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Topics in representations of orbifold and surface groups

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Abstract

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This thesis presents three projects whose common thread is the study of representations of orbifold and surface groups. These projects move from studying orbifold groups in $PSL(2,\mathbb{R})$ using tools from hyperbolic geometry to exploring surface subgroups in $PSL(n,\mathbb{R})$ with $n > 2$ using algebraic, dynamic, and geometric group theoretic approaches. In the first project, we build a new infinite family of non-commensurable pseudomodular groups obtained via the jigsaw method. The second project is concerned with obtaining families of Zariski dense rational surface group representations into $SL(n,\mathbb{R})$ for odd $n > 2$ by bending orbifold representations. The final project uses the composition of Hitchin representations into $PSL(3,\mathbb{R})$ with a generalization of the irreducible representation from $PSL(2,\mathbb{R})$ to $PSL(n,\mathbb{R})$ to construct families of Anosov representations outside the Hitchin component.

Overview

The results in this dissertation lie broadly in the fields of low-dimensional topology and geometric group theory, with a focus on hyperbolic geometry. The general question motivating them is:

what is the relation between the geometry of a space
and the representations of its fundamental group?

This query with roots in Thurston's geometrization conjecture has powered research in several areas of mathematics for over 30 years. In particular, the study of representations of the fundamental group of a surface S of negative Euler characteristic into a semisimple Lie group of higher rank is a current topic of particular interest with deep connections to geometry, dynamics, algebraic geometry, geometric group theory, and analysis. In this context, this dissertation details results about the existence and classification of representations of surface and orbifold groups with properties like modularity (thms. 2.4 and 2.5), Zariski density (thm. 3.1), and P-Anosov (thm. 4.1).

In the seventies, Thurston recast discrete and faithful representations of the fundamental group $\pi_1(S)$ of a surface S of negative Euler characteristic into $PSL(2,\mathbb{R})$ as objects encoding hyperbolic geometric structures on S. By the end of the 20th century Hitchin established fundamental results about the connected components of the representation space $\mathrm{Rep}^+(\pi_1(S), PSL(n,\mathbb{R}))$. The so-called *Hitchin component*, which contains an embedded copy of Teichmüller space, was introduced in this work. A unifying geometric interpretation of surface group representations in the Hitchin component came more than a decade later when Labourie built on the work of Choi and Goldman to define *Anosov representations* as holonomies of Anosov dynamical structures on the surface S.

This dissertation takes place against this backdrop and branches into three projects:

1. Classifying pseudomodular groups (chap. 2).

2. Constructing rational Zariski dense surface groups representations (chap. 3).

3. Exploring surface Anosov representations outside the Hitchin component (chap. 4).

These projects stem from ideas and problems introduced by Long, Reid, Lou, Tan, and Vo for (1), Long, Reid, and Thistlethwaite for (2), and Guichard, Wienhard, Kapovich, Leeb, and Porti for (3), among others. Their common thread is:

how can we control the geometry of a space to obtain specific properties on the representations of its fundamental group?

To approach this question we use tools from geometric group theory, representation theory and low dimensional topology, buildings and orbifold geometry, as well as software coding for computer exploration.

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Chapter 1

Hyperbolic objects

1.1 The hyperbolic plane

Consider the set

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

If $\gamma(t) = (x(t), y(t))$ with $t \in [a, b]$ is a curve in $\mathbb{H}^2$ then the *hyperbolic length* of γ is

$$\ell(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

This endows $\mathbb{H}^2$ with a metric d such that for any $p, q \in \mathbb{H}^2$

$$d(p, q) = \inf\{\ell(\gamma) \mid \gamma \text{ is a curve connecting } p \text{ and } q\}.$$

The *upper half-plane model of the hyperbolic plane* is the metric space $(\mathbb{H}^2, d)$. Geodesics in $\mathbb{H}^2$ are either half-circles whose center is on the real axis or vertical lines. The *boundary at infinity* of $\mathbb{H}^2$ is $\partial_\infty \mathbb{H}^2 \equiv \mathbb{R} \cup \{\infty\}$.

The group $\mathrm{Isom}(\mathbb{H}^2)$ of isometries of $\mathbb{H}^2$ consists of functions of the form

$$z \mapsto \frac{az+b}{cz+d} \text{ with } ad-bc=1, \text{ or } z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} \text{ with } ad-bc=-1.$$

The first kind of isometries form the subgroup $\mathrm{Isom}^+(\mathbb{H}^2)$ of orientation preserving isometries. The group $\mathrm{Isom}(\mathbb{H}^2)$ can be identified with $PGL(2,\mathbb{R}) = GL(2,\mathbb{R})/Z(GL(2,\mathbb{R}))$, while $\mathrm{Isom}^+(\mathbb{H}^2)$ is isomorphic to $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm Id\}$.

The action on $\mathbb{H}^2$ of an isometry $A \in PSL(2,\mathbb{R})$ extends to $\partial_\infty \mathbb{H}^2$. We can classify A into one of three types of isometries depending on its fixed points in $\mathbb{H}^2 \sqcup \partial_\infty \mathbb{H}^2$. We have that A is

- *elliptic* if A has a fixed point in $\mathbb{H}^2$, this is the unique fixed point in $\mathbb{H}^2 \sqcup \partial_\infty \mathbb{H}^2$;
- *parabolic* if A has a single fixed point in $\partial_\infty \mathbb{H}^2$; or
- *hyperbolic* if A has two fixed points in $\partial_\infty \mathbb{H}^2$.

It is not hard to check that A will be elliptic if and only if $|\mathrm{tr}(A)| < 2$, parabolic if and only if $|\mathrm{tr}(A)| = 2$ and hyperbolic if and only if $|\mathrm{tr}(A)| > 2$.

1.2 Hyperbolic surfaces

A *hyperbolic structure* on a topological surface S is an atlas of charts $\{\phi_i \colon U_i \to \mathbb{H}^2\}_{i\in\Delta}$ where each U_i is an open set in S, such that for any two charts ϕ_i, ϕ_j the composition $\phi_j \circ \phi_i^{-1} \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is the restriction of an orientation-preserving isometry of $\mathbb{H}^2$.

Any closed orientable surface S of genus ≥ 2 admits a hyperbolic structure. Geodesics on S are defined locally by using the chart maps to pull back the geodesics in $\mathbb{H}^2$. The metric in S also comes from pulling back the metric in $\mathbb{H}^2$. The universal cover $\tilde{S}$ of a compact surface is a complete, connected, simply-connected hyperbolic surface. This implies $\tilde{S}$ is isometric to $\mathbb{H}^2$. For details on these results we refer the reader to chapter 2 of [1].

A choice of isometry $D\colon \tilde{S} \to \mathbb{H}^2$ is called a *developing map*. This gives us a way to translate the action of $\pi_1(S)$ on $\tilde{S}$ by deck transformations into an isometric action of $\pi_1(S)$ on $\mathbb{H}^2$. In other words, if $D\colon \tilde{S} \to \mathbb{H}^2$ is a developing map then there exists a group homomorphism

$$\rho\colon \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2) \equiv PSL(2,\mathbb{R})$$

called a *holonomy* such that for every $\gamma \in \pi_1(S)$ the following diagram commutes

$$\begin{tikzcd} \tilde{S} \arrow[r, "D"] \arrow[d, "\gamma"'] & \mathbb{H}^2 \arrow[d, "\rho(\gamma)"] \\ \tilde{S} \arrow[r, "D"] & \mathbb{H}^2 \end{tikzcd}$$

Given a hyperbolic structure on S its developing map is uniquely defined up to compositions with isometries of $\mathbb{H}^2$, which in turn corresponds to conjugating ρ by an element of $PGL(2,\mathbb{R})$. Holonomies are the first examples we give of representations, one of the main objects of study in this work.

Definition 1.1 A *representation* of a group Γ is a group homomorphism $\rho\colon \Gamma \to G$ where G is a matrix Lie group. A representation ρ is *faithful* if $\mathrm{Ker}(\rho) = \{Id\}$ and it is *discrete* if $\rho(\Gamma)$ is a discrete subset of G.

A holonomy ρ is a faithful representation because the action of $\pi_1(S)$ on $\tilde{S}$ by deck transformations is free. Moreover the action of $\rho(\pi_1(S))$ on $\mathbb{H}^2$ turns the hyperbolic plane into a covering space for S, so holonomies are also discrete. Thus every hyperbolic structure on S defines a discrete and faithful representation $\pi_1(S) \to PSL(2,\mathbb{R})$, and this is unique up to conjugation. On the other hand, given a discrete and faithful representation $\rho\colon \pi_1(S) \to PSL(2,\mathbb{R})$ we can define a hyperbolic structure on S via the identification $S \equiv \mathbb{H}^2/\rho(\pi_1(S))$.

Definition 1.2 A *Fuchsian representation* is a discrete and faithful representation $\rho\colon \pi_1(S) \to PSL(2,\mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{H}^2)$, where S is a closed orientable surface of genus ≥ 2. The set of $PGL(2,\mathbb{R})$ conjugacy classes of Fuchsian representations is called the *Teichmüller space* of the surface S.

We usually refer to the fundamental group of a closed orientable surface of genus ≥ 2 as a *surface group*. In section 3.1 we will look more closely at the space of representations of surface groups into $PSL(n,\mathbb{R})$ for $n > 2$ and introduce the so-called Hitchin component, a higher rank analogue of Teichmüller space.

It is worth noting that if $\rho\colon \pi_1(S) \to PSL(2,\mathbb{R})$ is a Fuchsian representation then $\rho(\gamma)$ is hyperbolic for every $\gamma \neq Id$ and thus it is conjugate to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ with $\lambda > 1$. In particular $\rho(\pi_1(S))$ has no torsion. In the upcoming section we will introduce a second kind of hyperbolic object which contains elliptic elements in its fundamental group.

1.3 Hyperbolic orbifolds

In this section we will consider the action on $\mathbb{H}^2$ of a discrete subgroup Γ of $PSL(2,\mathbb{R})$ in which the stabilizer for any point of $\mathbb{H}^2$ is finite. The quotient $\mathbb{H}^2/\Gamma$ still inherits a hyperbolic metric from $\mathbb{H}^2$, but it can have singular points of three types: cone points, reflexion lines or corner reflectors. The resulting space is called a *hyperbolic orbifold*. In terms of charts a point in an orbifold which is not a singularity will have a neighborhood around it which is homeomorphic to an open set of $\mathbb{H}^2$, while a singularity will have a neighborhood homeomorphic to the quotient of an open set of $\mathbb{H}^2$ by the action of a finite group of isometries. For a precise definition of orbifolds in terms of charts we refer to section 2 of [2].

In practice we can also define a topological orbifold by specifying an underlying topological surface and its singularities. In the following chapters we will only use of orbifolds whose only singularities are cone point singularities. If $\mathcal{O}$ is such an orbifold, with underlying topological space a closed orientable surface of genus g and n cone points with cone angle $2\pi/p_i$, then a presentation for $\pi_1(\mathcal{O})$ is

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n \mid x_i^{p_i} = 1, \prod_{i=1}^{g}[a_i, b_i]x_i \ldots x_n = 1 \right\rangle.$$

Intuitively, if we have a surface S and include a cone point p with cone angle $2\pi/k$ then this cone point will create an obstruction to $\pi_1(S)$ in which a loop going around p can only by retracted if it loops around p a multiple of k times.

Given integers $l, m, n \geq 2$ such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ there is (up to isometry) a unique hyperbolic triangle T with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$, $\frac{\pi}{n}$. The triangle T generates a tiling of $\mathbb{H}^2$ with symmetry group $\Delta^*(l, m, n) \subset \mathrm{Isom}(\mathbb{H}^2) \equiv PSL(2, \mathbb{R})$. This group is generated by the reflections a, b, c on the sides of T and can be given the presentation

$$\Delta^*(l, m, n) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^l = (ca)^m = (bc)^n = 1\rangle.$$

Let $\Delta(l, m, n) < \Delta^*(l, m, n)$ be the subgroup of orientation preserving symmetries. Its generators are the rotations $x = ab$ and $y = ca$ by $\frac{2\pi}{l}$ and $\frac{2\pi}{m}$ around the corresponding vertices of the triangle T. The product $z = xy = cb$ is the rotation by $\frac{2\pi}{n}$ around the remaining vertex. A presentation for this group is

$$\Delta(l,m,n) = \langle x, y \mid x^l = y^m = (xy)^n = 1 \rangle.$$

A fundamental domain for the action of $\Delta(l,m,n)$ on $\mathbb{H}^2$ is the quadrilateral $T \cup b(T)$. This is a double cover of the triangle T, from which we get that $[\Delta^*(l,m,n) : \Delta(l,m,n)] = 2$. The quotient $\mathbb{H}^2/\Delta(l,m,n)$ is homeomorphic to the orbifold $S^2(l,m,n)$ whose underlying topological space is S^2 and has three cone points of orders l, m and n. By standard orbifold theory $\pi_1(S^2(l,m,n))$ is isomorphic to $\Delta(l,m,n)$. Up to conjugation, the isomorphism $\pi_1(S^2(l,m,n)) \to \Delta(l,m,n)$ is the unique discrete and faithful representation of $\pi_1(S^2(l,m,n))$ into $PSL(2,\mathbb{R})$. In other words $\pi_1(S^2(l,m,n))$ does not admit any non-trivial deformations inside $PSL(2,\mathbb{R})$.

Covering spaces for orbifolds are defined similarly as for surfaces, just taking some extra care in the covering of singular points. As we are used to, every subgroup of $\pi_1(\mathcal{O})$ corresponds to a covering space of the orbifold $\mathcal{O}$. All but four types of 2-dimensional orbifolds are covered by a surface ([2] thm. 2.3), those which are not are called *bad* and none of them will appear in our work.

1.4 Hyperbolic groups

Fundamental groups of hyperbolic surfaces and hyperbolic orbifolds are both examples of *hyperbolic groups*. The definition of these groups is based on two beautiful ideas in geometric group theory: that a group can be naturally seen as a metric space and that there is an expanded notion of curvature that does not depend on a space being a manifold. In this section we review these concepts and introduce some of the objects we will use in chapters 3 and 4. For a more detailed introduction to these topics we refer to chapters 7 and 9 of [3].

Let G be a group and S a generating set for G. The *word length* of an element $g \in G$ with respect to S is the shortest word in $S \cup S^{-1}$ that is equal to g. For example, the identity in G has length 0. This allows us to define the distance between two elements $g, h \in G$ as the word length of $g^{-1}h$ and thus obtain a *word metric* on G. Changing the generating set of G does not necessarily give isometric word metrics on G. To bypass this change in the metrics we need a weaker notion of isometry that allows for dilation and possible bad behavior on a controlled small scale.

Definition 1.3 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is called a *quasi-isometric embedding* if there are constants $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for all $x_1, x_2 \in X$. A quasi-isometric embedding $f\colon X \to Y$ is a *quasi-isometry* if there is a constant $D > 0$ such that for every point $y \in Y$ there is an $x \in X$ such that $d_Y(f(x), y) \leq D$.

If G is a finitely generated group then the word metrics on G with respect to any two finite generating sets are quasi-isometric. In general, quasi-isometries will allow us to compare whether two spaces have similar coarse or large scale geometric properties.

Theorem 1.4 (Milnor-Schwarz lemma) *Let G be a group and X be a proper geodesic metric space. Suppose that G acts properly discontinuously, cocompactly and by isometries on X. Then G is finitely generated and G is quasi-isometric to X.*

The previous fundamental result in geometric group theory tells us, for example, that both surface groups and fundamental groups of compact hyperbolic orbifolds (*orbifold groups*) are quasi-isometric to $\mathbb{H}^2$.

Definition 1.5 Consider a metric space X. Let $\mathcal{N}_\delta(A)$ denote the δ-neighborhood of the set A. A geodesic triangle with sides α, β, γ is *δ-thin* if $\alpha \subset \mathcal{N}_\delta(\beta \cup \gamma)$ and the other two similar inclusions hold. A metric space X is *δ-hyperbolic* if every geodesic triangle in X is δ-thin.

Some well known examples of δ-hyperbolic spaces are trees and the hyperbolic plane $\mathbb{H}^2$. For length metric spaces δ-hyperbolicity is preserved under quasi-isometries. Then it makes sense to say a group is *hyperbolic* if it is δ-hyperbolic for some $\delta > 0$ and some finite generating set. Since surface and orbifold groups are quasi-isometric to $\mathbb{H}^2$, these are examples of hyperbolic groups.

Definition 1.6 A *geodesic ray* in a metric space X is an isometry $\gamma\colon [0,\infty) \to X$ such that each segment $\gamma\colon [0,t) \to X$ is a path of shortest length from $\gamma(0)$ to $\gamma(t)$. Two geodesic rays γ_1, γ_2 are defined to be equivalent if there is a constant $k > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < k$ for all t. The *visual boundary* of X is the set of equivalence classes under the previous relation:

$$\partial_\infty X = \{[\gamma] \mid \gamma \text{ geodesic ray in } X\}.$$

There are different ways to endow $\partial_\infty X$ with a topology. In what follows we will assume, unless stated otherwise, that $\partial_\infty X$ has the *cone topology* in which intuitively two geodesic rays are close if for every $m > 0$ the rays stay for a long time within distance m. For a more detailed discussion of this topic see chapter II.8 in [4]. For example, in the case of the upper half-plane model of $\mathbb{H}^2$ the visual boundary is precisely the boundary at infinity $\partial_\infty \mathbb{H}^2 \equiv \mathbb{R} \cup \{\infty\}$. This is homeomorphic to S^1 and $\mathbb{H}^2 \sqcup \partial_\infty \mathbb{H}^2$ is homeomorphic to the closed unit disk in $\mathbb{R}^2$. This last homeomorphism is easier to appreciate in the Poincaré disk model of hyperbolic space. For δ-hyperbolic spaces the visual boundary is a large-scale geometric feature preserved under quasi-isometries. So surface and orbifold groups also have visual boundary homeomorphic to S^1.

Chapter 2

New examples from the jigsaw group construction

A *pseudomodular group* is a discrete subgroup $\Gamma \leq PGL(2, \mathbb{Q})$ which is not commensurable with $PSL(2, \mathbb{Z})$ and has cusp set precisely $\mathbb{Q} \cup \{\infty\}$. The existence of such groups was proved by Long and Reid. Later, Lou, Tan and Vo constructed two infinite families of non-commensurable pseudomodular groups which they called *jigsaw groups*. In this chapter we construct a new infinite family of non-commensurable pseudomodular groups obtained via this jigsaw construction. We also find that infinitely many of the simplest jigsaw groups are not pseudomodular, providing a partial answer to questions posed by the aforementioned authors.

2.1 Background

A *Fuchsian group* Γ is a discrete subgroup of $PSL(2, \mathbb{R})$. Such a group acts properly discontinuously by fractional linear transformations on $\mathbb{H}^2$, the upper half-plane model of hyperbolic space. This Γ-action extends to the boundary at infinity $\partial_\infty \mathbb{H}^2 \equiv \mathbb{R} \cup \{\infty\}$.

If an isometry $\gamma \in \Gamma$ has a fixed point on $\partial_\infty \mathbb{H}^2$ then it is either one of a pair of fixed points, in which case γ is called *hyperbolic*, or the fixed point is unique and we say γ is *parabolic*. The *cusps* of a Fuchsian group Γ are the points in $\partial_\infty \mathbb{H}^2$ fixed by parabolic elements of Γ.

An example where the cusp set is easily calculated is when $\Gamma = PSL(2, \mathbb{Z})$ is the modular group. Consider first the parabolic element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in PSL(2, \mathbb{Z})$. For every $x \in \partial_\infty \mathbb{H}^2$ we have that $T(x) = x + 1$, so ∞ is a fixed point of T and thus a cusp of $PSL(2, \mathbb{Z})$. For any $\frac{p}{q} \in \mathbb{Q}$ with $\gcd(p, q) = 1$ we can find $a, b \in \mathbb{Z}$ such that $bp + aq = 1$. Then $M = \begin{pmatrix} p & -a \\ q & b \end{pmatrix} \in PSL(2, \mathbb{Z})$ is such that $M(\infty) = \frac{p}{q}$. Since the trace is preserved by conjugation, MTM^{-1} is a parabolic in $PSL(2, \mathbb{Z})$ that fixes $\frac{p}{q}$. Thus $\mathbb{Q} \cup \{\infty\}$ is contained in the cusps of $PSL(2, \mathbb{Z})$. On the other hand, solving the equation $\frac{ax+b}{cx+d} = x$ we get that $x \in \mathbb{Q} \cup \{\infty\}$. Therefore $\text{cusps}(PSL(2, \mathbb{Z})) = \mathbb{Q} \cup \{\infty\}$.

Two Fuchsian groups are *commensurable* if they share a common subgroup that has finite index in both. It is known that commensurable Fuchsian groups have the same cusp set. In [5] Long and Reid explore the converse question: if Γ_1 and Γ_2 are finite covolume subgroups of $PSL(2, \mathbb{R})$ with the same cusp set, are they commensurable? The answer was on the negative and in theorem 1.2 of [5] they produced several examples of finite covolume Fuchsian groups with cusp set $\mathbb{Q} \cup \{\infty\}$ which are not commensurable with the modular group $PSL(2, \mathbb{Z})$. This motivates the following definition:

Definition 2.1 A *pseudomodular group* is a discrete subgroup $\Gamma \leq PGL(2, \mathbb{Q})$ which is not commensurable with $PSL(2, \mathbb{Z})$ and has cusp set precisely $\mathbb{Q} \cup \{\infty\}$.

Subsequently Ayaka and Tan [6] found another isolated example of a pseudomodular group and later Lou, Tan and Vo [7] constructed two infinite families of non-commensurable pseudomodular groups which they called *jigsaw groups*. In this chapter we examine a new infinite family of non-commensurable pseudomodular groups obtained via the jigsaw construction. We also find that infinitely many of the simplest jigsaw groups, called Weierstrass groups, are not pseudomodular.

To describe the jigsaw construction from [7] first let Δ_n, for $n \in \mathbb{N}$, be the ideal oriented triangle in $\mathbb{H}^2$ with vertices $\infty, -1$ and 0, and marked points

$$x_1 = -1 + i, \quad x_{\frac{1}{n}} = \frac{-n + i\sqrt{n}}{n+1}, \quad x_n = i\sqrt{n} \tag{2.1}$$

on the sides $[\infty, -1]$, $[-1, 0]$ and $[0, \infty]$ respectively. A *tile of type n* is any isometric transformation of Δ_n, keeping track of the images of marked points. The sides of a tile of type n will be called of *type 1*, *type $\frac{1}{n}$* or *type n* according to which has the image of x_1, $x_{\frac{1}{n}}$ and x_n. Consider the π-rotations ρ_i about the marked points x_i, represented here as elements of $PSL(2, \mathbb{R})$:

$$\rho_1 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad \rho_{\frac{1}{n}} = \sqrt{n} \begin{pmatrix} 1 & 1 \\ \frac{-n-1}{n} & -1 \end{pmatrix}, \quad \rho_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 0 & n \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Definition 2.2 The *n-th Weierstrass group W_n* is the discrete group $W_n = \langle \rho_1, \rho_{\frac{1}{n}}, \rho_n \rangle$.

For any $n \in \mathbb{N}$ the quotient surface $\mathbb{H}^2/W_n$ is an orbifold with a single cusp and three cone points of degree 2. Given the choice of marked points the element $\rho_1\rho_{\frac{1}{n}}\rho_n \in W_n$ is parabolic. Since the vertices of Δ_n are in $\mathbb{Q} \cup \{\infty\}$, then $W_n \leq PSL(2, \mathbb{Q})$ and all the vertices of the tiling of $\mathbb{H}^2$ generated by the action of W_n on Δ_n are in $\mathbb{Q} \cup \{\infty\}$. In the notation of [7] $\Delta_n = \Delta(1, 1/n, n)$ and $W_n = \Gamma(1, 1/n, n)$.

By gluing different tiles together we can create groups that are more complex than the Weierstrass groups. If we have two tiles Δ and Δ' with sides s_1, s_2, s_3 and s_1', s_2', s_3', and marked points x_1, x_2, x_3 and x_1', x_2', x_3' respectively, we say the sides s_i and s_j' match if both sides are of the same type. As explained in definition 2.2 of [7] this means that if we glue Δ to Δ' along s_i and s'_j by identifying x_i to x'_j, then the π-rotation about $x_i = x'_j$ will send Δ to Δ'. In this way, by gluing finitely many tiles we obtain a triangulated ideal polygon with marked points on the interior and exterior sides of the triangulation, such a polygon is called a *jigsaw*.

Definition 2.3 The *jigsaw group* Γ_J associated to a jigsaw J is the Fuchsian group generated by the π-rotations about the marked points of the (exterior) sides of J.

As a convention we will require that the jigsaw J used to define the jigsaw group Γ_J has a tile Δ_n with vertices ∞, -1 and 0 in it. The balancing condition on each tile of the jigsaw J ensures the quotient $\mathbb{H}^2/\Gamma_J$ is a complete orbifold with a single cusp and $N+2$ cone points of order 2, where N is the number of tiles that make up the jigsaw. Then Γ_J generates a tiling of $\mathbb{H}^2$ and J is a fundamental domain of the action of the group.

In theorems 2.4 and 2.5 of [7] Lou, Tan and Vo examine jigsaw groups composed of tiles of types 1, 2 and 3. They prove that jigsaws composed only of tiles of types 1 and 2 have cusp set equal to $\mathbb{Q}\cup\{\infty\}$, and those that consist of a single tile of type 2 and n tiles of type 1 are all pseudomodular and pairwise non-commensurable. On the other hand, they prove jigsaws made with tiles of type 1 and type 3 produce both an infinite family of pseudomodular groups and an infinite family of non-pseudomodular groups. Here we examine the groups generated by jigsaws made of tiles of types 1 and 4.

Theorem 2.4 *Let $J_{m,n}$ be the jigsaw formed by the Δ_1 tile followed by $m-1 \geq 0$ tiles of type 1 glued to the left and $n \geq 1$ tiles of type 4 glued to the right of Δ_1, so that all tiles in $J_{m,n}$ share ∞ as a common vertex (see figure 2.1). Then the associated jigsaw group $\Gamma_{m,n}$ has cusp set $\mathbb{Q} \cup \{\infty\}$. The infinite families $\Gamma_{1,n}$ and $\Gamma_{m,1}$ are pseudomodular and pairwise non-commensurable.*

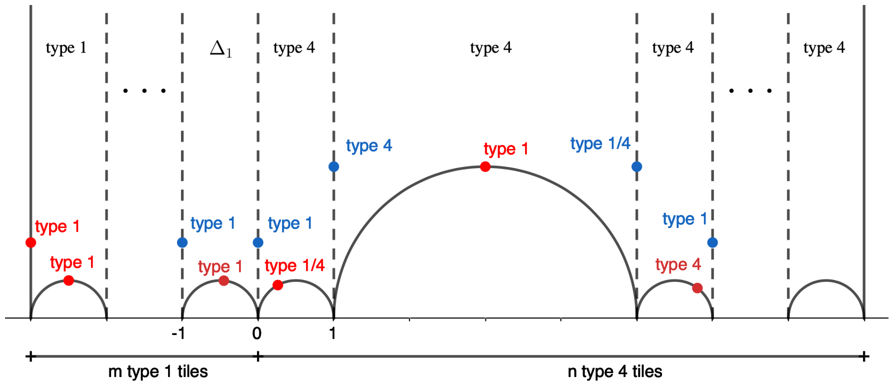

Figure 2.1: Jigsaw $J_{m,n}$

To prove this we refine the process followed in [7]. We first see that the cusp set of these groups is $\mathbb{Q} \cup \{\infty\}$ by finding an explicit covering of $\mathbb{R}$ by *killer intervals*, a tool introduced by Long and Reid in [5] (see definition 2.6). Then we check they are non-commensurable by proving that each jigsaw group in the given families is non-arithmetic and equals its commensurator. By carefully analyzing the combinatorics of gluing tiles together we can extend the examples of pseudomodular groups to jigsaws with more than one tile of type $n > 1$.

Since the submission for publication of the work in this chapter Lou, Tan and Vo have expanded the results of the families $\Gamma_{1,n}$ for $n \geq 5$ (see theorem 1.2 in [8]).

In the final section of this chapter we investigate whether there are only finitely many pseudomodular Weierstrass groups W_n, a question posed by Lou, Tan and Vo in section 9 of [7] which is a particular instance of the first open question posed by Long and Reid section 6 of [5]. The following result provides a partial answer to these questions.

Theorem 2.5 *The groups W_n with $n \geq 6$ and congruent to 0, 2 or 6 modulo 8 are not pseudomodular.*

For small values of $n \equiv 4 \pmod 8$ we have found that W_n is not pseudomodular. To construct these examples we have developed a computer program which tries to determine whether a given jigsaw group has cusp set equal to $\mathbb{Q} \cup \{\infty\}$ or contains a hyperbolic element fixing two rational points in $\partial_\infty \mathbb{H}^2 \equiv \mathbb{R} \cup \{\infty\}$. This python library is available in the following GitHub repository: `https://github.com/carmengg/pseudomodular_groups`. A survey of whether W_n is pseudo-modular or not for $n \leq 28$ can be found at the end of section 2.4.

2.2 Cusp set of the $\Gamma_{m,n}$ jigsaw groups

Let $\Gamma < PSL(2, \mathbb{Q})$ be a Fuchsian group such that the quotient $\mathbb{H}^2/\Gamma$ has a single cusp. Assume that ∞ is fixed by a parabolic element in Γ, so that the orbit of ∞ under the action of Γ equals the cusp set of Γ. Since $\Gamma \leq PSL(2, \mathbb{Q})$ then $\Gamma \cdot \infty \subseteq \mathbb{Q} \cup \{\infty\}$. Therefore to prove cusps$(\Gamma) = \mathbb{Q} \cup \{\infty\}$ we only need to see that $\mathbb{Q} \subseteq \Gamma \cdot \infty$. To check this Long and Reid introduced the following concept in example 1 of [5].

Definition 2.6 Let $p \in \mathbb{Q}$ be a cusp of Γ. A *killer interval I around p* is an interval $I \subset \mathbb{R}$ with $p \in I$ for which there exists $\gamma \in \Gamma$ such that if $k \in I$ is a rational number, then the absolute value of the denominator of $\gamma(k)$ is strictly smaller than that of k.

If $\mathbb{R}$ can be covered by killer intervals then for every $k \in \mathbb{Q}$ there will be a $\gamma \in \Gamma$ such that $\gamma(k) = \infty$. It is easy to see that every rational cusp of Γ has a killer interval around it. In detail, if $\gamma \in \Gamma < PSL(2, \mathbb{Q})$ is parabolic then we can always find a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbb{Q})$ such that $a, b, c, d \in \mathbb{Z}$, $\gcd(a, b, c, d) = 1$ and both γ and g have the same action on $\mathbb{H}^2$. Then $\frac{a}{c}$ is a cusp of Γ and $(\frac{a}{c} - \frac{1}{c}, \frac{a}{c} + \frac{1}{c})$ is a killer interval around it with associated map γ.

Definition 2.7 Let $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and suppose Γ contains some power of L. Then

$$\ell(\Gamma) = \min\{k \in \mathbb{Z}_{>0} \mid L^k \in \Gamma\}$$

is the *fundamental length of* Γ. A *fundamental interval* for Γ is any interval $[k, k + \ell(\Gamma)]$ with $k \in \mathbb{Z}$.

If $I = [k, k+\ell(\Gamma)]$ is a fundamental interval for Γ then every $x \in \mathbb{R}$ can be moved into I by a power of $L^{\ell(\Gamma)}$. Translating by a power of L does not increase the denominator of a rational number. Then to prove $\mathbb{Q} \subseteq \Gamma \cdot \infty$ we just have to cover a fundamental interval of Γ with killer intervals.

Now let $J_{m,n}$ be the jigsaw formed by the Δ_1 tile with vertices -1, 0 and ∞ followed by $m - 1 \geq 0$ tiles of type 1 glued to its left and $n \geq 1$ tiles of type 4 glued to its right, so that all tiles in $J_{m,n}$ share ∞ as a common vertex (see figure 2.1). Let $N = m+n$ and $v_0, v_1, \ldots, v_{N+1}$ be the cyclically ordered vertices of $J_{m,n}$, so that $v_0 = \infty, v_1 < v_2 < \ldots < v_{N+1}$. For each $0 \leq i \leq N$ let x_i be the marked point on the side $[v_i, v_{i+1}]$ and x_{N+1} be the marked point on $[v_{N+1}, v_0]$. Let $\Gamma_{m,n}$ be the jigsaw group associated to $J_{m,n}$. If ρ_i is the π-rotation around x_i then $\Gamma_{m,n} = \langle \rho_0, \rho_1, \ldots, \rho_{N+1} \rangle$. Clearly the vertices $v_1, \ldots, v_{N+1}$ are in the orbit of v_0 so $\mathbb{H}^2/\Gamma_{m,n}$ has a single cusp. Proposition 4.5 in [7] proves that

$$\rho_{N+1}\rho_N \ldots \rho_0 = \begin{pmatrix} 1 & \ell(\Gamma_{m,n}) \\ 0 & 1 \end{pmatrix} \tag{2.3}$$

and $\ell(\Gamma_{m,n}) = 3m + 6n$. Then $v_0 = \infty$ is fixed by a parabolic element of $\Gamma_{m,n}$. This implies that every vertex of a tile in the triangulation of $\mathbb{H}^2$ induced by $J_{m,n}$ is a cusp of $\Gamma_{m,n}$.

In the following let $\Delta(a, b, c)$ be the ideal triangle with vertices a, b, c and sides $[a, b]$, $[b, c]$ and $[c, a]$. Denote the π-rotation about a point $(x, y) \in \mathbb{H}^2$ by $R_{x,y}$. When $x, y \in \mathbb{Q}$ it is possible to represent $R_{x,y}$ as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbb{Q})$ with $\gcd(a, b, c, d) = 1$, this will allow us to calculate lengths of killer intervals.

In the triangulation of $\mathbb{H}^2$ produced by a jigsaw a *vertical tile* is one that has ∞ as a vertex. A *vertical side* of the triangulation induced by a jigsaw on $\mathbb{H}^2$ is one that has ∞ as an endpoint, it can be interior or exterior.

Proposition 2.8 (4.3 in [3]) *Let $T = \Delta(\infty, x_1, x_2)$ be a vertical tile of type 4 in the triangulation of $\mathbb{H}^2$ produced by a jigsaw. Let the sides of T be $e_1 = [\infty, x_1]$, $e_2 = [x_1, x_2]$ and $e_3 = [x_2, \infty]$, so that e_i has type k_i and marked point p_i. Then there are three possible configurations for T:*

- *if $k_1 = 1$ then $k_2 = \frac{1}{4}$, $k_3 = 4$ and $x_2 = x_1 + 1$. The marked points are $p_1 = (x_1, 1)$, $p_2 = (x_1 + \frac{1}{5}, \frac{2}{5})$ and $p_3 = (x_1 + 1, 2)$. The vertical tile to the right of T has type 4.*

- *if $k_1 = 4$ then $k_2 = 1$, $k_3 = \frac{1}{4}$ and $x_2 = x_1 + 4$. The marked points are $p_1 = (x_1, 2)$, $p_2 = (x_1 + 2, 2)$ and $p_3 = (x_1 + 4, 2)$. The vertical tiles to the right and left of T have type 4.*

- *if $k_1 = \frac{1}{4}$ then $k_2 = 4$, $k_3 = 1$ and $x_2 = x_1 + 1$. The marked points are $p_1 = (x_1, 2)$, $p_2 = (x_1 + \frac{4}{5}, \frac{2}{5})$ and $p_3 = (x_1 + 1, 1)$. The vertical tile to the left of T has type 4.*

All vertical tiles of type 1 are of the form $\Delta(\infty, x, x+1)$ with $m \in \mathbb{Z}$. The marked points on the sides $[\infty, x]$, $[x, x+1]$ and $[x+1, \infty]$ are $(x, 1)$, $(x+\frac{1}{2}, \frac{1}{2})$ and $(x+1, 1)$ respectively.

In all figures a solid line indicates an exterior side of a tile and a dashed line indicates an exterior side. Dotted lines indicate sides that could be either interior or exterior.

Theorem 2.9 *Let $J_{m,n}$ be a jigsaw as in theorem 2.4 and Γ its associated jigsaw group. Then cusps$(\Gamma) = \mathbb{Q} \cup \{\infty\}$.*

Proof. We will prove there is a covering of $\mathbb{R}$ by killer intervals of cusps of Γ. Consider the triangulation of $\mathbb{H}^2$ generated by the action of Γ on the triangulated jigsaw $J_{m,n}$. Since all tiles in $J_{m,n}$ are of type 1 or 4, proposition 4.3 in [7] implies that the vertices of a vertical tile that lie on $\mathbb{R}$ are integers at distance 1 or 4 from each other. Then $\mathbb{R}$ can be divided into consecutive intervals of lengths one and four, with each endpoint being an integer.

If v is an endpoint of a vertical side with $v \neq \infty$, then v is a cusp of the jigsaw group and by proposition 4.6 in [7] the killer interval around v is $(v-1, v+1)$. Then to cover $\mathbb{R}$ with killer intervals it will be enough to cover the gaps of length 4 between cusps.

By proposition 2.8 a vertical tile T_0 with vertices m and $m+4$ has to be a tile of type 4 where the side $[m, m+4]$ is of type 1. Without loss of generality we may translate this tile and assume $T_0 = \Delta(\infty, 0, 4)$, its marked points are $(0, 2)$, $(2, 2)$ and $(4, 2)$. Let $T_1 = R_{2,2}(T_0)$, so T_1 is adjacent to T_0 along the side $[0, 4]$ and has vertices 4, 0 and 2.

Case 1: T_1 is a tile of type 4. For this case see figure 2.2. Since 0 and 4 are endpoints of a vertical side the killer intervals around these cusps are $(-1, 1)$ and $(3, 5)$. The tile T_1 is type 4, so by proposition 4.7 in [7] the killer interval around 2 is $(1, 3)$. Then to cover the interval $[0, 4]$ it will be enough to check that 1 and 3 are cusps of Γ. We will use the following matrices for calculations:

$$R_{2,2} = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix}, \; R_{3,1} = \begin{pmatrix} -3 & 10 \\ -1 & 3 \end{pmatrix}, \; R_{\frac{16}{5},\frac{2}{5}} = \begin{pmatrix} -16 & 52 \\ -5 & 16 \end{pmatrix}.$$

Since $T_1 = R_{2,2}(T_0)$, then the side $[2, 4]$ of T_1 is type 4 with marked point $R_{2,2}(0, 2) =$

$(3,1)$. The tile adjacent to T_1 along $[2,4]$ is $T_2 = R_{3,1}(T_1) = \Delta(4,2,\frac{10}{3})$, it is of type 4 as well. The side $[2,\frac{10}{3}]$ has type 1 with marked point $R_{3,1}(2,2) = (\frac{16}{5},\frac{2}{5})$. Finally, consider the tile T_3 that is adjacent to T_2 along $[2,\frac{10}{3}]$. We have that $T_3 = R_{\frac{16}{5},\frac{2}{5}}(T_2) = \Delta(2,\frac{10}{3},3)$. This proves 3 is a vertex of a tile in the triangulation and therefore a cusp of Γ. By a similar argument we can prove 1 is a cusp of Γ. Notice this case also covers all jigsaws of the form $J_{0,n}$.

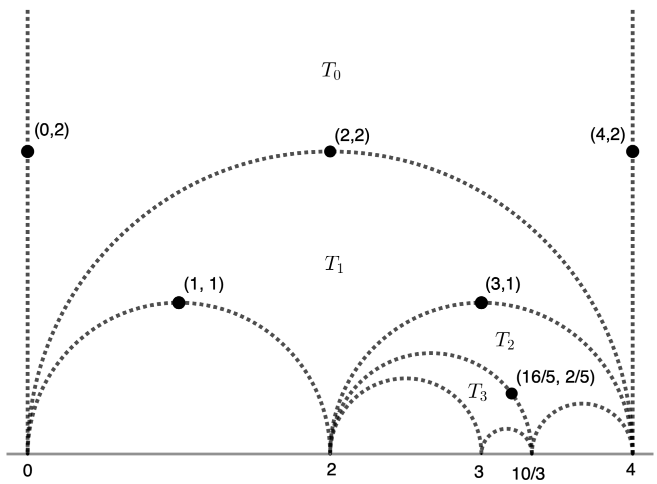

Figure 2.2: Tile T_1 is of type 4

Case 2: T_1 is a tile of type 1. Since 0 and 4 are endpoints of vertical sides, the killer intervals around them still are $(-1,1)$ and $(3,5)$. The tile T_1 now has type 1, so by proposition 4.7 [7] the killer interval around 2 is $(\frac{3}{2},\frac{5}{2})$. We will see that $(1,\frac{5}{3})$ and $(\frac{7}{3},3)$ are killer intervals for $\frac{4}{3}$ and $\frac{8}{3}$ respectively. Then the killer intervals for 0, $\frac{4}{3}$, 2, $\frac{8}{3}$ and 4 will cover $[0,1]\setminus\{1,3\}$. To finish it will only be necessary to check that 1 and 3 are cusps of Γ.

If the side $[0,4]$ of T_0 was exterior, then by rotating around the marked point $(2,2)$ we would get that T_1 is also of type 4. Thus it must be that $[0,4]$ is an interior side. Then T_0 is in the Γ-orbit of the unique tile T_0' of type 4 in the initial jigsaw $J_{m,n}$ that shares an interior side with a tile of type 1. Since T_0' has an exterior side of type $\frac{1}{4}$ (see figure 2.1) then in T_0 the side $[4, \infty]$, which has type $\frac{1}{4}$, must be exterior too. The side $[0, \infty]$ of T_0 is only exterior when $n = 1$. For a jigsaw $J_{1,n}$ the tiling follows the pattern shown in figure 2.3 and for a jigsaw $J_{m,n}$ with $m \geq 2$ the tiling is as in figure 2.4.

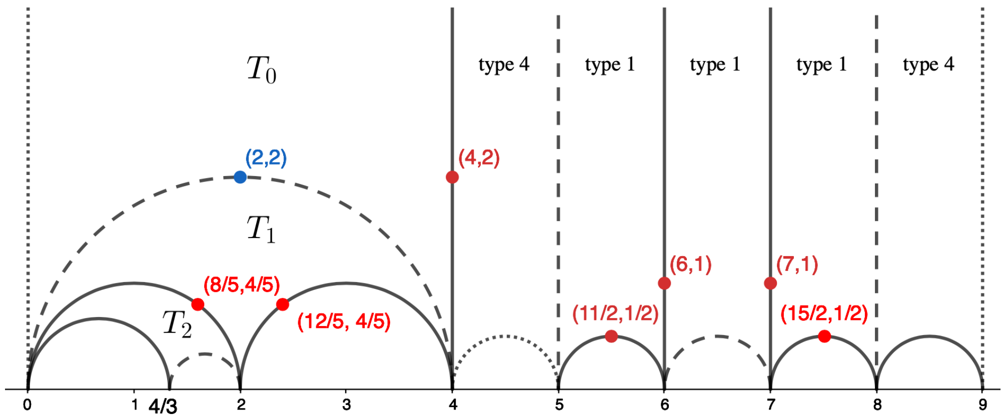

Figure 2.3: Vertical tiles for a jigsaw $J_{1,n}$

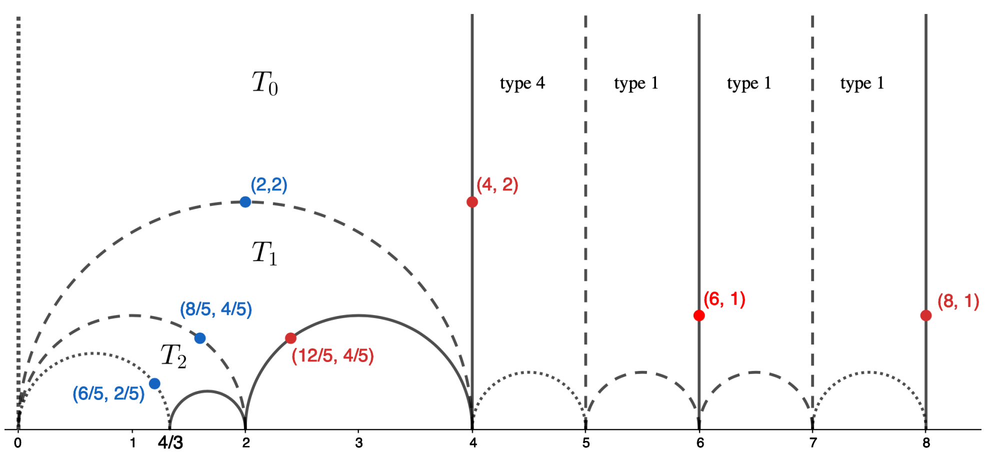

Figure 2.4: Vertical tiles for a jigsaw $J_{m,n}$ with $m \geq 2$

• 3 is a cusp. Notice that $[4, \infty]$ is an exterior side with marked point $(4, 2)$ and 8 is a vertex of the tiling for every $J_{m,n}$. Then $R_{4,2}(3) = 8$ implies 3 is a cusp of Γ. To find an element of Γ that sends ∞ to 3 we will have to consider two cases. For a jigsaw

$J_{1,n}$ the sides $[\infty, 7]$ and $[7, 8]$ are exterior of type 1 with marked points $(7, 1)$ and $(\frac{15}{2}, \frac{1}{2})$ respectively. Rotating around these points we get that $S = R_{4,2} R_{\frac{15}{2},\frac{1}{2}} R_{7,1} \in \Gamma$ sends ∞ to 3. For $J_{m,n}$ with $m \geq 2$ the side $[8, \infty]$ is exterior with marked point $(8, 1)$. Therefore $S' = R_{4,2} R_{8,1} \in \Gamma$ sends ∞ to 3. It can be calculated that

$$S = \begin{pmatrix} 12 & -64 \\ 4 & -21 \end{pmatrix} \quad \text{and} \quad S' = \begin{pmatrix} 12 & -100 \\ 4 & -33 \end{pmatrix},$$

so in both cases we obtain that $(3 - \frac{1}{4}, 3 + \frac{1}{4})$ is a killer interval around 3.

• $\frac{4}{3}$ is a cusp. The side $[2, 4]$ is exterior with marked point $(\frac{12}{5}, \frac{4}{5})$ in every $J_{m,n}$, so $R_{\frac{12}{5},\frac{4}{5}} \in \Gamma$. Since 3 is a cusp of Γ, then $\frac{4}{3} = R_{\frac{12}{5},\frac{4}{5}}(3)$ is too. For $J_{1,n}$ we get that $R_{\frac{12}{5},\frac{4}{5}} S \in \Gamma$ sends ∞ to $\frac{4}{3}$. For $J_{m,n}$ with $m \geq 2$ we see that $R_{\frac{12}{5},\frac{4}{5}} S' \in \Gamma$ does the same. By making $R_{\frac{12}{5},\frac{4}{5}} = \begin{pmatrix} 12 & -32 \\ 5 & -12 \end{pmatrix}$ we have that

$$R_{\frac{12}{5},\frac{4}{5}} S = \begin{pmatrix} 4 & -24 \\ 3 & -17 \end{pmatrix} \quad \text{and} \quad R_{\frac{12}{5},\frac{4}{5}} S' = \begin{pmatrix} 4 & -36 \\ 3 & -26 \end{pmatrix}.$$

This shows the killer interval around $\frac{4}{3}$ is $(1, \frac{5}{3})$.

• 1 is a cusp. The marked point on the side $[0, 2]$ of T_1 is $(\frac{8}{5}, \frac{4}{5})$, then $T_2 = R_{\frac{8}{5},\frac{4}{5}}(T_1) = \Delta(0, \frac{4}{3}, 2)$ is in the triangulation. For every $J_{m,n}$ the tile T_2 is of type 1, so the marked point on the side $[0, \frac{4}{3}]$ is $(\frac{6}{5}, \frac{2}{5}) = R_{\frac{8}{5},\frac{4}{5}}((2, 2))$. Therefore $1 = R_{\frac{6}{5},\frac{2}{5}}(2)$ is a cusp.

• $\frac{8}{3}$ is a cusp. Since $R_{\frac{12}{5},\frac{4}{5}} \in \Gamma$ for all $J_{m,n}$, then $\frac{8}{3} = R_{\frac{12}{5},\frac{4}{5}}(0)$ is a vertex of the tiling and a cusp of Γ. To find an element in Γ that sends ∞ to $\frac{8}{3}$ recall that T_0 is in the Γ-orbit of the unique tile of type 4 in $J_{m,n}$ that has an interior side adjacent to a tile of type one. Then there must be an $n \in \mathbb{N}$ and $G \in \Gamma$ so that the tile $T = \Delta(\infty, n, n + 1)$ is in the triangulation, has sides $[\infty, n]$, $[n, n + 1]$ and $[n + 1, \infty]$ of types 1, $\frac{1}{4}$ and 4 respectively, and $G(T) = T_0$. In particular we have that $G(\infty) = 0$. A direct calculation shows we can write $G = \begin{pmatrix} 0 & 4 \\ -1 & n+1 \end{pmatrix}$. Thus $R_{\frac{12}{5},\frac{4}{5}}G = \begin{pmatrix} 8 & 4-8n \\ 3 & 2-3n \end{pmatrix} \in \Gamma$ sends ∞ to $\frac{8}{3}$. This shows the killer interval around $\frac{8}{3}$ is $(\frac{7}{3}, 3)$. $\square$

2.3 Non-commensurability of the $\Gamma_{1,n}$ and $\Gamma_{m,1}$ jigsaw groups

The *commensurator* of a subgroup Γ of $PSL(2,\mathbb{R})$ is the subgroup

$$\mathrm{Comm}(\Gamma) = \{g \in PSL(2,\mathbb{R}) \mid g\Gamma g^{-1} \text{ commensurable with } \Gamma\}.$$

It is a theorem by Margulis [9] that if Γ is non-arithmetic then $\mathrm{Comm}(\Gamma)$ is the unique maximal element (with respect to subgroup inclusion) in the commensurability class of Γ. Following sections 7 and 8 in [7], to see that jigsaw groups Γ of the form $\Gamma_{1,n}$ and $\Gamma_{m,1}$ are pairwise non-commensurable we will check that each Γ is non-arithmetic and $\Gamma = \mathrm{Comm}(\Gamma)$. To prove the latter we analyze the location of tangency points on the maximal horocycle of the orbifold $\mathbb{H}^2/\Gamma$.

2.3.1 Non-arithmeticity.

By Takeuchi [10] if a non-compact Fuchsian group $\Gamma \leq PSL(2,\mathbb{R})$ of finite covolume, with no elements of order 2 and with invariant trace field $\mathbb{Q}$ is arithmetic, then $\mathrm{tr}(\gamma^2) \in \mathbb{Z}$ for all $\gamma \in \Gamma$. Since $\mathrm{tr}(\gamma^2) = (\mathrm{tr}\gamma)^2 - 2$ it is enough to see whether $(\mathrm{tr}\gamma)^2 \in \mathbb{Z}$.

Let J be a jigsaw as in theorem 2.4 with associated jigsaw group Γ, and let $\rho_0, \ldots, \rho_{N+1}$ be the generators of Γ as in (2.3). We will see the subgroup of index two $\Gamma^{(2)}$ consisting of all elements of Γ with even word length is non-arithmetic, and therefore Γ is non-arithmetic too. Notice the group $\Gamma^{(2)}$ still has finite covolume, a fundamental domain for $\Gamma^{(2)}$ is $J \cup \rho_0(J)$.

Proposition 2.10 *Let $J_{m,n}$ be a jigsaw as in theorem 2.4 and Γ its associated jigsaw group. Then Γ is non-arithmetic.*

Proof. It is enough to see that there exists $\gamma \in \Gamma^{(2)}$ such that $\mathrm{tr}(\gamma)^2 \notin \mathbb{Z}$. Let $e_j = [x_j, \infty]$ and $e_k = [x_k, \infty]$ be exterior vertical sides in the tiling of $\mathbb{H}^2$ induced by $J_{m,n}$. Assume that e_j is of type 4 and e_k is of type 1, so their marked points are $(x_j, 2)$ and $(x_k, 1)$ respectively. Since e_j and e_k are exterior sides the π-rotations

$$R_{x_j,2} = \frac{1}{2}\begin{pmatrix} x_j & -(x_j^2+4) \\ 1 & -x_j \end{pmatrix} \text{ and } = R_{x_k,1} = \begin{pmatrix} x_k & -(x_k^2+1) \\ 1 & -x_k \end{pmatrix}$$

are elements of Γ. We have that $(\mathrm{tr}(R_{x_j,2}R_{x_k,2}))^2 = \frac{1}{4}(-(x_k - x_j)^2 - 5)^2$, so

$$\begin{aligned} (\mathrm{tr}(R_{x_j,2}R_{x_k,2}))^2 \in \mathbb{Z} &\Leftrightarrow (-(x_k - x_j)^2 - 5)^2 \equiv 0 \pmod 4 \\ &\Leftrightarrow -(x_k - x_j)^2 - 5 \equiv 0 \pmod 2 \\ &\Leftrightarrow (x_k - x_j)^2 \equiv 1 \pmod 2 \\ &\Leftrightarrow x_k - x_j \equiv 1 \pmod 2. \end{aligned}$$

Then if $\Gamma^{(2)}$ is arithmetic the distance between the real vertex of a vertical side of type 1 and the real vertex of a vertical side of type 4 must be odd. However, the jigsaw $J_{m,n}$ has a tile T of type 1 with two exterior sides, so there is a tile in the orbit $\Gamma \cdot T$ where both exterior type 1 sides are vertical and at distance one from each other. Therefore one of these consecutive exterior vertical sides of type 1 will be at even distance from a vertical side of type 4.

□

2.3.2 Tangency points of maximal horocycle

Let Γ be the jigsaw group associated to a jigsaw $J = J_{m,n}$ as in theorem 2.4. Then the orbifold $\mathcal{O} = \mathbb{H}^2/\Gamma$ has $N = m + n$ cone points of order 2, a cusp and finite volume. Let $\pi\colon \mathbb{H}^2 \to \mathcal{O}$ be the corresponding quotient map. The lift of the cone points of $\mathcal{O}$ to $\mathbb{H}^2$ is the set of all marked points on exterior sides in the tiling $\Gamma \cdot J$ of $\mathbb{H}^2$. Since $J_{m,n}$ only has tiles of type 1 and type 4, by proposition 2.8 all the marked points in the tiling are on or below the line $y = 2$.

Recall that a *horocycle in* $\mathbb{H}^2$ centered at $\xi \in \partial_\infty \mathbb{H}^2 \equiv \mathbb{R} \cup \{\infty\}$ is a curve $\alpha \setminus \{\xi\} \subset \mathbb{H}^2$ where α is a Euclidean circle tangent to $\mathbb{R}$ at ξ, if $\xi \in \mathbb{R}$, or α is a line parallel to the x-axis if $\xi = \infty$. A curve C in $\mathcal{O}$ is a horocycle if C is the image under π of a horocycle in $\mathbb{H}^2$ and does not self-cross. For $t > 0$ let α_t be the line $y = t$. When $t > 2$ the horocycle $\pi(\alpha_t)$ loops once around the cusp of $\mathcal{O}$ without self-intersecting and the length of $\pi(\alpha_t)$ goes to 0 as t goes to ∞. The maximal horocycle in $\mathcal{O}$ is then $C = \pi(\alpha_2)$. The curve C is tangent to itself at the cone points of $\mathcal{O}$ that are projections of marked points of exterior sides in $\mathbb{H}^2$ with y-coordinate equal to 2. The lift $\tilde{C} = \pi^{-1}(C)$ to $\mathbb{H}^2$ is formed by the horizontal horocycle α_2, horocycles of radius 1 which are tangent to α_2, and smaller horocycles based at the other cusps which are disjoint from α_2.

To prove that $\Gamma_{m,n} = \mathrm{Comm}(\Gamma_{m,n})$ when $m = 1$ or $n = 1$ we will analyze the location of tangency points of $\tilde{C}$ along α_2. This will be used to see that $\mathbb{H}^2/\Gamma_{m,n}$ cannot be a proper finite cover of the orbifold $\mathbb{H}^2/\mathrm{Comm}(\Gamma_{m,n})$. Recall that the horizontal translation in $\mathbb{H}^2$ by $\ell(\Gamma_{m,n}) = 3m + 6n$ is the smallest horizontal translation that is an element of Γ.

Lemma 2.11 *Let $J = J_{m,1}$ with $m \geq 1$ and associated jigsaw group Γ. Let T be a horizontal translation by less than $\ell(\Gamma)$. Then there is a pair of tangency points p_1, p_2 of $\tilde{C}$ such that, if $L \neq Id$ is a horizontal translation by less than $\ell(\Gamma)$, then $L(p_1)$ and $L(p_2)$ are no longer tangency points of $\tilde{C}$.*

Proof. The tile $E = \Delta(\infty, 0, 1)$ is the unique tile of type 4 in J. Its sides $[0, 1]$ and $[1, \infty]$ are exterior of type $\frac{1}{4}$ and 4 respectively, the marked point on $[1, \infty]$ is $p_1 = (1, 2)$. Then $E' = R_{1,2}(E) = \Delta(\infty, 1, 5)$ is of type 4 and has $[5, \infty]$ as an exterior side with marked point $p_2 = (5, 2)$. The next tile $E'' = R_{5,2}(E') = \Delta(\infty, 5, 6)$ is of type 4 but now $[6, \infty]$ is an interior side and therefore there is no (exterior) marked point on it. Since E is the unique tile of type 4 in J, the tiles E, E' and E'' are the only vertical tiles of type 4 with vertices on the fundamental interval $I = [0, 3m + 6]$. Therefore p_1 and p_2 are the only two tangency points of $\tilde{C}$ at height 2 on $I \times [0, \infty)$. If L is a translation by $0 < k < \ell(\Gamma)$ where $L(p_1)$ is a tangency point of $\tilde{C}$, then it must be that $k = 4$ and $L(p_1) = p_2$. But $L(p_2)$ is on a vertical type 1 side, so it cannot be a tangency point.

□

To prove a similar result for jigsaws $J_{1,n}$ with $n > 1$ we will need not a pair but a triple of tangency points on α_2. To find these we examine patterns of consecutive vertical tiles.

Definition 2.12 The *width* of a vertical tile is the distance between its vertices on the x-axis. For $i = 1, \ldots, k$ let $T_i = \Delta(\infty, x_i, x_{i+1})$ be a vertical tile with $x_i < x_{i+1}$ and width w_i. The *width pattern* of the consecutive tiles $T_1, \ldots, T_m$ is the tuple $(w_1, \ldots, w_m)$.

By proposition 2.8, tiles of type 1 always have width 1 and tiles of type 4 have width either 1 or 4. In the proof of 2.13 we will also use *half tiles*, these are translations of either $\Delta(\infty, 0, 4) \cap ([0, 2] \times \mathbb{R})$ or $\Delta(\infty, 0, 4) \cap ([2, 4] \times \mathbb{R})$. Half tiles have width 2 and can be included in a list of adjacent vertical tiles to generate a width pattern. The width pattern $(2,2)$ is allowed to indicate two halves of the same tile of type 4 and width 4. We will need half tiles to account for marked points on the non-vertical sides of tiles of type 4. Finally, notice that not any tuple with coordinates 1, 2 or 4 corresponds to a width pattern. For example $(4,4)$, $(2,4)$ and $(4,2)$ would indicate two adjacent tiles of type 4 and width 4, which cannot be by proposition 2.8. And since half tiles are actually part of a "full" tile, we cannot have $(1,2,1)$ in any width pattern.

Lemma 2.13 *Let $J = J_{1,n}$ with $n > 1$ and associated jigsaw group Γ. Then there is a triple of tangency points p_1, p_2, p_3 of $\tilde{C}$ such that, if $L \neq Id$ is a horizontal translation by less than $\ell(\Gamma)$, then $L(p_1)$, $L(p_2)$ and $L(p_3)$ are no longer tangency points of $\tilde{C}$.*

Proof. Let $J = J_{m,n}$ be a jigsaw as in theorem 2.4, so that every tile in J is vertical. Let E be the unique tile of type 4 in J that has two exterior sides. We will consider cases depending on the congruency of n modulo 3.

- Case 1: $n \equiv 0 \pmod 3$, $n \geq 3$. The tangency points in $\alpha_2 \cap J$ are the points $(3 + 6(j-1), 2)$ with $j = 1, \ldots, \frac{n}{3}$ (see figure 2.5). Consider $p_1 = (2n-3, 2)$, the last of these points. The vertical tile adjacent to J to the right of $[2n, \infty]$ is of type 4, and since $n \equiv 0 \pmod 3$ it has width 1. The sides $[\infty, 2n]$ and $[2n+1, \infty]$ are exterior of types 1 and 4 respectively, so $p_2 = (2n+1, 2)$ is the next tangency point on α_2. The vertical tile to the right of $[2n+1, \infty]$ is of type 4 and width 4. Since the side $[2n+1, 2n+5]$ is exterior of type 1, the next tangency point on α_2 is $p_3 = (2n+3, 2)$. Let L be a horizontal translation by less than $\ell(\Gamma)$ and suppose $L(p_i) = p'_i \in \alpha_2$ are tangency points of $\tilde{C}$. Since $d_{euc}(p'_2, p'_3) = d_{euc}(p_2, p_3) = 2$ the width pattern of the tiles between p'_2 and p'_3 is $(1,1)$ or (2).

To get the width pattern (1,1) with the desired tangency points p'_2 and p'_3 we need two tiles $\Delta(\infty, k, k+1)$ and $\Delta(\infty, k+1, k+2)$ of type 4 and width 1 with exterior vertical

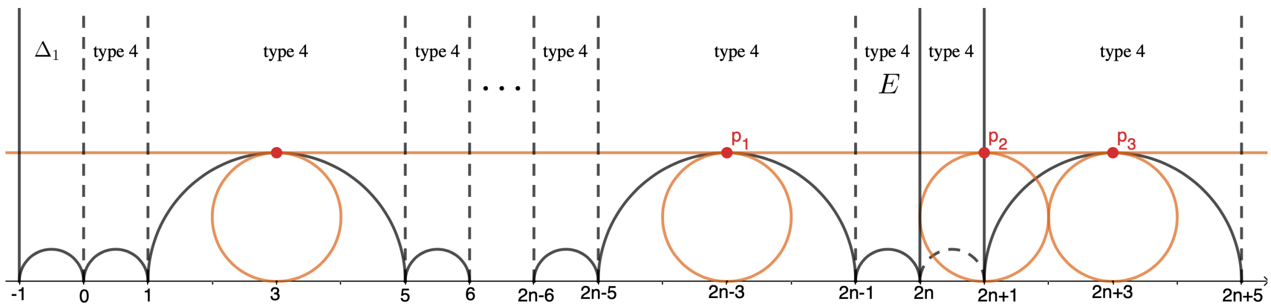

Figure 2.5: Tangency points for $n \equiv 0 \pmod 3$

sides $[k, \infty]$ and $[k+2, \infty]$ of types $\frac{1}{4}$ and 4 respectively. Since the tile E of J has interior side of type $\frac{1}{4}$, the tile $\Delta(\infty, k, k+1)$ is not in the orbit $\Gamma \cdot E$. Thus the sides $[k, k+1]$ and $[k+1, \infty]$ are interior. Then in the adjacent tile $\Delta(\infty, k-4, k)$ the sides $[k-4, \infty]$ and $[k-4, k]$ are interior. Since $4 = d_{euc}(p_1, p_2) = d_{euc}(p_1', p_2')$, then $p_1' = (k-4, 2)$ which is not a tangency point.

If the width pattern between p_2' and p_3' is (2) we must have a tile $T = \Delta(\infty, k, k+4)$ of type 4 and exterior sides $[\infty, k]$ and $[k, k+4]$. The two tiles that follow T to the left must be of type 4 and width 1, with exterior sides $[k-1, \infty]$ and $[k-2, k-1]$. Thus the tile $\Delta(\infty, k-2, k-1)$ is a vertical tile in the orbit $\Gamma \cdot E$. This implies we obtain $\Delta(\infty, k-2, k-1)$ by translating E by a multiple of $\ell(\Gamma)$ and L must be this translation. Therefore $L = Id$.

• Case 2: $n \equiv 1 \pmod 3$ and $n \geq 4$. In this case the tangency points in $\alpha_2 \cap J$ which are not on vertical sides are the points $(3+6(j-1), 2)$, with $j = 1 \ldots \frac{n-1}{3}$. Let $p_1 = (2n-5, 2)$ be the last of such tangency points. Since $n \equiv 1 \pmod 3$ the next vertical tile to the right of J is of type 4 and width 4. The sides $[\infty, 2n-1]$ and $[2n+3, \infty]$ are exterior with types 4 and $\frac{1}{4}$ respectively, so $p_2 = (2n-1, 2)$ and $p_3 = (2n+3, 2)$ are tangency points of the horocycle (see figure 2.6). As before, assume L is a horizontal translation by less than $\ell(\Gamma)$ and $p_i' = L(p_i)$ are tangency points. Since $d_{euc}(p_1', p_2') = d_{euc}(p_1, p_2) = 4$ the possible width patterns for tiles between p_1' and p_2' are (4), (1,1,2) , (2,1,1) and (1,1,1,1).

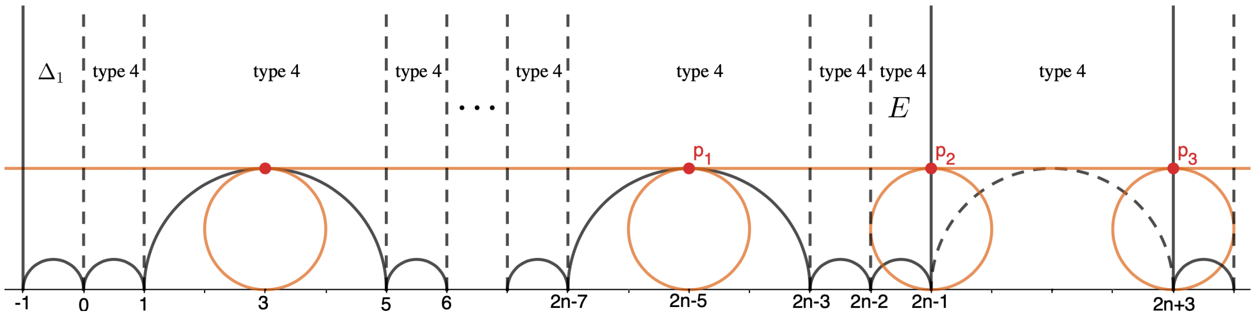

Figure 2.6: Tangency points for $n \equiv 1 \pmod 3$ for $n \geq 4$

To get the width pattern (4) between p_1' and p_2' we need a tile $E' = \Delta(\infty, k, k+4)$ of type 4 with vertical external sides, so E' is in the Γ-orbit of E (see figure 2.7). Since J has more than one tile of type 4, the tile $T_1 = \Delta(k, k+4, k+2)$ adjacent to E' must be of type 4 with external side $[k+2, k+4]$. Let $R \in \Gamma$ be the π-rotation about the marked point p_2' on $[k+4, \infty]$. Then $T_2 = R(T_1) = \Delta(\infty, k+5, k+6)$ has $[k+6, \infty]$ as an exterior vertical side of type 4. It follows that the tile adjacent to T_2 to the right has width 4 with interior side $[k+6, k+10]$. Since $d_{euc}(p_2', p_3') = d_{euc}(p_2, p_3) = 4$, then $p_3' = (k+8, 2)$ which is not a tangency point.

If the width pattern between p_1' and p_2' is (1,1,2) then we must have a tile $T_1 = \Delta(\infty, k, k+4)$ of type 4 where the side $[k, k+4]$ is exterior. Since the tile E in J has interior side of type 1, $T_1 \notin \Gamma \cdot E$. Then the sides $[\infty, k]$ and $[k+4, \infty]$ are interior (see figure 2.8). The tiles $T_2 = \Delta(\infty, k-1, k)$ and $T_3 = \Delta(\infty, k-2, k-1)$ to the left of T_1 must be of type 4 and width one, with $[k-1, \infty]$ interior of type 1 and $[k-2, \infty]$ exterior of type $\frac{1}{4}$. Having two interior vertical sides implies that T_2 is in a translation of the initial jigsaw J, and thus T_3 is too. However in J the exterior vertical sides have types 1 and 4, so this tile configuration is not possible.

If we have width pattern (2,1,1) or (1,1,1,1) between p_1' and p_2' then there is a tile

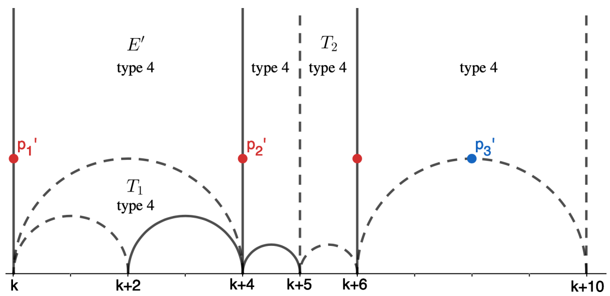

Figure 2.7: Width pattern (4)

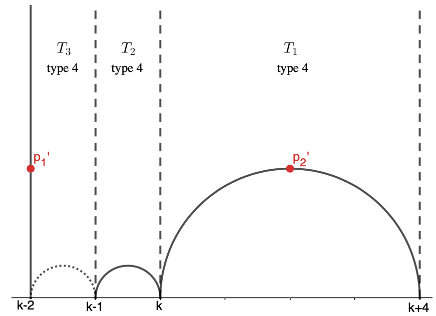

Figure 2.8: Width pattern (1,1,2)

$T_1 = \Delta(\infty, k, k+1)$ of type 4 whose side $[k+1, \infty]$ is exterior of type 4 and has p_2' in it. The tile adjacent to T_1 to the right has to be of type 4 and width 4, let this be $T_2 = \Delta(\infty, k+1, k+5)$. Because $d_{euc}(p_2', p_3') = 4$, we have that $p_3' \in [k+5, \infty]$, so this side must be exterior. This implies $[k, \infty]$ is exterior too and so $T_3 = \Delta(\infty, k-1, k)$ is of type 4 with exterior sides $[k-1, k]$ and $[k, \infty]$. Thus T_3 is obtained by translating the tile E in J by a multiple of $\ell(\Gamma)$. Since L must be this translation, we get that $L = Id$.

- Case 3: $n \equiv 2 \pmod 3$. The last tangency points on $\alpha_2 \cap J$ are $p_1 = (2n-1, 2)$ and $p_2 = (2n+1, 2)$. The next tangency point along α_2 is $p_3 = (2n+7, 2)$ (see figure 2.9). Let L be a horizontal translation by less than $\ell(\Gamma)$ and suppose $L(p_i) = p_i' \in \alpha_2$ are tangency points of $\tilde{C}$. Since $d_{euc}(p_1', p_2') = d_{euc}(p_1, p_2) = 2$ the width pattern of the tiles between p_1' and p_2' is (1,1) or (2).

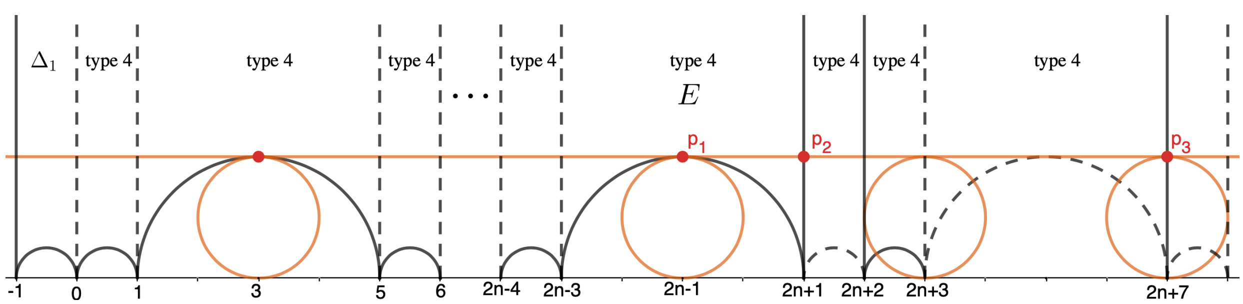

Figure 2.9: Tangency points for $n \equiv 2 \pmod 3$

Suppose the width pattern between p_1' and p_2' is (1,1). Then we have two adjacent tiles $\Delta(\infty, k, k+1)$ and $T_1 = \Delta(\infty, k+1, k+2)$ with exterior vertical sides $[k, \infty]$ and $[k+2, \infty]$ of types $\frac{1}{4}$ and 4 respectively. Their common side $[k+1, \infty]$ is interior of type 1. Since the tile E in J has an exterior sides of type 1, these tiles are not in the Γ-orbit of E. Thus the sides $[k, k+1]$ and $[k+1, k+2]$ must be interior. The tile to the right of T_1 has type 4 and width 4, with exterior sides $[k+2, k+6$ and $[k+6, \infty]$. Then the tile $T_2 = \Delta(\infty, k+6, k+7)$ must be of type 4 with side $[k+6, \infty]$ of type $\frac{1}{4}$. Notice that if v is a vertex of a tile in J which is not E or Δ_1, then three sides of tiles in J meet at v, two exterior and one interior. Then by looking at the vertex ∞ of T_2 we get that the side $[k+7, \infty]$ is exterior of type 1. Then $\Delta(\infty, k+7, k+8)$ is a tile of type 4 in the tiling and its side $[k+8, \infty]$ is exterior. Since $d_{euc}(p_2', p_3') = d_{euc}(p_2, p_3) = 6$ we have that $p_3' = (k+8, 2)$ is not a tangency point.

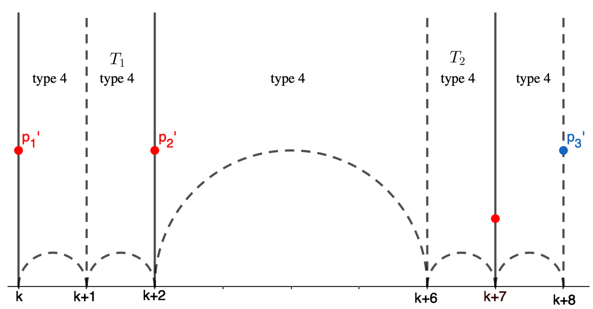

Figure 2.10: Width pattern (1,1)

If (2) is the width pattern between p_1' and p_2' then these tangency points are on a tile $\Delta(\infty, k, k+4)$ with exterior sides $[k, k+4]$ and $[k+4, \infty]$ of types 1 and $\frac{1}{4}$. This tile must then be a translation of E in J by a multiple of $\ell(\Gamma)$. As before this leads to $L = Id$.

□

Corollary 2.14 *Let $J_{m,n}$ with $m = 1$ or $n = 1$ be a jigsaw as in theorem 2.4 and $\Gamma_{m,n}$ its associated jigsaw group. Let $\tilde{C}$ be the preimage in $\mathbb{H}^2$ of the maximal horocycle of the* *orbifold* $\mathcal{O} = \mathbb{H}^2/\Gamma_{m,n}$ *and* L *a horizontal translation of* $\mathbb{H}^2$ *by less than* $\ell(\Gamma_{m,n})$*. Then* $L(\tilde{C}) \neq \tilde{C}$*.*

Proof. A horizontal translation that preserves $\tilde{C}$ must preserve the set of tangency points of $\tilde{C}$. Lemmas 2.11 and 2.13 show this is not possible for a translation by less than $\ell(\Gamma_{m,n})$.

□

Proposition 2.15 *Two distinct groups in the families* $\Gamma_{1,n}$ *and* $\Gamma_{m,1}$ *are non-commensurable.*

Proof. Let $\Gamma = \Gamma_{m,n}$ be a jigsaw group with $m = 1$ or $n = 1$ and let $[\Gamma]$ be its commensurability class. The group Γ is non-arithmetic by proposition 2.10, so its commensurator $\mathrm{Comm}(\Gamma)$ is the unique maximal element in $[\Gamma]$ [9]. In terms of covering spaces $\mathrm{Comm}(\Gamma)$ is the fundamental group of a unique minimal orbifold $\mathcal{O}' = \mathbb{H}^2/\mathrm{Comm}(\Gamma)$ which is finitely covered by any other orbifold $\mathbb{H}^2/G$ with $G \in [\Gamma]$.

Suppose that Γ is a proper subgroup of $\mathrm{Comm}(\Gamma)$ and let $\mathcal{O} = \mathbb{H}^2/\Gamma$. Let C be the maximal horocycle in $\mathcal{O}'$ and $\tilde{C}$ be the preimage of C to $\mathbb{H}^2$. The orbifold $\mathcal{O}$ has a single cusp, so $\tilde{C}$ is the preimage of the maximal horocycle in $\mathcal{O}$ too. Since $\mathcal{O}$ covers $\mathcal{O}'$ the lift $\tilde{C}$ must be invariant under a horizontal translation by k where $0 < k < \ell(\Gamma)$. However, this would contradict corollary 2.14. Therefore $\Gamma = \mathrm{Comm}(\Gamma)$.

□

Theorem 2.9 and proposition 2.15 complete the proof of theorem 1.

2.4 Non-pseudomodular Weierstrass groups

In this final section we prove theorem 2.5 which states the Weierstrass groups W_n with $n \geq 6$ congruent to 0, 2 or 6 mod 8 are not pseudomodular.

By definition a pseudomodular group Γ is discrete, so no element in $\mathbb{R}\cup\{\infty\} \equiv \partial_\infty \mathbb{H}^2$ can be simultaneously fixed by a parabolic and a hyperbolic element in Γ. Then to see that a given $\Gamma < PGL(2,\mathbb{Q})$ is not pseudomodular it suffices to find a hyperbolic element in Γ that fixes a rational number. Following [5] we call such a hyperbolic element a *special element* of Γ, and its fixed points *special points* in $\mathbb{Q}$. By constructing special elements we prove infinitely many of the W_n jigsaw groups are not pseudomodular. This result provides a partial answer to whether all but finitely many Weierstrass groups are non-pseudomodular, a question posed at the end of [7].

Proposition 2.16 *The Weierstrass groups W_n with $n > 2$ and congruent to 0, 2 or 6 mod 8 contain a special element.*

Proof. From definition 2.2 the generators of W_n can be represented by the matrices

$$a = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad b = \sqrt{n}\begin{pmatrix} 1 & 1 \\ \frac{-n-1}{n} & -1 \end{pmatrix}, \quad c = \frac{1}{\sqrt{n}}\begin{pmatrix} 0 & n \\ -1 & 0 \end{pmatrix}$$

in $PSL(2,\mathbb{R})$. Recall that a matrix in $PSL(2,\mathbb{R})$ represents a hyperbolic element of the isometries of $\mathbb{H}^2$ if its trace is bigger than 2. Let us examine each congruency class separately.

- Case 1: $n = 8k$, for $k \geq 1$. Consider $A = cba = \begin{pmatrix} 1 & 8k+2 \\ 0 & 1 \end{pmatrix} \in W_n$. A direct calculation shows that $cabaA^{k-1}caba$ has trace $4k + \frac{1}{4k} > 2$ and its fixed points are the integers $-4k$ and $4k-2$, so this is a special element in W_{8k}.

- Case 2: $n = 8k+2$, for $k \geq 1$. Let $A = abc = \begin{pmatrix} 1 & -8k-4 \\ 0 & 1 \end{pmatrix} \in W_{8k+2}$. It can be directly calculated that the rationals $\frac{2}{8k+1}$ and $\frac{-8k-2}{4k+3}$ are fixed points of $cA^{4k-1}ababaA^{-k+1}ca$, thus this is a special element in W_{8k+2}.

- Case 3: $n = 8k + 6$, for $k \geq 0$. Let $A = cba = \begin{pmatrix} 1 & 8k+8 \\ 0 & 1 \end{pmatrix} \in W_{8k+6}$. A direct computation shows that the element $aA^k cababac$ fixes 1 and has trace $9 + 24k + 16k^2 + \frac{1}{(3+4k)^2}$, which is greater than 2. Thus we have found a special element in W_{8k+6}.

$\square$

Theorem 2 immediately follows from the previous proposition.

We have found that W_n has a special element for small values of $n \equiv 4 \pmod 8$, though no clear pattern in the fixed points or the word of the element is clear. The computer program we developed to obtain these examples tries to determine whether a given rational is a special point by exploring its Γ-orbit. A survey of whether W_n has a special for $n \leq 28$ follows.

Group	**Pseudomodular or special**	**Group**	**Pseudomodular or special**
W_1	pseudomodular [7]	W_{15}	contains a special
W_2	pseudomodular [7]	W_{16}	contains a special
W_3	contains a special	W_{17}	contains a special
W_4	pseudomodular (theorem 2.4)	W_{18}	contains a special
W_5	contains a special	W_{19}	could not be determined
W_6	contains a special	W_{20}	contains a special
W_7	contains a special	W_{21}	contains a special
W_8	contains a special	W_{22}	contains a special
W_9	contains a special	W_{23}	contains a special
W_{10}	contains a special	W_{24}	contains a special
W_{11}	could not be determined	W_{25}	could not be determined
W_{12}	contains a special	W_{26}	contains a special
W_{13}	could not be determined	W_{27}	contains a special
W_{14}	contains a special	W_{28}	contains a special

Table 2.1: Survey of small Weierstrass groups

Chapter 3

Zariski dense surface subgroups in $SL(n, \mathbb{Q})$

Constructing Zariski dense surface subgroups in $SL(n, \mathbb{R})$ has attracted attention as a step to finding *thin groups*, these are infinite index subgroups of a lattice in $SL(n, \mathbb{R})$ which are Zariski dense. Finding thin subgroups inside lattices in a variety of Lie groups has been a topic of significant interest in recent years, in part from the connections thin groups have to expanders and the affine sieve of Bourgain, Gamburd, and Sarnak [11][12].

Though thin subgroups are in a sense generic [13][14], finding particular specimens of thin surface subgroups in a given lattice remains a difficult task. In this direction Long, Reid and Thistlethwaite [15] produced in 2011 the first infinite family of nonconjugate thin surface groups in $SL(3, \mathbb{Z})$. Their approach relies on parametrizing a family of representations ρ_t of the triangle group $\Delta(3, 3, 4)$ in the *Hitchin component*, so that for every $t \in \mathbb{Z}$ the subgroup $\rho_t(\Delta(3, 3, 4))$ is in $SL(3, \mathbb{Q})$ and has integral traces. By results of Bass [16] these two properties together with $\rho_t(\Delta(3, 3, 4))$ being non-solvable and finitely generated guarantee that it is conjugate to a subgroup of $SL(3, \mathbb{Z})$. In 2018 Long and Thistlethwaite [17] used a similar approach to obtain an infinite family of non-conjugate

Zariski dense surface subgroups in $SL(4,\mathbb{Z})$ and $SL(5,\mathbb{Z})$.

Ballas and Long [18] in turn used the idea of "bending" a representation of the fundamental group of a hyperbolic n-manifold $\pi_1(N)$ along an embedded totally geodesic and separating hypersurface to obtain thin groups in $SL(n+1,\mathbb{R})$ which are isomorphic to $\pi_1(N)$. The goal of this chapter is to combine the aforementioned approaches to construct a family of Zariski dense rational surface group representations by bending orbifold representations. Our main result is the following:

Theorem 3.1 *For every surface S finitely covering the orbifold $\mathcal{O}_{3,3,3,3}$ and every odd $n>1$ there exists a path of discrete, faithful and irreducible representations $\rho_t\colon \pi_1(S)\to SL(n,\mathbb{R})$, so that*

1. $\rho_0(\pi_1(S)) < SL(n,\mathbb{Z})$,
2. ρ_t *is Zariski dense for every* $t>0$ *and*
3. $\rho_t(\pi_1(S)) < SL(n,\mathbb{Q})$ *for every* $t\in\mathbb{Q}$.

Every representation ρ_t in theorem 3.1 is a surface Hitchin representation. Several of its properties are derived from the seminal work of Labourie [19] on Anosov representations, the classification of Zariski closures of surface Hitchin representations by Guichard [20] and the recent introduction of orbifold Hitchin representations by Alessandrini, Lee and Schaffhauser [21]. We provide an overview of these results in sections 3.1 and 3.2. At the end of section 3.2 we also prove the following criterion for Zariski density, which will be subsequently used to discard Zariski closures.

Proposition 3.2 *Let $\rho\colon \pi_1(\mathcal{O})\to PSL(n,\mathbb{R})$ be an orbifold Hitchin representation such that*

- *if $n=2k$ is even then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSp(2k,\mathbb{R})$ or,*

- *if $n = 2k+1$ is odd then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSO(k, k+1)$.*

Then $\rho(H)$ is Zariski dense in $PSL(n, \mathbb{R})$ for every finite index subgroup H of $\pi_1(\mathcal{O})$.

In section 3.3 we give a general construction to obtain a path of representations as in theorem 3.1. This is based on bending the fundamental group $\pi_1(\mathcal{O})$ of a hyperbolic 2-dimensional orbifold along a simple closed curve in $\mathcal{O}$ with infinite order as an element of $\pi_1(\mathcal{O})$. Theorem 3.1 then follows from applying the results in section 2 to a suitable representation of the fundamental group of the orbifold $\mathcal{O}_{3,3,3,3}$ whose underlying topological space is S^2 and has four cone points of order 3. This final step is covered in section 3.4.

Remark. During the finalization of this project, Long and Thistlethwaite used bending to construct thin surface groups in $SL(n, \mathbb{Z})$ for every odd n [22], the even case remains open.

3.1 Hitchin representations

In this section we give an introduction to surface and orbifold Hitchin representations. We will make extensive use of these representations in this and the upcoming chapters.

Recall a subgroup $H < GL(n, \mathbb{R})$ is *irreducible* if the only invariant subspaces for the action of H on $\mathbb{R}^n$ are $\{0\}$ and $\mathbb{R}^n$. A representation $\rho\colon \Gamma \to GL(n, \mathbb{R})$ is said to be *irreducible* if the image subgroup $\rho(\Gamma)$ is irreducible, and it is is *strongly irreducible* if the restriction of ρ to every finite index subgroup is irreducible. These characteristics are defined similarly for projective representations $\rho\colon \Gamma \to PGL(n, \mathbb{R})$.

3.1.1 Spaces of representations

Let G be a Lie group and let Γ be a group with a finite presentation $\langle \alpha_1, \ldots, \alpha_k \mid r_1, \ldots, r_m \rangle$. Then every relator r_i defines a map $R_i \colon G^k \to G$. If we let $\mathrm{Hom}(\Gamma, G) = \cap_{i=1}^m R_i^{-1}(Id)$, then the map $\phi \mapsto (\phi(\alpha_1), \ldots, \phi(\alpha_k))$ is a bijection between the set of all group homomorphisms from Γ to G and $\mathrm{Hom}(\Gamma, G)$. We will regard $\mathrm{Hom}(\Gamma, G)$ as having the subspace topology from G^k. In general $\mathrm{Hom}(\Gamma, G)$ may have singularities and not be a manifold.

The group G acts by conjugation on $\mathrm{Hom}(\Gamma, G)$, though this action may not be proper or free. Thus the orbit space

$$\mathrm{Rep}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/G$$

is possibly non-Hausdorff or may have orbifold singularities. In the case of $G = PSL(n, \mathbb{R})$ (or more generally when G is a reductive group ([23] sec. 1.6)) it is useful to restrict the G-action to a smaller set of $\mathrm{Hom}(\Gamma, G)$ to obtain better topological properties in the orbit space. Let $\mathrm{Hom}^+(\Gamma, G)$ be the subset of representations in $\mathrm{Hom}(\Gamma, G)$ which decompose as a direct sum of irreducible representations and let

$$\mathrm{Rep}^+(\Gamma, G) = \mathrm{Hom}^+(\Gamma, G)/G$$

be the quotient space by the conjugation action. With the quotient topology $\mathrm{Rep}^+(\Gamma, G)$ has the structure of an algebraic variety ([23] sec. 5.2)

3.1.2 The irreducible representation $\omega_n\colon PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$

In the following we will make extensive use of the representation

$$\tilde{\omega}_n\colon SL(2,\mathbb{R}) \to SL(n,\mathbb{R}) \tag{3.1}$$

given by the action of $SL(2,\mathbb{R})$ on the vector space $\mathcal{P}$ of homogeneous polynomials in 2 variables of degree $n-1$. To construct this representation fix the ordered basis $\mathcal{B} = \{x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1}\}$ of $\mathcal{P}$. If we identify the variables x, y with the canonical unit vectors e_1, e_2 in $\mathbb{R}^2$ then for any matrix in $SL(2,\mathbb{R})$ we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = ax + by \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y = bx + dy.$$

For any $g \in SL(2,\mathbb{R})$ we define $\tilde{\omega}_n(g) \in GL(\mathcal{P})$ by its action on the basis $\mathcal{B}$:

$$\tilde{\omega}_n(g)(x^{n-1-i}y^i) = (g \cdot x)^{n-1-i}(g \cdot y)^i.$$

If $n = 2k$ is even, the image of $\tilde{\omega}_n$ is contained in the symplectic group $Sp(2k,\mathbb{R})$, and if $n = 2k+1$ is odd, it is contained in a group isomorphic to $SO(k, k+1)$.

It is well known that the representation $\tilde{\omega}_n$ is absolutely irreducible. For completeness we sketch a proof of this fact (for details see [24] sec. 4.2). Since $SL(2,\mathbb{R})$ is a connected Lie group, the representation $\tilde{\omega}_n$ is irreducible if and only if the associated representation of Lie algebras $w_n\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R})$ is irreducible. Furthermore, w_n is irreducible if and only if the unique complex-linear extension $w_n\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ is irreducible. The matrices $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ form a basis for $\mathfrak{sl}(2,\mathbb{R})$ and it can be calculated that for every element $x^{n-1-k}y^k$ in the basis of the homogeneous polynomial space $\mathcal{P}$ we have that

$$w_n(X)(x^{n-1-k}y^k) = -(n-1-k)x^{n-2-k}y^{k+1} \text{ and } w_n(Y)(x^{n-1-k}y^k) = -kx^{n-k}y^{k-1}.$$

If W is a non-zero invariant subspace of $w_n(\mathfrak{sl}(2,\mathbb{R}))$ and $p \in W$ is a non-zero polynomial, then we just need to apply $w_n(X)$ enough times to p to get a non-zero multiple of y^{n-1} in W, and thus $y^{n-1} \in W$. Then applying $w_n(Y)$ to y^{n-1} we will obtain that each vector in the basis of $\mathcal{P}$ is in W. Up to conjugation $\tilde{\omega}_n$ is the unique irreducible representation from $SL(2,\mathbb{R})$ into $SL(n,\mathbb{R})$ ([24] thm. 4.32).

The representation $\tilde{\omega}_n \colon SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ induces a *projective representation* $\omega_n \colon PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ which is also irreducible and unique up to conjugation (see [25] def. 16.45).

3.1.3 Hitchin representations of surface groups

Let S be a closed surface of genus $g > 1$. In 1988 Goldman proved that $\mathrm{Rep}^+(\pi_1(S), PSL(2,\mathbb{R}))$ has $4g-3$ connected components, two of which are diffeomorphic to $\mathbb{R}^{6g-6}$ and called these *Teichmüller spaces* ([26] thm. A, see also note at end of thm. 10.2 in [27]). The two Teichmüller spaces $\mathcal{T}^{\pm}(S)$ are precisely the sets of conjugacy classes by $PSL(2,\mathbb{R})$ of *Fuchsian representations*, which are discrete and faithful representations $\rho \colon \pi_1(S) \to PSL(2,\mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{H}^2)$. As explained in section 1.2, Teichmüller spaces can be identified with equivalence classes of holonomies of hyperbolic structures on S, each component corresponding to whether the associated developing map preserves fixed (arbitrary) orientations on S and $\mathbb{H}^2$. When we look at the bigger representation space $\mathrm{Rep}^+(\pi_1(S), PGL(2,\mathbb{R}))$ then the equivalence classes of discrete and faithful representations $\pi_1(S) \to PGL(2,\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^2)$ form a single component $\mathcal{T}(S)$, also known as Teichmüller space ([23] sec. 4.3).

Definition 3.3 For $n > 2$ a representation $r\colon \pi_1(S) \to PSL(n,\mathbb{R})$ is called *Fuchsian* if it can be decomposed as $r = \omega_n \circ r_0$ where $r_0\colon \pi_1(S) \to PSL(2,\mathbb{R})$ is discrete and faithful, and $\omega_n\colon PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ is the unique irreducible representation introduced in subsection 3.1.2

Definition 3.4 The *Fuchsian locus* is the set of all $PSL(n,\mathbb{R})$ conjugacy classes of Fuchsian representations, namely the set $\omega_n(\mathcal{T}^{\pm}(S))$.

In 1992 Hitchin ([27], thm. 10.2) used the theory of Higgs bundles to prove that for $n > 2$ the space $\mathrm{Rep}^+(\pi_1(S), PSL(n,\mathbb{R}))$ has three topological connected components if n is odd and 6 if n is even. The Fuchsian locus is contained in one component in the odd case and in two components in the even case. Moreover, he proved each of these distinguished components, which are now called *Hitchin components*, is diffeomorphic to $\mathbb{R}^{(1-n^2)(1-g)}$. When $n > 2$ is even both Hitchin components are related by an inner automorphism of $PSL(n,\mathbb{R})$. In the odd case, where there is only one, we will denote the Hitchin component by $\mathrm{Hit}(\pi_1(S), PSL(n,\mathbb{R}))$.

Definition 3.5 Let S be a closed surface of genus greater than one. A representation $r\colon \pi_1(S) \to PSL(n,\mathbb{R})$ is a *surface Hitchin representation* if its $PSL(n,\mathbb{R})$-conjugacy class belongs to a Hitchin component of $\mathrm{Rep}^+(\pi_1(S), PSL(n,\mathbb{R}))$.

In the introduction of [27] Hitchin noted that the analytical point of view he used to examine the topology of $\mathrm{Rep}^+(\pi_1(S), PSL(n,\mathbb{R}))$ does not cast light on the geometric meaning of surface Hitchin representations, and so the parallelism between the Hitchin component and Teichmüller space remained incomplete. A year later, in 1993, Choi and Goldman proved that $\mathrm{Hit}(\pi_1(S), PSL(3,\mathbb{R}))$ parametrizes convex real projective structures on S [28][29]. More than a decade later Labourie defines *P-Anosov representations* as holonomies of Anosov dynamical structures on the unit tangent bundle of the surface

S which depend on a choice of parabolic subgroup P of $PSL(n,\mathbb{R})$ ([19] sec. 2.0.1). In chapter 4 we will examine an equivalent definition of Anosov representations from the perspective of geometric group theory. For now we limit ourselves to using these representations and their characteristics. In [19] Labourie proves that surface Hitchin representations are B-Anosov where B is any Borel subgroup of $PSL(n,\mathbb{R})$ and uses this geometric condition to prove important properties of Hitchin representations.

Definition 3.6 ([30] sec. 2.2) A matrix $A \in SL(n,\mathbb{R})$ is *purely loxodromic* if it is diagonalizable over $\mathbb{R}$ with eigenvalues of distinct modulus. If $A \in PSL(n,\mathbb{R})$ then we say A is *purely loxodromic* if any lift of A to an element of $SL(n,\mathbb{R})$ is purely loxodromic.

Theorem 3.7 ([19] thm. 1.5, lemma 10.1) *A surface Hitchin representation $r\colon \pi_1(S) \to PSL(n,\mathbb{R})$ is discrete, faithful and strongly irreducible. Moreover, the image of every non-trivial element of $\pi_1(S)$ under r is purely loxodromic.*

3.1.4 Hitchin representations of orbifold groups

Now let $\mathcal{O}$ be a 2-dimensional closed orbifold of negative orbifold Euler characteristic $\chi(\mathcal{O})$ and let $\pi_1(\mathcal{O})$ be its orbifold fundamental group. For a definition of orbifold Euler characteristic see section 13.3 of [31]. In [31] Thurston proves there is a connected component of the representation space $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(2,\mathbb{R}))$ which is homeomorphic to a ball of dimension $-3|\chi(\mathcal{O})| + 2k + l$, where k is the number of cone points and l the number of corner reflectors in $\mathcal{O}$. Moreover, he showed this component parametrizes hyperbolic structures on $\mathcal{O}$ and consists precisely of conjugacy classes of discrete and faithful representations of $\pi_1(\mathcal{O})$ into $PGL(2,\mathbb{R}) \equiv \mathrm{Isom}(\mathbb{H}^2)$, which we will call *Fuchsian representations* too. By analogy with surfaces, this component is called the *Teichmüller space* of the orbifold $\mathcal{O}$, we will denote it by $\mathcal{T}(\mathcal{O})$.

In 2019 Alessandrini, Lee and Schaffhauser used the irreducible representation ω_n to define the *Hitchin component* $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ as the unique connected component in this representation space which contains the connected Fuchsian locus $\omega_n(\mathcal{T}(\mathcal{O}))$ ([21] def. 2.3). They prove $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ is homeomorphic to an open ball and give a formula for its dimension in terms of the cone points and corner reflectors of $\mathcal{O}$ and the exponents of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$([21] thm. 1.2).

Definition 3.8 ([21] def. 2.4) Let $\mathcal{O}$ be a 2-dimensional connected closed orbifold with negative orbifold Euler characteristic. A representation $r\colon \pi_1(\mathcal{O}) \to PGL(n, \mathbb{R})$ is an *orbifold Hitchin representation* if its $PGL(n, \mathbb{R})$-conjugacy class belongs to the Hitchin component $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$.

Guichard and Wienhard in 2012 ([32] def. 2.10) generalized the definition of P-Anosov representations to allow for representations of word hyperbolic groups into semisimple Lie groups and proved they share many of the properties that Labourie's P-Anosov surface groups representations have. With this new definition it is natural to inquire whether orbifold Hitchin representations are Anosov. This property follows from the fact that if $\Gamma' < \Gamma$ is a finite index subgroup, then $r\colon \Gamma \to G$ is P-Anosov if and only if $r\colon \Gamma' \to G$ is P-Anosov ([33] prop. 2.8) and that every orbifold $\mathcal{O}$ of negative Euler characteristic is finitely covered by a surface of genus greater than one. Indeed, just as their surface counterparts, orbifold Hitchin representations are also B-Anosov where B is a Borel subgroup of $PGL(n, \mathbb{R})$ ([21] prop. 2.16). This geometric characterization has been an important tool for examining them.

Theorem 3.9 ([21] thm. 1.1) *An orbifold Hitchin representation $r\colon \pi_1(\mathcal{O}) \to PGL(n, \mathbb{R})$ is discrete, faithful and strongly irreducible. Moreover, the image of every infinite order element of $\pi_1(\mathcal{O})$ under r is purely loxodromic.*

3.2 Zariski dense Hitchin representations

In this section we focus on Zariski density of Hitchin representations and prove corollary 3.17 which gives a criterion to determine when the image of a finite index subgroup of an orbifold group under a Hitchin representation is Zariski dense.

3.2.1 Zariski dense representations

Let G be an algebraic matrix Lie group, then G has both its standard topology as a subset of some $\mathbb{R}^N$ and the Zariski topology, where closed sets are defined by zero sets of polynomials. If X is a subset of G then its *Zariski closure* is the closure of X in G with respect to the Zariski topology. The Zariski closure of a subgroup $H < G$ is the smallest algebraic subgroup of G which contains H ([34] lemma 2.1), and is a closed Lie subgroup of G. We say a subgroup $H < G$ is *Zariski dense* in G if its Zariski closure equals G. A representation $r\colon \Gamma \to G$ is *Zariski dense* if $r(\Gamma)$ is Zariski dense in G. For a survey of this subject see part 1 of [34] and chapter 1 in [35].

Proposition 3.10 *Let V be an n-dimensional vector space, $H < GL(V)$ a subgroup and $\overline{H}$ its Zariski closure. Then H is irreducible if and only if $\overline{H}$ is irreducible.*

Proof. If $\overline{H}$ has an invariant non-zero proper subspace $W < V$ then W is also an invariant subspace for $H < \overline{H}$.

Now suppose that H has an invariant subspace W of dimension $0 < k < n$. We can extend a basis for W to a basis of V to get a matrix $g \in GL(V)$ such that the conjugated subgroup gHg^{-1} is block triangular, i.e. for every $h \in H$

$$ghg^{-1} = \begin{pmatrix} A_{k\times k}(h) & * \\ 0 & B_{(n-k)\times(n-k)}(h) \end{pmatrix}.$$

Let $p_{ij} \in \mathbb{F}[x_1, \ldots, x_{n^2}]$ be such that $p_{ij}(x) = x_{ij}$ for every $x = (x_{ij}) \in GL(V)$ and denote its zero set in $GL(V)$ by Z_{ij}, each Z_{ij} is a closed set in the Zariski topology. Since $gHg^{-1} \subset Z_{ij}$ for every $k+1 \leq i \leq n$ and $1 \leq j \leq k$, then $\overline{gHg^{-1}} \subset Z_{ij}$ for the same indices. Since conjugation by g is a homeomorphism in the Zariski topology we have that $g\overline{H}g^{-1} = \overline{gHg^{-1}}$. Therefore $g\overline{H}g^{-1}$ has the same block triangular form as gHg^{-1} and thus has W as an invariant non-zero proper subspace.

□

Corollary 3.11 *Let $r\colon H \to GL(V)$ be a representation. Then r is irreducible if and only if the Zariski closure of its image $\overline{r(H)} < GL(V)$ is irreducible.*

3.2.2 Zariski closures of Hitchin representations

The image of the irreducible representation $\omega_n\colon PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ is contained in a conjugate of the projectivization of the symplectic group $PSp(n,\mathbb{R})$ if n is even, and if $n = 2k+1$ is odd it is contained in a conjugate of the orthogonal group $SO(k,k+1) = PSO(k,k+1)$. This implies that the images of Fuchsian representations are contained in (a conjugate of) $PSp(n,\mathbb{R})$ or $SO(k,k+1)$ depending on n, in particular they are not Zariski dense. More generally, for surface Hitchin representations Guichard [20] has announced a classification of Zariski closures of their lifts. An alternative proof of this result has been given recently by Sambarino ([36] cor. 1.5). The version of this result we cite here comes from theorem 11.7 in [37].

Theorem 3.12 ([20], [36]) *If $r\colon \pi_1(S) \to SL(n,\mathbb{R})$ is the lift of a surface Hitchin representation and H is the Zariski closure of $r(\pi_1(S))$, then*

- *If $n = 2k$ is even, H is conjugate to either $\omega_n(SL(2,\mathbb{R}))$, $Sp(2k,\mathbb{R})$ or $SL(2k,\mathbb{R})$.*

- *If $n = 2k+1$ is odd and $n \neq 7$, then H is conjugate to either $\omega_n(SL(2,\mathbb{R}))$, $SO(k,k+1)$ or $SL(2k+1,\mathbb{R})$.*
- *If $n = 7$, then H is conjugate to either $\omega_7(SL(2,\mathbb{R}))$, G_2, $SO(3,4)$ or $SL(7,\mathbb{R})$.*

Now consider $\mathcal{O}$ a 2-dimensional closed orientable orbifold of negative orbifold Euler characteristic, finitely covered by a closed surface S. Since $\pi_1(S)$ is a normal finite index subgroup of $\pi_1(\mathcal{O})$, given an orbifold Hitchin representation $r\colon \pi_1(\mathcal{O}) \to PGL(n,\mathbb{R})$ the identity components of the Zariski closures of $r(\pi_1(\mathcal{O}))$ and $r(\pi_1(S))$ are equal. For this context we can apply Guichard's classification of Zariski closures for surface Hitchin representations to orbifold groups. In theorem 6.3 of [21] Alessandrini, Lee and Schaffhauser use this to classify orbifold groups for which all Hitchin representations into a specific $PGL(n,\mathbb{R})$ have image in (a conjugate of) the same proper algebraic subgroup. This rigidity phenomenon contrasts with surface groups, for which Zariski dense representations are dense in the Hitchin component [20].

3.2.3 A criterion for Zariski density

Here we prove proposition 3.2 which gives us a criterion to find Zariski dense Hitchin representations.

Lemma 3.13 *Let $\rho\colon \pi_1(\mathcal{O}) \to PSL(n,\mathbb{R})$ with n even be an orbifold Hitchin representation. Then for every $[\alpha] \in \pi_1(\mathcal{O})$ of infinite order there is a lift $A \in SL(n,\mathbb{R})$ of $\rho([\alpha])$ which has n positive distinct eigenvalues.*

Proof. First consider a Fuchsian representation $\sigma\colon \pi_1(\mathcal{O}) \to PSL(2,\mathbb{R})$ and $[\alpha]$ an infinite order element of $\pi_1(\mathcal{O})$. Since $\mathcal{O}$ is a hyperbolic orbifold, $\sigma([\alpha])$ is conjugate to a hyperbolic element $\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \in PSL(2,\mathbb{R})$. This element can be lifted to a matrix

$\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in SL(2,\mathbb{R})$ with $\lambda > 0$. Let $\tilde{\omega}_n \colon SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ be the unique irreducible representation in (3.1), then $\tilde{\omega}_n \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in SL(n,\mathbb{R})$ has n distinct positive eigenvalues $\lambda^{n-1}, \lambda^{n-3}, \ldots, \lambda^{-(n-3)}, \lambda^{-(n-1)}$ and is a lift of $\omega_n \circ \sigma([\alpha]) \in PSL(n,\mathbb{R})$.

Now consider a Hitchin representation $\rho\colon \pi_1(\mathcal{O}) \to PSL(n,\mathbb{R})$. Let ρ_t be a path of Hitchin representations such that ρ_0 is Fuchsian and $\rho_1 = \rho$. This induces a path $\rho_t([\alpha]) \subset PSL(n,\mathbb{R})$. By the previous argument we may lift $\rho_t([\alpha])$ to a path $\tilde{A}_t \in SL(n,\mathbb{R})$ such that $\tilde{A}_0$ has n distinct positive eigenvalues. Since each eigenvalue of $\tilde{A}_t$ varies continuously and $\det \tilde{A}_t \neq 0$, all eigenvalues of $\tilde{A}_t$ are positive. Moreover, by theorem 3.9 the absolute values of the eigenvalues of $\rho_t([\alpha])$ are distinct. This in turn implies all the eigenvalues of $\tilde{A}_t$ are distinct. Therefore $\tilde{A}_1 \in SL(n,\mathbb{R})$ is a lift of $\rho([\alpha])$ with n positive distinct eigenvalues.

□

To prove our criterion for Zariski density (propositions 3.15 and 3.16) we will make use of the following theorem by Culver.

Theorem 3.14 ([38] thm. 2) *Let C be a real square matrix. Then the equation $C = \exp(X)$ has a unique real solution X if and only if all the eigenvalues of C are positive real and no elementary divisor (Jordan block) of C belonging to any eigenvalue appears more than once.*

Proposition 3.15 *Let $\rho\colon \pi_1(\mathcal{O}) \to PSL(n,\mathbb{R})$ with n even be an orbifold Hitchin representation so that $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSp(n,\mathbb{R})$. If S is a surface finitely covering $\mathcal{O}$ then $\rho(\pi_1(S))$ is Zariski dense.*

Proof. Let S be a surface finitely covering $\mathcal{O}$ and suppose that $\rho(\pi_1(S))$ is conjugate to a subgroup of $PSp(n,\mathbb{R}) = Sp(n,\mathbb{R})/\pm I$. Then there exists an alternating form

$\Omega \in SL(n,\mathbb{R})$ such that

$$Sp(\Omega) = \{g \in SL(n,\mathbb{R}) \mid g^T \Omega g = \Omega\}$$

and $\rho(\pi_1(S)) \subset PSp(\Omega) = Sp(\Omega)/\pm I$.

Let $[\alpha] \in \pi_1(\mathcal{O})$ be an infinite order element. By lemma 3.13 we can lift $\rho([\alpha]) \in PSL(n,\mathbb{R})$ to a matrix $A \in SL(n,\mathbb{R})$ with n positive distinct eigenvalues. Since $\pi_1(S)$ has finite index in $\pi_1(\mathcal{O})$ there exists a $k \in \mathbb{N}$ such that $\rho([\alpha])^k \in \rho(\pi_1(S))$. Then A^k is a lift of $\rho([\alpha])^k$ and $A^k \in Sp(\Omega)$. Given that A has n positive distinct eigenvalues, by theorem 3.14 there is a unique $X \in M_{n\times n}(\mathbb{R})$ such that $\exp(X) = A$. Then using that $\exp(kX) = A^k$ preserves Ω we get that

$$\begin{aligned} \exp(kX)^T \Omega \exp(kX) = \Omega \quad &\Rightarrow \quad \Omega^{-1} \exp(kX)^T \Omega = \exp(kX)^{-1} \\ &\Rightarrow \quad \exp(\Omega^{-1}(kX)^T \Omega) = \Omega^{-1} \exp(kX)^T \Omega = \exp(-kX). \end{aligned}$$

Applying theorem 3.14 now to $\Omega^{-1} \exp(kX)^T \Omega$ we obtain that

$$\begin{aligned} \Omega^{-1}(kX)^T \Omega = -kX \quad &\Rightarrow \quad -\Omega(kX)^T \Omega = -kX \\ &\Rightarrow \quad \Omega(kX)^T \Omega = kX. \end{aligned}$$

This implies that $kX \in \mathfrak{sp}(\Omega)$ and thus $A = \exp(X) \in Sp(\Omega)$. Given that A is a lift of $\rho([\alpha])$, we have that $\rho([\alpha]) \in PSp(\Omega)$. Since $\pi_1(\mathcal{O})$ is generated by its infinite order elements we get that $\rho(\pi_1(\mathcal{O})) \subset PSp(\Omega)$, a contradiction. So it cannot be that $\rho(\pi_1(S))$ is conjugate to a subgroup of $PSp(n,\mathbb{R})$. In particular, if r is a lift of the Hitchin surface representation $\rho|_{\pi_1(S)}$ then the Zariski closure of $r(\pi_1(S))$ cannot be conjugate to a subgroup of $Sp(n,\mathbb{R})$. By theorem 3.12 it must be that the Zariski closure of $r(\pi_1(S))$

is $SL(n,\mathbb{R})$. Therefore the Zariski closure of $\rho(\pi_1(S))$ is $PSL(n,\mathbb{R})$.

□

In the case when $n = 2k+1$ is odd, by theorem 3.12 the Zariski closure of $\rho(\pi_1(S))$ where ρ is a surface Hitchin representation is either conjugate to a subgroup of $SO(k,k+1)$ or equals $SL(n,\mathbb{R})$. By assuming there exists a symmetric bilinear form J such that $\rho(\pi_1(S)) \subset SO(J)$ we have an analogous proof to that of 3.15 to get a criterion for Zariski density of surface Hitchin representations in the odd case.

Proposition 3.16 *Let $\rho\colon \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ with n odd be an orbifold Hitchin representation such that there is no real quadratic form J for which $\rho(\pi_1(\mathcal{O})) \subset SO(J)$. If S is a surface finitely covering $\mathcal{O}$ then $\rho(\pi_1(S))$ is Zariski dense.*

Given that any finite index subgroup of $\pi_1(\mathcal{O})$ contains a surface subgroup which has finite index in $\pi_1(\mathcal{O})$ we obtain the following result.

Proposition 3.17 *Let $\rho\colon \pi_1(\mathcal{O}) \to PSL(n,\mathbb{R})$ be an orbifold Hitchin representation such that*

- *if $n = 2k$ is even then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSp(2k,\mathbb{R})$ or,*
- *if $n = 2k+1$ is odd then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSO(k,k+1)$.*

Then for every finite index subgroup H of $\pi_1(\mathcal{O})$ the image $\rho(H)$ is Zariski dense in $PSL(n,\mathbb{R})$.

3.3 Bending representations of orbifold groups

Theorem 3.21 in this section gives a general construction of a path ρ_t of Zariski dense Hitchin surface representations into $SL(n,\mathbb{R})$ for odd n. By requiring that the initial representation ρ_0 has image inside $SL(n,\mathbb{Q})$ we obtain corollary 3.22, in which every representation ρ_t with $t \in \mathbb{Q}$ also has image in $SL(n,\mathbb{Q})$.

3.3.1 Bending representations

Let $\mathcal{O}$ be a 2-dimensional orientable connected closed orbifold of negative orbifold Euler characteristic and $\mathcal{O}_L$, $\mathcal{O}_R$ be open connected suborbifolds with connected intersection $\mathcal{O}_L \cap \mathcal{O}_R$. Then $\mathcal{O}_L$ and $\mathcal{O}_R$ satisfy the assumptions of Van Kampen's theorem for orbifolds ([39] thm. 4.7.1) and we have that $\pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\pi_1(\mathcal{O}_L\cap\mathcal{O}_R)} \pi_1(\mathcal{O}_R)$. Given a representation $\rho\colon\ \pi_1(\mathcal{O}) \to G$ there is a standard way of bending ρ by an element δ of the centralizer in G of $\rho(\pi_1(\mathcal{O}_L \cap \mathcal{O}_R))$. To do so we define a representation $\tilde{\rho}_\delta\colon\ \pi_1(\mathcal{O}_L) * \pi_1(\mathcal{O}_R) \to G$ by

$$\tilde{\rho}_\delta(\alpha) = \begin{cases} \rho(\alpha), & \alpha \in \pi_1(\mathcal{O}_L) \\ \delta\rho(\alpha)\delta^{-1}, & \alpha \in \pi_1(\mathcal{O}_R) \end{cases}.$$

Now let $i_L\colon\ \pi_1(\mathcal{O}_L\cap\mathcal{O}_R) \to \pi_1(\mathcal{O}_L)$ be the morphism induced by the inclusion $\mathcal{O}_L\cap\mathcal{O}_R \to \mathcal{O}_L$. To ease the notation, for any $\gamma \in \pi_1(\mathcal{O}_L \cap \mathcal{O}_R)$ let $\gamma_L = i_L(\gamma) \in \pi_1(\mathcal{O}_L)$. Similarly define $\gamma_R = i_R(\gamma) \in \pi_1(\mathcal{O}_R)$. Then

$$\begin{aligned} \tilde{\rho}_\delta(\gamma_L\gamma_R^{-1}) &= \tilde{\rho}_\delta(\gamma_L)\tilde{\rho}_\delta(\gamma_R)^{-1} \\ &= \rho(\gamma)(\delta\rho(\gamma)\delta^{-1})^{-1} \\ &= \rho(\gamma)(\rho(\gamma)\delta\delta^{-1})^{-1} \\ &= e. \end{aligned}$$

This shows $H = \langle \gamma_L\gamma_R^{-1} \mid \gamma \in \pi_1(L \cap R)\rangle \subset \mathrm{Ker}(\tilde{\rho}_\delta)$ and so the normal closure N of H is also contained in $\mathrm{Ker}(\tilde{\rho}_\delta)$. Thus we may pass to the quotient to obtain a representation of the orbifold fundamental group

$$\begin{aligned} \rho_\delta\colon \pi_1(\mathcal{O}) \simeq (\pi_1(L) * \pi_1(R))/N &\to G \\ \alpha N &\mapsto \tilde{\rho}_\delta(\alpha). \end{aligned}$$

From this definition we also obtain that

$$\rho_\delta(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta\rho(\pi_1(\mathcal{O}_R))\delta^{-1} \rangle.$$

From now onwards we will consider the case where there is a simple closed curve $\gamma \subset \mathcal{O}$, not parallel to a cone point, that divides $\mathcal{O}$ into two orbifolds $\mathcal{O}_L$ and $\mathcal{O}_R$ which share γ as their common boundary, so that

$$\pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle [\gamma] \rangle} \pi_1(\mathcal{O}_R).$$

Proposition 3.18 *Let $\rho\colon \pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle [\gamma] \rangle} \pi_1(\mathcal{O}_R) \to SL(n,\mathbb{Q})$ be a representation for which $\rho([\gamma])$ has n distinct positive eigenvalues. Then there exists a path of representations $\rho_t\colon \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ with $t \geq 0$ such that*

1. *$\rho_0 = \rho$,*
2. *$\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1} \rangle$ for some $\delta_t \in SL(n,\mathbb{R})$ which commutes with $\rho([\gamma])$, and*
3. *ρ_t has image in $SL(n,\mathbb{Q})$ for every $t \in \mathbb{Q}$.*

Proof. The matrix $\rho([\gamma])$ is conjugate to a diagonal matrix D with entries $\lambda_1, \ldots, \lambda_n > 0$ along its diagonal. Now for every $t > 0$ define

$$\delta_t = (t\rho([\gamma]) + I)\det(t\rho([\gamma]) + I)^{-\frac{1}{n}} \tag{3.2}$$

Notice that $\det(t\rho([\gamma])+I) = \det(tD+I) = \Pi_{k=1}^{n}(t\lambda_i+1) > 0$, so $t\rho([\gamma])+I$ is invertible for all t. Then each δ_t is in $SL(n,\mathbb{R})$ and we can check δ_t commutes with $\rho([\gamma])$. Since ρ is a rational representation, whenever $t \in \mathbb{Q}$ the matrix $t\rho([\gamma])+I$ has rational entries and non-zero determinant, thus $t\rho([\gamma])+I \in SL(n,\mathbb{Q})$ if $t \in \mathbb{Q}$.

Let $\rho_t\colon \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ be the representation such that $\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1}\rangle$. Notice that $\rho_0 = \rho$ and that for every $t \in \mathbb{Q}$ the representation ρ_t has image in $SL(n,\mathbb{Q})$.

□

3.3.2 Discarding Zariski closures

For the rest of section 3.3 we will focus on the case where $n = 2k+1$ is odd. Recall that in this case $SL(n,\mathbb{R}) \equiv PSL(n,\mathbb{R})$.

Lemma 3.19 *Let $\rho\colon \Gamma \to SL(n,\mathbb{R})$ be an irreducible representation and suppose there is a quadratic form J such that $\rho(\Gamma) \subset SO(J)$. Then J is unique up to scaling.*

Proof. Suppose $\rho(\Gamma) < SO(J_1) \cap SO(J_2)$. Then for any $\rho(\gamma) \in \rho(\Gamma)$ we have that

$$J_1^{-1}\rho(\gamma)J_1 = \rho(\gamma)^{-T} = J_2^{-1}\rho(\gamma)J_2,$$

which implies that

$$\rho(\gamma)J_1J_2^{-1} = J_1J_2^{-1}\rho(\gamma).$$

Since n is odd, $J_1J_2^{-1}$ has a real eigenvalue λ. Then $\mathrm{Ker}(J_1J_2^{-1} - \lambda I)$ is a non-zero invariant subspace for the irreducible representation ρ, which implies $J_1 = \lambda J_2$.

□

Proposition 3.20 *Let $\rho\colon \pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R) \to SL(n,\mathbb{R})$ be a representation in which the restrictions $\rho|_{\pi_1(\mathcal{O}_L)}$ and $\rho|_{\pi_1(\mathcal{O}_R)}$ are irreducible and $\rho([\gamma])$ has n positive distinct eigenvalues. Suppose there is a quadratic form J such that $\rho(\pi_1(\mathcal{O})) \subset SO(J)$. Then there exists a path of representations $\rho_t\colon \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ such that*

1. *$\rho_0 = \rho$ and*

2. *for each $t > 0$ there is no quadratic form $\tilde{J}$ such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(\tilde{J})$.*

Proof. By proposition 3.18 we can use $\delta_t = (t\rho([\gamma]) + I)\det(t\rho([\gamma]) + I)^{-\frac{1}{n}}$ to construct a path of representations $\rho_t\colon \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ such that $\rho_0 = \rho$ and $\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1}\rangle$.

Now fix $t > 0$. Suppose there exists a quadratic form $\tilde{J}$ such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(\tilde{J})$. Since $\rho(\pi_1(\mathcal{O})) \subset SO(J)$ in particular $\rho_t(\pi_1(\mathcal{O}_L)) = \rho_0(\pi_1(\mathcal{O}_L)) \subset SO(J) \cap SO(\tilde{J})$. The restriction $\rho_t|_{\pi_1(\mathcal{O}_L)}$ is irreducible, so by lemma 3.19 J is a real multiple of $\tilde{J}$. Similarly, by construction $\rho_t(\pi_1(\mathcal{O}_R)) \subset SO(\delta_t J \delta_t^T) \cap SO(\tilde{J})$ and $\rho_t|_{\pi_1(\mathcal{O}_R)}$ is irreducible too. Thus $\delta_t J \delta_t^T$ is also a multiple of $\tilde{J}$. This implies there is a $\lambda \in \mathbb{R}$ such that $\lambda J = \delta_t J \delta_t^T$ and then $\lambda^n = \det(\delta_t)^2 = 1$. Since n is odd it must be that $\lambda = 1$ and we obtain $\delta_t \in SO(J)$. Given that

$$(t\rho([\gamma]) + I)J(t\rho([\gamma])^T + I) \;=\; t^2 J \;+\; tJ(\rho([\gamma])^T)^{-1} \;+\; tJ\rho([\gamma])^T \;+\; J,$$

having $J = \delta_t J \delta_t^T$ would imply that $\mu I = \rho([\gamma])^{-1} + \rho([\gamma])$ for some $\mu \in \mathbb{R}$. Recall that $\rho([\gamma])$ is conjugate to a diagonal matrix D whose eigenvalues are all distinct. If $\mu I = \rho([\gamma])^{-1} + \rho([\gamma])$ then by conjugating we would obtain that $\mu I = D^{-1} + D$, which is not the case given that $n > 2$.

$\square$

3.3.3 Representations of surface groups

Recall we are assuming that $\mathcal{O}$ is a 2-dimensional orientable connected closed orbifold of negative orbifold Euler characteristic. Such orbifolds are always finitely covered by a surface S of genus greater than one, so $\pi_1(S)$ is a finite index subgroup of $\pi_1(\mathcal{O})$. Given a representation $\rho\colon \pi_1(\mathcal{O}) \to G$ we will denote the restriction of ρ to $\pi_1(S)$ by ρ^S.

Theorem 3.21 *Suppose $\pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R)$ with $[\gamma]$ an infinite order element. Let $\rho\colon \pi_1(\mathcal{O}) \to SL(n, \mathbb{R})$ be an orbifold Fuchsian representation such that the restrictions $\rho|_{\pi_1(\mathcal{O}_L)}$ and $\rho|_{\pi_1(\mathcal{O}_R)}$ are irreducible. If S is a surface finitely covering $\mathcal{O}$ then there exists a path of representations $\rho_t^S\colon \pi_1(S) \to SL(n, \mathbb{R})$ such that $\rho_0^S = \rho^S$ and ρ_t^S is a Zariski dense surface Hitchin representation for each $t > 0$.*

Proof. Since $\rho\colon \pi_1(\mathcal{O}) \to SL(n, \mathbb{R})$ is an orbifold Hitchin representation with odd $n = 2k+1$ and $[\gamma]$ has infinite order, then $\rho([\gamma])$ has n positive distinct real eigenvalues. Moreover, since ρ is Fuchsian its image is contained in a conjugate of $SO(k, k+1)$. Using proposition 3.20 we obtain a path of representations $\rho_t\colon \pi_1(\mathcal{O}) \to SL(n, \mathbb{R})$ such that $\rho_0 = \rho$ and for each $t > 0$ there is no real quadratic form J such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(J)$. By proposition 3.16 each $\rho_t(\pi_1(S))$ is Zariski dense in $SL(n, \mathbb{R})$.

Now consider the continuous path $[\rho_t] \in \mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ for $t \geq 0$. Its image is connected so all $PGL(n, \mathbb{R})$-conjugacy classes $[\rho_t]$ are contained in the same connected component of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$. Because the representation $\rho_0 = \rho$ is Fuchsian, $[\rho_0]$ is in the Hitchin component $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ and so is every $[\rho_t]$. Thus, by theorem 3.9, each ρ_t is discrete, faithful and strongly irreducible. Since $\pi_1(S)$ has finite index in $\pi_1(\mathcal{O})$, each restriction $\rho_t^S\colon \pi_1(S) \to SL(n, \mathbb{R})$ is irreducible. In particular ρ_0^S is a surface Fuchsian representation. Then $[\rho_t^S]$ is a continuous path in $\mathrm{Rep}^+(\pi_1(S), SL(n, \mathbb{R}))$ with $[\rho_0^S] \in \mathrm{Hit}(\pi_1(S), SL(n, \mathbb{R}))$. Since the Hitchin component is path connected $[\rho_t^S] \in \mathrm{Hit}(\pi_1(S), SL(n, \mathbb{R}))$ for all $t \geq 0$.

$\square$

To finish this section notice that the construction of the path of Zariski dense representations in the previous theorem is based on proposition 3.18, so we may add the assumption of $\rho(\pi_1(\mathcal{O})) \subset SL(n, \mathbb{Q})$ to obtain that the image of every ρ_t is in $SL(n, \mathbb{Q})$ for every $t \in \mathbb{Q}$.

Corollary 3.22 *Let $\rho\colon \pi_1(\mathcal{O}) \to PSL(n, \mathbb{Q})$ be a representation satisfying the assumptions of theorem 3.21. If S is a surface finitely covering $\mathcal{O}$ then there exists a path $\rho_t^S\colon \pi_1(S) \to SL(n, \mathbb{R})$ of Hitchin representations such that $\rho_0^S = \rho^S$, ρ_t^S is Zariski dense for each $t > 0$ and ρ_t^S has image in $SL(n, \mathbb{Q})$ for every $t \in \mathbb{Q}$.*

3.4 Representations of $\pi_1(\mathcal{O}_{3,3,3,3})$

In this section we look at the orbifold $\mathcal{O}_{3,3,3,3}$ and find a Fuchsian representation $\rho\colon \pi_1(\mathcal{O}_{3,3,3,3}) \to SL(n, \mathbb{Z})$ satisfying the assumptions of corollary 3.22.

3.4.1 The orbifold $\mathcal{O}_{3,3,3,3}$

In what follows we focus on the triangle group $\Delta(3, 4, 4) \subset PSL(2, \mathbb{R})$. Triangle groups were introduced in section 1.3. The generators of $\Delta(3, 4, 4)$ are the rotations x and y by $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ around the corresponding vertices of the triangle T with angles $\{\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}\}$. This group has presentation

$$\Delta(3, 4, 4) = \langle x, y \mid x^3 = y^4 = (xy)^4 = 1\rangle. \tag{3.3}$$

The quotient $\mathbb{H}^2/\Delta(3, 4, 4)$ is homeomorphic to the orbifold $S^2(3, 4, 4)$ whose underlying topological space is S^2 and has three cone points of orders 3, 4 and 4. This defines up to conjugation an isomorphism $\pi_1(S^2(3, 4, 4)) \to \Delta(3, 4, 4) \subset PSL(2, \mathbb{R})$.

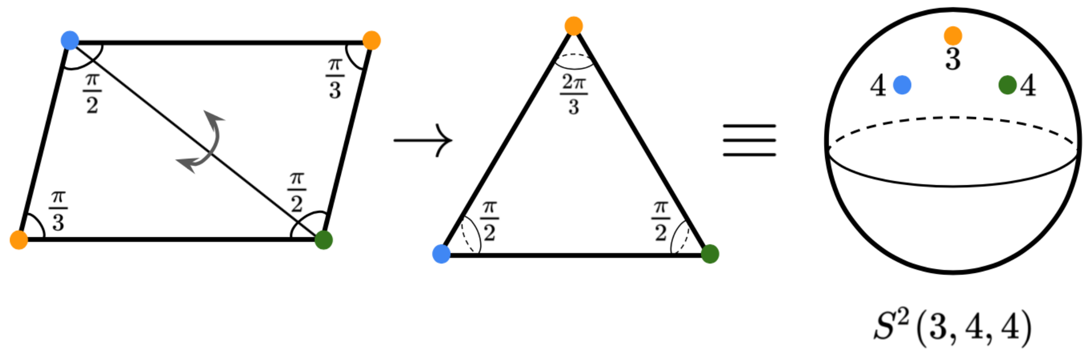

Figure 3.1: Orbifold $S^2(3, 4, 4)$

For our purposes we will need a representation of $\pi_1(S^2(3, 4, 4))$ with image in $SL(n, \mathbb{Z})$, which can be obtained by finding an integral representation of the triangle group $\Delta(3, 4, 4)$.

Proposition 3.23 ([22] thm. 2.1) *Let $\omega_n \colon PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ be the unique irreducible representation between these groups. Then for every odd n the restriction $\phi_n = \omega_n|_{\Delta(3,4,4)}$ is conjugate to a representation $\rho_n \colon \Delta(3, 4, 4) \to PSL(n, \mathbb{Z})$.*

By gluing four copies of a fundamental domain of $\Delta(3, 4, 4)$ around one of its vertices with angle $\frac{\pi}{2}$ we obtain an equilateral hyperbolic octagon K with angles alternating between $\frac{2\pi}{3}$ and $\frac{\pi}{2}$. Let $v_1, \dots, v_4$ be the vertices of K with angle $\frac{2\pi}{3}$ ordered cyclically and let θ_i be the rotation by $\frac{2\pi}{3}$ around v_i. By using the generators x, y of $\Delta(3, 4, 4)$ in the presentation 3.3 we let

$$\theta_1 = x \quad \text{and} \quad \theta_i = y\theta_{i-1}y^{-1} \quad \text{for} \quad i = 2, 3, 4. \tag{3.4}$$

Then $\langle \theta_1, \dots, \theta_4 \rangle$ is a subgroup of $\Delta(3, 4, 4)$ whose action on $\mathbb{H}^2$ has the octagon K as a fundamental domain. The quotient of $\mathbb{H}^2$ by the action of $\langle \theta_1, \dots, \theta_4 \rangle$ is homeomorphic to the orbifold $\mathcal{O}_{3,3,3,3}$ with underlying topological space S^2 and 4 cone points of order 3.

By construction we obtain that $\mathcal{O}_{3,3,3,3}$ is an index four orbifold covering of $S^2(3,4,4)$. If $\gamma_1, \ldots, \gamma_4$ are loops around the cone points of $\mathcal{O}_{3,3,3,3}$, then the orbifold fundamental group has the presentation

$$\pi_1(\mathcal{O}_{3,3,3,3}) = \langle \gamma_1, \ldots, \gamma_4 \mid \gamma_1^3 = \ldots = \gamma_4^3 = \gamma_1\gamma_2\gamma_3\gamma_4 = 1 \rangle. \tag{3.5}$$

Identifying each γ_i with the rotation θ_i gives an isomorphism $\pi_1(\mathcal{O}_{3,3,3,3}) \cong \langle \theta_1, \ldots, \theta_4 \rangle$ which defines (up to conjugation) a discrete and faithful representation

$$\sigma\colon\ \pi_1(\mathcal{O}_{3,3,3,3}) \to \Delta(3,4,4) < PSL(2, \mathbb{R}). \tag{3.6}$$

Lemma 3.24 *The representation $\sigma\colon\ \pi_1(\mathcal{O}_{3,3,3,3}) \to PSL(2,\mathbb{R})$ defined in (3.6) is Zariski dense.*

Proof. We will check that the group $\sigma(\pi_1(\mathcal{O}_{3,3,3,}))= \langle \theta_1, \ldots, \theta_4 \rangle < \Delta(3,4,4)$ is Zariski dense. Hyperbolic triangles with the same angles are isometric, so for our purposes we will fix the hyperbolic triangle with angles $\{\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}\}$ by placing it symmetrically along the y-axis in the upper-half plane. By having the generators x, y of $\Delta(3,4,4)$ defined in (3.3) in rational canonical form we obtain that:

$$x = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & -1-\sqrt{2} \\ -1+\sqrt{2} & \sqrt{2} \end{bmatrix}. \tag{3.7}$$

This choice of generators fixes a representative in the conjugacy class of the representation σ. Notice that $\theta_2\theta_1 = yxy^{-1}x$ is an infinite order element in $\Delta(3,4,4)$ and is therefore hyperbolic. By using the matrices in (3.7) we can explicitly find $P, D \in PGL(n, \mathbb{R})$ with D diagonal so that $P^{-1}(\theta_2\theta_1)P = D$. It suffices then to see that the conjugated representation $P^{-1}\sigma P$ is Zariski dense. Let H be the Zariski closure of $P^{-1}\sigma(\pi_1(\mathcal{O}_{3,3,3,3}))P$ in $PSL(2,\mathbb{R})$ and $\mathfrak{h}$ its Lie algebra. First notice that the Zariski closure of $\langle D\rangle$ is the algebraic torus whose Lie algebra is the span of $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Taking $X_2 = \mathrm{Ad}_{P^{-1}\theta_1\theta_2 P}(X_1)$ and $X_3 = \mathrm{Ad}_{P^{-1}\theta_1^2\theta_2 P}(X_1)$ we obtain three linearly independent vectors in $\mathfrak{h}$. Then $\dim(\mathfrak{h}) = 3 = \dim(\mathfrak{sl}(2,\mathbb{R}))$ so the two algebras must coincide and so $H = PSL(2,\mathbb{R})$.

□

3.4.2 Rational representations of $\pi_1(\mathcal{O}_{3,3,3,3})$

We will now focus on the case $n = 2k+1$ and the representation

$$\omega_n \circ \sigma \colon \pi_1(\mathcal{O}_{3,3,3,3}) \to SL(n,\mathbb{R}), \tag{3.8}$$

where σ is the representation defined in (3.6) and $\omega_n \colon PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R}) = SL(n,\mathbb{R})$ the irreducible representation introduced in 3.1.2. Since $\omega_n \circ \sigma$ is an orbifold Fuchsian representation, it is in particular irreducible. By proposition 3.23 we can conjugate $\omega_n \circ \sigma$ to obtain an integral representation

$$\rho \colon \pi_1(\mathcal{O}_{3,3,3,3}) \to SL(n,\mathbb{Z}) < SL(n,\mathbb{R}). \tag{3.9}$$

Now let $\gamma \subset \mathcal{O}_{3,3,3,3}$ be a simple closed loop dividing $\mathcal{O}_{3,3,3,3}$ into two orbifolds $\mathcal{O}_L$ and $\mathcal{O}_R$ which share γ as their common boundary and have two cone points of order 3 each. Then $[\gamma] \in \pi_1(\mathcal{O}_{3,3,3,3})$ is an infinite order element and $\pi_1(\mathcal{O}_{3,3,3,3}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R)$.

Proposition 3.25 *Let $\rho \colon \pi_1(\mathcal{O}_{3,3,3,3}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R) \to PSL(n,\mathbb{Z})$ be the representation defined in (3.9). Then the restrictions of ρ to $\pi_1(\mathcal{O}_L)$ and $\pi_1(\mathcal{O}_R)$ are irreducible.*

Proof. To see that $\rho|_{\pi_1(\mathcal{O}_L)}$ is irreducible it suffices to see that the restriction of $\omega_n \circ \sigma$ to $\pi_1(\mathcal{O}_L)$ is irreducible. By the proof of lemma 3.24 we have that $\sigma(\pi_1(\mathcal{O}_L))$ is Zariski dense in $PSL(2, \mathbb{R})$. By corollary 3.11, to see the representation $\omega_n \colon \sigma(\pi_1(\mathcal{O}_L)) \to PSL(n, \mathbb{R})$ is irreducible is equivalent to checking that the Zariski closure of its image is irreducible. This holds since $\omega_n \colon PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ is an irreducible representation and a morphism of algebraic groups, so $\omega_n(PSL(2, \mathbb{R})) = \omega_n(\overline{\sigma(\pi_1(\mathcal{O}_L)})) \subseteq \overline{\omega_n \circ \sigma(\pi_1(\mathcal{O}_L))}$.

To see $\rho|_{\pi_1(\mathcal{O}_R)}$ is irreducible it is enough to notice that the proof of 3.24 also holds for $\pi_1(\mathcal{O}_R)$ by using the generators θ_3 and θ_3 instead of θ_1 and θ_2.

$\square$

Knowing that ρ is an integral orbifold Fuchsian representation, the previous proposition shows ρ satisfies the assumptions of theorem 3.21. Thus we obtain the following application of corollary 3.22.

Theorem 3.26 *For every surface S finitely covering the orbifold $O_{3,3,3,3}$ and every odd $n > 1$ there exists a path of Hitchin representations $\rho_t \colon \pi_1(S) \to SL(n, \mathbb{R})$, so that*

1. $\rho_0(\pi_1(S)) \subset SL(n, \mathbb{Z})$,
2. ρ_t *is Zariski dense for every* $t > 0$ *and*
3. $\rho_t(\pi_1(S)) \subset SL(n, \mathbb{Q})$ *for every* $t \in \mathbb{Q}$.

Chapter 4

Constructing non-Hitchin Anosov representations

In 2006 Labourie introduced *Anosov representations* to generalize Fuchsian representations of surface groups and encode Anosov dynamic structures on a closed hyperbolic surface. The definition and properties of Anosov representations have since been expanded from surface groups to hyperbolic groups by Guichard and Wienhard [32]. These representations are now widely studied and have proven central to gaining a deeper understanding of the Hitchin component $\mathrm{Hit}(\pi_1(S), PSL(n, \mathbb{R}))$, a connected component of the space of representations $\mathrm{Rep}^+(\pi_1(S), PSL(n, \mathbb{R}))$ in which every representation is Anosov (see sec. 3.1).

More recently Kapovich, Leeb and Porti [40][41][42] defined uniformly regular and undistorted (URU) representations. URU representations are equivalent to Anosov representations, but their definition is based on the dynamic at infinity of the action induced by a representation $\rho\colon \pi_1(S) \to PSL(n, \mathbb{R})$ on the symmetric space X_n of $PSL(n, \mathbb{R})$. The role of parabolic subgroups in Anosov representations as defined by Labourie is now played by simplices in the spherical building at infinity of X_n.

In this chapter, we use the URU approach to construct Anosov representations of surface groups outside $\mathrm{Hit}(\pi_1(S), PSL(n, \mathbb{R}))$. The images of these representations contain matrices which are not purely loxodromic, and therefore the representations are contained in a component of $\mathrm{Rep}^+(\pi_1(S), PSL(n, \mathbb{R}))$ different from the Hitchin component (see thm. 3.7). This investigation of Anosov representations outside $\mathrm{Hit}(\pi_1(S), PSL(n, \mathbb{R}))$ is motivated by the work of Barbot [43], who in 2010 constructed a connected family of non-Hitchin Anosov representations into $PSL(3, \mathbb{R})$ and inquired whether the set of Anosov representations not in the Hitchin component was connected.

To construct these non-Hitchin representations we look at the composition of a Hitchin representation $\rho\colon \pi_1(S) \to PSL(3, \mathbb{R})$ with a generalization of the irreducible representation $PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ (see sec. 3.1.2). To verify these representations are URU, in section 4.3 we introduce *signatures*, a combinatorial tool for encoding the location of a simplex in the boundary at infinity of the symmetric space. Our main result in this chapter is the following (definitions will follow):

Theorem 4.1 *Let $\sigma_f^3 \subset \partial_\infty X_3$ be the fundamental chamber and $\rho\colon \Gamma \to PSL(3, \mathbb{R})$ be a σ_f^3-URU representation. Fix $n = \binom{d+2}{3}$ for some $d \in \mathbb{N}$, and let $\tau_f^n \subset \partial_\infty X_n$ be the simplex of signature $(1, 1, n-4, 1, 1)$ in the fundamental chamber of $\partial_\infty X_n$. Let $\omega_n\colon PSL(3, \mathbb{R}) \to PSL(n, \mathbb{R})$ be the representation induced by the action of $PSL(3, \mathbb{R})$ on the space of homogeneous polynomials of degree d in three variables. Then $\omega_n \circ \rho\colon \Gamma \to PSL(n, \mathbb{R})$ is a τ_f^n-URU representation and $\omega_n \circ \rho$ is not in $\mathrm{Hit}(\pi_1(S), PSL(n, \mathbb{R}))$.*

In sections 4.1 and 4.2 we give an overview of buildings, symmetric spaces and their boundaries at infinity. The goal of these two sections is to introduce all the objects and properties needed to define uniformly regular and undistorted representations in section 4.4. The construction of the non-Hitchin Anosov representations and the proof of theorem 4.1 are in section 4.5.

4.1 Buildings and symmetric spaces

A *building* is a simplicial complex Δ that can be expressed as a union of subcomplexes called *apartments* with the following axioms:

(1) each apartment Σ is isomorphic to a Coxeter complex,

(2) for any two simplices $\tau, \tau' \in \Delta$ there is an apartment Σ that contains τ and τ', and

(3) if Σ and Σ' are apartments containing simplices τ, τ' then there is an isomorphism between Σ and Σ' that fixes τ and τ' pointwise.

A *chamber* in a building is a simplex of maximal dimension. A properly contained simplex in a chamber σ is a *face* of σ. In particular, codimension 1 faces are the *walls* of the chamber. We often single out a specific apartment in the building and call it the *fundamental apartment*. Inside the fundamental apartment we fix a chamber σ_f which is called the *fundamental chamber*.

There are various ways to construct a building associated to a group G. In this section we give an overview of the building $\Delta(X)$ on the boundary at infinity $\partial_\infty X$ of the symmetric space X of G.

4.1.1 The building $\Delta(X)$

Let G be a semisimple connected Lie group with finite center and a left-invariant Riemannian metric. A *symmetric space* for G is a quotient $X = G/K$ where K is a maximal compact Lie subgroup of G. We assume X is of non-compact type. Since the symmetric space is a quotient manifold, it inherits a Riemannian metric from G. In the following, we explain how to construct a building associated with G on the boundary at infinity of its symmetric space.

A *geodesic ray* in X is an isometry $\gamma\colon [0,\infty) \to X$ such that each segment $\gamma\colon [0,t) \to X$ is a path of shortest length from $\gamma(0)$ to $\gamma(t)$. Two geodesic rays γ_1, γ_2 are defined to be equivalent if there is a constant $k > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < k$ for all t. The *boundary at infinity* of X is the set of equivalence classes under this relation:

$$\partial_\infty X = \{[\gamma] \mid \gamma \text{ geodesic ray in } X\}.$$

The union $X \sqcup \partial_\infty X$ admits a compact topology called the *cone topology* that restricts in X to its usual topology (see [4] part II, chap. 8). Moreover, for a fixed point $x \in X$, the symmetric space X is diffeomorphic to $T_x X$ via the exponential map, and $\partial_\infty X = \{[\gamma] \mid \gamma \text{ geodesic ray in } X \text{ starting at } x\}$. Then $\partial_\infty X$ can be identified with the sphere at infinity of $T_x X$ and $X \sqcup \partial_\infty X$ is homeomorphic to a closed Euclidean ball (see [44] sec. 3.1).

Definition 4.2 A *flat* F in X is a totally geodesic submanifold of X that is isometric to Euclidean space. A *maximal flat* is a flat of maximal dimension. The dimension of a maximal flat in X is called the *rank* of G.

For example, if G has rank one then maximal flats in its symmetric space are simply geodesics. In general, groups of rank one are better understood, but their properties fail to generalize easily to higher rank groups.

If f is an isometry of X then it sends geodesics to geodesics preserving equivalence classes, so f can be continuously extended to $\partial_\infty X$ by defining $f([\gamma]) = [f(\gamma)]$. The Lie group G acts by isometries on its symmetric space X, so the G-action extends continuously to $\partial_\infty X$. However, when rank > 1, the G-action on $\partial_\infty X$ is not transitive as it is on X. To see this, first notice that any geodesic ray in X is contained in a maximal flat.

Definition 4.3 A geodesic ray is *singular* if it is contained in more than one maximal flat, otherwise it is *regular*. If γ is a singular (resp. regular) geodesic ray we say that $[\gamma]$ is a *singular point* (resp. *regular point*) in $\partial_\infty X$.

The G-action on $\partial_\infty X$ sends singular points to singular points and regular points to regular points. Groups of rank one are the only ones where all points in $\partial_\infty X$ are regular.

To give $\partial_\infty X$ a building structure, fix a maximal flat $F \subset X$. The action of the Weyl group W of G on F induces a chamber decomposition of F where the walls of the chambers are fixed points of elements of W. This, in turn, creates a simplicial decomposition of $\partial_\infty F$ corresponding to the triangulation of a sphere. Since G acts transitively on the set of all maximal flats we can define a simplicial structure on all of $\partial_\infty X$ by looking at the G-orbit of $\partial_\infty F$. Define an apartment in $\partial_\infty X$ to be any image $g \cdot \partial_\infty F$. Then $\partial_\infty X$ with this system of apartments forms the building $\Delta(X)$ associated to the symmetric space of G. Since G acts transitively by isometries on X the building structure of $\partial_\infty X$ does not depend on the initial choice of maximal flat.

Definition 4.4 ([41] sec. 2.2) The *interior of a simplex* $\tau \subset \Delta(X)$ is the complement in τ of the union of subsimplices not containing τ, we denote it by $\mathrm{int}(\tau)$.

Points in the interior of a chamber are regular points, while points in the boundary of a chamber are singular points (see [44] sec. 3.8).

4.1.2 Types of points and simplices

As mentioned before, when the rank of G is at least two the G-action on $\partial_\infty X$ is not transitive as it is on X. However, the group G acts *strongly transitively* on the building $\Delta(X) \equiv \partial_\infty X$, this means:

(1) G acts transitively on the set of pairs (Σ, σ) where Σ is an apartment and $\sigma \in \Sigma$ is a chamber, and

(2) the action preserves the dimension of the simplices in σ.

In particular, for any $x \in \partial_\infty X$ the orbit $G \cdot x$ intersects every chamber of $\Delta(X)$ once, so the orbit space $\partial_\infty X/G$ can be identified with the fundamental chamber $\sigma_f \subset \Delta(X)$.

Definition 4.5 ([45] sec. 2.3) The *type map* is the projection

$$\theta\colon\ \partial_\infty X \to \sigma_f. \tag{4.1}$$

- The *type of a point* $x \in \partial_\infty X$ is its image $\theta(x) \in \sigma_f$.
- On every chamber of $\Delta(X)$ the type map restricts to an isometry. Then the *type of a simplex* $\tau \subset \Delta(X)$ is the simplex $\theta(\tau) \subset \sigma_f$.

The type map θ gives a convenient way of selecting a region of $\partial_\infty X$ by looking at the preimage of a certain subset of the fundamental chamber $\sigma_f \subset \Delta(X)$. For example, the set of regular points in $\partial_\infty X$ equals $\theta^{-1}(\mathrm{int}(\sigma_f))$. To allow for generalizations of regularity we will consider regularity relative to a simplex $\tau_f \subset \sigma_f$.

Definition 4.6 ([45] sec. 5.1) Let τ_f be a simplex in the fundamental chamber σ_f of $\Delta(X)$.

- The *open star of τ_f in σ_f* is the subset of $\partial_\infty X$ formed by the union of the interiors of all simplices in σ_f that contain τ_f, we denote it by $\mathrm{ost}(\tau_f)$.
- The *τ_f-regular part of $\partial_\infty X$* is the set

$$\partial_\infty^{\tau_f - reg} X = \theta^{-1}(\mathrm{ost}(\tau_f)) \subset \partial_\infty X.$$

4.2 The symmetric space of $PSL(n, \mathbb{R})$

For the rest of the chapter we focus on the case where $G = PSL(n, \mathbb{R}) = SL(n, \mathbb{R})/\{\pm Id\}$. Notice that when n is odd $PSL(n, \mathbb{R}) = SL(n, \mathbb{R})$. The group $PSL(n, \mathbb{R})$ is a non-compact semisimple Lie group of dimension $n^2 - 1$. We can give a $PSL(n, \mathbb{R})$-invariant Riemannian metric to $PSL(n, \mathbb{R})$ by letting $\langle A, B \rangle_g = \operatorname{tr}((g^{-1}A)(g^{-1}B)^T)$ for any $g \in G$ and $A, B \in T_g G$. On its Lie algebra $\mathfrak{sl}(n, \mathbb{R})$, which is the vector space of $n \times n$ traceless matrices, the metric simplifies to $\langle A, B \rangle_{Id} = \operatorname{tr}(AB^T)$.

A maximal compact subgroup of $PSL(n, \mathbb{R})$ is $K = PSO(n, \mathbb{R})$, so the symmetric space for $PSL(n, \mathbb{R})$ is the quotient space $PSL(n, \mathbb{R})/PSO(n, \mathbb{R})$. Let X_n be the space of $n \times n$ positive definite real symmetric matrices of determinant one. Then $PSL(n, \mathbb{R})$ acts on X_n by

$$g \cdot X = gXg^T.$$

The action is well defined since $(-g) \cdot X = g \cdot X$ for any $g \in SL(n, \mathbb{R})$. This action is transitive and the stabilizer of Id is $PSO(n, \mathbb{R})$, so we may identify

$$X_n \equiv PSL(n, \mathbb{R})/PSO(n, \mathbb{R}).$$

The symmetric space X_n can be seen as a submanifold of $PSL(n, \mathbb{R})$, and as such it inherits a Riemannian metric.

Let

$$F_n = \{A \in X_n \mid A \text{ is diagonal with positive entries } \}. \tag{4.2}$$

This is a maximal flat in X_n and

$$\begin{aligned} \log\colon F_n &\to \mathbb{R}^n \\ \operatorname{diag}(\lambda_1, \ldots, \lambda_n) &\mapsto (\log(\lambda_1), \ldots, \log(\lambda_n)) \end{aligned}$$

is an isometric embedding of this flat into $\mathbb{R}^n$. Therefore $\operatorname{rank}(PSL(n, \mathbb{R})) = \dim(F_n) = n - 1$.

Example 4.7 When $n = 2$, the symmetric space X_2 of $PSL(2, \mathbb{R})$ is diffeomorphic to $\mathbb{H}^2$. Geodesics in $\mathbb{H}^2$ are the maximal flats. Since any two points in $\partial_\infty X_2 = \partial_\infty \mathbb{H}^2$ are the endpoints of a geodesic, in the building $\Delta(X_2)$ any pair of points form an apartment, and every point is a chamber. In other words, the building structure $\Delta(X_2)$ is trivial. Moreover, since any geodesic ray is part of a unique geodesic, every point in $\partial_\infty X_2$ is a regular point.

4.2.1 Apartments in $\Delta(X_n)$

In the following, we focus on the case where $n \geq 3$. To better understand the structure of an apartment in $\Delta(X_n)$ consider the *Weyl group* of $PSL(n, \mathbb{R})$:

$$W = \{n \times n \text{ permutation matrices}\} \simeq S_n.$$

This group creates a chamber decomposition of F_n in the following way. For each $1 \leq i < j \leq n$ let $\sigma_{i,j} \in W$ be the matrix whose action on X_n exchanges the i-th and the j-th entries in the diagonal of an element in F_n. Let $H_{i,j} = \operatorname{Fix}(\sigma_{i,j}) \cap F_n$. If $x \in H_{i,j}$ then the i-th and j-th entries in its diagonal are equal, and the geodesic ray joining Id and x is a singular geodesic. Each set $H_{i,j}$ is a totally geodesic subspace of X_n and thus a flat that forms a wall of a chamber in F_n.

This chamber decomposition of F_n induces a simplicial decomposition of $\partial_\infty F_n$ isomorphic to the Coxeter complex of S_n. We choose this triangulation of $\partial_\infty F_n$ as the fundamental apartment in $\Delta(X_n)$. Consider the cone

$$C_n = \{D = (d_{ij}) \in F_n \mid d_{11} \geq d_{22} \geq \ldots \geq d_{nn} \}. \tag{4.3}$$

The flats $H_{i,i+1}$ for $i = 1, \ldots, n$ define the walls of C_n, and lower dimensional faces of C_n are intersections of these $H_{i,i+1}$. As fundamental chamber in $\Delta(X_n)$ we fix $\sigma_f = \partial_\infty C_n$. For each $1 \leq i < n$ define the geodesic ray $\gamma_i \colon [0, \infty) \to C_n$ as

$$\gamma_i(t) = \mathrm{diag}(\underbrace{e^t, \ldots, e^t}_{i}, \underbrace{e^{-t\frac{i}{n-i}}, \ldots, e^{-t\frac{i}{n-i}}}_{n-i}). \tag{4.4}$$

Then the vertices of the fundamental chamber σ_f are the points at infinity $[\gamma_1], \ldots, [\gamma_{n-1}]$.

Example 4.8 Consider the symmetric space $X_3 = SL(3, \mathbb{R})/SO(3, \mathbb{R})$ and the maximal flat F_3 in it. This is a 2-dimensional flat, so its walls $H_{i,j}$ are geodesics. The walls of F_3 are

$$\begin{aligned} H_{1,2} &= \{\mathrm{diag}(e^{\lambda}, e^{\lambda}, e^{-2\lambda}) \mid \lambda \in \mathbb{R}\}, \\ H_{2,3} &= \{\mathrm{diag}(e^{-2\lambda}, e^{\lambda}, e^{\lambda}) \mid \lambda \in \mathbb{R}\}, \\ H_{1,3} &= \{\mathrm{diag}(e^{\lambda}, e^{-2\lambda}, e^{\lambda}) \mid \lambda \in \mathbb{R}\}, \end{aligned}$$

they are shown in figure 4.1. Each wall is the union of two singular geodesic rays starting at the identity and going in opposite directions. The apartment we get as the triangulation of $\partial_\infty F_3$ is a hexagon where each vertex is a singular point and each point in the interior of an edge is regular.

Consider the geodesic rays $\gamma_1(t) = \mathrm{diag}(e^{2t}, e^{-t}, e^{-t})$ and $\gamma_2(t) = \mathrm{diag}(e^t, e^t, e^{-2t})$ with

$t > 0$, then $v_1 = [\gamma_1]$ and $v_2 = [\gamma_2]$ are the vertices of the fundamental chamber σ_f. For any simplex $\tau \subset \Delta(X_3)$ we have that its type $\theta(\tau)$ equals v_1, v_2 or σ_f. Following definition 4.6 we have that $\operatorname{ost}(v_i) = \{v_i\} \cup \operatorname{int}(\tau_f)$ and $\operatorname{ost}(\tau_f) = \operatorname{int}(\tau_f)$.

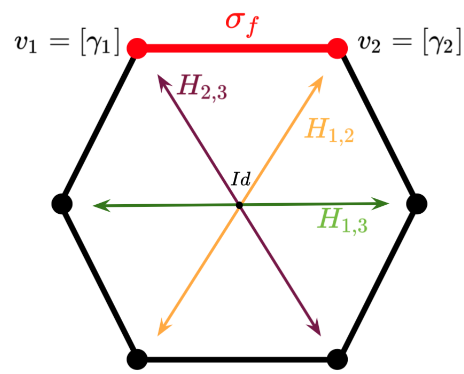

Figure 4.1: Walls in $F_3 \subset X_3$ and the fundamental apartment in $\Delta(X_3)$

4.3 Signatures

The goal of this section is to combinatorially encode the location of a simplex in the building $\Delta(X_n)$ relative to another. This approach will simplify the way we check representations are uniformly regular and undistorted in section 4.5. The following definitions are motivated by the terms used for flags, which are increasing sequences of subspaces of a finite dimensional vector space. Flags in $\mathbb{R}^n$ form a building which is isomorphic to $\Delta(X_n)$.

Definition 4.9 The *set of signatures* for the building $\Delta(X_n)$ is the set of tuples

$$\mathcal{S}_n = \left\{ (m_1, \ldots, m_K) \mid 2 \leq K \leq n,\ \sum m_i = n \right\}.$$

If τ is a simplex in the fundamental chamber σ_f of $\Delta(X_n)$, then there are indices $i_1 < \ldots < i_k$ such that τ is the convex hull of the points at infinity $[\gamma_{i_1}], \ldots, [\gamma_{i_k}]$ defined in (4.4). Let

$$m_1 = i_1,\ m_2 = i_2 - i_1,\ \ldots,\ m_{k+1} = n - i_k.$$

Then the map

$$\begin{aligned} \pi\colon \{\text{simplices in } \sigma_f\} &\to \mathcal{S}_n \\ \tau &\mapsto (m_1, \ldots, m_{k+1}) \end{aligned} \tag{4.5}$$

is a bijection. Injectivity follows because a simplex in $\Delta(X_n)$ is completely determined by its vertices. To see π is onto take a tuple $(m_1, \ldots, m_K) \in \S_n$ and let $i_k = \sum_{j=1}^{k} m_j$ for $1 \leq k \leq K-1$. Then the simplex $\tau \subset \sigma_f$ whose vertices are $[\gamma_{i_1}], \ldots, [\gamma_{i_{K-1}}]$ is such that $\pi(\tau) = (m_1, \ldots, m_K)$.

To assign a signature to every point in $\partial_\infty X_n$ and to every simplex in $\Delta(X_n)$ recall that given a simplex $\tau \subset \Delta(X_n)$, the unique simplex in σ_f that is in the $PSL(n,\mathbb{R})$-orbit of τ is its type, denoted by $\theta(\tau)$.

Definition 4.10 Since $\Delta(X_n)$ is a simplicial complex, every point $p \in \partial_\infty X_n$ is contained in the interior of a unique simplex $\tau_p \subset \Delta(X_n)$. Define the *signature map* as

$$\begin{aligned} s\colon \partial_\infty X_n &\to \mathcal{S}_n \\ p &\mapsto \pi \circ \theta(\tau_p), \end{aligned}$$

where π is the map defined in (4.5).

- The *signature of a point* $p \in \partial_\infty X_n$ is the tuple $s(p)$.

- The *signature of a simplex* $\tau \subset \Delta(X_n)$ is the tuple $s(p)$ where p is any point in the interior of τ, we denote it as $s(\tau)$.

Notice that the signature map is invariant under the $PSL(n,\mathbb{R})$-action on $\partial_\infty X_n$ and $\Delta(X_n)$.

Example 4.11 Using the notation of example 4.8 we see that the set of signatures of $\Delta(X_3)$ is

$$\mathcal{S}_3 = \{s(v_1) = (1,2),\ s(v_2) = (2,1),\ s(\sigma_f) = (1,1,1)\}.$$

If p is a point in the interior of a chamber in $\Delta(X_3)$ then $s(p) = (1,1,1)$. If v is a vertex in $\Delta(X_3)$ then $s(v) = (1,2)$ or $s(v) = (2,1)$. In figure 4.2 we can see the signatures for points in the fundamental apartment.

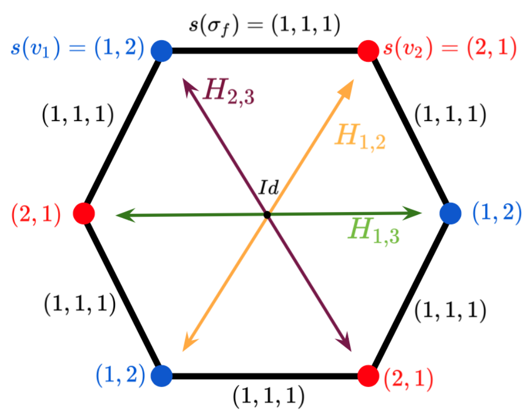

Figure 4.2: Signatures for points in the fundamental apartment of $\Delta(X_3)$.

For every point $p \in \partial_\infty X_n$ there is a geodesic ray $\gamma\colon [0,\infty) \to X_n$ with $\gamma(0) = Id$ such that $p = [\gamma]$. The following proposition shows how γ can be used to calculate the signature of p. With this viewpoint $s(p)$ is encoding the type of the smallest simplex in $\Delta(X_n)$ that contains p and the relative order and multiplicities of the eigenvalues along γ.

Proposition 4.12 *Let $\gamma\colon [0,\infty) \to X_n$ be a geodesic ray such that $\gamma(0) = Id$. Suppose that for some $t_0 > 0$ the eigenvalues of $\gamma(t_0)$ are $\lambda_1 > \ldots > \lambda_k$, with corresponding multiplicities $m_1, \ldots, m_k$. The following hold*

(i) for all $t > 0$ the matrix $\gamma(t)$ has k distinct eigenvalues $\lambda'_1 > \ldots > \lambda'_k$ with corresponding multiplicities $m_1, \ldots, m_k$,

(ii) $s([\gamma]) = (m_1, \ldots, m_k)$ for $[\gamma] \in \partial_\infty X_n$.

Proof. (i) Since $\gamma(t_0)$ is a symmetric positive definite matrix, there exists a matrix $M \in SO(n,\mathbb{R})$ such that $M \cdot \gamma(t_0) = M\gamma(t_0)M^T$ is diagonal with positive diagonal entries, this means $M \cdot \gamma(t_0)$ is in the maximal flat F_n. Next we can use a permutation matrix $w \in SO(n,\mathbb{R})$ to organize the diagonal entries of $M \cdot \gamma(t_0)$ in non-increasing order starting at the top-left entry, so that $(wM) \cdot \gamma(t_0) \in C_n$. We get that $(wM) \cdot \gamma(0) = Id$ and $(wM) \cdot \gamma(t_0)$ is a point in the interior of the cone C_n or a point in the interior of a face f of C_n, where

$$f = \left(\bigcap_{i=1}^{m_1-1} H_{i,i+1} \right) \cap \left(\bigcap_{i=m_1+1}^{m_2-1} H_{i,i+1} \right) \cap \ldots \cap \left(\bigcap_{i=m_1+\ldots+m_{k-1}+1}^{n-1} H_{i,i+1} \right). \tag{4.6}$$

The action of wM is isometric, so in the latter case $(wM) \cdot \gamma(t)$ is a geodesic with two points in f. Because f is geodesically embedded in X_n it must be that $(wM) \cdot \gamma(t) \subset f$. Moreover, since $(wM) \cdot \gamma(0) = Id$ is the vertex of f and $(wM) \cdot \gamma(t_0)$ is in the interior of f, we have that $(wM) \cdot \gamma(t)$ is in the interior of f for all $t > 0$. Notice that all matrices in the interior of f are diagonal with k distinct eigenvalues $\lambda'_1 > \ldots > \lambda'_k$ with corresponding multiplicities $m_1, \ldots, m_k$, and for every t we have that $(wM) \cdot \gamma(t)$ is conjugate to $\gamma(t)$. When $(wM) \cdot \gamma(t_0)$ is in the interior of C_n a similar argument shows $\gamma(t)$ has n distinct eigenvalues.

(ii) From part (1) we know there is a $g \in SL(n, \mathbb{R})$ such that $g \cdot \gamma(t)$ is a geodesic ray pointing towards the interior of C_n or towards the interior of a face f as defined in (4.6). In the first case $[g \cdot \gamma] \in \partial_\infty X_n$ is a point in the interior of the fundamental chamber σ_f and so

$$s([\gamma]) = s(g[\gamma]) = s([g \cdot \gamma]) = \pi(\sigma_f) = (\underbrace{1, \ldots, 1}_{n}).$$

In the case when $[g \cdot \gamma] \in \mathrm{int}(\partial_\infty f)$, by definition $\partial_\infty f$ is a simplex in the fundamental chamber of $\Delta(X)$ and we obtain that

$$s([\gamma]) = s(g[\gamma]) = s([g \cdot \gamma]) = \pi(\partial_\infty f) = (m_1, \ldots, m_k).$$

□

4.3.1 τ_f-regularity in terms of signatures

Simplices in the fundamental chamber σ_f of $\Delta(X_n)$ have a partial order given by inclusion. The set of signatures $\mathcal{S}_n$ inherits a partial order via the bijection $\pi\colon \{\text{simplices in } \sigma_f\} \to \mathcal{S}$ defined in (4.5).

Proposition 4.13 *Let $(m_1, \ldots, m_k)$ and $(\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}})$ be signatures in $\mathcal{S}_n$. Then $(m_1, \ldots, m_k) \leq (\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}})$ if and only if there exist $s_1, \ldots, s_k \in \mathbb{Z}_+$ with $s_1 + \ldots + s_k = \tilde{k}$ such that*

$$\begin{aligned} m_1 &= \tilde{m}_1 + \ldots + \tilde{m}_{s_1}, \\ m_2 &= \tilde{m}_{s_1+1} + \ldots + \tilde{m}_{s_1+s_2}, \\ &\vdots \\ m_k &= \tilde{m}_{s_1+\ldots+s_{k-1}+1} + \ldots + \tilde{m}_{\tilde{k}}. \end{aligned}$$

Proof. Let σ_f be the fundamental chamber of $\Delta(X_n)$ and $\tau, \tilde{\tau} \subset \sigma_f$ be simplices such that $\pi(\tau) = (m_1, \ldots, m_k)$ and $\pi(\tilde{\tau}) = (\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}})$. For $i = 1, \ldots, n-1$ let γ_i be the geodesic ray in X_n defined in equation (4.4), so that $v_i = [\gamma_i] \in \partial_\infty X_n$ are the vertices of σ_f. Then τ has vertices $v_{i_1}, \ldots, v_{i_{k-1}}$ where $i_l = m_1 + \ldots + m_l$, and $\tilde{\tau}$ has vertices $v_{j_1}, \ldots, v_{j_{\tilde{k}-1}}$ with $j_l = \tilde{m}_1 + \ldots + \tilde{m}_l$. By definition, $(m_1, \ldots, m_k) \leq (\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}})$ if and only if τ is contained in $\tilde{\tau}$, and this happens if and only if the vertices of τ are a subset of the vertices of $\tilde{\tau}$.

Suppose $\{v_{i_l}\}_{l=1}^{k-1} \subseteq \{v_{j_l}\}_{l=1}^{\tilde{k}-1}$. Then there is a sequence $0 < S_1 < S_2 < \ldots < S_{k-1} < k \leq \tilde{k}$ of indices such that $i_l = j_{S_l}$. This implies that

$$\begin{aligned} m_1 &= \tilde{m}_1 + \ldots + \tilde{m}_{S_1}, \\ m_1 + m_2 &= \tilde{m}_1 + \ldots + \tilde{m}_{S_2}, \\ &\vdots \\ m_1 + \cdots + m_{k-1} &= \tilde{m}_1 + \ldots + \tilde{m}_{S_{k-1}}. \end{aligned}$$

Making $s_1 = S_1$, $s_l = S_l - S_{l-1}$ for $1 < l < k$ and $s_k = \tilde{k} - S_{k-1}$ we satisfy the conditions of the proposition. Conversely, if we have $s_1, \ldots, s_k$ as in the statement then

$$v_{i_1} = v_{j_{s_1}},\ v_{i_2} = v_{j_{s_1+s_2}},\ \ldots,\ v_{i_{k-1}} = v_{s_1+\ldots+s_{k-1}}.$$

$\square$

Proposition 4.14 *Let p be a point in $\partial_\infty X_n$ and τ_f be a simplex in the fundamental chamber σ_f. The following are equivalent:*

(i) p is in the τ_f-regular part of $\partial_\infty X_n$,

(ii) $\theta(p) \in ost(\tau_f)$,

(iii) $s(\tau_f) \leq s(p)$.

Proof. Statements (i) and (ii) are equivalent by definition 4.6. Given $p \in \partial_\infty X_n$, its type $\theta(p)$ is in the interior of a unique simplex $\tau_{\theta(p)}$ contained in the fundamental chamber σ_f. Then $\theta(p) \in \mathrm{ost}(\tau_f)$ if and only if $\tau_f \subseteq \tau_{\theta(p)}$. Using the G-invariance of the signature map, the latter happens if and only if $s(\tau_f) \leq s(\tau_{\theta(p)}) = s(\theta(p)) = s(p)$.

□

Example. Consider the set of signatures for X_4, this is

$$\mathcal{S}_4 = \{(1,1,1,1), (2,1,1), (1,2,1), (1,1,2), (2,2), (1,3), (3,1)\}.$$

Using proposition 4.13 it is easy to see that $(1,3) = (1, 1+1+1)$ so $(1,3) < (1,1,1,1)$. Similarly $(1,3) = (1,1+2) = (1,2+1)$ so $(1,3) < (1,1,2)$ and $(1,3) < (1,2,1)$. By 4.13 there are no other signatures that greater or equal to $(1,3)$. Now let τ_f be the simplex in the fundamental chamber with signature $(1,3)$, this is an edge of the chamber. By proposition 4.14, a point p is in the τ_f regular part of $\partial_\infty X_n$ if and only if $s(p)$ equals $(1,3)$, $(1,1,2)$ or $(1,1,1,1)$.

4.3.2 Opposition in terms of signatures

A *Cartan involution* is an isometry f of a symmetric space X such that $f^2 = Id$ and has an isolated fixed point $x \in X$. In particular $df_x = -Id$ on $T_x X$. Like all isometries, a Cartan involution extends to a simplicial isomorphism of the building $\Delta(X) \equiv \partial_\infty X$.

Definition 4.15 ([45]) Two simplices in the building $\Delta(X)$ are *opposite simplices* if a Cartan involution of X can exchange them.

Notice that opposition of simplices is preserved under the $PSL(n, \mathbb{R})$-action on $\Delta(X)$. Given two opposite simplices $\tau_+, \tau_- \subset \Delta(X_n)$ we are interested in the relation between their signatures $s(\tau_\pm)$.

Definition 4.16 A pair of signatures $(m_1, \ldots, m_k)$, $(\tilde{m}_1, \ldots, \tilde{m}_k) \in \mathcal{S}_n$ are *complementary* if for each $1 \leq i \leq k-1$ we have

$$m_1 + \ldots + m_i + \tilde{m}_{k-i} + \ldots \tilde{m}_1 = n.$$

Notice that a first requirement for a pair of signatures to be complementary is that as tuples they both have the same length. It is easy to see that a signature $(m_1, \ldots, m_k)$ has a unique complementary signature and this is $(m_k, \ldots, m_1)$.

Definition 4.17 A signature is *self-opposite* if it is complementary to itself.

Notice that any two chambers in $\Delta(X_n)$ have signature $\underbrace{(1, \ldots, 1)}_{n}$, which is self-opposite.

Example 4.18 Consider the vertex $[\gamma_i]$ of the fundamental chamber, where γ_i is the geodesic ray defined in (4.4). We have that $s([\gamma_i]) = (i, n-i)$ by proposition 4.12. The map $f\colon\ x \mapsto (x^{-1})^T$ is a Cartan involution of X_n that sends the fundamental apartment to itself. Then $f(\gamma_i)\colon\ [0, \infty) \to X_n$ is the geodesic ray

$$f(\gamma_i)(t) = \mathrm{diag}(\underbrace{e^{-t}, \ldots, e^{-t}}_{i}, \underbrace{e^{t\frac{i}{n-i}}, \ldots, e^{t\frac{i}{n-i}}}_{n-i}),$$

and $s([f(\gamma_i)]) = s([\gamma_{n-i}]) = (n-i, i)$ by proposition 4.12. Therefore $[\gamma_i]$ and its opposite $[f(\gamma_i)]$ have complementary signatures.

We wish to extend the previous observation to any pair of opposite simplices in $\Delta(X_n)$.

Lemma 4.19 *Let f_1, f_2 be two Cartan involutions of X with isolated fixed points p_i. Suppose there exist maximal flats F_i in X such that $p_i \in F_i$ and $\partial_\infty F_1 = \partial_\infty F_2$ as subsets of $\partial_\infty X$. Then $f_1(\xi) = f_2(\xi)$ for every $\xi \in \partial_\infty F_i$.*

Proof. Because the flats F_1 and F_2 are isometric with $\mathbb{R}^k$ and have the same boundary at infinity, there exists an isometry $T\colon F_1 \to F_2$ so that $T(p_1) = p_2$ and for every $v \in T^1_{p_1} F_1$ we have that $dT_{p_1} v$ is the unique vector in $T^1_{p_2} F_2$ for which $[\exp(tv)] = [\exp(t(dT_{p_1} v))]$. This means that the geodesic ray starting at p_1 with direction v and the geodesic ray starting at p_2 with direction $dT_{p_1} v$ define the same point in $\partial_\infty X$. In particular, for any $\xi \in \partial_\infty F_i$ there is a unique $v \in T^1_{p_1} X$ such that $\xi = [\exp(tv)]$. Then

$$\begin{aligned}
f_1(\xi) &= f_1[\exp(tv)] \\
&= [\exp(td(f_1)_{p_1} v)] \\
&= [\exp(-tv)] \\
&= [\exp(-tdT_{p_1} v)] \\
&= [\exp(td(f_2)_{p_2} dT_{p_1} v)] \\
&= f_2[\exp(t(dT_{p_1} v))] \\
&= f_2(\xi).
\end{aligned}$$

□

Corollary 4.20 *A simplex in $\Delta(X)$ has exactly one opposite simplex in a given apartment.*

Proof. Let σ be an apartment in $\Delta(X)$, it is enough to see that every point $\xi \in \sigma$ has a unique opposite point in σ. Suppose f_1 and f_2 are Cartan involutions of X such that $f_1(\xi)$ and $f_2(\xi)$ are in the apartment σ. Let p_i be isolated fixed points of f_i, there are maximal flats F_i such that $p_i \in F_i$ and $\partial_\infty F_i = \sigma$. Then by lemma 4.19 we have that $f_1(\xi) = f_2(\xi)$.

□

Proposition 4.21 *Let $\tau_\pm \subset \Delta(X_n)$ be opposite simplices. Then the signatures $s(\tau_+)$ and $s(\tau_-)$ are complementary.*

Proof. Recall that the $PSL(n, \mathbb{R})$-action is strongly transitive on $\Delta(X_n)$ and leaves the signature map $s\colon \Delta(X_n) \to \mathcal{S}_n$ invariant. Then to calculate the signatures of τ_+ and τ_- we may assume without loss of generality that $\tau_\pm$ are opposite simplices in the fundamental apartment and τ_+ is a simplex in the fundamental chamber. Let τ_+ be the convex hull of the vertices $[\gamma_{i_1}], \ldots, [\gamma_{i_k}]$ as defined in (4.4), so that $i_1 < \ldots < i_k$. Then $s(\tau_+) = (i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_k)$. Since the Cartan involution $f\colon\, x \mapsto (x^{-1})^T$ sends the fundamental apartment to itself, by corollary 4.20 it must be that $f(\tau_+) = \tau_-$. Given that $\theta([f(\gamma_i)]) = \theta([\gamma_{n-i}])$, the type of τ_- is the simplex in the fundamental chamber with vertices $[\gamma_{n-i_k}], \ldots, [\gamma_{n-i_1}]$. Therefore $s(\tau_-) = (n - i_k, i_k - i_{k-1}, \ldots, i_2 - i_1, i_1)$ is the signature complementary to $s(\tau_+)$.

□

Notice that two simplices τ and τ' having complementary signatures is a necessary but not sufficient condition for τ and τ' to be opposite. For example, any two chambers c, c' in $\Delta(X_n)$ have signature $(\underbrace{1, \ldots, 1}_{n})$, which is a self-opposite signature, but two adjacent chambers in the same apartment are not opposite.

4.4 Uniformly regular and undistorted representations

If τ_f is a simplex in the fundamental chamber $\sigma_f \subset \partial_\infty X$, selecting the τ_f-regular part of $\partial_\infty X$ allows controlling the dynamics at infinity of the action of a discrete group $\Gamma < G$ on the symmetric space X. For such a subgroup Γ its *visual limit set* $\Lambda(\Gamma)$ is the set of accumulation points of an(y) Γ-orbit $\Gamma \cdot x \subset X$, this is

$$\Lambda(\Gamma) = \overline{\Gamma \cdot x} \cap \partial_\infty X.$$

Definition 4.22 ([40] def. 2.33) A discrete subgroup $\Gamma < G$ is *uniformly τ_f-regular* if

$$\theta(\Lambda(\Gamma)) \subset \mathrm{ost}(\tau_f),$$

this is if the visual limit set of Γ is contained in the τ_f-regular part of $\partial_\infty X$.

Though we will not make use of it, we want to mention that there is also a notion of *non-uniform τ_f-regularity* defined in section 2.2.1 of [46] as follows. Fix a point $p \in X$ and for any $\Omega \subset \sigma_f$ let $V(p, \Omega) \subset X$ be the union of all geodesic rays γ starting at p such that $[\gamma] \in \Omega$, this is the *cone on* Ω. Let $\tau_f \subset \sigma_f$ be a simplex and $\partial\mathrm{st}(\tau_f)$ be the union of the simplices in σ_f which do not contain τ_f. Then a sequence $\{x_n\}$ in $C = V(p, \sigma_f)$ is *τ_f-regular* if it drifts away from $V(p, \partial\mathrm{st}(\tau_f)) \subset \partial C$, this is

$$d(x_n, V(p, \partial\mathrm{st}(\tau_f))) \to \infty.$$

A sequence $\{x_n\}$ in C is *uniformly τ_f-regular* if it drifts away at linear rate with respect to $||x_n||$, this is

$$\liminf_{n\to\infty} d(x_n, V(p, \partial\mathrm{st}(\tau_f)))/||x_n|| > 0.$$

An arbitrary sequence $\{x_n\}$ in X is (uniformly) τ_f-regular if the sequence $d_C(p, x_n)$ of C-valued distances is (uniformly) τ_f-regular. A discrete subgroup $\Gamma < G$ is (uniformly) τ_f-regular if any of its orbits is. This last definition of uniformly τ_f-regular is equivalent to the one given in definition 4.22.

Definition 4.23 ([40] def. 3.22) For a simplex τ_f with self-opposite signature, a finitely generated discrete subgroup $\Gamma < G$ is *uniformly τ_f-regular and undistorted* (τ_f-URU) if it is

(i) *undistorted*, i.e. an(y) orbit map $\Gamma \to \Gamma \cdot x \subset X$ is a quasi-isometric embedding with respect to a word metric on Γ, and

(ii) uniformly τ_f-regular.

If $\rho\colon \Gamma \to G$ is a representation from a hyperbolic group Γ to G, then ρ is a *uniformly τ_f-regular and undistorted representation* (τ_f-URU) if the group $\rho(\Gamma) < G$ is τ_f-URU. If ρ is τ_f-URU for some simplex τ_f we simply say it is uniformly regular and undistorted (URU).

Example 4.24 Let S be a closed surface of genus > 2 and consider a discrete and faithful representation $\rho\colon \pi_1(S) \to PSL(2,\mathbb{R})$. It is well known that by fixing a point $x \in \mathbb{H}^2$ the orbit map $\pi_1(S) \to \pi_1(S)\cdot x$ is a quasi-isometry between $\pi_1(S)$ and $\mathbb{H}^2$. Thus the representation ρ is undistorted. Recall from example 4.7 that the building structure of $\Delta(X_2) \equiv \partial_\infty\mathbb{H}^2$ is trivial: every point is a chamber and every point in $\partial_\infty\mathbb{H}^2$ is a regular point. Since $\Lambda(\pi_1(S)) = \partial_\infty\mathbb{H}^2$, we obtain that ρ is uniformly σ_f-regular, where σ_f is the fundamental chamber of $\Delta(X_2)$. Therefore ρ is σ_f-URU. As we will see in the coming section, the theory becomes richer for representations into higher rank groups.

Uniformly τ_f-regular and undistorted representations, defined by Kapovich, Leeb and Porti in 2014, are equivalent to *Anosov representations* as defined by Guichard and Wienhard ([32] def. 2.10) based on the work of Labourie [19]. A representation $\rho\colon \Gamma \to G$ is τ_f-URU if and only if it is P-Anosov, where the parabolic group P is the stabilizer of τ_f under the G-action on $\partial_\infty X$. In the case that concerns us, which is that of representations $\pi_1(S) \to PSL(n,\mathbb{R})$, being τ_f-URU where τ_f is a simplex of signature $(m_1, \dots, m_k)$ is equivalent to being P-Anosov when $P < PSL(n,\mathbb{R})$ is the subgroup that stabilizes the flag $\langle e_1, \dots, e_{m_1}\rangle < \langle e_1, \dots, e_{m_1+m_2}\rangle < \dots < \langle e_1, \dots, e_{m_1+\dots+m_k}\rangle < \mathbb{R}^n$, where $e_1, \dots, e_k$ is the canonical basis for $\mathbb{R}^n$.

Example 4.25 Let $\rho\colon \pi_1(S) \to PSL(n,\mathbb{R})$ be a representation in the Hitchin component of $\mathrm{Rep}^+(\pi_1(S), PSL(n,\mathbb{R}))$. By work of Labourie we know that ρ is B-Anosov, where B is a Borel subgroup of $PSL(n,\mathbb{R})$. We can take B to be the group of upper triangular matrices in $PSL(n,\mathbb{R})$, this is the stabilizer of the complete flag $\langle e_1\rangle < \langle e_1, e_2\rangle < \dots < \langle e_1, \dots, e_{n-1}\rangle < \mathbb{R}^n$. This flag corresponds to the signature $\underbrace{(1, \dots, 1)}_{n}$, therefore ρ is σ_f-URU where σ_f is the fundamental chamber of $\Delta(X_n)$.

4.5 The representation $\omega_n\colon PSL(3,\mathbb{R}) \to PSL(n,\mathbb{R})$

In this section we construct a generalization of the irreducible representation of $PSL(2,\mathbb{R})$ into $PSL(n,\mathbb{R})$ we used in section 3.1.2.

Let $\mathcal{P}_d$ be the vector space of homogeneous polynomials of degree d in three variables. This space has dimension $n = \binom{d+2}{2}$. An ordered basis for $\mathcal{P}_d$ is

$$\mathcal{B} = \{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d, x^{d-1}z, x^{d-2}yz, \dots, xy^{d-2}z, y^{d-1}z, \dots, xz^{d-1}, yz^{d-1}, z^d\}.$$

If we identify the variable x with the canonical unit vector $(1,0,0)$ in $\mathbb{R}^3$ then for any matrix $g = (a_{ij}) \in SL(3,\mathbb{R})$ we have that $g \cdot x = a_{11}x + a_{21}y + a_{31}z$. By identifying y with $(0,1,0)$ and z with $(0,0,1)$ we can calculate $g \cdot y$ and $g \cdot z$ similarly. Then for any $g \in SL(3,\mathbb{R})$ we define $\omega_n(g) \in GL(\mathcal{P}_d)$ by its action on the basis $\mathcal{B}$:

$$\omega_n(g)(x^i y^j z^k) = (g \cdot x)^i (g \cdot y)^j (g \cdot z)^k.$$

By identifying $\mathcal{P}_d$ with $\mathbb{R}^n$ using the basis $\mathcal{B}$ we obtain the representation

$$\omega_n \colon SL(3,\mathbb{R}) \to SL(n,\mathbb{R}). \tag{4.7}$$

It will also be useful to construct the representation ω_n in a different way. Let $V = \langle x, y, z \rangle$ and, as before, identify V with $\mathbb{R}^3$ by making $x \equiv (1,0,0)$, $y \equiv (0,1,0)$ and $z \equiv (0,0,1)$. Given $d \geq 2$ consider the tensor product $\tilde{V} = \underbrace{V \otimes \cdots \otimes V}_{d}$. The symmetric group on d letters S_d acts on $\tilde{V}$ and the subspace W that is invariant under this action is isomorphic to the space of homogeneous polynomials $\mathcal{P}_d$. The action of $SL(3,\mathbb{R})$ on V extends to $\tilde{V}$: if $A \in SL(3,\mathbb{R})$ then $A \cdot (v_1 \otimes \cdots \otimes v_d) = (A \cdot v_1) \otimes \cdots \otimes (A \cdot v_d)$. When we restrict the action of $SL(3,\mathbb{R})$ to the subspace W we obtain the same action of $SL(3,\mathbb{R})$ on $\mathcal{P}_d$ we had previously constructed. To obtain a matrix $\omega_n(A) \in SL(n,\mathbb{R})$ from this action it is necessary to fix a basis for W and look at the action of A on this basis. For every $0 \leq i, j \leq d$ such that $i + j \leq d$ let

$$w_{ij} = \underbrace{x \otimes \cdots \otimes x}_{i} \otimes \underbrace{y \otimes \cdots \otimes y}_{j} \otimes \underbrace{z \otimes \cdots \otimes z}_{d-(i+j)}$$

and $\lambda_{ij} = \frac{1}{1+|S_n - \mathrm{Stab}(w_{ij})|}$. Consider the basis $\mathcal{C}$ for W given by the vectors

$$\lambda_{ij}\left(w_{ij} + \sum_{\sigma \notin \mathrm{Stab}(w_{ij})} \sigma \cdot w_{ij}\right).$$

Each of these vectors can be identified with the polynomial $x^i y^j z^{d-(i+j)} \in \mathcal{P}_d$, this way we can order the basis $\mathcal{C}$ by matching the order of the basis $\mathcal{B}$ of $\mathcal{P}_d$. Then for every $A \in SL(3,\mathbb{R})$ the matrix representation of the action of A on the basis $\mathcal{C}$ is the matrix $\omega_n(A) \in SL(n,\mathbb{R})$.

From the identification of V with $\mathbb{R}^3$ we get that V inherits the usual inner product $\langle\ ,\ \rangle$ in $\mathbb{R}^3$, making $\{x, y, z\}$ an orthonormal basis. Consider now $\tilde{V}$ with the inner product

$$\beta(v_1 \otimes \cdots \otimes v_d\ ,\ w_1 \otimes \cdots \otimes w_d) = \prod_{i=1}^{d} \langle v_i, w_i \rangle.$$

The basis $\mathcal{C}$ for W is orthogonal, so when we look at β restricted to the subspace W we obtain that $\beta(v,w) = ([v]_{\mathcal{C}})^T D [w]_{\mathcal{C}}$, where D is a diagonal matrix with positive diagonal entries and $[v]_{\mathcal{C}}$ is the coordinate vector of $v \in W$ with respect to the basis $\mathcal{C}$. Since any matrix $A \in SO(3,\mathbb{R})$ preserves the inner product β, when we look at the action of A on the subspace W we obtain that $\omega_n(A)^T D \omega_n(A) = D$. By adjusting the matrix D so that it has determinant one we obtain the following result.

Proposition 4.26 *There is a diagonal matrix $J \in SL(n,\mathbb{R})$ with positive diagonal entries such that*

$$\omega_n(SO(3,\mathbb{R})) \subset SO(J) = \{g \in SL(n,\mathbb{R}) \mid gJg^T = J\}.$$

In what follows we let $\omega_n \colon PSL(3,\mathbb{R}) \to PSL(n,\mathbb{R})$ be the projectivization of the representation we have defined.

4.5.1 The induced map $\tilde{\omega}\colon X_3 \to X_n$

Let $J \in PSL(n,\mathbb{R})$ be as in proposition 4.26 and $M \in PSL(n,\mathbb{R})$ such that $J = MM^T$. Then $\omega_n(PSO(3,\mathbb{R}))$ is contained in $PSO(J) = MSO(n,\mathbb{R})M^{-1}$, which is a maximal compact subgroup of $PSL(n,\mathbb{R})$. Knowing where the image of $PSO(3,\mathbb{R})$ goes to, we can construct a well defined map between quotient spaces:

$$\begin{aligned} PSL(3,\mathbb{R})/PSO(3,\mathbb{R}) &\to PSL(n,\mathbb{R})/PSO(J) \\ gPSO(3,\mathbb{R}) &\mapsto \omega_n(g)PSO(J). \end{aligned}$$

We wish to transform this into a map $\tilde{\omega}_n\colon X_3 \to X_n$ between the symmetric spaces X_3 and X_n. To do so recall that for every $A \in X_3$ there is a matrix $g \in PSL(3,\mathbb{R})$ such that $A = gg^T$. If $\tilde{g} \in PSL(3,\mathbb{R})$ is another matrix such that $A = \tilde{g}\tilde{g}^T$ then $\tilde{g} = gh$ for some $h \in PSO(3,\mathbb{R})$. Thus we obtain the following identification

$$\begin{aligned} X_3 &\to PSL(3,\mathbb{R})/PSO(3,\mathbb{R}) \\ gg^T &\mapsto gPSO(3,\mathbb{R}). \end{aligned}$$

Now consider the action of $PSL(n,\mathbb{R})$ on X_n where $g \cdot A = gAg^T$ for any $g \in PSL(n,\mathbb{R})$ and $A \in X_n$. The group $PSO(J)$ is the stabilizer of $J \in X_n$ and

$$\begin{aligned} PSL(n,\mathbb{R})/PSO(J) &\to X_n \\ gPSO(J) &\mapsto g \cdot J = gJg^T \end{aligned}$$

is a diffeomorphism between both spaces (see [47] chap. 21). Then the induced map $\tilde{\omega}_n$ on symmetric spaces is

$$\begin{aligned} \tilde{\omega}_n\colon X_3 &\to X_n \\ gg^T &\mapsto \omega_n(g)J\omega_n(g)^T = \omega_n(g)\cdot J. \end{aligned} \tag{4.8}$$

Proposition 4.27 *The map $\tilde{\omega}_n\colon X_3 \to X_n$ defined in (4.8) is such that*

(i) $\tilde{\omega}_n(h\cdot X) = \omega_n(h)\cdot\tilde{\omega}_n(X)$ for every $h \in SL(3,\mathbb{R})$ and $X \in X_3$, and

(ii) $\tilde{\omega}_n$ sends geodesics rays to geodesics rays.

Proof. To check (i) let $X \in X_3$ and take $g \in SL(3,\mathbb{R})$ such that $X = gg^T$. For any $h \in SL(3,\mathbb{R})$ we have that $h\cdot X = hgg^Th^T = (hg)(hg)^T$. Then

$$\tilde{\omega}_n(h\cdot X) = \omega_n(hg)\cdot J = \omega_n(h)\cdot(\omega_n(g)\cdot J) = \omega_n(h)\cdot\tilde{\omega}_n(X).$$

To prove (ii) first consider the geodesic ray

$$\begin{aligned} \gamma_r\colon [0,\infty) &\to X_3 \\ t &\mapsto \begin{pmatrix} e^{t(2-r)} & & \\ & e^{t(2r-1)} & \\ & & e^{t(-r-1)} \end{pmatrix} \end{aligned} \tag{4.9}$$

with $r \in [0,1]$. For each $t \geq 0$ there exists a diagonal matrix $g_t \in PSL(3,\mathbb{R})$ such that $\gamma_r(t) = (g_t)^2$. Similarly for J there exists a diagonal matrix $M \in PSL(n,\mathbb{R})$ such that

$J = M^2$. Since $\omega_n(g_t)$ and M are both diagonal and commute we have that

$$\begin{aligned}
\tilde{\omega}_n(\gamma_r(t)) &= \omega_n(g_t) J \omega_n(g_t)^T \\
&= \omega_n(g_t) M^2 \omega_n(g_t) \\
&= M\omega_n(g_t)\omega_n(g_t)M \\
&= M\omega_n(\gamma_r(t))M^T \\
&= M \cdot \omega_n(\gamma_r(t)).
\end{aligned}$$

For any $r \in [0,1]$ and $t \geq 0$ the matrix $\omega_n(\gamma_r(t))$ is in the cone C_n and it is easy to see that $\alpha_r \colon [0,\infty) \to X_n$ such that $\alpha_r(t) = \omega_n(\gamma_r(t))$ is a geodesic ray in X_n starting at the identity. Since the action of M on X_n is by isometries and $\tilde{\omega}_n(\gamma_r) = M \cdot \alpha_r$, then $\tilde{\omega}_n(\gamma_r)$ is a geodesic ray in X_n.

Now consider any geodesic ray $\beta \colon [0,\infty) \to X_3$. There is an $h \in PSL(3,\mathbb{R})$ such that $h \cdot \beta = \gamma_r$ for some $r \in [0,1]$. Then by (i) we have that

$$\tilde{\omega}_n(\beta) = \omega_n(h^{-1}) \cdot \tilde{\omega}_n(h \cdot \beta) = \omega_n(h^{-1}) \cdot \tilde{\omega}_n(\gamma_r).$$

Since $\tilde{\omega}_n(\gamma_r)$ is a geodesic ray and $\omega_n(h^{-1})$ acts by isometries then $\tilde{\omega}_n(\beta)$ is also a geodesic ray.

□

Lemma 4.28 *There is a $K > 0$ such that $d_{X_3}(Id, D) = K d_{X_n}(Id, \omega_n(D))$ for every diagonal matrix D in X_3 with positive diagonal entries.*

Proof. Recall that the set of diagonal matrices in X_m with positive diagonal entries forms a maximal flat $F_m \subset X_m$ where $\log \colon F_m \to \mathbb{R}^m$ given by $\log(\mathrm{diag}(\lambda_1, \ldots, \lambda_m)) =$

$(\log(\lambda_1), \dots \log(\lambda_m))$ is an isometric embedding. Then for any $D \in F_m$ we have that

$$d_{X_m}(Id, D) = d_{\mathbb{R}^m}(0, \log(D)) = ||\log(D)||.$$

Let $D = \mathrm{diag}(e^a, e^b, e^{-(a+b)}) \in F_3$, then

$$||\log(D)||^2 = ||(a, b, -(a+b))||^2 = 2(a^2 + b^2 + ab).$$

On the other hand, every vector in the basis $\mathcal{B}$ for the space of polynomials $\mathcal{P}_d$ has the form $x^i y^j z^{d-(i+j)}$ with $0 \le i, j \le d$ and $i + j \le d$. Then

$$D \cdot x^i y^k z^{d-(i+j)} \quad = \quad e^{a(2i+j-d)+b(2j+i-d)} x^i y^k z^{d-(i+j)}.$$

It can be checked that

$$\sum_{i=0}^{d} \sum_{j=0}^{d-i} (i-j)^2 = \sum_{\substack{0 \le i,j \le d \\ i+j \le d}} (2i+j-d)^2 = \sum_{\substack{0 \le i,j \le d \\ i+j \le d}} (2j+i-d)^2 = \sum_{\substack{0 \le i,j \le d \\ i+j \le d}} 2(2i+j-d)(2j+i-d).$$

Therefore

$$||\log(\omega_n(D))||^2 = \sum_{\substack{0 \le i,j \le d \\ i+j \le d}} (a(2i+j-d)+b(2j+i-d))^2 = \left(\sum_{i=0}^{d} \sum_{j=0}^{d-i} (i-j)^2 \right) (a^2+b^2+ab).$$

The statement follows from this equality.

$\square$

Proposition 4.29 *The map* $\tilde{\omega}_n \colon X_3 \to X_n$ *induced by the representation* $\omega_n \colon PSL(3, \mathbb{R}) \to PSL(n, \mathbb{R})$ *is a bi-Lipschitz embedding.*

Proof. Consider two matrices $A_1, A_2 \in X_3$, there exists a $g \in PSL(3, \mathbb{R})$ such that

$g \cdot A_1 = Id$ and $D = g \cdot A_2$ is in the maximal flat F_3. The matrix D is a diagonal matrix with positive diagonal entries. Since F_3 is geodesically embedded in X_3, the shortest path connecting D and Id belongs to the flat F_3. Then the distance between D and Id in X_3 is the distance these two points have in F_3. Therefore by lemma 4.28

$$d_{X_3}(A_1, A_2) = d_{X_3}(Id, D) = K d_{X_n}(Id, \tilde{\omega}_n(D)) = K d_{X_n}(\tilde{\omega}_n(A_1), \tilde{\omega}_n(A_2)).$$

□

4.5.2 Extending $\tilde{\omega}_n$ to $\partial_\infty X_n$

From propositions 4.27 and 4.29, we know that $\tilde{\omega}_n \colon X_3 \to X_n$ sends geodesic rays to geodesic rays, and it is a bi-Lipschitz embedding. Therefore it is possible to extend $\tilde{\omega}_n \colon \partial_\infty X_3 \to \partial_\infty X_n$ by making $\tilde{\omega}_n[\gamma] = [\tilde{\omega}_n(\gamma)]$. In this section, we focus on calculating what is the image under $\tilde{\omega}_n$ of the fundamental chamber σ_f^3 in the building $\Delta(X_3)$ and finding the signatures of the points in $\tilde{\omega}_n(\sigma_f^3) \subset \partial_\infty X_n$.

Lemma 4.30 *For every $r \in [0,1]$ let $D_r = \operatorname{diag}(e^{2-r}, e^{2r-1}, e^{-r-1}) \in PSL(3,\mathbb{R})$. Let $d \in \mathbb{N}$, $n = \binom{d+2}{3}$ and $\omega_n \colon PSL(3,\mathbb{R}) \to PSL(n,\mathbb{R})$ be the representation defined in (4.7). Then*

(i) $\omega_n(D_r)$ has repeated eigenvalues if and only if $r = \frac{p}{q} \in \mathbb{Q}$ with $\gcd(p,q) = 1$ is such that $q \leq d$, and

(ii) the largest, second largest, smallest and second smallest eigenvalues of $\omega_n(D_r)$ all have multiplicity one.

Proof. It is easy to calculate that $\omega_n(D_r)$ is a diagonal matrix whose diagonal entries are

$$\{e^{a(2-r)+b(2r-1)+c(-r-1)} \mid a, b, c \in [0,d] \cap \mathbb{Z} \text{ and } a+b+c = d\}. \tag{4.10}$$

Since we are only interested in the multiplicity and relative order of the diagonal entries in (4.10), we can instead consider their logarithms: $a(2-r)+b(2r-1)+c(-r-1)$. Making $c = d-(a+b)$ the previous expression simplifies to $3(a+rb)-d(r+1)$ with the conditions

$$a, b \in [0, d] \cap \mathbb{Z}, \tag{4.11}$$

$$a+b \leq d. \tag{4.12}$$

To prove (i) notice that $\omega_n(D_r)$ will have repeated eigenvalues if and only if there are two distinct tuples a, b and a', b' that satisfy (4.11), (4.12) and such that $3(a+rb)-d(r+1) = 3(a'+rb')-d(r+1)$, which is equivalent to $r(b-b') = a'-a$. Assume such a pair of tuples exists. Having $b-b'=0$ would force $a'-a=0$, so let $b > b'$. Then $r = \frac{a'-a}{b-b'} \in \mathbb{Q}$. Since $0 \leq b' < b \leq d$, the denominator $b-b'$ is between 1 and d as desired. For the converse suppose $r = \frac{p}{q} \in [0,1] \cap \mathbb{Q}$ is in lowest terms and $q \leq d$. Then the tuples $a = 0,\ b = d$ and $a' = p,\ b' = d-q$ satisfy the desired conditions.

To prove (ii) take $r \in (0,1)$, assume $\omega_n(D_r)$ has k distinct eigenvalues and let m_i be the multiplicity of its i-th largest eigenvalue. Each pair a, b satisfying (4.11) and (4.12) corresponds to a diagonal entry $e^{3(a+rb)-d(r+1)}$ of $\omega_n(D_r)$. Then finding extrema among the diagonal entries amounts to analyzing the extrema of $E(a,b) = a+rb$ on the set $\mathbb{Z}^2 \cap T$, where $T \subset \mathbb{R}^2$ is the triangle with vertices $(0,0)$, $(0,d)$ and $(d,0)$. The function E with domain T attains its absolute maximum and minimum once, it does so at $(d,0)$ and $(0,0)$ respectively. Thus $m_1 = m_k = 1$. Since $r < 1$ the second smallest value of E on $\mathbb{Z}^2 \cap T$ is $E(0,1) = r$. It is easy to see $E(a,b) > r$ for any other $(a,b) \in (\mathbb{Z}^2 \cap T) \setminus \{(0,0),(0,1)\}$, then $m_{k-1} = 1$. If $a+b < d$ then $a+rb < d-1+r < E(0,d)$. Along the segment from $(0,d)$ to $(d,0)$ the function increases as we go towards $(d,0)$. Therefore the second biggest value of E on $\mathbb{Z}^2 \cap T$ is attained once at $(d-1,1)$ and we get $m_2 = 1$.

□

Proposition 4.31 *For every $r \in [0,1]$ let γ_r be the geodesic ray*

$$\begin{aligned} \gamma_r \colon [0,\infty) &\to X_3 \\ t &\mapsto \begin{pmatrix} e^{t(2-r)} & & \\ & e^{t(2r-1)} & \\ & & e^{t(-r-1)} \end{pmatrix}. \end{aligned}$$

and consider the map between symmetric spaces $\tilde{\omega}_n \colon X_3 \to X_n$ defined in (4.8). Then the signature of $\tilde{\omega}_n([\gamma_r]) \in \partial_\infty X_n$ satisfies

(i) $s(\tilde{\omega}_n([\gamma_r])) < (\underbrace{1,\dots,1}_{n})$ *if and only if $r = \frac{p}{q} \in \mathbb{Q}$ with $\gcd(p,q) = 1$ such that $q \leq d$, and*

(ii) $(1,1,n-4,1,1) \leq s(\tilde{\omega}_n([\gamma_r]))$ *for $r \neq 0,1$.*

Proof. From the proof of (ii) in proposition 4.27 we know there is a matrix $M \in PSL(n,\mathbb{R})$ such that $\tilde{\omega}_n(\gamma_r(t)) = M \cdot \omega_n(\gamma_r(t))$. Since the signature is invariant under the $PSL(n,\mathbb{R})$ action we get that

$$s(\tilde{\omega}_n([\gamma_r])) = s([\tilde{\omega}_n(\gamma_r)]) = s([M \cdot \omega_n(\gamma_r)]) = s(M \cdot [\omega_n(\gamma_r)]) = s([\omega_n(\gamma_r)]).$$

Let m_i be the multiplicity of the i-th largest diagonal entry of $\omega_n(\gamma_r(1)) = \omega_n(\mathrm{diag}(e^{2-r}, e^{2r-1}, e^{-r-1}))$, then $s([\omega_n(\gamma_r)]) = (m_1,\dots,m_k)$ by proposition 4.12. To prove (i) notice that $s([\omega_n(\gamma_r)]) < (1,\dots,1)$ is equivalent to $\omega_n(\gamma_r(1))$ having repeated diagonal entries. For part (ii), by proposition 4.13 we have that $(1,1,n-4,1,1) \leq s([\omega_n(\gamma_r)]) = (m_1,\dots,m_k)$ if and only if $m_1 = m_2 = m_{k-1} = m_k = 1$. These two statements follow from lemma 4.30.

□

Proposition 4.32 *Let $\sigma_f^3 \subset \partial_\infty X_3$ be the fundamental chamber of $\Delta(X_3)$, and let $\tau_f^n \subset \partial_\infty X_n$ be the simplex of signature $(1,1,n-4,1,1)$ in the fundamental chamber of $\Delta(X_n)$. Let $\partial_\infty^{\sigma_f^3-reg} X_3$ be the σ_f^3-regular part of $\partial_\infty X_3$, and $\partial_\infty^{\tau_f^n-reg} X_n$ be the τ_f^n-regular part of $\partial_\infty X_n$. Then*

$$\tilde{\omega}_n(\partial_\infty^{\sigma_f^3-reg} X_3) \subseteq \partial_\infty^{\tau_f^n-reg} X_n.$$

Proof. Recall that C_3 is the cone made of all diagonal matrices in X_3 with positive diagonal entries in non-increasing order and $\sigma_f^3 = \partial_\infty C_3$. For every $r \in [0,1]$ define the geodesic ray $\gamma_r\colon [0,\infty) \to C_3$ as in proposition 4.31. Then $\mathrm{int}(\sigma_f^3) = \mathrm{int}(\partial_\infty C_3) = \{[\gamma_r] \mid r \in (0,1)\}$. Take $\tilde{\omega}_n(x) \in \tilde{\omega}_n(\partial_\infty^{\sigma_f^3-reg} X_3)$. By proposition 4.14 we know that $\tilde{\omega}_n(x)$ is in $\partial_\infty^{\sigma_f^3-reg} X_3$ if and only if $(1,1,n-4,1,1) = s(\tau_f) \leq s(\tilde{\omega}_n(x))$. Since x is in the σ_f^3-regular part of $\partial_\infty X_3$ there is a $g \in SL(3,\mathbb{R})$ such that $x = g \cdot [\gamma_r]$ for some $r \in (0,1)$. We then have that

$$s(\tilde{\omega}_n(x)) = s(\tilde{\omega}_n(g \cdot [\gamma_r])) = s(\omega_n(g) \cdot \tilde{\omega}_n([\gamma_r])) = s(\tilde{\omega}_n([\gamma_r])).$$

From part (ii) of proposition 4.31 we know that $(1,1,n-4,1,1) \leq s(\tilde{\omega}_n([\gamma_r]))$.

□

4.6 Composition of representations

In this section we examine the composition of Hitchin representations $\rho\colon \pi_1(S) \to PSL(3,\mathbb{R})$ with the representation $\omega_n\colon PSL(3,\mathbb{R}) \to PSL(n,\mathbb{R})$ defined in (4.7). The question of when a composition of representations is Anosov has been previously addressed by Guichard and Wienhard (see [32] sec. 4.1). Their work uses parabolic subgroups instead of simplices at infinity. We further examine where these compositions are located within the representation space. Theorem 4.34 and proposition 4.35 in this section complete the proof of theorem 4.1.

Proposition 4.33 *Let $\omega_n\colon PSL(3,\mathbb{R}) \to PSL(n,\mathbb{R})$ be the representation defined in (4.7) and $\tilde{\omega}\colon X_3 \to X_n$ its induced map between symmetric spaces as defined in (4.8). Then for any representation $\rho\colon \Gamma \to PSL(3,\mathbb{R})$ we have that $\Lambda_{\omega_n\circ\rho} = \tilde{\omega}_n(\Lambda_\rho)$.*

Proof. Recall that for any representation $r\colon \Gamma \to PSL(m,\mathbb{R})$ its visual limit set is $\Lambda_r = \overline{o_r(\Gamma)} \cap \partial_\infty X_m$, where $o_r\colon \Gamma \to X_m$ is any orbit map for r. Let $o_\rho\colon \Gamma \to X_3,\ \ \gamma \mapsto \rho(\gamma)\cdot Id$ be the orbit map of ρ, and $o_{\omega_n\circ\rho}\colon \Gamma \to X_n,\ \ \gamma \mapsto \omega_n(\rho(\gamma))\cdot J$ be the orbit map of $\omega_n \circ \rho$. Notice that $o_{\omega_n\circ\rho} = \tilde{\omega}_n \circ o_\rho$.

From propositions 4.27 and 4.29 we know that $\tilde{\omega}_n\colon X_3 \to X_n$ sends geodesic rays to geodesic rays and is bi-Lipschitz. Then we may extend $\tilde{\omega}_n$ to a topological embedding $\tilde{\omega}_n\colon X_3 \sqcup \partial_\infty X_3 \to X_n \sqcup \partial_\infty X_n$, where both compactifications have the cone topology. Given that $\tilde{\omega}_n$ is injective,

$$\tilde{\omega}_n(\Lambda_\rho) = \tilde{\omega}_n(\overline{o_\rho(\Gamma)} \cap \partial_\infty X_3) = \tilde{\omega}_n(\overline{o_\rho(\Gamma)}) \cap \tilde{\omega}_n(\partial_\infty X_3).$$

Since $X_3 \cup \partial_\infty X_3$ is compact, then $\tilde{\omega}_n$ restricts to a homeomorphism onto its image. Then $\tilde{\omega}_n(\overline{o_\rho(\Gamma)}) = \overline{\tilde{\omega}_n(o_\rho(\Gamma))}$, where this closure is taken with respect to $\tilde{\omega}_n(X_3 \sqcup \partial_\infty X_3)$. The image $\tilde{\omega}_n(X_3 \sqcup \partial_\infty X_3)$ is closed in $X_n \sqcup \partial_\infty X_n$ because $X_3 \sqcup \partial_\infty X_3$ is compact and $X_n \sqcup \partial_\infty X_n$ Hausdorff. Then the closure of $\tilde{\omega}_n(o_\rho(\Gamma))$ with respect to $\tilde{\omega}_n(X_3 \sqcup \partial_\infty X_3)$ and with respect to $X_n \sqcup \partial_\infty X_n$ are the same. We have then that

$$\begin{aligned} \tilde{\omega}_n(\Lambda_\rho) &= \tilde{\omega}_n(\overline{o_\rho(\Gamma)}) \cap \tilde{\omega}_n(\partial_\infty X_3) \\ &= \overline{\tilde{\omega}_n(o_\rho(\Gamma))} \cap \tilde{\omega}_n(\partial_\infty X_3) \\ &= \overline{o_{\omega_n\circ\rho}(\Gamma)} \cap \tilde{\omega}_n(\partial_\infty X_3). \end{aligned}$$

Now since $\overline{o_{\omega_n \circ \rho}(\Gamma)} = \tilde{\omega}_n(\overline{o_\rho(\Gamma)}) \subset \tilde{\omega}_n(X_3 \cup \partial_\infty X_3)$ then

$$\overline{o_{\omega_n \circ \rho}(\Gamma)} \cap \partial_\infty X_n \subset \tilde{\omega}_n(X_3 \cup \partial_\infty X_3) \cap \partial_\infty X_n = \tilde{\omega}_n(\partial_\infty X_3).$$

Therefore

$$\tilde{\omega}_n(\Lambda_\rho) = \overline{o_{\omega_n \circ \rho}(\Gamma)} \cap \tilde{\omega}_n(\partial_\infty X_3) = \overline{o_{\omega_n \circ \rho}(\Gamma)} \cap \partial_\infty X_n = \Lambda_{\omega_n \circ \rho}.$$

□

Theorem 4.34 *Let $\sigma_f^3 \subset \partial_\infty X_3$ be the fundamental chamber of $\Delta(X_3)$, and let $\tau_f^n \subset \partial_\infty X_n$ be the simplex of signature $(1, 1, n-4, 1, 1)$ in the fundamental chamber of $\Delta(X_n)$. Let $\rho\colon \Gamma \to SL(3, \mathbb{R})$ be a σ_f^3-URU representation and $\omega_n\colon SL(3, \mathbb{R}) \to SL(n, \mathbb{R})$ be the representation defined in (4.7). Then $\omega_n \circ \rho\colon \Gamma \to SL(n, \mathbb{R})$ is a τ_f^n-URU representation.*

Proof. Let o_ρ and $o_{\omega_n \circ \rho} = \tilde{\omega}_n \circ o_\rho$ be the orbit maps for ρ and $\omega_n \circ \rho$ used in the proof of 4.33. Since ρ is σ_f^3-URU in particular it is undistorted, so o_ρ is a quasi-isometric embedding. From proposition 4.29 we know that $\tilde{\omega}_n$ is bi-Lipschitz, therefore the composition $\tilde{\omega}_n \circ o_\rho = o_{\omega_n \circ \rho}$ is a quasi-isometric embedding too. This proves $\omega_n \circ \rho$ is undistorted.

Next we verify that $\Lambda_{\omega_n \circ \rho}$ is contained in $\partial_\infty^{\tau_f^n - reg} X_n$, the τ_f^n-regular part of $\partial_\infty X_n$. By proposition 4.33 we know that $\Lambda_{\omega_n \circ \rho} = \tilde{\omega}_n(\Lambda_\rho)$. Since ρ is σ_f^3-regular then Λ_ρ is contained in $\partial_\infty^{\sigma_f^3 - reg} X_3$, the σ_f^3-regular part of $\partial_\infty X_3$. Therefore using proposition 4.32 we obtain that $\Lambda_{\omega_n \circ \rho} = \tilde{\omega}_n(\Lambda_\rho) \subseteq \tilde{\omega}_n(\partial_\infty^{\sigma_f^3 - reg} X_3) \subseteq \partial_\infty^{\tau_f^n - reg} X_n$.

□

Remark. Proposition 4.3 in [32] can be used to see that the composition of representations constructed in 4.34 is τ_f-URU where $\tau_f \subset \partial_\infty X_n$ is a simplex of signature $(1, n-2, 1)$.

Proposition 4.35 *Let $\rho\colon \pi_1(S) \to PSL(3,\mathbb{R})$ be a representation in the Hitchin component Hit$(\pi_1(S), PSL(3,\mathbb{R}))$ and $\omega_n\colon SL(3,\mathbb{R}) \to SL(n,\mathbb{R})$ be the representation defined in (4.7). Then $\omega_n \circ \rho\colon \pi_1(S) \to PSL(n,\mathbb{R})$ is not in Hit$(\pi_1(S), PSL(n,\mathbb{R}))$.*

Proof. Consider first a Fuchsian representation $\omega_{3,2} \circ r\colon \pi_1(S) \to PSL(3,\mathbb{R})$, where $r\colon \pi_1(S) \to PSL(2,\mathbb{R})$ is discrete and faithful and $\omega_{3,2}\colon PSL(2,\mathbb{R}) \to PSL(3,\mathbb{R})$ is the unique irreducible representation between $PSL(2,\mathbb{R})$ and $PSL(3,\mathbb{R})$. For any $\gamma \in \pi_1(S)$ we have that $r(\gamma)$ is conjugate to a matrix $D = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, and so $\omega_{3,2}(D) = \begin{pmatrix} \lambda^2 & & \\ & 1 & \\ & & \frac{1}{\lambda^2} \end{pmatrix}$. Let $\rho_0 = \omega_{3,2} \circ r$. We can calculate that $\omega_n \circ \rho_0(\gamma)$ has 1 as an eigenvalue with multiplicity greater than one. Since every representation in $\mathrm{Hit}(\pi_1(S), PSL(n,\mathbb{R}))$ is purely loxodromic (see def. 3.6), then $\omega_n \circ \rho_0$ is not in $\mathrm{Hit}(\pi_1(S), PSL(n,\mathbb{R}))$. Now let $\rho\colon \pi_1(S) \to PSL(3,\mathbb{R})$ be a representation in $\mathrm{Hit}(\pi_1(S), PSL(3,\mathbb{R}))$. There exists a path of representations $\rho_t\colon [0,1] \to \mathrm{Hit}(\pi_1(S), PSL(3,\mathbb{R}))$ such that ρ_0 is a Fuchsian representation and $\rho_1 = \rho$. Since the path of representations $\omega_n \circ \rho_t$ is connected and $\omega_n \circ \rho_0$ is not in $\mathrm{Hit}(\pi_1(S), PSL(n,\mathbb{R}))$, then $\omega_n \circ \rho$ is not in the Hitchin component either. $\square$

Theorem 4.1 follows now from theorem 4.34 and proposition 4.35.

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