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Hypersurfaces with nonegative Ricci curvature in $\mathbb{H}^{n+1}$

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Abstract
Based on properties of n-subharmonic functions we show that a complete, noncompact, properly embedded hypersurface with nonnegative Ricci curvature in hyperbolic space has an asymptotic boundary at infinity of at most two points. Moreover, the presence of two points in the asymptotic boundary is a rigidity condition that forces the hypersurface to be an equidistant hypersurface about a geodesic line in hyperbolic space. This gives an affirmative answer to the question raised by Alexander and Currier (Proc Symp Pure Math 54(3):37–44, 1993).

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1 Introduction

For immersed hypersurfaces $\phi : M^n \to \mathbb{H}^{n+1}$ with appropriate orientation we recall the following successively stronger pointwise convexity conditions determined by the principal curvatures $\kappa_1, \ldots, \kappa_n$: For all $i \neq j \in \{1, \ldots, n\}$

$$\begin{array}{ll} \kappa_i > 0 & \textit{(strict) convexity} \\ \kappa_i \left(\sum_{l=1}^{n} \kappa_l \right) - \kappa_i^2 \geq n-1 & \textit{nonnegative Ricci curvature} \\ \kappa_i \kappa_j \geq 1 & \textit{nonnegative sectional curvature} \\ \kappa_i \geq 1 & \textit{horospherical convexity} \end{array}$$

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The influence of curvature conditions on the geometry and the asymptotic boundary of complete noncompact hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ has been studied by Epstein, Alexander and Currier, and the authors in [1,2,4,8–10]. In Epstein [10] it is shown that the asymptotic boundary of a complete proper embedding of $\mathbb{R}^2$ into $\mathbb{H}^3$ with nonnegative Gaussian curvature has a single point asymptotic boundary. The same is true for any complete noncompact horospherically convex hypersurface immersed in $\mathbb{H}^{n+1}$. In fact, in [8] it is shown by Currier that the only complete noncompact horospherically convex hypersurface immersed in $\mathbb{H}^{n+1}$ is a horosphere. While in Alexander and Currier [1,2] it is shown that a complete, noncompact, embedded hypersurface $\phi : M^n \to \mathbb{H}^{n+1}$ with nonnegative sectional curvature has at most two points in its asymptotic boundary. Moreover the presence of two points in the boundary at infinity is a rigidity condition that forces $\phi(M)$ to be an equidistant hypersurface. Recently, in [4] it is shown by the authors that the same conclusion as in [1,2] holds for immersed hypersurfaces.

In [1,2] it is observed that a properly embedded strictly convex hypersurface in hyperbolic space can be realized as a global vertical graph of a height function over a domain in a horosphere and that the height function is subharmonic when restricted to any 2-plane when the hypersurface has nonnegative sectional curvature. Based on the theory of subharmonic functions Alexander and Currier then managed to show that the asymptotic boundary is totally disconnected. In [2] the question was raised as to whether or not nonnegative Ricci curvature suffices for their asymptotic boundary theorem. In this note we affirmatively answer their question.

Main Theorem *For $n \geq 3$, suppose that Σ is an n-dimensional complete, noncompact hypersurface properly embedded in hyperbolic space $\mathbb{H}^{n+1}$ with nonnegative Ricci curvature. Then $\partial_\infty \Sigma$ consists of at most two points. The case that $\partial_\infty \Sigma$ consists of two points is a rigidity condition that forces Σ to be an equidistant hypersurface about a geodesic line.*

The classification of complete noncompact Riemannian manifolds with nonnegative Ricci curvature is very interesting and complicated subject (cf. Shen and Sormani [14], for example). On the other hand, from our main theorem and its proof we are able to easily classify those hypersurfaces that are properly embedded in hyperbolic space. In fact, there are only two classes: one is the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ and the other consists of nonnegative Ricci curvature metrics on $\mathbb{R}^n$.

Corollary *Suppose that (M^n, g) is a complete and noncompact Riemannian manifold with nonnegative Ricci curvature that can be properly isometrically embedded in hyperbolic space $\mathbb{H}^{n+1}$. Then (M^n, g) is either the standard cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ or it is a complete nonnegative Ricci curvature metric on $\mathbb{R}^n$.*

As suggested for embedded hypersurfaces in [2], in Sect. 2 we realize the rigidity result of our main theorem ultimately as consequence of the Cheeger–Gromoll splitting theorem [7], and the Gauss and Codazzi equations. Our proof of the rigidity part is local in nature and therefore does not need the embeddedness assumption. In fact, we can further strengthen the result by only assuming that the boundary at infinity has more than one connected component (cf. Theorem 2.1) as the Cheeger–Gromoll splitting theorem [7] will guarantee that such a hypersurface has exactly two ends. The key to our proof is to show that Ricci flat directions are in fact principal directions of the hypersurface for $n \geq 3$ (cf. Lemma 2.1). To resolve this issue we appeal to the fact that the Ricci operator and the shape operator of a hypersurface in any space form are pointwise simultaneously diagonalizable thanks to Bourguignon [5]. Then from the Gauss and Codazzi equations we are able to show that the principal curvatures are constant reciprocals with multiplicities 1 and $n - 1$. The fact that such a hypersurface must be an equidistant hypersurface then follows from the following classification theorem of so-called isoparametric hypersurfaces in hyperbolic space due to Cartan [6].

Cartan Theorem ([6]) *An isoparametric hypersurface in $\mathbb{H}^{n+1}$ must be either a sphere $\mathbb{S}^n$, a hyperbolic space $\mathbb{H}^n$, a Euclidean space $\mathbb{R}^n$ (called a horosphere), all three are totally umbilic, or a cylinder $\mathbb{S}^k \times \mathbb{H}^{n-k}$, where each factor is a space form.*

The most difficult part in proving the main theorem is to show that a connected asymptotic boundary can only be a single point. To do this, we pursue an avenue closely related to [1,2]. However, one nontrivial observation of ours is that problems related to Ricci curvature are sometimes reduced to the n-Laplacian equation. This was discovered in the context of conformal geometry, which can be applied to hypersurfaces in hyperbolic space as well. In Sect. 3 we prove that hypersurfaces embedded in hyperbolic space with nonnegative Ricci curvature give rise to height functions that are Euclidean n-subharmonic. Then in Sect. 4 we apply the theory of n-subharmonic functions to show that hypersurfaces embedded in hyperbolic space with nonnegative Ricci curvature must have asymptotic boundaries of Hausdorff dimension zero and are therefore a single point when connected. It is rather surprising that our calculation for Ricci curvature in dimensions larger than 2 (cf. Theorem 3.1) goes perfectly in line with what was observed in [1, Theorem 2.1] for Gaussian curvature in dimension 2.

For convenience of the reader, we conclude this section with a brief explanation of the curvature conditions under consideration. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of principal directions of an immersed hypersurface $\phi : M^n \to \mathbb{H}^{n+1}$. Due to the Gauss equations, the sectional curvatures of $\Sigma = \phi(M)$ are given by $K(e_i, e_j) = \kappa_i\kappa_j - 1$ for $i \neq j$ and therefore nonnegative sectional curvature is equivalent to the principal curvature condition $\kappa_i\kappa_j \geq 1$ for $i \neq j$ for hypersurfaces in $\mathbb{H}^{n+1}$. Clearly then all principal curvatures of a hypersurface $\phi : M^n \to \mathbb{H}^{n+1}$ with nonnegative sectional curvature are nonzero and of the same sign. The same is true for hypersurfaces with nonnegative Ricci curvature. Indeed, writing the Ricci curvature as the sum of sectional curvatures we see

$$Ric(e_i) = \sum_{j\neq i} K(e_i, e_j) = \sum_{j\neq i}(\kappa_i\kappa_j - 1) = \kappa_i \left(\sum_{j=1}^{n} \kappa_j\right) - \kappa_i^2 - (n-1).$$

Therefore, for hypersurfaces in $\mathbb{H}^{n+1}$, nonnegative Ricci curvature is equivalent to the principal curvature condition

$$\kappa_i \left(\sum_{j=1}^{n} \kappa_j\right) - \kappa_i^2 \geq n - 1 > 0$$

for all $i = 1, \ldots, n$, which clearly implies that all principal curvatures of ϕ are nonzero. Moreover, since

$$\kappa_i \left(\sum_{l=1}^{n} \kappa_l\right) - \kappa_i^2 \geq n - 1 > 0 \quad \text{and} \quad \kappa_j \left(\sum_{l=1}^{n} \kappa_l\right) - \kappa_j^2 \geq n - 1 > 0,$$

if $\kappa_i < 0$ and $\kappa_j > 0$ for some $i \neq j$, we arrive at the contradiction

$$\sum_{l=1}^{n} \kappa_l < \kappa_i < 0 \quad \text{and} \quad \sum_{l=1}^{n} \kappa_l > \kappa_j > 0.$$

Hence, the principal curvatures of a hypersurface in hyperbolic space with nonnegative Ricci curvature are all nonzero and of the same sign as claimed.

The orientation we take on hypersurfaces with nonnegative Ricci curvature is the orientation for which the second fundamental form of the hypersurface is positive definite. That is, we take the orientation so that all principal curvatures of the hypersurface are positive. With this orientation we may view nonnegative Ricci curvature as intermediate curvature condition between strict convexity and nonnegative sectional curvature. In particular, with our choice of orientation we have

$$Ric \geq 0 \Rightarrow k_i > 0, \quad \forall i. \tag{1.1}$$

Finally, we echo the question raised by Alexander and Currier [2] as to whether or not the Main Theorem in this paper still holds for immersed hypersurfaces.

2 Asymptotic boundary of multiple components

In this section we show that complete noncompact hypersurfaces immersed in hyperbolic space with nonnegative Ricci curvature and multiple component asymptotic boundaries are in fact equidistant hypersurfaces. Our approach is very much local in nature, hence we do not need to assume the hypersurfaces are embedded.

Let (M^n, g) be a complete Riemannian manifold with nonnegative Ricci curvature that can be isometrically immersed into $\mathbb{H}^{n+1}$. If $\partial_\infty M$ has more than one connected component, then (M^n, g) has a line. Then by the Cheeger–Gromoll splitting theorem [7] (see also Toponogov [13] for dimension 2), M splits isometrically as the product $M \cong \mathbb{R} \times N^{n-1}$ where (N^{n-1}, g_N) is a complete $(n-1)$-manifold with nonnegative Ricci curvature. Naturally, the product structure carries to the level of the tangent bundle and the Levi-Civita connection ∇ on M, which forces the Riemannian curvature tensor of (M^n, g) to split accordingly. Hence, the factor $\mathbb{R}$ of the product $M \cong \mathbb{R} \times N$ represents a flat direction in M.

To be more precise, let $(x_1, x_2, \ldots, x_n)$ denote local coordinates on a neighborhood of M adapted to the product structure $M \cong \mathbb{R} \times N$ where $x_1 = t$ is the coordinate corresponding to distance in the factor $\mathbb{R}$ and $(x_2, \ldots, x_n)$ are local coordinates on N. Then locally the metric

$$g = dt^2 + g_N$$

where g_N is independent of t and $\frac{\partial}{\partial t}$ is a flat direction. The Riemannian curvature tensor

$$R_{ijkt} = 0 \tag{2.1}$$

for all $i, j, k \in \{1, \ldots, n\}$. Therefore,

$$R_{it} = 0 \tag{2.2}$$

for all $i \in \{1, \ldots, n\}$. In other words, the flat direction $\frac{\partial}{\partial t}$ is pointwise an eigendirection for the Ricci curvature operator corresponding to the eigenvalue 0.

It turns out that the key to establish rigidity is to know that the flat direction is a principal direction of the hypersurface. A pleasantly surprising fact due to Bourguignon [5] (see also [3] Corollary 16.17) is that the Ricci curvature form and the second fundamental form commute since the second fundamental form of a hypersurface in a space form is always a Codazzi tensor.

Lemma 2.1 *Suppose that $\phi : M^n \to \mathbb{H}^{n+1}$ is an isometric immersion where M^n has nonnegative Ricci curvature and splits as $\mathbb{R} \times N$. Then, for $n \geq 3$, the flat direction is a principal direction for ϕ.*

Proof It is well-known that the second fundamental form of a hypersurface in a space form is a Codazzi tensor. Due to Bourguignon [5] (see also Besse [3] Corollary 16.17), it follows that the Ricci operator and the shape operator then commute. Hence, the Ricci operator and the shape operator preserve each other's invariant subspaces and are therefore pointwise simultaneously diagonalizable. Let V_0 denote the eigenspace of the Ricci operator that corresponds to the eigenvalue 0 at a point on the hypersurface. Clearly, $\dim(V_0) \geq 1$ since it contains at least the flat direction.

Let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis of principal directions with the principal curvatures κ_i at the point. Up to linear combinations of the principal directions in their respective eigenspaces, we may assume that $\{e_1, \ldots, e_n\}$ simultaneously diagonalizes the Ricci curvature operator. Moreover, since $\dim(V_0) \geq 1$, up to reordering we may assume $V_0 = \text{span}\{e_1, \ldots, e_k\}$ for some $1 \leq k \leq n$. Clearly, if $k = 1$, then the flat direction is a principal direction. Otherwise, let us assume $k \geq 2$. Then, for each $i = 1, \ldots, k$,

$$0 = Ric(e_i) = \kappa_i \left(\sum_{j=1}^{n} \kappa_j \right) - \kappa_i^2 - (n-1) = \kappa_i H - \kappa_i^2 - (n-1), \tag{2.3}$$

where $H = \sum_{j=1}^{n} \kappa_j$ is the mean curvature. From (2.3) we see

$$\kappa_i = \frac{H \pm \sqrt{H^2 - 4(n-1)}}{2} \quad \text{for } i = 1, \ldots, k.$$

But then, since $n \geq 3$, $\kappa_i > 0$ for all $i = 1, 2, \ldots, n$, and $k \geq 2$, we must have

$$\kappa_i = \kappa_0 = \frac{H - \sqrt{H^2 - 4(n-1)}}{2} \quad \text{for } i = 1, \ldots, k.$$

Therefore, every vector in V_0 is a principal direction associated with the principal curvature κ_0. Thus, the flat direction is a principal direction at any point on the hypersurface. □

It is interesting to notice that Lemma 2.1 works only for dimensions larger than 2. For flat cases in dimension 2 one needs Volkov and Vladimirova [15] instead (please see [4] for an alternative proof in dimension 2). We are now in a position to apply the Codazzi equations to establish the rigidity result.

Theorem 2.1 *For $n \geq 3$, let $\phi : M^n \to \mathbb{H}^{n+1}$ be an isometric immersion of a complete noncompact manifold (M^n, g) with nonnegative Ricci curvature. If the asymptotic boundary at infinity $\partial_\infty \phi(M)$ has more than one connected component, then $\phi(M)$ is an equidistant hypersurface about a geodesic line.*

Proof Let $X_i = \phi_*(\frac{\partial}{\partial x_i})$ denote the local frame on the hypersurface adapted to the product structure. From the discussion above we may assume that $X_t = \phi_*(\frac{\partial}{\partial x_1})$ is a unit length flat direction that is orthogonal to $X_2, \ldots, X_n$. In addition, due to Lemma 2.1, we may also assume that X_t is a principal direction with principal curvature κ_0. Then, from (2.1) and Gauss equations we have

$$\begin{aligned} 0 = R_{itjt} &= R^{\mathbb{H}}_{itjt} + II_{ij} II_{tt} - II_{it} II_{tj} \\ &= -g_{ij} + \kappa_0 II_{ij} \quad \text{for } i, j = 2, \ldots, n \end{aligned} \tag{2.4}$$

and therefore

$$II = \frac{1}{\kappa_0} g_N \tag{2.5}$$

when restricted to directions tangential to N. That is, $\kappa_i = \frac{1}{\kappa_o}$ for all $i = 2, \dots, n$. Now since $g = dt^2 + g_N$ with g_N independent of t, it follows that the Christoffel symbols for g satisfy

$$\Gamma^j_{it} = \Gamma^j_{ti} = \Gamma^t_{ij} = 0 \quad \text{for any } i, j \in \{1, \dots, n\}. \tag{2.6}$$

Furthermore, from (2.5), we see

$$\nabla_{X_t} II_{ii} = X_t \left(\frac{1}{\kappa_0} (g_N)_{ii} \right) = ||X_i||^2_{g_N} X_t(\kappa_i) \tag{2.7}$$

for any $i \in \{2, \dots, n\}$. Moreover, from the Codazzi equations, we find

$$\nabla_{X_t} II_{ii} = \nabla_{X_i} II_{ti} = -\Gamma^l_{ii} II_{lt} + \Gamma^l_{ti} II_{li} = 0. \tag{2.8}$$

Meanwhile,

$$X_i(\kappa_0) = \nabla_{X_i} II_{tt} = \nabla_{X_t} II_{ti} = -\Gamma^l_{tt} II_{li} + \Gamma^l_{ti} II_{lt} = 0. \tag{2.9}$$

Thus, from (2.7), (2.8) and (2.9) it follows the principal curvatures κ_0 and $\kappa_i = \frac{1}{\kappa_0}$ are constant.

Due to Currier [8, Theorem B] it follows that $\kappa_0 \neq \kappa_i$ for $i \neq 1$ since otherwise $\kappa_0 = \kappa_i = 1$ so the hypersurface is horospherically convex and therefore a horosphere, which contradicts the assumption that the hypersurface has more than one end. Therefore, locally the hypersurface has exactly two distinct constant principal curvatures κ_0 of multiplicity 1 and $\frac{1}{\kappa_0}$ of multiplicity $n - 1$. It then follows from Cartan Theorem (cf. Cartan Theorem in the introduction) that the hypersurface is an equidistant hypersurface about a geodesic line. □

3 Calculations for vertical graphs in hyperbolic space

In [1,2,10] it is observed by Epstein, Alexander and Currier that a complete, noncompact, properly embedded, strictly convex hypersurface in hyperbolic space can be realized globally in Busemann coordinates as a graph of a height function over a domain in a horosphere. Moreover, in [1,2] it is shown that embedded hypersurfaces with nonnegative sectional curvature give rise to height functions that are subharmonic with respect to the Euclidean metric when restricted to any 2-plane. Then, as a consequence of the theory of subharmonic functions on domains in the plane, in [1,2] it is concluded that a hypersurface embedded in hyperbolic space with nonnegative sectional curvature must have a single point asymptotic boundary when the asymptotic boundary is connected.

Moving on to the situations when only Ricci curvature is assumed to be nonnegative, the theory of subharmonic functions in dimension 2 is not applicable and the method in [1,2] fails in dimensions larger than 2. Our approach here is to employ the theory of n-subharmonic functions instead of subharmonic functions in dimensions $n > 2$.

Consider the upper half-space model $\mathbb{R}^{n+1}_+$ of hyperbolic space with standard coordinates $(x_1, \dots, x_n, x_{n+1})$ and hyperbolic metric

$$g_{\mathbb{H}} = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}.$$

In the upper half-space model of hyperbolic space we note that

$$\nabla^{\mathbb{H}}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \delta_{ij} \frac{1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}} \quad \text{and} \quad \nabla^{\mathbb{H}}_{\frac{\partial}{\partial x_\alpha}} \frac{\partial}{\partial x_{n+1}} = -\frac{1}{x_{n+1}} \frac{\partial}{\partial x_\alpha}.$$

Note that in our convention Greek letters run from $1, 2, \ldots, n+1$ while Latin letters run from $1, 2, \ldots, n$. Let Σ be the vertical graph of a function $x_{n+1} = f(x_1, \ldots, x_n)$ over a domain Ω in

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_+ : x_{n+1} = 0\}.$$

Denote the induced tangent vectors on Σ by

$$X_i = \frac{\partial}{\partial x_i} + f_i \frac{\partial}{\partial x_{n+1}}$$

where $f_i = \frac{\partial f}{\partial x_i}$. Then the induced metric on Σ as a hypersurface in $\mathbb{H}^{n+1}$ is given by

$$g := f^{-2}(\delta_{ij} + f_i f_j) dx^i dx^j$$

with inverse

$$g^{ij} = f^2 \left(\delta^{ij} - \frac{f_i f_j}{1 + |Df|^2} \right),$$

where we have denoted the Euclidean norm squared of the Euclidean gradient of f by

$$|Df|^2 = \delta^{ij} f_i f_j = \sum_{i=1}^n f_i^2.$$

Then a straightforward computation gives

$$\nabla^{\mathbb{H}}_{X_i} X_j = f^{-1} \left((\delta_{ij} + f f_{ij} - f_i f_j) \frac{\partial}{\partial x_{n+1}} - f_i \frac{\partial}{\partial x_j} - f_j \frac{\partial}{\partial x_i} \right).$$

Hence, with respect to unit normal

$$\nu = \frac{f}{(1 + |Df|^2)^{\frac{1}{2}}} (-f_1, -f_2, \ldots, -f_n, 1)$$

on Σ, we compute the second fundamental form of Σ

$$II_{ij} = \langle \nabla^{\mathbb{H}}_{X_i} X_j, \nu \rangle = \frac{1}{f^2 (1 + |Df|^2)^{\frac{1}{2}}} (\delta_{ij} + f_i f_j + f f_{ij}). \tag{3.1}$$

Moreover, denoting the Euclidean Laplacian of f by Δf, it follows that the mean curvature of Σ is

$$\begin{aligned} H &= \frac{1}{(1 + |Df|^2)^{\frac{1}{2}}} \sum_{i,j=1}^n \left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) (\delta_{ij} + f_i f_j + f f_{ij}) \\ &= \frac{1}{(1 + |Df|^2)^{\frac{1}{2}}} \left(n + f \Delta f - \frac{f}{1 + |Df|^2} \sum_{i,j=1}^n f_{ij} f_i f_j \right). \end{aligned}$$

Now at any point $x \in \mathbb{R}^n$ where $h = \log f$ is finite and $Df(x) \neq 0$, we may choose local coordinates where $\frac{\partial}{\partial x_1} = \frac{Df}{|Df|}$ is the Euclidean unit vector in the direction of Df and with $f_j(x) = \frac{\partial f}{\partial x_j}(x) = 0$ for all $j \neq 1$. In such coordinates $f_1^2 = |Df|^2$ so we may write the mean curvature of Σ at such a point x as

$$H = \frac{f}{(1 + f_1^2)^{\frac{3}{2}}} \left(f_{11} + \frac{1 + f_1^2}{f} \right) + \frac{f}{(1 + f_1^2)^{\frac{1}{2}}} \sum_{i=2}^n \left(f_{ii} + \frac{1}{f} \right). \tag{3.2}$$

Next we calculate the Ricci curvature for the vertical graph Σ in hyperbolic space via Gauss equations

$$R^{\Sigma}_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}) + (II_{ik}II_{jl} - II_{il}II_{jk}).$$

From (3.1) it follows that the Ricci curvature tensor has components

$$\begin{aligned}
R_{ik} &= -(n-1)g_{ik} + \frac{1}{f^2(1+|Df|^2)}\sum_{j,l=1}^{n}\left(\delta_{jl} - \frac{f_j f_l}{1+|Df|^2}\right)\\
&\quad \times ((\delta_{ik} + f_i f_k + f f_{ik})(\delta_{jl} + f_j f_l + f f_{jl}) - (\delta_{il} + f_i f_l + f f_{il})(\delta_{jk} + f_j f_k + f f_{jk}))\\
&= -(n-1)g_{ik} + \frac{1}{f^2(1+|Df|^2)}\\
&\quad \times \left((\delta_{ik} + f_i f_k + f f_{ik})\left(n + f\Delta f - \frac{f}{1+|Df|^2}\sum_{j,l=1}^{n} f_{jl} f_j f_l\right)\right.\\
&\quad \left. - \sum_{l=1}^{n}(\delta_{il} + f_i f_l + f f_{il})\left(\delta_{lk} + f f_{lk} - \frac{f}{1+|Df|^2}\sum_{j=1}^{n} f_{jk} f_j f_l\right)\right).
\end{aligned} \tag{3.3}$$

Now, let us consider the gradient of f with respect to the induced metric g

$$\nabla^g f = g^{ij} f_i X_j = f^2\left(\delta^{ij} - \frac{f_i f_j}{1+|Df|^2}\right) f_i X_j = \frac{f^2}{1+|Df|^2}\sum_{j=1}^{n} f_j X_j,$$

and its normalization

$$\frac{\nabla_g f}{||\nabla_g f||_g} = \frac{f}{|Df|(1+|Df|^2)^{\frac{1}{2}}}\sum_{j=1}^{n} f_j X_j.$$

Denoting the components of the normalized gradient of f by

$$\bar{f}^i = \frac{f}{|Df|(1+|Df|^2)^{\frac{1}{2}}} f_i,$$

from (3.3) we calculate the Ricci curvature in the direction of the normalized gradient of f

$$\begin{aligned}
R_{ik}\bar{f}^i\bar{f}^k &= -(n-1) + \frac{1}{|Df|^2(1+|Df|^2)^2}\sum_{i,k=1}^{n} f_i f_k\\
&\quad \times \left((\delta_{ik} + f_i f_k + f f_{ik})\left(n + f\Delta f - \frac{f}{1+|Df|^2}\sum_{j,l=1}^{n} f_{jl} f_j f_l\right)\right.\\
&\quad \left. - \sum_{l=1}^{n}(\delta_{il} + f_i f_l + f f_{il})\left(\delta_{lk} + f f_{lk} - \frac{f}{1+|Df|^2}\sum_{j=1}^{n} f_{jk} f_j f_l\right)\right)\\
&= -(n-1) + \frac{1}{|Df|^2(1+|Df|^2)^2}\\
&\quad \times \left(\left(|Df|^2 + |Df|^4 + f\sum_{i,k=1}^{n} f_{ik} f_i f_k\right)\right.
\end{aligned}$$

$$\times\left(n+f\Delta f-\frac{f}{1+|Df|^2}\sum_{j,l=1}^{n}f_{jl}f_jf_l\right)$$
$$-\sum_{l=1}^{n}\left(f_l+|Df|^2f_l+f\sum_{i=1}^{n}f_{il}f_i\right)$$
$$\times\left(f_l+f\sum_{k=1}^{n}f_{lk}f_k-\frac{f}{1+|Df|^2}\sum_{j,k=1}^{n}f_{jk}f_jf_lf_k\right)\Bigg). \tag{3.4}$$

For convenience, we denote

$$\sum_{i,j=1}^{n}f_{ij}f_if_j=H_1(f)\quad\text{and}\quad\sum_{i,j,k=1}^{n}f_{ik}f_{kj}f_if_j=H_2(f).$$

Then we may write (3.4) as

$$\begin{aligned}R_{ik}\bar f^i\bar f^k&=-(n-1)+\frac{1}{|Df|^2(1+|Df|^2)^2}(|Df|^2(1+|Df|^2)+fH_1(f))\\&\quad\times\left(n+f\Delta f-\frac{fH_1(f)}{1+|Df|^2}\right)-\Bigg(|Df|^2(1+|Df|^2)+2fH_1(f)\\&\quad+f^2H_2(f)-\frac{f^2(H_1(f))^2}{1+|Df|^2}\Bigg)\Bigg)\\&=-(n-1)+\frac{1}{|Df|^2(1+|Df|^2)^2}\Bigg(n|Df|^2(1+|Df|^2)+nfH_1(f)\\&\quad+f\Delta f|Df|^2(1+|Df|^2)+f^2H_1(f)\Delta f-fH_1(f)|Df|^2-\frac{f^2(H_1(f))^2}{1+|Df|^2}\\&\quad-\left(|Df|^2(1+|Df|^2)+2fH_1(f)+f^2H_2(f)-\frac{f^2(H_1(f))^2}{1+|Df|^2}\right)\Bigg)\\&=-(n-1)\frac{|Df|^2}{1+|Df|^2}+\frac{f}{|Df|^2(1+|Df|^2)^2}((n-2)H_1(f)\\&\quad+\Delta f|Df|^2(1+|Df|^2)+fH_1(f)\Delta f-H_1(f)|Df|^2-fH_2(f)).\end{aligned}\tag{3.5}$$

Now, as above, at any given point where $h=\log f$ is finite and $Df\neq 0$, we choose a local normal coordinate such that $\frac{\partial}{\partial x_1}$ is a Euclidean unit vector in the direction of Df. Then pointwise we may simplify (3.5) as follows:

$$\begin{aligned}R_{ik}\bar f^i\bar f^k&=-(n-1)\frac{f_1^2}{1+f_1^2}+\frac{f}{(1+f_1^2)^2}\Bigg((n-2)f_{11}+\Delta f(1+f_1^2)\\&\quad+ff_{11}\Delta f-f_{11}f_1^2-f\sum_{i=1}^{n}f_{1i}^2\Bigg)\\&=-(n-1)\frac{f_1^2}{1+f_1^2}+\frac{f}{(1+f_1^2)^2}\Bigg((n-1)\left(f_{11}+\frac{1+f_1^2}{f}\right)-(n-1)\frac{1+f_1^2}{f}\\&\quad+f\left(\frac{1+f_1^2}{f}+f_{11}\right)\sum_{i=2}^{n}f_{ii}-f\sum_{i=2}^{n}f_{1i}^2\Bigg)\end{aligned}$$

$$= \frac{f^2}{(1+f_1^2)^2}\left(\frac{1+f_1^2}{f}+f_{11}\right)\sum_{i=2}^{n}\left(f_{ii}+\frac{1}{f}\right)-(n-1)-\frac{f^2}{(1+f_1^2)^2}\sum_{i=2}^{n}f_{1i}^2. \tag{3.6}$$

But then, since the Ricci curvature is nonnegative, it follows that $R_{ik}\bar{f}^i\bar{f}^k \geq 0$ so

$$\left[\frac{f}{(1+f_1^2)^{\frac{3}{2}}}\left(\frac{1+f_1^2}{f}+f_{11}\right)\right]\left[\frac{f}{(1+f_1^2)^{\frac{1}{2}}}\sum_{i=2}^{n}\left(f_{ii}+\frac{1}{f}\right)\right] \geq (n-1). \tag{3.7}$$

Note that the sum of the two factors in (3.7) is the mean curvature in the light of (3.2).

Lemma 3.1 *On a hypersurface in hyperbolic space with nonnegative Ricci curvature the mean curvature of the hypersurface $H \geq n$.*

Proof From the assumption that the Ricci is nonnegative, for each $i = 1, \ldots, n$, one has

$$\kappa_i H \geq n - 1 + \kappa_i^2$$

where κ_i denote the principal curvatures. Therefore, with our choice of orientation, $\kappa_i > 0$ and

$$H^2 \geq n(n-1) + \sum_{i=1}^{n}\kappa_i^2 \geq n(n-1) + \frac{1}{n}H^2$$

which implies that $H \geq n$. □

Since both sum and product are positive, the two factors on the left of the Eq. (3.7) are both positive. Therefore,

$$\sqrt{(n-1)\left(\frac{1+f_1^2}{f}+f_{11}\right)} \cdot \sqrt{\sum_{i=2}^{n}\left(f_{ii}+\frac{1}{f}\right)} \geq (n-1)\frac{1+f_1^2}{f}. \tag{3.8}$$

Theorem 3.1 *Suppose that Σ is a vertical graph of a function $x_{n+1} = f(x_1, \ldots, x_n)$ in the upper half-space model of hyperbolic space with $f \in C^2$ wherever the hyperbolic height function $h = \log f$ is finite. If Σ has nonnegative Ricci curvature, then the height function is Euclidean n-subharmonic. That is,*

$$\Delta_n \log f = Div(|D\log f|^{n-2}D\log f) \geq 0 \tag{3.9}$$

wherever $h = \log f$ is finite.

Proof One may focus on the points where $Df \neq 0$. From (3.8) and Young's inequality, we have

$$2(n-1)\frac{1+f_1^2}{f^2} \leq (n-1)\left(\frac{f_{11}}{f}+\frac{1+f_1^2}{f^2}\right)+\sum_{i=2}^{n}\frac{f_{ii}}{f}+(n-1)\frac{1}{f^2}$$

which implies

$$\begin{aligned} 0 &\leq (n-1)\frac{f_{11}}{f}-(n-1)\frac{f_1^2}{f^2}+\sum_{i=2}^{n}\frac{f_{ii}}{f} = (n-1)(\log f)_{11}+\sum_{i=2}^{2}(\log f)_{ii} \\ &= (n-2)|D\log f|^{-2}\sum_{i,j=1}^{n}(\log f)_{ij}(\log f)_i(\log f)_j + \Delta(\log f) \\ &= |D\log f|^{-(n-2)}\Delta_n \log f \end{aligned} \tag{3.10}$$

and completes the proof. □

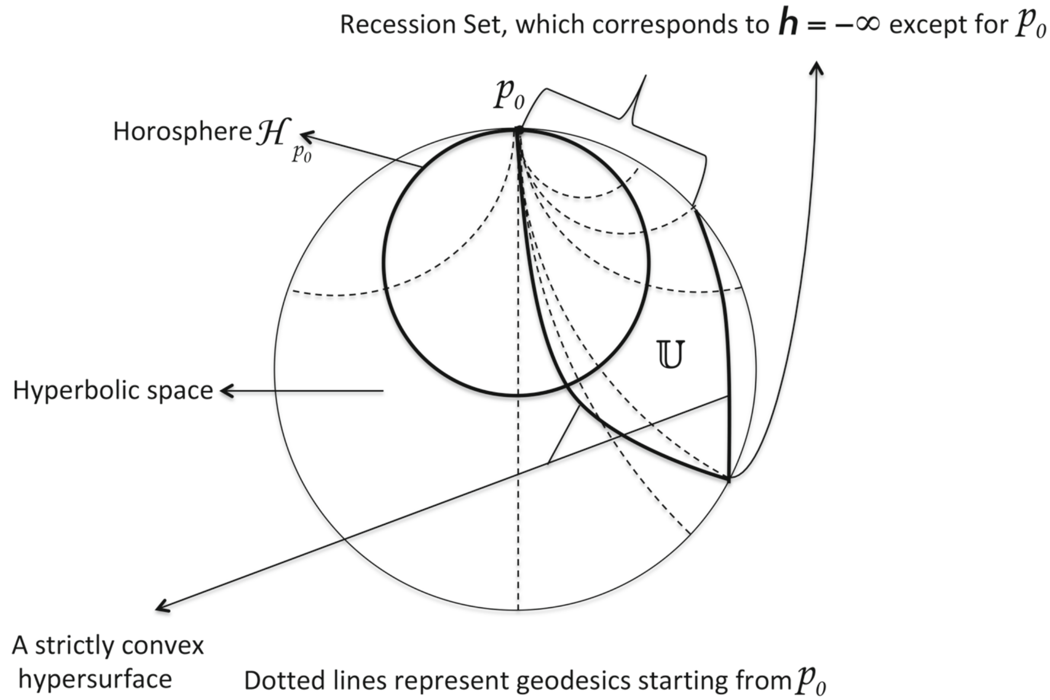
4 n-Subharmonic functions and proof of main theorem

Let Σ be a complete, noncompact, properly embedded hypersurface in $\mathbb{H}^{n+1}$ with nonnegative Ricci curvature. Then from (1.1) Σ is strictly convex and it is known that Σ is the boundary of a strictly convex body $\mathbb{U}$ in hyperbolic space. Then the recession set $R(\Sigma)$ for Σ is the collection of end points at infinity of all geodesic rays which lie entirely inside $\mathbb{U}$. Thanks to Epstein, Alexander and Currier [1,2,10], it is also known that Σ can be realized as a vertical graph of a height function over a domain in any horosphere centered at a point in the recession set (cf. [1,2,10]). Let us state a lemma to collect some useful facts for us.

Lemma 4.1 (cf. [1, Proposition 2.2]) *Suppose that Σ is a complete, noncompact, properly embedded, strictly convex hypersurface in hyperbolic space. Then Σ is a graph of a height function $h : \Omega \to \mathbb{R} \cup \{-\infty\}$. Here $\Omega \subset \mathcal{H}_{p_0}$, where $\mathcal{H}_{p_0}$ is a horosphere centered at some point p_0 in the recession set $R(\Sigma)$. Moreover the following hold:*

- *The domain Ω is a convex and open subset of $\mathcal{H}_{p_0}$.*
- *The height function h is continuous and locally bounded from above in Ω.*
- *$\mathbb{P}(\{h = -\infty\}) \cup \{p_0\}$ is the recession set $R(\Sigma)$, where $\mathbb{P}$ is the simple orthogonal projection when using the half space model taking p_0 as the infinity.*

Proof One can refer to the graph below for the notions. Note that all geodesic lines from p_0 have one-to-one correspondence with the horosphere $\mathcal{H}_{p_0}$. From (1.1), $\mathbb{U}$ is strictly convex. So all geodesic lines from the point p_0 are exclusively of three kinds: those lying entirely inside $\mathbb{U}$; those lying entirely outside $\mathbb{U}$; those intersecting Σ transversally. The intersection points of $\mathcal{H}_{p_0}$ with those geodesic lines which are not lying entirely outside $\mathbb{U}$ make up the domain Ω. Since $\mathbb{U}$ is open and convex, it is easy to prove that Ω is open and convex in $\mathcal{H}_{p_0}$. Let $(x_1, \ldots, x_n, x_{n+1} = e^y)$ be the Busemann coordinate and assume $\mathcal{H}_{p_0}$ is given by $y = 0$. In this coordinate, Σ together with the recession set $R(\Sigma)\backslash\{p_0\}$ can be viewed as the graph of the function $y = h(x_1, \ldots, x_n)$ over Ω. Σ corresponds to where h is finite and $R(\Sigma)\backslash\{p_0\}$ corresponds to where $h = -\infty$. It is not difficult to prove that h is continuous in Ω from the fact the hypersurface is smooth and convex. So it is locally bounded from above in Ω since $h < +\infty$ in Ω. The third item is also obvious from our construction.



In this section, based on the theory of n-subharmonic functions and n-polar sets in [11,12] by Heinonen, Kilpelainen, and Martio, and Lindqvist, we present an argument here to show that for a complete properly embedded hypersurface with nonnegative Ricci curvature, the set

$$\{h = -\infty\} \subset \Omega$$

is totally disconnected. In particular, p_0 is a connected component of the recession set, since if there were other points in the connected component containing p_0, these points are totally disconnected, which is absurd. Then if the hypersurface is not an equidistant hypersurface, $\partial_\infty \Sigma = \{p_0\}$. So we will complete the proof of the main theorem.

For the convenience of the readers we recall some of the basics in the theory of p-subharmonic functions on domains in $\mathbb{R}^n$. Our introduction here is mostly based on [11,12], therefore readers are referred to [11,12] for details and proofs. First we recall Definition 7.1 of [11] (see also Definition 5.1 of [12]), which defines viscosity p-subharmonic functions in terms of the comparison principle.

Definition 4.1 ([11, Definition 7.1] [12, Definition 5.1]) A function $u : W \to \mathbb{R} \cup \{-\infty\}$ is called viscosity p-subharmonic in a domain $W \subset \mathbb{R}^n$, if

(1) u is upper semi-continuous in W;
(2) $u \not\equiv -\infty$ in W;
(3) For each $W_1 \subset\subset W$, the comparison principle holds: if $v \in C(\overline{W}_1)$ is p-harmonic in W_1 and $v|_{\partial W_1} \geq u|_{\partial W_1}$, then $v \geq u$ in W_1.

The most important analytic tools for us are Theorems 10.1 and 2.26 in [11], which we state as follows:

Theorem 4.1 ([11, Theorems 10.1 and 2.26]) *Suppose that u is a viscosity p-subharmonic function defined in a domain $W \subset \mathbb{R}^n$. Then its p-polar set $\{u = -\infty\} \subset W$ is of* *zero p-capacity and of Hausdorff dimension at most $n - p$. Particularly, for a viscosity n-subharmonic function u, the set $\{u = -\infty\}$ is of zero n-capacity and*

$$dim_{\mathscr{H}}(\{u = -\infty\}) = 0.$$

Therefore, the main issue in proving the Main Theorem is to verify that the height functions for complete properly embedded hypersurfaces in hyperbolic space with nonnegative Ricci curvature are viscosity n-subharmonic in $\Omega \subset \mathbb{R}^n$. In the light of Definition 4.1, we only need to verify the comparison principle. Here we make a note that viscosity p-subharmonic functions may not belong to $W^{1,p}_{loc}$. However, if it is locally bounded from below, then it belongs to $W^{1,p}_{loc}$. For our purpose we introduce the notion of weakly p-subharmonic functions.

Definition 4.2 ([12, Definition 2.12]) For $p \geq 1$ and a domain $W \subset \mathbb{R}^n$, a function $u \in W^{1,p}_{loc}(W)$ satisfying

$$\int \langle |Du|^{p-2} Du, D\eta \rangle dx \leq 0 \quad \text{for each } \eta \in C_0^\infty(W) \text{ and } \eta \geq 0 \tag{4.1}$$

is called a weakly p-subharmonic function in W.

From Theorem 2.15 in [12] and subsequent remarks we have the following comparison principle for weakly p-subharmonic functions.

Theorem 4.2 ([12, Theorem 2.15]) *Suppose that u is a weakly p-subharmonic function and v is a p-harmonic function in a bounded domain $W \subset \mathbb{R}^n$. If for every $\zeta \in \partial W$*

$$\limsup_{x \to \zeta} u(x) \leq \liminf_{x \to \zeta} v(x) \tag{4.2}$$

with the possibilities $\infty \leq \infty$ and $-\infty \leq -\infty$ excluded, then $u \leq v$ almost everywhere in Ω.

Consequently, due to Theorem 3.1 in the previous section, away from the recession set, the height function h is clearly weakly n-subharmonic and satisfies the comparison principle. Now we are ready to prove our main theorem.

Proof of the Main Theorem We claim that the height function $h = \log f$ is viscosity n-subharmonic in its domain Ω as defined in Lemma 4.1. It is clear that $h \not\equiv -\infty$ and that h is upper semi-continuous. One only needs to verify the Comparison Principle in (3) of Definition 4.1. Assume otherwise, that condition (3) does not hold for h in Ω. Let $v \in C(\overline{W})$ be an n-harmonic function in $W \subset\subset \Omega$ with $v \geq h$ on ∂W but $h > v$ in some nonempty open subset $W_0 \subset W$ with $h = v$ on ∂W_0. Then it is easily seen that $W_0 \cap \{h = -\infty\} = \emptyset$. That is to say the height function h is finite in W_0 and therefore satisfies the comparison principle Theorem 4.2 on W_0, which is a contradiction. Thus, the height function is indeed viscosity n-subharmonic.

In the light of Theorem 4.1, we know that $\dim_{\mathscr{H}}(R(\Sigma)) = 0$. So the asymptotic boundary is totally disconnected, that is, every connected component of the asymptotic boundary can only be a single point. If the asymptotic boundary has more than one connected component, then we know $\partial_\infty \Sigma$ consists of exactly two points by the Cheeger–Gromoll splitting theorem [7] and the discussion in Sect. 2. Hence, by Theorem 2.1, it follows that Σ is an equidistant hypersurface. Otherwise, the asymptotic boundary must consist of a single point. So the proof of the Main Theorem is complete.

Our corollary now follows easily from our main theorem and the discussion above. From our main theorem one easily sees that a manifold (M^n, g) with nonnegative Ricci curvature that can be isometrically embedded in $\mathbb{H}^{n+1}$ can either be embedded as an equidistant hypersurface and is therefore diffeomorphic to a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ or it can be embedded as a hypersurface with single point boundary at infinity. In the latter case, one finds in that the hypersurface can be realized as the graph of a smooth (nonsingular) height function over a convex open domain Ω in a horosphere and is therefore diffeomorphic to $\mathbb{R}^n$. □

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