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A FINITE ELEMENT METHOD FOR THE SOLUTION OF A POTENTIAL THEORY INTEGRAL EQUATION

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ABSTRACT

This paper discusses a finite element approximation for an integral equation of the second kind deduced from a potential theory boundary value problem in two variables. The equation is shown to admit a unique solution, to be variational and coercive in the Hilbert space of functions $\underline{\sigma} \varepsilon H^{\frac{1}{2}}(\Gamma), \int_{\Gamma} \sigma d\gamma = 0$. *The Galerkin method with finite elements as trial functions is shown to lead to an optimal rate of convergence.*

1 - INTRODUCTION.

In [5], [10], *the integral equation method has been used to solve the boundary value problem*

$$(1.1)\quad \begin{cases} \Delta u = 0 \quad in \quad \Omega\ , \\ \Delta u = 0 \quad in \quad \Omega'\ , \\ (1-\lambda)u^- - (1+\lambda)u^+ = 2g \ on \quad \Gamma\ , \ |\lambda| \leq 1\ , \\ \dfrac{\partial u^+}{\partial n} = \dfrac{\partial u^-}{\partial n} \quad on \quad \Gamma\ , \\ |\nabla u| = O(|x|^{-2}) \quad as \quad x \to \infty\ , \end{cases}$$

where $x = (x_1, x_2)$, Ω *is a bounded domain with a sufficiently smooth boundary curve* Γ *in the plane* R_2, $\Omega' = R_2 \setminus \overline{\Omega}$, $\overline{\Omega} = \Omega \cup \Gamma$. *We denote by* u^- *the limit of* u *when* x *approaches* Γ *from the interior and by* u^+ *the corresponding exterior limit. We also adopt the convention that the normal direction of* Γ *is towards the interior of* Ω. *We denote by* $\|.\|_s$, $\|.\|_{s,\Omega}$ *the norms in* $H^s(\Gamma)$, $H^s(\Omega)$ *for all real* s *(see* [4] *for the definition of these spaces). Let us also set* $\overset{0}{H}{}^s(\Omega) = H^s(\Omega) \cap H_0^{s-\frac{1}{2}}(\Gamma)$, $H_0^s(\Gamma) = \left\{\sigma \in H^s(\Gamma),\ \langle\sigma, 1\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} = 0\right\}$, $C_0^{1,\alpha}(\Gamma) = C^{1,\alpha}(\Gamma) \cap H_0^0(\Gamma)$, *where the notation* $C^{1,\alpha}(0 < \alpha \leq 1)$ *stands for the space of Hölder continuously differentiable functions on* Γ.

The particular cases of the problem (1.1) are the exterior Dirichlet problem $(\lambda = 1)$ *and the interior one* $(\lambda = -1)$. *For* $|\lambda| < 1$, *the problems of computation of electrostatic field and magnetostatic one in piecewise homogeneous media lead to this problem* [10]. *Let us remark that these problems admit other formulation* [10] ; *in* [8], *the advantages of formulation (1.1) for the magnetostatic problems are considered.*

The solution is given in the form [5], [10], [8], *of a double-layer potential*

$$(1.2)\qquad (U\sigma)(x) \equiv \left(\frac{1}{\pi}\right) \int_\Gamma \sigma(\xi)\left[\left(\frac{\partial}{\partial n_\xi}\right) Log\ \left(\frac{1}{r}\right) - \frac{\pi}{\gamma_0}\right] d\gamma_\xi \quad , \ \forall\ x \in \Omega \cup \Omega'\ ,$$

By using

(1.3) $(U\sigma)(\pm) = (\pm)\sigma + K\sigma$,

where

(1.4) $(K\sigma)(x) = (\frac{1}{\pi})\int_{\Gamma} \sigma(\xi) (\frac{\partial}{\partial n_{\xi}}) \text{Log}\left[(\frac{1}{r}) - \frac{\pi}{\gamma_0}\right] d\gamma_{\xi}, \ \forall x \in \Gamma$,

the problem (1.1) is reduced to the Fredholm integral equation of the second kind in terms of the unknown dipole density σ on Γ

(1.5) $A\sigma \equiv \sigma - \lambda K\sigma = g$.

Note that the constant $-\frac{\pi}{\gamma_0}$ is added in (1.2) to exclude the eigenvalue $\lambda = -1$ from the spectrum of the operator K [5]. In [5], [10], (1.5) has been solved numerically in $C(\Gamma)$ by a finite difference approximation.

This paper discusses a finite element approximation of (1.5) suggested in [8]. In Section 2, we study (1.5). First, by applying the a prior estimates of potential theory, we prove that (1.5) is an isomorphism in $H^s(\Gamma)$ for all real s . Note that a similar result has been proved in [7]. Then, we introduce an operator Q by (2.2), and by using the imbedding theorems [4] and an interpolation theorem [1], we show that the inner product $(Q\sigma,\alpha)_0$ is equivalent to the Hilbert space inner product in $H_0^{\frac{1}{2}}(\Gamma)$. After finishing the paper, the author was informed that this result has been obtained in [7] from another point of view. Then, we show that A is a self-adjoint operator with respect to this inner product, and we give the variational formulation of (1.5) in $H_0^{\frac{1}{2}}(\Gamma)$. In Section 3, we apply the Galerkin method with finite elements satisfying the inverse assumption (3.2) and the convergence property (3.1), and we show that the rate of convergence is optimal.

In [11], a finite difference scheme similar to a double-layer potential has been used to solve some boundary value problems in a bounded domain. A Galerkin method for (1.5) (with $|\lambda| = 1$) has been considered in [7]. We refer to [3] for other references on the solution of Fredholm integral equations of the second kind.

Throughout the paper, c denotes a constant independent from h ,

2 - VARIATIONAL PRINCIPLE FOR THE PROBLEM IN $H_0^{\frac{1}{2}}(\Gamma)$.

THEOREM 2.1 : *In $H^s(\Gamma)$, $s \geq 0$, K is compact and its spectrum is within the interval $(-\Lambda,\Lambda)$, $0<\Lambda<1$. (1.5) has a unique solution in $H^s(\Gamma)$ for all real s, and the following estimate holds :*

$$c_1 \|\mu\|_s \leq \|A\mu\|_s \leq c_2 \|\mu\|_s \quad . \tag{2.1}$$

Proof : The operator K is compact in $C(\Gamma)$ and in $H^s(\Gamma)$, $s \geq 0$ [3, p.458]. For $\mu \in H^0(\Gamma)$, $K\mu \in C(\Gamma)$. It follows that the spectrum of K in $H^s(\Gamma)$, $s \geq 0$, is a subset of its spectrum in $C(\Gamma)$. By [5] the maximum absolute value of eigenvalues of K in $C(\Gamma)$ satisfies $\Lambda < 1$. Since $|\lambda|^{-1} \geq 1$, (1.5) has a unique solution, $\forall g \in H^s(\Gamma)$, $s \geq 0$; and therefore, (2.1) is valid for $s \geq 0$. For $s<0$, by using a classical duality argument and taking into account that (2.1) holds also when A is replaced by its adjoint A^*, we have

$$\|A\mu\|_s = \sup_{\|\sigma\|_{-s}\leq 1} |(\sigma,A\mu)_0| = \sup_{\|\sigma\|_{-s}\leq 1} |(A^*\sigma,\mu)_0| \leq \sup_{\|\sigma\|_{-s}\leq 1} \|A^*\sigma\|_{-s}\|\mu\|_s \leq c_2 \|\mu\|_s$$

Similarly, one can show the left-hand inequality in (2.1) :

$$\|\mu\|_s = \sup_{\|\sigma\|_{-s}\leq 1} |(\sigma,\mu)_0| = \sup_{\|\sigma\|_{-s}\leq 1} |(A^{-1}\sigma,A^*\mu)| \leq \sup_{\|\sigma\|_{-s}\leq 1} \|A^{-1}\sigma\|_{-s}\|A^*\mu\|_s \leq c_1^{-1}\|A\mu\|_s$$

Let us define an operator Q :

$$Q\sigma = \frac{\partial u}{\partial n}, \quad u = U\sigma, \quad \forall \sigma \in C_0^{1,\alpha}(\Gamma). \tag{2.2}$$

LEMME 2.1 : *On $C_0^{1,\alpha}(\Gamma)$, Q is linear, symmetric, and satisfies*

$$c\|\sigma\|^2_{\frac{1}{2}} \leq (Q\sigma,\sigma)_0, \quad \forall \sigma \in C_0^{1,\alpha}(\Gamma). \tag{2.3}$$

Proof : Q is linear. By using (1.3) and integrating by parts, it is easy to see that Q is symmetric. Setting $U\sigma = u$, $U\mu = v$, we have

$$(2.4) \qquad (Q\sigma,\mu)_0 = \int_\Gamma \frac{\partial u}{\partial n}\,\mu\, d\gamma = 2^{-1}\int_\Gamma \frac{\partial u}{\partial n}\,(v^- - v^+)\, d\gamma = 2^{-1}\int_\Gamma \left(\frac{\partial u^-}{\partial n} v^- - \frac{\partial u^+}{\partial n} v^+\right) d\gamma$$

$$= 2^{-1}\int_{R^2} \nabla u\, \nabla v\, dx = (\sigma, Q\mu)_0 \,.$$

From (2.4), we have

$$(2.5) \qquad (Q\sigma,\sigma)_0 \geqslant 2^{-1}\int_\Omega (\nabla u)^2\, dx \,.$$

The norm in $H^1(\Omega)$ *can be defined by* [6]

$$(2.6) \qquad \|u\|^2_{1,\Omega} = \int_\Omega (\nabla u)^2\, dx + \left(\int_\Gamma u^- d\gamma\right)^2 .$$

From the trace theorem [4] *and Theorem 2.1, we have for* $|\lambda| = 1$, $s = \frac{1}{2}$,

$$(2.7) \qquad \|u\|_{1,\Omega} \geqslant c\,\|u^-\|_{\frac12} \geqslant c_1\,\|\sigma\|_{\frac12} \,.$$

For $\sigma \in C_0^{1,\alpha}(\Gamma)$, $u^- = \sigma + K\sigma \in C_0^{1,\alpha}(\Gamma)$, *and therefore (2.6) is written as*

$$(2.8) \qquad \|u\|_{1,\Omega} = \int_\Omega (\nabla u)^2\, dx \,, \quad \forall\, \sigma \in C_0^{1,\alpha}(\Gamma).$$

Now, (2.3) follows from (2.5), (2.7), (2.8).

Let $H_Q(\Gamma)$ *be the Hilbert space obtained by the completion of* $C_0^{1,\alpha}(\Gamma)$ *with the norm* $\|\sigma\|^2_Q = (Q\sigma,\sigma)_0$. *By* [6, p. 79], Q *can be extended to a self-adjoint operator in* $H_Q(\Gamma)$. *Let* $Q^{\frac12}$ *be its positive square root.*

<u>*LEMMA 2.2*</u> *: The inner product*

$$(2.9) \qquad \|\sigma\|^2_Q \equiv \|Q^{\frac12}\sigma\|^2_0 = 2^{-1}\int_{R^2} (\nabla u)^2\, dx \,, \quad u = U\sigma \,, \quad \forall\, \sigma \in H_0^{\frac12}(\Gamma)$$

is equivalent to the norm in $H_0^{\frac12}(\Gamma)$.

<u>*Proof*</u> *: By Theorem 2.1, for* $\sigma \in H_0^{1}(\Gamma)$, $u^- = \sigma + K\sigma \in H_0^{1}(\Gamma)$. *From the imbedding theorems for a harmonic function* [4, Section 7.3], *it follows that* $u \in \overset{0}{H}{}^{\frac32}(\Omega)$ *and* $Q\sigma = \frac{\partial u}{\partial n} \in H_0^0(\Gamma)$. *Since all the mappings* $\sigma \to u^- \to Q\sigma$ *are continuous, then* $\|Q\sigma\|_0 \leqslant c\,\|\sigma\|_1$. *Using the obvious inequality* $\|Q^0\sigma\|_0 \leqslant \|\sigma\|$

and the definition of $H^s(\Gamma)$; an interpolation theorem [1, p. 254] gives $\|Q^{\frac{1}{2}}\sigma\|_0 \leq c\,\|\sigma\|_{\frac{1}{2}}$, $\forall\,\sigma \in H_0^{\frac{1}{2}}(\Gamma)$. From (2.3), it follows [6, p. 68] that $c_1\,\|\sigma\|_{\frac{1}{2}} \leq \|Q^{\frac{1}{2}}\sigma\|_0$, $\forall\,\sigma \in H_Q(\Gamma)$. The statement of the lemma thus follows.

THEOREM 2.2 : A is self-adjoint in $H_0^{\frac{1}{2}}(\Gamma)$ with respect to the inner product (2.9) and the following estimate holds

$$(2.10)\qquad (1-|\lambda|\ \Lambda)\ \|\mu\|_Q^2 \leq a(\mu,\mu) \leq (1+|\lambda|\ \Lambda)\ \|\mu\|_Q^2\ ,\ \forall\,\mu \in H_0^{\frac{1}{2}}(\Gamma)\ ;$$

there exists a unique element $\sigma \in H_0^{\frac{1}{2}}(\Gamma)$ such that

$$(2.11)\qquad a(\sigma,\mu) = (g,\mu)_Q\ ,\forall\,\mu \in H_0^{\frac{1}{2}}(\Gamma),\ g \in H_0^{\frac{1}{2}}(\Gamma),$$

where

$$(2.12)\qquad a(\sigma,\mu) = (A\sigma,\mu)_Q\ .$$

Proof : Let us verify that K is self-adjoint with respect to the inner product $(.,.)_Q$. From Theorem 2.1, K is bounded in $H_0^{\frac{1}{2}}(\Gamma)$. It is therefore sufficient to verify that it is symmetric for smooth σ,μ . By setting $u = U\sigma$, $v = U\mu$ and using (1.3) and (2.4), we have

$$(2.13)\qquad (K\mu,\sigma)_Q = 2^{-1}(v^- + v^+, \frac{\partial u}{\partial n}) = 2^{-1}\left[\int_\Omega \nabla v\ \nabla u\ dx - \int_{\Omega'} \nabla v\ \nabla u\ dx\right] = (\mu,K\sigma)_Q\ .$$

From Theorem 2.1, K is also compact. It follows $\|K\|_Q = \Lambda$. Now (2.10) follows easily from (1.5), (2.12). The second statement of the theorem can be deduced from the first one.

COROLLARY 2.1 : For $g \in H_0^{\frac{1}{2}}(\Gamma)$, the problem (1.1) has a unique solution $u(x) = (U\sigma)(x),\ x \in R^2 \setminus \Gamma\ ; |\nabla u| \in H^0(\Omega) \times H^0(\Omega')$.

Proof : Existence follows from Theorem 2.2. Let u be the difference between two solutions of (1.1). Then

$$(1-\lambda)\int_\Omega (\nabla u)^2\ dx + (1+\lambda)\int_{\Omega'} (\nabla u)^2\ dx = 0\ .$$

3 - THE RATE OF CONVERGENCE IN THE FINITE ELEMENT METHOD.

Let $H_h \subset H^m(\Gamma)$, $m > 0$, *integer*, $0 < h < 1$, *be a regular finite element space satisfying the following conditions :*

- Convergence property : $\forall\, \mu \in H^s(\Gamma)$, $\exists\, \tilde{\mu}_h \in H_h$ *such that for* $k \leq s$, *with* $-m-1 \leq k \leq m$, $-m \leq s \leq m+1$, *we have*

$$\|\mu - \tilde{\mu}_h\|_k \leq c\, h^{s-k}\, \|\mu_h\|_s . \tag{3.1}$$

- Inverse assumption : $\forall\, \mu_h \in H_h$ *for* $k \leq s$, *with* $|k|, |s| \leq m$, *we have*

$$\|\mu_h\|_s \leq c\, h^{k-s}\, \|\mu_h\|_k \tag{3.2}$$

Remark : The particular choice of finite elements satisfying (3.1), (3.2) is considered in [3].

The approximate solution σ_h *of (2.11) is obtained from*

$$a(\sigma_h, \mu_h) = (g, \mu_h)_Q \; , \; \forall\, \mu_h \in H_h . \tag{3.3}$$

We adopt the notation $u = U\sigma$, $u_h = U\sigma_h$, *where* σ *is the solution of (2.11) and* σ_h *the solution of (3.3).*

THEOREM 3.1 : Let $g \in H_0^s(\Gamma)$, $|k| \leq m$, $k \leq s \leq m+1$. *Then the error in the finite element method satisfies*

$$\|\sigma - \sigma_h\|_k \leq c_1\, h^{s-k}\, \|\sigma\|_s \leq c\, h^{s-k}\, \|g\|_s . \tag{3.4}$$

Proof :

(i) $\frac{1}{2} \leq k \leq m$. *The proof is similar to that of [3, Theorem 4] , and follows from (2.1), (2.10), the equivalence of norms* $\|.\|_Q$ *and* $\|.\|_{\frac{1}{2}}$, *(3.1), (3.2), (3.3).*

(ii) $-m \leq k \leq \frac{1}{2}$. By using Nitsche's trick [9, pp. 166-167], we have

$$(3.5) \qquad (f, \sigma - \sigma_h)_Q \leq c \, \|\mu - \delta_h\|_{\frac{1}{2}} \, \|\sigma - \sigma_h\|_{\frac{1}{2}} \, , \ \forall \, \delta_h \in H_h \, ,$$

where $f \in H_0^{\frac{1}{2}}(\Gamma)$, and

$$(3.6) \qquad a(\mu, \delta) = (f, \delta)_Q \, , \ \forall \, \delta \in H_0^{\frac{1}{2}}(\Gamma).$$

From (3.6), (3.1) for $k = \frac{1}{2}$, and from Theorems 2.1, 2.2, for $\delta_h = \tilde{\mu}_h$, we hav

$$(3.7) \qquad \|\mu - \delta_h\|_{\frac{1}{2}} \leq c \, h^{t-\frac{1}{2}} \, \|f\|_t \, , \quad \frac{1}{2} \leq t \leq m+1 \, .$$

Substituting (3.7) and (3.4) for $k = \frac{1}{2}$ in (3.5),

$$(f, Q(\sigma - \sigma_h))_0 \leq c \, h^{s+t-1} \, \|f\|_t \, \|\sigma\|_s \, .$$

By duality in the definition of negative norms,

$$(3.8) \qquad \|Q(\sigma - \sigma_h)\|_{-t} = \sup_{\|f\|_t \leq 1} |(f, Q(\sigma - \sigma_h))_0| \leq c \, h^{s+t-1} \, \|\sigma\|_s \, .$$

From the imbedding theorems for a harmonic function [4, Section 7.3], we have for $Q(\sigma - \sigma_h) \equiv (\frac{\partial}{\partial n})(u - u_h) \in H_0^{-t}(\Gamma)$, $u - u_h \in \overset{0}{H}{}^{\frac{3}{2}-t}(\Omega)$, and therefore $u^- - u_h^- \in H_0^{1-t}(\Gamma)$; then, by Theorem 2.1, for $|\lambda| = 1$, $s = 1-t$, $\sigma - \sigma_h \in H^{1-t}(\Gamma)$. Since all the mappings $Q(\sigma - \sigma_h) \rightarrow u^- - u_h^- \rightarrow \sigma - \sigma_h$ are continuous, we have

$$(3.9) \qquad \|\sigma - \sigma_h\|_{1-t} \leq c \, \|Q(\sigma - \sigma_h)\|_{-t} \, .$$

Then, (3.4) follows from (3.8) and (3.9).

<u>THEOREM 3.2</u> : For $1 \leq s \leq m+1$, the potentials converge uniformly in $\overline{\Omega}$ and in $\overline{\Omega}'$ so that

$$(3.10) \qquad |u(x) - u_h(x)| \leq c \, h^{s-\frac{1}{2}} \, \|g\|_s \, ;$$

in Ω and in Ω' we have the following estimate

$$(3.11) \qquad |u(x) - u_h(x)| \leq \left(\frac{c}{e(x,\Gamma)}\right) h^{s+m} \, \|g\|_s \, ,$$

$$(e(x,\Gamma))^{-1} = ((dist(x,\Gamma))^{-2} + \pi^2 \gamma_c^{-2})^{\frac{1}{2}} + \sum_{n=1}^{m} (dist(x,\Gamma))^{-n-1} \, .$$

Proof : From the maximum principle,

$$\max_{\overline{\Omega}} |u - u_h| = \max_{\Gamma} |u - u_h| = \max |u^- - u_h^-|$$

By using the following inequality

$$|\sigma(s)| \leq c\,(h^{-\frac{1}{2}} \|\sigma\|_0 + h^{\frac{1}{2}} \|\sigma\|_1),\ \forall\ \sigma \in H^1(\Gamma),$$

Theorem 2.1 for $|\lambda| = 1$, $s = 0,1$, and Theorem 3.1 for $k = 0,1$, we obtain (3.10).
The case $x \in \overline{\Omega}'$ is considered in the same way (see also [3, Theorem 8]).
(3.11) follows from the following inequality

$$|u(x) - u_h(x)| \leq c\ \|\sigma - \sigma_h\|_{-m}\ \|(\frac{\partial}{\partial n_\xi})(\mathrm{Log}\ (\frac{1}{r}) - \frac{\pi}{\gamma_0})\|_m$$

and Theorem 3.1 for $k = m$ (see also [7, p. 110]).

This way of taking the problem seems to lead to rather complex coefficients to compute. Here numerical studies are required. In a next paper we will consider the approximation of the boundary due to [7] and deal with this problem. It seems useful to give a simple remark here. In numerical computations, when the integral operators are replaced by matrices by using some quadrature rules, if we need to compute $AB\,\overline{a}$ (A,B being matrices and a a vector), we compute $A(B\,\overline{a})$ and not $(AB)\,\overline{a}$. It would be interesting to compare our method for the solution of (1.5) and the method used in [7] and [11].

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