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Abstract

The degree to which any deregulated market functions efficiently often depends on the ability of market agents to respond quickly to fluctuating conditions. Many restructured electricity markets, however, experience high prices caused by supply shortages and little demand-side response. We examine the implications for market operations when a risk-averse retailer's end-use consumers are allowed to perceive real-time variations in the electricity spot price. Using a market-equilibrium model, we find that price elasticity both *increases* the retailer's revenue risk exposure and *decreases* the spot price. Since the latter induces the retailer to reduce forward electricity purchases, while the former has the opposite effect, the overall impact of price responsive demand on the electricity forward price is ambiguous. Indeed, each retailer's response depends on the relative magnitudes of its risk exposure and end-user price elasticity. Nevertheless, price elasticity decreases *cumulative* electricity consumption. By extending the analysis to allow for early settlement of demand, we find that forward stage end-user price responsiveness decreases the electricity forward price relative to the case with price-elastic demand only in real time. Moreover, we find that only if forward stage end-user demand is price elastic will the equilibrium electricity forward price be reduced.

Keywords: Ancillary services, forward contracts, price-elastic demand.

1. Introduction

In most organized economies, infrastructure industries, e.g., those involving energy, telecommunications, and transportation, have traditionally been subject to government regulation. Such oversight has been predicated on the "natural monopoly" characteristics of these industries, which imply that costs decline with output and that a single extensive network is necessary to deliver the final product to consumers. Hence, the need for multiple firms or duplicate transportation networks within a geographic region is obviated. Indeed, it is economically inefficient to build several parallel roads between any two destinations when one road conveys traffic just as effectively.

Within the set of infrastructure industries, electricity was especially suited to government regulation due to its lack of storability, the complex nature of its transmission, and to a lesser extent, economies of scale in its generation. In particular, electricity transmission, unlike other transportation networks, requires coördinated behavior to ensure that injections and withdrawals of electricity are continuously balanced. This coördination is necessitated by Kirchhoff's laws, which state that alternating current (AC) follows the path of least impedance along a transmission system. Consequently, events occurring in one region have implications for the entire system, i.e., actions on the grid are not localized. Therefore, any decentralized system would prove to be impracticably complex because it would have to be balanced in real-time to prevent its collapse. As a result, electricity supply functions, such as generation and transmission, were kept *vertically integrated* under the auspices of a regulated entity that exclusively provided all services within a given geographic region.

While vertical integration of generation and transmission internalized many operating and investment complementarities, viz., the coördination of efficient electricity dispatch, it, nevertheless, turned the generation sector into a *de facto* monopoly. As a consequence, a potentially competitive generation sector[1] was encumbered by government regulation and its associated inefficiencies. According to [13], some of these inefficiencies included low productivity of facilities and labor, excessive long-run investment in generating capacity, and a gap between regulated retail prices and wholesale market prices. Along with technological advancements, such as the development of low-cost, small-scale generation plants using combined-cycle methods and the introduction of decentralized coördination facilitated by advanced telecommunications, the existence of the aforementioned economic inefficiencies has provided the impetus for many state and federal governments to deregulate their electric power sectors.

The desire to provide incentives for efficient operation of the electricity industry has, thus, led to its restructuring in many countries. In general, this has meant unbundling of the various electricity services so that they may be provided by specialized firms subject to light regulation instead of heavily regulated investor-owned utilities (IOUs). As identified in [16], the four main electricity supply functions provided by an IOU were:

[1]There exists little evidence that large companies are necessary to exploit economies of scale in generation(see [15]).

- **generation**: conversion of primary energy to electricity.
- **transmission**: transportation of electricity along meshed high-voltage wires to substations.
- **distribution**: transportation of electricity along low-voltage wires to customer meters.
- **retailing**: arrangements for billing, on-site support, and demand management.

In order to facilitate deregulation, regulatory agencies worldwide have begun to introduce competition into areas of the electricity industry that are technologically amenable to it. This has meant that the generation and retailing sectors have seen the promotion of competition because economies of scale are either exhausted at current levels of production or are not applicable at all here (see [28]). These services are, thus, to be provided through the markets. For the IOUs, this has generally implied divestiture of their generation assets. The transmission and distribution sectors, however, continue to be regulated because of their "natural monopoly" characteristics. Outside of these very general guidelines, the actual paths taken by electricity industry restructuring movements vary considerably across states and countries.

The contours of these reforms, as traced out in [27], touch upon the two extremes in electricity market design. One approach is highly centralized in that it seeks to emulate the tight control of the vertically integrated paradigm by exploiting the complementarities between generation and transmission. In this environment, an independent system operator (ISO) not only manages the transmission system, but also conducts market operations with centralized dispatch. According to [27], such a framework works best when there is ample competition and accurate information is available for the ISO's optimization problem. At the other extreme is a decentralized approach in which the participation of agents is not even required. Indeed, instead of there being a centralized dispatch, market agents can transact bilaterally or through the markets, such as a day-ahead power exchange (PX), with the ISO charged only with protecting system reliability and operating real-time (spot) markets for imbalances. Since there is no explicit coördination of energy, reserves, or transmission markets, such an environment requires a profusion of trading opportunities in order to function smoothly.

Regardless of the form of electricity industry deregulation, it has been documented that introduction of competition into electricity generation sectors leads to some improvements in social welfare. For example, in the England and Wales (E&W) electricity spot market, even though average prices are higher than system marginal costs, they are not as high as theoretically predicted (see [30]). Moreover, the E&W competitive generation sector saw marked improvements in labor productivity within three years of deregulation. Along with greater economic efficiency in the generation sector, however, deregulation has also introduced new problems into an industry that was once insulated from competitive forces.

A prominent issue with deregulated electricity industries has been the exercise of market power by generators. Evidence exists that both the E&W and California wholesale electricity markets have had at least the conditions that reward the withholding of generation capacity from the market, thereby leading to market-clearing prices well in excess of marginal costs (see [6], [7], and [8]). In most cases, the ability of generators to exercise market power is linked to deficiencies in market structure or rules (see [28]). Although the existence of market power does not negate the benefits of deregulation, its presence has attracted the attention of oversight boards.

On a related note, while some price volatility is to be expected in any competitive commodity market due to supply and demand conditions, seasonality, and lack of storability, there has been evidence that some price volatility may have been caused or exacerbated by firms exercising market power (see [10]). Of course, some of the risks associated with price volatility may be effectively managed through the many financial instruments available to market participants; however, if the market is really being "gamed" as believed by some, then the effectiveness of these financial methods is questionable.

Another issue receiving attention now is the maintenance of system reliability. Commonly referred to as "ancillary services" (AS), reserve generation resources that can be quickly dispatched in order to meet real-time contingencies serve to support the transmission of electricity and to maintain reliable operation. Whereas previously AS were provided through the advice of regional reliability councils (RRCs), now they are the duty of the ISO. While the way in which AS are now procured varies across regions (e.g., in California, AS can either be procured competitively or self-provided), the fact remains that there exists tremendous risk (in terms of prices well as consequences for the grid) if AS are not obtained in a timely and efficient manner.

The overarching problem that almost all electricity deregulation efforts have failed to address, however, is on the demand side. Indeed, the bill for electricity usage paid by most end-use consumers is not related to the real-time (spot) price of electricity, even in most deregulated industries. In E&W, for example, about 5% of the total system load in 1996 was purchased by end-users who experienced periodic variations in the real-time price (see [28]). In California, retail rates were frozen in order to allow IOUs to recover sunk costs of investments in generating plants that had been made before the 1996 restructuring (so-called "stranded assets"), but would not be viable after deregulation. By not allowing demand to be price elastic, many electricity deregulation efforts stretch generation resources to the point where system stability is threatened. Indeed, due to an unresponsive demand side, electricity demand has to be met regardless of the cost. This, in turn, facilitates the exercise of market power, amplifies price volatility, and creates difficulties for the ISO in procuring AS.

Unlike other competitive commodity markets, deregulated electricity markets had virtually no demand-side response because end-use consumers were exposed to a constant retail rate independent of market conditions. Under certain circumstances, the ISOs could act to reduce AS purchases and exercise IL contracts, but in the extreme case of California, these mechanisms were only modestly successful at reducing load during 2000 because the high frequency of outages decreased customer response (see [17]). Consequently, when the supply side experienced shocks, the absence of price elasticity on the demand side resulted in wholesale prices that were substantially higher than both their historical levels and the frozen retail rate. Other factors, such as the lack of inventories, the long lead time required to add new generating capacity, and evidence of market power, exacerbated the problem and also contributed to the shutdown of the CalPX.

In theory, this problem could have been averted if end-use consumers were exposed to real-time prices from the onset. A case for the adoption of real-time prices is made in [5], which postulates that their effect is to reduce the demand for electricity during peak hours. This then lowers the electricity spot price and reduces the need to build more power plants. Furthermore, the strategic role of generator hedging alone in reducing price volatility and mitigating market power is explicated in [2] and [29]. Thus, the combination of real-time pricing with long-term hedge contracts for electricity, which decrease the ability of generators to exercise market power, could have enabled the California markets to function cost-effectively.

In general, because deregulated electricity industries essentially inherited outright the basic service tariffs from the vertically integrated era, few measures exist to promote a price-responsive demand side. As pointed out in [27], such protocols would be to the benefit of retailers, especially in risk management. However, beyond contracts for differences (CFDs), which are utilized in the E&W market and Australian Victoria Pool (VicPool), for instance, generators and retailers have little recourse for risk sharing in response to price volatility. Indeed, in spite of the promise for increased competition, the retailing sectors of deregulated electricity industries are prevented from responding to external shocks. The lack of any demand-side price response and limited retail hedging opportunities are indicative of such restrictions.

While hedging instruments are commonplace in electricity markets and metering technology for real-time pricing is becoming technologically feasible, in practice, there are restrictions on their usage. As a result, the impact of real-time pricing on forward markets is unclear. The objective of this paper is to assess the implications on electricity forward market operations, viz., the equilibrium price and optimal quantity traded, when a risk-averse retailer's end-use consumers are allowed to respond to real-time price signals. We model a perfectly competitive electricity industry, in which a spot market for electricity and forward markets for both electricity and AS exist. Using two specifications of end-use consumer response, we find that real-time pricing impacts the electricity forward price through the retailer's relative magnitudes of end-use consumer price elasticity and risk exposure.

The structure of this paper is as follows:

- in **section 2**, we review the literature pertaining to hedging strategies in forward markets and price responsiveness.
- in **section 3**, we describe the model of electricity production and markets.

- in **section 4**, we solve for the equilibrium prices and quantities in each market when end-use consumer demand depends solely on the spot price.
- in **section 5**, we examine the implications of allowing end-use consumer demand to respond to both the electricity forward and spot prices.
- in **section 6**, we summarize the main results and give direction for future research in this area.

2. Survey of the Relevant Literature

The use of hedging is common in competitive markets with volatile prices. Nevertheless, price volatility itself is desirable because it transmits signals to agents about market conditions. Indeed, the ultimate causes of price volatility include the variable costs of inputs to production as well as fluctuations in supply and demand. In the electricity industry, this volatility is amplified due to the unstorability of electricity. Traditional utility regulation, however, levelized retail rates so that consumers perceive long-run costs in order to make rational purchasing decisions regarding appliances (see [4]). This lack of price volatility in the retail sector of the electricity industry obscures the true cost of electricity from end-use consumers and prevents its efficient allocation. In many deregulated electricity industries, retailers have faced considerable risk due to the combination of purchasing electricity at volatile prices and selling it to end-use consumers at stable rates. In certain markets, such as those for AS, this problem has been compounded by the market structure which did not prevent generators from exercising market power, and thus, increased price volatility even further (see [21]).

While the literature on hedging is extensive, we focus here on some of its particular applications to electricity markets. In [24], hedging is described as a way of insuring against the increased price volatility in the electricity industry resulting from the introduction of competition. A survey of hedging techniques by generators, retailers, and end-users employing futures, options, and swaps provides insight into managing risk from price volatility. The pricing of futures is explained using the "no arbitrage" approach. Furthermore, the risks associated with unregulated use of financial instruments themselves are highlighted through case studies, and regulators are cautioned to prevent speculation.

Indeed, financial instruments are used for strategic purposes, in addition to risk management. The interaction between hedging and strategic motives is examined in [1] by modeling an oligopolistic industry producing a durable good in which transactions can take place in either the forward or spot market. Depending on the form of conjectural variation, i.e., Cournot competition or constant market share, producers use forward trading strategically (as well as for hedging if they are risk averse) in order to improve their situation in the spot market. In some cases, producers are induced to become net buyers in the forward market. This occurs in the case of constant market share when the strategic incentive outweighs the desire to hedge because the marginal revenue of producers decreases with forward sales. This result is amplified in [2], which examines a duopoly model of forward trading, but without uncertainty. By ignoring any risk hedging motives on part of the producers, it is shown that in a two period environment the one producer allowed to trade forward benefits from this "first mover" advantage. When both producers are allowed to trade forward, a *prisoner's dilemma* emerges because both producers would like to trade forward; however, when they both do so, they both end up worse off than in the case without forward markets. By extending the number of forward trading periods to $N > 2$, it is found that the resulting outcome converges to the case with perfect competition.

In [3], the market-equilibrium approach used in [1] and [2] is applied to electricity markets, with risk-averse retailers purchasing electricity from risk-averse generators in a perfectly competitive setting. The demand that retailers face in their area is stochastic, but price inelastic, which implies that they are interested in purchasing electricity forward in order to hedge against spot price risk. In particular, it is found that the equilibrium forward price deviates from the expected spot price by two risk-related terms. The overall deviation of the forward price from the spot price is ambiguous, however, because one term (related to the variance of system demand) decreases the forward price from the spot price, while the other (related to the skewness of system demand) increases it relative to the spot price. Within the context of risk management, the first term corresponds to the retailers' desire to hedge spot price risk by increasing forward sales. Alternatively, positive skewness implies extreme demand realizations, which result in high spot market prices, and thus, make forward purchases attractive to retailers.

The relationship between electricity and AS is developed in [22], where reliability concerns are to be met through purchases of AS from generators by the ISO. A market-equilibrium model within a perfectly competitive industry is used to analyze the decisions taken by market agents. This approach is extended in [20], in which risk-averse generators and retailers (facing stochastic and completely price-inelastic demands), along with an AS-procuring ISO, interact through an electricity spot market and forward markets for both electricity and AS. Each type of agent tries to maximize its own expected utility of wealth. The resulting optimal reaction functions, together with the market-clearing conditions (which require that each market has zero net demand), yield equilibrium prices and quantities traded. The resulting equilibrium forward price for AS is composed of two terms: one which compensates the generator for *opportunity costs* (due to foregone revenues from potential energy sales), and the other which provides an *energy payment* for actual electricity production.

While many of the intuitive properties of the electricity forward price from [3] are preserved, the inclusion of AS into the model implies that the ISO absorbs some of the risk faced by retailers. Consequently, retailers' revenues are not as highly correlated with the spot price as in [3], which then induces them to reduce their forward sales of electricity. This increases the electricity forward price, while energy payments from AS calls increase the spot price. Empirical analysis provides preliminary corroboration of this effect, especially during off-peak hours when the spot price is low enough to encourage generators to offer their capacity into the AS market instead.

The impact of hedging on the behavior of producers is the subject of [29], in which the optimal bidding and hedging strategies of generators in a deregulated electricity industry are studied. A theoretical analysis of the industry indicates that if generators decrease the number of hedge contracts sold, then their best-response price offer increases. Empirical evidence from the Australian National Electricity Market (NEM1) verifies this outcome: reduced levels of forward contracting lead to increased profits for generators. In spite of this, wholesale prices in NEM1 remained at the level of marginal costs due to high levels of hedging. Such overcontracting is explained in part by the high level of generating capacity in NEM1 and the highly price-elastic residual demands faced by generators, which promote aggressive bidding behavior, and therefore, decrease expected future wholesale prices. The impact of hedging on NEM1 wholesale prices is compelling enough to suggest that regulators of recently deregulated electricity industries force a large enough quantity of hedge contracts on privatized generators in order to deter their exercising market power.

Although forward contracting reduces risk due to volatile spot prices and mitigates the exercise of market power, in [5] it is argued that hedging by itself is unable to reduce electricity prices. The unstorability of electricity and hard supply constraints are particular characteristics of the electricity industry that imply price spikes if end-users have no incentives to alter their consumption patterns. Indeed, the single, stable retail rate that most end-users face, even in a decentralized deregulated electricity industry like California's, offers end-users no economic rationale for reducing consumption during hours of peak demand. Consequently, more generation capacity is needed to satisfy peaking demand, which, thus, entails plant construction costs for end-users through higher future retail rates.

With recent advances in real-time metering technology, it is feasible to implement a protocol through which end-users are exposed to the hourly variability in electricity prices while maintaining stable monthly bills. As described in [5], this is possible if retailers hedge a large part of the monthly demand and then *ex post* charge end-users the spot price plus the return (or loss) per unit from the long-term contract. The hedging eliminates most of the monthly variability from electricity bills, but real-time pricing transmits the hourly fluctuations in market conditions to end-users. As a result, because end-users realize that their electricity usage rate is proportional to the spot price, they are motivated to reduce consumption during hours with high spot prices. Hence, together with long-term contracting, real-time pricing has the potential to reduce demand during peak hours, which lowers the overall spot price and results in decreased forward contract prices.

Examples of pricing tailored to the demand side in the electricity industry are addressed in the literature. In [18], time-varying end-user demand for electricity is modeled as separate, but interrelated, demands for distinct products. The welfare-maximizing prices formalize the concept of product differentiation: even though the same underlying product is being consumed during the various hours, its pricing differs according to its attributes. Product differentiation is exploited further in [12] to derive the result that service dispatch according to priority classes results in efficiency gains. Instead of there being a uniform level of service achieved through random rationing, the emergence of priority classes enables end-users who value service more to receive it with greater reliability. In the absence of transaction costs, it is shown that priority service is as efficient a rationing mechanism as spot pricing. This view is reinforced in [11] since many of the welfare gains are realized through a few, i.e., two or three, priority classes.

The concept of electricity product differentiation is extended in [23] to include interruptible service. The resulting pricing structure provides incentives to end-users to self-select their service options from a menu that is designed to minimize the sum of losses from interruptions. It is shown that a utility can design and implement such a price structure by knowing only the probability distribution of outage costs for the total system, rather than for individual end-users. In [25] and [26], this analysis is extended to allow for early notification, in which an end-user has the option of paying a fee in order to receive advance warning of interruption.

Empirical work on price-elastic demand in electricity markets reveals the extent of the impact of a fully responsive demand-side. In [6], an empirical analysis of market power in California indicates that the elasticity of demand is a significant factor in mitigating the degree of market power. The extent of price-elastic demand in reducing prices and consumption is investigated in [9]. In the service territory of San Diego Gas & Electric (SDG&E), the retail rates to which end-users were exposed increased during the summer of 2000. It is estimated that a doubling of the retail rate results in a modest reduction in demand (approximately two percent). The fact that end-users were exposed to wholesale prices with a five-week lag and that a retroactive rate-freeze had been promised by politicians implied that the actual rate increase was not substantial. However, the inelastic nature of the electricity supply-side for high levels of production implies that even modest shifts in demand will result in substantially lower prices. In this study, we formalize the theoretical relationship between price-elastic demand and equilibrium prices and consumption that is sketched in the literature.

3. Electricity Markets and Production

In this section, we model the markets for electricity and AS in order to assess the impact of price elastic end-user demand. We assume perfectly competitive[2] spot and forward markets for electricity and a forward market for one type of AS (as opposed to the four or more that actually exist in most markets). We analyze production decisions for only a single future time period because the non-storability of electricity creates markets that are effectively independent over time. For simplicity, we assume that all uncertainty is resolved before spot market decisions are made. Underlying this assumption is the

[2] The degree to which the electricity markets are competitive is open to debate. Our concern, however, is more with pricing once market mechanisms are fully in place.

fact that power companies are able to forecast demand in the immediate future, i.e., the next hour, with precision. Here, we also abstract from transmission constraints by supposing that electricity can be transmitted costlessly. Of course, in reality transmission bottlenecks play a significant role in determining the pattern of electricity generation and pricing. However, our focus is on the short-term strategies of market agents that will determine equilibrium prices rather than on congestion pricing. In addition, we ignore ramping constraints and unit commitment issues in order to focus solely on pricing decisions.

Although market agents are assumed to face no uncertainty while making decisions in the real-time spot market, this assumption is invalid at the forward market stage. This supposition, together with risk aversion on part of market agents, implies that there will be demand for forward trading as agents try to hedge their spot market positions.[3] As in [1], we formalize the notion of risk-averse agents by assuming that the objective of each market agent i is to maximize its expected utility of profit function, which is $E_\omega[U(\pi_i(\omega))] \equiv E[\pi_i(\omega)] - \frac{A_i}{2} Var(\pi_i(\omega))$. Here, ω is a random variable that depicts the state of the world, which is unknown to the market agent when making forward market decisions but is realized before making spot market decisions. Naturally, agent i's profit $\pi_i(\omega)$ depends on the state of the world. $A_i > 0$ is a risk-aversion parameter that can differ across agent types.

Within this framework, we have three distinct types of agents who have various interests in the markets:

- $n \in \mathcal{Z}_+$ **generators**: generator p_i has $\alpha_{p_i} > 0$ megawatts (MW) of production capacity available for any given period.[4] It can use this capacity either to generate electricity and sell it into the electricity markets or to reserve the capacity and sell it into the AS forward market. For selling the output from $X_{p_i}^S$ MW of capacity into the electricity spot market, generator p_i receives the endogenously determined electricity spot price P_X^S. At the forward stage, if the generator sell the output from $X_{p_i}^F$ MW of capacity into the electricity forward market, it receives the endogenously determined electricity forward price P_X^F. If it sells $Y_{p_i}^F$ MW of capacity into the AS forward market, the generator receives the endogenously determined per MW AS forward price P_Y^F.

- $m \in \mathcal{Z}_+$ **retailers**: retailer r_j purchases electricity from the spot and forward markets and sells it to end-use consumers in its exclusive franchise area at a fixed unit price of $P_{r_j} \geq 0$. The total retail demand for electricity in its area, $X_{r_j}(P_X^S)$, is uncertain at the time of the decision to purchase forward and must be satisfied. However, the dependence of total retail demand on the electricity spot price formalizes the fact that end-users respond to real-time fluctuations in the spot price. This

[3] If we assume that market agents are risk neutral, then there is little incentive for them to use forward contracts. Furthermore, risk aversion on part of the agents enables the model to capture intangibles that affect strategic decisions, such as financial distress costs.

[4] This is not really a maximal capacity, but is a parameter that indexes production costs.

approximates how end-users can be induced to perceive spot prices as suggested in [5] even if in California, for example, most end-use consumers are guaranteed fixed per unit prices. The retailer takes the risk of purchasing from a volatile market, which would seem to imply that retailers would like to purchase forward contracts to lock in their purchase prices. Hence, retailer r_j's purchases in the spot and forward markets ($X^S_{r_j}$ and $X^F_{r_j}$, respectively) are used to meet its retail demand.

- **an ISO**: the ISO procures enough AS from the forward market to comply with the minimum levels required for reliability, Y_I. Usually, this implies that the amount of AS procured by the ISO is approximately a fixed percentage of overall electricity demand. The ISO, thus, acquires enough AS from the forward market (Y^F_I) to meet its requirements.

As we shall show in section 4, all agents act out of self-interest in order to maximize their respective expected utilities of wealth. Their interaction in the markets then determines equilibrium prices and positions for electricity and AS, which we analyze to determine how they are affected by allowing for price-elastic end-user demand.

4. Market Trading with Single-Stage Price Settlement

Here, we solve the optimization problems of the agents introduced in section 3. We use the market-equilibrium approach developed in [3] and extended in [20] to incorporate AS trading. With two types of markets, i.e., forward and spot, we have two time stages in the model. At the forward market stage, we assume that agents maximize their respective expected utilities of wealth without knowledge of spot market conditions. Only at the spot market stage is ω revealed, and given the forward market transactions, the agents conduct spot market transactions. In order to solve this model, we proceed backwards by first evaluating the agents' spot market problems given that uncertainty has been resolved and that forward transactions are fixed. We then step back in time to determine the optimal forward quantities traded and the equilibrium prices.

Whereas in [3] and [20] the demand faced by any given retailer was inelastic, here we incorporate the approach of [1] and [2] by allowing demand to vary with price. In order to keep the analysis tractable, we specify demand to be a linear function of price. In this section, we use a *single-stage settlement* rule by allowing only the spot price to affect demand. Later on in section 5, we allow for a *two-stage settlement* rule in which both forward and spot electricity prices affect demand.

Our model addresses AS trading by requiring the ISO to purchase a certain amount of them in order to maintain system reliability during grid contingencies. While the generators have enough capacity to satisfy the total system retail demand, they are, nevertheless, induced to reserve some capacity for reliability purposes through the AS market. As in most deregulated electricity industries, we assume that the ISO purchases

AS equal to a fixed percentage of expected total retail demand. If any contingencies occur, the ISO calls upon these reserves to generate. Hence, AS resemble call options for electricity that generators and the ISO uses to mitigate physical and financial risk.[5]

4.1. Spot Trading

At the spot market stage, ω has been revealed so agents approach their optimization problems without any uncertainty. Furthermore, because all forward positions ($X_{p_i}^{F*}$, $Y_{p_i}^{F*}$, $X_{r_j}^{F*}$, and Y_I^{F*}) and prices (P_X^{F*} and P_Y^{F*}) have been determined, we treat them as fixed. Hence, the only decision to be taken at this stage is how much to transact in the spot market.

Applying the notation and assumptions of section 3, the profit-maximization problem of **generator** p_i as follows:

$$\pi_{p_i}^*(\omega, X_{p_i}^{F*}, Y_{p_i}^{F*}) = \max_{X_{p_i}^S}\{P_X^S X_{p_i}^S + P_X^{F*} X_{p_i}^{F*} + P_Y^{F*} Y_{p_i}^{F*} - \frac{\theta}{2\alpha_{p_i}}(X_{p_i}^S + X_{p_i}^{F*} + fY_{p_i}^{F*})^2\} \quad (1)$$

where $\pi_{p_i}^*(\cdot)$ is the maximized profit level, $\theta > 0$ is the per MW input (e.g., fuel) cost, and $0 \leq f \leq 1$ denotes the fraction of AS capacity sold that is called upon to generate.[6] Fuel cost is incurred only for actual electricity generation, i.e., to produce electricity sold as energy, and to operate any AS capacity that is specifically required by the ISO to generate in response to grid contingencies. Furthermore, the cost term exhibits the quadratic form, which implies increasing marginal costs of generation. Intuitively, this models the fact that as demand increases, less efficient sources of generation are brought on line. For the purposes of this model, we assume that continuous quadratic functions reasonably approximate generation costs, even though actual generation costs may be discontinuous.

The profit-maximization problem of **retailer** r_j is:

$$\begin{aligned} \pi_{r_j}^*(\omega, X_{r_j}^{F*}) &= \max_{X_{r_j}^S} \{P_{r_j} X_{r_j} - P_X^S X_{r_j}^S - P_X^{F*} X_{r_j}^{F*}\} \\ &\quad \text{subject to } \; X_{r_j}^S + X_{r_j}^{F*} \geq X_{r_j}(P_X^S) \qquad (2) \end{aligned}$$

where $\pi_{r_j}^*(\cdot)$ is the maximized profit level, and $X_{r_j}(P_X^S) \equiv a_{r_j} - b_{r_j} P_X^S$ is the realized total electricity demand in the franchise area of retailer r_j. This demand is linear in the spot price and deviates from its maximum possible value, a_{r_j}, in proportion to end-user responsiveness, $b_{r_j} > 0$.[7] While a_{r_j} is stochastic at the forward stage, b_{r_j} is always

[5] Whether or not the "option" is exercised, however, is dependent solely upon grid conditions. Unlike a call option in the financial literature, this "option" can only be exercised by the ISO during a grid contingency. In fact, during contingencies, the option *must* be exercised.

[6] Usually, per MW input costs vary with the level of production, but we abstract from that in order to maintain the tractability of the model. In addition, we assume that generators have sufficient capacity to meet system demand.

[7] This indexes the price elasticity of retail demand.

deterministic. As in most decentralized systems, the AS in our model are procured by the **ISO** to fulfill system requirements, and if they are called upon to generate, the output from the AS reserves are used by the ISO to satisfy grid contingencies. Since the ISO has no role in real-time, we defer the presentation of its optimization problem to section 4.2.

Since generator p_i's problem is to decide how much electricity to sell into the spot market in order to maximize its profit, its **first-order necessary condition** is:

$$\begin{aligned} \frac{\partial \pi^*_{p_i}(\omega, X^{F*}_{p_i}, Y^{F*}_{p_i})}{\partial X^S_{p_i}} &= 0 \\ \Rightarrow P^S_X - \frac{\theta}{\alpha_{p_i}}(X^{S*}_{p_i} + X^{F*}_{p_i} + fY^{F*}_{p_i}) &= 0 \\ \Rightarrow X^{S*}_{p_i} = \frac{\alpha_{p_i}}{\theta} P^S_X - X^{F*}_{p_i} - fY^{F*}_{p_i} \end{aligned} \quad (3)$$

The **second-order sufficiency condition** for this problem is also satisfied:

$$\frac{\partial^2 \pi^*_{p_i}(\omega, X^{F*}_{p_i}, Y^{F*}_{p_i})}{\partial X^{S\,2}_{p_i}} = -\frac{\theta}{\alpha_{p_i}} < 0. \quad (4)$$

Hence, there is a global maximum to generator p_i's problem, which is achieved by offering $X^{S*}_{p_i}$ MWs of electricity for sale in the spot market. Retailer r_j, on the other hand, has little choice in selecting its purchase quantity because it *must* satisfy the retail demand in its area, $a_{r_j} - b_{r_j} P^S_X$. Its spot market purchases are, therefore, equal to the retail demand in its area less its forward purchases of electricity, i.e., $X^{S*}_{r_j} = a_{r_j} - b_{r_j} P^S_X - X^{F*}_{r_j}$.

Using equation 3 together with the retailer's purchase requirement, we now solve for the equilibrium electricity spot price. In our model, the **market-clearing conditions** are:

$$\sum_{i=1}^{n} X^{S*}_{p_i} + \sum_{i=1}^{n} fY^{F*}_{p_i} = \sum_{j=1}^{m} X^{S*}_{r_j} \quad (5)$$

$$\sum_{i=1}^{n} X^{F*}_{p_i} = \sum_{j=1}^{m} X^{F*}_{r_j} \quad (6)$$

$$\sum_{i=1}^{n} Y^{F*}_{p_i} = Y^{F*}_{I} \quad (7)$$

Equation 5 states that in order for an equilibrium to occur in the electricity spot market, the total sales by the generators *plus* the total AS calls equal the total purchases by the retailers. Similarly, equations 6 and 7 ensure that total supply equals total demand in the forward markets for electricity and AS, respectively.

Solving for the equilibrium spot price, we obtain:

$$P^{S*}_X = \frac{\theta a}{\alpha + \theta b} \quad (8)$$

where $\alpha \equiv \sum_{i=1}^{n} \alpha_{p_i}$, $a \equiv \sum_{i=1}^{n} a$, $b \equiv \sum_{i=1}^{n} b$, and total system retail demand is $X_R \equiv \sum_{j=1}^{m} X_{r_j} = \frac{\alpha}{\alpha+\theta b} a$. The details of this derivation are left for appendix A. Intuitively, the electricity spot price is simply the pro-rated cost of meeting the overall electricity retail demand. Since this is a perfectly competitive market with uncertainty having been resolved, we would expect all generators to be compensated at the marginal cost as they are here. The implication of a price-elastic demand at this stage is that the equilibrium spot price is lower here than in [20] as end-users respond to it by reducing consumption. By letting $b_{r_j} = 0, for j = 1, \ldots, m$, we recover the spot price from [20].

By substituting equation 8 into equation 3 and the retailer's purchase requirement, we obtain the optimal quantities sold and purchased in the spot market by generator p_i and retailer r_j, respectively:

$$X_{p_i}^{S*} = \frac{\alpha_{p_i} a}{\alpha + \theta b} - X_{p_i}^{F*} - f Y_{p_i}^{F*} \tag{9}$$

and

$$X_{r_j}^{S*} = a_{r_j} - \frac{b_{r_j} \theta a}{\alpha + \theta b} - X_{r_j}^{F*} \tag{10}$$

Compared to the case with no price response, here both quantities are reduced. In particular, generator p_i's equilibrium output is its pro-rated share of the total system retail demand (which is now reduced due to price elasticity) less its forward commitments. Similarly, retailer r_j's equilibrium purchase is the retail demand in its area (again, this is lower than in the case without price elasticity) less the quantity purchased forward.

4.2. Forward Trading

After having analyzed the spot market transactions, we now step back in time to evaluate the agents' forward transactions. By maximizing their respective expected utilities of profit, the agents reveal the quantities of electricity and AS that they transact through the forward market. Applying the market- equilibrium conditions, we then assess equilibrium forward prices for both electricity and AS to examine the impact of price-elastic demand. Unlike spot market trading, forward market analysis is conducted in face of uncertainty about the state of the world. Specifically, the random variable ω is not known at this stage.

Accounting for this uncertainty, we express generator p_i's profit as:

$$\begin{aligned} \pi_{p_i}(\omega) = \quad & P_X^{S*}(\omega) X_{p_i}^{S*}(\omega) + P_X^F X_{p_i}^F + P_Y^F Y_{p_i}^F \\ & - \frac{\theta}{2\alpha_{p_i}} (X_{p_i}^{S*}(\omega) + X_{p_i}^F + f(\omega) Y_{p_i}^F)^2 \end{aligned} \tag{11}$$

Setting $X_{p_i}^F = 0$ and $Y_{p_i}^F = 0$, we define the *unhedged* profit level:

$$\rho_{p_i}^*(\omega) = P_X^{S*}(\omega) X_{p_i}^{S*}(\omega) - \frac{\theta}{2\alpha_{p_i}} X_{p_i}^{S*^2}(\omega) \tag{12}$$

Substituting in equation 9 with $X^{F*}_{p_i} = 0$ and $Y^{F*}_{p_i} = 0$, we obtain:

$$\rho^*_{p_i}(\omega) = \frac{\theta}{\alpha} X_R(\omega) \frac{\alpha_{p_i}}{\alpha} X_R(\omega) - \frac{\theta}{2\alpha_{p_i}} (\frac{\alpha_{p_i}}{\alpha} X_R(\omega))^2$$

$$\Rightarrow \quad \rho^*_{p_i}(\omega) = \frac{\alpha_{p_i}\theta}{2\alpha^2} X_R^2(\omega) \tag{13}$$

By using equations 8 and 13 together with equation 9 as usual, we obtain:

$$\pi_{p_i}(\omega) = \quad \rho^*_{p_i}(\omega) + X^F_{p_i}(P^F_X - P^{S*}_X(\omega)) + Y^F_{p_i}(P^F_Y - f(\omega)P^{S*}_X(\omega)) \tag{14}$$

Employing the expected utility of profit function with the same absolute risk-aversion parameter $A_P > 0$ for all generators, we express generator p_i's forward stage optimization problem:

$$\max_{X^F_{p_i}, Y^F_{p_i}} \{E[\rho^*_{p_i}(\omega)] + X^F_{p_i}(P^F_X - E[P^{S*}_X(\omega)]) + Y^F_{p_i}(P^F_Y - E[f(\omega)P^{S*}_X(\omega)])$$
$$-\frac{A_P}{2} Var(\rho^*_{p_i}(\omega) + X^F_{p_i}(P^F_X - P^{S*}_X(\omega)) + Y^F_{p_i}(P^F_Y - f(\omega)P^{S*}_X(\omega)))\} \tag{15}$$

The first-order necessary conditions imply:

$$X^{F*}_{p_i} = \frac{P^F_X - E[P^{S*}_X(\omega)]}{A_P Var(P^{S*}_X(\omega))} + \frac{Cov(\rho^*_{p_i}(\omega), P^{S*}_X(\omega))}{Var(P^{S*}_X(\omega))} - \frac{Y^{F*}_{p_i} Cov(P^{S*}_X(\omega), f(\omega)P^{S*}_X(\omega))}{Var(P^{S*}_X(\omega))} \tag{16}$$

and

$$Y^{F*}_{p_i} = \frac{P^F_Y - E[f(\omega)P^{S*}_X(\omega)]}{A_P Var(f(\omega)P^{S*}_X(\omega))} + \frac{Cov(\rho^*_{p_i}(\omega), f(\omega)P^{S*}_X(\omega))}{Var(f(\omega)P^{S*}_X(\omega))} - \frac{X^{F*}_{p_i} Cov(P^{S*}_X(\omega), f(\omega)P^{S*}_X(\omega))}{Var(f(\omega)P^{S*}_X(\omega))} \tag{17}$$

Intuitively, equation 16 indicates that generator p_i increases its forward sales of electricity either:

- in response to an increase in the forward price relative to the expected spot price, or
- to reduce the covariation of its unhedged profit with the spot price

Moreover, its electricity forward sales decrease if it increases its AS commitments. Analogously, its forward sales of AS as described by equation 17 are motivated by the desire for higher mean profit and lower variance of profit. We leave the derivation of these equations for appendix B along with verification of the second-order sufficiency conditions. Note that in order to satisfy these, we assume that the fraction of AS required to generate, $f(\omega)$, is independent of the total system retail demand, $X_R(\omega)$.

Solving equations 16 and 17 simultaneously, we isolate expressions for the amount of electricity and AS sold forward by generator p_i:

$$X^{F*}_{p_i} = \frac{1}{Z(f(\omega), a(\omega), \theta, \alpha, b)} [\frac{(P^F_X - E[P^{S*}_X(\omega)]) Var(f(\omega)a(\omega))}{A_P}$$
$$+ Cov(\rho^*_{p_i}(\omega), P^{S*}_X(\omega)) Var(f(\omega)a(\omega)) - \frac{(P^F_Y - E[f(\omega)P^{S*}_X(\omega)]) Cov(a(\omega), f(\omega)a(\omega))}{A_P}$$
$$- Cov(\rho^*_{p_i}(\omega), f(\omega)P^{S*}_X(\omega)) Cov(a(\omega), f(\omega)a(\omega))] \tag{18}$$

and

$$
\begin{aligned}
Y_{p_i}^{F*} = \tfrac{1}{Z(f(\omega),a(\omega),\theta,\alpha,b)}[\tfrac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(a(\omega))}{A_P} \\
+Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Var(a(\omega)) - \tfrac{(P_X^F - E[P_X^{S*}(\omega)])Cov(a(\omega),f(\omega)a(\omega))}{A_P} \\
-Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Cov(a(\omega), f(\omega)a(\omega))]
\end{aligned} \tag{19}
$$

where

$$
Z(f(\omega), a(\omega), \theta, \alpha, b) \equiv \frac{\theta^2}{(\alpha + \theta b)^2}\left[Var(a(\omega))Var(f(\omega)a(\omega)) - Cov^2(a(\omega), f(\omega)a(\omega))\right]
$$

(which also equals $\frac{\theta^2}{(\alpha+\theta b)^2}Var(f(\omega))Var(a(\omega))E[a^2(\omega)]$). These expressions amplify the intuition of equations 16 and 17: generator p_i increases forward sales of one product if either its forward price increases relative to its expected spot price or the covariance between its spot price and unhedged profits increases. Conversely, it reduces forward sales of one product if the other product either becomes relatively more profitable or offers greater relative risk hedging opportunities. We leave derivation of these equations for appendix C and now consider retailer r_j's forward trading decisions.

Using the description from section 4.1 and substituting in the binding constraint from equation 2, we express retailer r_j's profit as:

$$
\begin{aligned}
\pi_{r_j}(\omega) = \quad & P_{r_j}X_{r_j}(P_X^{S*}(\omega)) - P_X^F X_{r_j}^F - P_X^{S*}(\omega)X_{r_j}^{S*}(\omega) \\
\Rightarrow \pi_{r_j}(\omega) = \quad & P_{r_j}X_{r_j}(P_X^{S*}(\omega)) - P_X^F X_{r_j}^F - P_X^{S*}(\omega)(a_{r_j}(\omega) \\
& -b_{r_j}P_X^{S*}(\omega) - X_{r_j}^F) \\
\Rightarrow \pi_{r_j}(\omega) = \quad & (P_{r_j} - P_X^{S*}(\omega))(a_{r_j}(\omega) - b_{r_j}P_X^{S*}(\omega)) + (P_X^{S*}(\omega) - P_X^F)X_{r_j}^F
\end{aligned} \tag{20}
$$

Letting $\rho_{r_j}^*(\omega) \equiv (P_{r_j} - P_X^{S*}(\omega))[a_{r_j}(\omega) - b_{r_j}P_X^{S*}(\omega)]$ be the unhedged profit level for retailer r_j, we rewrite equation 20 as:

$$
\pi_{r_j}(\omega) = \quad \rho_{r_j}^*(\omega) + (P_X^{S*}(\omega) - P_X^F)X_{r_j}^F \tag{21}
$$

The optimization problem of retailer r_j is to select the amount of electricity to purchase (or sell) forward in order to maximize its expected utility of profit, where $E_\omega[U(\pi_{r_j}(\omega))] \equiv E[\pi_{r_j}(\omega)] - \frac{A_R}{2}Var(\pi_{r_j}(\omega))$ and $A_R > 0$, the retail analog of A_P, is common to all retailers. Mathematically, this becomes:

$$
\begin{aligned}
\max_{X_{r_j}^F}\{E[\rho_{r_j}^*(\omega)] + X_{r_j}^F(E[P_X^{S*}(\omega)] - P_X^F) - \frac{A_R}{2}[Var(\rho_{r_j}^*(\omega)) \\
+X_{r_j}^{F^2}Var(P_X^{S*}(\omega)) + 2X_{r_j}^F Cov(\rho_{r_j}^*(\omega), P_X^{S*}(\omega))]\}
\end{aligned} \tag{22}
$$

The resulting first-order necessary condition implies:

$$
X_{r_j}^{F*} = \tfrac{E[P_X^{S*}(\omega)] - P_X^F}{A_R Var(P_X^{S*}(\omega))} - \tfrac{Cov(\rho_{r_j}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))} \tag{23}
$$

Similar to equation 16, equation 23 indicates that retailer r_j's forward purchases increase in response to the bias in the spot price over the forward price. Furthermore, its forward purchases are reduced (increased) if there exists positive (negative) covariance between its unhedged profits and the electricity spot price. The second-order sufficiency condition is also satisfied:

$$\frac{\partial^2 E_\omega[U(\pi_{r_j}(X^F_{r_j}))]}{\partial X^{F^2}_{r_j}} = -A_R Var(P^{S*}_X(\omega)) < 0. \tag{24}$$

The ISO's optimization problem is different from that of generator p_i or retailer r_j. Unlike other agents, the ISO has no active role in the spot market. It merely procures enough AS from the forward market so that it equals a certain percentage, γ, of expected total retail demand. Then, if exogenous grid contingencies arise (just before spot market trading occurs), the ISO orders a fraction, $f(\omega)$, of the AS reserves to generate.[8] Therefore, because the ISO faces no tradeoff between spot and forward trading, its optimization problem is not affected by risk aversion and is simply:

$$\begin{aligned} \pi^*_I(Y^F_I) &= \max_{Y^F_I} \{-P^F_Y Y^F_I\} \\ &\quad \text{subject to } \; Y^F_I \geq Y_I \equiv \gamma E[X_R(\omega)] \end{aligned} \tag{25}$$

where $\pi^*_I(\cdot)$ is the maximized profit level, Y^F_I is the amount of AS purchased by the ISO from the forward market, $0 \leq \gamma \leq 1$ is the AS requirement as a fraction of expected total retail demand, and Y_I is its total purchase requirement. By inspecting equation 25, we see that the ISO only has to purchase enough AS forward to satisfy the forecasted reserve requirements. Hence, the ISO's transaction is:

$$\begin{aligned} Y^{F*}_I &= \gamma E[X_R(\omega)] \\ \Rightarrow Y^{F*}_I &= \frac{\gamma\alpha}{\alpha + \theta b} E[a(\omega)] \end{aligned} \tag{26}$$

From the preceding analysis, we observe that the use of electricity and AS forwards by market agents is influenced by the covariance between unhedged profits and the spot price. Indeed, equations 15 and 22 imply that generators and retailers can reduce risk as long as their respective covariance terms are non-negative. To trace the effects of these covariance terms, we evaluate them explicitly:

Lemma 1

$$Cov(\rho^*_{p_i}(\omega), P^{S*}_X(\omega)) = \frac{\alpha_{p_i}\theta^2}{2(\alpha + \theta b)^3} Cov(a^2(\omega), a(\omega))$$

[8] We assume that this generation from AS reserves is equal in proportion across all generators, i.e., if generator p_i sold $Y^{F*}_{p_i}$ MWs of AS, then the ISO orders it to generate $f(\omega)Y^{F*}_{p_i}$ MWs of electricity in the spot market (in addition to its electricity sales through the spot and forward energy markets, $X^{S*}_{p_i}$ and $X^{F*}_{p_i}$, respectively).

Proof: Using equation 8 together with equation 13, we obtain:

$$Cov(\rho^*_{p_i}(\omega), P_X^{S*}(\omega)) = Cov(\frac{\alpha_{p_i}\theta}{2(\alpha+\theta b)^2}a^2(\omega), \frac{\theta}{\alpha+\theta b}a(\omega)) \tag{27}$$

The result follows. ■

Lemma 2

$$Cov(\rho^*_{p_i}(\omega), f(\omega)P_X^{S*}(\omega)) = \frac{\alpha_{p_i}\theta^2}{2(\alpha+\theta b)^3}Cov(a^2(\omega), f(\omega)a(\omega))$$

Proof: This follows from lemma 1. ■

Lemma 3

$$\begin{aligned} Cov(\rho^*_{r_j}(\omega), P_X^{S*}(\omega)) = \; & \frac{\theta P_{r_j}}{\alpha+\theta b}Cov(a_{r_j}(\omega), a(\omega)) \\ & -\frac{\theta^2}{(\alpha+\theta b)^2}Cov(a_{r_j}(\omega)a(\omega), a(\omega)) \\ & +\frac{\theta^3 b_{r_j}}{(\alpha+\theta b)^3}Cov(a^2(\omega), a(\omega)) \\ & -\frac{\theta^2 b_{r_j}P_{r_j}}{(\alpha+\theta b)^2}Var(a(\omega)) \end{aligned}$$

Proof:

$$\begin{aligned} \rho^*_{r_j}(\omega) &= (P_{r_j} - P_X^{S*}(\omega))[a_{r_j}(\omega) - b_{r_j}P_X^{S*}(\omega)] \\ \Rightarrow \rho^*_{r_j}(\omega) &= (P_{r_j} - \frac{\theta}{\alpha+\theta b}a(\omega))[a_{r_j}(\omega) - b_{r_j}P_X^{S*}(\omega)] \\ \Rightarrow Cov(\rho^*_{r_j}(\omega), P_X^{S*}(\omega)) &= Cov(P_{r_j}a_{r_j}(\omega) - \frac{\theta}{\alpha+\theta b}a(\omega)a_{r_j}(\omega) \\ & \quad +\frac{\theta^2 b_{r_j}}{(\alpha+\theta b)^2}a^2(\omega) - \frac{b_{r_j}P_{r_j}\theta}{\alpha+\theta b}a(\omega), \frac{\theta}{\alpha+\theta b}a(\omega)) \end{aligned} \tag{28}$$

The result then follows. ■

By substituting lemmas 1, 2, and 3 into equations 18, 19, and 23, we obtain optimal reaction functions for generator p_i and retailer r_j:

$$X_{p_i}^{F*} = \frac{1}{Z(f(\omega), a(\omega), \theta, \alpha, b)}[\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)a(\omega))}{A_P}$$

$$
\begin{aligned}
&+\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega))Var(f(\omega)a(\omega))}{2(\alpha+\theta b)^3} \\
&-\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P} \\
&-\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}] \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
Y_{p_i}^{F*} \;=\; & \frac{1}{Z(f(\omega), a(\omega), \theta, \alpha, b)}[\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(a(\omega))}{A_P} \\
&+\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Var(a(\omega))}{2(\alpha+\theta b)^3} \\
&-\frac{(P_X^F - E[P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P} \\
&-\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}] \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
X_{r_j}^{F*} \;=\; & \frac{E[P_X^{S*}(\omega)] - P_X^F}{A_R Var(P_X^{S*}(\omega))} - \frac{(\alpha+\theta b)P_{r_j} Cov(a_{r_j}(\omega), a(\omega))}{\theta Var(a(\omega))} \\
&+\frac{Cov(a(\omega)a_{r_j}(\omega), a(\omega))}{Var(a(\omega))} - \frac{\theta b_{r_j} Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))} + b_{r_j}P_{r_j} \qquad (31)
\end{aligned}
$$

Intuitively, generator p_i increases its forward electricity sales if either the forward price is higher than the expected spot price or spot market profit covaries positively with the spot market price. Since the latter implies extreme positive retail demand realizations, i.e., intervals during which retailers avoid purchases at high spot prices, generator p_i is induced to increase its forward sales. Similarly, it reduces forward electricity sales if AS are relatively more lucrative. Its AS forward sales are analogously affected by the relative values of the AS forward price and expected spot price, the desire for removing covariation in spot market profit, and the relative attraction of electricity forward sales.

Meanwhile, retailer r_j increases electricity forward purchases if either the electricity forward price is less than the expected spot price or extreme positive demand realizations in its area covary positively with those for the entire industry. Alternatively, it reduces its forward purchases if retail revenues increase with the spot price. The effect of price elasticity is to induce both an increase (because its retail revenues now covary more with the spot price) and a decrease (because end-users reduce consumption) in the demand for forwards. As in [3], retailers have some degree of differentiation due to their varying levels of end-user price responsiveness and correlation of local demand with industry-wide demand. Producers, on the other hand, are homogenous because they all face the same marginal cost of production. This heterogeneity among retailers is what drives their desire for risk reduction. We now examine the impact this has on forward prices.

4.3. Equilibrium Forward Prices

By using the market-clearing conditions (equations 5, 6, and 7) together with the agents' optimal forward reaction functions (equations 29, 30, 31, and 26), we assess the equilibrium forward prices for electricity and AS:

$$\begin{aligned} P_X^{F*} &= E[P_X^{S*}(\omega)] + \frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta} \\ &\quad -\frac{\theta}{\alpha\eta}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] Var(X_R(\omega)) \end{aligned} \tag{32}$$

and

$$\begin{aligned} P_Y^{F*} &= E[f(\omega)P_X^{S*}(\omega)] + \frac{E[f(\omega)]\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta} \\ &\quad -\frac{\theta}{\alpha\eta}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] E[f(\omega)]Var(X_R(\omega)) \\ &\quad +\frac{\theta\gamma E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]}{\eta'\alpha} \end{aligned} \tag{33}$$

Here, $\eta \equiv \frac{n}{A_P} + \frac{m}{A_R}$ reflects the number of firms trading in the electricity markets and their degree of risk aversion; $\eta' \equiv \frac{n}{A_P}$ reflects the number of generators and their degree of risk aversion; $\beta_{r_j} \equiv \frac{Cov(X_{r_j}(\omega), X_R(\omega))}{Var(X_R(\omega))}$ is the extent to which demand in retailer r_j's franchise area is correlated with total retail demand; and

$$\beta'_{r_j} \equiv \beta_{r_j}\left(\frac{\alpha+\theta b}{\alpha}\right) - \frac{\theta}{\alpha}b_{r_j} \tag{34}$$

The latter term accounts for the change in the covariation of retailer r_j's demand with total retail demand due to price elasticity. We leave derivation of the prices for appendix D and now discuss their intuitive properties.

Equation 32 is similar in structure to the electricity forward price in [20]. Specifically, the forward price differs from the expected spot price by two terms related to statistical aspects of the total retail demand. The skewness of total retail demand increases the forward price from the expected spot price because of the retailers' desire to avoid spot market purchases during times of extreme positive demand realizations. Indeed, if the skewness term were positive, then such realizations would be more likely to occur than demand realizations that were below the mean. As a result, retailers would be more likely to make spot market purchases precisely when then spot price is high (because by equation 8, the spot price varies directly with total retail demand). Consequently, retailers respond to this by shifting their electricity purchases from the spot to the forward market. This then induces an increase in the quantity of electricity supplied forward by generators and results in the forward price's being increased from the expected spot price.

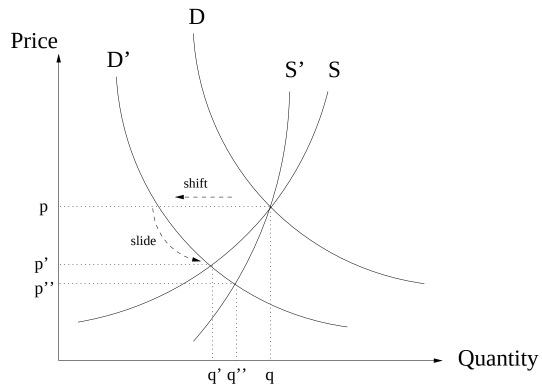

Figure 1: Impact of Increased Retailer Spot Market Profit on Forward Price

In contrast, more profitable spot market retailing decreases the forward price from the expected spot price. This is because retailers try to hedge risky spot market conditions by selling forward (or equivalently, decreasing their forward purchases) to create an offsetting exposure. As retailers decrease forward purchases, the forward price decreases in proportion to the variability of industry-wide demand. Intuitively, the greater the volatility in total retail demand (and by extension, in the spot price), the less likely are generators to decrease their quantity of electricity supplied forward when its demand decreases. Hence, without an offsetting decrease in the quantity of electricity supplied in the forward market, the forward price of electricity plummets in comparison to a case in which generator are more willing to reduce forward output. Figure 1 illustrates this effect, with supply curve S' representing the state of the world with a more volatile spot market. The shift in demand from D to D' causes quantity supplied to decrease more and price to decrease less with supply curve S, the one that represents a less volatile spot market, than with S'. In the case of low spot market volatility, generators are more willing to offset the decreased demand for forwards, so the equilibrium point (p'', q'') is reached by sliding down demand curve D' until supply curve S' is encountered.

By maintaining the reliability of the grid through AS purchases, the ISO indirectly impacts the electricity forward price. When the AS reserves procured by the ISO are called upon to generate in the spot market, generators are compensated for this production by retailers, who pay an extra $\gamma E[f(\omega)]E[P_X^{S*}(\omega)]$ for spot market electricity. This induces generators to shift some of their capacity away from the production of forward electricity and towards the provision of AS. Hence, the contraction in supply of forward electricity puts upward pressure on the electricity forward price which then increases the quantity of electricity supplied forward by generators. In equilibrium, then, the electricity forward price is higher than it is in [3].

The effect of end-user price elasticity on the electricity forward price is twofold:

- a *direct* effect which increases the forward price
- an *indirect* effect which decreases the forward price

From equation 34, we note that the former effect arises directly from the fact that end-users now respond to fluctuations in the spot price, corresponding to the $\frac{\theta}{\alpha} b_{r_j}$ term. This in turn makes retail revenues more dependent on the spot price, which induces retailers to increase forward purchases of electricity to offset this increased spot market risk exposure. The resulting increase in demand for forward electricity then drives up its equilibrium price. The consequence of price responsiveness, however, is that the electricity spot price is now lower than it would be with inelastic demand. Therefore, this decreases electricity demanded forward, which then decreases the equilibrium forward price in proportion to the correlation of local demand with industry-wide demand. Indeed, the more its local demand varies with industry-wide demand, the more retailer r_j is affected by the decrease in the spot price. This industry-wide phenomenon can be traced to the $\frac{\alpha+\theta b}{\alpha}$ term in equation 34, which reflects the decreased spot price. Hence, the effect of price elasticity on the electricity forward price is ambiguous since it depends entirely upon whether the direct or indirect effect is stronger.

We can, however, determine under which circumstances either effect will dominate. By thinking of $\frac{b_{r_j}}{b}$ as the *relative* price elasticity of end-users in retailer r_j's area and using equation 34, we conclude that the indirect effect will dominate if $\beta_{r_j} > \frac{b_{r_j}}{b}$. In such a scenario, the correlation of local demand with industry-wide demand is greater than the relative price elasticity. As a result, the decrease in spot price impacts retailer r_j more than it does a retailer with either a relatively insulated or price responsive end-user demand, thereby leading to a decrease in forward purchases. Conversely, if $\beta_{r_j} < \frac{b_{r_j}}{b}$, then retailer r_j has either relatively price-elastic or relatively insulated end-user demand. This implies its retail revenues vary more with the spot price, so it increases forward purchases to offset this exposure.

Intuitively, if β_{r_j} is large relative to $\frac{b_{r_j}}{b}$, then the price responsiveness of end-users in *other* retailers' areas is chiefly responsible for the decrease in spot price. In effect, retailer r_j is put in a position of simply reacting to the price responsiveness of others by reducing its own forward purchases. The high correlation of its local demand with industry-wide demand forces it to do so. By contrast, if β_{r_j} is small relative to $\frac{b_{r_j}}{b}$, then retailer r_j has the luxury of not being forced to react to the decreased spot price. In this case, it is more motivated by the direct effect of price elasticity, which increases the risk exposure of its retail revenues to the spot price. Indeed, its own end-users reduce consumption and decrease the spot price, therefore, inducing it to increase its forward purchases.

In addition, end-user price responsiveness impacts the forward price by decreasing the total retail electricity demand compared to its level in [20]. This then reduces the impact of both the skewness and variance terms in equation 32. By letting $b_{r_j} \to 0$ for $j = 1, \ldots, m$ in equation 32, we recover the result of [20] in which there is no end-user price response.

Since its structure is similar to that of equation 32, equation 33 retains many of the aforementioned intuitive properties. In order to gain more insight into AS pricing, we rewrite equation 33 as:

$$P_Y^{F*} = \frac{\theta\gamma}{\eta'\alpha} E[P_X^{S*}(\omega)] Var(f(\omega)) E[X_R^2(\omega)] + E[f(\omega)] P_X^{F*} \tag{35}$$

We observe that the per MW AS forward price has two terms, the first of which is a *capacity payment* that compensates the generator (at the electricity spot price) for its opportunity costs. This reflects the fact that by reserving capacity instead of offering electricity in the spot market, the generator loses revenue due to foregone electricity sales from the reserves that are not called. As compensation, regardless of whether or not the generator is called, it is paid an amount proportional to $E[P_X^{S*}(\omega)]$, which is the average market price that it could have earned from the reserved capacity that was not called by the ISO. In order to account for uncertainty, this payment is scaled by $Var(f(\omega))E[X_R^2(\omega)]$. Note that equation 35 implies that no capacity payment is necessary in a deterministic world. Indeed, in such a situation, the generator knows with certainty at the forward stage how much reserve capacity will be called by the ISO, and thus, incurs no opportunity costs. Similarly, the capacity payment is zero if the generator is risk neutral, i.e., it does not care about the volatility of its profit.

Conversely, the second term compensates the generator for electricity actually called upon to generate. Towards that end, the generator is paid the forward market electricity price when it is called upon to generate from its AS reserves. The generator is compensated at the forward price because the ISO is effectively contracting forward for energy. Moreover, because on average only a fraction $E[f(\omega)]$ of the reserves sold into the AS forward market will be called upon to generate, this payment compensates the generator for an equivalent amount of energy sold into the forward electricity market. Hence, by thinking of AS as call options, we interpret P_Y^{F*} in terms of the classic two-part call option payoff, which includes a guaranteed up-front payment and a contingent payment if the option is exercised.

Some sensitivity analysis reveals the extent of the impact of $f(\omega)$ on the AS price. Recall that in a deterministic world, i.e., $Var(f(\omega)) = 0$, there is no capacity payment. Consequently, if AS reserve calls are rare events, then the AS forward price decreases to zero. Intuitively, if generators are asked to reserve part of their capacity, but are never called upon to generate from it, then they have no incentive to offer AS. Indeed, in a world without grid contingencies, there is no need for an ISO. To see this, set $Var(f(\omega)) = 0$ and then take $\lim_{E[f(\omega)]\to 0} P_Y^{F*}$, using equation 35. At the other extreme, if grid contingencies occur frequently in a deterministic world, then the AS price converges to the electricity forward price. The rationale for this is that electricity and AS forwards become perfect substitutes in such a situation, hence their prices equilibrate. We arrive at this conclusion by taking $\lim_{E[f(\omega)]\to 1} P_Y^{F*}$ and using equation 35 with $Var(f(\omega)) = 0$.

4.4. Optimal Forward Positions

Using the equilibrium forward prices, we derive the optimal forward positions taken by agents in our model as derived in appendix E:

$$
\begin{aligned}
X_{p_i}^{F*} &= \frac{\alpha_{p_i}}{\alpha} E[X_R(\omega)] + \left[\frac{1}{2\eta A_P} + \frac{\alpha_{p_i}}{2\alpha}\right] \frac{Skew(X_R(\omega))}{Var(X_R(\omega))} \\
&\quad - \frac{\alpha}{\eta\theta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] \\
&\quad - \frac{\gamma E[f(\omega)] E[X_R(\omega)]}{n}
\end{aligned} \tag{36}
$$

$$
Y_{p_i}^{F*} = \frac{\gamma E[X_R(\omega)]}{n} \tag{37}
$$

$$
\begin{aligned}
X_{r_j}^{F*} &= E[X_{r_j}(\omega)] + \frac{\alpha}{\theta\eta A_R}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] \\
&\quad + \frac{Coskew(X_{r_j}(\omega), X_R(\omega))}{Var(X_R(\omega))} - \frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))} \\
&\quad - \frac{\alpha}{\theta}\beta'_{r_j}\left[P_{r_j} - E[P_X^{S*}(\omega)]\right]
\end{aligned} \tag{38}
$$

$$
Y_I^{F*} = \gamma E[X_R(\omega)] \tag{39}
$$

where β'_{r_j} is as defined in equation 34.

The decisions of all risk-averse agents are affected in part by motives for hedging, which then have primary and feedback implications for the forward positions. Consider generator p_i's forward sales of electricity in equation 36: they differ from its pro-rated share of expected total retail demand by three risk-related terms. The first, proportional to skewness of total retail demand, increases its forward sales because positively skewed demand spurs retailers to shift purchases into the forward market, thereby putting upward pressure on the forward price. This is relieved when generators increase the quantity of electricity sold forward. In contrast, the second term arises out of retailers' desire to avoid spot market risk by selling forward (or, reducing forward demand). This, however, decreases the forward price in proportion to retailers' spot market profitability, and in equilibrium, reduces the quantity of electricity sold forward by generators. Finally, the third term represents the AS reserves called upon to generate, and thus, decreases the electricity available to sell forward. The equilibrium quantity of AS reserves sold by generator p_i simply equals its pro-rated share of AS requirements, as indicated in equation 37. In sum, it equals the total AS purchased by the ISO in equation 39.

Equation 36 also captures the effect of incorporating AS and price elasticity into the model. The former was already shown to increase the forward price above its level without AS trading because the amount of electricity sold forward decreases. Due to the resulting upward pressure on the electricity price, an increase in quantity supplied forward is induced. The overall effect of AS, however, is to decrease forward sales of electricity compared to a model without AS. To see this, note that the fourth term of equation 36 accounts for the shift in the supply curve due to AS sales, whereas the last part of the third term, $\frac{\alpha}{\eta\theta A_P}\gamma E[f(\omega)]E[P_X^{S*}(\omega)]$ captures the increase in quantity supplied forward. Their sum, $\frac{-mA_P\gamma E[f(\omega)]E[X_R(\omega)]}{n^2 A_R+mnA_P}$, is negative, leading to the conclusion that AS trading reduces generator p_i's forward sales of electricity. Figure 2 describes the dynamics of the shift in equilibrium. The effect of price elasticity is less straightforward, however. On the one hand, the direct effect is to spur retailers to increase forward purchases in order to offset increased retail revenue risk exposure, which increases the forward price and induces an increase in the quantity of electricity supplied forward. The indirect effect of price elasticity, on the other hand, reduces the spot price, which decreases forward purchases by retailers and results in decreased quantity of electricity supplied forward. Overall, the effect of price elasticity is ambiguous as discussed in section 4.3.

Similarly, retailer r_j's forward purchases of electricity are motivated by risk hedging. Equation 38 indicates that retailer r_j's forward purchases deviate from the expected local demand in its area by four terms. First, its forward purchases are increased by the coskewness of local demand with industry-wide demand because a higher coskewness implies greater spot market purchase costs. In other words, retailer r_j will purchase most on the spot market precisely when the spot price is highest. It is, therefore, beneficial for it to increase forward purchases to offset the risk from such events. The cumulative effect of such a response, however, is to bid up the electricity forward price. Consequently, retailer r_j reduces its quantity of electricity purchased forward. To see this, note that $Skew(X_R(\omega)) \equiv \sum_{j=1}^{m} Coskew(X_{r_j}(\omega), X_R(\omega))$. Thus, the skewness term captures the industry-wide effect of behavior motivated by the coskewness term, as illustrated by figure 3. Here, demand shifts outward to D', thereby leading to an upward pressure on price, which is corrected as the new equilibrium point (p', q') is reached by sliding up along D' until supply curve S is reached. As discussed in section 4.3, retailer r_j offsets the risks due to increased retail profitability by selling more electricity forward, which is equivalent to decreasing its forward purchases. The fifth term of equation 38 captures this effect. However, in decreasing its forward purchases, retailer r_j puts downward pressure on the forward price which results in an increase in the quantity of electricity demanded forward. This feedback effect accounted for by the second term of equation 38 and illustrated in figure 1, where (p', q') represents the new equilibrium.

AS trading reduces retailer r_j's quantity of electricity purchased forward because generators optimally reduce their electricity forward sales in order to shift some capacity into AS provision. In terms of figure 2, the new equilibrium, (p', q'), results in decreased forward trading of electricity and a higher forward price in comparison to the levels in [3], (p, q). The effect of price elasticity not as clear, however. Its direct effect is to dampen

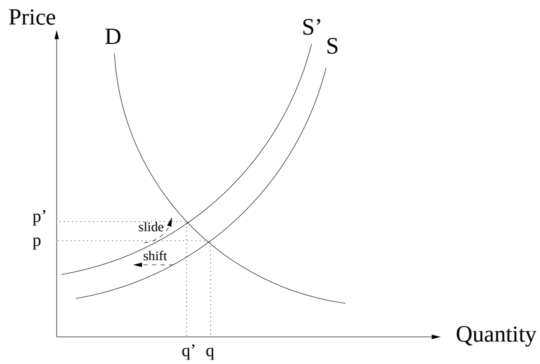

Figure 2: Effect of AS on Forward Price

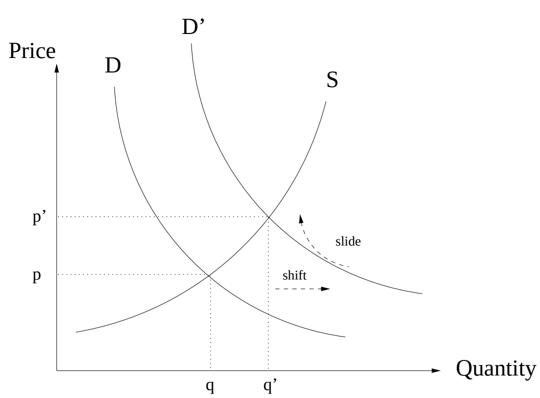

Figure 3: Impact of Coskewness of Local Demand with Industry-wide Demand on Forward Price

the fifth term in equation 38 and amplify the second one. The former arises because price elasticity induces a dependency in retailer r_j's revenues with its costs (the spot price) which it offsets by diversifying its costs, i.e., purchasing more electricity forward. Meanwhile, the latter occurs because price elasticity reduces downward pressure on the forward price, which causes the quantity of electricity purchased forward to decrease relative to the case in [20] with inelastic end-user demands. Indirectly, price elasticity increases the effect of the fifth term in equation 38 and diminishes that of the second one. The former follows from the fact that price elasticity reduces the spot price, thereby causing an even greater decrease in forward purchases which hedges the risk due to increased spot market profitability. As a feedback effect, this increases the downward pressure on the electricity forward price, hence leading to the latter effect, i.e., an increase in the quantity of electricity purchased forward. While the effect of price elasticity *may* be to increase forward purchases, recall from equation 10 that any such increases are offset by decreased spot purchases. Hence, the overall effect of price elasticity is to decrease total consumption of electricity.

5. Market Trading with Two-Stage Price Settlement

In the previous section, we examined the consequences of price elasticity on equilibrium forward prices and positions using a demand specification with single-stage settlement. This implied that end-user demand was price responsive only at the spot market stage, something that retailers had to anticipate while making forward market decisions. In this section, we allow for a two-stage settlement protocol, in which end-user demand is responsive to both the forward and spot price. Compared to section 4, the sole change is that now end-user demand depends only on the forward price at the forward stage and only on the spot price at the spot market stage. Specifically, the end-user demand encountered by retailer r_j at the forward stage is the sum of the forward and spot demand:

$$X_{r_j}(X_{F_{r_j}}(P_X^F), X_{S_{r_j}}(P_X^{S*}(\omega))) = \; X_{F_{r_j}}(P_X^F) + X_{S_{r_j}}(P_X^{S*}(\omega)) \tag{40}$$

where

$$X_{F_{r_j}}(P_X^F) = \; a_{F_{r_j}} - b_{F_{r_j}} P_X^F \tag{41}$$

and

$$X_{S_{r_j}}(P_X^{S*}(\omega)) = \; a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega) \tag{42}$$

Here, $a_{F_{r_j}} > E[a_{S_{r_j}}(\omega)]$ and $b_{F_{r_j}} P_X^F > b_{S_{r_j}}$ so that forward stage demand is more price elastic than real-time demand.

In addition, to account for how retailer r_j uses forward purchases, we partition $X_{r_j}^F$ into two variables: $X_{r_j}^{F_f}$ and $X_{r_j}^{F_s}$. The former is electricity purchased forward that is used to satisfy forward demand, $X_{F_{r_j}}(P_X^F)$, whereas the latter is that which is used to meet the residual demand occurring in the spot market, $X_{S_{r_j}}(P_X^{S*}(\omega))$. In effect, this provides retailers with an opportunity to "lock in" part of their local demand at the forward price. Doing so removes the retail revenue risk exposure described in section 4.3 that arises out of spot market price elasticity, which results in the dependence of both revenues and costs of retailers on the spot price. We now examine how two-stage demand settlement can provide retailers with a mechanism to reduce spot market risk.

5.1. Spot Market Analysis

In real-time, generator p_i's optimization problem is unaffected by the change in demand specification. As a result, equation 3 still accurately describes its spot market production. In contrast, retailer r_j's problem has to be modified to reflect the fact that not all forward purchases are used to meet real-time end-user demand. This implies that its new problem is now:

$$\begin{aligned} \pi^*_{r_j}(X^S_{r_j}) \quad &= \quad \max_{X^S_{r_j}} \{P_{r_j} X_{r_j} - P^S_X X^S_{r_j} - P^{F*}_X (X^{F_s}_{r_j} + X^{F_f}_{r_j})\} \\ &\quad\quad \text{subject to } \; X^S_{r_j} + X^{F_f*}_{r_j} \geq X_{S_{r_j}}(P^S_X) \end{aligned} \tag{43}$$

Hence, retailer r_j's optimal spot market purchase quantity is modified to:

$$X^{S*}_{r_j} = a_{S_{r_j}} - b_{S_{r_j}} P^{S*}_X - X^{F_f*}_{r_j} \tag{44}$$

In solving for the equilibrium spot price, we make use of the same market-equilibrium conditions described by equations 5, 6, and 7, except that now equation 6 is modified to:

$$\sum_{i=1}^{n} X^{F*}_{p_i} = \sum_{j=1}^{m} (X^{F_f*}_{r_j} + X^{F_s*}_{r_j}) \tag{45}$$

By inserting equations 3 and 44 into equation 45, we find that:

$$P^{S*}_X = \left[\frac{\theta}{\alpha + \theta b_S}\right] \left(a_S + X_{R_f}(P^{F*}_X)\right) \tag{46}$$

where $X_{R_f}(P^{F*}_X) \equiv \sum_{j=1}^{m} X^{F_f*}_{r_j}$. Similar to equation 8, equation 46 also states that the equilibrium spot price is proportional to total retail demand, $X_R = \frac{\alpha(a_S + X_{R_f}(P^{F*}_X))}{\alpha + \theta b_S}$. The difference is that with a two stage demand settlement protocol, total retail demand decreases from its level in the single-stage demand settlement scenario due to price responsiveness of demand at the forward stage.

By inserting equation 46 into equation 44, we also derive the equilibrium spot market purchases by retailer r_j:

$$X^{S*}_{r_j} = a_{S_{r_j}} - \frac{b_{S_{r_j}} \theta (a_S + X_{R_f}(P^{F*}_X))}{\alpha + \theta b_S} - X^{F_s*}_{r_j} \tag{47}$$

Again, this is similar to equation 47, but modified to reflect the fact that part of the total retail demand is being settled forward and is dependent on the forward price. Generator p_i's optimal spot market sales are:

$$X_{p_i}^{S*} = \frac{\alpha_{p_i}}{\alpha + \theta b_S}(a_S + X_{R_f}(P_X^{F*})) - X_{p_i}^{F*} - fY_{p_i}^{F*} \tag{48}$$

5.2. Forward Market Analysis

Similar to the approach taken in section 4.2, we formulate generator p_i's forward stage optimization problem. By using equations 46 and 48, we derive the two-stage demand settlement analog of equation 13:

$$\rho_{p_i}^*(\omega) = \frac{\alpha_{p_i}\theta}{(\alpha + \theta b_S)^2}a_S^2(\omega) - \frac{\theta}{2\alpha_{p_i}}(\frac{\alpha_{p_i}}{\alpha + \theta b_S}a_S(\omega))^2$$

$$\Rightarrow \quad \rho_{p_i}^*(\omega) = \frac{\alpha_{p_i}\theta}{2(\alpha + \theta b_S)^2}a_S^2(\omega) \tag{49}$$

We employ equations 46 and 49 together with equation 48 to obtain:

$$\begin{aligned}\pi_{p_i}(\omega) = \quad & \rho_{p_i}^*(\omega) + X_{p_i}^F(P_X^F - P_X^{S*}(\omega)) + Y_{p_i}^F(P_Y^F - f(\omega)P_X^{S*}(\omega)) \\ & + \frac{\alpha_{p_i}\theta}{(\alpha + \theta b_S)^2}X_{R_f}(P_X^{F*})a_S(\omega) + \frac{\alpha_{p_i}\theta}{2(\alpha + \theta b_S)^2}X_{R_f}^2(P_X^{F*})\end{aligned} \tag{50}$$

Putting this through the expected utility of profit function, generator p_i's optimization problem becomes:

$$\begin{aligned}& \max_{X_{p_i}^F, Y_{p_i}^F}\{E[\rho_{p_i}^*(\omega)] + X_{p_i}^F(P_X^F - E[P_X^{S*}(\omega)]) + Y_{p_i}^F(P_Y^F - E[f(\omega)P_X^{S*}(\omega)]) \\ & + \frac{\alpha_{p_i}\theta}{(\alpha + \theta b_S)^2}X_{R_f}(P_X^{F*})E[a_S(\omega)] + \frac{\alpha_{p_i}\theta}{2(\alpha + \theta b_S)^2}X_{R_f}^2(P_X^{F*}) \\ & - \frac{A_P}{2}Var(\rho_{p_i}^*(\omega) + X_{p_i}^F(P_X^F - P_X^{S*}(\omega)) + Y_{p_i}^F(P_Y^F - f(\omega)P_X^{S*}(\omega)) \\ & + \frac{\alpha_{p_i}\theta}{(\alpha + \theta b_S)^2}X_{R_f}(P_X^{F*})a_S(\omega) + \frac{\alpha_{p_i}\theta}{2(\alpha + \theta b_S)^2}X_{R_f}^2(P_X^{F*}))\}\end{aligned} \tag{51}$$

This yields the following first-order necessary conditions:

$$\begin{aligned}X_{p_i}^{F*} = & \frac{P_X^F - E[P_X^{S*}(\omega)]}{A_P Var(P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))} - \frac{Y_{p_i}^{F*}Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))} \\ & + \frac{\alpha_{p_i}\theta X_{R_f}(P_X^{F*})}{(\alpha+\theta b_S)^2 Var(P_X^{S*}(\omega))}Cov(P_X^{S*}(\omega), a_S(\omega))\end{aligned} \tag{52}$$

and

$$\begin{aligned}Y_{p_i}^{F*} = & \frac{P_Y^F - E[f(\omega)P_X^{S*}(\omega)]}{A_P Var(f(\omega)P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))} - \frac{X_{p_i}^{F*}Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))} \\ & + \frac{\alpha_{p_i}\theta X_{R_f}(P_X^{F*})}{(\alpha+\theta b_S)^2 Var(f(\omega)P_X^{S*}(\omega))}Cov(f(\omega)P_X^{S*}(\omega), a_S(\omega))\end{aligned} \tag{53}$$

Verification of the second-order sufficiency conditions is similar to that in appendix B.

By solving equations 52 and 53 simultaneously, we obtain:

$$X_{p_i}^{F*} = \frac{1}{Z'(f(\omega), P_X^{S*}(\omega))}[\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)P_X^{S*}(\omega))}{A_P}$$
$$+Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - \frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{A_P}$$
$$-Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))] + \frac{\alpha_{p_i}}{\alpha + \theta b_S} X_{R_f}(P_X^{F*}) \quad (54)$$

and

$$Y_{p_i}^{F*} = \frac{1}{Z'(f(\omega), P_X^{S*}(\omega))}[\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(P_X^{S*}(\omega))}{A_P}$$
$$+Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Var(P_X^{S*}(\omega)) - \frac{(P_X^F - E[P_X^{S*}(\omega)])Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{A_P}$$
$$-Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))] \quad (55)$$

where

$$Z'(f(\omega), P_X^{S*}(\omega)) \equiv Var(f(\omega))Var(P_X^{S*}(\omega))E[P_X^{S*^2}(\omega)]$$

While equation 55 is similar to equation 19, generator p_i's forward electricity sales here differ from those in section 4.2 by the amount of electricity demand that is settled forward.

Retailer r_j's problem is similarly modified to reflect changes in the specification of end-user demand. In particular, equation 20 becomes:

$$\begin{aligned}
\pi_{r_j}(\omega) = \quad & P_{r_j} X_{r_j}(X_{F_{r_j}}(P_X^F), X_{S_{r_j}}(P_X^{S*}(\omega))) - P_X^F(X_{r_j}^{F_f} + X_{r_j}^{F_s}) - P_X^{S*}(\omega)X_{r_j}^{S*}(\omega) \\
\Rightarrow \pi_{r_j}(\omega) = \quad & P_{r_j}[(a_{F_{r_j}} - b_{F_{r_j}} P_X^F) + (a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega))] \\
& -P_X^F(X_{r_j}^{F_f} + X_{r_j}^{F_s}) - P_X^{S*}(\omega)[(a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega)) - X_{r_j}^{F_s}] \\
\Rightarrow \pi_{r_j}(\omega) = \quad & P_{r_j}(a_{F_{r_j}} - b_{F_{r_j}} P_X^F) + (P_{r_j} - P_X^{S*}(\omega))(a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega)) \\
& +(P_X^{S*}(\omega) - P_X^F)X_{r_j}^{F_s} - P_X^F X_{r_j}^{F_f} \quad (56)
\end{aligned}$$

We now let $\rho_{r_j}^*(\omega) \equiv (P_{r_j} - P_X^{S*}(\omega))[a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega)]$ and rewrite equation 56 as:

$$\pi_{r_j}(\omega) = \quad \rho_{r_j}^*(\omega) + (P_{r_j} - P_X^F)X_{r_j}^{F_f} + (P_X^{S*}(\omega) - P_X^F)X_{r_j}^{F_s} \quad (57)$$

Implicit in the derivation of equation 57 is another market-clearing condition:

$$X_{F_{r_j}}(P_X^{F*}) = X_{r_j}^{F_f *}, j = 1, \ldots, m \quad (58)$$

Somewhat tautologically, equation 58 states that, in equilibrium, all end-user demand settled at the forward stage is satisfied by $X_{r_j}^{F_f *}$.

Using equation 57, we now express retailer r_j's optimization problem:

$$\max_{X_{r_j}^{F_s}} \{E[\rho_{r_j}^*(\omega)] + (P_{r_j} - P_X^F)X_{r_j}^{F_f} + X_{r_j}^{F_s}(E[P_X^{S*}(\omega)] - P_X^F)$$
$$-\frac{A_R}{2}[Var(\rho_{r_j}^*(\omega)) + X_{r_j}^{F_s^2} Var(P_X^{S*}(\omega)) + 2X_{r_j}^{F_s} Cov(\rho_{r_j}^*(\omega), P_X^{S*}(\omega))]\} \quad (59)$$

The first-order necessary condition implies:

$$X_{r_j}^{F_s*} = \frac{E[P_X^{S*}(\omega)] - P_X^F}{A_R Var(P_X^{S*}(\omega))} - \frac{Cov(\rho_{r_j}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))} \tag{60}$$

The ISO's problem is still accurately reflected by equation 25, but because the definition of $X_R(\omega)$ is now different, its AS purchases become:

$$\begin{aligned} Y_I^{F*} &= \gamma E[X_R(\omega)] \\ \Rightarrow Y_I^{F*} &= \frac{\gamma\alpha}{\alpha + \theta b_S}(E[a_S(\omega)] + X_{R_f}(P_X^{F*})) \end{aligned} \tag{61}$$

As in section 4.2, we now evaluate the covariances between unhedged profits and the spot price. Since the definitions of both terms are now slightly different, we modify lemmas 1, 2, 3 as follows:

Lemma 4

$$Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega)) = \frac{\alpha_{p_i}\theta^2}{2(\alpha + \theta b_S)^3} Cov(a_S^2(\omega), a_S(\omega))$$

Proof: This follows from lemma 1. ■

Lemma 5

$$Cov(\rho_{p_i}^*(\omega), f(\omega) P_X^{S*}(\omega)) = \frac{\alpha_{p_i}\theta^2}{2(\alpha + \theta b_S)^3} Cov(a_S^2(\omega), f(\omega) a_S(\omega))$$

Proof: This also follows from lemma 1. ■

Lemma 6

$$\begin{aligned} Cov(\rho_{r_j}^*(\omega), P_X^{S*}(\omega)) = \;& \frac{\theta P_{r_j}}{\alpha + \theta b_S} Cov(a_{S_{r_j}}(\omega), a_S(\omega)) \\ & - \frac{\theta^2 X_{R_f}(P_X^{F*})}{(\alpha + \theta b_S)^2} Cov(a_{S_{r_j}}(\omega), a_S(\omega)) \\ & - \frac{\theta^2}{(\alpha + \theta b_S)^2} Cov(a_{S_{r_j}}(\omega) a_S(\omega), a_S(\omega)) \\ & + \frac{\theta^3 b_{S_{r_j}}}{(\alpha + \theta b_S)^3} Cov(a_S^2(\omega), a_S(\omega)) \\ & - \frac{\theta^2 b_{S_{r_j}} P_{r_j}}{(\alpha + \theta b_S)^2} Var(a_S(\omega)) \\ & + \frac{2\theta^3 b_{S_{r_j}} X_{R_f}(P_X^{F*})}{(\alpha + \theta b_S)^3} Var(a_S(\omega)) \end{aligned}$$

Proof:

$$\begin{aligned}
\rho^*_{r_j}(\omega) &= (P_{r_j} - P_X^{S*}(\omega))[a_{S_{r_j}}(\omega) - b_{S_{r_j}} P_X^{S*}(\omega)] \\
\Rightarrow \rho^*_{r_j}(\omega) &= (P_{r_j} - \frac{\theta}{\alpha + \theta b_S}(a_S(\omega) + X_{R_f}(P_X^{F*}))[a_{r_j}(\omega) - b_{r_j} P_X^{S*}(\omega)] \\
\Rightarrow Cov(\rho^*_{r_j}(\omega), P_X^{S*}(\omega)) &= Cov(P_{r_j} a_{S_{r_j}}(\omega) - \frac{\theta}{\alpha + \theta b_S}(a_S(\omega) + X_{R_f}(P_X^{F*}))a_{S_{r_j}}(\omega) \\
&\quad + \frac{\theta^2 b_{S_{r_j}}}{(\alpha + \theta b_S)^2}[a_S(\omega) + X_{R_f}(P_X^{F*})]^2 \\
&\quad - \frac{b_{S_{r_j}} P_{r_j} \theta}{\alpha + \theta b_S}(a_S(\omega) + X_{R_f}(P_X^{F*})), \\
&\quad \frac{\theta}{\alpha + \theta b_S}(a_S(\omega) + X_{R_f}(P_X^{F*})))
\end{aligned} \tag{62}$$

The result then follows. ■

We now substitute lemmas 4, 5, and 6 into equations 54, 55, and 60 to obtain the optimal reaction functions for generator p_i and retailer r_j in the case of a two stage demand settlement system:

$$\begin{aligned}
X_{p_i}^{F*} &= \frac{1}{Z'(f(\omega), P_X^{S*}(\omega))}[\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)P_X^{S*}(\omega))}{A_P} \\
&\quad + \frac{\alpha_{p_i}\theta^2 Cov(a_S^2(\omega), a_S(\omega))Var(f(\omega)P_X^{S*}(\omega))}{2(\alpha + \theta b_S)^3} \\
&\quad - \frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(P_X^{S*}(\omega))}{A_P} \\
&\quad - \frac{\alpha_{p_i}\theta^2 Cov(a_S^2(\omega), a_S(\omega))Var(P_X^{S*}(\omega))(E[f(\omega)])^2}{2(\alpha + \theta b_S)^3}] \\
&\quad + \frac{\alpha_{p_i}}{\alpha + \theta b_S} X_{R_f}(P_X^{F*})
\end{aligned} \tag{63}$$

$$\begin{aligned}
Y_{p_i}^{F*} &= \frac{1}{Z'(f(\omega), P_X^{S*}(\omega))}[\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(P_X^{S*}(\omega))}{A_P} \\
&\quad + \frac{\alpha_{p_i}\theta^2 Cov(a_S^2(\omega), f(\omega)a_S(\omega))Var(P_X^{S*}(\omega))}{2(\alpha + \theta b_S)^3} \\
&\quad - \frac{(P_X^F - E[P_X^{S*}(\omega)])Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{A_P} \\
&\quad - \frac{\alpha_{p_i}\theta^2 Cov(a_S^2(\omega), a_S(\omega))Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{2(\alpha + \theta b_S)^3}]
\end{aligned} \tag{64}$$

$$
\begin{aligned}
X_{r_j}^{F_s*} &= \frac{E[P_X^{S*}(\omega)] - P_X^F}{A_R Var(P_X^{S*}(\omega))} - \frac{\theta P_{r_j} Cov(a_{S_{r_j}}(\omega), a_S(\omega))}{(\alpha + \theta b_S) Var(P_X^{S*}(\omega))} \\
&+ \frac{\theta^2 X_{R_f}(P_X^{F*}) Cov(a_{S_{r_j}}(\omega), a_S(\omega))}{(\alpha + \theta b_S)^2 Var(P_X^{S*}(\omega))} \\
&+ \frac{\theta^2 Cov(a_S(\omega) a_{S_{r_j}}(\omega), a_S(\omega))}{(\alpha + \theta b_S)^2 Var(P_X^{S*}(\omega))} - \frac{\theta^3 b_{S_{r_j}} Cov(a_S^2(\omega), a_S(\omega))}{(\alpha + \theta b_S)^3 Var(P_X^{S*}(\omega))} \\
&+ b_{S_{r_j}} P_{r_j} - \frac{2\theta b_{S_{r_j}} X_{R_f}(P_X^{F*})}{\alpha + \theta b_S} \qquad (65)
\end{aligned}
$$

Equation 63 differs slightly from equation 29 in that it reflects the forward settlement of some end-user demand. Similarly, equation 65 captures the impact of forward settlement on forward purchases for spot market use by retailers. Relative to equation 31, the first additional term, proportional to $X_{R_f}(P_X^{F*}) Cov(a_{S_{r_j}}(\omega), a_S(\omega))$, indicates that retailer r_j increases forward purchases for spot market use if it increases forward settlement of demand. Intuitively, increases in end-user demand settled forward increase the spot price (due to the dependence of the spot price on total retail demand), thereby increasing the chance that large spot market purchases will occur precisely when the spot price is high. In order to hedge against this risk, retailer r_j optimally increases its purchases of forwards that are to be used for real-time settlement of end-user demand.

The second additional term, $\frac{2\theta b_{S_{r_j}} X_{R_f}(P_X^{F*})}{\alpha + \theta b_S}$, reflects the fact that increased forward settlement of end-user demand reduces forward purchases needed for spot market use in proportion to retailer r_j's spot market end-user price elasticity, $b_{S_{r_j}}$. This outcome arises because when retailer r_j "locks in" more end-user demand at the forward stage, the less there is left over for the spot market stage. Hence, the need to purchase forward in order to settle demand at the spot market stage decreases. Furthermore, the more price elastic the spot market end-user demand, the greater the reduction in forwards purchased for spot market use. Indeed, spot market retail revenues decrease if spot market end-user demand is relatively more price elastic, thus making forward settlement more attractive than spot market settlement.

We now solve for $P_X^{F*'}$ and $P_Y^{F*'}$ (so denoted to distinguish them from those prices derived in section 4.3) under the two-stage demand settlement protocol by inserting equations 63 and 65 into equation 45 and equations 64 and 61 into equation 7, respectively:

$$
\begin{aligned}
P_X^{F*'} &= [\alpha b_F(1 + \gamma E[f(\omega)]) Var(P_X^{S*}(\omega)) + \eta(\alpha + \theta b_S + \theta b_F)]^{-1} [\eta\theta(E[a_S(\omega)] + a_F) \\
&+ \frac{\alpha \theta^2 Skew(a_S(\omega))}{2(\alpha + \theta b_S)^2} - \theta[\sum_{j=1}^{m} P_{r_j}(\beta_{r_j} - \frac{\theta}{\alpha + \theta b_S} b_{S_{r_j}}) \\
&- \frac{\alpha\theta}{(\alpha + \theta b_S)^2}(E[a_S(\omega)] + a_F)(1 + \gamma E[f(\omega)])] Var(a_S(\omega))] \qquad (66)
\end{aligned}
$$

and

$$
P_Y^{F*'} = [(\alpha + \theta b_S)\eta' Var(P_X^{S*}(\omega))]^{-1} [Z'(f(\omega), P_X^{S*}(\omega)) \gamma \alpha (E[a_S(\omega)] + a_F - b_F P_X^{F*'})
$$

$$+E[f(\omega)]P_X^{F*'}] \tag{67}$$

where $\beta_{r_j} \equiv \frac{Cov(a_{S_{r_j}}(\omega), a_S(\omega))}{Var(a_S(\omega))}$, $a_F \equiv \sum_{j=1}^m a_{F_{r_j}}$, and $b_F \equiv \sum_{j=1}^m b_{F_{r_j}}$.

The inclusion of forward stage end-user demand settlement decreases the forward price relative to the case in section 4.3. Recall that allowing for forward settlement not only shifts end-user demand to the forward stage, but also makes it more price elastic. The latter follows from the fact that we assume $b_{F_{r_j}} > b_{S_{r_j}}$. Consequently, in equation 66, the aggregate price elasticity of end-user demand at the forward stage, b_F, appears in the denominator, thereby scaling down the forward price in comparison to its level in equation 32. To see this, note that if we eliminate forward stage price elasticity, we essentially recover the result of section 4.3. Indeed, $\lim_{b_F \to 0} P_X^{F*'} = P_X^{F*}$, where $E[a_S(\omega) + a_F] = E[a(\omega)]$.

From the preceding discussion, we conclude that the effect of forward settlement is to reduce the electricity forward price. However, this result holds only if end-user demand at the forward stage is also price elastic. On the other hand, if it is inelastic, then end-users do not reduce demand if the forward price increases. In that case, the electricity forward price would be unaffected by forward settlement of end-user demand. Ostensibly, it is forward market price elasticity that drives the results obtained in this section.

Price elasticity at the forward stage also impacts the AS forward price. Note from equation 67 that the energy payment to generators is decreased in proportion to $b_F P_X^{F*'}$. Since forward stage price elasticity reduces total retail demand, the ISO procures less AS than it does in the case with only single stage settlement of demand. Hence, it reduces its energy payments in proportion to how much total retail demand was decreased due to price-elastic forward settlement. Again, by eliminating price elasticity at the forward stage, we recover the result of section 4.3: $\lim_{b_F \to 0} P_Y^{F*'} = P_Y^{F*}$.

6. Conclusions

One of the problems with deregulated electricity markets is hypothesized to be the absence of price responsiveness on the demand side. Unlike other competitive entities, many deregulated electricity industries are characterized as being incomplete since only suppliers are able to receive and respond to fluctuations in market prices. The resulting failure to allocate electricity without often resorting to random rationing, i.e., rolling blackouts in California, was thought to be a natural consequence of this deficiency. Allowing end-users to respond to price signals, however, is considered as a means to alleviate this inefficiency. Specifically, end-user price elasticity is supposed to decrease electricity consumption during peak hours and reduce the electricity forward price.

In order to determine the effect of price elasticity on forward prices, we use a market-equilibrium model based loosely on a decentralized paradigm, incorporating AS requirements and an ISO. Under the assumption of perfect competition, we introduce price elasticity into the demand side by allowing end-users of electricity retailers to perceive and respond to the real-time (spot) price via a linear relationship. The electricity forward price is affected by this change through two related channels. First, increased covariation between retailers' revenues and costs in the spot market induces them to increase forward purchases of electricity in order to remove this additional risk exposure. Second, price elasticity at the real-time stage reduces the spot price relative to its level with no price elasticity, which engenders retailers to decrease forward purchases since spot purchases become more attractive. Hence, the overall effect is ambiguous because the latter response decreases the electricity forward price, while the former increases it. Nevertheless, total consumption of electricity decreased as a result of price elasticity.

We next extend the model to allow end-user demand to be price responsive at the forward stage as well. This feature allows a retailer to "lock in" part of the demand in its area, thereby reducing the need to use electricity forwards for hedging purposes. The impact of this change is to decrease the equilibrium forward price relative to the case with only real-time settlement of end-user demand. This reduction is contingent, however, upon the price elasticity of forward stage end-user demand. Indeed, completely inelastic end-user demand at the forward stage implies that the retailer simply transfers part of its real-time end-user demand to the forward stage. On the other hand, price elasticity at the forward stage allows end-users to respond to fluctuations in the forward price, thus, reducing the electricity forward price. This effect also has an impact on the equilibrium AS price because it reduces the AS required to generate in response to grid contingencies.

For future work, we would like to use experimental economics to determine under which circumstances each effect of price elasticity dominates. From a theoretical point of view, our model could benefit from the introduction of an oligopolistic supply-side with a competitive fringe. Indeed, the results in this paper are driven primarily by differences among retailers, hence variation among generators is likely to add to the explanatory power of the model.

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Appendix A: Solving for the Equilibrium Spot Market Price

Substituting equation 3 and the retailers' purchase requirements into equation 5, we obtain:

$$\begin{aligned}\sum_{i=1}^{n}\tfrac{\alpha_{p_i}}{\theta}P_X^S - \sum_{i=1}^{n}X_{p_i}^{F*} - \sum_{i=1}^{n}fY_{p_i}^{F*} + \sum_{i=1}^{n}fY_{p_i}^{F*} \\ = \sum_{j=1}^{m}(a_{r_j} - b_{r_j}P_X^S) - \sum_{j=1}^{m}X_{r_j}^{F*}\end{aligned}$$

Making use of equation 6, and letting $\alpha \equiv \sum_{i=1}^{n}\alpha_{p_i}$, we arrive at the following:

$$(\tfrac{\alpha}{\theta} + b)P_X^S = a \tag{68}$$

This is equivalent to equation 8.

Appendix B: Forward Market Optimization

The first-order conditions for the generator's forward market optimization problem are:

$$\begin{aligned}&\frac{\partial E_\omega[U(\pi_{p_i}(X_{p_i}^F, Y_{p_i}^F))]}{\partial X_{p_i}^F} = 0 \\ &\Rightarrow P_X^F - E[P_X^{S*}(\omega)] - A_P X_{p_i}^F Var(P_X^{S*}(\omega)) + A_P Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega)) \\ &\quad - A_P Y_{p_i}^F Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) = 0 \\ &\Rightarrow A_P X_{p_i}^F Var(P_X^{S*}(\omega)) = P_X^F - E[P_X^{S*}(\omega)] + A_P Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega)) \\ &\quad - A_P Y_{p_i}^F Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) \\ &\Rightarrow X_{p_i}^{F*} = \frac{P_X^F - E[P_X^{S*}(\omega)]}{A_P Var(P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))} - \frac{Y_{p_i}^{F*}Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))}\end{aligned} \tag{69}$$

and

$$\begin{aligned}&\frac{\partial E_\omega[U(\pi_{p_i}(X_{p_i}^F, Y_{p_i}^F))]}{\partial Y_{p_i}^F} = 0 \\ &\Rightarrow P_Y^F - E[f(\omega)P_X^{S*}(\omega)] - A_P Y_{p_i}^F Var(f(\omega)P_X^{S*}(\omega))\end{aligned}$$

$$
\begin{aligned}
&+A_P Cov(\rho^*_{p_i}(\omega), f(\omega)P_X^{S*}(\omega)) \\
&-A_P X^F_{p_i} Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) = 0 \\
&\Rightarrow A_P Y^F_{p_i} Var(f(\omega)P_X^{S*}(\omega)) = P_Y^F - E[f(\omega)P_X^{S*}(\omega)] \\
&+A_P Cov(\rho^*_{p_i}(\omega), f(\omega)P_X^{S*}(\omega)) - A_P X^F_{p_i} Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) \\
&\Rightarrow Y^{F*}_{p_i} = \frac{P_Y^F - E[f(\omega)P_X^{S*}(\omega)]}{A_P Var(f(\omega)P_X^{S*}(\omega))} + \frac{Cov(\rho^*_{p_i}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))} \\
&-\frac{X^{F*}_{p_i} Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))}
\end{aligned} \tag{70}
$$

In order for the **second-order sufficiency conditions** to be satisfied, we make the assumption that $f(\omega)$ is independent of $P_X^{S*}(\omega)$ (and therefore, of both $X_R(\omega)$ and $a(\omega)$). Intuitively, there is no reason to believe that the *fraction* of reserves called upon to generate in real-time is affected by the real-time load. Proceeding with the analysis, we see that the hessian matrix, $H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}$, is negative definite, i.e., the determinants of the principal minors are nonzero and alternate in sign with the first ones being negative:

$$
\begin{aligned}
&H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})} \\
&= \begin{bmatrix} -A_P Var(P_X^{S*}(\omega)) & -A_P Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) \\ -A_P Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) & -A_P Var(f(\omega)P_X^{S*}(\omega)) \end{bmatrix}
\end{aligned} \tag{71}
$$

This yields the following determinants:

$$
\det(-A_P Var(P_X^{S*}(\omega))) = -A_P Var(P_X^{S*}(\omega)) < 0 \tag{72}
$$

$$
\det(-A_P Var(f(\omega)P_X^{S*}(\omega))) = -A_P Var(f(\omega)P_X^{S*}(\omega)) < 0 \tag{73}
$$

and

$$
\begin{aligned}
&\det(H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}) = A_P^2 Var(P_X^{S*}(\omega)) Var(f(\omega)P_X^{S*}(\omega)) \\
&-A_P^2 Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega)) \\
&\Rightarrow \det(H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}) = A_P^2 [Var(P_X^{S*}(\omega)) Var(f(\omega)P_X^{S*}(\omega)) \\
&-Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))] \\
&\Rightarrow \det(H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}) = A_P^2 [Var(P_X^{S*}(\omega))[Var(f(\omega)) Var(P_X^{S*}(\omega)) \\
&+(E[f(\omega)])^2 Var(P_X^{S*}(\omega)) + (E[P_X^{S*}(\omega)])^2 Var(f(\omega))] - (E[f(\omega)]E[P_X^{S*^2}(\omega)] \\
&-E[f(\omega)](E[P_X^{S*}(\omega)])^2)^2] \\
&\Rightarrow \det(H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}) = A_P^2 [Var(f(\omega))(Var(P_X^{S*}(\omega)))^2 \\
&+(E[f(\omega)])^2 (Var(P_X^{S*}(\omega)))^2 + (E[P_X^{S*}(\omega)])^2 Var(f(\omega)) Var(P_X^{S*}(\omega)) \\
&-(E[f(\omega)])^2 (Var(P_X^{S*}(\omega)))^2] \\
&\Rightarrow \det(H_{\pi_{p_i}(X^{F*}_{p_i}, Y^{F*}_{p_i})}) = A_P^2 [Var(f(\omega))(Var(P_X^{S*}(\omega)))^2
\end{aligned}
$$

$$+(E[P_X^{S*}(\omega)])^2 Var(f(\omega))Var(P_X^{S*}(\omega))]$$
$$\Rightarrow \det(H_{\pi_{p_i}(X_{p_i}^{F*},Y_{p_i}^{F*})}) = A_P^2 Var(f(\omega))Var(P_X^{S*}(\omega))[Var(P_X^{S*}(\omega)) + (E[P_X^{S*}(\omega)])^2]$$
$$\Rightarrow \det(H_{\pi_{p_i}(X_{p_i}^{F*},Y_{p_i}^{F*})}) = A_P^2 Var(f(\omega))Var(P_X^{S*}(\omega))E[P_X^{S*2}(\omega)]$$
$$\Rightarrow \det(H_{\pi_{p_i}(X_{p_i}^{F*},Y_{p_i}^{F*})}) > 0 \qquad (74)$$

Hence, there is a global maximum to generator p_i's problem.[9]

Appendix C: Solving for the Generator's Optimal Reaction Functions

We first solve simultaneously for $X_{p_i}^{F*}$ and $Y_{p_i}^{F*}$ by inserting equation 17 into equation 16:

$$X_{p_i}^{F*} = \frac{P_X^F - E[P_X^{S*}(\omega)]}{A_P Var(P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))}[\frac{P_Y^F - E[f(\omega)P_X^{S*}(\omega)]}{A_P Var(f(\omega)P_X^{S*}(\omega))}$$
$$+\frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))} - \frac{X_{p_i}^{F*} Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))}]$$
$$\Rightarrow \quad X_{p_i}^{F*} = \frac{P_X^F - E[P_X^{S*}(\omega)]}{A_P Var(P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}$$
$$+X_{p_i}^{F*}\frac{Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}$$
$$\Rightarrow \quad X_{p_i}^{F*}\left[\frac{Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}\right]$$
$$= \frac{P_X^F - E[P_X^{S*}(\omega)]}{A_P Var(P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}$$

[9]We make use of two facts concerning expressions with independent random variables A and B:

1. $Cov(A, AB) = E[B]Var(A)$.
2. $Var(AB) = Var(A)Var(B) + (E[B])^2 Var(A) + (E[A])^2 Var(B)$ (see page 89 of [19]).

$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}$$

$$\Rightarrow \quad X_{p_i}^{F*} = \frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)P_X^{S*}(\omega))}{A_P[Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))]}$$
$$+\frac{Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega))}{[Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))]}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P[Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))]}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{[Var(P_X^{S*}(\omega))Var(f(\omega)P_X^{S*}(\omega)) - Cov^2(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))]} \tag{75}$$

Then by using the fact that $Z(f(\omega), a(\omega), \theta, \alpha, b) \equiv \frac{\theta^2}{(\alpha+\theta b)^2}Var(f(\omega))Var(a(\omega))E[a^2(\omega)]$ and employing equation 8, we arrive at equation 18.

Next, by inserting equation 18 into equation 17, we obtain:

$$Y_{p_i}^{F*} = \frac{P_Y^F - E[f(\omega)P_X^{S*}(\omega)]}{A_P Var(f(\omega)P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))}$$
$$-\frac{Cov(P_X^{S*}(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))Z(f(\omega), a(\omega), \theta, \alpha, b)}[\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)a(\omega))}{A_P}$$
$$+Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Var(f(\omega)a(\omega))$$
$$-\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P}$$
$$-Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Cov(a(\omega), f(\omega)a(\omega))]$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{P_Y^F - E[f(\omega)P_X^{S*}(\omega)]}{A_P Var(f(\omega)P_X^{S*}(\omega))} + \frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)P_X^{S*}(\omega))}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))}{Var(f(\omega)a(\omega))Z(f(\omega), a(\omega), \theta, \alpha, b)}[\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)a(\omega))}{A_P}$$
$$+Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))Var(f(\omega)a(\omega))$$
$$-\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P}$$
$$-Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Cov(a(\omega), f(\omega)a(\omega))]$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{\alpha^2(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{\theta^2 A_P Var(f(\omega)a(\omega))} + \frac{\alpha^2 Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{\theta^2 Var(f(\omega)a(\omega))}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))(P_X^F - E[P_X^{S*}(\omega)])}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)}$$

$$+\frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov^2(a(\omega), f(\omega)a(\omega))}{A_P Var(f(\omega)a(\omega))Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$+\frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Cov^2(a(\omega), f(\omega)a(\omega))}{Var(f(\omega)a(\omega))Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P Var(f(\omega)a(\omega))}\left[\frac{\alpha^2}{\theta^2} + \frac{Cov^2(Xa(\omega), f(\omega)a(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)}\right]$$
$$+\frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))}{Var(f(\omega)a(\omega))}\left[\frac{\alpha^2}{\theta^2} + \frac{Cov^2(a(\omega), f(\omega)a(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)}\right]$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))(P_X^F - E[P_X^{S*}(\omega)])}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(a(\omega))}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$+\frac{Cov(\rho_{p_i}^*(\omega), f(\omega)P_X^{S*}(\omega))Var(a(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))(P_X^F - E[P_X^{S*}(\omega)])}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{Cov(a(\omega), f(\omega)a(\omega))Cov(\rho_{p_i}^*(\omega), P_X^{S*}(\omega))}{Z(f(\omega), a(\omega), \theta, \alpha, b)} \tag{76}$$

This is equivalent to equation 19.

Appendix D: Solving for Equilibrium Forward Prices

We now solve for P_X^{F*} by inserting equations 29 and 31 into equation 6:

$$\frac{n(P_X^F - E[P_X^{S*}(\omega)])Var(f(\omega)a(\omega))}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), a(\omega))Var(f(\omega)a(\omega))}{2(\alpha+\theta b)^3 Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$-\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3 Z(f(\omega), a(\omega), \theta, \alpha, b)}$$
$$= \frac{m(E[P_X^{S*}(\omega)] - P_X^F)}{A_R Var(P_X^{S*}(\omega))} - \frac{\alpha\sum_{j=1}^m P_{r_j} Cov(a_{r_j}(\omega), a(\omega))}{\theta Var(a(\omega))}$$
$$+\frac{\sum_{j=1}^m Cov(a_{r_j}(\omega)a(\omega), a(\omega))}{Var(a(\omega))} - \frac{\theta b Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))}$$

$$+\sum_{j=1}^{m} b_{r_j} P_{r_j}$$

By letting $\eta \equiv n/A_P + m/A_R$, $\eta' \equiv n/A_P$, and $\beta_{r_j} \equiv \frac{Cov(a_{r_j}(\omega),a(\omega))}{Var(a(\omega))}$, and using the fact that $Cov(a^2(\omega), a(\omega)) \equiv Skew(a(\omega)) + 2E[a(\omega)]Var(a(\omega))$ and $P_X^{S*}(\omega) = \frac{\theta}{\alpha+\theta b}a(\omega)$, we obtain:

$$(P_X^F - E[P_X^{S*}(\omega)])\left[\frac{nVar(f(\omega)a(\omega))}{A_P Z(f(\omega),a(\omega),\theta,\alpha,b)} + \frac{m\alpha^2}{A_R\theta^2 Var(a(\omega))}\right]$$
$$= \frac{Cov(a(\omega), f(\omega)a(\omega))}{Z(f(\omega),a(\omega),\theta,\alpha,b)}[\frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}] + Cov(a^2(\omega), a(\omega))[\frac{1}{Var(a(\omega))}$$
$$-\frac{\alpha\theta^2 Var(f(\omega)a(\omega))}{2(\alpha+\theta b)^3 Z(f(\omega),a(\omega),\theta,\alpha,b)} - \frac{\theta b}{(\alpha+\theta b)Var(a(\omega))}]$$
$$-\frac{(\alpha+\theta b)\sum_{j=1}^m P_{r_j}\beta_{r_j}}{\theta} + \sum_{j=1}^{m} b_{r_j}P_{r_j}$$

$$\Rightarrow \quad \frac{(P_X^F - E[P_X^{S*}(\omega)])}{Z(f(\omega),a(\omega),\theta,\alpha,b)}\left[\frac{nVar(f(\omega)a(\omega))}{A_P} + Var(f(\omega))E[a^2(\omega)]\frac{m}{A_R}\right]$$
$$= \frac{Cov(a(\omega), f(\omega)a(\omega))}{Z(f(\omega),a(\omega),\theta,\alpha,b)}[\frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}]$$
$$+\frac{Cov(a^2(\omega), a(\omega))}{Z(f(\omega),a(\omega),\theta,\alpha,b)}[\frac{\theta^2 Var(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^2} - \frac{\alpha\theta^2 Var(f(\omega)a(\omega))}{2(\alpha+\theta b)^3}$$
$$-\frac{\theta^3 bVar(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^3}] - \frac{(\alpha+\theta b)}{\theta}\sum_{j=1}^{m} P_{r_j}\beta_{r_j} + \sum_{j=1}^{m} b_{r_j}P_{r_j}$$

$$\Rightarrow \quad (P_X^F - E[P_X^{S*}(\omega)])[\frac{n}{A_P}(Var(f(\omega))Var(a(\omega)) + (E[f(\omega)])^2 Var(a(\omega))$$
$$+(E[a(\omega)])^2 Var(f(\omega))) + \frac{m}{A_R}Var(f(\omega))E[a^2(\omega)]]$$
$$= Cov(a(\omega), f(\omega)a(\omega))[\frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])}{A_P}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}] + Cov(a^2(\omega), a(\omega))[\frac{\theta^2 Var(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^2}$$
$$-\frac{\theta^3 bVar(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^3} - \frac{\theta^2\alpha}{2(\alpha+\theta b)^3}(Var(f(\omega))Var(a(\omega))$$
$$+(E[f(\omega)])^2 Var(a(\omega)) + (E[a(\omega)])^2 Var(f(\omega)))]$$

$$-\frac{(\alpha+\theta b)Z(f(\omega),a(\omega),\theta,\alpha,b)}{\theta}\sum_{j=1}^{m}P_{r_j}\beta_{r_j}+Z(f(\omega),a(\omega),\theta,\alpha,b)\sum_{j=1}^{m}b_{r_j}P_{r_j}$$

$$\Rightarrow\quad (P_X^F-E[P_X^{S*}(\omega)])[\frac{n}{A_P}Var(f(\omega))E[a^2(\omega)]+\frac{n}{A_P}(E[f(\omega)])^2Var(a(\omega))$$
$$+\frac{m}{A_R}Var(f(\omega))E[a^2(\omega)]]$$
$$=\frac{n(P_Y^F-E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2Cov(a^2(\omega),a(\omega))(E[f(\omega)])^2Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$+Cov(a^2(\omega),a(\omega))[(\frac{\theta^2}{(\alpha+\theta b)^2}-\frac{\theta^3 b}{(\alpha+\theta b)^3}-\frac{\alpha\theta^2}{2(\alpha+\theta b)^3})Var(f(\omega))E[a^2(\omega)]$$
$$-\frac{\alpha\theta^2(E[f(\omega)])^2Var(a(\omega))}{2(\alpha+\theta b)^3}]$$
$$-\frac{(\alpha+\theta b)Z(f(\omega),a(\omega),\theta,\alpha,b)}{\theta}\sum_{j=1}^{m}P_{r_j}\beta_{r_j}+Z(f(\omega),a(\omega),\theta,\alpha,b)\sum_{j=1}^{m}b_{r_j}P_{r_j}$$

$$\Rightarrow\quad (P_X^F-E[P_X^{S*}(\omega)])[\eta Var(f(\omega))E[a^2(\omega)]+\eta'(E[f(\omega)])^2Var(a(\omega))]$$
$$=\frac{n(P_Y^F-E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2Cov(a^2(\omega),a(\omega))(E[f(\omega)])^2Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$-\frac{\alpha\theta^2Cov(a^2(\omega),a(\omega))(E[f(\omega)])^2Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$+\frac{\theta^2Cov(a^2(\omega),a(\omega))Var(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^2}[1-\frac{\theta b}{\alpha+\theta b}-\frac{\alpha}{2(\alpha+\theta b)}]$$
$$-\frac{(\alpha+\theta b)Z(f(\omega),a(\omega),\theta,\alpha,b)}{\theta}\sum_{j=1}^{m}P_{r_j}\beta_{r_j}+Z(f(\omega),a(\omega),\theta,\alpha,b)\sum_{j=1}^{m}b_{r_j}P_{r_j}$$

$$\Rightarrow\quad (P_X^F-E[P_X^{S*}(\omega)])[\eta Var(f(\omega))E[a^2(\omega)]+\eta'(E[f(\omega)])^2Var(a(\omega))]$$
$$=\frac{n(P_Y^F-E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2Var(f(\omega))E[a^2(\omega)]Skew(a(\omega))}{2(\alpha+\theta b)^3}$$
$$+\frac{\alpha\theta^2Var(f(\omega))E[a^2(\omega)]E[a(\omega)]Var(a(\omega))}{(\alpha+\theta b)^3}$$
$$+Z(f(\omega),a(\omega),\theta,\alpha,b)\sum_{j=1}^{m}P_{r_j}\left[b_{r_j}-\beta_{r_j}\frac{(\alpha+\theta b)}{\theta}\right]$$

$$\Rightarrow\quad (P_X^F-E[P_X^{S*}(\omega)])[\eta Var(f(\omega))E[a^2(\omega)]+\eta'(E[f(\omega)])^2Var(a(\omega))]$$

$$= \frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2 Var(f(\omega))E[a^2(\omega)]Skew(X_R(\omega))}{2(\alpha+\theta b)^3}$$
$$-\frac{(\alpha+\theta b)Z(f(\omega),a(\omega),\theta,\alpha,b)}{\theta}\left[\sum_{j=1}^{m} P_{r_j}\beta_{r_j} - E[P_X^{S*}(\omega)]\right]$$
$$-bZ(f(\omega),a(\omega),\theta,\alpha,b)E[P_X^{S*}(\omega)] + Z(f(\omega),a(\omega),\theta,\alpha,b)\sum_{j=1}^{m} b_{r_j}P_{r_j}$$
$$\Rightarrow \quad P_X^{F*} = E[P_X^{S*}(\omega)] + \frac{n(P_Y^{F*} - E[f(\omega)P_X^{S*}(\omega)])E[f(\omega)]Var(a(\omega))}{A_P(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$+\frac{\alpha\theta^2 Var(f(\omega))E[a^2(\omega)]Skew(a(\omega))}{2(\alpha+\theta b)^3(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$-\frac{(\alpha+\theta b)Z(f(\omega),a(\omega),\theta,\alpha,b)}{\theta(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}\left[\sum_{j=1}^{m} P_{r_j}\beta_{r_j} - E[P_X^{S*}(\omega)]\right]$$
$$-\frac{Z(f(\omega),a(\omega),\theta,\alpha,b)}{\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega))}\left[bE[P_X^{S*}(\omega)] - \sum_{j=1}^{m} b_{r_j}P_{r_j}\right] \quad (77)$$

We now arrive at a similar expression for P_Y^{F*} by inserting equations 30 and 26 into equation 7:

$$\frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$-\frac{n(P_X^F - E[P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P}$$
$$-\frac{\alpha\theta^2 Cov(a^2(\omega), a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}$$
$$= \frac{\alpha\gamma}{\alpha+\theta b}E[a(\omega)]Z(f(\omega),a(\omega),\theta,\alpha,b)$$
$$\Rightarrow \quad \frac{n(P_Y^F - E[f(\omega)P_X^{S*}(\omega)])Var(a(\omega))}{A_P} = \frac{\alpha\gamma E[a(\omega)]Z(f(\omega),a(\omega),\theta,\alpha,b)}{\alpha+\theta b}$$
$$-\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$+\frac{n(P_X^F - E[P_X^{S*}(\omega)])Cov(a(\omega), f(\omega)a(\omega))}{A_P}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}$$

$$\Rightarrow \quad P_Y^F - E[f(\omega)P_X^{S*}(\omega)] = \frac{\alpha\gamma E[a(\omega)]Z(f(\omega), a(\omega), \theta, \alpha, b)}{(\alpha+\theta b)\eta' Var(a(\omega))}$$
$$+\frac{(P_X^F - E[P_X^{S*}(\omega)])Var(a(\omega))E[f(\omega)]}{Var(a(\omega))}$$
$$+\frac{\alpha\theta^2 Cov(a^2(\omega), a(\omega))Var(a(\omega))E[f(\omega)]}{2(\alpha+\theta b)^3\eta' Var(a(\omega))}$$
$$-\frac{\alpha\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Var(a(\omega))}{2(\alpha+\theta b)^3\eta' Var(a(\omega))}$$
$$\Rightarrow \quad P_Y^{F*} = E[f(\omega)P_X^{S*}(\omega)] + \frac{\alpha\gamma\theta^2 E[a(\omega)]Var(f(\omega))E[a^2(\omega)]}{\eta'(\alpha+\theta b)^3}$$
$$+E[f(\omega)](P_X^{F*} - E[P_X^{S*}(\omega)]) \tag{78}$$

By solving equations 77 and 78 simultaneously, we arrive at equations 32 and 33:

$$P_X^{F*} = E[P_X^{S*}(\omega)] + \frac{\alpha\theta^2 Var(f(\omega))E[a^2(\omega)]Skew(a(\omega))}{2(\alpha+\theta b)^3(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$-\frac{(\alpha+\theta b)Z(f(\omega), a(\omega), \theta, \alpha, b)}{\theta(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}\left[\sum_{j=1}^{m} P_{r_j}\beta_{r_j} - E[P_X^{S*}(\omega)]\right]$$
$$-\frac{Z(f(\omega), a(\omega), \theta, \alpha, b)}{\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega))}\left[bE[P_X^{S*}(\omega)] - \sum_{j=1}^{m} b_{r_j}P_{r_j}\right]$$
$$+\frac{nE[f(\omega)]Var(a(\omega))}{A_P(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$\left[\frac{\alpha\gamma\theta^2 E[a(\omega)]Var(f(\omega))E[a^2(\omega)]}{\eta'(\alpha+\theta b)^3} + E[f(\omega)](P_X^{F*} - E[P_X^{S*}(\omega)])\right]$$
$$\Rightarrow \quad (P_X^{F*} - E[P_X^{S*}(\omega)])\left[\frac{A_P\eta Var(f(\omega))E[a^2(\omega)]}{A_P(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}\right]$$
$$= \frac{\alpha\theta^2 Var(f(\omega))E[a^2(\omega)]Skew(a(\omega))}{2(\alpha+\theta b)^3(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$+\frac{nE[f(\omega)]Var(a(\omega))\gamma\alpha\theta^2 E[a(\omega)]Var(f(\omega))E[a^2(\omega)]}{\eta'(\alpha+\theta b)^3 A_P(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}$$
$$-\frac{(\alpha+\theta b)Z(f(\omega), a(\omega), \theta, \alpha, b)}{\theta(\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega)))}\left[\sum_{j=1}^{m} P_{r_j}\beta_{r_j} - E[P_X^{S*}(\omega)]\right]$$
$$-\frac{Z(f(\omega), a(\omega), \theta, \alpha, b)}{\eta Var(f(\omega))E[a^2(\omega)] + \eta'(E[f(\omega)])^2 Var(a(\omega))}\left[bE[P_X^{S*}(\omega)] - \sum_{j=1}^{m} b_{r_j}P_{r_j}\right]$$
$$\Rightarrow \quad (P_X^{F*} - E[P_X^{S*}(\omega)])\eta Var(f(\omega))E[a^2(\omega)]$$
$$= \frac{\alpha\theta^2 Var(f(\omega))E[a^2(\omega)]Skew(a(\omega))}{2(\alpha+\theta b)^3}$$

$$+\frac{E[f(\omega)]Var(a(\omega))\gamma\alpha\theta^2 E[a(\omega)]Var(f(\omega))E[a^2(\omega)]}{(\alpha+\theta b)^3}$$

$$-\frac{\theta}{\alpha+\theta b}Var(f(\omega))Var(a(\omega))E[a^2(\omega)]\left[\sum_{j=1}^{m}P_{r_j}\beta_{r_j}-E[P_X^{S*}(\omega)]\right]$$

$$-\frac{\theta^2}{(\alpha+\theta b)^2}Var(f(\omega))Var(a(\omega))E[a^2(\omega)]\left[bE[P_X^{S*}(\omega)]-\sum_{j=1}^{m}b_{r_j}P_{r_j}\right]$$

$$\Rightarrow\quad P_X^{F*}-E[P_X^{S*}(\omega)]=\frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta}+\frac{E[f(\omega)]Var(X_R(\omega))\gamma\theta^2 E[X_R(\omega)]}{\alpha^2\eta}$$

$$-\frac{\theta}{\alpha\eta}\left[\sum_{j=1}^{m}P_{r_j}\beta_{r_j}-E[P_X^{S*}(\omega)]\right]Var(X_R(\omega))-\frac{\theta^2 b}{\eta\alpha^2}\sum_{j=1}^{m}P_{r_j}\beta_{r_j}Var(X_R(\omega))$$

$$+\frac{\theta^2 Var(X_R(\omega))}{\eta\alpha^2}\sum_{j=1}^{m}b_{r_j}P_{r_j}$$

$$\Rightarrow\quad P_X^{F*}=E[P_X^{S*}(\omega)]+\frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta}-\frac{\theta}{\alpha\eta}[\sum_{j=1}^{m}P_{r_j}\left(\beta_{r_j}(1+\frac{\theta b}{\alpha})-\frac{\theta}{\alpha}b_{r_j}\right)$$

$$-E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])]Var(X_R(\omega)) \tag{79}$$

This is identical to equation 32. By substituting this into equation 78, we obtain equation 33:

$$P_Y^{F*}=E[f(\omega)P_X^{S*}(\omega)]+\frac{\gamma\theta^2 E[X_R(\omega)]Var(f(\omega))E[X_R^2(\omega)]}{\eta'\alpha^2}$$

$$+\frac{E[f(\omega)]\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta}-\frac{\theta}{\alpha\eta}[\sum_{j=1}^{m}P_{r_j}\left(\beta_{r_j}(1+\frac{\theta b}{\alpha})-\frac{\theta}{\alpha}b_{r_j}\right)$$

$$-E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])]E[f(\omega)]Var(X_R(\omega))$$

$$\Rightarrow\quad P_Y^{F*}=E[f(\omega)P_X^{S*}(\omega)]+\frac{\gamma\theta E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]}{\eta'\alpha}$$

$$+\frac{E[f(\omega)]\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta}-\frac{\theta}{\alpha\eta}[\sum_{j=1}^{m}P_{r_j}\left(\beta_{r_j}(1+\frac{\theta b}{\alpha})-\frac{\theta}{\alpha}b_{r_j}\right)$$

$$-E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])]E[f(\omega)]Var(X_R(\omega))$$

$$\Rightarrow\quad P_Y^{F*}=E[f(\omega)P_X^{F*}(\omega)]+\frac{\gamma\theta E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]}{\eta'\alpha} \tag{80}$$

Appendix E: Solving for Equilibrium Forward Positions

Here, we derive the equilibrium forward positions. To obtain $X_{p_i}^{F*}$, we substitute Equations 32 and 33 into Equation 29:

$$X_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)}[\frac{Var(f(\omega)a(\omega))\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta A_P}$$

$$-\frac{\theta}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] Var(f(\omega)a(\omega))Var(X_R(\omega))$$

$$+\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega))Var(f(\omega)a(\omega))}{2(\alpha+\theta b)^3}$$

$$-\frac{E[f(\omega)]\theta^2 Skew(X_R(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2\alpha^2\eta A_P}$$

$$-\frac{\gamma\theta E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]Cov(a(\omega), f(\omega)a(\omega))}{\eta'\alpha A_P}$$

$$+\frac{\theta}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right](E[f(\omega)])^2(Var(a(\omega)))^2$$

$$-\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), f(\omega)a(\omega))Cov(a(\omega), f(\omega)a(\omega))}{2(\alpha+\theta b)^3}]$$

$$\Rightarrow \quad X_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)}[\frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta A_P}[Var(f(\omega)a(\omega))$$

$$-(E[f(\omega)])^2 Var(a(\omega))] + \frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega))}{2(\alpha+\theta b)^3}[Var(f(\omega)a(\omega))$$

$$-(E[f(\omega)])^2 Var(a(\omega))] + \frac{\theta}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right]$$

$$Var(X_R(\omega))[(E[f(\omega)])^2 Var(a(\omega)) - Var(f(\omega)a(\omega))]$$

$$-\frac{\gamma\theta E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]E[f(\omega)]Var(a(\omega))}{\eta'\alpha A_P}]$$

$$\Rightarrow \quad X_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)}[\frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2\eta A_P}Var(f(\omega))E[a^2(\omega)]$$

$$+\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega))}{2(\alpha+\theta b)^3}Var(f(\omega))E[a^2(\omega)]$$

$$-\frac{\theta}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] Var(X_R(\omega))Var(f(\omega))E[a^2(\omega)]$$

$$-\frac{\gamma\theta E[P_X^{S*}(\omega)]Var(f(\omega))E[X_R^2(\omega)]E[f(\omega)]Var(a(\omega))}{\eta'\alpha A_P}]$$

$$\Rightarrow \quad X_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)}[\frac{Skew(X_R(\omega))(\alpha+\theta b)^2 Z(f(\omega), X_R(\omega), \theta, \alpha, b)}{2\alpha^2 \eta A_P Var(a(\omega))}$$
$$+\frac{\alpha_{p_i} Skew(a(\omega)) Z(f(\omega), X_R(\omega), \theta, \alpha, b)}{2(\alpha+\theta b) Var(a(\omega))} + \frac{\alpha_{p_i} E[a(\omega)] Z(f(\omega), X_R(\omega), \theta, \alpha, b)}{\alpha+\theta b}$$
$$-\frac{\alpha}{\theta\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] Z(f(\omega), X_R(\omega), \theta, \alpha, b)$$
$$-\frac{\gamma\alpha E[f(\omega)] E[P_X^{S*}(\omega)] Z(f(\omega), X_R(\omega), \theta, \alpha, b)}{n\theta}]$$
$$\Rightarrow \quad X_{p_i}^{F*} = \frac{\alpha_{p_i}}{\alpha} E[X_R(\omega)] + \left[\frac{1}{2\eta A_P} + \frac{\alpha_{p_i}}{2\alpha}\right] \frac{Skew(X_R(\omega))}{Var(X_R(\omega))}$$
$$-\frac{\alpha}{\eta\theta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] - \frac{\gamma E[f(\omega)] E[X_R(\omega)]}{n} \tag{81}$$

Similarly, by substituting Equations 32 and 33 into Equation 30, we obtain $Y_{p_i}^{F*}(\omega)$:

$$Y_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)}[\frac{\theta^2 E[f(\omega)] Skew(X_R(\omega)) Var(a(\omega))}{2\alpha^2 \eta A_P}$$
$$+\frac{\gamma\theta E[P_X^{S*}(\omega)] Var(f(\omega)) E[X_R^2(\omega)] Var(a(\omega))}{n\alpha}$$
$$-\frac{\theta}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] E[f(\omega)] Var(X_R(\omega)) Var(a(\omega))$$
$$+\frac{\alpha_{p_i}\theta^2 E[f(\omega)] Cov(a^2(\omega), a(\omega)) Var(a(\omega))}{2(\alpha+\theta b)^3}$$
$$-\frac{\theta^2 Skew(X_R(\omega)) E[f(\omega)] Var(a(\omega))}{2\alpha^2\eta A_P}$$
$$+\frac{\theta E[f(\omega)] Var(a(\omega))}{\alpha\eta A_P}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] Var(X_R(\omega))$$
$$-\frac{\alpha_{p_i}\theta^2 Cov(a^2(\omega), a(\omega)) E[f(\omega)] Var(a(\omega))}{2(\alpha+\theta b)^3}]$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{1}{Z(f(\omega), X_R(\omega), \theta, \alpha, b)} \frac{\gamma\theta E[X_R(\omega)] Var(f(\omega)) E[X_R^2(\omega)] Var(a(\omega))}{n\alpha}$$
$$\Rightarrow \quad Y_{p_i}^{F*} = \frac{\gamma E[X_R(\omega)]}{n} \tag{82}$$

By substituting Equation 32 into Equation 31 and making use of the fact that for any random variables A and B, $Cov(AB, B) = Coskew(A, B) + E[B]Cov(A, B) + E[A]Var(B)$, we obtain the expression for $X_{r_j}^{F*}$:

$$X_{r_j}^{F*} = -\frac{\theta^2 Skew(X_R(\omega))}{2\alpha^2 \eta A_R Var(P_X^{S*}(\omega))} - \frac{(\alpha+\theta b) P_{r_j}\beta'_{r_j}}{\theta}$$

$$+\frac{\theta Var(X_R(\omega))}{\alpha\eta A_R Var(P_X^{S*}(\omega))}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right]$$

$$+\frac{Cov(a(\omega)a_{r_j}(\omega), a(\omega))}{Var(a(\omega))} - \frac{\theta b_{r_j} Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))} + b_{r_j}P_{r_j}$$

$$\Rightarrow \quad X_{r_j}^{F*} = -\frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))} - \frac{(\alpha+\theta b)P_{r_j}\beta_{r_j}}{\theta}$$

$$+\frac{\alpha}{\theta\eta A_R}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right]$$

$$+\frac{Cov(a(\omega)a_{r_j}(\omega), a(\omega))}{Var(a(\omega))} - \frac{\theta b_{r_j} Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))} + b_{r_j}P_{r_j}$$

$$\Rightarrow \quad X_{r_j}^{F*} = \frac{\alpha}{\theta}\left[\frac{1}{\eta A_R}\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - P_{r_j}\beta'_{r_j}\right] - \frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))}$$

$$-\frac{\alpha(1+\gamma E[f(\omega)])}{\theta\eta A_R}E[P_X^{S*}(\omega)] + \frac{Cov(a(\omega)a_{r_j}(\omega), a(\omega))}{Var(a(\omega))}$$

$$-\frac{\theta b_{r_j} Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))}$$

$$\Rightarrow \quad X_{r_j}^{F*} = \frac{\alpha}{\theta}\left[\frac{1}{\eta A_R}\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - P_{r_j}\beta'_{r_j}\right] - \frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))}$$

$$-\frac{E[X_R(\omega)]}{\eta A_R} + \frac{Cov(a(\omega)a_{r_j}(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))}$$

$$-\frac{\gamma E[f(\omega)]E[X_R(\omega)]}{\eta A_R} - \frac{\theta b_{r_j} Cov(a^2(\omega), a(\omega))}{(\alpha+\theta b)Var(a(\omega))}$$

$$\Rightarrow \quad X_{r_j}^{F*} = \frac{\alpha}{\theta}\left[\frac{1}{\eta A_R}\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - P_{r_j}\beta'_{r_j}\right] - \frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))}$$

$$-\frac{\alpha(1+\gamma E[f(\omega)])}{\theta\eta A_R}E[P_X^{S*}(\omega)] + \frac{Coskew(X_{r_j}(\omega), X_R(\omega))}{Var(X_R(\omega))} + E[X_{r_j}(\omega)]$$

$$+\frac{E[X_R(\omega)]Cov(X_{r_j}(\omega), X_R(\omega))}{Var(X_R(\omega))}$$

$$\Rightarrow \quad X_{r_j}^{F*} = E[X_{r_j}(\omega)] - \frac{\alpha}{\theta}\beta'_{r_j}\left[P_{r_j} - E[P_X^{S*}(\omega)]\right]$$

$$+\frac{Coskew(X_{r_j}(\omega), X_R(\omega))}{Var(X_R(\omega))} - \frac{Skew(X_R(\omega))}{2\eta A_R Var(X_R(\omega))}$$

$$+\frac{\alpha}{\theta\eta A_R}\left[\sum_{j=1}^{m} P_{r_j}\beta'_{r_j} - E[P_X^{S*}(\omega)](1+\gamma E[f(\omega)])\right] \qquad (83)$$