in the asymptotic distributions, we notice that the asymptotic behaviors of $\hat{\pi}_i$ and t_i, $i = 1, 2$, have identical forms as in Phillips and Perron (1988). In light of their approach, we can construct pivotal versions of $\hat{\pi}_i$ and t_i, $i = 1, 2$, that converge in distribution to standard Dickey-Fuller distributions (Dickey and Fuller, 1979). More specifically, for $i = 1, 2$, we can substitute any consistent estimator for λ_i^2 and $\tilde{\gamma}(0)$ below:

$$(4T)\hat{\pi}_i - \frac{\frac{1}{2}(\lambda_i^2 - \tilde{\gamma}(0))}{(4T)^{-2}\sum_t Y_{i,t-1}^2} \Rightarrow \frac{\int_0^1 W_i(r)dW_i(r)}{\int_0^1 W_i^2(r)dr},$$

$$\frac{\sqrt{\tilde{\gamma}(0)}}{\lambda_i} t_i - \frac{\frac{1}{2}(\lambda_i^2 - \tilde{\gamma}(0))}{\lambda_i^2 \sqrt{(4T)^{-2}\sum_t Y_{i,t-1}^2}} \Rightarrow \frac{\int_0^1 W_i(r)dW_i(r)}{\sqrt{\int_0^1 W_i^2(r)dr}}.$$

Remark 4.5. However, there is no easy way to construct pivotal statistics for $\hat{\pi}_3$, t_3, $\hat{\pi}_4$, t_4, and F-statistics such as F_{34}. The difficulties are two-fold. Firstly the denominators of the asymptotic distributions of these statistics contain weighted sums with unknown weights λ_3^2 and λ_4^2; secondly W_3 and W_4 are in general correlated standard Brownian motions as in Theorem 3.1.

Remark 4.6. The result in Theorem 4.1 can be generalized. Suppose $\{Y_{4t+s}\}$ is not generated by $H_0^{1,2,3,4}$, and only has some of the seasonal unit roots. Let $U_t = (1 - L^4)Y_t$, and $\boldsymbol{U}_t = (U_{4t-3}, U_{4t-2}, U_{4t-1}, U_{4t})'$. Then we can find $H(z)$ such that $\boldsymbol{U}_t = \boldsymbol{H}(B)\boldsymbol{\epsilon}_t$. The asymptotic distributions of the t-statistics and the F-statistics have the same forms as those in Theorem 4.1, with $\boldsymbol{\Theta}(1)$ substituted by $\boldsymbol{H}(1)$, and $\tilde{\gamma}$ based on $\{U_t\}$.

4.3 Seasonal block bootstrap algorithm

Since many of the asymptotic distributions delivered in Theorem 4.1 are non-standard, non-pivital, and not directly pivotable, we propose the application of bootstrap. Since the regression error $\{V_{4t+s}\}$ of (4.1) is seasonally stationary, we in particular apply the seasonal block bootstrap of Dudek et al. (2014). The algorithm of seasonal block bootstrap seasonal unit root test is illustrated below.

Algorithm 4.1. *Step 1: get the OLS estimators $\hat{\pi}_1$, $\hat{\pi}_2$, t-statistics t_1, t_2, and the F-statistics F from the regression of the unaugmented HEGY test*

$$(1-L^4)Y_t = \sum_{j=1}^{4} \hat{\pi}_j Y_{j,t-1} + e_t, \quad t = 1, ..., 4T;$$

Step 2: record residual $\hat{V}_t$ from regression

$$(1-L^4)Y_{4t+s} = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y_{j,4t+s-1} + \hat{V}_{4t+s};$$

Step 3: let $\check{V}_{4t+s} = \hat{V}_{4t+s} - \frac{1}{T}\sum_{t=1}^{T} \hat{V}_{4t+s}$, choose a integer block size b, and let $l = \lfloor 4T/b \rfloor$. For $t = 1, b+1, ..., (l-1)b+1$, let

$$(V_t^*, ..., V_{t+b-1}^*) = (\check{V}_{I_t}, ..., \check{V}_{I_t+b-1}),$$

where $\{I_t\}$ is a sequence of iid uniform random variables taking values in $\{t-4R_{1,n}, ..., t-4, t, t+4, ..., t+4R_{2,n}\}$ with $R_{1,n} = \lfloor (t-1)/4 \rfloor$ and $R_{2,n} = \lfloor (n-b-t+1)/4 \rfloor$;

Step 4: set the $\hat{\pi}_{j,s}$ corresponding to the null hypothesis to be zero. For example, set $\pi_{3,s} = \pi_{4,s} = 0$ for all s when testing roots at $\pm i$. Generate $\{Y_t^*\}$ by

$$(1-L^4)Y_{4t+s}^* = \sum_{j=1}^{4} \hat{\pi}_{j,s} Y_{j,4t+s-1}^* + V_{4t+s}^*;$$

Step 5: get OLS estimates $\hat{\pi}_1^*$, $\hat{\pi}_2^*$, t-statistics t_1^*, t_2^*, and F-statistics F^* from regression

$$(1-L^4)Y_t^* = \sum_{j=1}^{4} \hat{\pi}_j^* Y_{j,t-1}^* + e_t^*, \quad t = 1, ..., 4T;$$

Step 6: run step 3, 4, and 5 for B times to get B sets of statistics $\hat{\pi}_1^*$, $\hat{\pi}_2^*$, t_1^*, t_2^*, and F^*. Count separately the numbers of $\hat{\pi}_1^*$, $\hat{\pi}_2^*$, t_1^*, t_2^*, and F^* than which $\hat{\pi}_1$, $\hat{\pi}_2$, t_1, t_2, and F are more extreme. If these numbers are higher than $B(1-level)$, then consider $\hat{\pi}_1$, $\hat{\pi}_2$, t_1, t_2 and F extreme, and reject the corresponding hypotheses.

4.4 Seasonal block bootstrap asymptotics

Theorem 4.2. *Let $S_T^*(u_1, u_2, u_3, u_4)'$*

$$= \frac{1}{\sqrt{4T}}\Big(\sum_{t=1}^{\lfloor 4Tu_1 \rfloor} V_t^*/\sigma_1^*, \sum_{t=1}^{\lfloor 4Tu_2 \rfloor} (-1)^t V_t^*/\sigma_2^*, \sum_{t=1}^{\lfloor 4Tu_3 \rfloor} \sqrt{2}\sin(\frac{\pi t}{2}) V_t^*/\sigma_3^*, \sum_{t=1}^{\lfloor 4Tu_4 \rfloor} \sqrt{2}\cos(\frac{\pi t}{2}) V_t^*/\sigma_4^*\Big)',$$

where

$$\sigma_1^* = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T} V_t^*], \qquad \sigma_2^* = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T} (-1)^t V_t^*],$$

$$\sigma_3^* = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}\sqrt{2}\sin(\frac{\pi t}{2})V_t^*], \qquad \sigma_4^* = Std^\circ[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}\sqrt{2}\cos(\frac{\pi t}{2})V_t^*].$$

If $b \to \infty$, $T \to \infty$, $b/\sqrt{T} \to 0$, then no matter which hypothesis is true, $S_T^* \Rightarrow W$ in probability, where $\boldsymbol{W}(t) = (W_1(t), W_2(t), W_3(t), W_4(t))'$ is a four-dimensional standard Brownian motion.

By the FCLT given by Theorem 4.2, the proof of Theorem 4.1, and the convergence of the bootstrap standard deviation σ_i^* (Dudek et al., 2014), we have that the conditional distribution of t_i^*, $i = 1, 2$, and F^* in probability converges to the limiting distribution of t_i, $i = 1, 2$, and F, respectively. Hence the justification of the seasonal block bootstrap unaugmented HEGY test.

Corollary 4.1. *Suppose the assumptions in Theorem 4.1 hold. If $b \to \infty$, $T \to \infty$, $b/\sqrt{T} \to 0$, then*

$$\sup_x |P^\circ(t_i^* \le x) - P(t_i \le x)| \xrightarrow{p} 0, \; i = 1, 2,$$

$$\sup_x |P^\circ(F^* \le x) - P(t \le x)| \xrightarrow{p} 0.$$

5 Simulation

5.1 Data generating process

We focus on the hypotheses test for root at 1 (H_0^1 against H_1^1), root at -1 (H_0^2 against H_1^2), and root at $\pm i$ ($H_0^{3,4}$ against $H_1^{3,4}$). In each hypothesis test, we equip one sequence with all nuisance unit roots at 1, -1, and $\pm i$, and the other with none of the nuisance unit roots. The detailed data generation processes are listed in Table 1. To produce power curves, we let parameter ρ =0, 0.004, 0.008, 0.012, 0.016, and 0.020. Notice that ρ is set to be seasonally homogeneous for the sake of simplicity. Further, we generate six types of innovations $\{V_{4t+s}\}$ according to Table 2, where $\epsilon_t \sim iid\, N(0, 1)$. The values of ϕ_s are assigned so that the misspecified constant parameter representation (see Section 2) of the "period" sequence has almost the same AR structure as the "ar" sequence.

Table 1: Data generation processes

Data Generating Processes		Nuisance Root: No	Nuisance Root: Yes
Root	1	$(1-(1-\rho)L)Y_t = V_t$	$(1+L)(1+L^2)(1-(1-\rho)L)Y_t = V_t$
	-1	$(1+(1-\rho)L)Y_t = V_t$	$(1-L)(1+L^2)(1+(1-\rho)L)Y_t = V_t$
	$\pm i$	$(1+(1-\rho)L^2)Y_t = V_t$	$(1+L)(1-L)(1+(1-\rho)L^2)Y_t = V_t$

Table 2: Types of noises

Noise Type	iid	$V_t = \epsilon_t$
	heter	$V_{4t+s} = \sigma_s \epsilon_{4t+s}$, $\sigma_1 = 10$, $\sigma_2 = \sigma_3 = \sigma_4 = 1$
	ma_{pos}	$V_t = \epsilon_t + 0.5\epsilon_{t-1}$
	ma_{neg}	$V_t = \epsilon_t - 0.5\epsilon_{t-1}$
	ar	$V_t = \epsilon_t + 0.5V_{t-1}$
	period	$V_{4t+s} = \epsilon_{4t+s} + \phi_s V_{4t+s-1}$, $\phi_1 = 0.2$, $\phi_2 = 0.45$, $\phi_3 = 0.65$, $\phi_4 = 0.8$

5.2 Testing procedure

Here we give additional details for the algorithms of the seasonal iid bootstrap augmented HEGY test (Algorithm 3.1) and the seasonal block bootstrap unaugmented HEGY test (Algorithm 4.1) used in the simulations.

5.2.1 Seasonal iid bootstrap augmented HEGY test

To improve the empirical performance of seasonal iid bootstrap algorithm (Algorithm 3.1), we select stepwise, truncate the coefficient estimators, and use (3.3) when testing roots at 1 or -1. Firstly, a stepwise selection procedure is applied to the regression in step 2 of Algorithm 3.1. To begin with, we choose a maximum lag k_{max}. k_{max} may be chosen by AIC, BIC, or modified information criteria (for further discussions, see Castro, Osborn, and Taylor, 2016). In our simulation we fix $k_{max} = 4$ for simplicity. Afterward, we apply a stepwise selection with Variance Inflating Factor (VIF) criterion to solve the multicollinearity between the regressors. In this selection, we locate the regressor with the largest VIF, remove this regressor from the regression if its VIF is larger than 10, and rerun the regression. Then we implement another stepwise selection on lags $(1 - L^4)Y_{4t+s-i}$, $i = 1, 2, ..., k$, by iteratively removing lags whose t-statistics have absolute values smaller than 1.65 (see also Burridge and Taylor, 2004). Then the estimated coefficients of the deleted regressors are set to be zero, while the estimated coefficients of the remaining regressors are recorded and used in step 2 and 4. The stepwise selection on the t-statistics of the lags are also applied to step 1 and 5.

Secondly, notice that in step 2, the true parameters $\pi_{j,s}$, $j = 1, 2, 3$, are smaller or equal to zero under both null and alternative hypotheses. However, the OLS estimators $\hat{\pi}_{j,s}$, $j = 1, 2, 3$, are often positive, especially when $\pi_{j,s} = 0$. This positivity not only renders the estimation of $\pi_{j,s}$ inaccurate, but also makes the equation in step 4 of Algorithm 3.1 non-causal, and the bootstrapped sequence $\{Y^{\star}_{4t+s}\}$ explosive. The solution of this problem is to truncate the OLS estimator. Let $\check{\pi}_{j,s} = \min(0, \hat{\pi}_{j,s})$, $j = 1, 2, 3$. Immediately we get $|\check{\pi}_{j,s} - \pi_{j,s}| \leq |\hat{\pi}_{j,s} - \pi_{j,s}|$. After we substitute $\check{\pi}_{j,s}$ for $\hat{\pi}_{j,s}$ in step 4, the empirical performance of seasonal iid bootstrap improves significantly.

Thirdly, we use the original step 4 of Algorithm 3.1 when testing roots at $\pm i$, but apply the alternative step (3.3) to the test of root at 1 or -1. (When apply the alternative step (3.3), we similarly select stepwise the lags and truncate the coefficients.) Unpublished simulation result shows an advantage of (3.3) when testing root at 1 or -1. This advantage occurs especially when all nuisance roots occur, or equivalently when all of the true $\pi_{j,s}$'s are zero, since in this case the inclusion of $Y^{\star}_{j,4t+s-1}$ in (3.3) becomes redundant.

5.2.2 Seasonal block bootstrap unaugmented HEGY test

To improve the empirical performance of seasonal block bootstrap algorithm (Algorithm 4.1), we truncate the coefficient estimators, taper the blocks, and optimize the block size. Firstly, as in the seasonal iid bootstrap algorithm, we let $\check{\pi}_{j,s} = \min(0, \hat{\pi}_{j,s})$, $j = 1, 2, 3$, and substitute $\check{\pi}_{j,s}$ for $\hat{\pi}_{j,s}$ in step 4.

Secondly, it is known that the bootstrapped data around the edges of the bootstrap blocks are not good imitations of the original data. To reduce this "edge effect", we apply tapered seasonal block bootstrap proposed by Dudek, Paparoditis, and Politis (2016), which put less weight on the bootstrapped data around the edges. In our simulation the weight function is set identical to the function in Dudek et al. (2016).

Thirdly, both test statistics $\hat{\pi}_j$ and t_j can be employed to run seasonal block bootstrap unaugmented HEGY test. So do various block sizes. In the following preliminary simulation we check the impact on empirical sizes of choices of test statistics and block sizes (for a thorough discussion on optimal block size, see Paparoditis and Politis, 2003). Let $\hat{\pi}^{(i)}$ indicates the bootstrap test based on coefficient estimator $\hat{\pi}$ with block size i, and $t^{(i)}$ indicates the bootstrap test based on t-statistics t with block size i. Set the sample size $T = 120$; in each test $B = 250$ bootstrap replicates are created; the nominal size $\alpha = 0.05$; the empirical sizes are calculated using $N = 300$ iterations. The results on the empirical sizes of the tests are included in Table 3, 4, and 5.

From Table 3, 4, and 5 we can see that the choice of statistics and block sizes does not affect the empirical sizes of the tests very much. (Indeed, unpublished simulations show that empirical powers are not much affected either.) We also find that the distortion of empirical size becomes the worst when testing root at -1 with nuisance roots and ma_{pos} noise. Noticing $t^{(4)}$ gives the best result in the worst scenario, we base the test on the t-statistics and let the block size be four in the succeeding simulations.

Table 3: Empirical sizes of tests for unit root at 1

Nuisance Root	Noise Type	Tests					
		$\hat{\pi}^{(4)}$	$\hat{\pi}^{(8)}$	$\hat{\pi}^{(12)}$	$t^{(4)}$	$t^{(8)}$	$t^{(12)}$
False	iid	0.067	0.047	0.043	0.067	0.050	0.040
	heter	0.057	0.067	0.050	0.053	0.063	0.040
	$\mathrm{ma}_{\mathrm{pos}}$	0.090	0.050	0.030	0.087	0.050	0.023
	$\mathrm{ma}_{\mathrm{neg}}$	0.080	0.073	0.093	0.080	0.060	0.093
	ar	0.043	0.047	0.063	0.047	0.053	0.060
	period	0.043	0.043	0.047	0.047	0.043	0.047
True	iid	0.137	0.123	0.110	0.117	0.110	0.110
	heter	0.160	0.160	0.193	0.160	0.150	0.190
	$\mathrm{ma}_{\mathrm{pos}}$	0.063	0.053	0.073	0.053	0.043	0.057
	$\mathrm{ma}_{\mathrm{neg}}$	0.517	0.500	0.570	0.527	0.500	0.567
	ar	0.010	0.023	0.033	0.010	0.020	0.030
	period	0.017	0.003	0.023	0.017	0.007	0.023

Table 4: Empirical sizes of tests for unit root at -1

Nuisance Root	Noise Type	Tests					
		$\hat{\pi}^{(4)}$	$\hat{\pi}^{(8)}$	$\hat{\pi}^{(12)}$	$t^{(4)}$	$t^{(8)}$	$t^{(12)}$
False	iid	0.040	0.043	0.053	0.040	0.047	0.050
	heter	0.040	0.073	0.040	0.047	0.060	0.033
	$\mathrm{ma}_{\mathrm{pos}}$	0.080	0.080	0.073	0.073	0.080	0.073
	$\mathrm{ma}_{\mathrm{neg}}$	0.060	0.063	0.043	0.063	0.067	0.043
	ar	0.040	0.047	0.050	0.047	0.047	0.053
	period	0.030	0.037	0.050	0.037	0.033	0.063
True	iid	0.143	0.127	0.127	0.143	0.120	0.130
	heter	0.123	0.147	0.177	0.120	0.140	0.173
	$\mathrm{ma}_{\mathrm{pos}}$	0.483	0.543	0.533	0.463	0.550	0.523
	$\mathrm{ma}_{\mathrm{neg}}$	0.070	0.083	0.077	0.070	0.070	0.077
	ar	0.240	0.313	0.343	0.233	0.313	0.333
	period	0.247	0.327	0.310	0.243	0.310	0.303

Table 5: Empirical sizes of tests for unit roots at $\pm i$

Nuisance Root	Noise Type	Tests		
		$F^{(4)}$	$F^{(8)}$	$F^{(12)}$
False	iid	0.053	0.050	0.047
	heter	0.067	0.090	0.073
	ma_{pos}	0.067	0.060	0.047
	ma_{neg}	0.073	0.040	0.083
	ar	0.047	0.030	0.030
	period	0.053	0.040	0.027
True	iid	0.017	0.020	0.017
	heter	0.013	0.020	0.010
	ma_{pos}	0.087	0.063	0.097
	ma_{neg}	0.060	0.067	0.123
	ar	0.113	0.147	0.120
	period	0.093	0.100	0.090

5.3 Results

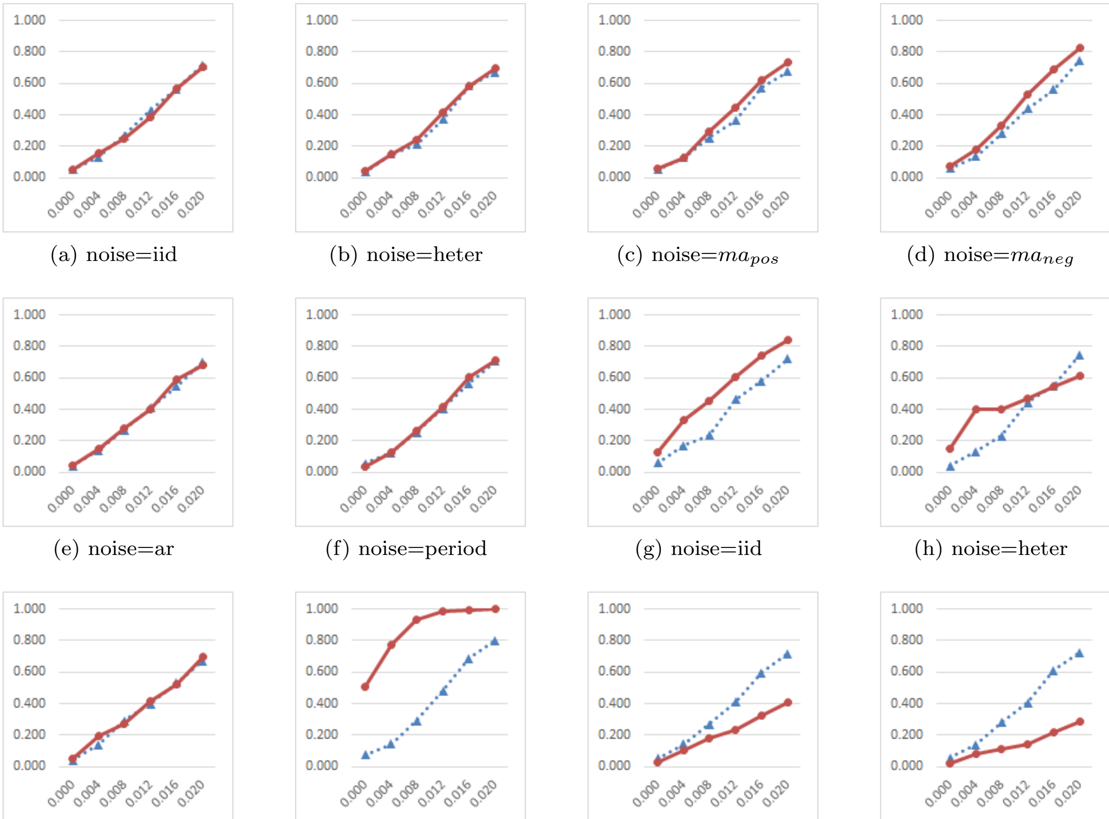
Now we present in Figure 1, 2, and 3 the main simulation result of the seasonal iid bootstrap augmented HEGY test and the seasonal block bootstrap unaugmented HEGY test. This simulation includes two cases of nuisance roots (see Table 1) and six types of noises (see Table 2), and sets sample size $T = 120$, number of bootstrap replicates $B = 500$, number of iterations $N = 600$, and nominal size $\alpha = 0.05$.

5.3.1 Root at 1

When our data have a potential root at 1, but no other nuisance roots at -1 or $\pm i$, the power curves of the both bootstrap tests almost overlap, according to (a)-(f) in Figure 1. Further, both power curves start at the correct size, $\alpha = 0.05$, and tend to one when ρ departs from zero. Hence both tests work well when no nuisance root occurs.

When data have a potential root at 1 and all nuisance roots at -1 and $\pm i$, the sizes of seasonal block bootstrap unaugmented HEGY test are distorted in (g), (h), (j), and (l) in Figure 1. These distortions may result from the errors in estimating $\pi_{j,s}$ and the need to recover $\{Y_{4t+s}\}$ with the estimated $\pi_{j,s}$. The size distortion in (j) is particularly serious, since the unit root filter $(1 - L)$ is partially cancelled by the Moving Average (MA) filter $(1 - 0.5L)$, and this cancellation cannot be handled well by block bootstrap (Paparoditis and Politis, 2003). In contrast, in (l) the filter $(1 - L)$ is enhanced by the AR filters $(1 - \phi_s L)$, thus the size is distorted toward zero.

On the other hand, seasonal iid bootstrap unaugmented HEGY test is free of the size distortions when data have nuisance roots. This is in part because the test recovers $\{Y_{4t+s}\}$ using the true values of $\pi_{j,s}$, namely zero, instead of using the estimated values. Moreover, when both HEGY tests have almost the correct sizes as in (i) and (k), seasonal iid bootstrap unaugmented HEGY test attains in general higher power. Therefore, when testing the root at 1, seasonal iid bootstrap unaugmented HEGY test is recommended.



(a) noise=iid (b) noise=heter (c) noise=ma_{pos} (d) noise=ma_{neg}

(e) noise=ar (f) noise=period (g) noise=iid (h) noise=heter

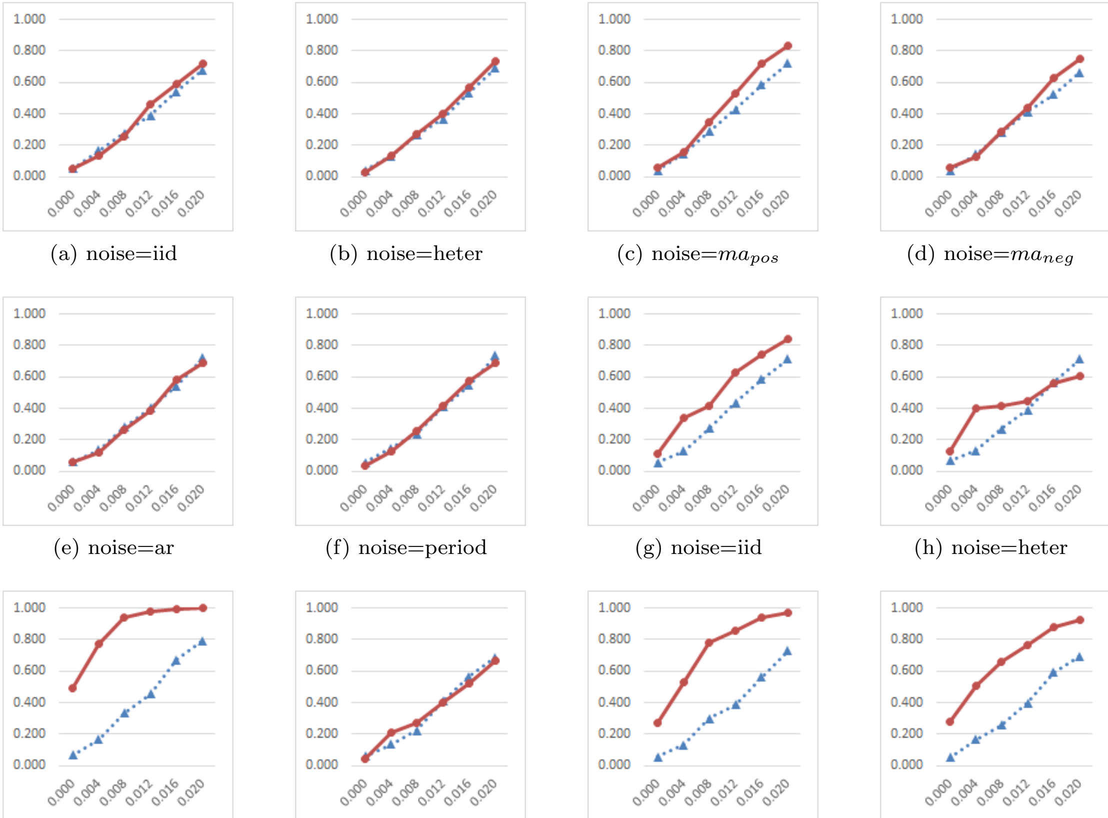
(i) noise=ma_{pos} (j) noise=ma_{neg} (k) noise=ar (l) noise=period

Figure 1: Powers as a function of ρ when testing roots at 1
(a)-(f) have no nuisance roots; (g)-(l) have all nuisance roots;
blue dotted curve is for seasonal iid bootstrap; red solid curve is for seasonal block bootstrap.

5.3.2 Root at -1

Now we come to the tests for root at -1. When none of the nuisance root at 1 or $\pm i$ exists, the power curves of the two tests are very close to each other, as (a)-(f) in Figure 2 indicate. This closeness of curves has been seen in (a)-(f) in Figure 1, and indicates the nice performance of both tests.

When nuisance roots are present, sizes of seasonal block bootstrap unaugmented HEGY test are distorted in nearly all scenarios in (g)-(l) in Figure 2. In particular, the size distortion in (i) is the worst, because of the partial cancellation of the seasonal unit root filter $(1 + L)$ and the MA filter $(1+0.5L)$. However, the power curves of seasonal iid bootstrap augmented HEGY test start around the nominal size 0.05 in all of (g)-(l). Further, these curves tend to 1, as ρ grows larger. Therefore, we recommend seasonal iid bootstrap test for testing root at -1.



(a) noise=iid (b) noise=heter (c) noise=ma_{pos} (d) noise=ma_{neg}

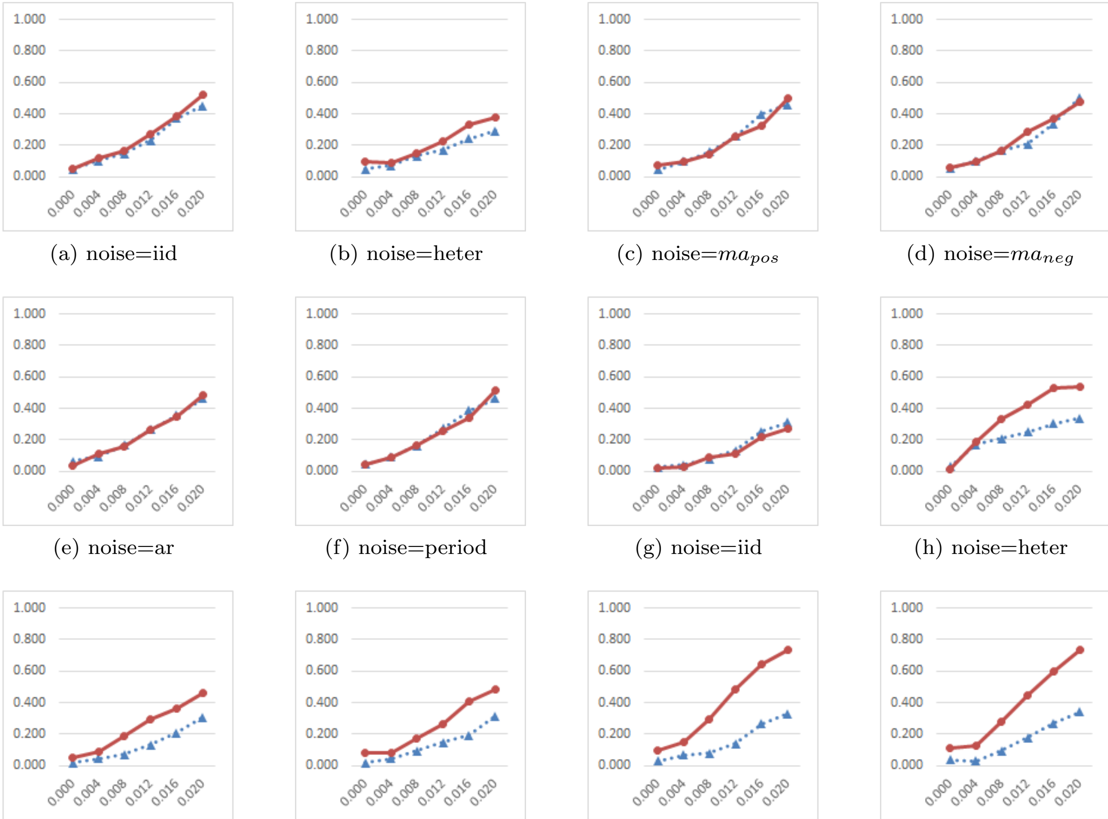
(e) noise=ar (f) noise=period (g) noise=iid (h) noise=heter

(i) noise=ma_{pos} (j) noise=ma_{neg} (k) noise=ar (l) noise=period

Figure 2: Powers as a function of ρ when testing roots at -1
(a)-(f) have no nuisance roots; (g)-(l) have all nuisance roots;
blue dotted curve is for seasonal iid bootstrap; red solid curve is for seasonal block bootstrap.

5.3.3 Root at $\pm i$

Finally we discuss the tests for roots at $\pm i$. With none of the nuisance root at 1 or -1, (a)-(f) in Figure 3 illustrate that both tests achieve sizes that are close to the nominal size, and powers that tend to one. When all of nuisance roots show up, both tests suffer from some size distortions. The empirical sizes of seasonal iid bootstrap augmented HEGY test are biased toward zero in (g)-(l); the sizes of seasonal block bootstrap unaugmented HEGY test are biased toward zero in (g) and (h), but are biased toward one in (j)-(l). On the other hand, seasonal block bootstrap unaugmented HEGY test's empirical powers prevail throughout (g)-(l), and therefore shall be recommended for testing roots at $\pm i$.



(a) noise=iid (b) noise=heter (c) noise=ma_{pos} (d) noise=ma_{neg}

(e) noise=ar (f) noise=period (g) noise=iid (h) noise=heter

(i) noise=ma_{pos} (j) noise=ma_{neg} (k) noise=ar (l) noise=period

Figure 3: Powers as a function of ρ when testing roots at $\pm i$
(a)-(f) have no nuisance roots; (g)-(l) have all nuisance roots;
blue dotted curve is for seasonal iid bootstrap; red solid curve is for seasonal block bootstrap.

6 Conclusion

In this paper we analyze the non-periodic augmented and unaugmented HEGY tests in the seasonal heterogeneous setting. Given root at 1 or -1, the asymptotic distributions of the test statistics are standard. However, given concurrent roots at 1 and -1, or roots at $\pm i$, the asymptotic distributions are not standard, pivotal, nor directly pivotable. Therefore, when seasonal heterogeneity exists, HEGY tests can be used to test the single roots at 1 or -1, but cannot be directly applied to any combinations of roots.

Bootstrap proves to be an effective remedy for HEGY tests in the seasonal heterogeneous setting. The two bootstrap approaches, namely 1) seasonal iid bootstrap augmented HEGY test and 2) seasonal block bootstrap unaugmented HEGY test, turn out to be both theoretically solid. In the comparative simulation study, seasonal iid bootstrap augmented HEGY test has better performance when testing roots at 1 or -1, but seasonal block bootstrap unaugmented HEGY test outperforms when testing roots at $\pm i$.

Therefore, when testing seasonal unit roots under seasonal heterogeneity, the aforementioned bootstrap HEGY tests become competitive alternatives of the Wald-test proposed by Ghysels et al. (1996). Further study will be needed to compare the theoretical and empirical efficiency of the two bootstrap HEGY tests and the Wald-test by Ghysels et al. (1996).

7 Appendix

The appendix includes the proof of the theorems in this paper. We first present the proof for the asymptotics of the unaugmented HEGY test, then the asymptotics of the augmented HEGY test, then the consistency of the seasonal iid bootstrap augmented HEGY test, and finally the consistency of the seasonal block bootstrap unaugmented HEGY test.

7.1 Proof of Theorem 4.1.

Lemma 1. *Suppose Assumption 2.A or Assumption 2.B hold. Let* $\boldsymbol{Y}_t = (Y_{4t-3}, Y_{4t-2}, Y_{4t-1}, Y_{4t})'$, $\boldsymbol{V}_t = (V_{4t-3}, V_{4t-2}, V_{4t-1}, V_{4t})'$. *Let* $\boldsymbol{\Gamma}_j = E[\boldsymbol{V}_t \boldsymbol{V}_{t-j}']$. *Let* $\boldsymbol{W}(t) = (W_1(t), W_2(t), W_3(t), W_4(t))'$ *be a four-dimensional standard Brownian motion. Let* $\int \boldsymbol{W} d\boldsymbol{W}'$ *denotes* $\int_0^1 \boldsymbol{W}(r) d\boldsymbol{W}(r)'$, *and* $\int \boldsymbol{W}\boldsymbol{W}'$ *denotes* $\int_0^1 \boldsymbol{W}(r)\boldsymbol{W}(r)'dr$. *Then, under* $H_0^{1,2,3,4}$,

$$T^{-1}\sum_{t=1}^{T} \boldsymbol{Y}_{t-1}\boldsymbol{V}_t' \Rightarrow \boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\{\int \boldsymbol{W} d\boldsymbol{W}'\}\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)' + \sum_{j=1}^{\infty} \boldsymbol{\Gamma}_j' \equiv \boldsymbol{Q}_1,$$

$$T^{-2}\sum_{t=1}^{T} \boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}' \Rightarrow \boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\{\int \boldsymbol{W}\boldsymbol{W}'\}\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)' \equiv \boldsymbol{Q}_2,$$

$$T^{-1}\sum_{t=1}^{T} \boldsymbol{V}_t\boldsymbol{V}_{t-j}' \xrightarrow{p} \boldsymbol{\Gamma}_j.$$

Proof. See Hamilton (1994, proposition 18.1) for the proof given iid innovations, Chan and Wei (1988) under Assumption 2.A, and De Jong and Davidson (2000) under Assumption 2.B. □

Lemma 2. *Let* $\boldsymbol{X}_{U,j} = (Y_{j,0}, ..., Y_{j,4T-1})'$, *and* $\boldsymbol{X}_U = (\boldsymbol{X}_{U,1}, \boldsymbol{X}_{U,2}, \boldsymbol{X}_{U,3}, \boldsymbol{X}_{U,4})$, *where* U *stands for unaugmented HEGY, and* $\{Y_{j,4t+s}\}$ *is defined in* (2.4). *Let* $\boldsymbol{V} = (V_1, ..., V_{4T})'$, $\boldsymbol{\Upsilon}$ *be the matrix generated by assigning zero to all entries of* $\boldsymbol{\Gamma}_0$ *but those above the main diagonal. Then, under* $H_0^{1,2,3,4}$,

(*a*)

$$(4T)^{-2}(\boldsymbol{X}_U'\boldsymbol{X}_U)_{11} \Rightarrow \frac{1}{4}\boldsymbol{c}_1'\boldsymbol{Q}_2\boldsymbol{c}_1 \equiv \eta_1,$$

$$(4T)^{-2}(\boldsymbol{X}_U'\boldsymbol{X}_U)_{22} \Rightarrow \frac{1}{4}\boldsymbol{c}_2'\boldsymbol{Q}_2\boldsymbol{c}_2 \equiv \eta_2,$$

$$(4T)^{-2}(\boldsymbol{X}_U'\boldsymbol{X}_U)_{33} \Rightarrow \frac{1}{8}(\boldsymbol{c}_3'\boldsymbol{Q}_2\boldsymbol{c}_3 + \boldsymbol{c}_4'\boldsymbol{Q}_2\boldsymbol{c}_4) \equiv \eta_3,$$

$$(4T)^{-2}(\boldsymbol{X}_U'\boldsymbol{X}_U)_{44} \Rightarrow \frac{1}{8}(\boldsymbol{c}_3'\boldsymbol{Q}_2\boldsymbol{c}_3 + \boldsymbol{c}_4'\boldsymbol{Q}_2\boldsymbol{c}_4) \equiv \eta_3,$$

$$(4T)^{-1}(\boldsymbol{X}_U'\boldsymbol{X}_U)_{ij} \xrightarrow{p} 0, \text{ for } i \neq j.$$

(*b*)

$$
\begin{aligned}
(4T)^{-1}\boldsymbol{X}'_{U,1}\boldsymbol{V} &\Rightarrow \frac{1}{4}(\boldsymbol{c}'_1\boldsymbol{Q}_1\boldsymbol{c}_1 + \boldsymbol{c}'_1\boldsymbol{\Upsilon}\boldsymbol{c}_1) \equiv \xi_1,\\
(4T)^{-1}\boldsymbol{X}'_{U,2}\boldsymbol{V} &\Rightarrow \frac{1}{4}(\boldsymbol{c}'_2\boldsymbol{Q}_1\boldsymbol{c}_2 + \boldsymbol{c}'_2\boldsymbol{\Upsilon}\boldsymbol{c}_2) \equiv \xi_2,\\
(4T)^{-1}\boldsymbol{X}'_{U,3}\boldsymbol{V} &\Rightarrow \frac{1}{4}(\boldsymbol{c}'_3\boldsymbol{Q}_1\boldsymbol{c}_3 + \boldsymbol{c}'_4\boldsymbol{Q}_1\boldsymbol{c}_4 + \boldsymbol{c}'_3\boldsymbol{\Upsilon}\boldsymbol{c}_3 + \boldsymbol{c}'_4\boldsymbol{\Upsilon}\boldsymbol{c}_4) \equiv \xi_3,\\
(4T)^{-1}\boldsymbol{X}'_{U,4}\boldsymbol{V} &\Rightarrow \frac{1}{4}(\boldsymbol{c}'_3\boldsymbol{Q}_1\boldsymbol{c}_4 - \boldsymbol{c}'_4\boldsymbol{Q}_1\boldsymbol{c}_3 + \boldsymbol{c}'_3\boldsymbol{\Upsilon}\boldsymbol{c}_4 - \boldsymbol{c}'_4\boldsymbol{\Upsilon}\boldsymbol{c}_3) \equiv \xi_4.
\end{aligned}
$$

Proof. For the proof of part (a), see the Lemma 3.2(a) of Burridge and Taylor (2001a) and its proof. For part (b), we only present the proof of the first statement. Other statements are proven in similar ways. By Lemma 1,

$$
\begin{aligned}
(4T)^{-1}\boldsymbol{X}'_{U,1}\boldsymbol{V} &= (4T)^{-1}\sum_{t=1}^{T}\sum_{s=-3}^{0} Y_{1,4t+s-1}V_{4t+s}\\
&= (4T)^{-1}\sum_{t=1}^{T}\sum_{s=-3}^{0}(\boldsymbol{c}'_1\boldsymbol{Y}_{t-1} + \sum_{i=-2}^{s} V_{4t-1+i})V_{4t+s}\\
&= (4T)^{-1}\sum_{t=1}^{T}\boldsymbol{c}'_1\boldsymbol{Y}_{t-1}\boldsymbol{V}'_t\boldsymbol{c}_1 + (4T)^{-1}\sum_{t=1}^{T}\sum_{s=-3}^{0}\sum_{i=-2}^{s} V_{4t-1+i}V_{4t+s}\\
&\Rightarrow \frac{1}{4}(\boldsymbol{c}'_1\boldsymbol{Q}_1\boldsymbol{c}_1 + \boldsymbol{c}'_1\boldsymbol{\Upsilon}\boldsymbol{c}_1) \qquad \square
\end{aligned}
$$

Proof of Theorem 4.1. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)'$, $\hat{\boldsymbol{\pi}} = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)'$, $\boldsymbol{t} = (t_1, t_2, t_3, t_4)'$, and $\hat{\sigma}^2 = (4T)^{-1}(\boldsymbol{V} - \boldsymbol{X}_U\hat{\boldsymbol{\pi}})'(\boldsymbol{V} - \boldsymbol{X}_U\hat{\boldsymbol{\pi}})$. Then

$$
\begin{aligned}
(4T)\hat{\boldsymbol{\pi}} &= (\boldsymbol{X}'_U\boldsymbol{X}_U)^{-1}\boldsymbol{X}'_U\boldsymbol{V} \Rightarrow [\mathrm{diag}(\eta_1, \eta_2, \eta_3, \eta_4)]^{-1}(\xi_1, \xi_2, \xi_3, \xi_4)' \text{ by Lemma 2},\\
\hat{\sigma}^2 &= (4T)^{-1}(\boldsymbol{V}'\boldsymbol{V} + 2(\boldsymbol{X}_U\hat{\boldsymbol{\pi}} - \boldsymbol{X}_U\boldsymbol{\pi})'(\boldsymbol{V} - \boldsymbol{X}_U\boldsymbol{\pi})\\
&\quad +(\boldsymbol{X}_U\hat{\boldsymbol{\pi}} - \boldsymbol{X}_U\boldsymbol{\pi})'(\boldsymbol{X}_U\hat{\boldsymbol{\pi}} - \boldsymbol{X}_U\boldsymbol{\pi}))\\
&= (4T)^{-1}\boldsymbol{V}'\boldsymbol{V} + o_p(1) \text{ by the consistency of } \hat{\boldsymbol{\pi}}\\
&\xrightarrow{p} \mathrm{tr}(\boldsymbol{\Gamma}_0)/4,\\
\boldsymbol{t} &= \hat{\sigma}^{-1}[\mathrm{diag}(\boldsymbol{X}'_U\boldsymbol{X}_U)^{-1}]^{-1/2}(\boldsymbol{X}'_U\boldsymbol{X}_U)^{-1}\boldsymbol{X}'_U\boldsymbol{V}\\
&\Rightarrow (\mathrm{tr}(\boldsymbol{\Gamma}_0)/4)^{-1/2}[\mathrm{diag}(\eta_1, \eta_2, \eta_3, \eta_4)]^{-1/2}(\xi_1, \xi_2, \xi_3, \xi_4)'.
\end{aligned}
$$

Further, the asymptotic distributions of F-statistics are identical with the asymptotic distributions of the averages of the squares of the corresponding t-statsitics, for example, $F_{34} - \frac{1}{2}(t_3^2 + t_4^2) \xrightarrow{p} 0$, due to the asymptotic orthogonality indicated by Lemma 2. $\square$

7.2 Proof of Theorem 3.1.

The proof follows the lines of Said and Dickey (1984) and contains two parts. Firstly, we show when $T \to \infty$ and $k = k_T \to \infty$ simultaneously, the statistic of interest tends to a limit free of k, and then we prove this limit tends to a certain distribution as $T \to \infty$.

To begin with, notice that when $k \to \infty$, the error term of regression (3.2) tends to a limit. Surprisingly, this limit is in general not ϵ_t, because the regression (3.2) falsely assumes seasonally homogeneous coefficients and thus in general cannot find the correct residuals ϵ_t. To find the limit, recall that $\{\tilde{V}_t\}$ is defined as a misspecified constant parameter representation of $\{V_{4t+s}\}$. Under Assumption 1.B, the spectral densities of $\{\tilde{V}_t\}$ are finite and positive everywhere, so $\{\tilde{V}_t\}$ has AR(∞) and MA(∞) expressions

$$\tilde{\psi}(L)\tilde{V}_t = \tilde{\zeta}_t \text{ and } \tilde{V}_t = \tilde{\theta}(L)\tilde{\zeta}_t, \tag{7.1}$$

where $\tilde{\psi}(z) = 1 - \sum_{i=1}^{\infty}\tilde{\psi}_i z^i$, $\tilde{\theta}(z) = 1 + \sum_{i=1}^{\infty}\tilde{\theta}_i z^i$. Let $\zeta_t^{(k)} = V_t - \sum_{i=1}^{k}\tilde{\psi}_i V_{t-i}$, and $\zeta_t = V_t - \sum_{i=1}^{\infty}\tilde{\psi}_i V_{t-i}$, where $\{\tilde{\psi}_i\}$ are the AR coefficients defined in (7.1). Since a misspecified constant parameter representation of ζ_t is $\tilde{V}_t - \sum_{i=1}^{\infty}\tilde{\psi}_i\tilde{V}_{t-i}$, which is exactly $\tilde{\zeta}_t$ defined in (7.1), no ambiguity arises. The following lemma provides two properties of $\{\zeta_t\}$, whose proof is left to the readers.

Lemma 3.

$$\textit{(a)}\ \frac{1}{4}\sum_{s=-3}^{0} Cov(\zeta_{4t+s-j}, \zeta_{4t+s}) = 0,\ \forall j = 1, 2, ...,$$

$$\textit{(b)}\ \frac{1}{4}\sum_{s=-3}^{0} Cov(V_{4t+s-j}, \zeta_{4t+s}) = 0,\ \forall j = 1, 2, ...$$

Now we show when $T \to \infty$ and $k \to \infty$ simultaneously, the statistics of interest tend to certain limits. Let $\boldsymbol{X}$ be the design matrix of regression equation (3.2), $\hat{\boldsymbol{\beta}} = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4, \hat{\phi}_1, ..., \hat{\phi}_k)'$ be the estimated coefficient vector of regression equation (3.2), $\boldsymbol{\beta} = (0, 0, 0, 0, \tilde{\psi}_1, ..., \tilde{\psi}_k)'$, $\boldsymbol{\zeta}^{(k)} = (\zeta_{1+k}^{(k)}, ..., \zeta_{4T}^{(k)})'$, and $\boldsymbol{\zeta} = (\zeta_{1+k}, ..., \zeta_{4T})'$. Define the $(4+k) \times (4+k)$ dimensional scaling matrix $\boldsymbol{D}_T = diag((4T-k)^{-1}, (4T-k)^{-1}, (4T-k)^{-1}, (4T-k)^{-1}, (4T-k)^{-1/2}, ..., (4T-k)^{-1/2})$. Then

$$\boldsymbol{D}_T^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = (\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T)^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}^{(k)}.$$

Let $||\cdot||$ be the L_2 induced norm of matrices. Now we want to define a diagonal matrix $\boldsymbol{R}$ such that $||\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T - \boldsymbol{R}||$ converges to 0 in probability. By the multivariate Beveridge-Nielson Decomposition (see Hamilton, 1994, pp. 545-546), since $(4T-k)^{-1}\sum(1-L^4)Y_{t-i}(1-L^4)Y_{t-j}$ converges in probability to the seasonal average of autocovariance of V_t of lag $|i-j|$, we let

$$\boldsymbol{R} = diag(R_1, R_2, R_3, R_4, \tilde{\boldsymbol{\Gamma}}),$$

where

$$R_1 = \frac{c_1'\boldsymbol{\Theta}(1)\sum \boldsymbol{S}_t\boldsymbol{S}_t'\boldsymbol{\Theta}(1)'c_1}{(4T-k)^2}$$
$$R_2 = \frac{c_2'\boldsymbol{\Theta}(1)\sum \boldsymbol{S}_t\boldsymbol{S}_t'\boldsymbol{\Theta}(1)'c_2}{(4T-k)^2}$$
$$R_3 = \frac{c_3'\boldsymbol{\Theta}(1)\sum \boldsymbol{S}_t\boldsymbol{S}_t'\boldsymbol{\Theta}(1)'c_3 + c_4'\boldsymbol{\Theta}(1)\sum \boldsymbol{S}_t\boldsymbol{S}_t'\boldsymbol{\Theta}(1)'c_4}{2(4T-k)^2}$$
$$R_4 = R_3, \quad \boldsymbol{S}_t = \sum_{i=1}^{t}\boldsymbol{\epsilon}_i, \quad \tilde{\boldsymbol{\Gamma}}_{i,j} = \tilde{\gamma}(|i-j|).$$

Following the definition of $\boldsymbol{R}$, we make the following decomposition:

$$\begin{aligned}\boldsymbol{D}_T^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &= (\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T)^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}^{(k)} \\ &= [(\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T)^{-1} - \boldsymbol{R}^{-1}]\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}^{(k)} \\ &+ \boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'(\boldsymbol{\zeta}^{(k)} - \boldsymbol{\zeta}) \\ &+ \boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}\end{aligned} \tag{7.2}$$

Notice the last term in the right hand side summation, $\boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}$, is free of k. Later we will find out its asymptotic distribution as $T \to \infty$. But now we need to prove the first two terms in the right hand side of (7.2) converge to zero as $T \to \infty$ and $k \to \infty$. Indeed,

$$||(\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T)^{-1} - \boldsymbol{R}^{-1}|| = o_p(k^{-1/2}), \tag{7.3}$$

$$||\boldsymbol{D}_T\boldsymbol{X}'(\boldsymbol{\zeta}^{(k)} - \boldsymbol{\zeta})|| = o_p(1), \tag{7.4}$$

$$||\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}|| = O_p(k^{1/2}), \tag{7.5}$$

$$||\boldsymbol{R}^{-1}|| = O_p(1). \tag{7.6}$$

Equation (7.3) can be proven straightforwardly (see Said and Dickey, 1984). For (7.4), notice

$$\begin{aligned}
&E||\boldsymbol{D}_T\boldsymbol{X}'(\boldsymbol{\zeta}^{(k)} - \boldsymbol{\zeta})||^2 \\
=&E[(4T-k)^{-2}\sum_{j=1}^{4}(\sum_t Y_{j,t-1}(\zeta_t^{(k)} - \zeta_t))^2 + (4T-k)^{-1}\sum_{i=1}^{k}(\sum_t V_{t-i}(\zeta_t^{(k)} - \zeta_t))^2].
\end{aligned}$$

Notice that $\zeta_t^{(k)} - \zeta_t = \sum_{i=k+1}^{\infty}\tilde{\psi}_i V_{t-i}$. Under assumption 1.B, $\{V_{4t+s}\}$ is a VARMA sequence with finite orders, thus $\{\tilde{V}_t\}$ also has an ARMA expression with finite orders (see Osborn, 1991),

$$\tilde{\varphi}(L)\tilde{V}_t = \tilde{\vartheta}(L)\tilde{\zeta}_t. \tag{7.7}$$

Hence, $\tilde{\psi}(L) = \tilde{\vartheta}(L)^{-1}\tilde{\varphi}(L)$ has exponentially decaying coefficient $\tilde{\psi}_i$. It follows straightforwardly that $E||\boldsymbol{D}_T\boldsymbol{X}'(\boldsymbol{\zeta}^{(k)} - \boldsymbol{\zeta})||^2 \to 0$. For (7.5), notice that

$$E||\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}||^2 = E[(4T-k)^{-2}\sum_{j=1}^{4}(\sum_{t=k+1}^{4T} Y_{j,t-1}\zeta_t)^2 + (4T-k)^{-1}\sum_{i=1}^{k}(\sum_{t=k+1}^{4T} V_{t-i}\zeta_t)^2].$$

By Lemma 3 and the stationarity of $\{\boldsymbol{\epsilon}_t\}$,

$$\begin{aligned}
&E[((4T-k)^{-1/2}\sum_{t=k+1}^{4T} V_{t-i}\zeta_t)^2] \\
&= \frac{1}{4}\sum_{s=-3}^{0}\sum_{h=-\infty}^{\infty} Cov(V_{4t+s-i}\zeta_{4t+s}, V_{4t+s-h-i}\zeta_{4t+s-h}) + o(1) \\
&= \frac{1}{4}\sum_{s=-3}^{0}\sum_{h=-\infty}^{\infty} Cov(V_{s-i}\zeta_s, V_{s-h-i}\zeta_{s-h}) + o(1)
\end{aligned}$$

Without loss of generality we can focus on $i = 1$ and $s = 0$. By writing V_t and ζ_t as linear combinations of ϵ_t,

$$\sum_{h=-\infty}^{\infty} Cov(V_{-1}\zeta_0, V_{-h-1}\zeta_{-h}) \leq \text{const. } sup_{i_1,j_1,i_2,j_2}\sum_{h=-\infty}^{\infty}|Cov(\epsilon_{i_1-1}\epsilon_{j_1}, \epsilon_{i_2-h-1}\epsilon_{j_2-h})|.$$

The right hand side of this inequality is assumed to be bounded under Assumption 2.A. On the other hand, the right hand side is also bounded under Assumption 2.B, by the lemma below.

Lemma 4. *Suppose (i) $\{z_t\}_{t=1}^n$ is a strictly stationary strong mixing time series with mean zero and finite $4+\delta$ moment for some $\delta > 0$, and (ii) $\{z_t\}$'s strong mixing coefficient $\alpha(h)$ satisfies $\sum_{h=1}^{\infty} h\alpha^{\delta/(4+\delta)}(h) < \infty$. Then $\exists K > 0$ such that for all i_1, i_2, j_1, and j_2,*

$$\sum_{h=-\infty}^{\infty} |Cov(z_{i_1} z_{j_1}, z_{i_2-h} z_{j_2-h})| < K.$$

Proof. Let $h_1 = h + i_1 - i_2$, $h_2 = h + j_1 - j_2$, $h_3 = h + i_1 - j_2$, $h_4 = h + j_1 - i_2$. By Lemma A.0.1 of Politis, Romano, and Wolf (1999),

$$\begin{aligned}
&\sum_{h=-\infty}^{\infty} |Cov(z_{i_1} z_{j_1}, z_{i_2-h} z_{j_2-h})| \\
&\leq \text{const.} \sum_{h=-\infty}^{\infty} (\alpha(min(|h_1|, |h_2|, |h_3|, |h_4|)))^{\frac{\delta}{4+\delta}} \\
&\leq \text{const.} \sum_{h=-\infty}^{\infty} (\alpha(|h_1|)^{\frac{\delta}{4+\delta}} + \alpha(|h_2|)^{\frac{\delta}{4+\delta}} + \alpha(|h_3|)^{\frac{\delta}{4+\delta}} + \alpha(|h_4|)^{\frac{\delta}{4+\delta}}) \\
&\leq \text{const.} \sum_{h=-\infty}^{\infty} (\alpha(|h|)).
\end{aligned}$$

□

We have proven that $E[((4T-k)^{-1/2} \sum_{t=k+1}^{4T} V_{t-i}\zeta_t)^2] = O(1)$. Similarly, it can be shown that $E[((4T-k)^{-1} \sum_{t=k+1}^{4T} Y_{j,t-1}\zeta_t)^2] = O(1)$. Hence, (7.5) follows. To justify (7.6), notice

$$\frac{\boldsymbol{c_i'}\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2} \sum \boldsymbol{S}_t \boldsymbol{S}_t' \boldsymbol{\Omega}^{1/2} \boldsymbol{\Theta}(1)' \boldsymbol{c_i}}{(4T-k)^2} \Rightarrow \boldsymbol{c_i'}\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2} \int \boldsymbol{W}\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c_i},$$

where $\boldsymbol{W}$ indicates standard four-dimensional Brownian Motion. Since

$$P(\boldsymbol{c_i'}\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2} \int \boldsymbol{W}\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c_i} = 0) = 0,$$

$\forall \epsilon > 0$, $\exists M_\epsilon > 0$, such that $P(\boldsymbol{c_i'}\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2} \int \boldsymbol{W}\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c_i} < M_\epsilon) < \epsilon$. (7.6) follows from the definition of $O_p(1)$.

Combining equations (7.3), (7.4), (7.5), and (7.6), we have

$$\begin{aligned}
||[(\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}_T)^{-1} - \boldsymbol{R}^{-1}]\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}^{(k)}|| &= o_p(1) \\
||\boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'(\boldsymbol{\zeta}^{(k)} - \boldsymbol{\zeta})|| &= o_p(1) \\
||\boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}|| &= O_p(k^{1/2}).
\end{aligned}$$

From these results, we can immediately show the consistency of $\hat{\boldsymbol{\beta}}$. Notice $\boldsymbol{D}_T^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = O_p(k^{1/2})$ by (7.2). The consistency follows from $||\boldsymbol{D}_T|| = O((4T-k)^{-1})$ and $k = o(T^{1/3})$. Further, the asymptotic distribution of $\hat{\boldsymbol{\beta}}$ can be derived with the asymptotic equivalence of $\boldsymbol{D}_T^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ and $\boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}$. Notice $\boldsymbol{R}^{-1}\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}$ is free of k. As $T \to \infty$, $\boldsymbol{R}^{-1}$ converges in distribution to a functional of Brownian motion, and the asymptotics of $\boldsymbol{D}_T\boldsymbol{X}'\boldsymbol{\zeta}$ can be found with the following lemma.

Lemma 5.

$$\frac{1}{4T}\sum_{t=1}^{4T}Y_{1,t-1}\zeta_t \Rightarrow Var(\tilde{\zeta}_t)\tilde{\theta}(1)\int_0^1 W_1(r)dW_1(r),$$
$$\frac{1}{4T}\sum_{t=1}^{4T}Y_{2,t-1}\zeta_t \Rightarrow Var(\tilde{\zeta}_t)\tilde{\theta}(-1)\int_0^1 W_2(r)dW_2(r),$$

$$\begin{aligned}&(\frac{1}{4T}\sum_{t=1}^{4T}Y_{3,t-1}\zeta_t)^2+(\frac{1}{4T}\sum_{t=1}^{4T}Y_{4,t-1}\zeta_t)^2\\ &\Rightarrow \frac{Var(\tilde{\zeta}_t)[\frac{1}{4}\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4\int W_4(r)dW_4(r)+\frac{1}{4}\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3\int W_3(r)dW_3(r)]^2}{\frac{1}{4}(\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4+\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3)}\\ &\quad+\frac{Var(\tilde{\zeta}_t)[\sqrt{\frac{1}{4}\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4\frac{1}{4}\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3}(\int_0^1 W_3(r)dW_4(r)-\int W_4(r)dW_3(r))]^2}{\frac{1}{4}(\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4+\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3)}.\end{aligned}$$

Proof of Lemma 5. Firstly we focus on the convergence of $\frac{1}{4T}\sum_{t=1}^{4T}Y_{1,t-1}\zeta_t$. The convergence of $\frac{1}{4T}\sum_{t=1}^{4T}Y_{2,t-1}\zeta_t$ can be proven analogously. Let $\xi_t=\tilde{\psi}(L)Y_t$, $\xi_{1,t}=\tilde{\psi}(L)Y_{1,t}$, $\boldsymbol{\xi}_t=(\xi_{4t-3},\xi_{4t-2},\xi_{4t-1},\xi_{4t})'$, $\boldsymbol{\zeta}_t=(\zeta_{4t-3},\zeta_{4t-2},\zeta_{4t-1},\zeta_{4t})'$. Then $B\boldsymbol{\xi}_t=\boldsymbol{\zeta}_t$, and

$$\begin{aligned}&\frac{1}{4T}\sum_{t=1}^{4T}Y_{1,t-1}\zeta_t\\ &=\tilde{\theta}(1)\frac{1}{4T}\sum_{t=1}^{4T}\sum_{s=-3}^{0}\xi_{1,4t+s-1}\zeta_{4t+s}\ \text{(by Beveridge-Nielson Decomposition, up to } o_p(1))\\ &=\tilde{\theta}(1)\frac{1}{4T}\sum_{t=1}^{T}[\boldsymbol{c}_1'\boldsymbol{\xi}_{t-1}\boldsymbol{\zeta}_t'\boldsymbol{c}_1+\sum_{s=-3}^{0}\sum_{k=-3}^{s-1}\zeta_{4t+k}\zeta_{4t+s}]\\ &\Rightarrow\frac{1}{4}\tilde{\theta}(1)\boldsymbol{c}_1'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int\boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_1\ \text{(by Lemma 3 and FCLT)}\\ &\quad+\frac{1}{4}\tilde{\theta}(1)[\sum_{s=-3}^{0}\sum_{k=-3}^{s-1}E\zeta_{4t+k}\zeta_{4t+s}+\boldsymbol{c}_1'\sum_{i=1}^{\infty}E\boldsymbol{\zeta}_{t-i}\boldsymbol{\zeta}_t'\boldsymbol{c}_1]\\ &=\frac{1}{4}\tilde{\theta}(1)\boldsymbol{c}_1'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int\boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_1\ \text{(since } \{\tilde{\zeta}_t\} \text{ is white noise)}\\ &=\frac{1}{4}\tilde{\theta}(1)(\tilde{\psi}(1))^2\boldsymbol{c}_1'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int\boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_1\ \text{(since } \boldsymbol{c}_1'\tilde{\boldsymbol{\Psi}}(1)=\tilde{\psi}(1)\boldsymbol{c}_1')\\ &=Var(\tilde{\zeta}_t)\tilde{\theta}(1)\int_0^1 W_1(r)dW_1(r)\\ &\text{(by Osborn (1991, p. 378), } \frac{1}{4}\boldsymbol{c}_1'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_1=Var(\tilde{\zeta}_t)\tilde{\theta}(1)^2).\end{aligned}$$

Secondly we show the convergence of $(\frac{1}{4T}\sum_{t=1}^{4T}Y_{3,t-1}\zeta_t)^2+(\frac{1}{4T}\sum_{t=1}^{4T}Y_{4,t-1}\zeta_t)^2$. Let $\xi_{3,t}=\tilde{\psi}(L)Y_{3,t}$, $\tilde{\psi}_a=(\tilde{\psi}(i)+\tilde{\psi}(-i))/2$, $\tilde{\psi}_b=(\tilde{\psi}(i)-\tilde{\psi}(-i))/2i$, $\tilde{\theta}_a=(\tilde{\theta}(i)+\tilde{\theta}(-i))/2$, and $\tilde{\theta}_b=(\tilde{\theta}(i)-\tilde{\theta}(-i))/2i$.

Then

$$
\begin{aligned}
&\frac{1}{4T}\sum_{t=1}^{4T} Y_{3,t-1}\zeta_t \\
&= \frac{1}{4T}\sum_{t=1}^{4T}(\tilde{\theta}_a \xi_{3,t-1} - \tilde{\theta}_b \xi_{4,t-1})\zeta_t \\
&\text{(by Beveridge-Nielson Decomposition, up to } o_p(1)) \\
&= \frac{1}{4T}\sum_{t=1}^{T}\tilde{\theta}_a[\boldsymbol{c}_3'\boldsymbol{\xi}_{t-1}\boldsymbol{\zeta}_t'\boldsymbol{c}_3 + \boldsymbol{c}_4'\boldsymbol{\xi}_{t-1}\boldsymbol{\zeta}_t'\boldsymbol{c}_4 - \sum_{s=-3}^{-2}\zeta_{4t+s}\zeta_{4t+s+2}] \\
&\quad -\frac{1}{4T}\sum_{t=1}^{T}\tilde{\theta}_b[\boldsymbol{c}_3'\boldsymbol{\xi}_{t-1}\boldsymbol{\zeta}_t'\boldsymbol{c}_4 - \boldsymbol{c}_4'\boldsymbol{\xi}_{t-1}\boldsymbol{\zeta}_t'\boldsymbol{c}_3 - \sum_{s=-3}^{-1}\zeta_{4t+s}\zeta_{4t+s+1} + \zeta_{4t-3}\zeta_{4t}] \\
&\Rightarrow \frac{1}{4}\tilde{\theta}_a[\boldsymbol{c}_3'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_3 \\
&\qquad\qquad +\boldsymbol{c}_4'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_4] \\
&\quad -\frac{1}{4}\tilde{\theta}_b[\boldsymbol{c}_3'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_4 \\
&\qquad\qquad -\boldsymbol{c}_4'\tilde{\boldsymbol{\Psi}}(1)\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\tilde{\boldsymbol{\Psi}}(1)'\boldsymbol{c}_3] \\
&\text{(by Lemma 3 and FCLT, the covariances of } \zeta_t \text{ cancel out since } \{\tilde{\zeta}_t\} \text{ is white noise)} \\
&= \frac{1}{4}\tilde{\theta}_a|\tilde{\psi}(i)|^2[\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4 + \boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3] \\
&\quad -\frac{1}{4}\tilde{\theta}_b|\tilde{\psi}(i)|^2[\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4 - \boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3] \\
&\text{(since } \boldsymbol{c}_3'\tilde{\boldsymbol{\Psi}}(1) = \tilde{\psi}_b\boldsymbol{c}_4' + \tilde{\psi}_a\boldsymbol{c}_3',\ \boldsymbol{c}_4'\tilde{\boldsymbol{\Psi}}(1) = \tilde{\psi}_a\boldsymbol{c}_4' - \tilde{\psi}_b\boldsymbol{c}_3',\ \text{and } \tilde{\psi}_a^2 + \tilde{\psi}_b^2 = |\tilde{\psi}(i)|^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{4T}\sum_{t=1}^{4T} Y_{4,t-1}\zeta_t \\
&= \frac{1}{4}\tilde{\theta}_b|\tilde{\psi}(i)|^2[\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4 + \boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3] \\
&\quad +\frac{1}{4}\tilde{\theta}_a|\tilde{\psi}(i)|^2[\boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4 - \boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}^{1/2}\int \boldsymbol{W}d\boldsymbol{W}'\boldsymbol{\Omega}^{1/2}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3]
\end{aligned}
$$

The lemma follows from $|\tilde{\psi}(i)|^2 = |\tilde{\theta}(i)|^{-2}$ and (Osborn, 1991)

$$
Var(\tilde{\zeta}_t)|\tilde{\theta}(i)|^2 = \frac{1}{4}(\boldsymbol{c}_4'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_4 + \boldsymbol{c}_3'\boldsymbol{\Theta}(1)\boldsymbol{\Omega}\boldsymbol{\Theta}(1)'\boldsymbol{c}_3). \qquad \square
$$

Now we come to the asymptotic distribution of the t-statistics and the F-statistics. Notice,

$$
t_i = \hat{\sigma}^{-1}[[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{ii}]^{-1/2}[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\zeta}^{(k)}]_i
$$

$$
\begin{aligned}
&= \hat{\sigma}^{-1}[[[(4T-k)^{-2}(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{ii}]^{-1/2} - [[\boldsymbol{R}^{-1}]_{ii}]^{-1/2}](4T-k)[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\zeta}^{(k)}]_i \\
&\quad + \hat{\sigma}^{-1}[[\boldsymbol{R}^{-1}]_{ii}]^{-1/2}((4T-k)(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\zeta}^{(k)} - \boldsymbol{R}^{-1}(4T-k)^{-1}\boldsymbol{X}'\boldsymbol{\zeta})_i \\
&\quad + \hat{\sigma}^{-1}[[\boldsymbol{R}^{-1}]_{ii}]^{-1/2}(\boldsymbol{R}^{-1}(4T-k)^{-1}\boldsymbol{X}'\boldsymbol{\zeta})_i \\
&= \hat{\sigma}^{-1}[[\boldsymbol{R}^{-1}]_{ii}]^{-1/2}(\boldsymbol{R}^{-1}(4T-k)^{-1}\boldsymbol{X}'\boldsymbol{\zeta})_i + o_p(1).
\end{aligned}
$$

By the consistency of $\hat{\boldsymbol{\beta}}$, we have $\hat{\sigma}^2 \xrightarrow{p} Var(\tilde{\zeta}_t)$. The asymptotic distributions of the t-statistics follows straightforwardly from Lemma 5. Further, the asymptotic distributions of the F-statistics are identical with the asymptotic distributions of the averages of the squares of the corresponding t-statistics because of the asymptotic orthogonality of the regression. Hence, the proof of Theorem 3.1 is complete.

7.3 Proof of Theorem 3.2.

Define $\{i_t\}$ and $\{I_t\}$ such that $\epsilon_t^\star = \check{\epsilon}_{i_t}$ and $\epsilon_{4t+s}^\star = \check{\epsilon}_{4I_t+s}$. By Algorithm 3.1, $\{i_t\}$ is a sequence of independent but not identical random variables, while $\{I_t\}$ is a sequence of iid random variables. Recall

$$
(1-L^4)Y_{4t+s} = \sum_{j=1}^{4} \pi_{j,s} Y_{j,4t+s-1} + \sum_{i=1}^{k} \phi_{i,s}(1-L^4)Y_{4t+s-i} + e_{4t+s},
$$

where $\{e_{4t+s}\}$ is the regression error. Let

$$
\begin{aligned}
\upsilon_{T,t}^{(1)} &= (e_{i_t} - E^\circ e_{i_t})/Std^\circ(e_{i_t}) \\
\upsilon_{T,t}^{(2)} &= (-1)^t(e_{i_t} - E^\circ e_{i_t})/Std^\circ((-1)^t e_{i_t}) \\
\upsilon_{T,t}^{(3)} &= \sqrt{2}\sin(\frac{\pi t}{2})(e_{i_t} - E^\circ e_{i_t})/Std^\circ(\sqrt{2}\sin(\frac{\pi t}{2})e_{i_t}) \\
\upsilon_{T,t}^{(4)} &= \sqrt{2}\cos(\frac{\pi t}{2})(e_{i_t} - E^\circ e_{i_t})/Std^\circ(\sqrt{2}\cos(\frac{\pi t}{2})e_{i_t})
\end{aligned}
$$

Let $R_T^\star$ be the partial sum of $\upsilon_{T,t}$ above. Formally,

$$
R_T^\star(u_1,u_2,u_3,u_4) = (\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1 \rfloor}\upsilon_{T,t}^{(1)}, \frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_2 \rfloor}\upsilon_{T,t}^{(2)}, \frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_3 \rfloor}\upsilon_{T,t}^{(3)}, \frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_4 \rfloor}\upsilon_{T,t}^{(4)})'.
$$

To justify theorem 3.2, it suffices to show

$$
\|S_T^\star - R_T^\star\| \xrightarrow{p} 0 \text{ uniformly in } u_1,\ u_2,\ u_3 \text{ and } u_4, \tag{7.8}
$$

$$
\text{and } R_T^\star \Rightarrow \boldsymbol{W} \text{ in probability,} \tag{7.9}
$$

because the unconditional convergence in (7.8) implies that in probability the conditional distribution of $\|S_T^\star - R_T^\star\|$ given $\{Y_{4t+s}\}$ converges to zero. To prove (7.8), we can without loss of generality focus on the uniform convergence of the first coordinate, that is, uniformly in u_1,

$$
|\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1 \rfloor}\epsilon_t^\star/\sigma_1^\star - \frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1 \rfloor}\upsilon_{T,t}^{(1)}| \xrightarrow{p} 0. \tag{7.10}
$$

Notice that uniformly in u_1,

$$\begin{aligned}
&\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1\rfloor}\epsilon_t^{\star}\\
=&\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}\epsilon_{4t+s}^{\star}+o_p(1)\\
=&\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}(\hat{\epsilon}_{4I_t+s}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}\hat{\epsilon}_{4t+s})+o_p(1)\\
=&\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}(e_{4I_t+s}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}e_{4t+s})\\
&-\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}\sum_{j=1}^{4}(\hat{\pi}_{j,s}-\pi_{j,s})(Y_{j,4I_t+s-1}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}Y_{j,4t+s-1})\\
&-\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}\sum_{i=1}^{k}(\hat{\phi}_{i,s}-\phi_{i,s})((1-L^4)Y_{4I_t+s-j}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}(1-L^4)Y_{4t+s-j})\\
&+o_p(1)\\
=&\frac{1}{\sqrt{4T}}\sum_{s=-3}^{0}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}(e_{4I_t+s}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}e_{4t+s})-B_T(u_1)-C_T(u_1)+o_p(1)\\
=&\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1\rfloor}(e_{i_t}-E^{\circ}e_{i_t})-B_T(u_1)-C_T(u_1)+o_p(1),
\end{aligned} \tag{7.11}$$

where $B_T(u_1)$ and $C_T(u_1)$ have obvious definitions. Now we show $B_T(u_1)\xrightarrow{p}0$, and $C_T(u_1)\xrightarrow{p}0$, uniformly in u_1.

For $B_T(u_1)$, notice if $\pi_{j,s}\neq 0$, then $\{Y_{j,4t+s}\}$ is weakly stationary, so $\hat{\pi}_{j,s}-\pi_{j,s}$ is $O_p(T^{-1/2})$ (see Berk, 1974), and it follows straightforwardly that $B_T(u_1)\xrightarrow{p}0$ uniformly in u_1. On the other hand, if $\pi_{j,s}=0$, then by Theorem 3.1, $\hat{\pi}_{j,s}-\pi_{j,s}=O_p(T^{-1})$. Let

$$Q_T(u_1)=\frac{1}{\sqrt{4T}}\sum_{t=\lfloor k/4\rfloor+1}^{\lfloor Tu_1\rfloor}(Y_{j,4I_t+s-1}-\frac{1}{T}\sum_{t=\lfloor k/4\rfloor+1}^{T}Y_{j,4t+s-1}).$$

It suffices to show that $\sup_{0\le u_1\le 1}Q_T(u_1)=o_p(T)$. By continuous mapping theorem, it suffices to prove $(4T)^{-1}Q_T(\cdot)\Rightarrow 0(\cdot)$, where $0(\cdot)\equiv 0$. It is straightforward to show the weak convergence of the finite dimensional distributions of $(4T)^{-1}Q_T(\cdot)$. Furthermore, $(4T)^{-1}Q_T(\cdot)$ is tight, since (see Billingsley, 1999, Theorem 14.1) $\forall r_1\le r\le r_2$,

$$\begin{aligned}
&E[(\frac{Q_T(r_2)}{T}-\frac{Q_T(r)}{T})^2(\frac{Q_T(r)}{T}-\frac{Q_T(r_1)}{T})^2]\\
=&E[Var^{\circ}[\frac{Q_T(r_2)}{T}-\frac{Q_T(r)}{T}]Var^{\circ}[\frac{Q_T(r)}{T}-\frac{Q_T(r_1)}{T}]]\to 0.
\end{aligned} \tag{7.12}$$

Hence $(4T)^{-1}Q_T(\cdot) \Rightarrow 0(\cdot)$, and $B_T(u_1) \xrightarrow{p} 0$ uniformly in u_1 follows. For $C_T(u_1)$, in light of the derivation of Theorem 3.1, it can be shown that $\hat{\phi}_{i,s} - \phi_{i,s} = O_p(T^{-1/2})$ holds not only under alternative hypotheses but also under the null. Hence, it follows that uniformly in u_1, $C_T(u_1) \xrightarrow{p} 0$. Therefore, recalling (7.11), we have

$$\frac{1}{\sqrt{4T}} \sum_{t=1}^{\lfloor 4Tu_1 \rfloor} \epsilon_t^\star - \frac{1}{\sqrt{4T}} \sum_{t=1}^{\lfloor 4Tu_1 \rfloor} (\epsilon_{i_t} - E^\circ \epsilon_{i_t}) \xrightarrow{p} 0.$$

Further, it is straightforward to show $E[B_T^2(1)] \xrightarrow{p} 0$, and $E[C_T^2(1)] \xrightarrow{p} 0$. Using the same decomposition as in (7.11), $\sigma_1^\star - Std^\circ(e_{i_t}) \xrightarrow{p} 0$. Hence we have proven (7.8).

Secondly we prove (7.9), Notice that the standard deviations in the definition of $\{\upsilon_{T,t}^{(j)}\}$ are bounded in probability. For example,

$$Std^\circ(e_{i_t}) = Std^\circ(e_{4I_t+s}) = Std(e_{4t+s}) + o_p(1) = Std(\epsilon_{4t+s}) + o_p(1),$$

Further, given $\{Y_{4t+s}\}$, for fixed $j = 1, ..., 4$, $\upsilon_{T,1}^{(j)}, \upsilon_{T,2}^{(j)}, ..., \upsilon_{T,T}^{(j)}$ are conditionally iid random variables. Finally, for all $u \geq 0$,

$$Var^\circ[\frac{1}{\sqrt{4T}} \sum_{m=1}^{\lfloor 4Tu \rfloor} \upsilon_{T,m}^{(j)}] \xrightarrow{p} u,$$

$$Cov^\circ(\frac{1}{\sqrt{4T}} \sum_{m=1}^{\lfloor 4Tu \rfloor} \upsilon_{T,m}^{(j)}, \frac{1}{\sqrt{4T}} \sum_{m=1}^{\lfloor 4Tu \rfloor} \upsilon_{T,m}^{(i)}) \xrightarrow{p} 0 \quad \text{for } i \neq j.$$

The convergence $R_T^\star$ of to $\boldsymbol{W}$ follows by generalizing (see Kreiss and Paparoditis, 2015) the real world result of Helland (1982, Theorem 3.3) to the bootstrap world.

7.4 Proof of Theorem 4.2

Proof. Without loss of generality, assume block size b is a multiple of four. Let $i_m = I_{(m-1)b+1}$. Then the mth block of $\{V_t^*\}$ starts from $\check{V}_{i_m}$. Let $\upsilon_{l,m}^{(j)}$ be the rescaled aggregation of the mth block, defined by

$$\upsilon_{l,m}^{(1)} = \frac{1}{\sqrt{b}} \sum_{h=1}^{b} (V_{i_m+h-1} - E^\circ V_{i_m+h-1}) / Std^\circ(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_m+h-1})$$

$$\upsilon_{l,m}^{(2)} = \frac{1}{\sqrt{b}} \sum_{h=1}^{b} (-1)^h (V_{i_m+h-1} - E^\circ V_{i_m+h-1}) / Std^\circ(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} (-1)^h V_{i_m+h-1})$$

$$\upsilon_{l,m}^{(3)} = \frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \sin(\frac{\pi h}{2})(V_{i_m+h-1} - E^\circ V_{i_m+h-1}) / Std^\circ(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \sin(\frac{\pi h}{2}) V_{i_m+h-1})$$

$$\upsilon_{l,m}^{(4)} = \frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \cos(\frac{\pi h}{2})(V_{i_m+h-1} - E^\circ V_{i_m+h-1}) / Std^\circ(\frac{1}{\sqrt{b}} \sum_{h=1}^{b} \sqrt{2} \cos(\frac{\pi h}{2}) V_{i_m+h-1})$$

Let R_T^* be the partial sum of the block aggregations above. Formally,

$$R_T^*(u_1, u_2, u_3, u_4) = (\frac{1}{\sqrt{l}} \sum_{m=1}^{\lfloor lu_1 \rfloor} \upsilon_{l,m}^{(1)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{\lfloor lu_2 \rfloor} \upsilon_{l,m}^{(2)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{\lfloor lu_3 \rfloor} \upsilon_{l,m}^{(3)}, \frac{1}{\sqrt{l}} \sum_{m=1}^{\lfloor lu_4 \rfloor} \upsilon_{l,m}^{(4)})'$$

To prove theorem 4.2, it suffices to show

$$\|S_T^* - R_T^*\| \xrightarrow{p} 0 \text{ uniformly in } u_1,\, u_2,\, u_3 \text{ and } u_4, \tag{7.13}$$

$$\text{and } R_T^\star \Rightarrow \boldsymbol{W} \text{ in probability,} \tag{7.14}$$

where $||\cdot||$ denotes the L_2 norm. To show (7.13), without loss of generality we focus on the uniform convergence of the first coordinate, that is, uniformly in u_1, $|\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1\rfloor} V_t^*/\sigma_1^* - \frac{1}{\sqrt{l}}\sum_{m=1}^{\lfloor lu_1\rfloor} \upsilon_{l,m}^{(1)}| \xrightarrow{p} 0$. Notice that,

$$\begin{aligned}\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1\rfloor} V_t^* &= \frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{B_m}\check{V}_{i_m+h-1} \\ &= \frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{b}\check{V}_{i_m+h-1} - \frac{1}{\sqrt{4T}}\sum_{h=B_{M(u_1)}+1}^{b}\check{V}_{i_{M(u_1)}+h-1}\end{aligned} \tag{7.15}$$

where $M(u_1) = \lceil \lfloor 4Tu_1\rfloor/b\rceil$ denotes the total number of the blocks, and $B_m = min(b, \lfloor 4Tu_1\rfloor - (m-1)b)$ is the length of the mth block. It suffices to only consider the first term in (7.15), since

$$\sup_{0\le u_1\le 1}|\frac{1}{\sqrt{4T}}\sum_{h=B_{M(u_1)}+1}^{b}\check{V}_{i_{M(u_1)}+h}| = O_p(\frac{1}{\sqrt{l}}\ln l).$$

By the definition of $\check{V}_t$,

$$\begin{aligned}&\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{b}\check{V}_{i_m+h-1} \\ =&\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{s=-3}^{0}\sum_{t=1}^{b/4}(V_{i_m+4t+s-1} - \frac{1}{T}\sum_{t=1}^{T}V_{4t+s}) \\ &-\sum_{j=1}^{4}\sum_{s=-3}^{0}(\hat{\pi}_{j,s}-\pi_{j,s})\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{t=1}^{b/4}(Y_{j,i_m+4t+s-1} - \frac{1}{T}\sum_{t=1}^{T}Y_{4t+s}).\end{aligned}$$

Now we show the second term on the right hand side of the equation above converges uniformly in u_1 to 0 in probability. Here we only present the result for j=1, s=0. Notice if $\pi_{1,s} \neq 0$ for some s, then $(\hat{\pi}_{1,0} - \pi_{1,0}) = o_p(1)$. Hence, the result follows the weakly stationarity of the vector sequence $\{\boldsymbol{Y}_t\}$. On the other hand, if $\pi_{1,s} = 0$ for all s, then $(\hat{\pi}_{1,0} - \pi_{1,0}) = O_p(T^{-1})$. Hence, we only need to show that

$$Q_T(u) \overset{def}{=} \frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{t=1}^{b/4}(Y_{1,i_m+4t-1} - \frac{1}{T}\sum_{t=1}^{T}Y_{4t})$$

has $\frac{Q_T(\cdot)}{T} \Rightarrow 0(\cdot)$, where $0(u_1) \equiv 0$. The convergence of finite dimensional distribution of $\frac{Q_T(\cdot)}{T}$ can be proven by the line of Politis and Paparaditis (2003, p. 841). Furthermore, it can be shown that $\frac{Q_T(\cdot)}{T}$ is tight using (7.12). Hence $\frac{Q_T(\cdot)}{T} \Rightarrow 0(\cdot)$. Therefore,

$$|\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{b}\check{V}_{i_m+h-1} - \frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{s=-3}^{0}\sum_{t=1}^{b/4}(V_{i_m+4t+s-1} - \frac{1}{T}\sum_{t=1}^{T}V_{4t+s})| \xrightarrow{p} 0$$

uniformly in u_1. Since it is straightforward to show

$$|\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{s=-3}^{0}\sum_{t=1}^{b/4}(V_{i_m+4t+s-1}-\frac{1}{T}\sum_{t=1}^{T}V_{4t+s})-\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{b}(V_{i_m+h-1}-E^{\circ}[V_{i_m+h-1}])|\xrightarrow{p}0$$

uniformly in u_1, and

$$|\frac{1}{\sqrt{4T}}\sum_{m=1}^{M(u_1)}\sum_{h=1}^{b}(V_{i_m+h-1}-E^{\circ}[V_{i_m+h-1}])-\frac{1}{\sqrt{4T}}\sum_{m=1}^{\lfloor lu_1\rfloor}\sum_{h=1}^{b}(V_{i_m+h-1}-E^{\circ}[V_{i_m+h-1}])|\xrightarrow{p}0$$

uniformly in u_1, we have obtained that

$$|\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor 4Tu_1\rfloor}V_t^*-\frac{1}{\sqrt{4T}}\sum_{m=1}^{\lfloor lu_1\rfloor}\sum_{h=1}^{b}(V_{i_m+h-1}-E^{\circ}[V_{i_m+h-1}])|\xrightarrow{p}0 \tag{7.16}$$

uniformly in u_1. Now we show that $Var^{\circ}[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}V_t^*]-Var^{\circ}[\frac{1}{\sqrt{b}}\sum_{h=1}^{b}V_{i_m+h-1}]\xrightarrow{p}0$. Notice,

$$\begin{aligned}
&\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}V_t^*=\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{h=1}^{b}\hat{V}_{i_m+h-1}\\
&=\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{s=-3}^{0}\sum_{t=1}^{b/4}(V_{i_m+4t+s-1}-\frac{1}{T}\sum_{t=1}^{T}V_{4t+s})\\
&-\sum_{j=1}^{4}\sum_{s=-3}^{0}(\hat{\pi}_{j,s}-\pi_{j,s})\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{t=1}^{b/4}(Y_{j,i_m+4t+s-1}-\frac{1}{T}\sum_{t=1}^{T}Y_{4t+s})\\
&=\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{h=1}^{b}(V_{i_m+h-1}-E^{\circ}[V_{i_m+h-1}])+\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{s=-3}^{0}\sum_{t=1}^{b/4}(E^{\circ}V_{i_m+4t+s-1}-\frac{1}{T}\sum_{t=1}^{T}V_{4t+s})\\
&-\sum_{j=1}^{4}\sum_{s=-3}^{0}(\hat{\pi}_{j,s}-\pi_{j,s})\frac{1}{\sqrt{4T}}\sum_{m=1}^{l}\sum_{t=1}^{b/4}(j,Y_{i_m+4t+s-1}-\frac{1}{T}\sum_{t=1}^{T}Y_{4t+s})\\
&=A_T+B_T-\sum_{j=1}^{4}C_{T,j}
\end{aligned}$$

where A_T, B_T and $C_{T,j}$, $j=1,...,4$ have obvious definitions. It is straightforward to show $E^{\circ}[B_T^2]\xrightarrow{p}0$, $E^{\circ}[C_{T,j}^2]\xrightarrow{p}0$ for $j=1,...,4$, and $Var^{\circ}[A_T]=Var^{\circ}[\frac{1}{\sqrt{b}}\sum_{h=1}^{b}V_{i_m+h-1}]$. Hence, we have

$$Var^{\circ}[\frac{1}{\sqrt{4T}}\sum_{t=1}^{4T}V_t^*]-Var^{\circ}[\frac{1}{\sqrt{b}}\sum_{h=1}^{b}V_{i_m+h-1}]\xrightarrow{p}0. \tag{7.17}$$

By (7.16) and (7.17), we have shown

$$|\frac{1}{\sqrt{4T}}\sum_{t=1}^{\lfloor lu_1\rfloor}V_t^*/\sigma_1^*-\sum_{m=1}^{\lfloor lu_1\rfloor}\upsilon_{l,m}^{(1)}|\xrightarrow{p}0$$

uniformly in u_1, and thus $\|S_T^* - R_T^*\| \xrightarrow{p} 0$ uniformly in u_1, u_2, u_3 and u_4.

Secondly we prove (7.14). Given assumption B.1, it is sufficient to show that the following three properties hold:

$$\sum_{m=1}^{\lfloor lu \rfloor} E^\circ[\upsilon_{l,m}^{(i)^2}] \xrightarrow{p} u, \ \forall u \geq 0, \ and \, \forall \, i = 1, ..., 4, \tag{7.18}$$

$$\sum_{m=1}^{\lfloor lu \rfloor} E^\circ[\upsilon_{l,m}^{(i)^2} 1(|\upsilon_{l,m}| > \epsilon)] \xrightarrow{p} 0, \ \forall u \geq 0, \ \ \forall \, i = 1, ..., 4, \tag{7.19}$$

$$\sum_{m=1}^{\lfloor lu \rfloor} E^\circ[\upsilon_{l,m}^{(i)} \upsilon_{l,m}^{(j)}] \xrightarrow{p} 0, \ \forall u \geq 0, \ \ \forall \, i, j \in \{1, 2, 3, 4\}, \ i \neq j. \tag{7.20}$$

Helland (1982) shows that if $\{\upsilon_{l,m}\}$ is a martingale difference array and the above three properties hold in real world, then $\sum_{m=1}^{\lfloor lu \rfloor} \upsilon_{l,m} \Rightarrow \boldsymbol{W}(u)$. By Beveridge-Neilson Decomposition (Hamilton, 1994, Proposition 17.2), Helland's result can be generalized to the case when $\{\upsilon_{l,m}\}$ is a convolution of a constant array and a martingale difference array. Further, Helland's result can be generalized to the bootstrap world (see Kreiss and Paparoditis, 2015). Hence the sufficiency of the three properties above.

To verify (7.18) and (7.19), notice that $\forall t \geq 0$, $i = 1, ..., 4$,

$$\sum_{m=1}^{\lfloor lt \rfloor} E^\circ[\upsilon_{l,m}^{(i)^2}] = \lfloor lt \rfloor / l \to t,$$

and, by the dominated convergence theorem,

$$\sum_{m=1}^{\lfloor lt \rfloor} E^\circ[\upsilon_{l,m}^{(i)^2} 1(|\upsilon_{l,m}| > \epsilon)] \xrightarrow{p} 0.$$

Hence, it remains to verify the (7.20), which indicates asymptotic independence between coordinates of R_T^*. Note that the third property need to be proved for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Here we cite as an example the case $i = 1$ and $j = 3$. The rest of cases can be shown by similar calculations. Notice,

$$\begin{aligned}\sum_{m=1}^{\lfloor lt \rfloor} E^\circ[\upsilon_{l,m}^{(1)} \upsilon_{l,m}^{(3)}] =& \frac{E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1} \frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}]}{Std^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1}] Std^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}]} \\ &- \frac{E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1}] E^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}]}{Std^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1}] Std^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}]}.\end{aligned}$$

Since

$$E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1}] \xrightarrow{p} 0, \ E^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}] \xrightarrow{p} 0,$$

and both $Std^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1}]$ and $Std^\circ[\frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}]$ converge in probability to constants (Dudek et al., 2014), we only need to show that

$$E^\circ[\frac{1}{\sqrt{b}} \sum_{h=1}^{b} V_{i_1+h-1} \frac{1}{\sqrt{b}} \sum_{r=1}^{b} \sqrt{2} \sin(\pi r/2) V_{i_1+r-1}] \xrightarrow{p} 0.$$

Notice,

$$
\begin{aligned}
&E^\circ[\frac{1}{\sqrt{b}}\sum_{h=1}^{b}V_{i_1+h-1}\frac{1}{\sqrt{b}}\sum_{r=1}^{b}\sqrt{2}\sin(\pi r/2)V_{i_1+r-1}] \\
&=\frac{\sqrt{2}}{b(T-b/4)}\sum_{i=1}^{T-b/4}\sum_{h=1}^{b}\sum_{r=1}^{b}\sin(\pi r/2)V_{4i+h-4}V_{4i+r-4} \\
&=-A+B+o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
A&=\frac{\sqrt{2}}{b(T-b/4)}\sum_{h=1}^{b/4}\sum_{j=1}^{T-b/4}V_{4j-3}V_{4j+4h-6}, \\
B&=\frac{\sqrt{2}}{b(T-b/4)}\sum_{h=1}^{b/4}\sum_{j=1}^{T-b/4}V_{4j-3}V_{4j+4h-4}.
\end{aligned}
$$

The proof under Assumption 1.B is complete after showing

$$
A\xrightarrow{p}0, B\xrightarrow{p}0. \tag{7.21}
$$

by the lemma 6 below. Now consider Assumption 2.B. Let $\boldsymbol{v}_{l,m}=(v_{l,m}^{(1)},v_{l,m}^{(2)},v_{l,m}^{(3)},v_{l,m}^{(4)})'$. Let $(\lambda_{l,1},\lambda_{l,2},\lambda_{l,3},\lambda_{l,4})'$ be the eigenvalues of $Var\sum_{m=1}^{l}\boldsymbol{v}_{l,m}$. It is sufficient (Wooldridge and White, 1988, Corollary 4.2) to show that the following two properties hold:

$$
E^\circ(\frac{1}{\sqrt{l}}\sum_{m=1}^{\lfloor lt\rfloor}v_{l,m}^{(i)})(\frac{1}{\sqrt{l}}\sum_{m=1}^{\lfloor lt\rfloor}v_{l,m}^{(j)})\xrightarrow{p}t\mathbb{1}\{i=j\},\ \forall t\geq 0,\forall\, i,j, \tag{7.22}
$$

$$
(\lambda_{l,1}^{-1},\lambda_{l,2}^{-1},\lambda_{l,3}^{-1},\lambda_{l,4}^{-1})=O(l^{-1}). \tag{7.23}
$$

Notice, to show (7.22), it suffices to show (7.21), which is already ensured by Lemma 4 and Lemma 6. Equation (7.23) follows from the continuity of the eigenvalue function. Hence we have completed the proof when block size b is a multiple of four.

When b is not a multiple of four, it is straightforward to show (7.13). For (7.14), let

$$
R_{T,s}^{*}=(\frac{1}{\sqrt{l/4}}\sum_{k=1}^{\lfloor\lfloor lu_1\rfloor/4\rfloor}v_{l,4k+s}^{(1)},\frac{1}{\sqrt{l/4}}\sum_{k=1}^{\lfloor\lfloor lu_2\rfloor/4\rfloor}v_{l,4k+s}^{(2)},\frac{1}{\sqrt{l/4}}\sum_{k=1}^{\lfloor\lfloor lu_3\rfloor/4\rfloor}v_{l,4k+s}^{(3)},\frac{1}{\sqrt{l/4}}\sum_{k=1}^{\lfloor\lfloor lu_4\rfloor/4\rfloor}v_{l,4k+s}^{(4)})'.
$$

Since $\{R_{T,s}^{*},\ s=-3,...,0\}$ are mutually independent with respect to P°, and $R_{T,s}^{*}\Rightarrow\boldsymbol{W}$ in probability for all $s=-3,...,0$, we have $R_T^{*}=\frac{1}{2}\sum_{s=-3}^{0}R_{T,s}^{*}+o_p(1)\Rightarrow\boldsymbol{W}$ in probability. □

Lemma 6. *Suppose (i) $\{z_t\}_{t=1}^{n}$ is a fourth-order stationary time series with finite $4+\delta$ moment for some $\delta>0$. (ii) $\exists K>0$, $\forall$ i, j, k, and l, $\sum_{h=-\infty}^{\infty}|Cov(z_iz_j,z_{k-h}z_{l-h})|<K$. Suppose $b\to\infty$ and $n\to\infty$. Then,*

$$
Var[\frac{1}{bn}\sum_{t=1}^{n}\sum_{j=1}^{b}z_tz_{t-j}]\to 0.
$$

Proof.

$$
\begin{aligned}
&Var[\frac{1}{bn}\sum_{t=1}^{n}\sum_{j=1}^{b} z_t z_{t-j}] \\
&= \frac{1}{b^2n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\sum_{j_1=1}^{b}\sum_{j_2=1}^{b} cov[z_0 z_{-j_1}, z_{t_2-t_1} z_{t_2-t_1-j_1}] \\
&= \frac{1}{b^2n^2}\sum_{h=1-n}^{n-1}(n-|h|)\sum_{j_1=1}^{b}\sum_{j_2=1}^{b} cov[z_0 z_{-j_1}, z_h z_{h-j_1}] \\
&< \frac{K}{n} \to 0.
\end{aligned}
$$

□

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