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Degenerate Diffusions with Advection

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by

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ABSTRACT OF THE DISSERTATION

Degenerate Diffusions with Advection

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Flow of an ideal gas through a homogeneous porous medium can be described by the well-known Porous Medium Equation (PME). The key feature is that the pressure is proportional to some powers of the density, which corresponds to the anti-congestion effect given by the degenerate diffusion. This effect is widely seen in fluids, biological aggregation and population dynamics. If adding an advection, the equation can be naturally contextualized as a population moving with preferences or fluids in a porous medium moving with wind. Furthermore we may consider drifts that depend on the solution itself by a non-local convolution, which describe the interaction between particles in a swarm model or a model for chemotaxis. In this dissertation, we study those PDEs.

In the first two chapters, we consider local advection transportation driven by a known vector field. Chapter 1 is devoted to investigating the Hölder regularity of solutions in terms of the L^p_x bounds of the vector field. By a scaling argument, we find that $p = d$ (spatial dimension) is critical. Along with a De Giorgi-Nash-Moser type arguments, we prove Hölder regularity of solutions after time 0 in the subcritical regime $p > d$. And we give examples showing the loss of uniform Hölder continuity of solutions in the critical regime even for divergence-free drifts.

In Chapter 2, we are interested in the geometric properties of the free boundary of the solution (u): $\partial\{u > 0\}$. First it is shown that, if the initial data has a super-quadratic

growth at the free boundary, then the support strictly expands relative to the streamline. We then proceed to show the nondegeneracy and $C^{1,\alpha}$ regularity of the free boundary, when the solution is locally monotone in space. The main challenge lies in establishing the non-degeneracy, which appears new even for the zero drift case.

In Chapter 3 and 4, we consider more general drifts given by a non-local convolution with u, representing the interaction between particles as swarms of locusts or cells. Chapter 3 discusses the vanishing viscosity limit of the equation in a bounded and convex domain. The limit agrees with the first-order system with a projection operator on the boundary proposed by Carrillo, Slepcev and Wu. Thus our result gives another justification of the first-order equation. We apply the gradient flow method and we explore bounded approximations of singular measures in the generalized Wasserstein distance.

Chapter 4 considers singular kernels of the form $(-\Delta)^{-s}u$ with $s \in (0, \frac{d}{2})$. With $s = 1$ we recover the well-known Patlak-Keller-Segel equation which is an macroscopic description of the chemotaxis phenomenon. The competition between the attractive interaction and the diffusion is one of the core of subject of diffusion-aggregation equations. We study well-posedness, boundedness and Hölder regularity of solutions in most of the subcritical regime. Several open questions will be discussed.

The dissertation of Yuming Zhang is approved.

Terence Tao

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To my parents

Bingyu Wang, Wenyuan Zhang

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CHAPTER 1

Regularity Properties of Solutions

1.1 Introduction

In this chapter we study nonnegative solutions u to the following problem:

$$u_t = \Delta u^m + \nabla \cdot (uV) \quad \text{in } \mathbb{R}^d \times [0, \infty) \quad \text{with } m > 1. \tag{1.1.1}$$

The advection term $V : \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be time-independent, though our results extend trivially to $V(x,t) \in L^\infty(L^p(\mathbb{R}^d); \mathbb{R}^+)$.

The $m > 1$ in the nonlinear diffusion term above represents anti-congestion effect (see [7, 28, 44, 71]), and has been considered in many physical applications, including fluids in porous medium and population dynamics. Our system (1.1.1) can be thus naturally contextualized as a population moving with preferences or fluids in a porous medium moving with wind (see e.g. [8, 25, 29, 44, 51, 58]). The goal of this chapter is to investigate well-posedness and regularity properties of (1.1.1) in terms of bounds of V in $L^p(\mathbb{R}^d)$.

When $m = 1$, our equation is the classical drift-diffusion equation where an extensive literature is available for the corresponding regularity results, as we will discuss below. When $V = 0$, (1.1.1) is the classical *porous medium equation* (see the book [70]) where initially integrable, nonnegative weak solutions exist, is unique and immediately become Hölder continuous for positive times. In contrast to these two cases, few regularity results are available for (1.1.1) with $m > 1$ and nonzero V, even in smooth settings. Below we discuss differences in local behaviors of solutions between our equation and the aforementioned cases by a scaling argument.

For given $a, r > 0$, let $u_{a,r}(x,t) := au(rx, r^2 a^{m-1} t)$. Then $\tilde{u} := u_{a,r}$ solves

$$\partial_t \tilde{u} = \Delta \tilde{u}^m + \nabla \cdot (\tilde{V} \tilde{u}) \text{ with } \tilde{V}(x) := a^{m-1} r V(rx).$$

When $V = 0$, the above scaling was used in [35, 38] along with De Giorgi-Nash-Moser iteration arguments to derive Hölder continuity results. Here $1/a$ is chosen to be the size of oscillation for $u_{a,r}$ in the unit neighborhood, and our goal is to show that this oscillation decays with a polynomial rate as $r \to 0$. Thus our interest is in the case when the oscillation is large, i.e. when $a \leq r^{-\epsilon}$ for arbitrary small $\epsilon > 0$. Note that

$$\|\tilde{V}(\cdot)\|_{L^p(\mathbb{R}^d)} = a^{m-1} r^{1-\frac{d}{p}} \|V(\cdot)\|_{L^p(\mathbb{R}^d)}.$$

Recalling that a is bounded by an arbitrarily small negative power of $r > 0$, it is plausible that if V is bounded in $L^p(\mathbb{R}^d)$ for some $p > d$, then solutions to (1.1.1) behave like the classical porous medium equation in small scales and generate bounded, Hölder continuous solutions. Indeed when $V \in L^p(\mathbb{R}^d)$ with $p > d$ we will show that weak solutions exist and stay bounded for all times, if the solutions are initially bounded.

These heuristics however pose serious challenges to deliver uniform regularity results for our equation. The most apparent difference from the linear case comes from the fact that our diffusion is degenerate at low densities. The key is to prove the oscillation reduction in spite of the competition between the singularity of the drift and the degeneracy of the diffusion in small scales. Perhaps for this difficulty, it stays open to show that solutions become immediately bounded when starting with merely integrable initial data, when $p > d$.

On the other hand we are able to show that when $p \leq d$, uniform Hölder estimates are impossible even among divergence-free vector fields, thus establishing half of the sharp threshold. This is again expected to hold from the above heuristics, however the corresponding result does not seem to be shown for the linear case $m = 1$ to the best of our knowledge. Our proof, based on barrier arguments akin to [65], uses the degeneracy of diffusion at low densities and thus cannot be extended to the linear case.

Most results in this chapter come from a joint work with Inwon Kim [50]. In [50], Theorem 1.2.1 is proved when $p > d + \frac{4}{d+2}$. In my current work with Sukjung Hwang, we improve it to $p > d$.

1.1.1 Summary of Results

Below we state two theorems that summarizes our main results.

Theorem 1.1.1 (Well-posedness and regularity)**.** *Let us consider* (1.1.1) *with nonnegative initial data* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and with* $\|V\|_{L^p(\mathbb{R}^d)} < \infty$.

(a) [Theorem 1.2.1] If $p > d$*, then there exists a weak solution* $u \in C([0,\infty), L^1(\mathbb{R}^d))$*. Moreover* u *is uniformly bounded for all* $t \in [0,\infty)$*,*

$$\sup |u| \leq C(m, d, \|u_0\|_\infty, \|u_0\|_1, p, \|V\|_p).$$

(b) [Theorem 1.2.4] The weak solution is unique if V *is uniformly* C^1 *in* $\mathbb{R}^d$*.*

(c) [Theoreml 1.3.1] If $p > d$*, and if* u *is a weak solution of* (1.1.1) *in* $\{|x| \leq 1\} \times [0,1]$ *that is also bounded, then* u *is Hölder continuous in* $\{|x| < 1\} \times (0,1)$*.*

As for (b), when V is not C^1, general uniqueness of weak solutions are open except between strong solutions: see Theorem 1.2.5.

Regarding (c), some relevant results for (1.1.1) are from [31,35], where integrability conditions are assumed on both V and ∇V. Let us also very briefly mention some results for the linear case $m = 1$ where the threshold $L^\infty_t L^d_x$ remains the same. In [40,64] it is shown that if $V \in L^\infty([0,T], BMO^{-1}(\mathbb{R}^d))$, then an initially integrable solution becomes immediately Hölder continuous. As for $L^p_x L^q_t$ type bounds, Lieberman (see [55], Ch. VI) proves Harnack's inequality (which implies Hölder continuity of solutions) for $V \in L^p_x L^q_t$ with $\frac{d}{p} + \frac{2}{q} < 1$. Later Nazarov and Ural'tseva extend the results to $\frac{d}{p} + \frac{2}{q} = 1, q < \infty$ in [60]. For the borderline case, the corresponding result for $L^d_x L^\infty_t$ drifts is open except for the stationary case, see [63]. The papers [61,73] consider this problem for divergence free drifts in spaces sharing the same scaling property with L^d and BMO^{-1}. In two dimensions, even L^1-bound for time independent divergence-free drift turns out to be sufficient to yield continuous solutions ([64], [65]). Corresponding regularity results for $m > 1$ in two dimensions remains open.

Next we state the singularity results for the threshold case, where $V \in L^d(\mathbb{R}^d)$.

Theorem 1.1.2 (Loss of regularity, Theorem 1.4.2)**.** *There exist sequences of vector fields $\{V_n\}_n$, which are uniformly bounded in $L^d(\mathbb{R}^d)$, along with sequences of compactly supported, uniformly bounded initial data $\{u_{0,n}\}$, such that the following holds: the solutions $\{u_n\}_n$ of* (1.1.1) *with V_n and initial data $u_{0,n}$ stays uniformly bounded, but they do not share any common mode of continuity.*

The sequence of drifts given in above theorem represents strongly compressive drifts concentrated near the origin. Thus one naturally asks whether the regularity of solutions are better with singular, but divergence-free drifts. It turns out that the critical norm for drifts stays the same for divergence-free drifts.

Theorem 1.1.3. *(Loss of regularity II) [Theorem 1.4.3] There is a sequence of vector fields $\{V_n\}_n$ that are uniformly bounded in $L^3(\mathbb{R}^3)$, and a sequence of uniformly smooth initial data $\{u_{n,0}\}_n$, such that the corresponding solutions $\{u_n\}$ of* (1.1.1) *are uniformly bounded in height but not bounded in any Hölder norm in a unit parabolic neighborhood.*

The proof of above theorem is motivated by the corresponding result in [65], where loss of continuity is shown for solutions of fractional diffusion with drift at critical regime. In contrast to [65] our example makes use of the degeneracy of diffusion in small density region, such as finite propagation properties or slow decay rate for the density heights. For linear diffusion the corresponding loss of Hölder regularity results appear to be open, to the best of our knowledge. Let us mention that for linear diffusion with L^1-drifts, [64] shows the existence of discontinuous solutions for $d = 3$, while in two dimensions time-dependent vector fields are needed to generate discontinuity in solutions (see [65]).

1.2 Priori Estimates

In this section several a priori estimates are obtained for solutions for (1.1.1).

Definition 1.2.1. *We say an integrable vector vector field* $V : \mathbb{R}^d \to \mathbb{R}^d$ *is admissible if* $V = V_1 + V_2$ *where*

$$\|V_1\|_\infty + \|V_2\|_p < \infty \text{ for some } p > d.$$

Let V be an admissible vector field given in Definition 1.2.1. For any $\epsilon > 0$, consider smooth vector fields $\{V_1^\epsilon, V_2^\epsilon\}$ such that, as $\epsilon \to 0$, V_1^ϵ converges to V_1 in $L^\infty(\mathbb{R}^d)$ and V_2^ϵ converges to V_2 in $L^p(\mathbb{R}^d)$. Denote $V^\epsilon := V_1^\epsilon + V_2^\epsilon$ and

$$\varphi_\epsilon(x) := x^m + \epsilon x.$$

For some large $r > 0$, we consider $u_{\epsilon,r}$ which solves the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u_{\epsilon,r} = \Delta\varphi_\epsilon(u_{\epsilon,r}) + \nabla\cdot(u_{\epsilon,r}V^\epsilon) = 0 & \text{in } B_r \times [0,T], \\ (\nabla\varphi_\epsilon(u_{\epsilon,r}) + u_{\epsilon,r}V^\epsilon)\cdot\nu = 0 & \text{on } (\partial B_r)\times[0,T], \\ u_{\epsilon,r}(x,0) = u_0(x) & \text{on } B_r \end{cases} \tag{1.2.1}$$

where ν denotes the outward unit normal on ∂B_r. Note that (1.2.1) is a uniformly parabolic quasi-linear equation with smooth coefficients, and thus $u_{\epsilon,r}$ exists and is smooth.

In the following theorem, we are going to prove that $u_{\epsilon,r}$ are uniformly bounded independent of ϵ and r. We use a refined iteration method of Lemma 5.1 [52].

Theorem 1.2.1. *Let* $u = u_{\epsilon,r}$ *solves* (1.2.1) *with initial data* $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ *and admissible vector fields* $V^\epsilon = V_1^\epsilon + V_2^\epsilon$. *Then* $u(x,t)$ *is uniformly bounded for all* $(x,t) \in \mathbb{R}^d \times [0,\infty)$. *The bound only depends on* $m, p, \|V_1^\epsilon\|_\infty, \|V_2^\epsilon\|_p, \|u_0\|_1$ *and* $\|u_0\|_\infty$.

Proof. Without loss of generality, let us suppose that the total mass of u_0 is 1 and so is the total mass of $u(\cdot,t)$ by the equation. Let us omit the script ϵ on V^ϵ and simply write $V = V_1 + V_2$.

Denote $u_1 := \max\{(u-1), 0\}$. Since u is smooth, we multiply u_1^{n-1} on both sides of (1.2.1) and find

$$\partial_t \int_{B_r} u_1^n dx = n \int_{B_r} u_t u_1^{n-1} dx \leq -mn \int_{B_r} u^{m-1} \nabla u \nabla u_1^{n-1} dx - n \int_{B_r} V u \nabla u_1^{n-1} dx.$$

Since in the region where $\nabla u_1 \neq 0$, $u \geq 1$, the above

$$\leq -c_m \int_{B_r} \left|\nabla u_1^{\frac{n}{2}}\right|^2 dx - 2(n-1) \int_{B_r} V u u_1^{\frac{n}{2}-1} \nabla u_1^{\frac{n}{2}} dx.$$

We have for any $\delta > 0$,

$$n \left| \int_{B_r} V u u_1^{\frac{n}{2}-1} \nabla u_1^{\frac{n}{2}} dx \right| \leq \delta \int_{B_r} \left|\nabla u_1^{\frac{n}{2}}\right|^2 dx + C n^2 \int_{\mathbb{R}^d \cap \{u \geq 1\}} \left| V u u_1^{\frac{n}{2}-1} \right|^2 dx.$$

Later we will fix a δ small enough such that the sum of the positive coefficients in front of $\int_{B_r} |\nabla u_1^{\frac{n}{2}}|^2 dx$ terms are bounded by c_m. The above shows

$$\partial_t \int_{B_r} u_1^n dx \lesssim -\int_{B_r} \left|\nabla u_1^{\frac{n}{2}}\right|^2 dx + n^2 \underbrace{\int_{\{u \geq 1\}} \left| V u u_1^{\frac{n}{2}-1} \right|^2 dx}_{X_n :=} \tag{1.2.2}$$

where the constant in " $\lesssim$ " depends only on m, δ. Next

$$X_n \lesssim \underbrace{\int_{\{u \geq 1\}} \left| V_1 (1+u_1) u_1^{\frac{n}{2}-1} \right|^2 dx}_{X_{n1} :=} + \underbrace{\int_{\{u \geq 1\}} \left| V_2 (1+u_1) u_1^{\frac{n}{2}-1} \right|^2 dx}_{X_{n2}}$$

and

$$X_{n1} \lesssim \int_{\{u \geq 1\}} \left| u_1^{n-1} + u_1^n \right|^2 dx.$$

By Hölder's inequality,

$$X_{n2} \lesssim \left(\int_{\{u \geq 1\}} V_2^{2q_1} dx \right)^{\frac{1}{q_1}} \left(\int_{\{u \geq 1\}} u_1^{nq_2} + u_1^{(n-2)q_2} dx \right)^{\frac{1}{q_2}} \lesssim \left(\int_{\{u \geq 1\}} u_1^{nq_2} + u_1^{(n-2)q_2} dx \right)^{\frac{1}{q_2}}$$

where $q_1 = \frac{p}{2}$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$. By the condition

$$1 > \frac{1}{q_2} > 1 - \frac{2}{d}. \tag{1.2.3}$$

Because u has total mass 1, the total volume of the set $\{u \geq 1\}$ is bounded by 1. So $X_{n1} \lesssim X_{n2}$ and we have

$$X_n \lesssim \left(\int_{\{u \geq 1\}} u_1^{nq_2} + u_1^{(n-2)q_2} dx \right)^{\frac{1}{q_2}} \lesssim \left(\int_{\{u \geq 1\}} u_1^{nq_2} + 1 dx \right)^{\frac{1}{q_2}} \lesssim \left\| u_1^{\frac{n}{2}} \right\|_{2q_2}^2 + 1$$

By Gagliardo-Nirenberg inequality,

$$\left\| u_1^{\frac{n}{2}} \right\|_{2q_2} \leq C_1 \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^{\gamma} \left\| u_1^{\frac{n}{2}} \right\|_1^{1-\gamma} + C_1 \left\| u_1^{\frac{n}{2}} \right\|_1$$

where $\frac{1}{2q_2} = \left(\frac{1}{2} - \frac{1}{d}\right)\gamma + (1-\gamma)$ and

$$\gamma = \left(1 - \frac{1}{2q_2}\right) \Big/ \left(\frac{1}{2} + \frac{1}{d},\right)$$

which belongs to $(0,1)$ due to (1.2.3), and C_1 only depends on p. By Young's inequality

$$X_n \leq \frac{\delta}{n^2} \left\| \nabla u_1^{\frac{n}{2}} \right\|^2 + C_\delta n^{c_\gamma} \left(\int u_1^{\frac{n}{2}} dx \right)^2 + C \tag{1.2.4}$$

with $c_\gamma = \frac{2\gamma}{1-\gamma}$.

Again using Galiardo-Nirenberg inequality and Young's inequality it follows

$$\left\| u_1^{\frac{n}{2}} \right\|_2 \lesssim \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^{\beta} \left\| u_1^{\frac{n}{2}} \right\|_1^{1-\beta} + \left\| u_1^{\frac{n}{2}} \right\|_1 \lesssim \left\| \nabla u_1^{\frac{n}{2}} \right\|_2 + \left\| u_1^{\frac{n}{2}} \right\|_1$$

with $\beta = \frac{1}{2} / \left(\frac{1}{2} + \frac{1}{d}\right)$. So for some universal $C, c > 0$

$$\left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2 \geq C \left\| u_1^{\frac{n}{2}} \right\|_2^2 - c \left\| u_1^{\frac{n}{2}} \right\|_1^2. \tag{1.2.5}$$

From (1.2.2), (1.2.4) and (1.2.5), we have

$$\partial_t \int_{B_r} u_1^n + c_0 \int_{B_r} u_1^n dx \leq C n^{c_\gamma + 2} \left(\int_{B_r} u_1^{\frac{n}{2}} dx \right)^2 + C n^2.$$

Now let $n_k = 2^k$ for $k = 0, 1, 2...$ and $A_k(t) = \int u_1^{n_k}(x,t) dx$. To conclude the proof we need the following lemma, whose proof will be given in the appendix.

Lemma 1.2.2. *Suppose $\{n_k\}$ is a sequence defined by*

$$n_0 = 1, \quad n_{k+1} := 2n_k + a \text{ for all } k \geq 0, \text{ where } a > -1. \tag{1.2.6}$$

Let $\{A_k(\cdot), k = 0, 1, ...\}$ be a sequence of differentiable, positive functions on $[0, \infty)$ that satisfies

$$\frac{d}{dt} A_k + C_0 A_k \leq C_1^{n_k} + {C_1}^k (A_{k-1})^{2 + C_1 n_k^{-1}},$$

for some constants C_0, C_1. Then $\{B_k(t) := A_k^{(n_k^{-1})}(t)\}$ are uniformly bounded for all $t > 0$ and k, given that $\{B_k(0)\}$ with respect to k and $\{B_0(t)\}$ are uniformly bounded with respect to $t > 0$.

From above lemma, $A_k^{n_k^{-1}}$ are uniformly bounded. We have that $\|u_1^n(\cdot,t)\|_n$ are uniformly bounded for all t and $n \in \{2^k, k=0,1,2...\}$. By interpolation, this shows that $\|u_1\|_p < \infty$ for $1 \le p \le \infty$. Since

$$\int_{B_r} u^n dx \le \int_{\{u\ge 2\}} (2u_1)^n dx + 2^{n-1} \int_{\{u \le 2\}} u\, dx \lesssim 2^n,$$

we find the L^∞ bound of u which is independent of r, ϵ. □

1.2.1 Existence

In this section, we show existence of solutions to (1.1.1) with $V \in L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ for some $p > d$. We use the following notion for solutions.

Definition 1.2.2. *Let $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be non-negative. We say that a non-negative function $u(x,t) : \mathbb{R}^d \times [0,T] \to [0,\infty)$ is a subsolution (resp. supersolution) to* (1.1.1) *if*

$$u \in C([0,T], L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times [0,T]),$$
$$uV \in L^2([0,T]\times\mathbb{R}^d) \ \ \textit{and} \ \ u^m \in L^2(0,T,\dot{H}^1(\mathbb{R}^d)).$$

And for all non-negative test functions $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$

$$\int_0^T \int_{\mathbb{R}^d} u\phi_t dxdt \ge (\textit{resp.} \ \le) \int_{\mathbb{R}^d} u_0(x)\phi(0,x)dx + \int_0^T \int_{\mathbb{R}^d} (\nabla u^m + uV)\nabla\phi \, dxdt.$$

We say u is a weak solution to (1.1.1) *if it is both sub- and supersolution of* (1.1.1).

Theorem 1.2.3. *Assume V is admissible. Then there exists a weak solution u to* (1.1.1) *with nonnegative initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.*

Proof. The proof is parallel to the previous works [5, 6, 8]. Recall that $u_{\epsilon,r}$ solve (1.2.1). Theorem 1.2.1 states that for all $t \in [0,T]$, $\{u_{\epsilon,r}\}$ are uniformly bounded in $L^1(B_r) \cap L^\infty(B_r)$ independent of ϵ, r.

Using $\varphi_\epsilon(u_{\epsilon,r})$ as the test function in (1.2.1), we obtain

$$\left(\int_{B_r} \frac{1}{m+1} u_{\epsilon,r}^{m+1} + \frac{\epsilon}{2}u^2 dx\right)\Bigg|_0^T =$$

$$-\iint_{B_r\times[0,T]} |\nabla\varphi_\epsilon(u_{\epsilon,r})|^2 dxdt - \iint_{B_r\times[0,T]} u_{\epsilon,r}V^\epsilon\cdot\nabla\varphi_\epsilon(u_{\epsilon,r})dxdt.$$

From Hölder and Young's inequality

$$\iint_{B_r\times[0,T]} |\nabla\varphi_\epsilon(u_{\epsilon,r})|^2 dxdt \le C + \iint_{B_r\times[0,T]} u_{\epsilon,r}^2|V^\epsilon|^2 dxdt \tag{1.2.7}$$

Let q be such that $\frac{2}{q}+\frac{2}{p}=1$. Then

$$\|u_{\epsilon,r}V^\epsilon\|^2_{L^2(B_r\times[0,T])} \le \|u_{\epsilon,r}V_1^\epsilon\|^2_{L^2(B_r\times[0,T])} + 2\,\|u_{\epsilon,r}\|^2_{L^q(B_r\times[0,T])}\,\|V_2^\epsilon\|^2_{L^p(B_r\times[0,T])}.$$

The two terms on the right hand side are uniformly bounded with respect to ϵ and r, since $\{u_{\epsilon,r}\}$ are uniformly bounded in $L^\infty(B_R)\cap L^1(B_R)$.

By (1.2.7), $\{\nabla\varphi_\epsilon(u_{\epsilon,r})\}$ are uniformly bounded in $L^2(B_r\times[0,T])$. As in Theorem 1 of [5], $\{u_{\epsilon,r}\}_{\epsilon>0}$ is precompact in $L^1(B_r\times[0,T])$. Along a subsequence as $\epsilon\to 0$, we obtain a weak solution u_r to (1.1.1) in $B_r\times[0,T]$ with no-flux boundary condition. Then following the proof of Theorem 1 [5], it follows that $u_r\in C([0,T],L^1(B_R))$.

Now we send $r\to\infty$. Notice that the $L^\infty([0,T],L^p(B_r)), p\in[1,\infty]$ bounds we have on $\{u_r\}$ and $L^2(B_r\times[0,T])$ bounds on $|\nabla u_r^m|$ are independent of r. These bounds yields sufficient compactness to yield a subsequential limit $u\in C([0,T],L^1(\mathbb{R}^d))$ which is a weak solution of (1.1.1). For complete details, we refer to Theorem 2 [5]. $\square$

1.2.2 Uniqueness

This section discusses two uniqueness results. First let us consider a relatively smooth vector field V and show comparison principle for weak solutions.

Theorem 1.2.4. *Write $V=(V^i)_{i=1,\ldots,d}$ and I_d as $d\times d$ identity matrix. Suppose for some $M>0$*

$$|V|<+\infty,\ -MI_d\le DV\le MI_d. \tag{1.2.8}$$

Let $\bar u, \underline{u}$ be respectively a subsolution and a supersolution of (1.1.1) *with initial functions $\bar u_0, \underline{u}_0$ such that $\bar u_0\le\underline{u}_0$. Then $\bar u\le\underline{u}$ for $t\ge 0$.*

Proof. Define $a(x,t) := (\underline{u}^m - \bar{u}^m)/(\underline{u} - \bar{u})$ if $\underline{u} \neq \bar{u}$, otherwise $a(x,t) := m\underline{u}^{m-1}$. Suppose $\epsilon > 0$ is small enough and N is large enough such that

$$\iint_{\mathbb{R}^d \times [0,1] \cap \{a \geq N\}} |\underline{u}^m - \bar{u}^m| dxdt \leq \epsilon^2. \tag{1.2.9}$$

Let $a_{N,\epsilon}$ be a smooth approximation of $a + \epsilon$ such that for $t \in [0,1]$

$$\epsilon \leq a_{N,\epsilon} \leq N, \quad \|a_{N,\epsilon}(\cdot,t) - \min\{a(\cdot,t), N\} - \epsilon\|_2 \leq \epsilon. \tag{1.2.10}$$

For any smooth non-negative compactly supported test function ξ, we consider the following dual problem to (1.1.1):

$$\begin{cases} \varphi_t + a_{N,\epsilon}\Delta\varphi - V \cdot \nabla\varphi + \xi = 0 & \text{in } \mathbb{R}^d \times [0,T]; \\ u(x,T) = 0 & \text{on } \mathbb{R}^d \end{cases} \tag{1.2.11}$$

for some $T \in (0,1]$ to be determined. Since $a_{N,\epsilon} \geq \epsilon$, there is a unique solution $\varphi \geq 0$ of (1.2.11) which is smooth.

We write $u = \underline{u} - \bar{u}$. Since $\underline{u}$ and $\bar{u}$ are respectively super and subsolutions, by the weak inequality satisfied by u with respect to test function φ, we deduce

$$0 \leq \iint_{\mathbb{R}^d \times [0,T]} u\varphi_t dxdt + \iint_{\mathbb{R}^d \times [0,T]} au\Delta\varphi dxdt - \iint_{\mathbb{R}^d \times [0,T]} uV\nabla\varphi dxdt - \int_{\mathbb{R}^d} u(x,0)\varphi(x,0)dx.$$

Using that $u(\cdot,0) \geq 0, \varphi \geq 0$ and (1.2.11), then

$$\begin{aligned} \iint_{\mathbb{R}^d \times [0,T]} u\xi dxdt &\leq \iint_{\mathbb{R}^d \times [0,T]} |u||a - a_{N,\epsilon}||\Delta\varphi| dxdt \\ &\leq \left(\iint_{\mathbb{R}^d \times [0,T]} a_{N,\epsilon}|\Delta\varphi|^2 dxdt\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^d \times [0,T]} \frac{|a - a_{N,\epsilon}|^2}{a_{N,\epsilon}}|u|^2 dxdt\right)^{\frac{1}{2}}. \end{aligned} \tag{1.2.12}$$

We want to obtain a priori estimate for the term $\Delta\varphi$.

Fix $\zeta(t)$ be a smooth function such that $1 \leq \zeta(t) \leq 2$ and $\zeta_t \geq 2dM + 4M + 1$ for $t \in [0,T]$ which can be done when T is small enough.

We multiply (1.2.11) by $\zeta\Delta\varphi$, after integration that is

$$\iint_{\mathbb{R}^d \times [0,T]} \zeta V \cdot \nabla\varphi\Delta\varphi dxdt =$$

$$\iint_{\mathbb{R}^d\times[0,T]} \varphi_t\zeta\Delta\varphi dxdt + \iint_{\mathbb{R}^d\times[0,T]} \zeta a_{N,\epsilon}|\Delta\varphi|^2 dxdt + \iint_{\mathbb{R}^d\times[0,T]} \zeta\xi\Delta\varphi dxdt.$$

Using integration by parts and Hölder's inequality in the first inequality, the above (see Theorem 6.5 [70] in 6.2.1 for details).

$$\geq \iint_{\mathbb{R}^d\times[0,T]} \frac{1}{2}\zeta_t|\nabla\varphi|^2 dxdt + \iint_{\mathbb{R}^d\times[0,T]} \zeta a_{N,\epsilon}|\Delta\varphi|^2 dxdt - \iint_{\mathbb{R}^d\times[0,T]} \zeta\nabla\xi\nabla\varphi dxdt$$

$$\geq (dM+2M)\iint_{\mathbb{R}^d\times[0,T]} |\nabla\varphi|^2 dxdt + \iint_{\mathbb{R}^d\times[0,T]} a_{N,\epsilon}|\Delta\varphi|^2 dxdt - C\iint_{\mathbb{R}^d\times[0,T]} |\nabla\xi|^2 dxdt.$$

Then $\iint_{\mathbb{R}^d\times[0,T]} a_{N,\epsilon}|\Delta\varphi|^2 dxdt \leq$

$$\iint_{\mathbb{R}^d\times[0,T]} \zeta V\cdot\nabla\varphi\Delta\varphi dxdt - (d+2)M\iint_{\mathbb{R}^d\times[0,T]} |\nabla\varphi|^2 dxdt + C\iint_{\mathbb{R}^d\times[0,T]} |\nabla\xi|^2 dxdt. \quad (1.2.13)$$

By (1.2.8), $-(V^i_{x_j}) \leq MI_d$ and $|\nabla\cdot V| \leq dM$,

$$\iint_{\mathbb{R}^d\times[0,T]} \zeta V\cdot\nabla\varphi\Delta\varphi dxdt = \frac{1}{2}\iint_{\mathbb{R}^d\times[0,T]} \zeta|\nabla\varphi|^2\nabla\cdot V dxdt - \iint_{\mathbb{R}^d\times[0,T]} \zeta\sum_{i,j}\varphi_{x_i}V^i_{x_j}\varphi_{x_j} dxdt$$

$$\leq (d+2)M\iint_{\mathbb{R}^d\times[0,T]} |\nabla\varphi|^2 dxdt.$$

Plugging the above inequality and (1.2.13) into (1.2.12), we get

$$\iint_{\mathbb{R}^d\times[0,T]} u\xi dxdt \leq C\|\nabla\xi\|_{L^2(\mathbb{R}^d\times[0,T])}\left(\iint_{\mathbb{R}^d\times[0,T]} \frac{|a-a_{N,\epsilon}|^2}{a_{N,\epsilon}}|u|^2 dxdt\right)^{\frac{1}{2}}.$$

Now we use (1.2.10) and find out

$$\iint_{\mathbb{R}^d\times[0,T]} |a-a_{N,\epsilon}|^2|u|^2 \leq 2\iint_{\mathbb{R}^d\times[0,T]} |\min\{a,N\}+\epsilon-a_{N,\epsilon}|^2|u|^2 dxdt +$$

$$2\iint_{\mathbb{R}^d\times[0,T]} \epsilon^2|u|^2 dxdt + 2\iint_{\{a>N\}} a^2|u|^2 dxdt$$

$$\leq \left(2\|u\|^2_\infty + 2\|u\|^2_{L^2(\mathbb{R}^d\times[0,T])} + 2\right)\epsilon^2 \leq C\epsilon^2.$$

So since $a_{N,\epsilon} \geq \epsilon$, by (1.2.12)

$$\iint_{\mathbb{R}^d\times[0,T]} u\xi dxdt \leq C\|\nabla\xi\|_{L^2(\mathbb{R}^d\times[0,T])}\epsilon^{\frac{1}{2}}.$$

Letting $\epsilon > 0$, we conclude that $\iint_{\mathbb{R}^d\times[0,T]} u\xi dxdt \leq 0$ for all arbitrary C^∞_c test function $\xi \geq 0$. And so $u \leq 0$ within time $[0,T]$.

Finally since T only depends on d, M, doing this repeatedly finishes the proof. □

Our second uniqueness result is a consequence of the following L^1 contraction, which holds between "strong solutions" if m is not too large depending on the singularity of V. The existence of strong solutions remain open, with the exception of zero drift case (see [1] and section 8.1.1 of [70]).

Theorem 1.2.5. *Suppose $\|V\|_p < \infty$ for some $p \geq 2$ and $1 < m < 1 + \frac{2}{p}$. Let u_1, u_2 be two nonnegative weak solutions to* (1.1.1) *with initial datas $u_{1,0}, u_{2,0}$ respectively. Assume in addition that*

$$\partial_t(u_1 - u_2) \in L^1(\mathbb{R}^d \times [0,T]).$$

Then the following holds:

$$\int_{\mathbb{R}^d} (u_1 - u_2)_+(t)dx \leq \int_{\mathbb{R}^d} (u_{1,0} - u_{2,0})_+ dx \quad \text{for } 0 \leq t \leq T.$$

Proof. Let $\varphi \in C^1(\mathbb{R})$ be such that $\varphi(s) = 0$ if $s \leq 0$ and $\varphi(s) = 1$ if $s \geq 1$, with $\varphi'(s) \in (0,2)$. Denote $\varphi_n(s) := \varphi(ns)$ for $n = 1, 2, ...$ By definition of the weak solution we have, with $w := u_1^m - u_2^m$,

$$\begin{aligned}\iint_{\mathbb{R}^d\times[0,T]} (u_1 - u_2)_t \varphi_n(w) dxdt &= -\iint_{\mathbb{R}^d\times[0,T]} \nabla w \nabla \varphi_n(w) dxdt \\ &\quad - \iint_{\mathbb{R}^d\times[0,T]} (u_1 - u_2) V \cdot \nabla \varphi(w) dxdt \\ = -\iint_{\mathbb{R}^d\times[0,T]} |\nabla w|^2 \varphi_n'(w) dxdt &- \iint_{\mathbb{R}^d\times[0,T]} (u_1 - u_2) V \cdot \nabla w \ \varphi_n'(w) dxdt.\end{aligned}$$

Since $\varphi_n' \leq 2n$,

$$-\iint_{\mathbb{R}^d\times[0,T]} (u_1 - u_2) V \nabla w \ \varphi_n'(w) dxdt \leq$$

$$\iint_{\mathbb{R}^d\times[0,T]} |\nabla w|^2 \varphi_n'(w) dx + 2n \iint_{0<w\leq\frac{1}{n}} |u_1 - u_2|^2 |V|^2 dxdt.$$

When $p > 2$, let q be such that $\frac{2}{q} + \frac{2}{p} = 1$. Note that when $w > 0$, $u_1 > u_2 \geq 0$ and thus

$$|u_1 - u_2|^m \leq |u_1^m - u_1^{m-1} u_2| \leq |w|.$$

Thus we have

$$\iint_{0<w\leq\frac{1}{n}} |u_1-u_2|^2|V|^2dxdt \leq \iint_{0<w\leq\frac{1}{n}} |w|^{\frac{1}{m}(2-\frac{2}{q})}|u_1-u_2|^{\frac{2}{q}}|V|^2dxdt.$$

$$\leq n^{-\frac{1}{m}(2-\frac{2}{q})}\left(\iint_{0<w\leq\frac{1}{n}} |u_1-u_2|dxdt\right)^{\frac{2}{q}}\left(\iint_{\mathbb{R}^d\times[0,T]} |V_2|^p dxdt\right)^{\frac{2}{p}}.$$

Then, since $|u_1-u_1|\leq C([0,T];L^1(\mathbb{R}^d))$, it follows that

$$\int_{\mathbb{R}^d}(u_1-u_2)_t\,\varphi_n(w)dx \leq Cn^{1-\frac{1}{m}(2-\frac{2}{q})} = Cn^{1-\frac{p+2}{pm}},$$

the right hand side of which goes to 0 as $n\to\infty$ due to $m<1+\frac{2}{p}$. Now we send $n\to\infty$ to derive the desired inequality:

$$\int_{\mathbb{R}^d}(u_1-u_2)_+(t)dx \leq \int_{\mathbb{R}^d}(u_{1,0}-u_{2,0})_+dx.$$

If $p=2$, parallel and easier proof yields the result. $\square$

1.3 Hölder Continuity

1.3.1 Interior Estimates

In this section we establish the Hölder continuity results for (1.1.1).

Theorem 1.3.1. *Suppose V is locally uniformly bounded in $L^p(\mathbb{R}^d)$ for some $p>d$. Let u be a non-negative weak solution to equation* (1.1.1) *in Q_1. If $u(\cdot,t)$ is uniformly bounded by M in Q_1, then $u(\cdot,\cdot)$ is Hölder continuous in $Q_{\frac{1}{2}}$. The Hölder norm only depends on M, m, p, d and $\|V\|_{L^p_{loc}}$.*

The proof of above theorem consists of several lemmas and propositions. We begin with notations. The scaled parabolic cylinders are denoted by

$$Q(r,c) := \{x,|x|<r\}\times(-cr^2,0) \text{ for } r,c>0. \tag{1.3.1}$$

The standard parabolic cylinder is denoted by $Q_r := Q(r,1)$.

For given p, we will use

$$\delta_1 := 2 - \frac{2d}{p},\ \delta_2 := \frac{1}{2}\delta_1,\ q_1 := 1 - \frac{2}{p},\ q_2 := 1 - \frac{1}{p}.$$

In particular if $p > d \geq 2$, $q_1 + \frac{2}{d} > 1, q_2 + \frac{2}{d+2} > 1$. Let us define a new variable

$$\nu := u^m. \tag{1.3.2}$$

Then ν satisfies

$$\frac{\partial}{\partial t}\nu^{\frac{1}{m}} = \Delta\nu + \nabla \cdot (\nu^{\frac{1}{m}} V). \tag{1.3.3}$$

Next we re-scale ν by

$$v(x,t) := \nu(rx, r^2 w^{-\alpha} t) \ \text{ in } Q(r, w^{-\alpha}) \quad \text{ with } \alpha := \frac{m-1}{m}. \tag{1.3.4}$$

Then v solves

$$w^\alpha (v^{\frac{1}{m}})_t = \Delta v + r\nabla \cdot (v^{\frac{1}{m}} \tilde{V}), \quad \text{ where } \tilde{V}(x,t) := V(rx, r^2 w^{-\alpha} t). \tag{1.3.5}$$

Also denote

$$v_k^+ := \max\{(v-k), 0\}, \quad v_k^- := \max\{(k-v), 0\}.$$

We begin with an energy inequality. The proof of the lemma below are in the same spirit of the ones in Theorem 1.2 in [38] and Lemma 6.5 [37] which applies to (1.1.1) with $V = 0$. We will emphasize on the differences in the proof that occurs due to the nonzero drift term.

Let S be a measurable set in $\mathbb{R}^d$. The indicator function $\chi_S(x)$ equals 1 if $x \in S$ and it equals 0 otherwise.

Lemma 1.3.2. *Suppose v satisfies* (1.3.5) *in a neighbourhood of Q_1 for some positive w, r such that $w \geq osc_{Q_1} v$. Suppose V is locally uniformly bounded in $L^p(\mathbb{R}^d)$ for some $p > 0$. Let $\zeta \in C_0^\infty(Q_1)$ be non-negative and*

$$\zeta \leq 1, \quad |\nabla \zeta| \leq C_1, \quad |\Delta \zeta^2| \leq C_1^2, \quad |\zeta_t| \leq C_2.$$

Denote $B' := B_1 \cap supp\{\zeta\}$ and for $q \in (0,1]$

$$B_{k;q} := \left(\int_{-1}^{0}\left(\int_{B'} \chi_{\{v(x,t)<k\}} dx\right)^{q} dt\right)^{\frac{1}{q}},$$

$$A_{k;q} := \left(\int_{-1}^{0}\left(\int_{B'} \chi_{\{v(x,t)>k\}} dx\right)^{q} dt\right)^{\frac{1}{q}}$$

and M^+, M^- as the supremum and infimum of v in Q_1 respectively.

If $\frac{w}{4} \geq M^-$, then for $t \in [-1,0], k \leq M^+$,

$$\int_{B_1\times\{t\}} |v_k^-\zeta|^2 dx + \int_{-1}^{t} \left\|\nabla\left(v_k^-\zeta\right)\right\|_{2,B_1\times\{s\}}^2 ds \lesssim (C_1^2 + C_2) w^2 B_{k;1} +$$

$$r^{\delta_1} w^{\frac{2}{m}} B_{k;q_1}^{q_1} + C_1 r^{\delta_2} w^{1+\frac{1}{m}} B_{k;q_2}^{q_2}.$$

For $t \in [-1,0], \frac{w}{4} \geq M^-, k \geq M^-$, we have

$$\int_{B_1\times\{t\}} |v_k^+\zeta|^2 dx + \int_{-1}^{t} \left\|\nabla\left(v_k^+\zeta\right)\right\|_{2,B_1\times\{s\}}^2 ds \lesssim (C_1^2 + C_2) w^2 A_{k;1} +$$

$$r^{\delta_1} w^{\frac{2}{m}} A_{k;q_1}^{q_1} + C_1 r^{\delta_2} w^{1+\frac{1}{m}} A_{k;q_2}^{q_2}. \tag{1.3.6}$$

Proof. Let us only prove the second inequality. After multiplying (1.3.5) by $v_k^+\zeta^2$ and doing integration in space as well as from 0 to t, we get

$$w^{\alpha} m^{-1} \int_{B_1\times\{t\}} \left(\int_0^{v_k^+} (k+\xi)^{-\alpha}\xi d\xi\right) \zeta^2 dx + \int_{-1}^{t} \left\|\nabla\left(v_k^+\zeta\right)\right\|_{2,B_1\times\{s\}}^2 ds$$

$$\leq 2C_1^2 \int_{-1}^{t} \left\|v_k^+\right\|_{2,B_1}^2 ds + 2C_2 w^{\alpha} m^{-1} \int_{-1}^{t}\int_{B_1} \left(\int_0^{v_k^+} (k+\xi)^{-\alpha}\xi d\xi\right) \zeta dx ds +$$

$$r \int_{-1}^{t}\int_{B_1} v^{\frac{1}{m}} \tilde{V} \nabla (v_k^+\zeta^2) dx ds + 2C_1 r \iint_{Q_1} v^{\frac{1}{m}} |\tilde{V}| v_k^+ \zeta dx ds.$$

Since $v_k^+ + k \leq w$, we know

$$\frac{1}{2} w^{-\alpha} (v_k^+)^2 \leq \int_0^{v_k^+} (k+\xi)^{-\alpha}\xi d\xi \leq \int_0^{v_k^+} (k+\xi)^{\frac{1}{m}} d\xi \leq w^{\frac{1}{m}} v_k^+. \tag{1.3.7}$$

The term $r\int_{-1}^{t}\int_{B_1} v^{\frac{1}{m}} \tilde{V}\nabla(v_k^+\zeta^2) dx ds$ is bounded by

$$2r^2 \int_{-1}^{t}\int_{B_1} v^{\frac{2}{m}} |\tilde{V}|^2 \zeta^2 \chi_{\{v>k\}} dx ds + \frac{1}{2}\int_{-1}^{t}\int_{B_1} |\nabla\left(v_k^+\zeta\right)|^2 dx ds + r \int_{-1}^{t}\int_{B_1} v^{\frac{1}{m}} |\tilde{V}| v_k^+ \zeta |\nabla\zeta| dx ds.$$

From the above inequality we deduce

$$\int_{B_1\times\{t\}}|v_k^+\zeta|^2dx+\int_{-1}^{t}\left\|\nabla\left(v_k^+\zeta\right)\right\|_{2,B_1\times\{s\}}^2ds\lesssim C_1^2\left\|v_k^+\right\|_{2,Q_1}^2+C_2w\iint_{Q_1}v_k^+dxdt+$$
$$r^2\iint_{Q_1}v^{\frac{2}{m}}|\tilde{V}|^2\zeta^2\chi_{\{v>k\}}dxdt+C_1r\iint_{Q_1}v^{\frac{1}{m}}|\tilde{V}|v_k^+\zeta dxds.$$

We denote the last two terms in the above by X. Note $v_k^+\lesssim w$, therefore

$$\left\|v_k^+\right\|_{2,Q_1}^2\le w^2A_{k;1},\ \left\|v_k^+\right\|_{1,Q_1}\le wA_{k;1}.$$

Recalling that $A_{k;1}=meas\{Q_1\cap\{v>k\}\}$, it follows that

$$\int_{B_1\times\{t\}}|v_k^+\zeta|^2(t)dx+\int_{-1}^{t}\left\|\nabla\left(v_k^+\zeta\right)\right\|_{2,B_1\times\{s\}}^2ds\lesssim(C_1^2+C_2)w^2A_{k;1}+X.\tag{1.3.8}$$

Now we bound the term X. Since $\tilde{V}(x,t)=V(rx,r^2w^{-\alpha}t)$, by the assumption, for each time t

$$\|\tilde{V}(\cdot,t)\|_p=r^{-\frac{d}{p}}\|V(\cdot,t)\|_p\lesssim r^{-\frac{d}{p}}.\tag{1.3.9}$$

Then recalling $q_1:=1-\frac{2}{p}$,

$$r^2\iint_{Q_1}|\tilde{V}|^2\zeta^2\chi_{\{v>k\}}dxdt\le r^2\int_{-1}^{0}\left(\int_{B_1}|\tilde{V}|^pdx\right)^{\frac{2}{p}}\left(\int_{B'}\chi_{v>k}dx\right)^{q_1}dt$$
$$\le\int_{-1}^{0}r^2\|\tilde{V}\|_p^2\left(\int_{B'}\chi_{v>k}dx\right)^{q_1}dt\lesssim r^{2-\frac{2d}{p}}A_{k;q_1}^{q_1}.$$

Similarly, for q_2 satisfying $\frac{1}{p}+q_2=1$ we have

$$r\iint_{Q_1}|\tilde{V}|\zeta\chi_{v>k}dxdt\lesssim r^{1-\frac{d}{p}}A_{k;q_2}^{q_2}.$$

Combining with (1.3.8), this immediately gives (1.3.6) by the assumptions. Parallel argument applies for the first inequality, except that instead of (1.3.7) we apply

$$\frac{1}{2}k^{-\alpha}(v_k^-)^2\le\int_0^{v_k^-}(k-\xi)^{-\alpha}\xi d\xi\le k^{\frac{1}{m}}v_k^-$$

and the bounds of $v,\tilde{V}$. $\square$

Corollary 1.3.3. *Under the assumptions of Lemma 1.3.2. If there exist some universal constants $c, \epsilon > 0$ such that*

$$k \geq cw,\ r^{\delta_1} w^{\frac{2}{m}} \leq r^{\epsilon} w^2,\ r^{\delta_2} w^{1+\frac{1}{m}} \leq r^{\epsilon} w^2.$$

Then we have

$$\int_{-1}^{0} \left\| \nabla \left(v_k^+ \zeta \right) \right\|_{2,B_1\times\{s\}}^2 ds \leq C|M^+ - k|^2 A_{k;1} + C r^{\epsilon} w^2 A_{k;1}^{q_1}. \tag{1.3.10}$$

Proof. The proof follows from a straightforward modification of the one of Lemma 1.3.2. First, by the assumptions we can replace the second and third inequalities in (1.3.7) by

$$\int_0^{v_k^+} (k+\xi)^{-\alpha} \xi d\xi \leq k^{-\alpha} \int_0^{v_k^+} \xi d\xi \lesssim w^{-\alpha} (v_k^+)^2 \lesssim w^{-\alpha} |M^+ - k|^2.$$

Second by Hölder's inequality it is not hard to see that $A_{k,q}$ is increasing in q for $q \in (0,1]$ i.e. $A_{k;q_1} \leq A_{k;q_2} \leq A_{k;1}$. With these two and the previous proof, we conclude with the clean expression (1.3.10). $\square$

The first energy inequality in Lemma 1.3.2 will be used in Proposition 1.3.4. The second one will be used in Lemma 1.3.10 and we will apply (1.3.10) in Lemma 1.3.9.

Next we prove two propositions which regards oscillation reduction. The first one implies that under a suitable assumption the solution is bounded away from 0 with certain amount. The other shows that if the assumption is not satisfied, then the supremum of the solution decreases once we look at a smaller parabolic neighborhood.

Proposition 1.3.4. *Let $p > d$, $\alpha = \frac{m-1}{m}$ and*

$$\delta_0 = \left(1 - \frac{1}{m}\right) / \left(1 - \frac{d}{p}\right).$$

Suppose ν solves (1.3.3) in a neighbourhood of $Q(r, w^{-\alpha})$ for some $r, w > 0$. Denote $M^- = \inf\{\nu, (x,t) \in Q(r, w^{-\alpha})\}$ and let us assume that

$$w \geq osc_{Q(r,w^{-\alpha})}\nu;\ \text{and}\ \ M^- \leq \frac{w}{4}. \tag{1.3.11}$$

Then there exists $c_0 \in (0,1)$ *that only depends on* m, p *and* $\|V\|_{L^p(Q(r,w^{-\alpha}))}$ *such that the following holds: for all* $0 < r < w^{\delta_0}$ *if*

$$meas\left\{(x,t) \in Q(r,w^{-\alpha}), \nu(x,t) \geq M^- + \frac{w}{2}\right\} \geq (1-c_0)|Q(r,w^{-\alpha})|, \tag{1.3.12}$$

then

$$\nu|_{Q(\frac{r}{2},w^{-\alpha})} \geq M^- + \frac{w}{4}.$$

Proof. Recall that $v(x,t)$ defined in (1.3.4) satisfies (1.3.5) in Q_1. Set

$$r_n := \frac{1}{2} + 2^{-n},\ \tilde{Q}_n = Q(r_n, 1),\ k_n := M^- + \frac{w}{4} + \frac{w}{2^{n+2}},$$

$$\tilde{B}_{n;q} = \left(\int_{-r_n^2}^0 \left(\int_{B_{r_n}} \chi_{v(x,t)<k_n} dx\right)^q dt\right)^{\frac{1}{q}}.$$

Pick $\zeta_n \in C_0^\infty(\tilde{Q}_n \cup \left(\tilde{Q}_n + (0, 2^{-n})\right))$ which equals its maximum 1 in $\tilde{Q}_{n+1}$. Since $r_n^2 - r_{n+1}^2 \sim 2^{-n}$, we can assume

$$|\nabla\zeta_n| \lesssim 2^n, \quad |\Delta\zeta_n^2| \lesssim 4^n, \quad |\partial_t\zeta_n| \lesssim 2^n.$$

Recall the notation $v_k^- := \max\{k - v, 0\}$. By Lemma 1.3.2, after integration we have

$$\begin{aligned}\operatorname*{ess\,sup}_{-r_{n+1}{}^2 \leq t \leq 0} \int_{B_{r_{n+1}} \times \{t\}} |v_{k_n}^-|^2 dx + \int_{-r_{n+1}}^t \left\|\nabla\left(v_{k_n}^-\zeta_n\right)\right\|_{2,B_{r_n}\times\{s\}}^2 ds \\ \lesssim 4^n w^2 \tilde{B}_{n;1} + r^{\delta_1} w^{\frac{2}{m}} \tilde{B}_{n;q_1}^{q_1} + 2^n r^{\delta_2} w^{1+\frac{1}{m}} \tilde{B}_{n;q_2}^{q_2}.\end{aligned}$$

Unravelling the definition and condition we have $r^{\delta_1} w^{\frac{2}{m}} \leq w^2, r^{\delta_2} w^{1+\frac{1}{m}} \leq w^2$. Therefore if taking supremum of $t \in [-r_{n+1}^2, 0]$ as well as $t = 0$, we obtain

$$\begin{aligned}\left\|v_{k_n}^-\zeta_n\right\|_{V^{1,0}}^2 := \operatorname*{ess\,sup}_{-r_{n+1}{}^2 \leq t \leq 0} \int_{B_{r_n}} |v_{k_n}^-\zeta_n|^2(\cdot,t) dx + \int_{-r_{n+1}^2}^0 \left\|\nabla\left(v_{k_n}^-\zeta_n\right)\right\|_{2,B_{r_n}\times\{s\}}^2 ds \\ \lesssim 4^n w^2 \tilde{B}_{n;1} + w^2 \tilde{B}_{n;q_1}^{q_1} + 2^n w^2 \tilde{B}_{n;q_2}^{q_2},\end{aligned} \tag{1.3.13}$$

By Sobolev type embedding (see page 76 in [54]),

$$\left\|v_{k_n}^-\zeta_n\right\|_{L^2(B_{r_n}\times[-r_{n+1}^2,0])}^2 \lesssim \left\|v_{k_n}^-\zeta_n\right\|_{V^{1,0}}^2 \times \tilde{B}_{n;1}^{\frac{2}{d+2}}.$$

So by (1.3.13), we get

$$\left\|v_{k_n}^- \zeta_n\right\|^2_{L^2(B_{r_n}\times[-r_{n+1}^2,0])} \lesssim \left(4^n \tilde{B}_{n;1} + \tilde{B}_{n;q_1}^{q_1} + 2^n \tilde{B}_{n;q_2}^{q_2}\right) w^2 \tilde{B}_{n;1}^{\frac{2}{d+2}}. \tag{1.3.14}$$

By definition, $v_{k_n}^- \geq \frac{w}{2^{n+3}}$ in $\tilde{Q}_{n+1} \cap \{v < k_{n+1}\}$. Then

$$\left\|v_{k_n}^- \zeta_n\right\|^2_{L^2(B_{r_n}\times[-r_{n+1}^2,0])} \geq \iint_{\tilde{Q}_{n+1}\cap\{v_{k_{n+1}}^- > 0\}} \left(w\ 2^{-n-3}\right)^2 dxdt \geq w^2 2^{-2n-6} \tilde{B}_{n+1;1}.$$

Notice the length of the time interval is bounded by 1. By definition, $\tilde{B}_{n;q}$ is monotone in $q \in [0,1]$. In particular since $q_1 < q_2 < 1$,

$$\tilde{B}_{n;q_1} \leq \tilde{B}_{n;q_2} \leq \tilde{B}_{n;1}.$$

Putting above computations together, we arrive at

$$\tilde{B}_{n+1;1} \lesssim 16^n \tilde{B}_{n;1}^{1+\frac{2}{d+2}} + 16^n Z_n^{1+\kappa} \tilde{B}_{n;1}^{\frac{2}{d+2}} \tag{1.3.15}$$

where

$$Z_n := \left(\int_{-r_n^2}^0 \left(\int_{B_{r_n}} \chi_{v(x,t)<k_n} dx\right)^{\frac{p-2}{p}} dt\right)^{\frac{1}{1+\kappa}} \qquad \text{and } \kappa := \frac{2(p-d)}{pd}.$$

Define

$$h_1 = 2(1+\kappa), \quad h_2 = \frac{2(1+\kappa)p}{p-2}$$

and then, after unravelling the definitions,

$$Z_n = \left[\int_{-r_n^2}^0 \left(A_-^n(t)\right)^{\frac{h_1}{h_2}}\ dt\right]^{\frac{2}{h_1}}.$$

We observe that

$$\begin{aligned} Z_{n+1} &\leq \left[\int_{-r_{n+1}^2}^0 \left(\int_{B_{n+1}} (v-\mu_- - k_n)_-^{h_2}\, dx\right)^{\frac{h_1}{h_2}} dt\right]^{\frac{2}{h_1}} \\ &\leq \left[\int_{-r_n^2}^0 \left(\int_{B_n} (v-\mu_- - k_n)_-^{h_2} \zeta_n^{h_2}\, dx\right)^{\frac{h_1}{h_2}} dt\right]^{\frac{2}{h_1}} = \|v_{k_n}^- \zeta_n\|^2_{L_t^{h_1} L_x^{h_2}(Q_n)} \end{aligned} \tag{1.3.16}$$

On the left side, we apply the Sobolev type embedding (Proposition A.1.4) and then use (1.3.13) to say

$$\|v_{k_n}^- \zeta_n\|^2_{L_t^{h_1} L_x^{h_2}(Q_n)} \leq C \|v_{k_n}^- \zeta_n\|^2_{V^2(\bar{Q}_n)} \leq w^2 \tilde{B}_{n;1} + 2^n w^2 Z_n^{1+\kappa}. \tag{1.3.17}$$

Then the combination of (1.3.16) and (1.3.17) provides

$$Z_{n+1} \le C16^n \tilde{B}_{n;1} + C4^n Z_n^{1+\kappa}. \tag{1.3.18}$$

With (1.3.15) and (1.3.18), we apply Lemma 1.3.5 to conclude that both $\tilde{B}_{n;1}$ and Z_n tend to zero as $n \to \infty$, provided

$$\tilde{B}_{0;1} + Z_0^{1+\kappa}$$

are small enough. $\tilde{B}_{\infty;1} = 0$ directly gives our conclusion.

□

We state a lemma concerning the geometric convergence of sequences of numbers. The proof is given in [36].

Lemma 1.3.5. *Let $\{Y_n\}$ and $\{Z_n\}$, $n = 0, 1, 2, \dots$, be sequences of positive numbers, satisfying the recursive inequalities*

$$\begin{cases} Y_{n+1} \le Cb^n \left(Y_n^{1+\alpha} + Z_n^{1+\kappa} Y_n^{\alpha}\right) \\ Z_{n+1} \le Cb^n \left(Y_n + Z_n^{1+\kappa}\right) \end{cases}$$

where $C, b > 1$ and $\kappa, \alpha > 0$ are given numbers. If

$$Y_0 + Z_0^{1+\kappa} \le (2C)^{-\frac{1+\kappa}{\sigma}} b^{-\frac{1+\kappa}{\sigma^2}}, \quad \text{where } \sigma = \min\{\kappa, \alpha\},$$

then $\{Y_n\}$ and $\{Z_n\}$ tend to zero as $n \to \infty$.

Now we proceed to the second proposition.

Proposition 1.3.6. *Let ν and c_0 be as in Proposition 1.3.4. Suppose* (1.3.11) *holds while* (1.3.12) *is not satisfied. Then there exist universal constants c_1, c_2 which only depends on c_0, p and $\|V\|_{L^p(Q_1)}$such that the following is true:*

For r satisfying $r < c_1 w^{c_2}$, there exists some $\eta \in (0, 1)$ such that

$$\nu|_{Q\left(\frac{r}{2}, \frac{c_0}{2} w^{-\alpha}\right)} \le \eta w.$$

The constant η is independent of w which may depend on c_0.

The proof of the proposition rests on a number of lemmas which are variants of Lemma 6.1, 6.2, 6.3, 6.4 in [37]. We will sketch the proof for some lemmas and emphasize on the differences.

Lemma 1.3.7–Lemma 1.3.10 stated below are proven under the conditions of Proposition 1.3.6 and, for c_0 given in Proposition 1.3.4 we have

$$M^+ - M^- \geq \frac{w}{2} + c_0 w, \tag{1.3.19}$$

where M^+ and M^- denote respectively the supremum and infimum of ν in $Q(r, w^{-\alpha})$.

Let $v(x,t)$ be as given in (1.3.4) and define

$$A_{k,R}(t) := \{x \in B_R : v(x,t) > k\}. \tag{1.3.20}$$

We denote $A_k(t) := A_{k,1}(t)$.

Lemma 1.3.7. *Let $k_1 = M^+ - c_0 w$. There exists $\tau \in (-1, -\frac{c_0}{2})$ such that*

$$|A_{k_1}(\tau)| \leq (1 - c_0)\left(1 - \frac{c_0}{2}\right)^{-1} |B_1|.$$

Proof. Observe that, by (1.3.19), $k_1 \geq M^- + \frac{w}{2}$. If the claim is false,

$$meas\left\{(x,t) : |x| \leq 1, t \in (-1, -\frac{c_0}{2}), v > M^- + \frac{w}{2}\right\} \geq \int_{-1}^{-\frac{c_0}{2}} |A_{k_1}(t)| dt > (1 - c_0)|B_1|$$

which agrees with (1.3.12) and thus contradicts with the condition of Proposition 1.3.6. □

Lemma 1.3.8. *Let c_0 as given in Proposition 1.3.4, M^+ in* (1.3.19) *and $A_k(t)$ in* (1.3.20)*. There exist universal constants $c_1, c_2 > 0$ and a sufficiently large positive integer $q = q(c_0)$ which is independent of w such that if $r < c_1 w^{c_2}$ then for $k_2 = M^+ - \frac{c_0}{2^q} w$ we have*

$$|A_{k_2}(t)| \leq \left(1 - \frac{c_0^2}{4}\right)|B_1| \text{ for } t \in [-\frac{c_0}{2}, 0].$$

Proof. Without loss of generality we may assume that $c_0 < 1$. We follow the outline of the proof for Lemma 6.2 in [37]. The additional ingredient is that we need to consider the effect of the drift term and give a clear description of how small r need to be. For $q > 3$, consider

$$\psi(x) = \log^+\left(\frac{c_0 w}{c_0 w - (x - (M^+ - c_0 w))^+ + \frac{c_0 w}{2^q}}\right).$$

Then

$$0 \le \psi(x) \le q \log 2, \text{ and } \psi'(x) \in \left[0, \frac{2^q}{c_0 w}\right] \quad \text{for } x \in [0, M^+]. \tag{1.3.21}$$

Let ζ be a cutoff function in B_1 that

$$\zeta = 1 \text{ in } B_{1-\lambda},\ \zeta \in [0,1],\ |\nabla \zeta| \le \frac{2}{\lambda}$$

where $\lambda \in (0,1)$ is to be determined.

Consider $\phi = (\psi^2)'(v)\zeta^2(x)$. Let τ be from Lemma 1.3.7 and set $Q^\tau := B_1 \times [\tau, t]$. By calculating $\iint_{Q^\tau} m v^\alpha v_t (\psi^2)'\zeta^2 dxdt$ and using equation (1.3.5), we find

$$w^\alpha \iint_{Q^\tau} \partial_t \phi dxdt = -\iint_{Q^\tau} m\left(\nabla v + r v^{\frac{1}{m}} \tilde{V}\right)\left(\nabla (v^\alpha (\psi^2)'\zeta^2)\right) dxdt.$$

Notice that $(\psi^2(t))'' = 2(1+\psi(t))(\psi'(t))^2$. So

$$\begin{aligned} & w^\alpha \int_{B_1 \times \{t\}} \psi^2(v)\zeta^2 dx - w^\alpha \int_{B_1 \times \{\tau\}} \psi^2(v)\zeta^2 dx \\ &+ \underbrace{\iint_{Q^\tau} v^{-\frac{1}{m}} (\psi^2)'|\nabla v|^2 \zeta^2 dxdt + \iint_{Q^\tau} v^\alpha (1+\psi)|\psi'|^2 |\nabla v|^2 \zeta}_{X_1 :=} \\ \lesssim_m & \underbrace{\iint_{Q^\tau} |\nabla v\ \psi' \psi^{\frac{1}{2}} v^{\frac{\alpha}{2}}| v^{\frac{\alpha}{2}} \psi^{\frac{1}{2}} \zeta \lambda^{-1}}_{X_2 :=} + \underbrace{r \iint_{Q^\tau} |\nabla (v^\alpha (\psi^2)'\zeta^2)| \cdot |\tilde{V} v^{\frac{1}{m}}| dxdt}_{X_3 :=}. \end{aligned}$$

Since $v \le M^+ \sim w$, $\psi \lesssim q$,

$$X_2 \le C w^\alpha q \lambda^{-2} + o(1) X_1.$$

From the Hölder inequality and the fact that $|\nabla \zeta| \lesssim \lambda^{-1}, \zeta \in [0,1]$,

$$\begin{aligned} X_3 &\le Cr \iint_{Q^\tau} \left(|\nabla v\ \psi^{\frac{1}{2}} (\psi')^{\frac{1}{2}} \zeta| \psi^{\frac{1}{2}} (\psi')^{\frac{1}{2}} |\tilde{V}| \zeta + v(1+\psi)|\psi'|^2 |\nabla v||\tilde{V}|\zeta^2 + \lambda^{-1} v \psi \psi' |\tilde{V}| \zeta \right) dxdt \\ &\lesssim o(1) X_1 + r^2 \iint_{Q^\tau} \left(v^{\frac{1}{m}} \psi \psi' + v^{2-\alpha}(1+\psi)|\psi'|^2 \right) |\tilde{V}|^2 dxdt + r \iint_{Q^\tau} \lambda^{-1} v \psi \psi' |\tilde{V}| dxdt. \end{aligned}$$

Recall (1.3.21) and that $v \lesssim w \lesssim 1$. Hence we obtain

$$X_3 \lesssim o(1) X_1 + (4^q q r^2 / c_0^2 w^\alpha) \iint_{Q^\tau} |\tilde{V}|^2 dxdt + (2^q q r / c_0 \lambda) \iint_{Q^\tau} |\tilde{V}| dxdt$$

Now by (1.3.9)

$$X_3 \lesssim o(1)X_1 + (4^q q r^{\delta_1}/c_0^2 w^\alpha) + (2^q q r^{\delta_2}/c_0\lambda).$$

Let $A_{k,R}(t)$ be as given in (1.3.20). Computations in the proof of Lemma 6.2 [37] yield

$$\int_{B_1\times\{t\}} \psi^2(v)\zeta^2 dx \geq ((q-1)\log 2)^2 |A_{k_2,1-\lambda}(t)|,$$
$$\int_{B_1\times\{\tau\}} \psi^2(v) dx \leq (q\log 2)^2 |A_{k_1}(\tau)|$$

where k_1 is as defined in Lemma 1.3.7. From the above,

$$|A_{k_2,1-\lambda}(t)| \leq \left(\frac{q}{q-1}\right)^2 |A_{k_1}(\tau)| + \frac{Cq}{\lambda^2(q-1)^2} + \frac{C4^q r^{\delta_1} q}{c_0^2(q-1)^2 w^{2\alpha}} + \frac{C2^q r^{\delta_2} q}{c_0\lambda(q-1)^2 w^\alpha}.$$

And we have

$$|A_{k_2,1-\lambda}(t)| \geq |A_{k_2}(t)| - |B_1 \backslash B_{1-\lambda}| \geq |A_{k_2}(t)| - Cd\lambda |B_1|.$$

By Lemma 1.3.7 and $q \geq 3$, we obtain

$$|A_{k_2}(t)| \leq \left(\left(\frac{q}{q-1}\right)^2 \frac{1-c_0}{1-\frac{1}{2}c_0} + C_0 d\lambda\right)|B_1| + C_1\left(\frac{1}{\lambda^2 q} + \frac{4^q r^{\delta_1}}{c_0^2 q w^{2\alpha}} + \frac{2^q r^{\delta_2}}{c_0 q\lambda w^\alpha}\right), \quad (1.3.22)$$

where C_0 and C_1 are universal constants.

Let us now choose λ and q such that

$$\lambda := \frac{1}{4C_0 d}c_0^2, \quad \left(\frac{q}{q-1}\right)^2 \leq \left(1 - \frac{c_0}{2}\right)(1+c_0), \; \frac{C_1}{\lambda q} \leq \frac{1}{4}c_0^2|B_1|.$$

It is possible to choose such q since for c_0 small, $(1-\frac{c_0}{2})(1+c_0) > 1$. Due to the drift term we require

$$(4^q r^{\delta_1}/c_0^2 w^{2\alpha} + 2^q r^{\delta_2}/c_0\lambda w^\alpha) \lesssim \frac{1}{4}qc_0^2|B_1|.$$

Since c_0 is fixed and $\lambda(c_0), q(c_0)$ are fixed, this condition is equivalent to $r \leq c_1 w^{c_2}$ for some fixed $c_1(c_0), c_2(c_0) > 0$.

Finally we can conclude with the right hand side of (1.3.22) $\leq (1 - (\frac{c_0}{2})^2)|B_1|$.

□

Lemma 1.3.9. *Let q be as given in Lemma 1.3.8. Then for any $\gamma \in (0,1)$ there exist $c(\gamma, c_0, q) > 0$ and $p_0(\gamma, c_0, q) > q$ such that the following holds: if r satisfies the assumption given in Lemma 1.3.8 and further satisfies $r \leq c$, then*

$$\left|\left\{(x,t) \in Q\left(1, \frac{c_0}{2}\right), v > M^+ - \frac{c_0}{2^{p_0}} w\right\}\right| \leq \gamma \left|Q(1, \frac{c_0}{2})\right|.$$

Proof. The lemma is a variant of Remark 6.1, Lemma 6.3, 6.4 [37].

Write $\{u > k\} := \{x \in B_1,\ u > k\}$ for any $u \in W^{1,2}(B_1)$. Lemma 6.3 [37] says that for any $l > k$,

$$(l-k)|\{u>k\}|^{1-\frac{1}{d}} \leq \frac{C}{|B_1| - |\{u>k\}|} \int_{\{u>k\}\backslash\{u>l\}} |\nabla u| dx. \tag{1.3.23}$$

We will chosider $l = k_{s+1}, k = k_s$ in above inequaltiy, wher $k_s := M^+ - \frac{c_0}{2^s} w$, where s is a sufficiently large integer to be determined below.

Let us choose $\lambda = \lambda(\gamma, c_0)$ such that

$$\left|Q\left(1, \frac{c_0}{2}\right) \backslash Q\left(\lambda, \frac{c_0}{2}\right)\right| \leq \frac{\gamma}{2} \left|Q\left(1, \frac{c_0}{2}\right)\right|. \tag{1.3.24}$$

With above choice of λ, let $0 \leq \zeta(x,t) \leq 1$ be a cut-off function compactly supported in $Q\left(1, \frac{c_0}{2}\right)$ which equals 1 in $Q(\lambda, \frac{c_0}{2})$.

Write

$$A_k^\zeta(t) := \{x \in B_1, v\zeta > k\},\ A_{k,c_0}^\zeta := \left\{(x,t) \in Q(1, \frac{c_0}{2}), v\zeta > k\right\},$$
$$A_{k,\lambda,c_0} := \left\{(x,t) \in Q\left(\lambda, \frac{c_0}{2}\right), v > k\right\}.$$

Then

$$\frac{wc_0}{2^{s+1}} |A_{k_{s+1}}^\zeta(t)|^{1-\frac{1}{d}} \leq \frac{C}{meas\left\{B_1 \backslash A_{k_s}^\zeta(t)\right\}} \int_{A_{k_s}^\zeta(t) \backslash A_{k_{s+1}}^\zeta(t)} |\nabla(v\zeta)| dx. \tag{1.3.25}$$

Recall that from (1.3.20) $A_{k,R}(t) = \{x \in B_R, v > k\}$ and $A_k(t) = A_{k,1}(t)$, so by definitions of the sets

$$A_{k,\lambda}(t) \subseteq A_k^\zeta(t) \subseteq A_k(t).$$

By Lemma 1.3.8, for any $t \in [-\frac{c_0}{2}, 0]$,

$$meas\left\{B_1 \backslash A_{k_s}^\zeta(t)\right\} \geq meas\{B_1 \backslash A_{k_s}(t)\} \geq \left(\frac{c_0}{2}\right)^2 |B_1|.$$

Since $|A^{\zeta}_{k_s}(t)|$ is bounded by $|B_1|$,

$$C|A^{\zeta}_{k_{s+1}}(t)|^{1-\frac{1}{d}} \geq |A^{\zeta}_{k_{s+1}}(t)| \geq |A_{k_{s+1},\lambda}(t)|.$$

Then

$$|A_{k_{s+1},\lambda,c_0}| \leq C\int_{-\frac{c_0\lambda}{2}}^{0} |A^{\zeta}_{k_{s+1}}(t)|^{1-\frac{1}{d}}dt.$$

After integrating (1.3.25), Hölder inequality yields that

$$\frac{wc_0}{2^{s+1}}|A_{k_{s+1},\lambda,c_0}| \leq \frac{C}{c_0^2}\int_{-\frac{c_0}{2}}^{0}\int_{A^{\zeta}_{k_s}(t)\backslash A^{\zeta}_{k_{s+1}}(t)} |\nabla(v\zeta)|dxdt$$

$$\leq \frac{C}{c_0^2}\left(\iint_{A^{\zeta}_{k_s,c_0}\backslash A^{\zeta}_{k_{s+1},c_0}} |\nabla(v\zeta)|^2dxdt\right)^{\frac{1}{2}}\left|A^{\zeta}_{k_s,c_0}\backslash A^{\zeta}_{k_{s+1},c_0}\right|^{\frac{1}{2}}.$$

Next according to (1.3.10)

$$\iint_{\zeta v>k_s,(x,t)\in Q(1,\frac{c_0}{2})} |\nabla(v\zeta)|^2dxdt \lesssim_{\lambda} \frac{c_0^2w^2}{2^{2s}}+r^{\epsilon}w^2.$$

Then

$$\frac{wc_0}{2^{s+1}}|A_{k_{s+1},\lambda}| \leq \frac{C}{c_0^2}\left(\frac{c_0^2w^2}{2^{2s}}+r^{\epsilon}w^2\right)^{\frac{1}{2}}\left|A^{\zeta}_{k_s}\backslash A^{\zeta}_{k_{s+1}}\right|^{\frac{1}{2}}.$$

Now we let r be small enough that $r^{\epsilon}4^{p_0}c_0^{-2} \leq 1$. Then for all $q \leq s \leq p_0-1$

$$|A_{k_{s+1},\lambda,c_0}|^2 \leq \frac{C_0}{c_0^4}|A^{\zeta}_{k_s,c_0}\backslash A^{\zeta}_{k_{s+1},c_0}|.$$

As in [37], since the sum of $|A^{\zeta}_{k_s,c_0}\backslash A^{\zeta}_{k_{s+1},c_0}|$ is uniformly bounded by $|B_1|$. If p_0 is large enough, there is $s_0 \in [q, p_0-1]$ that

$$C_0\left|A^{\zeta}_{k_{s_0},c_0}\backslash A^{\zeta}_{k_{s_0+1},c_0}\right| \leq \frac{Cc_0^4}{p_0-q-1} \leq c_0^4\left(\frac{c'\gamma}{2}\right)^{\frac{1}{2}}$$

with $c' = |Q\left(1,\frac{c_0}{2}\right)|$. Let us choose $s = s_0$. Then

$$|A_{k_{s_0+1},\lambda,c_0}| \leq \frac{c'\gamma}{2} = \frac{\gamma}{2}\left|Q(1,\frac{c_0}{2})\right|.$$

Consequently

$$\left|\left\{v > M^+ - \frac{c_0}{2^{p_0}}w \text{ in } Q\left(\lambda,\frac{c_0}{2}\right)\right\}\right| = \left|A_{k_{p_0},\lambda,c_0}\right| \leq |A_{k_{s_0+1},\lambda,c_0}| \leq \frac{\gamma}{2}\left|Q\left(1,\frac{c_0}{2}\right)\right|.$$

Note that p_0 can be determined by c_0, q, γ and so we only need $r \leq c(c_0, q, \gamma)$.

Finally from (1.3.24)

$$\left|\left\{v > M^+ - \frac{c_0}{2^{p_0}} w \text{ in } Q\left(1, \frac{c_0}{2}\right)\right\}\right| \leq \left|\left\{v > M^+ - \frac{c_0}{2^{p_0}} w \text{ in } Q\left(\lambda, \frac{c_0}{2}\right)\right\}\right|$$
$$+ \left|Q\left(1, \frac{c_0}{2}\right) \backslash Q(\lambda, \frac{c_0}{2})\right| \leq \gamma \left|Q\left(1, \frac{c_0}{2}\right)\right|.$$

□

The following lemma helps finding the value of $\gamma(c_0, p_0)$. The proof is parallel to Proposition 1.3.4.

Lemma 1.3.10. *Let p_0 be as given in Lemma 1.3.9. Suppose $p > d$. There exists $\gamma \in (0,1)$ independent of w, r, p_0 such that if $r < c_1 w_2^c$ for some c_1, c_2 depending on c_0, p_0 and*

$$\left|\left\{(x,t) \in Q(1, \frac{c_0}{2}), v > \left(M^+ - \frac{c_0}{2^{p_0}} w\right)\right\}\right| \leq \gamma \left|Q(1, \frac{c_0}{2})\right|,$$

then

$$\left|\left\{(x,t) \in Q\left(\frac{1}{2}, \frac{c_0}{2}\right), v > \left(M^+ - \frac{c_0}{2^{p_0+1}} w\right)\right\}\right| = 0.$$

Proof of Proposition 1.3.6

Without loss of generality, we may assume $c_0 < \frac{1}{8}$. If $M^+ - M^- \leq \frac{w}{2} + c_0 w$, then v is bounded by $M^+ \leq (\frac{3}{4} + c_0) w$. In this case, taking $\eta \leq (\frac{3}{4} + c_0)$ finishes the proof of the Proposition with, for instance, $c_1 = c_2 = 1$.

Otherwise condition (1.3.19) is satisfied. In this case we fix c_1, c_2 as given in Lemma 1.3.7, p_0 as in Lemma 1.3.9 and γ as in Lemma 1.3.10. By Lemma 1.3.7 and Lemma 1.3.8, we know that the conclusion of Lemma 1.3.9 is valid for the range of r satisfying $r < c_1 w^{c_2}$. By Lemma 1.3.9, we know that the condition in Lemma 1.3.10 is satisfied for the specific choice of p_0. By Lemma 1.3.10 we proved that if (1.3.12) is not satisfied, the solution goes down from above (from M^+ to $M^+ - c_0 2^{-p_0-1} w$) if restricted to the smaller box $Q(1/2, c_0/2)$. This yields the conclusion with $\eta = 1 - c_0 2^{-p_0-1}$.

Proof of Theorem 1.3.1 The proof follows an iteration process which was described in the proof of Theorem 7.17 [70], based on Propositions 1.3.4 and 1.3.6.

Recall that $M := \sup_{Q_1} u$ and $\alpha = \frac{m-1}{m}$. Fix $(x_0, t_0) \in Q_{\frac{1}{2}}$, without loss of generality we can assume it is $(0, 0)$, and let $\nu := u^m$.

The goal for the argument below is to obtain

$$\eta^k w \geq osc_{Q(a^k r, b^{2k})} \nu \text{ for all integers } k, \tag{1.3.26}$$

where $a, b, \eta \in (0, 1)$ only depends on M, m, p, $\|V\|_{L^p(Q_1)}$ and the dimension d.

We start with some $Q(r, w^{-\alpha})$ for some $w > 0, 0 < r \leq \frac{1}{2}$ such that

$$Q(r, w^{-\alpha}) \subset Q_{\frac{1}{2}},\ w \geq osc_{Q(r,w^{-\alpha})} \nu. \tag{1.3.27}$$

For example we can take $w = M$.

Let us start with a given pair of (r_0, w_0) that satisfies (1.3.27). Below we will generate a sequence of pairs (r_n, w_n) that satisfies (1.3.27). For each n and the given pair (r_n, w_n) let us denote

$$M_n^- := \inf_{Q(r_n, w_n^{-\alpha})} \nu, \quad M_n^+ := \sup_{Q(r_n, w_n^{-\alpha})} \nu.$$

Let c_1 and c_2 be as given in Proposition 1.3.6. For each given pair (r_n, w_n) the next pair (r_{n+1}, w_{n+1}) is generated depending on the following cases.

Case 1: if $r_n > c_1 w_n^{c_2}$, the situation is in some sense better since the oscillation is under control. In order to apply the preceding scheme, let $w_{n+1} = w_n, r_{n+1} = \frac{1}{2} r_n$, and we repeat until it falls into Case 2 or 3.

Case 2: if $r_n \leq c_1 w_n^{c_2}$ and either $M_n^- \geq \frac{w_n}{4}$ or (1.3.12) holds, we claim $\nu \in [w_n/4, M_n^+]$ in $Q(\frac{3r_n}{4}, w_n^{-\alpha})$. This is trivial if $M_n^- \geq \frac{w_n}{4}$, otherwise with the help of (1.3.12) we can apply Proposition 1.3.4. Then from classical regularity theory for parabolic equations, it follows that (1.3.26) holds for $k \geq n$.

Case 3: We are left with the case $r_n \leq c_1 w_n^{c_2}$, $M_n^- < \frac{w_n}{4}$ and (1.3.12) fails. In this case Proposition 1.3.6 yields constants $0 < c_0, \eta < 1$ which are independent of w such that

$$osc_{Q(\frac{r_n}{2}, \frac{c_0}{2} w_n^{-\alpha})} \nu \leq \eta w_n. \tag{1.3.28}$$

We choose

$$w_{n+1} := \eta w_n, \quad r_{n+1} := c_3 r_n.$$

Here $c_3^2 := \frac{1}{8}\eta^\alpha c_0$ is chosen such that $Q(r_{n+1}, w_{n+1}^{-\alpha}) \subset Q(\frac{r_n}{2}, \frac{c_0}{2} w_n^{-\alpha})$. From this choice of c_3 and (1.3.28) it follows that (1.3.27) holds for (r_{n+1}, w_{n+1}).

Suppose Case 3 is iterated for n times. Then inside $\{|x| < c_3^n r, t \in (-c_5 c_4^{2n} r^2, 0)\}$, the oscillation of ν is bounded by $\eta^n w$ and here $c_4 = \frac{1}{4}c_0, c_5 = c_0 w^{-\alpha}$. This yields (1.3.26) for $k = n$.

1.4 Loss of Regularity: Examples

In this section we show by examples that the regularity results obtained in section 3 and 4 are false for drifts in $L^d(\mathbb{R}^d)$. We will discuss examples with both potential vector fields and divergence-free vector fields.

1.4.1 Loss of Continuity for Potential Drifts

First let us recall the description of stationary solutions for (1.1.1) with potential vector fields.

Theorem 1.4.1. *[[8], [51]] For a radially symmetric, increasing potential $\Phi \in C^\infty(\mathbb{R}^d)$, the following is true:*

1. *The unique stationary solution of* (1.1.1)*, with a prescribed mass M, is of the form*

$$\rho_M = \left(C(M) - \frac{m-1}{m}\Phi \right)_+^{\frac{1}{m-1}}.$$

2. *Let ρ solve* (1.1.1) *with $V = \nabla\Phi$ and with smooth compactly supported initial data ρ_0 with $\int \rho_0 = M$. Then the support of ρ stays bounded for all times, and $\|\rho(\cdot,t) - \rho_M(\cdot)\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$.*

Based on above theorem, we can show that $L^d(\mathbb{R}^d)$ bound on drifts does not guarantee any modulus of continuity for solutions of (1.1.1) even when the solutions are uniformly bounded.

Theorem 1.4.2. *There exist a family of potentials Φ_A such that $\nabla\Phi_A \in L^d(\mathbb{R}^d)$ and a family of initial data u_0^A which are uniformly bounded in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that the following holds: The solutions u^A of* (1.1.1) *with $V = \nabla\Phi_A$ with initial datas u_0^A stays uniformly bounded but lacks any uniform modulus of continuity as $A \to \infty$.*

Proof. Let $\phi(x) = |x|^2$, and let ρ be a stationary solution of (1.1.1) ρ given in Theorem 1.4.1, with a sufficiently small mass such that ρ is supported inside of the unit ball. Let $\phi_A(x) := \phi(Ax)$, and $\rho^A(x) := \rho(Ax)$, which is a stationary solution for ϕ_A. Let us next modify ϕ_A so that $\nabla\phi_A$ is uniformly bounded in $L^d(\mathbb{R}^d)$, let Φ_A satisfy

1. $\Phi_A = \phi_A$ if $|x| \leq 1/A$.

2. $|\nabla\Phi_A| \leq |\nabla\phi_A|$ if $|x| \leq 2/A$.

3. $|\nabla\Phi_A| \leq \min\{1, |x|^{-1}\}$ if $|x| \geq 2/A$.

4. Φ_A is smooth, radially symmetric and increasing.

Then $\nabla\Phi_A$ is uniformly bounded in $L^d(\mathbb{R}^d)$ and ρ_A is still a stationary solution for the modified potential Φ_A.

For $A > 1$, consider a sequence of functions $u_0^A \geq 0$ such that they are uniformly bounded in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\int u_0^A dx = \int \rho^A dx = CA^{-d}$. By Theorem 1.4.1, the solution u^A of (1.1.1) with initial data u_0^A and with $V = \nabla\Phi_A$ converges uniformly to $\rho^A = \rho(Ax)$ and ρ^A converges pointwise to a discontinuous function ρ^∞ which is 1 at $x = 0$ and zero for sufficiently small $|x|$. It follows that u^A cannot share any uniform modulus of continuity.

We are left to show that u^A is bounded. To see this let $v^A(x,t) := u^A\left(A^{-1}x, A^{-2}t\right)$. Then

$$v_t^A = \Delta\left(v^A\right)^m + \nabla\cdot\left(v^A A^{-1}\left(\nabla\Phi_A\right)\left(A^{-1}x\right)\right),$$

$$\text{and } \left\|A^{-1}\left(\nabla\Phi_A\right)\left(A^{-1}x\right)\right\|_{d+1}^{d+1} = A^{-1}\left\|\nabla\Phi_A\right\|_{d+1}^{d+1}$$

$$\lesssim A^{-1}\left(\int_0^{\frac{2}{A}}|A^2x|^{d+1}dx + \int_1^\infty |x|^{-d-1}dx + 1\right) < \infty.$$

The vector field $A^{-1}\left(\nabla\Phi_A\right)\left(A^{-1}x\right)$ are uniformly bounded in L^{d+1} and $v^A(0)$ are uniformly bounded in $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. By previous Theorem 1.2.1, v^A are uniformly bounded and so are u^A. □

1.4.2 Loss of Hölder Regularity for Divergence-Free Drifts

In previous subsection we have seen that drifts bounded in $L^d(\mathbb{R}^d)$ and initial data that are bounded in $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ are insufficient to yield uniform mode of continuity for solutions of (1.1.1). In this section we will show that the loss of regularity continues to be true for divergence-free vector fields, though here we are only able to present loss of Hölder estimates. Our example leaves open the possibility of weaker modulus of continuity.

Let us recall that, for u solving (1.1.1) with divergence-free V, the pressure variable $v := \frac{m}{m-1}u^{m-1}$ solves

$$v_t - (m-1)v\Delta v - |\nabla v|^2 + V\cdot\nabla v = 0. \tag{1.4.1}$$

We will prove the following theorem by constructing barriers for the pressure equation above.

Theorem 1.4.3. *There exist a sequence of bounded vector fields $\{V_n\}$ which are uniformly bounded in $L^3(\mathbb{R}^3)$ and a sequence $\{u_n\}_n$ of solutions for (1.4.1) with V_n that satisfies the following:*

1. *$\{u_n(x,0)\}$ are uniformly bounded in $C^k(\mathbb{R}^3)$ for any $k>0$;*
2. *$\{u_n\}$ are uniformly bounded in $\mathbb{R}^3\times[0,1]$;*
3. *For any $\delta>0$ we have $\sup_n[u_n]_\delta = \infty$, where $[f]_\delta$ denotes the C^δ semi-norm of f in $\mathbb{R}^3\times[0,1]$.*

∘ *Construction of vector fields*

Let us denote $x = (x_1, x_2, y) \in \mathbb{R}^3$. For $s \in (0,1)$, define

$$\psi(x_1, x_2, y) := (-(y + x_1 - x_2)^s + (y + x_1 + x_2)^s, (y - x_1 + x_2)^s - (y + x_1 + x_2)^s,\ 0\ ).$$

For $\epsilon > 0$, let κ and μ_ϵ be two smooth cut-off functions satisfying

$$\chi_{[-\frac{1}{3},\frac{1}{3}]} \le \kappa \le \chi_{[-\frac{1}{2},\frac{1}{2}]} \tag{1.4.2}$$

and

$$\chi_{[2\epsilon,10]} \le \mu_\epsilon \le \chi_{[\epsilon,20]}, \quad |\mu_\epsilon'|(x) \le \frac{2}{|x|}. \tag{1.4.3}$$

Now we define $V := \nabla \times F$ with

$$F(x) := \frac{1}{4}(s)^{\frac{1}{3}} \psi(x) \kappa\left(\frac{x_1 - x_2}{y}\right) \kappa\left(\frac{x_1 + x_2}{y}\right) \mu_\epsilon(|x|). \tag{1.4.4}$$

We claim that for all $s, \epsilon \in (0,1)$ any small, V is bounded uniformly in $L^3(\mathbb{R}^3)$. To show this, by symmetry it is enough to consider the following regions:

$$S_1 := \{(x_1, x_2, y) \in B_1,\ y \ge 3\max\{|x_1 + x_2|, |x_1 - x_2|\}, |(x_1, x_2, y)| \ge 2\epsilon\},$$

$$S_2 := \left\{(x_1, x_2, y) \in B_1,\ \frac{1}{2}y \ge x_1 + x_2 \ge \frac{1}{3}y > 0, |(x_1, x_2, y)| \ge 2\epsilon\}\right\},$$

$$S_3 := \{(x_1, x_2, y) \in B_1,\ |(x_1, x_2, y)| \le 2\epsilon\}.$$

In S_1, $\kappa = \mu_\epsilon = 1$ and $\|\nabla \times \psi\| \le Cs|y|^{s-1}$. Therefore

$$\|V(x)\|^3_{L^3(S_1)} \le C \int_0^1 \int_0^{\frac{1}{3}y} \int_0^{\frac{1}{3}y} s^4 |y|^{3(s-1)} dx_1 dx_2 dy \le s^3 C.$$

In S_2, since $|\kappa'|, |\kappa|$ are bounded and $\mu_\epsilon = 1$, each component in $\nabla \times F$ is bounded by $Cs^{\frac{1}{3}}|y|^{s-1}$. Since $\mu_\epsilon' \le \frac{2}{|x|}$ similar bound holds in S_3, and we have

$$\|V(x)\|^3_{L^3(S_2 \cup S_3)} \le C \int_0^1 \int_0^{\frac{1}{2}y} \int_0^{\frac{1}{2}y} s|y|^{3(s-1)} dx_1 dx_2 dy \le C.$$

We will prove Theorem 1.4.3 by comparison principle.

For $\epsilon > 0$, let us define the parameters

$$M := s^{-\frac{4}{3}} \quad \text{and} \quad T := M(1-(4\epsilon)^{2-s})/(2-s). \tag{1.4.5}$$

Define

$$z(t) := (1-(2-s)M^{-1}t)^{\frac{1}{2-s}}, \quad t \in [0,T] \tag{1.4.6}$$

so that z satisfies

$$z' = -M^{-1}z^{s-1}, \quad z(0) = 1, \quad z(T) = 4\epsilon.$$

We can write $V = s^{\frac{4}{3}}(V_1, V_2, V_3)$, inside S_1

$$\begin{aligned} V_1 &= -\frac{1}{4}(y - x_1 + x_2)^{s-1} + \frac{1}{4}(y + x_1 + x_2)^{s-1}, \\ V_2 &= -\frac{1}{4}(y + x_1 - x_2)^{s-1} + \frac{1}{4}(y + x_1 + x_2)^{s-1}, \\ V_3 &= -\frac{1}{4}(y - x_1 + x_2)^{s-1} - \frac{1}{4}(y + x_1 - x_2)^{s-1} - \frac{1}{2}(y + x_1 + x_2)^{s-1}. \end{aligned}$$

∘ *Construction of subsolution*

Let $\varphi \in C_0^\infty(B_1(0))$ be a smooth, non-negative, radially symmetric and decreasing function with $|\Delta\varphi| \leq C$ for some $C > 0$. For $r \in (0, \frac{1}{9})$ and a constant c_s, define

$$\bar{u}(x,t) := c_s z^s(t)\Phi(x,t) := c_s z^s(t)\varphi\left(\frac{x_1, x_2, y - z(t)}{rz(t)}\right).$$

Then the support of $\bar{u}$ lies inside the upper cone S_1.

Lemma 1.4.4. *Let $\bar{u}$ be defined as above. Then there exist $r_s > 0$ independent of ϵ and a universal constant $C > 0$ such that for $r \leq r_s$ and $c_s = Cs^{\frac{7}{3}}r^2$, $\bar{u}$ is a subsolution to* (1.4.1)*. Furthermore $\bar{u}(0,0,4\epsilon,T) \geq c_s(4\epsilon)^s$.*

Proof. We need to check that

$$\bar{u}_t - (m-1)\bar{u}\Delta\bar{u} + V \cdot \nabla\bar{u} \leq 0$$

inside the support of $\bar{u}$, which lies in B_{rz}. Since $|\Delta\varphi| \leq C$, it suffices to show that

$$(z^s)' \leq -\frac{C(m-1)}{r^2z^2}c_s z^{2s} \tag{1.4.7}$$

and

$$\partial_t \Phi + V \nabla \Phi \leq 0 \tag{1.4.8}$$

in B_{rz}.

Since (1.4.7) is equivalent to $C(m-1)c_s \leq sr^2 M^{-1}$, it holds when

$$c_s := \frac{sr^2}{C(m-1)M} \lesssim s^{\frac{7}{3}} r^2.$$

Next notice

$$\partial_t \Phi + \tilde{V} \cdot \nabla \Phi = 0, \text{ with } \tilde{V} = -M^{-1} z^{s-2}(x_1, x_2, y).$$

Hence to show (1.4.8), it suffices to show $(\tilde{V}-V)\cdot\nabla\Phi \geq 0$ for $t \in [0,T]$ and for $(x_1, x_2, y-z) \in B_{rz}$.

Recall that $V = s^{\frac{4}{3}}(V_1, V_2, V_3)$, $M = s^{-\frac{4}{3}}$. Since $\nabla\Phi$ is parallel to (x_1, x_2, y), it suffices to show that

$$((V_1, V_2, V_3) + z^{s-2}(x_1, x_2, y)) \cdot (x_1, x_2, y-z) \geq 0 \text{ for } \{x : x_1^2 + x_2^2 + (y-z)^2 \leq z^2 r^2\}.$$

By $(s-1)$-homogeneity of V, this is equivalent to

$$((V_1, V_2, V_3) + (x_1, x_2, y)) \cdot (x_1, x_2, y-1) \geq 0 \text{ for } \{x : x_1^2 + x_2^2 + (y-1)^2 \leq r^2\}, \tag{1.4.9}$$

The left handside of (1.4.9) can be written as

$$\begin{aligned} f(x_1, x_2, y) = &-\frac{1}{4}|y - x_1 + x_2|^{s-1}(x_1 + y - 1) - \frac{1}{4}|y + x_1 - x_2|^{s-1}(x_2 + y - 1) \\ &- \frac{1}{4}|y + x_1 + x_2|^{s-1}(-x_1 - x_2 + 2(y-1)) + x_1^2 + x_2^2 + y(y-1). \end{aligned} \tag{1.4.10}$$

Straightforward computation yields

$$f(0,0,1) = f_{x_1}(0,0,1) = f_{x_2}(0,0,1) = f_y(0,0,1),$$

$$f_{x_i x_i} = 1 + s, f_{yy} = 4 - 2s, f_{x_1 x_2} = 0, f_{x_i y} = -\frac{1}{2}(s-1).$$

So $(0,0,1)$ is a local minimum of f. Hence there exists $r_s > 0$ which only depends on s such that (1.4.9) holds for $(x_1, x_2, y-z) \in B_{r_s z}$, thus we conclude that v is a subsolution of (1.4.1) when c_s and r_s are sufficiently small. In particular observe that

$$\bar{u}(0,0,4\epsilon,T) \sim c_s z^s(T) = c_s(4\epsilon)^s \text{ with } c_s \leq C s^{\frac{7}{3}} r^2,$$

where C is a universal constant which is independent of s, ϵ.

$\square$

∘ *Construction of supersolution*

Let us consider a smooth function $\varphi(R) : [0,\infty) \to [0,1]$ with the following properties:

1. φ is increasing with $\varphi(0) = \varphi'(0) = 0$ and $\varphi \equiv 1$ for $R \geq 1$.
2. There exists a constant $C^* > 0$ that

$$(m-1)\varphi(\varphi'' + \frac{1}{R}\varphi') + |\varphi'|^2 \leq C^*\varphi \tag{1.4.11}$$

To construct such φ, for instance we can choose $\varphi = R^2$ for $|R| \leq 1/2$ and extend it to a smooth function satisfying 1,2. With the above φ, define

$$\Phi((x,y),t) := \varphi\left(\left|\frac{(x, y + z(t))}{rz(t)}\right|\right)$$

and

$$k(t) := \left(C_0 - C^* \frac{Mr^{-2}}{s} z(t)^s\right)^{-1}, \tag{1.4.12}$$

where

$$C_0 := 2C^* M r^{-2} s^{-1} (4\epsilon)^{-s} \sim r^{-2} s^{-\frac{7}{3}} \epsilon^{-s}. \tag{1.4.13}$$

The choice of C_0 is to ensure that $k(t)$ stays nonnegative for $0 \leq t \leq T$ and $k(0) = \frac{2}{C_0} = \frac{4^s}{C^*} r^2 s^{\frac{7}{3}} \epsilon^s$. Also $k' \geq C z^{-2} r^{-2} k^2$ for $t \in [0,T]$.

For $r < \frac{1}{9}$, we define

$$\underline{u}(x,t) = k(t)\Phi(x,t) := k(t)\varphi\left(\frac{(x_1, x_2, y + z(t))}{rz(t)}\right).$$

Lemma 1.4.5. *Let $\underline{u}$ be defined as above. There exist $r_s > 0$ independent of ϵ and a universal constant $C_1 > 0$. If $r \leq r_s$ and $k(0) \leq C_1 r^2 s^{\frac{7}{3}} \epsilon^s$, then $\underline{u}$ is a supersolution to* (1.4.1) *in the time interval* $[0,T]$*. Furthermore $\underline{u}(0,0,-4\epsilon,T) = 0$ and $\underline{u}(x,0) \geq C r^2 s^{\frac{7}{3}} \epsilon^s$ for some universal C.*

Proof. Consider the region S_1. Showing that $\underline{u}$ is a supersolution is equivalent to

$$(k'\Phi + k\partial_t\Phi) - (m-1)k^2\Phi\Delta\Phi - k^2|\nabla\Phi|^2 + kV\cdot\nabla\Phi \geq 0.$$

By (1.4.11),

$$Ck^2z^{-2}r^{-2}\Phi \geq (m-1)k^2\Phi\Delta\Phi + k^2|\nabla\Phi|^2.$$

Thus it suffices to show

$$k'\Phi \geq Cz^{-2}r^{-2}k^2\varphi, \quad \text{and} \quad \partial_t\Phi + V\cdot\nabla\Phi \geq 0 \tag{1.4.14}$$

in $S := \{(x_1, x_2, y),\ -y \geq \max\{|x_1+x_2|, |x_1-x_2|\},\ |x| \geq 2\epsilon\}$.

The first inequality in (1.4.14) holds, as before, due to (1.4.11) and the definition of $k(t)$. To show the second inequality, write $V = M^{-1}(V_1, V_2, V_3)$. In $y < 0$

$$M\left(\partial_t\Phi + V\cdot\nabla\Phi\right) = \nabla\varphi\cdot(x_1, x_2, y+z)z^{-3+s} + \nabla\varphi\cdot(V_1, V_2, V_3)z^{-1}.$$

By definition in the region S

$$\begin{aligned}
V_1 &= \frac{1}{4}(y - x_1 + x_2)^{s-1} - \frac{1}{4}(y + x_1 + x_2)^{s-1},\\
V_2 &= \frac{1}{4}(y + x_1 - x_2)^{s-1} - \frac{1}{4}(y + x_1 + x_2)^{s-1},\\
V_3 &= \frac{1}{4}(y - x_1 + x_2)^{s-1} + \frac{1}{4}(y + x_1 - x_2)^{s-1} + \frac{1}{2}(y + x_1 + x_2)^{s-1}.
\end{aligned}$$

As before we only need to verify that there exists $r = r_s$ such that inside $|x_1|^2 + |x_2|^2 + |y+1|^2 \leq r$

$$\begin{gathered}
|x_1|^2 + |x_2|^2 + (y+1)y + \frac{1}{4}|y - x_1 + x_2|^{s-1}(x_1 + y + 1) + \frac{1}{4}|y + x_1 - x_2|^{s-1}(x_2 + y + 1)\\
+\frac{1}{4}|y + x_1 + x_2|^{s-1}(-x_2 - x_1 + 2y + 2) \geq 0.
\end{gathered}$$

Recall f defined in (1.4.10), then the above is equivalent to

$$f(-x_1, -x_2, -y) \geq 0$$

near $(0, 0, -1)$ which has already been verified when r is small enough (depending only on s). Hence $\underline{u}$ is a supersolution. Note that $k(0) \geq Cr^2 s^{\frac{7}{3}}\epsilon^s$, and thus we conclude.

□

Proof. (of Theorem 1.4.3). Let $\delta > 0$ be any small and fix one module of holder continuity $w(\tau) = C\tau^\delta$. Let us select $s < \frac{1}{2}\delta$ and ϵ arbitrarily small. Let r be sufficiently small so that Lemma 1.4.4 and Lemma 1.4.5 applies. Let C_1 be the constant in Lemma 1.4.5.

Consider a smooth function $v_0 : \mathbb{R}^d \to \mathbb{R}$ be supported in the upper half plane and

1. $v_0 \geq \frac{C_1}{2} r_s^2 s^{\frac{7}{3}} \epsilon^s$ in $B_{r_s}(0,1)$;
2. $v_0 \leq C_1 r_s^2 s^{\frac{7}{3}} \epsilon^s$.

Let $v_s = v_{s,\epsilon}$ solve (1.4.1) with initial data v_0. Let us choose ϵ small enough so that $\bar{u}$ given in Lemma 1.4.4 with $c_s := \frac{C_1}{2} r_s^2 s^{\frac{7}{3}} \epsilon^s$ is a subsolution of (1.4.1). From comparison principle, the solution to (1.4.1) with initial data v_0 satisfies

$$v(0, 4\epsilon, T) \geq C r_s^2 s^{\frac{7}{3}} \epsilon^{2s}. \tag{1.4.15}$$

Next let $\underline{u}$ be the supersolution as given in Lemma 1.4.5. Then we have $\underline{u}(\cdot, 0) \geq v_0$, thus by comparison principle it follows that $v_2 \geq v$. Then at time T,

$$v_s(0, -4\epsilon, T) \leq \underline{u}(0, -4\epsilon, T) = 0. \tag{1.4.16}$$

Putting (1.4.15) and (1.4.16) together, it follows that

$$|v_s(0, 4\epsilon, T) - v_s(0, -4\epsilon, T)| / |8\epsilon|^\delta \geq C r_s^2 s^{\frac{7}{3}} \epsilon^{2s-\delta} = C(s)\epsilon^{2s-\delta}. \tag{1.4.17}$$

Finally, let us normalize parameters so that the singular time T is comparable to 1.

$$u_{s,\epsilon}(x,t) = v_{s,\epsilon}(M^{\frac{1}{2}} x, Mt).$$

Let us normalize T by

$$\tilde{T} = T/M = \frac{1 - (4\epsilon)^{2-s}}{2-s},$$

which is close to 1/2 for all s, ϵ close to 0. Notice $u_{s,\epsilon}$ solves equation (1.4.1) with V replaced by

$$\tilde{V}(x) = \tilde{V}_s := M^{\frac{1}{2}} V(M^{\frac{1}{2}} x),$$

where V is defined in (1.4.4). Then $\{\tilde{V}_s\}$ are uniformly bounded in $L^3(\mathbb{R})$ for all s.

From (1.4.17), it follows that

$$|u_s(0, 4\epsilon/M^{\frac{1}{2}}, \tilde{T}) - u_s(0, -4\epsilon/M^{\frac{1}{2}}, \tilde{T})|/|8\epsilon|^{\delta} \geq C_s \epsilon^{2s-\delta}.$$

Then as $\epsilon \to 0$, any C^{δ}- norm with $\delta > 2s$ again grows to infinity at time $\tilde{T}$ which is uniformly bounded this time. Thus we can conclude our theorem if we choose

$$u_n := u_{1/n, \epsilon_n}.$$

where ϵ_n is chosen sufficiently small such that the C^n norm of $u_{1/n,\epsilon_n}(x, 0)$ is bounded.

$\square$

CHAPTER 2

Free Boundary Regularity

2.1 Introduction

This chapter, which is a joint work with Inwon Kim [49], continues the discussion about the degenerate-diffusion-drift equation (1.1.1). Due to the degenerate diffusion, it can be shown that the solution u is compactly supported for all times if the initial data u_0 is compactly supported (see [51]). Our interest is on the regularity of the *free boundary:* $\partial\{u > 0\}$.

(1.1.1) can be written in the form of continuity equation,

$$u_t - \nabla \cdot ((\nabla\rho + V)u) = 0, \quad Q = \mathbb{R}^d \times [0, \infty)$$

where

$$\rho = \frac{m}{m-1}u^{m-1}. \tag{2.1.1}$$

Hence formally the normal velocity for the free boundary can be written as

$$\vec{b} = -(\nabla\rho + V) \cdot \vec{n} = |\nabla\rho| - V \cdot \vec{n} \quad \text{ on } (x,t) \in \Gamma := \partial\{\rho > 0\}, \tag{2.1.2}$$

where $\vec{n} = \vec{n}_{x,t}$ is the outward normal vector at given boundary points. Given that u solves a diffusion equation, it would be natural to expect that the free boundary is regularized by the pressure gradient $|\nabla\rho|$ if V is smooth, as long as ρ stays non-degenerate near the free boundary and topological singularities are ruled out. In general neither can be guaranteed even with zero drift.

2.1.1 Literature

When $V = 0$ (1.1.1) is the well-known *Porous Medium Equation* (PME), for which a vast amount of literature is available. We refer to the book [70]. Let us mention several classical results that are relevant in this chapter. The semi-convexity estimate $\Delta\rho > -\infty$ for $t > 0$ was shown by Aronson and Benilan [3] and played a fundamental role in the regularity theory of (PME). When the initial data $\rho_0 = \rho(\cdot, 0)$ has super-quadratic growth near the free boundary, Caffarelli and Friedman [16] showed that the support of solution strictly expands in time. While nondegeneracy is not obtained in this scenario, they prove a weaker description on the expansion rate of the free boundary by showing that its free boundary can be represented as $t = S(x)$ where S is Hölder continuous. To discuss further regularity results, it is natural to require some geometric properties of the solution to rule out topological singularities such as merging of two fingers. The $C^{1,\alpha}$ regularity of the free boundary is established by Caffarelli and Wolanski [18], under the assumption of nondegeneracy and Lipschitz continuity of solutions. These assumptions are shown to hold after a finite time $T_0 > 0$ by Caffarelli, Vazquez and Wolanski [17], where T_0 is the first time the support of solution expands to contain its initial convex hull. More recently, Kienzler explored the stability of solutions that are close to the flat traveling wave fronts to (PME) [46]. Kienzler, Koch and Vazquez [47] improved this result and showed that solutions that are locally close to the traveling waves are smooth: see further discussion on their result in comparison to ours below Theorem 2.1.3.

Few results are available for qualitative properties of (1.1.1). With the exception of $V = x$, there appears to be no change of coordinates that eliminates the drift dependence in (1.1.1). Well-posedness is shown in [9] and [35] for weak solutions and in [51] for viscosity solutions. Asymptotic convergence to equilibrium of (1.1.1) is shown in [20] using energy dissipation when V is the gradient of a convex potential. A recent result in [50] shows Hölder continuity of solutions for uniformly bounded drift. Our aim in this article is to study the free boundary regularity of (1.1.1). This is closely related to the nondegeneracy property of the pressure variable ρ, as seen in the motion law (2.1.2). It appears that free boundary regularity exhibits some fundamental differences from the zero drift case. Numerical experiments in [57] present the interesting possibility that an initially planar solution with smooth drift could develop corners without topological changes. The free boundary regularity or nondegeneracy of pressure is unknown even for traveling wave solutions in $\mathbb{R}^2$ [59].

2.1.2 Summary of Results

In this chapter, we will always assume

$$V = V(x,t) \in C^{3,1}_{x,t}(Q) \tag{2.1.3}$$

and $u_0 \geq 0$ is continuous and compactly supported.

For our analysis, we will consider the pressure variable (2.1.1) and the equation it satisfies:

$$\rho_t = (m-1)\rho\,\Delta\rho + |\nabla\rho|^2 + \nabla\rho\cdot V + (m-1)\rho\,\nabla\cdot V \tag{2.1.4}$$

in $Q = \mathbb{R}^d \times (0,\infty)$.

We first show the semi-convexity (Aronsson-Benilan) estimate through a barrier argument on $\Delta\rho$. This is where we use the C^3_x norm of V.

Theorem 2.1.1. *[Theorem 2.2.2] Let u solve* (1.1.1) *in Q with* (2.1.3)*, and let ρ be the corresponding pressure variable given by* (2.1.1)*. Then for some $\sigma > 0$, $\Delta\rho > -\frac{\sigma}{t} - \sigma$ in the sense of distribution for all $t > 0$.*

Next we discuss a weak nondegeneracy property in the event of no initial waiting time. With zero drift this corresponds to the strict expansion property of the positive set, [16]. In our case this property needs to be understood in terms of the streamlines. Let us define the *streamline* $X(t) = X(x_0, t_0; t)$ to be as the unique solution of the ODE:

$$\begin{cases} \partial_t X(t) = -V(X(t), t_0 + t), \quad t \in \mathbb{R}, \\ X(0) = x_0. \end{cases} \tag{2.1.5}$$

We will use the notation $\Omega := \{(x,t),\, u(x,t) > 0\}$ and $\Omega_t := \{x,\, u(\cdot,t) > 0\}$. While the streamlines are a natural coordinate for us to measure the "strict expansion" of Ω_t over time, the coordinate does not cope well with the diffusion term in the equation. The most delicate scenario occurs with degenerate pressure, where the time range we need to observe is much larger than the space range. To deal with such case we need to carefully localize V and utilize the actual streamline instead of its linear approximations.

Theorem 2.1.2. *[Theorem 2.3.4] Let ρ be as given in Theorem 2.2.2, and fix $(x_0, t_0) \in \Gamma := \partial\{\rho > 0\} \cap \{t > 0\}$. Then either of the following holds:*

(Type one) $X(-s) := X(x_0, t_0; -s) \in \Gamma$ for $s \in [0, t_0]$;

(Type two) There exist $C_*, \beta > 1$ and $h > 0$ such that for $s \in (0, h)$

$$\rho(x, t_0 - s) = 0 \quad \text{if } |x - X(-s)| \leq C_* s^\beta, \quad \rho(x, t_0 + s) > 0 \quad \text{if } |x - X(s)| \leq C_* s^\beta.$$

Moreover, if ρ_0 satisfies the near-boundary growth estimate

$$\rho_0(x) \geq \gamma(dist(x, \Omega_0^C))^{2-\varsigma} \ \text{ for some } \gamma, \varsigma > 0 \ \text{ and } \ \Delta u_0 > -\infty, \tag{2.1.6}$$

then any point on Γ is of type two.

The growth condition in (2.1.6) is optimal, since there is a stationary solution to (1.1.1) with a corner on its free boundary and with quadratic growth (see Theorem 2.6.3).

Next we proceed to show the nondegeneracy property of ρ, as it is essential for the regularity of its free boundary. This step presents the most challenging and novel part of our analysis. To illustrate the difficulties, let us briefly go over the main components of the celebrated arguments in [17], which provides non-degeneracy of solutions for (PME) for times $t > T_0$. One key ingredient in their analysis was the scale invariance of the equation under the transformation

$$\rho_{\epsilon,A}(x,t) := \frac{1 + A\epsilon}{(1+\epsilon)^2}\rho((1+\epsilon)x, (1 + A\epsilon)t + B) \quad \text{for any } A, B, \epsilon > 0,$$

In [17] $\rho_{\epsilon,A}$ was compared to ρ to obtain the space-time directional monotonicity

$$x \cdot \nabla\rho + (At + B)\rho_t \geq 0 \quad \text{on } \Gamma. \tag{2.1.7}$$

Applying (2.1.2) with $V = 0$ we then have

$$|\nabla u| = \frac{u_t}{|\nabla u|} \geq \frac{1}{(At+B)}\nu \cdot (\frac{x}{|x|}) \quad \text{on } \Gamma,$$

where the first equality is from (2.1.2), the second equality is due to the level set formulation of the normal velocity, and the last inequality is due to (2.1.7) and the fact that $\nabla\rho$ is parallel to the negative normal $-\nu$ on the free boundary. Thus non-degeneracy follows if we know that the free boundary is a Lipschitz graph with respect to the radial direction. This was shown in [17] for $t > T_0$ by the celebrated moving planes arguments, and thus we can conclude.

For nonzero drift, neither scaling invariance nor the moving planes method is available due to the inhomogeneity in V. In fact it is not reasonable to expect consistent free boundary behavior for large times, except possibly when V is a potential vector field. Still, it is reasonable to expect that, without topological singularites and waiting time, the diffusive nature of the equation (2.1.4) regularizes the free boundary. With this in mind we show a local non-degeneracy result under the assumption of directional monotonicity and zero waiting time.

Let us define the spatial cone of directions

$$W_{\theta,\mu} := \{y \in \mathbb{R}^d : |\frac{y}{|y|} - \mu| \leq 2\sin\frac{\theta}{2}\} \quad \text{with axis } \mu \in \mathcal{S}^{d-1} \text{ and } \theta \in (0, \pi/2]. \tag{2.1.8}$$

We say ρ is *monotone* with respect to $W_{\theta,\mu}$ if $\rho(\cdot, t)$ is non-decreasing along directions in $W_{\theta,\mu}$. We also denote $Q_r := \{|x| \leq r\} \times (-r, r)$.

Theorem 2.1.3. *[Local Nondegeneracy, Theorem 2.4.6] Let ρ be a solution of* (2.1.4) *in $Q = \mathbb{R}^d \times (0, \infty)$ with its initial data $\rho_0 = \rho(x, 0)$ satisfying* (2.1.6). *Fix $(x_0, T_0) \in \Gamma$ with $T_0 > 2$. Suppose that ρ is monotone with respect to $W_{\theta,\mu}$ in $Q_2 + (x_0, T_0)$ for some θ and μ. Then there exists $\kappa_* > 0$ such that*

$$\liminf_{\epsilon \to 0^+} \frac{\rho(x + \epsilon\mu, t)}{\epsilon} \geq \kappa_* \quad \text{for } (x,t) \in \Gamma \cap (Q_1 + (x_0, T_0)).$$

Above theorem is of local nature, with minimal conditions on the initial data that rules out waiting time. For the proof we adopt a local perturbation argument introduced in [30].

In the zero drift case, [47] considered solutions that are locally close to a planar traveling wave solution, which endows a discrete small-scale nondegeneracy and flatness on the solution. It was shown there that over time the flatness improves in its scale to yield the usual nondegeneracy and smoothness of the solutions. It was conjectured there whether a cone monotonicity could replace the planar barriers, given that waiting time could be ruled out. While we do not pursue improvement of flatness in scale, our result yields a positive partial answer to this question.

Building on the above non-degeneracy result, we proceed to study the free boundary regularity. To prevent sudden changes in the evolution caused by changes in the far-away region, we assume that, in the weak sense,

$$\rho_t \leq A\,(\mu \cdot \nabla\rho + \rho + 1) \quad \text{ in } Q_1 + (x_0, T_0) \text{ for some } A > 0. \tag{2.1.9}$$

Theorem 2.1.4. *Let ρ be given as in Theorem 2.1.3. If in addition* (2.1.9) *holds, then ρ is Lipschitz continuous and Γ is $C^{1,\alpha}$ in $Q_{1/2} + (x_0, T_0)$.*

As for the proof, we largely follow the iterative argument given in [18], which compares in different scales the solution with its shifted version. For nonzero drifts (2.1.4) changes under coordinate shifts, and thus a notable modification is necessary in the iteration procedure.

The following is an application of the above theorem.

Theorem 2.1.5. *[Theorem 2.6.1]. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a smooth and bounded function. Let ρ solve* (2.1.4) *in $Q = \mathbb{R}^2 \times (0, \infty)$ with $V = (\alpha(x_2), 0)$ and the initial data $\rho_0(x) = \rho(x, 0) = (x_1)_+$, under linear growth condition at infinity. Then Γ is locally uniformly $C^{1,\alpha}$ in Q.*

In [59] the existence of traveling wave solutions are shown with the above choice of V. The free boundary regularity remains open for the traveling waves, and possible formulation of corners has been observed in numerical experiments [57]. We consider the initially planar solution that was used in [57] to approximate the travelling waves. Our argument rules out the possibility of finite time singularity of the free boundary, but leaves open the possibility of asymptotic singularity.

We finish with examples which illustrate possible singular behaviors of free boundaries that exhibit differences from the zero drift case.

Theorem 2.1.6. *[Theorem 2.6.3]. There is $V \in C^3_x(\mathbb{R}^d)$ such that* (2.1.4) *has a stationary profile with a corner on its free boundary.*

2.2 Regularity of the Pressure

In our analysis it is often convenient to work with classical solutions of (1.1.1), which is made possible by the following result.

Lemma 2.2.1 (Section 9.3 [70])**.** *Let $U := B_1$ or $\mathbb{R}^d$. Suppose ϱ solves* (1.1.1) *in $U \times [0,1]$ with $\varrho_0 \in L^1(U) \cap L^\infty(U)$. Then there exists a sequence of ϱ_k (or u_k) which are strictly positive, classical solutions of* (1.1.1) *and $\varrho_k \to \varrho$ locally uniformly in $U \times (0,1]$ as $k \to \infty$.*

Let ϱ be a solution to (1.1.1) with condition (2.1.3). By Theorem 1.2.1,

$$\|\varrho\|_\infty \leq \sigma(\|V\|_\infty, \|\varrho_0\|_1 + \|\varrho_0\|_\infty) \text{ for all } (x,t) \in Q.$$

With this, we prove the fundamental estimate (for the pressure variable $u = \frac{m}{m-1}\varrho^{m-1}$) below.

Theorem 2.2.2. *There exists a universal constant σ such that*

$$\Delta u > -\frac{\sigma}{\tau} - \sigma \quad \textit{in } \mathbb{R}^d \times [\tau, \infty) \tag{2.2.1}$$

in the sense of distribution.

Proof. By Lemma 2.2.1, we only need to consider the smooth solutions. If (2.2.1) holds for the approximated smooth solutions, it holds for general solutions in the sense of distribution.

Now we assume that u is smooth and consider $p = \Delta u$. Then by differentiating (2.1.4) twice, we get

$$\begin{aligned} p_t &= (m-1)u\Delta p + 2m\nabla u \nabla p + (m-1)p^2 + 2\Sigma\, u_{ij}u_{ij} \\ &\quad + \nabla p \cdot V + 2\Sigma\, u_{ij}b^i_j + \nabla u \cdot \Delta V + (m-1)\left(p\nabla \cdot V + 2\nabla u \cdot \nabla(\nabla \cdot V) + u\Delta(\nabla \cdot V)\right) \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}\left|(m-1)p\nabla\cdot V + 2\Sigma\, u_{ij}b^i_j\right| &\leq \frac{m}{2}p^2 + \Sigma|u_{ij}|^2 + \sigma m\\ &\leq \frac{(m-2)}{2}p^2 + 2\Sigma\,|u_{ij}|^2 + \sigma m,\\ |\nabla u\cdot\Delta V + 2(m-1)(\nabla u\cdot\nabla(div\, V)| &\leq m|\nabla u|^2 + \sigma m,\\ (m-1)\,(u\Delta(div\, V)) &\leq \sigma m.\end{aligned}$$

Thus we obtain

$$p_t - (m-1)u\Delta p - 2m\nabla u\cdot\nabla p - \frac{m}{2}p^2 - \nabla p\cdot V + m|\nabla u|^2 + \sigma m \geq 0.$$

Viewing u as a known function, we may write the above quasilinear parabolic operator of p as $\mathcal{L}_0(p)$ and so we have $\mathcal{L}_0(p)\geq 0$. Below will construct a barrier for this operator to obtain a lower bound for p.

Let $w := -\frac{\sigma_1}{t+\tau} + u - \sigma_2$ for some $\tau, \sigma_1, \sigma_2 > 0$ to be determined later. Then

$$\begin{aligned}\mathcal{L}_0(w) = \frac{\sigma_1}{(t+\tau)^2} + u_t - (m-1)u\Delta u - 2m|\nabla u|^2 - \frac{m}{2}\left(-\frac{\sigma_1}{t+\tau} + u - \sigma_2\right)^2\\ -\nabla u\cdot V + m|\nabla u|^2 + Cm.\end{aligned}$$

Now we use the equation (2.1.4) to obtain

$$\begin{aligned}\mathcal{L}_0(w) &\leq \frac{\sigma_1}{(t+\tau)^2} - (m-1)|\nabla u|^2 - \frac{m}{2}\left(-\frac{\sigma_1}{t+\tau} + u - \sigma_2\right)^2 + \sigma m\\ &\leq \frac{\sigma_1}{(t+\tau)^2} - \frac{m}{2}\frac{\sigma_1}{(t+\tau)^2} - \frac{m}{2}(\sigma_2 - u)^2 + \sigma m \leq 0,\end{aligned}$$

if we choose $\sigma_1 \geq 2/m$ and $\sigma_2 \geq \|u\|_\infty + (2\sigma)^{1/2}$. Hence $\mathcal{L}_0(w) \leq 0 \leq \mathcal{L}_0(p)$, and by comparison principle we conclude that

$$\Delta u = p \geq w \geq -\frac{\sigma_1}{t} - \sigma_2.$$

□

As a remark, if $\Delta u_0 > -\infty$ in the sense of distribution, then there exists C such that $\Delta u \geq -C$ for all time.

Next we prove a useful property about the support of solutions: if $x_0 \in \Omega_{t_0}$ for some t_0, then $X(x_0, t_0; t) \in \Omega_t$ for all $t \geq t_0$. The proof is parallel to the proof of Lemma 3.5 [48] where they used a barrier argument. We will consider for $t > 0$, since at $t = 0$, u may not even be continuous. Again we are using smooth approximations.

Lemma 2.2.3. *The set* $\overline{\{u > 0\}} \cap \{t > 0\}$ *is non-decreasing along the streamlines.*

Proof. Choose $t \geq \eta_0 > 0$. Recall (2.1.5), we denote $X(x, t; s)$ with $s \geq 0$ as the streamline starting at (x, t). By (2.2.1) and the equation, denoting $C_0 = \frac{\sigma}{\eta_0} + \sigma$, we have

$$\begin{aligned} \partial_s u(X(x, t; s), t + s) &= (u_t + \nabla u \cdot V)(X(x, t; s), t + s) \\ &\geq (-C_0(m-1)u + |\nabla u|^2 + (m-1)u\nabla \cdot V)(X(x, t; s), t + s) \\ &\geq -Cu(X(x, t; s), t + s) \end{aligned}$$

where $C = (m-1)(C_0 + \|\nabla \cdot V\|_\infty)$. Thus for all $s \geq 0$

$$e^{Cs} u(X(x, t; s), t + s) \geq u(x, t). \tag{2.2.2}$$

In particular, if $x \in \Omega_t$

$$u(X(x, t; s), t + s) > 0 \quad \text{ for all } s \geq 0.$$

$\square$

2.3 Regularity of the Free Boundary

Here we study the propagation of the free boundaries along streamlines. The central idea in [16] was to measure the time the free boundary moves away from a given point by distance R, in terms of the average pressure in a ball of size R, making it sufficient to track the size of the pressure average over time instead of the free boundary movement.

The first lemma states that if the average pressure is low around a free boundary point, then the support cannot expand out too fast.

Lemma 2.3.1. *Suppose $t_0 \geq \eta_0 > 0$. There exist τ_0, c_0 such that for any $R > 0$ and $\tau \in (0, \tau_0)$, if*

$$u(x, t_0) = 0 \text{ for } x \in B(x_0, R),$$
$$\fint_{B(X(x_0,t_0;\tau),R)} u(x, t_0 + \tau)dx \leq \frac{c_0 R^2}{\tau},$$

then

$$u(x, t_0 + \tau) = 0 \text{ for } x \in B(X(x_0, t_0; \tau), R/6).$$

Proof. Note that Theorem 2.2.2 yields

$$\Delta u \geq -C_0 := -\frac{\sigma}{\eta_0} - \sigma \quad \text{for } t \geq \eta_0. \tag{2.3.1}$$

For simplicity, suppose $x_0 = 0, t_0 = 0$, and consider the rescaled function

$$\tilde{u}(x, t) = \frac{\tau}{R^2} u(Rx, \tau t). \tag{2.3.2}$$

Then $\tilde{u}$ satisfies

$$\tilde{u}_t = (m-1)\tilde{u}\Delta\tilde{u} + |\nabla\tilde{u}|^2 + \nabla\tilde{u} \cdot \vec{b}' + (m-1)\tilde{u}\,\nabla\vec{b}'.$$

with $\vec{b}'(x, t) := \frac{\tau}{R} V(Rx, \tau t)$, and the corresponding streamline $\tilde{X}(t) = \frac{1}{R} X(0, 0; \tau t)$.

Set $\epsilon := C_0 \tau_0$ and then $\Delta\tilde{u} = \tau\Delta u \geq -\epsilon$. Here ϵ can be arbitrarily small if τ_0 is small. From our assumption, it follows that

$$\fint_{B(\tilde{X}(1),1)} \tilde{u}(x, 1)dx \leq c_0.$$

Using this and that $\tilde{u} + \epsilon|x|^2/(2d)$ is subharmonic, we find for $x \in B(\tilde{X}(1), \frac{1}{2})$,

$$\begin{aligned} \tilde{u}(x, 1) &\leq -\frac{\epsilon|x|^2}{2d} + \fint_{B(\tilde{X}(1),1/2)} \tilde{u}(y, 1) + \frac{\epsilon|y|^2}{2d} dy \\ &\leq 2^d \fint_{B_1} \tilde{u}(y, 1)dy + \sigma\epsilon \leq 2^d c_0 + \sigma\epsilon. \end{aligned} \tag{2.3.3}$$

Now consider

$$v(x, t) := \tilde{u}(x + \tilde{X}(t), t).$$

By (2.3.1), we know $\Delta v \geq -\epsilon$. From the equation, v satisfies

$$\begin{aligned}
v_t(x,t) &= (m-1)v\Delta v + |\nabla v|^2 + \nabla v \cdot (\vec{b}'(x+\tilde{X},t) - \vec{b}'(\tilde{X},t)) + (m-1)v\nabla \cdot \vec{b}'(x+\tilde{X},t) \\
&\geq -\epsilon(m-1)v + |\nabla v|^2 - \sigma\tau|\nabla v||x| - \sigma\tau v \\
&\geq -\epsilon(m-1)v - \sigma\tau v - \sigma\tau^2|x|^2,
\end{aligned}$$

where the first inequality is due to the fact that for some universal σ

$$|\nabla\vec{b}'| \leq \sigma\tau \quad \text{and} \quad |\vec{b}'(x+\tilde{X},t) - \vec{b}'(\tilde{X},t)| \leq \sigma\tau|x|, \tag{2.3.4}$$

while in the second inequality, we applied Hölder's inequatlity.

Since $\epsilon = C_0\tau$ and $C_0 \geq 1$, we obtain

$$v_t(x,t) \geq -\sigma\epsilon\, v - \sigma\epsilon^2|x|^2. \tag{2.3.5}$$

Hence we get in $B_{\frac{1}{2}} \times (0,1)$

$$\begin{aligned}
v(x,1) &\geq e^{\sigma\epsilon(t-1)}v(x,t) - \sigma(1-e^{\sigma\epsilon(t-1)})\epsilon\,|x|^2 \\
&\geq e^{-\sigma\epsilon}v(x,t) - \sigma\epsilon.
\end{aligned}$$

Using (2.3.3) and taking ϵ to be small, we conclude that

$$v(x,t) \leq e^{\sigma\epsilon}(2^d c_0 + \sigma\epsilon) \text{ in } B_{\frac{1}{2}} \times (0,1). \tag{2.3.6}$$

To conclude we need to proceed with a barrier argument to put an upper bound for the support of v. To this end observe that

$$\begin{aligned}
v_t - (m-1)v\Delta v &= |\nabla v|^2 + \nabla v \cdot (\vec{b}'(x+\tilde{X},t) - \vec{b}'(\tilde{X},t)) + (m-1)v\nabla \cdot \vec{b}'(x+\tilde{X},t) \\
&\leq |\nabla v|^2 + \sigma\tau|\nabla v|\,|x| + \sigma\tau v
\end{aligned}$$

and thus

$$\mathcal{L}_1(v) := v_t - (m-1)v\Delta v - 2|\nabla v|^2 - \sigma\tau^2|x|^2 - \sigma\tau v \leq 0. \tag{2.3.7}$$

Define

$$\varphi(x,t) := \lambda\left(\frac{t}{36} + \frac{(|x|-1/3)}{6}\right)_+.$$

By direct computations, $\mathcal{L}_1(\varphi) \geq 0$ for $\frac{1}{3} - \frac{t}{6} \leq |x| \leq \frac{1}{2}$ if

$$\frac{1}{\lambda} \geq \left(\frac{t}{6} + |x| - \frac{1}{3}\right)\left((m-1)(d-1)|x|^{-1} + \frac{\sigma\epsilon}{6\lambda}\right) + 2 + \frac{9\epsilon}{\lambda^2}.$$

This is valid for $t \in (0,1)$ provided that we take λ to be small and then ϵ to be small.

By the assumption we have $v(x,0) = 0$ in $B_{\frac{1}{2}}$ and thus $v \leq \varphi$ on $|x| \leq 1/2, t = 0$. On the lateral boundary $|x| = 1/2, t \in (0,1)$, by (2.3.6) if c_0, ϵ are small enough depending on universal constants

$$v \leq e^{\sigma\epsilon}(2^d c_0 + \sigma\epsilon) \leq \frac{\lambda}{36} \leq \varphi.$$

Hence by comparison in $B_{\frac{1}{2}} \times (0,1)$ we have $v \leq \varphi$. In particular

$$\tilde{u}(x + \tilde{X}(1), 1) = v(x,1) \leq \varphi(x,1) = 0$$

for $|x| < \frac{1}{6}$ and we proved the lemma.

□

The following says sufficient average pressure pushes the support to expand out relative to the streamline.

Proposition 2.3.2. *Suppose $t_0 \geq \eta_0 > 0$. For any $c_1 > 0$, there exist $\lambda, c_2, \tau_0 > 0$ such that the following holds. For $R > 0$ and $\tau \leq \tau_0$, if If*

$$\fint_{B(x_0,R)} u(x,t_0)dx \geq c_1 \frac{R^2}{\tau}$$

then

$$u(X(x_0,t_0;\lambda\tau), t_0 + \lambda\tau) \geq c_2 \frac{R^2}{\tau}.$$

Proof. 1. Let C_0 be as in (2.3.1), and set $(x_0,t_0) = (0,0)$ by shifting coordinates. We consider the corresponding density variable $\varrho(x,t) = (\frac{m-1}{m}u(x,t))^{\frac{1}{m-1}}$ and its rescaled version

$$\tilde{\varrho}(x,t) = (\frac{\tau}{R^2})^{\frac{1}{m-1}} \varrho(Rx, \tau t),$$

which then solves

$$\tilde{\varrho}_t = \Delta\tilde{\varrho}^m + \nabla\cdot(\tilde{\varrho}\,\vec{b}') \text{ where } \vec{b}'(x,t) := \frac{\tau}{R}V(Rx,\tau t).$$

Since $\Delta u \geq -C_0, \tau \leq \tau_0$, choosing $\epsilon = C_0\tau_0$ yields $\Delta\tilde{\varrho}^m \geq -\epsilon\tilde{\varrho}$. As before we set

$$\tilde{X} = \tilde{X}(t) := \frac{1}{R}X(0,0;\tau t) \text{ and } \rho(x,t) := \tilde{\varrho}(x+\tilde{X},t).$$

2. For $Y(t) := \int_{B_1} \rho^m(x,t)dx$. First let us show that $Y(\lambda)$ stays sufficiently positive if $\epsilon\lambda$ is small.

Note $\tilde{X}(0) = 0$, therefore by our assumption

$$\begin{aligned} Y(0) &= \fint_{B_1} \rho^m(x,0)dx = \sigma(\frac{\tau}{R^2})^{\frac{m}{m-1}} \fint_{B(0,R)} \varrho^m(x+\tilde{X}(0),0)dx \\ &= \sigma \fint_{B(0,R)} (\frac{\tau}{R^2}u)^{\frac{m}{m-1}}(x,0)dx \\ &\geq \sigma \left(\frac{\tau}{R^2}\fint_{B(0,R)} u(x,0)dx\right)^{\frac{m}{m-1}} \\ &\geq \sigma c_1^{\frac{m}{m-1}} =: c_1'. \end{aligned}$$

where σ is universal. Due to (2.3.5) and $v(x,t) = \frac{m}{m-1}\rho^{m-1}(x,t)$, for ϵ small enough and $|x| \leq 1$

$$(\rho^m)_t \geq -\sigma\epsilon\rho^m - \sigma\epsilon^2|x|^2\rho \geq -\sigma\epsilon\rho^m - \sigma\epsilon. \tag{2.3.8}$$

Consequently

$$Y(t) \geq e^{-\sigma\epsilon t}Y(0) - \sigma\epsilon t \geq e^{-\sigma\epsilon\lambda}c_1' - \sigma\epsilon\lambda > \frac{c_1'}{2} \sim c_1^{\frac{m}{m-1}} \tag{2.3.9}$$

for $t \in (0,\lambda]$ if $\epsilon\lambda <<_\sigma 1$.

3. Next we will establish a lower bound on the growth rate of $Z(t) := \int_0^t Y(s)ds$, using the weak solution formulation of $\tilde{\varrho}$.

Lemma 2.3.3. *For universal constants σ_1, σ_2 and γ,*

$$e^{-\sigma_1\epsilon t}\int_0^t Y ds \leq \sigma_2 \int_0^t \rho^m(0,s)ds + \sigma_2\epsilon^\gamma + \sigma_2|Y|^{\frac{1}{m}} \tag{2.3.10}$$

Proof. As in [16], we introduce the Green's function in a unit ball so that G solves

$$\Delta G = -\sigma_d\delta(x) + \sigma_d I_{B_1} \quad \text{and} \quad G = 0, \nabla G = 0 \text{ on } \partial B_1. \tag{2.3.11}$$

Let us only discuss the dimension $d \geq 3$, where G is defined as

$$G(x) = |x|^{2-d} - 1 - \frac{d-2}{2}(1 - |x|^2). \tag{2.3.12}$$

We want to differentiate $\int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}(x,t)dx$ with respect to t. Since $G(x - \tilde{X}) = 0$ for x on the boundary of $B(\tilde{X}, 1)$, we have

$$\begin{aligned}
&\left(\int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}(x,t)dy\right)' \\
&= \int_{B(\tilde{X},1)} \nabla G(x - \tilde{X}) \cdot \vec{b}'(\tilde{X})\tilde{\varrho}\, dx + \int_{B(\tilde{X},1)} G(x - \tilde{X})\, \tilde{\varrho}_t\, dx \\
&= \int_{B(\tilde{X},1)} \nabla G(x - \tilde{X}) \cdot (\vec{b}'(\tilde{X}) - \vec{b}'(x))\tilde{\varrho}\, dx + \int_{B(\tilde{X},1)} \Delta G(x - \tilde{X})\, \tilde{\varrho}^m\, dx =: A_1 + A_2.
\end{aligned} \tag{2.3.13}$$

Recall the definition of $\vec{b}'$ and bounds on V, we know $\nabla \vec{b}' \geq -\sigma\epsilon I_d$. Then for $x, y \in \mathbb{R}^d$,

$$(\vec{b}'(x) - \vec{b}'(y)) \cdot (x - y) \geq -\sigma\epsilon|x - y|^2.$$

By (2.3.12), we know $\nabla G(x) = -(d-2)(|x|^{-d} - 1)x$. Thus,

$$\begin{aligned}
A_1 &= -\int_{B(\tilde{X},1)} (d-2)(|x - \tilde{X}|^{-d} - 1)(x - \tilde{X}) \cdot (\vec{b}'(\tilde{X}) - \vec{b}'(x))\tilde{\varrho}\, dx \\
&\geq -\sigma\epsilon \int_{B(\tilde{X},1)} (d-2)(|x - \tilde{X}|^{-d} - 1)|x - \tilde{X}|^2 \tilde{\varrho}\, dx \\
&\geq -\sigma\epsilon \int_{B(\tilde{X},1)} G(x - \tilde{X})\tilde{\varrho}\, dx
\end{aligned} \tag{2.3.14}$$

As for A_2, applying (2.3.11), we obtain

$$A_2 = -\sigma_d\, \tilde{\varrho}^m(\tilde{X}, t) + \sigma \int_{B(\tilde{X},1)} \tilde{\varrho}^m(x,t)\, dx. \tag{2.3.15}$$

Using (2.3.14), (2.3.15), we find for some universal $\sigma > 0$

$$\begin{aligned}
\left(\int_{B(\tilde{X},t)} G(x - \tilde{X})\tilde{\varrho}(x,t)dy\right)' \geq -\sigma_d\, \tilde{\varrho}^m(\tilde{X}, t) + \sigma \int_{B(\tilde{X},t)} \tilde{\varrho}^m(x,t)\, dx \\
- \sigma\epsilon \int_{B(\tilde{X},t)} G(x - \tilde{X})\, \tilde{\varrho}(x,t)dx.
\end{aligned}$$

Hence we derive

$$e^{\sigma\epsilon t} \int_{B_1} G(|x|)\rho(x,t)dx \geq -\sigma_d \int_0^t e^{\sigma\epsilon s}\rho^m(0,s)ds + \sigma \int_0^t \int_{B_1} e^{\sigma\epsilon s}\rho^m(x,s)\, dxds,$$

which simplifies to

$$\int_0^t e^{-\sigma\epsilon t} Y(s)ds \le \sigma \int_{B_1} G(|x|)\rho(x,t)dx + \sigma \int_0^t \rho^m(0,s)ds. \tag{2.3.16}$$

Next following the proof of Lemma 2.3 [16], using (2.3.16) and the integrability property of G, we can bound $\int_{B_1} G\rho\, dx$ by the sum of $Y^{\frac{1}{m}}$ and powers of ϵ. We conclude the proof of the lemma. $\square$

4. Now let us show that a contradiction occurs if our statement is false and

$$u(X(0,\lambda\tau),\lambda\tau) \le c_2 \frac{R^2}{\tau}.$$

In terms of ρ, we have

$$\rho^m(0,\lambda) \le \sigma(m)\, c_2^{\frac{m}{m-1}}.$$

Assume $\epsilon\lambda << 1$ and apply (2.3.8) again, we obtain for some universal σ and $t \in (0,\lambda]$

$$\rho^m(0,t) \le \sigma e^{\sigma\epsilon\lambda} c_2^{\frac{m}{m-1}} + \sigma\epsilon\lambda.$$

Let us assume for some σ large enough

$$c_1^{\frac{1}{m-1}} \ge \sigma\epsilon^\gamma + \sigma(e^{\sigma\epsilon\lambda} c_2^{\frac{m}{m-1}} \lambda + \epsilon\lambda^2), \tag{2.3.17}$$

and then we have for $t \in (0,\lambda]$ and some universal σ_3

$$\sigma_2 \int_0^t \rho^m(0,s)ds + \sigma_2\epsilon^\gamma \le \sigma_3 Y(t)^{\frac{1}{m}}.$$

Here we used (2.3.9). Thus in this situation, by (2.3.10), for $t \in (0,\lambda]$

$$e^{-\sigma_1\epsilon t} \int_0^t Y ds \le (\sigma_2+\sigma_3) Y^{\frac{1}{m}}.$$

Writing $Z(t) = \int_0^t Y(s)ds$, we have for some universal σ

$$Z' \ge \sigma e^{-\sigma\epsilon t} Z^m, \quad \text{with } Z(\frac{\lambda}{2}) \ge c_3\lambda \tag{2.3.18}$$

where the second inequality comes from (2.3.9), and

$$2c_3 = e^{-\sigma\epsilon\lambda} c_1' - \epsilon\lambda \sim c_1^{\frac{m}{m-1}}.$$

Solving the ODE (2.3.18) (with inequalities replaced by equalities) shows $t \in (0, \frac{\lambda}{2}]$

$$Z(t+\frac{\lambda}{2}) \geq \left((c_3\lambda)^{1-m} - f(t)\right)^{\frac{1}{1-m}}, \tag{2.3.19}$$

where for some universal $\sigma_4 > 0$

$$f(t) := \int_{\lambda/2}^{t+\lambda/2} \sigma e^{-\sigma\epsilon s} ds = \sigma e^{-\sigma\lambda\epsilon/2} \frac{(e^{\sigma\epsilon t} - 1)}{\sigma\epsilon} \sim \sigma_4 t + \sigma\epsilon t^2$$

since we can assume $\sigma\epsilon << 1$.

It is direct that f is monotone increasing in t. Notice the right hand side of (2.3.19) goes to $+\infty$ as

$$t \to f^{-1}((c_3\lambda)^{1-m})$$

which is impossible provided that $f^{-1}((c_3\lambda)^{1-m}) \leq \frac{\lambda}{2}$, which is equivalent to

$$(c_3\lambda)^{1-m} \leq f(\frac{\lambda}{2}) \sim \sigma_4\lambda + \sigma\epsilon\lambda^2. \tag{2.3.20}$$

It is not hard to see that (2.3.20) holds if $\lambda \geq C(c_1, \sigma)$ and $\sigma\epsilon\lambda << 1$.

We have proved the proposition with $\tau_0 = \epsilon/C_0$, λ satisfying (2.3.20), and c_2 satisfying (2.3.17).

$\square$

For any $(x_0, t_0) \in \Gamma$, consider the streamline segment ending at point (x_0, t_0). We use the notation

$$\Upsilon(x_0, t_0) := \{(X(x_0, t_0; -s), t_0 - s), \quad s \in (0, t_0)\}.$$

We have the following theorem:

Theorem 2.3.4. *Suppose $t_0 \geq \eta_0 > 0$ and fix any point $(x_0, t_0) \in \Gamma$. Then the following is true:*

(1) Either $\Upsilon(x_0, t_0) \subset \Gamma$ or $\Upsilon(x_0, t_0) \cap \Gamma = \emptyset$.

(2) Suppose the second case, then there exist positive constants C_*, β, h such that for all $s \in (0,h)$

$$\varrho(x, t_0 - s) = 0 \quad \text{if} \quad |x - X(x_0, t_0; -s)| \le C_* s^\beta,$$
$$\varrho(x, t_0 + s) > 0 \quad \text{if} \quad |x - X(x_0, t_0; s)| \le C_* s^\beta.$$

If the second case holds for $(x_0, t_0) \in \Gamma$, we say (x_0, t_0) is of the second type. Here β only depends d, η_0 and $\|V\|_{C^{3,0}_{x,t}}$.

Part (2) is a quantitative description of the second alternative in part (1). The proof is essentially given by Theorems 3.1-3.2 [16] based on the Lemmas 2.3.1 - 2.3.2. Let us only sketch the proof for part (1) below.

If the assertion of (1) is not true, then without loss of generality we can find $t_0 > t_1 > t_2 > 0$ such that $t_0 - t_1 >> t_1 - t_2$ and

$$x_0 \in \Gamma_{t_0},\ x_1 := X(x_0, t_0; t_1 - t_0) \in \Gamma_{t_1},\ x_2 := X(x_0, t_0; t_2 - t_0) \notin \Gamma_{t_2}.$$

Consequently for some R, $u(\cdot, t_2) = 0$ in $B(x_2, R)$. Since $x_1 = X(x_2, t_2; t_1 - t_2)$, by Lemma 2.3.1,

$$\fint_{B(x_1,R)} u(x, t_1)dx \ge \frac{c_0 R^2}{t_1 - t_2}.$$

Since $t_0 - t_1 >> (t_1 - t_2)$, Lemma 2.3.2 yields that $u(x_0, t_0) = u(X(x_1, t_1; t_0 - t_1), t_0) > 0$, which leads to the contradiction.

When the initial data grows faster than quadratically, it is possible to characterize the constants C_*, h in above theorem in terms of time variable. By a compactness argument, iteratively using Theorem 2.3.4 and arguing as in the remark on Theorem 3.2 in [16], we have the following theorem.

Theorem 2.3.5. *Suppose* (2.1.6). *Then any point $x_0 \in \Gamma_{t_0}$ with $t_0 \le T$ is of the second type and the constants C_*, h in Theorem 2.3.4 (2) only depend on the conditions.*

2.4 Monotonicity Implies Nondegeneracy

In this section, we discuss nondegeneracy property of solutions. We start with the following theorem.

Theorem 2.4.1. *Let u solve (2.1.4) in Q_2. Suppose $\Delta u \geq -C_0$ and Γ is of type two in Q_2. Suppose in addition that there exist $\theta \in (0, \pi/2)$ and $\mu \in \mathcal{S}^{d-1}$ such that u is cone monotone with respect to $W_{\theta,\mu}$ in Q_2. Then there exists a constant $C > 0$ such that for sufficiently small $\epsilon > 0$ we have*

$$u(X(x, t; C\epsilon) - \epsilon\mu, t + C\epsilon) > 0 \text{ for } (x, t) \in \Gamma \cap Q_1. \tag{2.4.1}$$

Remark 2.4.2. The constant C in Theorem 2.4.1 depend on

$$\{\theta, C_*, h, \beta, C_0 \text{ and universal constants }\},$$

where C_*, h, β are constants given in Theorem 2.3.4. An estimate of C_0 can be found in Theorem 2.2.2.

The main ingredient in the proof of the theorem, motivated from [30], is the construction of a supersolution for the following operator associated with $v(x, t) := u(x + X(t), t)$.

$$\mathcal{L}_2 v := \partial_t v - (m-1)v\Delta v - |\nabla v|^2 - \nabla v \cdot (V(x+X(t), t) - V(X(t), t)) - (m-1)v\nabla \cdot V(x+X(t), t). \tag{2.4.2}$$

Since the supersolution to be constructed is a rescaled inf-convolution of v, comparison of the two leads to space-time monotonicity of v.

Let ψ be a positive smooth function in B_2 and $0 < \psi < \frac{1}{2}$ and $v \in C^\infty(B_2)$ be non-negative. Consider

$$f(x) := \inf_{B(x,\psi(x))} v(y).$$

We have the following two properties.

Lemma 2.4.3. *If for some $\sigma_1 = \sigma_1(d) > 0$ large enough*

$$\Delta\psi = \frac{\sigma_1 |\nabla\psi|^2}{|\psi|}$$

in B_2, then for some universal σ_2, in B_1

$$\Delta f(x) - \Delta v(y) \leq \sigma_2 \|\nabla\psi\|_\infty \max\{\Delta v(y), 0\} \quad \text{if } f(x) = v(y).$$

Without loss of generality, we can take $\sigma_2 \geq 3$.

Lemma 2.4.4. *For $x \in B_1$,*

$$|\nabla f(x) - \nabla v(y)| = |\nabla v(y)||\nabla\psi(x)| \quad \text{if } f(x) = v(y).$$

We postpone the proofs of the two Lemmas to the appendix.

Let $\varphi : \mathbb{R}^d \to (0, \infty)$ be a smooth function and σ_2 be as given above. For some constants $\alpha, A_0, M_0 \geq 1$ we define

$$w(x,t) := (1 + A_0\epsilon t) \inf_{y \in B(x, R_\epsilon(x,t))} u(y + r\epsilon\mu + X(0,0;\zeta(t)), \zeta(t)) \tag{2.4.3}$$

where

$$R_\epsilon(x,t) := \epsilon\varphi(x)(1 - \alpha t) \tag{2.4.4}$$

$$\zeta(t) := (1 + \sigma_2 M_0 \epsilon)(t + \frac{A_0\epsilon t^2}{2}). \tag{2.4.5}$$

We will choose A_0 and α in Proposition 2.4.5 and M_0, r in the proof of Theorem 2.4.1.

Proposition 2.4.5. *Suppose $\Delta u \geq -C_0$ in Q_2. Fix $M_0 \geq 1$ and consider $\varphi : B_2 \to \mathbb{R}$ such that*

$$\begin{cases} \Delta\varphi &= \frac{\sigma_1|\nabla\varphi|^2}{|\varphi|}, \\ & \frac{r}{M_0} \leq \varphi(\cdot) \leq rM_0, \quad \|\nabla\varphi\|_\infty \leq M_0 \text{ for some } r \in (0,1), \end{cases} \tag{2.4.6}$$

Then there exist $A_0, \alpha, \tau, \epsilon_0$ depending only on M_0 and universal constants such that for all $\epsilon \leq \epsilon_0$ the function w defined in (2.4.3) satisfies

$$\mathcal{L}_2 w \geq 0 \text{ in } B_r \times (0, \tau).$$

Proof. Below and within this section, we will use the notation

$$X_t := X(0,0;t), \quad X_\zeta := X(0,0;\zeta(t)).$$

By Lemma 2.2.1, we may assume that u is smooth.

Denote $v(x,t) = u(x + X_t, t)$ which solves $\mathcal{L}_2 v = 0$, and suppose

$$\tau \leq \min\{1/A_0,\, 1/(\sigma_2 M_0),\, 1/(5\alpha)\}, \tag{2.4.7}$$

where A_0, α will be determined in (2.4.15). Define $g(t) := 1 + A_0 \epsilon t$. By definition of ζ,

$$\partial_t \zeta(t) = (1 + \sigma_2 M_0 \epsilon)\, g(t),$$

$$0 \leq \zeta - t \leq \sigma \sigma_2 M_0 t \epsilon \leq \sigma \epsilon \quad (\text{ by } (2.4.7)) \tag{2.4.8}$$

for some universal σ. By definition of w there is $z(x)$ such that $w(x,t) = g(t) v(z(x), \zeta(t))$ and

$$|z - x| \leq \epsilon (M_0 + 1) r. \tag{2.4.9}$$

We will write $w = w(x,t)$ and $v = v(z(x), \zeta(t))$. Computing as in [48] it follows that

$$\begin{aligned} \partial_t w &\geq A_0 \epsilon v - \partial_t R_\epsilon \, |\nabla w| + (\zeta') g \, v_t \\ &= A_0 \epsilon v + \varphi \alpha \epsilon \, g |\nabla v| + (1 + \sigma_2 M_0 \epsilon) g^2 v_t \quad (\text{ by } (2.4.4)) \\ &\geq A_0 \epsilon v + \frac{r \alpha \epsilon}{M_0} |\nabla v| + (1 + \sigma_2 M_0 \epsilon) g^2 v_t \quad (\text{ by } (2.4.6)). \end{aligned} \tag{2.4.10}$$

By Lemma 2.4.3,

$$\begin{aligned} -\Delta w &\geq -g \Delta v - \sigma_2 \|\nabla R_\epsilon\|_\infty |\Delta v| \\ &\geq -(1 + \sigma_2 M_0 \epsilon) g \Delta v - C_1 \epsilon \quad (\text{ since } \Delta v \geq -C_0) \end{aligned}$$

where $C_1 = \sigma C_0 M_0$. By Lemma 2.4.4,

$$|\nabla w - g(\nabla v)| = |\nabla R_\epsilon| |g \nabla v| \leq \epsilon M_0 |\nabla v|.$$

Because $\sigma_2 \geq 3$ and ϵ is small,

$$|\nabla w|^2 \leq (1 + \sigma_2 M_0 \epsilon) g^2 |\nabla v|^2.$$

For $x \in B_r$, we have

$$|V(x + X_\zeta, \zeta) - V(X_\zeta, \zeta)| \leq \|\nabla V\|_\infty r \leq \sigma r,$$

and thus

$$(\nabla w - g\nabla v)\cdot(V(x+X_\zeta,\zeta)-V(X_\zeta,\zeta)) \le \sigma M_0 r\epsilon|\nabla v|. \tag{2.4.11}$$

By definition of $g(t)$,

$$|g-(1+\sigma_2 M_0\epsilon)g^2| \lesssim \sigma_2 M_0\epsilon.$$

Hence

$$\left|g\nabla v-(1+\sigma_2 M_0\epsilon)g^2\nabla v\right| |V(x+X_\zeta,\zeta)-V(X_\zeta,\zeta)| \le \sigma\sigma_2 M_0 r\epsilon|\nabla v|. \tag{2.4.12}$$

By (2.4.9),

$$|V(x+X_\zeta,\zeta)-V(z+X_\zeta,\zeta)| \le \sigma M_0 r\epsilon. \tag{2.4.13}$$

Therefore, by (2.4.11)-(2.4.13)

$$\begin{aligned}&\left|\nabla w\cdot(V(x+X_\zeta,\zeta)-V(X_\zeta,\zeta))-(1+\sigma_2 M_0\epsilon)g^2\nabla v\cdot(V(z+X_\zeta,\zeta)-V(X_\zeta,\zeta))\right|\\ &\le \sigma(1+\sigma_2)M_0 r\epsilon|\nabla v| := C_2 r\epsilon|\nabla v|.\end{aligned}$$

Similarly

$$\left|g(\nabla\cdot V(x+X_\zeta)-(1+CM_0\epsilon)g^2(\nabla\cdot V(z+X_\zeta)\right| \le C_2 r\epsilon.$$

Putting together above estimates we get

$$\begin{aligned}\widetilde{\mathcal{L}}_2 w :=\, &\partial_t w-(m-1)w\Delta w-|\nabla w|^2-\nabla w\cdot(V(x+X_\zeta,\zeta)-V(X_\zeta,\zeta)\\ &-(m-1)w\,(\nabla\cdot V)(x+X_\zeta,\zeta)\\ \ge\, &A_0\epsilon v+\frac{\alpha r\epsilon}{2M_0}|\nabla v|+(1+\sigma_2 M_0\epsilon)g^2(v_t-(m-1)v\Delta v-|\nabla v|^2)\\ &-(1+\sigma_2 M_0\epsilon)g^2\nabla v\cdot(V(z+X_\zeta,\zeta)-V(X_\zeta,\zeta))\\ &-(m-1)(1+\sigma_2 M_0\epsilon)g^2 v(\nabla\cdot V(z+X_\zeta,\zeta))\\ &-C_1\epsilon v-C_2 r\epsilon|\nabla v|-C_2 r\epsilon v.\end{aligned}$$

Using (2.4.2) we obtain

$$\begin{aligned}\widetilde{L}_2 w &\ge A_0\epsilon v+\frac{\alpha r\epsilon}{2M_0}|\nabla v|+(1+CM_0\epsilon)g^2\mathcal{L}_2 v(z,\zeta)-(C_1+C_2 r)\epsilon v-C_2 r\epsilon|\nabla v|\\ &= A_0\epsilon v+\frac{\alpha r\epsilon}{2M_0}|\nabla v|-(C_1+C_2 r)\epsilon v-C_2 r\epsilon|\nabla v|\end{aligned}$$

Write

$$2C_3 := A_0 - C_1 - C_2 r, \quad 2C_4 := \frac{\alpha}{2M_0} - C_2.$$

Assume ϵ to be small enough and we have

$$\begin{aligned}\mathcal{L}_2(w) &\geq \mathcal{L}_2(w) - \widetilde{\mathcal{L}}_2(w) + 2C_4 r|\nabla v| + 2C_3\epsilon v \\ &\geq C_3\epsilon w + C_4 r\epsilon|\nabla w| - |\nabla w|V_0 \\ &\quad - (m-1)w\,|\nabla V(x+X_\zeta,\zeta) - \nabla V(x+X_t,t)|\end{aligned} \tag{2.4.14}$$

$$\text{where} \quad V_0 := |V(x+X_\zeta,\zeta) - V(X_\zeta,\zeta) - (V(x+X_t,t) - V(X_t,t))|.$$

First, we estimate V_0:

$$\begin{aligned}V_0 &= \left|\int_t^\zeta \partial_s V(x+X_s,s) - \partial_s V(X_s,s)ds\right| \\ &\leq \int_t^\zeta |((\nabla V)(x+X_s,s) - (\nabla V)(X_s,s))V(X)| + |(\partial_t V)(x+X_s,s) - (\partial_t V)(X_s,s)|\,ds \\ &\leq \sigma|x|\int_t^\zeta \left\|D^2 V\right\|_\infty \|V\|_\infty + \|D\partial_t V\|_\infty\,ds \leq \sigma r\epsilon.\end{aligned}$$

Next

$$|\nabla V(x+X_\zeta,\zeta) - \nabla V(x+X_t,t)| \leq \sigma\epsilon.$$

Thus by (2.4.14), if C_3, C_4 are taken sufficiently large depending on universal constants, it follows that $\mathcal{L}_2(w) \geq 0$. This is possible if we choose A_0, α such that

$$A_0 = \sigma M_0(1+C_0), \quad \alpha = \sigma M_0^2 \tag{2.4.15}$$

where σ is universal.

□

Proof of Theorem 2.4.1

Let Φ be the unique solution of

$$\begin{cases}\Delta(\Phi^{-\sigma_1+1}) = 0 & \text{in } B_{\frac{1}{2}}\backslash B_{\sin\theta/10} \\ \Phi = A_{d,\theta} & \text{on } \partial B_{\sin\theta/10} \\ \Phi = \frac{1}{2}\sin\theta & \text{on } \partial B_{\frac{1}{2}},\end{cases}$$

where σ_1 is as in Proposition 2.4.5 and $A_{d,\theta}$ is large enough that $\Phi(\frac{\mu}{5}) \geq 3$. Then for some $M_0(\theta, d) \geq 1$

$$\frac{1}{M_0} \leq \Phi \leq M_0, \quad \|\nabla\Phi\|_\infty \leq M_0 \quad \text{in } B_{1/2}.$$

With above M_0, let A_0, α, τ be as given in Proposition 2.4.5.

Next fix any $(\hat{x}, \hat{t}) \in Q_1 \cap \Gamma$ and let C^*, h, β be as given in Theorem 2.3.4. Choose t_1 such that

$$0 < t_1 \leq \min\{\tau, h, \sigma/(1 + C_0)\} \tag{2.4.16}$$

and set $r := \min\{C_* t_1^\beta, \frac{1}{4}\} > 0$ so that, due to Theorem 2.3.4,

$$u(x, \hat{t} - t_1) = 0 \text{ for all } x \in B(X(\hat{x}, \hat{t}; -t_1), r). \tag{2.4.17}$$

In the proof we set the point $(X(\hat{x}, \hat{t}; -t_1), \hat{t} - t_1)$ to be the origin for simplicity. Recall the notation $X_t := X(0, 0; t)$ and that in our setting $(X_{t_1}, t_1) \in \Gamma$.

Let us consider

$$v(x, t) := u(x + X_t, t),$$

which then solves $\mathcal{L}_2 v = 0$. By (2.4.16) and (2.4.7), we have $\{(x+X_t, t), x \in B_1, t \in [0, t_1]\} \subset Q_1$.

It follows from (2.4.17) that

$$v(x, 0) = 0 \quad \text{in } B_r. \tag{2.4.18}$$

For $P := -\frac{r}{5}\mu$ and $r_\theta := \frac{r}{10}\sin\theta$, we define $\varphi(x) := r\Phi(\frac{x-P}{r})$. Recall w defined from (2.4.3):

$$w(x, t) = (1 + A_0 \epsilon t) \inf_{B(0, \epsilon\varphi(x)(1-\alpha t))} v(y + r\epsilon\mu, \zeta(t)).$$

Now we define the cylindrical domain

$$\Sigma := (B(P, r/2) \setminus B(P, r_\theta)) \times [0, t_1]$$

and compare v with w in this domain. Observe that (2.4.6) is satisfied inside Σ. By Proposition 2.4.5,

$$\mathcal{L}_2(w) \geq 0 \text{ in } \Sigma.$$

We claim that $w \geq v$ on the parabolic boundary of Σ. First from (2.4.18) it follows that

$$w(x,0) \geq 0 = v(x,0) \text{ on } B(P, \frac{r}{2}).$$

Next observe that, since $v(0,t_1) = u(X_{t_1}, t_1) = 0$,

$$v(0,t) = u(X_t, t) = 0 \text{ for } t \in [0, t_1].$$

Due to the cone monotonicity and definition of v,

$$w(\cdot, t) \geq v(\cdot, t) = 0 \text{ in } B(P, r_\theta) \subset B(-\frac{r}{5}\mu, \frac{r}{5}\sin\theta), \text{ for } t \in [0, t_1].$$

Next, since $\varphi(x) = \frac{r}{2}\sin\theta$ on $\partial B(P, r/2)$,

$$\begin{aligned} w(x,t) &\geq (1+\sigma_2 M_0 \epsilon) \inf_{B(x, r\epsilon(1-\alpha t)\frac{\sin\theta}{2})} v(y + r\epsilon\mu, \zeta) \\ &= (1+\sigma_2 M_0\epsilon) \inf_{B(x, \frac{r\epsilon}{2}\sin\theta)} u(y + r\epsilon\mu + X_\zeta, \zeta) \end{aligned} \tag{2.4.19}$$

Hence to prove our claim it remains to show that $w \geq v$ on $\partial B(P, \frac{r}{2}) \times [0, \tau_1]$.

By (2.4.8), we know $s := \zeta(t) - t \leq \sigma M_0 \epsilon t \leq \sigma\epsilon$. For $(z,t) \in \Sigma$, (2.2.2) yields that

$$e^{Cs} u(X(z,t;s), t+s) \geq u(z,t) \text{ where } C = (m-1)(C_0 + \|\nabla \cdot V\|_\infty). \tag{2.4.20}$$

Set $\vec{b}_* := V(0,0)$. Since V is smooth, $|X(z,t;s) - z| \leq \sigma s$. By (2.4.8), we have

$$\begin{aligned} |X(z,t;s) - z + \vec{b}_* s| &= |\int_0^s b(X(z,t;y), y) - \vec{b}_* dy| \\ &\leq (|\nabla V||X(z,t;s) - z| + |\partial_t V| s)\, s \\ &\leq \sigma s^2 \leq \frac{r\epsilon}{4}\sin\theta \end{aligned}$$

if ϵ is small enough ($\epsilon \leq \frac{r\sin\theta}{4\sigma}$). Also we can obtain

$$|X_\zeta - X_t + \vec{b}_* s| \leq \frac{r\epsilon}{4}\sin\theta.$$

Thus

$$|X(z,t;s) - z - X_\zeta + X_t)| \leq \frac{r\epsilon}{2}\sin\theta.$$

Therefore fix t, let $z(y) = y + r\epsilon\mu + X(t)$ and take infimum of $y \in B(x, \frac{r\epsilon}{2}\sin\theta)$

$$\begin{aligned}\inf_{y\in B(x,\frac{r\epsilon}{2}\sin\theta)} u(y + r\epsilon\mu + X_\zeta, \zeta) &= \inf_{y\in B(x,\frac{r\epsilon}{2}\sin\theta)} u(z(y) - X_\zeta + X_t, t+s) \\ &\geq \inf_{y\in B(x,r\epsilon\sin\theta)} u(X(z(y), t; s), t+s).\end{aligned}$$

From (2.4.20),

$$\inf_{y\in B(x,\frac{r\epsilon}{2}\sin\theta)} e^{Cs}u(y + r\epsilon\mu + X_\zeta, \zeta) \geq \inf_{y\in B(x,r\epsilon\sin\theta)} u(y + r\epsilon\mu + X(t), t).$$

Due to (2.4.19), we derive

$$w(x,t) \geq (1 + \sigma_2 M_0\epsilon)e^{-C(\zeta-t)} \inf_{B(x,r\epsilon\sin\theta)} u(y + r\epsilon\mu + X_t, t)$$

with $C = \sigma(C_0 + 1)$. By (2.4.8) and (2.4.16), $\zeta - t \leq \sigma M_0\epsilon t$. Since $t \leq \sigma/(1 + C_0)$, if σ is smaller than a universal constant,

$$(1 + \sigma_2 M_0\epsilon)e^{-C(\zeta-t)} \geq (1 + \sigma_2 M_0\epsilon)e^{-\sigma_2 M_0\epsilon/2} \geq 1.$$

Now by cone monotonicity and (2.4.16), for small ϵ,

$$w(x,t) \geq u(x + X_t, t) = v(x,t) \text{ on } \partial B(P, r/2) \times [0, t_1],$$

and we have proved our claim, i.e. $w \geq v$ on the parabolic boundary of Σ. Now comparison principle yields $w \geq v$ in Σ.

Note that by (2.4.7) $(1 - \alpha t_1) \geq 4/5$. Since $\varphi(0) \geq 3r$, For $|x| \leq r\epsilon/5$ we have

$$-r\epsilon\mu \in B(x, \frac{12}{5}r\epsilon) + r\epsilon\mu \subset B(x, \varphi(0)\epsilon(1 - \alpha t_1)) + r\epsilon\mu$$

and

$$(1 + A_0\epsilon t_1)\, v(-r\epsilon\mu, \zeta(t_1)) \geq w(x, t_1) \geq v(x, t_1).$$

Since $0 \in \Gamma_{t_1}(v)$ we have

$$(1 + A_0\epsilon t_1)\, v(-r\epsilon\mu, \zeta(t_1)) \geq \sup_{x\in B(0,\frac{r\epsilon}{5})} v(x, t_1) > 0.$$

We can now conclude.

Theorem 2.4.6. *Let u be a solution of* (2.1.4) *in Q with u_0 satisfying* (2.1.6)*. Fix $(x_0, T_0) \in \Gamma$ with $T_0 > 2$. Suppose that u is monotone with respect to $W_{\theta,\mu}$ in $Q_2 + (x_0, T_0)$. Then there exists a constant $\kappa_* > 0$ such that*

$$u(x + \epsilon\mu, t) \geq \kappa_* \epsilon \tag{2.4.21}$$

for all $(x, t) \in \Gamma \cap (Q_1 + (x_0, T_0))$ and sufficiently small $\epsilon > 0$.

Proof. Theorem 2.2.2 yields that $\Delta u > -\infty$ in $Q_2 + (x_0, T_0)$. Furthermore, when (2.1.6) holds for u_0, Theorem 2.3.5 yields that Γ is always of type two. Thus if we consider

$$\tilde{u}(x, t) := u(x + x_0, t + T_0),$$

this $\tilde{u}$ satisfies all the conditions in the previous Theorem 2.4.1 and therefore (2.4.1) holds for $\tilde{u}$. After shifting coordinates we may assume that $x_0 = 0, T_0 = 0$. For simplicity we will denote $\tilde{u}$ by u.

We claim that there is $\kappa > 0$ such that for all ϵ sufficiently small

$$\sup_{y \in B(x,\epsilon)} u(y, t) \geq \kappa\epsilon \quad \text{ for } (x, t) \in \Gamma \cap Q_1. \tag{2.4.22}$$

Now fix one $(\hat{x}, \hat{t}) \in \Gamma \cap Q_1$. Suppose (2.4.22) fails for $(x, t) = (\hat{x}, \hat{t})$ and we want to obtain a contradiction.

By Theorem 2.4.1, there exists C such that for all ϵ sufficiently small

$$x_1 := X(\hat{x} + \epsilon\mu, \hat{t}; -C\epsilon) \notin \Omega_{\hat{t} - C\epsilon}.$$

By the cone monotonicity condition,

$$x_1 - \mu\epsilon + B\left(0, \sin\theta\, \epsilon\right) \notin \Omega_{\hat{t} - C\epsilon}. \tag{2.4.23}$$

Set

$$x_2 := X(\hat{x}, \hat{t}; -C\epsilon), \quad f(t) := X(\hat{x} + \epsilon\mu, \hat{t}; t) - X(\hat{x}, \hat{t}; t)$$

and then

$$f(0) = \epsilon\mu, \quad f(-C\epsilon) = x_1 - x_2.$$

By the equation of streamlines, we have $|f'(t)| \leq \sigma |f(t)|$ and thus

$$|x_1 - x_2 - \epsilon\mu| = |f(-C\epsilon) - f(0)| \leq \epsilon(e^{\sigma C\epsilon} - 1) \leq \sin\theta\,\epsilon/2 \tag{2.4.24}$$

if $C^2\epsilon^2$ is sufficiently small compared to $\sin\theta$.

According to (2.4.23) and (2.4.24), we have

$$R := dist(x_2, \Gamma_{\hat{t}-C\epsilon}) \geq \sin\theta\,\epsilon/2.$$

Therefore

$$u(\cdot, \hat{t} - C\epsilon) = 0 \text{ in } B(x_2, R).$$

By the assumption, the failure of (2.4.22) implies that

$$\fint_{B(X(x_2,\hat{t}-C\epsilon;C\epsilon),R)} u(x,\hat{t})dx = \fint_{B(\hat{x},R)} u(x,\hat{t})dx \leq \kappa\epsilon.$$

Now let c_0 be from Lemma 2.3.1, and we take $\kappa = \kappa(c_0, C, \theta)$ to be small enough such that for all small $\epsilon > 0$

$$\kappa\epsilon \leq c_0 \left(\sin\theta/2\right)^2 \frac{\epsilon}{C} \leq c_0 \frac{R^2}{C\epsilon}.$$

Hence Lemma 2.3.1 shows

$$u(x, \hat{t}) = 0 \text{ in } B(\hat{x}, R/6),$$

which contradicts with the fact that $\hat{x} \in \Gamma_{\hat{t}}$. We proved the claim.

Now for any $(x,t) \in \Gamma_1$ and for any $\gamma \in (0,1)$, by (2.4.22), there exist $\kappa_* := \kappa\gamma, \epsilon_0(\gamma) > 0$ such that

$$\sup_{y \in B(x,\gamma\epsilon)} u(y,t) \geq \kappa_*\epsilon \text{ for any } \epsilon \in (0, \epsilon_0).$$

Therefore we can find $y \in B(x, \gamma\epsilon)$ that $u(y,t) \geq \kappa_*\epsilon$. From the geometry, if $\gamma = \gamma(\theta)$ is small enough, $x + \epsilon\mu \in y + W_{\theta,\mu}$. Due to the condition that u is cone monotone, we can conclude

$$u(x + \epsilon\mu, t) \geq \kappa_*\epsilon \text{ for any } (x,t) \in \Gamma \cap Q_1 \text{ and } \epsilon \in (0, \epsilon_0).$$

$\square$

2.5 Flatness Implies Smoothness

In this section we study regularity properties of Γ and we are going to prove Theorem 2.1.4. Proceeding as in the proof of Theorem 2.4.6, it is enough for us to show the following theorem.

Theorem 2.5.1. *Let u be as given in Theorem 2.4.1, and assume* (2.1.9) *in Q_2. Then there exists $\alpha \in (0,1)$ such that $\Gamma \cap Q_1$ is a d-dimensional $C^{1,\alpha}$ surface.*

The cone monotonicity and (2.1.9) provide sufficient monotonicity properties for the solution to rule out topological singularities and to localize the regularization phenomena driven by the diffusion in the interior of the domain. For the proof we follow the outline for the zero drift built on [18] and [17], while we elaborate on the differences. First we establish Lipschitz regularity of solutions as well as nondegeneracy at the free boundary.

Lemma 2.5.2. *Let u be a solution of* (2.1.4) *in Q_2. Suppose* (2.1.9)*, the cone mononicity and $\Delta u \geq -C_0$ hold in Q_2. Then u is Lipschitz continuous in Q_1 and $\Gamma \cap Q_{\frac{1}{2}}$ is a d-dimensional Lipschitz continuous surface.*

Proof. First let us prove that $|\nabla u|$ is bounded in Q_1. From the equation and $\Delta u \geq -C_0$

$$u_t \geq |\nabla u|^2 - \sigma(C_0+1)u + \nabla u \cdot V. \tag{2.5.1}$$

Above estimate combined with condition (2.1.9) yields

$$(A+\sigma)|\nabla u| + C(C_0, A, \sigma)\, u + A \geq |\nabla u|^2,$$

which turns into a bound on $|\nabla u|$. The bound on u_t now follows and (2.1.9). Hence for some L depending on A, C_0 and universal constants, we have

$$|\nabla u| + |u_t| \leq L \quad \text{ in } Q_1. \tag{2.5.2}$$

Next, the cone monotonicity implies that Γ is Lipschitz in space, and hence it remains to show that Γ is Lipschitz in time. Lemma 2.2.3 and smoothness of V indicate that, for any $\tau > 0$ and $u^\tau(\cdot, t) := u(\cdot, t+\tau)$

$$(\Omega(u) \cap Q_{1/2}) \subset (\ C\tau\text{-neighbourhood of } \Omega(u^\tau)) \cap Q_1.$$

Thus it remains to show the other inclusion.

For any $(x,t) \in \Gamma$ and $C_1 \geq 1$, by the cone monotonicity

$$u(\cdot,t) = 0 \text{ in } B(y,R),$$

where $y := x - C_1\tau\mu$ and $R := C_1 \sin\theta\,\tau$. By (2.5.2) and the fact that $|X(y,t;\tau) - y| \leq \|V\|_\infty \tau$,

$$\begin{aligned}\sup_{z\in B(X(y,t;\tau),R)} u(z,t+\tau) &\leq u(X(y,t;\tau),t) + L(R+\tau)\\ &\leq u(y,t) + \sigma L\tau + L(1+C_1\sin\theta)\tau\\ &\leq C\,C_1\,\tau,\end{aligned}$$

where C depends on $L, \|V\|_\infty$. Thus

$$\fint_{B(X(y,t;\tau),R)} u(z,t+\tau)dz \leq CC_1\tau \leq c_0\frac{R^2}{\tau} = (c_0\,C_1^2\sin^2\theta)\,\tau.$$

The last inequality holds if C_1 is large enough compared to $L, \|V\|_\infty, \theta$. Lemma 2.3.1 yields

$$u(x - C_1\tau\mu, t+\tau) = 0$$

and therefore for some C

$$(\Omega(u^\tau) \cap Q_{1/2}) \subset (\ C\tau\text{-neighbourhood of } \Omega(u)) \cap Q_1.$$

$\square$

The following proposition strengthens the nondegeneracy in Theorem 2.4.6.

Proposition 2.5.3. *Under the conditions of Lemma 2.5.2, there exist $\delta < \frac{1}{2}$ and $c_1 > 0$ such that*

$$\nabla_\mu u(x,t) \geq c_1 \quad \text{in } Q_\delta \cap \Omega(u).$$

Proof. Fix any $(\hat{x},\hat{t}) \in \{u > 0\} \cap Q_\delta$ for some $\delta \leq \epsilon_0$ small enough to be determined. Let $h := dist(\hat{x}, \Gamma_{\hat{t}}) < \delta$. From Lemma 6.1, $\Gamma(u)$ is Lipschitz continuous. Let us denote the Lipschitz constant as L_1, and choose $C_2 = C_2(L_1) \geq 2$ such that

$$dist(\hat{x}, \Gamma_{\hat{t}-h}) \leq (C_2 - 1)h.$$

Denote (y, s) such that $s = \hat{t} - h, y \in \Gamma_s$ and $dist(\hat{x}, y) = dist(\hat{x}, \Gamma_s)$. Thus $B(y, h) \subset B(\hat{x}, C_2 h)$.

By the fundamental theorem of calculus and (2.4.21),

$$\fint_{B(y,h)\cap\{u>0\}} \nabla_\mu u(x, s)dx \geq \frac{\sigma}{h} \fint_{\partial B(y,h)\cap\{u>0\}} u(x, s)dx \geq \kappa_*'$$

for some $\kappa_*' > 0$ only depending on κ_* and L_1.

Let us define

$$\Omega^r := \{(x, t) \in \Omega, dist((x, t), \partial\Omega) > r\}.$$

Fix $\kappa \in (0, 1)$ to be a small constant only depending on κ_*' such that

$$\fint_{B(y,h)\cap\Omega^{\kappa h}} \nabla_\mu u(x, s)dx \geq \frac{\kappa_*'}{2}.$$

Therefore there exists

$$z \in B(y, h) \cap \Omega^{\kappa h} \subset B(\hat{x}, C_2 h) \cap \Omega^{\kappa h}$$

such that

$$\nabla_\mu u(z, s) \geq \frac{\kappa_*'}{2}. \tag{2.5.3}$$

Differentiating (2.1.4), it follows that $\phi := \nabla_\mu u$ satisfies the following parabolic equation

$$\phi_t = (m - 1)\phi\Delta u + (m - 1)u\Delta\phi + (2\nabla u + V) \cdot \nabla\phi + (m - 1)\phi\nabla \cdot V + f.$$

where

$$f := \nabla u \cdot \nabla_\mu V + (m - 1)u\nabla \cdot \nabla_\mu V.$$

Since u is Lipschitz continuous and V is smooth, f is uniformly bounded. Then

$$\phi' := \phi e^{C_3(t-s)} + \|f\|_\infty(t - s) \quad \text{ with } C_3 = (m - 1)(C_0 + \|\nabla \cdot V\|_\infty)$$

satisfies

$$\phi_t' \geq (m - 1)u\Delta\phi' + (2\nabla u + V) \cdot \nabla\phi'.$$

Let

$$\Sigma := \Omega^{\kappa h} \cap \left(B(0, C_2 h) \times (-h, 0) + (\hat{x}, \hat{t})\right). \tag{2.5.4}$$

For any $(x,t) \in \Sigma \cap Q_1$ which is κh away from Γ, by the cone monotonicity and (2.4.21) we have

$$u \geq c\kappa h \quad \text{for some } c > 0 \text{ independent of } h. \tag{2.5.5}$$

The rescaled version of ϕ, $w(x,t) := \phi'(xh + \hat{x}, th + s)$ satisfies

$$w_t \geq (m-1)\frac{u}{h}\Delta w + (2\nabla u + V) \cdot \nabla w \quad \text{in } \Sigma_h := (\Sigma - (\hat{x}, s))/h. \tag{2.5.6}$$

Since $\frac{u}{h} \geq c\rho > 0$ in Σ_h due to (2.5.5), the operator in (2.5.6) is uniformly parabolic. Let us apply the Harnack inequality to w and write it in terms of ϕ. We find for some $C > 0$,

$$\phi(\hat{x}, \hat{t})e^{C_3(\hat{t}-s)} + \|f\|_\infty(\hat{t} - s) \geq \frac{1}{C}\phi(z,s) \geq \frac{\kappa_*'}{2C},$$

where the last inequality follows from (2.5.3).

Since $\hat{t} - s = h \leq \delta$, further assuming δ to be small enough, we can get $\phi(\hat{x}, \hat{t}) \geq \frac{\kappa_*'}{4C} > 0$. Finally we conclude that $\nabla_\mu u \geq c_1 > 0$ for some $c_1 > 0$ in $\Omega \cap Q_\delta$.

$\square$

Next we show the strict monotonicity of u along the streamlines.

Lemma 2.5.4. *Let u be given as in Proposition 2.5.3. Consider $v(x,t) = u(x + X, t)$ with $X := X(0,0;t)$. Then there exist $\delta < \frac{1}{2}$ and $c_2 > 0$ such that*

$$v_t \geq c_2 \quad \text{in } Q_\delta \cap \{v > 0\}.$$

Proof. By definition, v solves (2.4.2) i.e. $\mathcal{L}_2(v) = 0$. In Q_δ, by (2.5.2), we know $v \leq L\delta$. Using the regularity of V and Lemma 2.5.4, we have

$$\begin{aligned}
\partial_t v &\geq -C_0(m-1)v + \frac{1}{2}|\nabla v|^2 - 4|V(x+X) - V(X)|^2 - (m-1)v\|\nabla V\|_\infty \\
&\geq -\sigma C_0\delta + \frac{c_1^2}{2} - 4|x|^2\|\nabla V\|_\infty^2 - \sigma\delta \\
&\geq -\sigma C_0\delta + \frac{c_1^2}{2} - \sigma\delta^2 - \sigma L\delta
\end{aligned}$$

for $(x,t) \in Q_\delta$, which is positive if δ is small enough compared to C_0, c_1, L and universal constants. $\square$

Now we are ready to follow the iteration procedure given in [18]. Their argument, by now classical, describes the enlargement of directional monotonicity for the rescaled solutions as we zoom in near a free boundary point. More precise discussions are below.

Recall $(0,0) \in \Gamma$. For $\delta > 0$, consider the rescaled v and its corresponding drift and streamline

$$v_\delta(x,t) := \frac{1}{\delta} v(\delta x, \delta t), \quad V_\delta(x,t) := V(\delta x, \delta t), \quad X_\delta := X(\delta t), \tag{2.5.7}$$

and let us reset v, V, X as $v_\delta, V_\delta, X_\delta$. Then there exists a universal σ that

$$\|V\|_\infty \le \sigma, \quad \|\nabla V\|_\infty + \|V_t\|_\infty \le \sigma\delta, \quad \|D^2 V\|_\infty + \|\nabla \partial_t V\|_\infty \le \sigma\delta^2. \tag{2.5.8}$$

From Lemmas 2.5.2 - 2.5.4, for sufficiently small $\delta > 0$ we have

$$0 \le v \le L, \quad 1/L \le |\nabla v|, \, \nabla_\mu v, \, v_t \le L, \quad \Delta v \ge -L\delta \quad \text{in } Q_1 \tag{2.5.9}$$

Now we introduce some notations. $\hat{\nabla} := (\nabla, \partial_t)$. We denote $f_i := \partial_{x_i} f$, $f_{ij} := \partial^2_{x_i x_j} f$. Let ν, μ be two vectors in $\mathbb{R}^{d+1}$ or $\mathbb{R}^d$. We denote the angle between them by

$$\langle \nu, \mu \rangle := \arccos\left(\frac{\nu \cdot \mu}{|\nu||\mu|}\right) \in [0, \pi].$$

Let $\mu \in \mathbb{R}^d$ and $\theta \in [0, \pi/2]$. Let $\nu \in \mathbb{R}^{d+1}$, define the space-time cone $\widehat{W}_{\theta,\nu}$ for $\theta \in [0, \pi/2]$ as

$$\widehat{W}_{\theta,\nu} := \{p \in \mathbb{R}^{d+1}, \quad \langle p, \nu \rangle \le \theta\}. \tag{2.5.10}$$

We say v has the *cone of monotonicity* $\widehat{W}_{\theta,\nu}$ if

$$\hat{\nabla}_p v \ge 0 \quad \text{in } Q_1 \text{ for all } p \in \widehat{W}_{\theta,\nu}.$$

The following lemma, yielding the initial cone of monotonicity for v, can be proven parallel to the Proposition 2.1 of [18]. Let us write the positive time direction as e_{d+1}.

Lemma 2.5.5. *Let v solve* (2.4.2), *and assume* (2.5.8)- (2.5.9) . *Let* $\mu_0 := (1/\sqrt{2})(\mu, 0) + (1/\sqrt{2})e_{d+1}$. *Then there exists* $\theta_0 > 0$ *such that*

$$\hat{\nabla}_p v \ge \frac{1}{2L} \quad \textit{in } Q_1 \quad \textit{for all unit } p \in \widehat{W}_{\theta_0,\mu_0}.$$

Now we begin our iteration procedure. Fix some $J(L) \in (0,1)$ to be chosen later, and define

$$v_k(x,t) := \frac{1}{J^k} v(J^k x, J^k t) \quad \text{for } k \in \mathbb{N}. \tag{2.5.11}$$

Then v_k satisfies equation (2.4.2) with $V(x,t)$ replaced by $\vec{b}_k := V(J^k x, J^k t)$. Let us write $X_k(t)$ to be the streamline generated by $\vec{b}_k$ starting at $(0,0)$. We have $X_k := X_k(0,0,t) = \frac{1}{J^k} X(0,0;J^k t)$.

Due to (2.5.8) - (2.5.9) we have in Q_1

(A_k) $0 \leq v_k \leq L$, $\Delta v_k \geq -L\delta$, $|\nabla v_k| + |\partial_t v_k| \leq L$;

(B_k) $\nabla_\mu v_k, \partial_t v_k \geq \frac{1}{L}$;

(C_k) $\|\vec{b}_k\|_\infty \leq \sigma$, $\|\nabla \vec{b}_k\|_\infty + \|\partial_t \vec{b}_k\|_\infty \leq \sigma\delta J^k$, $\|D^2 \vec{b}_k\|_\infty + \|\nabla\partial_t \vec{b}_k\|_\infty \leq \sigma\delta^2 J^{2k}$.

In [18], they show inductively that the cone of monotonicity $\widehat{W}_{\theta_k,\mu_k}$ for v_k has strictly increasing θ_k, converging to $\pi/2$ as $k \to \infty$. This and the rate of increasing angles leads to the $C^{1,\alpha}$ regularity of the free boundary.

However for us the competition with drift term requires a stronger inductive property than the cone of monotonicity, see the remark below Lemma 2.5.8. We make an extra observation that follows from the enlargement of cones as well as the nondegeneracy of the solution:

(D_k) There exist $\mu_k, \theta_k \geq \theta_0$ such that $\hat{\nabla}_p v_k \geq c_* J^k$ for all unit $p \in \widehat{W}_{\theta_k,\mu_k}$ in Q_1.

We will proceed with several lemmas that leads to the enlargement of cones in Proposition 2.5.10. The proofs of the lemmas will be postponed until after the Proposition.

For simplicity of notations, we write $v := v_k, \vec{b} := \vec{b}_k, X = X_k$. First we show that some improvements on monotonicity can be obtained on the set $\{v = \epsilon\}$. And this is one place we need to use (D_k).

Lemma 2.5.6. *[Enlargement of Cones] Suppose $(A_k) - (D_k)$ holds for v. For any $\epsilon \in (0,1)$, there exist $r \leq \frac{1}{10}, \delta_0, C$ only depending on ϵ, L such that for any $\gamma \in (0,\epsilon)$, $\delta \in (0,\delta_0)$, unit*

$p \in \widehat{W}_{\theta_k,\mu_k} \cap \mathcal{S}^d$ *and* $\tau := C\epsilon^{-1}\cos\langle\, p, \hat{\nabla} v(\mu, -2r)\,\rangle$, *we have*

$$v((x,t)+\gamma p) \geq (1+\tau\gamma)v \ \ on \ \ (B_{\frac{3}{4}} \times (-2r, 2r)) \cap \{v = \epsilon\}$$

$$and \quad v \leq \epsilon \ in \ Q_{2r}.$$

Next we show that this improvement can propagate to the zero level set of v.

Lemma 2.5.7. *Suppose v is a solution of* (2.4.2) *and* $(A_k) - (D_k)$ *hold for v, and w is a supersolution of* (2.4.2). *Let $\delta(\epsilon), r(\epsilon), \tau(\epsilon)$ be as given in Lemma 2.5.6. Suppose that $w \geq v$ and $w \geq (1+\tau\gamma)v$ in $(B_{\frac{1}{2}} \times (-2r, 2r)) \cap \{v = \epsilon\}$. Then, if ϵ is small enough (independently of r, δ, τ),*

$$w \geq (1+\tau\gamma)v \ in \ (B_{\frac{1}{4}} \times (-2r, 2r)) \cap \{v \leq \epsilon\}.$$

Lastly we further improve the monotonicity in a smaller domain.

Lemma 2.5.8. *Let v, w, τ be as in Lemma 2.5.7. Consider a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^+$ such that ϕ is supported in B_{2r} and $\phi, |\nabla\phi|, |D^2\phi| \leq \kappa\tau\gamma$ for a sufficiently small $\kappa > 0$. If $v \leq \epsilon$ in Q_{2r} then we have*

$$w(x,t) \geq v(x + (t+2r)\phi(x)\mu, t) \ in \ Q_{2r}.$$

Remark 2.5.9. Above lemma was shown for the zero drift case in [18] based on the invariance of (PME) under coordinate translations. This invariance does not hold for us, and thus we modify the iterative arguments as follows. In each step we construct a barrier of the form $w(x,t) = v((x,t)+p) + e(t)$ for some $e > 0$. The last inductive property (D_k) ensures that this extra term $e(t)$ can be chosen small enough at each iteration, to derive the improvement of cone monotonicity up to the free boundary.

Now we give the main proposition.

Proposition 2.5.10. *Let v be a solution to* (2.4.2). *Suppose* $(0,0) \in \Gamma$ *and* (2.5.8)- (2.5.9). *Then there exist $J, S \in (0,1)$ and a monotone family of cones $\widehat{W}_{\theta_k,\mu_k}$ with $\theta_k = \theta_{k-1} + S(\frac{1}{2}\pi - \theta_{k-1})$ such that*

$$\hat{\nabla}_p v(x,t) \geq (2L)^{-1} J^k \ in \ Q_{J^k} \quad for\ any\ unit\ p \in \widehat{W}_{\theta_k,\mu_k}.$$

Proof. Let ϵ, r, δ_0 be as given in Lemmas 2.5.6-2.5.7, which only depend on L and universal constants. For some $\delta < \delta_0$, we reset $v, \vec{b}, X$ as $v_\delta, \vec{b}_\delta, X_\delta$. Next let v_k be as in (2.5.11) and similarly as before set $\vec{b}_k, X_k$. We will take some $J \le r$ to be determined. It is straightforward that for all $k \ge 0$, $(A_k)-(C_k)$ hold. When $k=0$, due to Lemma 2.5.5, (D_0) holds for $v = v_0$.

Let us suppose that (D_k) holds for some $k \ge 0$ with $\mu_k, \theta_k \ge \theta_0$ i.e. the hypothesis of Lemmas 2.5.6- 2.5.8 are satisfied. We will show (D_{k+1}).

For any $\gamma \in (0, \epsilon)$ and a unit vector $p \in \widehat{W}_{\theta_k, \mu_k}$, define

$$\widetilde{w}(x,t) := v_k((x,t) + \gamma p)$$

which satisfies $\mathcal{L}_2(\widetilde{w}) \ge -E\gamma$ in Q_1 where E is an upper bound of

$$|\nabla \widetilde{w}||\hat{\nabla}_p \vec{b}_k(x + X_k)| + (m-1)\widetilde{w}|\nabla \cdot \hat{\nabla}_p \vec{b}_k|.$$

By the condition $(A_k)(C_k)$ and the fact that $|\partial_t X_k| \le \sigma$, we can set

$$E = \sigma L \delta J^k.$$

Thus by Lemma 2.5.6, $w := \widetilde{w} + E(t+2r)\gamma$ satisfies the hypothesis of Lemmas 2.5.7-2.5.8.

According the lemmas, we can select $r \in (0, \frac{1}{10})$ only depending on L, σ. Next take one ϕ satisfying the condition of Lemma 2.5.8 and we assume

$$\phi \ge \sigma r^2 \kappa \tau \gamma \quad \text{ in } B_r \text{ for some universal } \sigma. \tag{2.5.12}$$

By the lemmas, we find that with $c' := \sigma r^3 \kappa / L$ and in Q_r

$$\begin{aligned} w(x,t) &\ge v_k(x + (t+2r)\phi(x)\mu, t) \\ &\ge v_k(x,t) + \frac{t+2r}{L}\phi(x) \qquad (\text{ by } (B_k)) \\ &\ge v_k(x,t) + c'\tau\gamma. \qquad (\text{ by } (2.5.12)) \end{aligned} \tag{2.5.13}$$

By the definition of τ in Lemma 2.5.6, we obtain in $Q_r \cap \{v_k > 0\}$

$$\begin{aligned}
\hat{\nabla}_p v_k(x,t) &= \lim_{\gamma\to 0} \frac{v_k((x,t)+\gamma p) - v_k(x,t)}{\gamma} \\
&\geq \lim_{\gamma\to 0} \frac{w(x,t) - v_k(x,t)}{\gamma} - 3Er \\
&\geq c'\tau - \sigma L\delta J^k r \qquad\qquad (\text{ by (2.5.13)}) \\
&= C_2 \cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle - \sigma L\delta J^k r
\end{aligned} \tag{2.5.14}$$

where $C_1 := c'C\epsilon^{-1}$ which only depends on L, σ. From (A_k) and (D_k)

$$\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle = \frac{\hat{\nabla}_p v_k}{|\hat{\nabla} v_k|}(\mu,-2r) \geq \frac{1}{L}\hat{\nabla}_p v_k(\mu,-2r) \geq \frac{1}{2L^2}J^k. \tag{2.5.15}$$

Taking δ to be small enough only depending on L and σ, (2.5.14) yields

$$\hat{\nabla}_p v_k(x,t) \geq \frac{C_1}{2}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle \quad \text{in } Q_r \cap \{v_k > 0\}.$$

Thus in the same region

$$\cos\langle\, p, \hat{\nabla} v_k(x,t)\rangle = \frac{\hat{\nabla}_p v_k}{|\hat{\nabla} v_k|}(x,t) \geq \frac{C_1}{2L}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle. \tag{2.5.16}$$

For $p \in \mathcal{S}^{d+1}$, set

$$\rho(p) := \frac{C_1}{4L}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle.$$

For any $q \in B(p,\rho(p))$ we have $\sin\langle\, p, q\,\rangle \leq \rho(p)$ and thus

$$\begin{aligned}
\cos\langle\, q, \hat{\nabla} v_k(x,t)\rangle &\geq \cos\langle\, p, \hat{\nabla} v_k(x,t)\rangle - \sin\langle\, p, q\,\rangle \\
&\geq \frac{C_1}{2L}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle - \rho(p) \quad (\text{ by (2.5.16)}) \\
&= \frac{C_1}{4L}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle.
\end{aligned} \tag{2.5.17}$$

It follows that

$$\begin{aligned}
\hat{\nabla}_q v_k(x,t) &\geq \frac{C_1}{4L^2}\cos\langle\, p, \hat{\nabla} v_k(\mu,-2r)\rangle \quad (\text{ by } (A_k) \text{ and (2.5.17)}) \\
&\geq \frac{C_1}{8L^4}J^k \quad (\text{ by (2.5.15)}).
\end{aligned} \tag{2.5.18}$$

Since (2.5.18) holds for all $q \in B(p,\rho(p))$, there exists a larger cone $\widehat{W}_{\theta_{k+1},\mu_{k+1}}$ for some $\mu_{k+1} \in \mathbb{R}^{d+1}$, $S \in (0,1)$ and $\theta_{k+1} = \theta_k + S(\frac{1}{2}\pi - \theta_k)$ such that

$$\hat{\nabla}_p v_k(x,t) \geq \frac{C_1}{8L^4}J^k \qquad \text{for all unit vector } p \in \widehat{W}_{\theta_{k+1},\mu_{k+1}} \text{ and } (x,t) \in Q_r.$$

Here S can be chosen in a way that it is independent of θ_k for all $\theta_k \geq \theta_0$. We refer readers to Proposition 2.5 in [18] and [15] for more details.

Let $J := \min\{C_1/(4L^3), r\}$. Recalling $v_{k+1}(x,t) = \frac{1}{J} v_k(Jx, Jt)$, we obtain for all unit $p \in \widehat{W}_{\theta_{k+1}, \mu_{k+1}}$

$$\hat{\nabla}_p v_{k+1}(x,t) = \hat{\nabla}_p v_k \geq \frac{C_1}{8L^4} J^k \geq c_* J^{k+1} \text{ in } Q_1.$$

We checked (D_{k+1}) and therefore by induction we conclude the proof of the theorem.

$\square$

2.6 Discussion of Traveling Waves and Potential Singularities

In this section we discuss evolution of solutions in two space dimensions, in several explicit scenario.

2.6.1 A Discussion on Traveling Waves

For simplicity, we restrict to two space dimensions $d = 2$. The drift is chosen as

$$\vec{b}(x_1, x_2) := (\alpha(x_2), 0), \text{ where } \alpha \text{ is Lipschitz and bounded.} \tag{2.6.1}$$

Our regularity analysis cannot address the traveling waves themselves, but we are able to say that such singularity, if at all, is of asymptotic nature. More precisely we show that dynamic solutions, used in [57] to approximate the travelling waves, stay smooth in any finite time interval.

Theorem 2.6.1. *Let u solve (2.1.4) in $\mathbb{R}^2 \times (0, \infty)$ with $\vec{b}$ given in (2.6.1) with the initial data $u_0(x) = (x_1)_+$. To find a unique solution we impose the growth at infinity $\frac{u(x,t)}{x_1} \to 1$ as $x_1 \to \infty$.*

For u given as above, the following holds:

(a) u is uniformly Lipschitz continuous in $\mathbb{R}^2 \times [0, \infty)$.

(b) For any fixed $T > 0$, there exists $\tau_0(T) > 0$ such that for all $t \in [0, T]$ and $\tau \leq \tau_0$

$$\partial_{x_1} u \pm \tau \partial_{x_2} u \geq 0.$$

(c) u is nondegenerate, and $\Gamma(u)$ is $C^{1,\alpha}$ in $\mathbb{R}^2 \times [0, T]$.

Proof. Let us rewrite (2.1.4) with our choice of $\vec{b}$:

$$\partial_t u - (m-1)u\,\Delta u - |\nabla u|^2 - \alpha(x_2)\,\partial_{x_1} u = 0. \tag{2.6.2}$$

Let $\varphi(x,t) := (x_1 + \sigma_1 t)_+$ with $\sigma_1 := \sup|\alpha| + 1$. Then φ_1 is a supersolution of (2.6.2) with the same initial data as u, and thus $u \leq \varphi$. In particular, for any $\epsilon > 0$

$$u(x - \sigma_1\epsilon e_1, \epsilon) \leq \varphi(x - \sigma_1\epsilon e_1, \epsilon) = (x_1)_+ = u(x, 0), \tag{2.6.3}$$

where we denote the positive x_1 direction as e_1.

Let $u^\epsilon(x,t) := u(x - \sigma_1\epsilon e_1, t + \epsilon)$ for $\epsilon > 0$. From (2.6.3), it follows that $u^\epsilon(\cdot, 0) \leq u_0$. Since u^ϵ also solves (2.6.2), by comparison principle it follows that $u^\epsilon \leq u$, and thus

$$u_t - \sigma_1 u_{x_1} \leq 0. \tag{2.6.4}$$

Above inequality with (2.5.1) yields that u is uniformly Lipschitz continuous in space and time.

Next to show (b), let us define, for $\epsilon > 0$ and $\sigma_2 = \sup|\partial_{x_2}\vec{b}|$,

$$w(x,t) := \sup_{|y| \leq \epsilon e^{-\sigma_2 t}} u(x + y - \epsilon e_1, t).$$

As done in Lemma 2.4.4, for some $y \in B(y, \epsilon e^{-\sigma_2 t})$

$$w_t(x,t) = (u_t - \sigma_2\epsilon e^{-\sigma_2 t}|\nabla u|)(y,t).$$

Therefore

$$\begin{aligned}
&w_t - (m-1)w\Delta w - |\nabla w|^2 - \nabla w \cdot \vec{b} - (m-1)w\nabla\cdot\vec{b} \\
&\leq -\sigma_2\epsilon e^{-\sigma_2 t}|\nabla w| + |\nabla w| \sup_{\vec{y} \in B(x, \epsilon e^{-\sigma_2 t})} |\vec{b}(y - \epsilon e_1) - \vec{b}(x)| \\
&\leq (-\sigma_2\epsilon e^{-\sigma_2 t} + \epsilon e^{-\sigma_2 t}\|\alpha'\|_\infty)|\nabla w| \leq 0,
\end{aligned}$$

where in the second equality above, we used the fact that $\vec{b}$ only depends on x_2. Since $w(x,0) \leq u_0$, again comparison principle for (2.6.2) yields $w \leq u$. By this ordering, for $t \leq T$

$$u(x,t) \geq \sup_{|y| \leq \epsilon e^{-\sigma_2 T}} u(x + y - \epsilon e_1, t)$$

which leads to part (b) with $\tau \leq \tan(arcsin(e^{-\sigma_2 T}))$. Since (a)-(b) implies (2.1.9) and that u is cone monotone, Proposition 2.5.3 and Theorem 2.5.1 yields (c).

□

Before stating more examples, we need the following technical lemma which is used for comparison.

Lemma 2.6.2. *For some $R > 0$, set $U := B_R$ or $\mathbb{R}^d$. Let ψ be a non-negative continuous function defined in $U \times [0,T]$ such that*

(a) ψ is smooth in its positive set and in the set $\psi_t - \Delta\psi^m - \nabla \cdot (\vec{b}\,\psi) \geq 0$,

(b) ψ^α is Lipschitz continuous for some $\alpha \in (0,m)$,

(c) $\Gamma(\psi)$ has Hausdorff dimension $d-1$.

Then

$$\psi_t - \Delta\psi^m - \nabla \cdot (\vec{b}\,\psi) \geq 0 \text{ in } U \times [0,T]$$

in the weak sense i.e. for all non-negative $\phi \in C_c^\infty(U \times [0,T))$

$$\int_0^T \int_{\mathbb{R}^d} \psi\, \phi_t dx dt \leq \int_{\mathbb{R}^d} \psi(0,x)\phi(0,x)dx + \int_0^T \int_{\mathbb{R}^d} (\nabla\psi^m + \psi\,\vec{b})\nabla\phi\; dx dt. \qquad (2.6.5)$$

We postpone the proof to the appendix.

Theorem 2.6.3. *There exist solutions u_1, u_2 to (2.1.4) in Q with bounded smooth spatial vector fields and non-negative, bounded and smooth (in its positive set) initial data such that the following can happen.*

1. u_1 is stationary and there is a corner on $\Gamma_0(u_1)$.

2. For a finite time, there is a corner of shrinking angles on $\Gamma_t(u_2)$.

Proof. Write (x, y) as the space coordinate. Let

$$\vec{b} := -\nabla\Phi(x, y) \text{ for some smooth function } \Phi,$$

and then it can be checked directly that

$$u_1 := \max\{\Phi, 0\}$$

is a stationary solution to (2.1.4). Notice $\Gamma_0(u_1)$ is the 0-level set of Φ and we claim that if Φ is degenerate, the interface can be non-smooth.

For example, we can take

$$\Phi(x, y) = g(x)g(y)$$

where g is a function on $\mathbb{R}$ that it is only positive in $(0, 1)$. Then $\partial\{u_1 > 0\}$ is a square. In particular, $\partial\{u_1 > 0\}$ contains a Lipschitz corner at the origin.

Next we show (2). Take $\vec{b} := (ax, by)$ (for a moment) and

$$\varphi(x, y, t) := \begin{cases} \lambda(t)(x^2 - k(t)y^2)_+ & \text{if } x > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\lambda(t) = e^{\sigma_1 t}, k(t) = k_0 e^t \text{ for some } \sigma_1, k_0 > 0.$$

Then the $\Gamma_t(\varphi)$ contains a corner with vertex at the origin.

Let us show that φ is a supersolution to (1.1.1) for $t \in (0, 1/\sigma_1)$. Due to Lemma 2.6.2, we only need to check this for $x > k^{1/2}|y|$.

$$\begin{aligned}
\mathcal{L}\varphi &:= \varphi_t - (m-1)\varphi\Delta\varphi - |\nabla\varphi|^2 - \nabla\varphi\cdot\vec{b} - (m-1)\varphi\nabla\cdot\vec{b} \\
&= (x^2 - ky^2)\lambda' - \lambda k' y^2 - (m-1)\lambda^2(x^2 - ky^2)(2 - 2k) - 4\lambda^2 x^2 - 4\lambda^2 k^2 y^2 \\
&- 2a\lambda x^2 + 2bk\lambda y^2 - (m-1)\lambda(x^2 - ky^2)(a + b) \\
&= (x^2 - ky^2)(\lambda' - \lambda^2(m-1)(2 - 2k) - \lambda(m-1)(a + b) - 2a - 4\lambda^2) \\
&\quad + \lambda y^2(2bk - k' - 4\lambda k - 4\lambda k^2 - 2ak) \\
&\geq (x^2 - ky^2)\lambda\left(\sigma_1 - \sigma(\lambda, m, k_0, a, b)\right) + \lambda y^2 k((2b - 1) - (4\lambda + 4\lambda k + 2a)).
\end{aligned} \tag{2.6.6}$$

Now we fix a and take b such that

$$2b - 1 \geq 4\lambda + 8\lambda k_0 + 2a \geq 4\lambda + 4\lambda k(t) + 2a,$$

if $\sigma_1 \geq 10$ and $t \leq 1/\sigma_1$. Next we further take σ_1 to be large enough such that, the first part of (2.6.6) is also non-negative. We conclude that for $t \in (0, 1/\sigma_1)$, φ is indeed a supersolution and its support contains a corner with angles shrinking from $2\arctan(k_0^{-\frac{1}{2}})$ to $2\arctan(k(t)^{-\frac{1}{2}})$.

Now consider a solution u_2 with initial data u_0 such that $u_0 = \varphi(x, y, 0)$ in B_1 and $u_0 \leq \varphi(x, y, 0)$. By comparison, $\varphi \geq u_2$ for all times and so

$$\Omega_t(u_2) \subset \Omega_t(\varphi) \subset \{x > k^{1/2}(t)|y|\}.$$

Since $\vec{b} = 0$ at the origin, the origin is a one-point streamline. By Lemma 2.2.3, $0 \in \overline{\Omega_t(u_2)}$ for all $t \geq 0$. Thus $\Gamma_t(u_2)$ has a shrinking corner for a short time. Lastly since u_2 is compactly supported, we can truncate $\vec{b}$ to be bounded which does not affect u_2 and its support.

$\square$

CHAPTER 3

Vanishing Viscosity Limit

3.1 Introduction

In this chapter, we consider a more general equation than (1.1.1):

$$\begin{cases} \dfrac{\partial}{\partial t}\mu - \epsilon\Delta\mu - \nabla\cdot(\mu(\nabla V + \nabla W * \mu)) = 0 & \text{in } \Omega\times[0,T], \\ (\epsilon\nabla\mu + \mu\nabla V + \mu\nabla W * \mu)\cdot n = 0 & \text{on } (\partial\Omega)\times[0,T] \\ \mu(x,0) = \mu_0(x) & \text{on } \Omega \end{cases} \tag{3.1.1}$$

where $\Omega \subset \mathbb{R}^d$, n is the outer normal direction of the boundary and $\mu(\cdot,t)$ is a probability measures supported in Ω.

The system describes the density of moving particles which are confined to some region and flow with a velocity field

$$v := -(\epsilon\nabla\mu/\mu + \nabla V + \nabla W * \mu)$$

inside of the domain. One part of the velocity field is generated from interactions between different particles represented by the interaction potential W, given by $(\nabla W * \mu)$. This type of problem arises in many applications with various interaction kernel W, such as in swarming models with $W(x) = -Ce^{-|x|}, W(x) = -Ce^{-|x|^2}$ and in models of chemotaxis with $W(x) = \frac{1}{2\pi}\log|x|$, see [26,27] for more references. At the same time, the particles are subject to an external potential $V(x)$. For the diffusion term, the model takes into account random movements of the particles.

The goal is to study the asymptotic behaviour of solutions as $\epsilon \to 0$. If simply replacing ϵ by 0 in (3.1.1), the equation becomes a first order equation. the solution of the first order equation can be very rough (for example the sum of delta masses), it does not make sense to assume Neumann or Dirichlet boundary condition. We find one candidate is the model proposed by Carrillo, Slepcev and Wu [27,72] where they considered the case with no random movements. In the model, in order to characterize the boundary behaviour, they define the following projection operator $P_x : \mathbb{R}^d \to \mathbb{R}^d$ as follows

$$P_x(v) = \begin{cases} v & \text{if } v \cdot n \leq 0, \\ v - (v \cdot n)n & \text{if } v \cdot n > 0. \end{cases} \tag{3.1.2}$$

The equation is formulated as:

$$\begin{cases} \dfrac{\partial}{\partial t}\mu(x,t) + \nabla \cdot \left(\mu P_x(-\nabla V - \nabla W * \mu)\right)(x,t) = 0 & \text{in } \overline{\Omega}_T, \\ \mu(x,0) = \mu_0(x) & \text{on } \overline{\Omega}. \end{cases} \tag{3.1.3}$$

Wellposedness of (3.1.3) is given in [27] and the solutions are shown to be both gradient flow solutions ([2]) and weak solutions. The generalization of such results to time-dependent domain can be found in [75].

We are going to show that (3.1.3) can be indeed obtained as the limit as $\epsilon \to 0$ of the diffusion equation (3.1.1), imposing the additional condition that the domain is bounded and spatially convex. This result is significant since it provides a natural justification for the first-order system (3.1.3).

This chapter belongs to the second part of the [75]. In the original paper [75], both the equations (3.1.1), (3.1.3) are studied in the space-time domain, possibly unbounded with some minor requirements on the regularity of the boundary. The main Theorem 3.3.1 is proved allowing the domain to be time-dependent ($\Omega_T = \cup_t(\Omega(t) \times \{t\})$) as long as each time slice $\Omega(t)$ is bounded and convex.

3.2 Wasserstein Gradient Flow

Let us always assume the following two assumptions in this chapter.

(O) $\Omega \subset \mathbb{R}^d$ is bounded and convex.

(C) $V(\cdot), W(\cdot) \in C^2(\mathbb{R}^d)$.

The following discussions are mainly from [2, 33].

3.2.1 Wasserstein Distance

Given a probability measure μ, we write $m_2(\mu) = \int_{\mathbb{R}^d} |x|^2 d\mu$ as the second moment of μ. The set of all probability measures on $\overline{\Omega}$ with finite second moment is denoted by $\mathcal{P}_2(\overline{\Omega})$. The set of absolutely continuous (with respect to Lebesgue measure) probability measures with finite second moment is written as $\mathcal{P}_2^a$. For $\mu \in \mathcal{P}_2^a$, we usually write $\mu = u\mathcal{L}^d$ where u is its density. For probability measures supported in $\overline{\Omega}$, we will think of them as measures in $\mathbb{R}^d$, extended by 0 outside $\overline{\Omega}$.

Now we discuss the Wasserstein metric and we refer readers to [2] for details. Suppose X, Y are measurable subsets of $\mathbb{R}^d$ and $\mu_1 \in \mathcal{P}_2(X), \mu_2 \in \mathcal{P}_2(Y)$. A plan between μ_1, μ_2 is any Borel measure γ on $X \times Y$ which has μ_1 as its first marginal and μ_2 as its second marginal. We write $\gamma \in \Gamma(\mu_1, \mu_2)$. It has been shown that there exists an optimal transport plan $\gamma \in \Gamma(\mu_1, \mu_2)$ such that

$$\int_{X\times Y} |x-y|^2 d\gamma(x,y) = \min\left\{\int_{X\times Y} |x-y|^2 d\gamma'(x,y), \gamma' \in \Gamma(\mu_1,\mu_2)\right\}.$$

The above quantity is defined to be the *2-Wasserstein distance* between μ_1, μ_2 (the Kantorovich's formulation). Throughout this chapter we use this distance for probability measures with notation $d_W(\cdot,\cdot)$ unless otherwise stated. And later by Wasserstein distance (metric) we mean 2-Wasserstein distance (metric). We denote the set of optimal transport plans between μ_1 and μ_2 by $\Gamma_0(\mu_1, \mu_2)$.

Let $\mu_2 \in \mathcal{P}_2(Y)$, a measurable function $\mathbf{t} : Y \to X$ transports μ_2 onto $\mu_1 \in \mathcal{P}_2(X)$ if $\mu_1(B) = \mu_2(\mathbf{t}^{-1}(B))$ for all measurable $B \subseteq X$, and we write $\mu_1 = \mathbf{t}_{\#}\mu_2$. If $\mu_2 \in \mathcal{P}_2^a(Y)$, then for any $\mu_1 \in \mathcal{P}_2(X)$ there is an optimal transport map $\mathbf{t}_{\mu_2}^{\mu_1} : Y \to X$ such that $\mathbf{t}_{\mu_2\,\#}^{\mu_1}\mu_2 = \mu_1$ (with reference to [56]). And we have, in Monge's formulation,

$$d_W^2(\mu_1,\mu_2) = \int_Y |\mathbf{t}_{\mu_2}^{\mu_1}(x) - x|^2 d\mu_2(x).$$

Given $\mu_1, \mu_2 \in \mathcal{P}_2(X), \mu \in \mathcal{P}_2^a(X)$. Let $\mathbf{t}_\mu^{\mu_1}, \mathbf{t}_\mu^{\mu_2}$ be an optimal transport maps from μ to μ_1 and μ_2 respectively. Then the *Pseudo-Wasserstein distance* with base μ is defined as

$$d_\mu^2(\mu_1, \mu_2) = \int_X |\mathbf{t}_\mu^{\mu_1} - \mathbf{t}_\mu^{\mu_2}|^2 d\mu.$$

By Proposition 1.15 [33], d_μ is a metric on

$$\mathcal{P}_\mu(X) := \{\mu' \in \text{probability measures on } X, d_W(\mu, \mu') < +\infty\}.$$

And we have for any μ, $d_W(\cdot, \cdot) \le d_\mu(\cdot, \cdot)$.

3.2.2 Gradient Flow Structure

Let us define the energy function $\phi^\epsilon(\mu)$ associated to (3.1.1) as

$$\begin{aligned} \phi^\epsilon(\mu) &= \mathcal{U}^\epsilon(\mu) + \mathcal{V}(\mu) + \mathcal{W}(\mu) \qquad (3.2.1) \\ &:= \epsilon \int_{\overline{\Omega}} u \log u dx + \int_{\overline{\Omega}} V(x) d\mu(x) + \frac{1}{2} \int_{\overline{\Omega}^2} W(x-y) d\mu(y) d\mu(x) \end{aligned}$$

and the energy associated to (3.1.3) as $\phi(\mu) := \phi^0(\mu)$.

In $\mathcal{U}^\epsilon$ term, u is the probability density function of μ if μ is absolutely continuous with respect to Euclidean measure. We set $\phi^\epsilon(\mu) = \infty$ if μ is not absolutely continuous. We will discuss the gradient flow structure later.

Define *the proper domain of functional* ϕ^ϵ is

$$Dom(\phi^\epsilon) := \left\{\mu \in \mathcal{P}_2(\overline{\Omega}),\ \phi^\epsilon(\mu) < +\infty\right\}.$$

Notice there is no difference between $\mu \in Dom(\phi^1)$ and $\mu \in Dom(\phi^\epsilon)$ for some $\epsilon > 0$. Next as a convention,

$$\frac{\nabla u}{u} := \begin{cases} \frac{\nabla u}{u} & \text{if } u \neq 0, \\ 0 & \text{if } \nabla u = 0, \\ +\infty & \text{if } \nabla u \neq 0, u = 0. \end{cases}$$

Definition 3.2.1. *(absolutely continuous curve). Given an interval $I \subset \mathbb{R}$. $\mu(\cdot) : I \to \mathcal{P}_2(\overline{\Omega})$ is absolutely continnous if there exists $m \in L^1(I)$ such that*

$$d_W(\mu(t), \mu(s)) \le \int_s^t m(r)dr \text{ for all } s, t \in I, s < t.$$

According to Theorem 8.3.1 [2], for any absolutely continuous $\mu(t)$, there exists a Borel vector field $v(x,t)$ such that

$$\partial_t \mu + \nabla \cdot (\mu v) = 0$$

holds in duality with $C_0^\infty(\mathbb{R}^d \times I)$. We call v the *velocity field* of μ.

Definition 3.2.2. *(subdifferential). Given $\phi^\epsilon : \mathcal{P}_2 \to \mathbb{R} \cup \{+\infty\}$ with $\epsilon \ge 0$ as the above. ξ belongs to the subdifferential of ϕ^ϵ at $\mu \in Dom(\phi^\epsilon)$ if for all $w \in \mathcal{P}_2(\overline{\Omega})$*

$$\phi^\epsilon(w) \ge \phi^\epsilon(\mu) + \int_{\overline{\Omega}} \langle \xi, \boldsymbol{t}_\mu^w - i \rangle d\mu + o(d_W(w, \mu)).$$

We write $\xi \in \partial \phi^\epsilon(\mu)$.

Definition 3.2.3. *We say that an absolutely continuous curve $\mu(t)$ is a gradient flow solution to* (3.1.1) *if*

$$\begin{aligned} v &= -(\epsilon \nabla u/u + \nabla V + \nabla W * \mu) && \text{when } \epsilon > 0, \\ v &= P_x(-(\nabla V + \nabla W * \mu)) && \text{when } \epsilon = 0, \end{aligned}$$

belongs to the velocity field of $\mu(t)$ and

$$-v(\cdot, t) \in \partial(\phi^\epsilon(\mu(t))) \text{ for } L^1\text{-a.e.} t > 0.$$

When $\epsilon = 0$, the existence and uniqueness of the gradient flow solution to (3.1.3) are solved in [27]. When $\epsilon > 0$, the well-posedness of gradient flow solution to (3.1.1) is by-now standard by the celebrated JKO scheme.

3.2.3 JKO Scheme

Without loss of generality, we consider the case when $\epsilon = 1$:

$$\begin{cases} \dfrac{\partial}{\partial t}\mu - \nabla \cdot (\nabla \mu + \nabla V \mu + (\nabla W * \mu)\mu) = 0 & \text{in } \Omega_T, \\ (\nabla \mu + \nabla V \mu + (\nabla W * \mu)\mu + c\mu) \cdot n = 0 & \text{on } \partial_l \Omega_T, \\ \mu = \mu_0 & \text{on } \Omega. \end{cases} \tag{3.2.2}$$

Suppose $\mu_0 \in Dom(\phi^1)$ and conditions (C)(O) hold. Fix a small time step $\tau > 0$, define $J_\tau : \mathcal{P}_2^a(\Omega) \to \mathcal{P}_2^a(\Omega)$ by

$$J_\tau(\mu) \in argmin_{v \in \mathcal{P}_2(\Omega)} \left\{ \frac{1}{2\tau} d_W^2(\mu, v) + \phi^1(v) \right\}. \tag{3.2.3}$$

First we show the existence of such minimizers. With the assumptions (C) on V, W, we have ϕ^1 is lower semi-continuous, coercive, compact. Then

$$\inf_{v \in \mathcal{P}_2(\Omega)} \left\{ \frac{1}{2\tau} d_W^2(\mu, v) + \phi^1(v) \right\}$$

is bounded below. And we can find a sequence of measures whose energy converges to the infimum and they all belong to $\mathcal{P}_2^a$ due to the internal energy. Then lower semi-continuity of ϕ^1 and compactness guarantee the existence of the limit. Details can be found in section 2.1 in [2] or Lemma 4.2 of [72]. Since $\{\Omega\}$ is convex, we have the uniqueness of the minimizer.

Set

$$\mu_\tau^k := J_\tau \circ ... \circ J_\tau(\mu_0) \in \mathcal{P}(\overline{\Omega}).$$

Define a discrete type solution with time step τ as

$$\mu_\tau(t) := \mu_\tau^k \quad \text{if } t \in ((k-1)\tau, k\tau]. \tag{3.2.4}$$

Let us fix $\mu_0 \in Dom(\phi^1)$ and any $T > 0$. If τ is small enough and $n\tau < T$, then it can be shown that there exists $C > 0$ independent of τ, k, n, T such that

$$\sum_{k=0}^{n-1} d_W^2(\mu_\tau^k, \mu_\tau^{k+1}) \le C\tau, \quad \phi^1(\mu_\tau^n) \le C.$$

This provides enough compactness of $\mu_\tau, \tau \in (0,1)$. As proved in [2, 45] that along the subsequence $\tau \to 0$,

$$\mu_\tau(\cdot) \to \mu(\cdot) \in C^0\left([0,T]; \mathcal{P}_2(\mathbb{R}^d)\right).$$

It can be shown that $\mu(t)$ is a gradient flow solution to (3.1.1) and furthermore it is also a weak solution in duality with $C^\infty(\overline{\Omega} \times [0,\infty))$.

3.3 Vanishing Viscosity Theorem

Theorem 3.3.1. *Assume* (C)(O) *hold,* $\mu_0 \in Dom(\phi^\epsilon)$ *for some* $\epsilon > 0$*, and it is supported in* Ω*. Let* $\mu^\epsilon(\cdot)$ *be the weak solution to equation* (3.1.1) *and* $\mu(\cdot)$ *be the weak solution to equation* (3.1.3) *with the same initial data* μ_0*. Then there exist constants* c, C *that*

$$d_W^2(\mu^\epsilon(t), \mu(t)) \le C\epsilon^{\frac{1}{d+2}} t e^{ct} \text{ for all } t \in [0,\infty).$$

Lastly let us mention that in [32], the vanishing viscosity limit problem in the whole domain was studied in the case when $V = 0$ and $-W$ is the Newtonian potential. Their proof heavily relies on the specific choice of kernel W, and also the fact that the domain is $\mathbb{R}^d$ which eliminates the task of determining the limiting boundary condition.

3.3.1 Proof of the Theorem

We consider equations (3.1.3) and (3.1.1) in bounded, convex domain in this section. Let μ^ϵ be the weak solution to (3.1.1) and μ be the weak solution to (3.1.3). We want to show that μ^ϵ converges to μ in Wasserstein metric as $\epsilon \to 0$.

Now we give two lemmas.

Lemma 3.3.2. *Suppose* (O)(C) *hold, and* $\mu_0 \in Dom(\phi^1)$*. Let*

$$v^\epsilon(x,t) = \left(\epsilon \frac{\nabla u^\epsilon}{u^\epsilon} + \nabla V + \nabla W * \mu^\epsilon\right)(x,t).$$

Then for any $0 < \epsilon < 1$*,*

$$\int_{\Omega_T} |v^\epsilon(x,t)|^2 u^\epsilon(x,t) dx dt \le C.$$

The proof of the lemma is standard (see Proposition 10.4.13 [2]). Recall the *internal energy* is denoted as

$$\mathcal{U}^\epsilon(\mu) = \epsilon \int_{\mathbb{R}^d} u \log u dx \text{ where } \mu = u\mathcal{L}^d.$$

We can have the following.

Corollary 3.3.3. *Settings are as above. For any* $0 < T' \leq T$, $\int_{0 \leq t \leq T'} \mathcal{U}^\epsilon(\mu^\epsilon(t))dt \to 0$ *as* $\epsilon \to 0$.

Proof. By the Euclidean Logarithmic Sobolev inequality (see [42] [34]) and the fact that $u^\epsilon(t)$ is supported in $\Omega(t)$,

$$\int_{\Omega(t)} u^\epsilon \log u^\epsilon dx \leq \frac{d}{2}\log(\frac{1}{2\pi de}\int_{\Omega(t)} \frac{|\nabla u^\epsilon|^2}{u^\epsilon}dx).$$

Write $\mathcal{U}^\epsilon(t) := \mathcal{U}^\epsilon(\mu^\epsilon(t))$. Then

$$\int_0^T \exp\left(\epsilon^{-1}\mathcal{U}^\epsilon(t)\right) dt \leq C\int_{\Omega_T} \frac{|\nabla u^\epsilon|^2}{u^\epsilon}dxdt \leq C\epsilon^{-2}.$$

We used Lemma 3.3.2 and the regularity of V, W in the last inequality. Now for ϵ small enough, assume $\epsilon^2 e^N \geq N$ holds for all $N \geq \epsilon^{-\frac{1}{2}}$. Thus

$$\int_0^T \epsilon^{-1}\mathcal{U}^\epsilon(t)dt \leq \int_0^T \epsilon^{-\frac{1}{2}}dt + \int_{\mathcal{U}^\epsilon(t)\geq\epsilon^{\frac{1}{2}}} \epsilon^{-1}\mathcal{U}^\epsilon(t)dt$$

$$\leq C\epsilon^{-\frac{1}{2}} + \int_0^T \epsilon^2\exp\left(\epsilon^{-1}\mathcal{U}^\epsilon(t)\right)dt \leq C\epsilon^{-\frac{1}{2}}$$

which finishes the proof. □

The next lemma is one important ingredient to the proof of the convergence. Note it is possible that $\mu \notin Dom(\phi^\epsilon)$, the plan is to regularize it and replace it by a $\tilde{\mu} \in \mathcal{P}_a^2$. We look for a $\tilde{\mu}$ with density function uniformly bounded by $\epsilon^{-\alpha}$ for some $0 < \alpha < 1$. Additionally we need $d_{\mu^\epsilon}(\mu, \tilde{\mu})$ to be small where $d_{\mu^\epsilon}(\cdot, \cdot)$ is the Pseudo-Wasserstein metric with base μ^ϵ. As a remark, this is stronger than requiring $d_W(\mu, \tilde{\mu})$ to be small.

Lemma 3.3.4. *Given any $\mu \in \mathcal{P}_2(\overline{\Omega}), v \in \mathcal{P}_2^a(\Omega)$ where Ω is a bounded, convex subset of $\mathbb{R}^d$. For any $s > 0$ small enough, there exists $\mu_s \in \mathcal{P}_a^2(\Omega)$ such that*

$$d_v(\mu, \mu_s) \leq Cs \text{ and}$$

$$\max\{\mu_s(x), x \in \Omega\} \leq s^{-d}.$$

The constant C only depends on the diameter and the volume of Ω.

We postpone the proof to the next section. Now we give the proof for our main theorem in this chapter.

Proof. (of Theorem 3.3.1).

For any $\omega_1 \in \mathcal{P}_2^a(\Omega(t))$, let $\mu_s := (s\mathbf{t}_{\mu^\epsilon}^{\omega_1} + (1-s)\mathbf{i})_\# \mu^\epsilon$ with $\mu^\epsilon = \mu^\epsilon(t)$. The convexity of the domain implies $\mu_s \in \mathcal{P}_2^a(\Omega(t))$. For any Fréchet subdifferential of ϕ^ϵ at μ^ϵ (see section 10 [2]) $\xi^\epsilon \in L^2(\mu^\epsilon; \mathbb{R}^d)$, we have

$$\liminf_{s \to 0} \frac{\phi^\epsilon(\mu_s) - \phi^\epsilon(\mu^\epsilon)}{s} \geq \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\omega_1} - \mathrm{i} \rangle d\mu^\epsilon.$$

By (C), ϕ^ϵ is $\tilde{\lambda}-$convex for $\tilde{\lambda} = \min\{\lambda, 3\lambda\}$. So by the Characterization by Variational inequalities and monotonicity in 10.1.1 [2],

$$\frac{\phi^\epsilon(\mu_s) - \phi^\epsilon(\mu^\epsilon)}{s} \leq \phi^\epsilon(\omega_1) - \phi^\epsilon(\mu^\epsilon) - \frac{\tilde{\lambda}}{2}(1-s) d_W^2(\omega_1, \mu^\epsilon).$$

Then we take $s \to 0$ and find

$$\phi^\epsilon(\omega_1) - \phi^\epsilon(\mu^\epsilon) \geq \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\omega_1} - \mathbf{i} \rangle d\mu^\epsilon + \frac{\tilde{\lambda}}{2} d_W^2(\omega_1, \mu^\epsilon). \tag{3.3.1}$$

By the JKO scheme, μ^ϵ is a gradient flow solution and we can choose $\xi^\epsilon = -v^\epsilon$, the tangent velocity field of μ^ϵ.

Similarly since μ is a gradient flow solution, $\xi := P_{x,t}(-\nabla V - \nabla W * \mu) = -v$ is one Fréchet subdifferential of ϕ at μ and then for any $\omega_2 \in \mathcal{P}_2(\overline{\Omega(t)})$

$$\phi(\omega_2) - \phi(\mu) \geq \int_{\Omega(t)} \langle \xi, \mathbf{t}_\mu^{\omega_2} - \mathbf{i} \rangle d\mu + \frac{\tilde{\lambda}}{2} d_W^2(\omega_2, \mu). \tag{3.3.2}$$

For each t we use Lemma 3.3.4 to modify μ. Take $v = \mu^\epsilon, s = \epsilon^{\frac{1}{d+2}}$ and let $\tilde{\mu} = \mu_s \in \mathcal{P}_2^a(\Omega(t))$ with $\tilde{\mu} = \tilde{u}\mathcal{L}^d$. Then for all $0 \leq t \leq T$

$$\max\{\tilde{u}(x,t)\} \leq \epsilon^{-\frac{d}{d+2}}, \quad d_{\mu^\epsilon}(\tilde{\mu}, \mu) \leq C\epsilon^{\frac{1}{d+2}}.$$

Plug in $\omega_1 = \tilde{\mu}$ in (3.3.1),

$$\phi^\epsilon(\tilde{\mu}(t)) - \phi^\epsilon(\mu^\epsilon(t)) \geq \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{i}\rangle d\mu^\epsilon + \frac{\tilde{\lambda}}{2} d_W^2(\tilde{\mu}, \mu^\epsilon)$$

$$\geq \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\mu} - \mathbf{i}\rangle d\mu^\epsilon + \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}\rangle d\mu^\epsilon + \frac{\tilde{\lambda}}{2}(d_W(\mu, \mu^\epsilon) + C\epsilon^{\frac{1}{d+2}})^2.$$

Let γ^ϵ be an optimal transport plan between μ, μ^ϵ. The above

$$\geq \int_{\Omega(t)^2} \langle \xi^\epsilon(y), x - y\rangle d\gamma^\epsilon + \int_{\Omega(t)} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}\rangle d\mu^\epsilon - C d_W^2(\mu, \mu^\epsilon) - C\epsilon^{\frac{2}{d+2}}. \tag{3.3.3}$$

Take $w_2 = \mu^\epsilon$ in (3.3.2),

$$\phi(\mu^\epsilon(t)) - \phi(\mu(t)) \geq \int_{\overline{\Omega(t)}} \langle \xi(x), y - x\rangle d\gamma^\epsilon + \frac{\tilde{\lambda}}{2} d_W^2(\mu, \mu^\epsilon). \tag{3.3.4}$$

Next by Hölder's inequality

$$\left|\int_{\Omega_T} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}\rangle d\mu^\epsilon dt\right| \leq \left(\int_{\Omega_T} |\xi^\epsilon|^2 d\mu^\epsilon dt\right)^{\frac{1}{2}} \left(\int_{\Omega_T} |\mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}|^2 d\mu^\epsilon dt\right)^{\frac{1}{2}}.$$

By Lemma 3.3.2, $\int_{\Omega_T} |\xi^\epsilon|^2 d\mu^\epsilon dt$ is uniformly bounded and

$$\left(\int_{\Omega(t)} |\mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}|^2 d\mu^\epsilon\right)^{\frac{1}{2}} = d_{\mu^\epsilon}(\mu, \tilde{\mu})$$

is the Pseudo-Wasserstein distance induced by $\mu^\epsilon \in \mathcal{P}_2^a$. So

$$\left|\int_{\Omega_T} \langle \xi^\epsilon, \mathbf{t}_{\mu^\epsilon}^{\tilde{\mu}} - \mathbf{t}_{\mu^\epsilon}^{\mu}\rangle d\mu^\epsilon dt\right| \leq C\int_0^T d_{\mu^\epsilon}(\mu, \tilde{\mu}) dt \leq C\epsilon^{\frac{1}{d+2}} T.$$

This inequality as well as (3.3.3) (3.3.4) gives for any $T' \in [0, T]$

$$\int_{\overline{\Omega_{T'}}^2} \langle -\xi(x) + \xi^\epsilon(y), x - y\rangle d\gamma^\epsilon dt \leq$$

$$\int_0^{T'} (\mathcal{U}^\epsilon(\tilde{\mu}(t)) - \mathcal{U}^\epsilon(\mu^\epsilon(t))) dt + C\int_0^{T'} d_W^2(\mu, \mu^\epsilon) dt + C\epsilon^{\frac{1}{d+2}} T'.$$

Because $\tilde{u}(x,t) \le \epsilon^{-\frac{d}{d+2}}$ pointwise and the domain is bounded, we have

$$\int_0^{T'} \mathcal{U}^\epsilon(\tilde{\mu})dt = \epsilon \int_{\Omega_{T'}} (\tilde{u}\log\tilde{u})(x,t)dxdt \le C\epsilon^{\frac{1}{d+2}}T'.$$

Also note $(u^\epsilon \log u^\epsilon)$ is bounded below, we have $-\mathcal{U}^\epsilon(\mu^\epsilon(t)) \le C\epsilon$. Then

$$\int_{\overline{\Omega_{T'}}^2} \langle -\xi(x) + \xi^\epsilon(y), x-y\rangle d\gamma^\epsilon dt \le C\int_0^{T'} d_W^2(\mu,\mu^\epsilon)dt + C\epsilon^{\frac{1}{d+2}}T'. \tag{3.3.5}$$

By Theorem 8.4.7 and Lemma 4.3.4 from [2], we find

$$\frac{d}{dt}d_W^2(\mu,\mu^\epsilon) \le 2\int_{\overline{\Omega(t)}^2} \langle v(x) - v^\epsilon(y), x-y\rangle d\gamma^\epsilon = 2\int_{\overline{\Omega(t)}^2} \langle \xi^\epsilon(y) - \xi(x), x-y\rangle d\gamma^\epsilon.$$

By (3.3.5) and $d_W^2(\mu,\mu^\epsilon)(0) = 0$, we deduce that

$$d_W^2(\mu,\mu^\epsilon)(T') \le C\int_0^{T'} d_W^2(\mu,\mu^\epsilon)dt + \delta(\epsilon)T'$$

for all $T' \in [0,T]$ and $\delta(\epsilon) = C\epsilon^{\frac{1}{d+2}}$ for some constant C depends only on the domain and universal constants. Then Gronwall's inequality finishes the proof that we have

$$d_W^2(\mu,\mu^\epsilon)(t) \le \delta(\epsilon)te^{ct}.$$

Actually if we keep track of the constants, $\delta(\epsilon) \le C\epsilon^\beta$ for all $\beta \in (0, \frac{1}{d+1})$ where C depends on β, λ, the volumes and diameters of $\Omega(t), t \in [0,T]$. $\square$

3.3.2 Modification of Measures in P-Wasserstein

Proof. (of Lemma 3.3.4)

Without loss of generality, suppose Ω has volume 1 in Euclidean measure. Let e be the Euclidean measure restricted in Ω and then $e \in \mathcal{P}_a^2(\Omega)$. Since v is absolutely continuous, $\mathbf{t}_v^e$ and $\mathbf{t}_v^\mu$ exist and $\mathbf{t}_v^e$ is one to one on Ω outside a v zero measure subset. Let

$$\mu_s := \left(((1-s)\mathbf{t}_v^\mu + s\mathbf{t}_v^e)_\# v\right)$$

be the generalized geodesic joining μ, e with base v, which is defined as in Definition 9.2.2 [2]. Due to the convexity of the domain, we have $\mu_s \in \mathcal{P}_2(\Omega)$. By Proposition 2.6.4 [33], the generalized geodesic is of constant speed in the sense that

$$d_v(\mu, \mu_s) = s d_v(\mu, e).$$

Since the domain is bounded, $d_v(\mu, e)$ is uniformly bounded for all probability measures v, μ, e. We deduce that $d_v(\mu, \mu_s) \leq Cs$.

Now we show the pointwise boundedness of μ_s. Let $\varphi = \chi_{B_r}(x)$ which equals 1 in B_r and 0 outside. Thus

$$\int_\Omega \varphi d\mu_s = \int_\Omega \varphi\left((1-s)\mathbf{t}_v^\mu + s\mathbf{t}_v^e\right) dv = v\left\{\left((1-s)\mathbf{t}_v^\mu + s\mathbf{t}_v^e\right)^{-1} B_r(x)\right\} \tag{3.3.6}$$

Write $S := \left((1-s)\mathbf{t}_v^\mu + s\mathbf{t}_v^e\right)^{-1} B_r(x)$. By definition

$$vol\left\{B_r(x)\right\} = vol\left\{\left((1-s)\mathbf{t}_v^\mu + s\mathbf{t}_v^e\right) S\right\}.$$

Now we apply Brunn-Minkowski inequality (Lemma A.1.8) to find the above

$$\geq vol\left\{s\mathbf{t}_v^e S\right\} = s^d v(S).$$

So

$$v(S) \leq s^{-d} vol\{B_r(x)\} = vol\left\{B_0(1)\right\}\left(\frac{r}{s}\right)^d.$$

By (3.3.6), for any $\varphi = \chi_{B_r}(x)$ we find out

$$\frac{1}{vol\left\{B_r(x)\right\}} \int_\Omega \varphi d\mu_s \leq s^{-d}.$$

This shows that u_s is an L^∞ function in Ω with bound s^{-d}. □

We make a remark that the modification of μ done in Lemma 3.3.4 can not be replaced by simply convoluting μ with a smooth, positive, compactly supported function. We want to show that, the difference between one measure and a "small perturbation" (including convolutions) of it can be large in the Pseudo-Wasserstein metric for some base measure. To illustrate the main idea, let us consider the following base measure v which is a sum of delta masses. And instead of convolution, we first consider small shifts.

Suppose in $\mathbb{R}^2$, $\epsilon > 0$,

$$v = \frac{1}{2}\delta_{(-1,0)} + \frac{1}{2}\delta_{(1,0)},\ \mu_1 = \frac{1}{2}\delta_{(-\epsilon,1)} + \frac{1}{2}\delta_{(\epsilon,-1)},\ \mu_2 = \frac{1}{2}\delta_{(\epsilon,1)} + \frac{1}{2}\delta_{(-\epsilon,-1)}.$$

Then the optimal transport maps from v to μ_i are

$$\mathbf{t}_v^{\mu_1}(x) = \begin{cases} (\epsilon,-1) & \text{when } x = (1,0), \\ (-\epsilon,1) & \text{when } x = (-1,0); \end{cases} \qquad \mathbf{t}_v^{\mu_2}(x) = \begin{cases} (\epsilon,1) & \text{when } x = (1,0), \\ (\epsilon,-1) & \text{when } x = (-1,0). \end{cases}$$

So

$$d_v^2(\mu_1,\mu_2) = \int_{\mathbb{R}^2} |\mathbf{t}_v^{\mu_1} - \mathbf{t}_v^{\mu_2}|^2 dv = 4.$$

For small ϵ, geometrically μ_2 is just a small perturbation of μ_1. This shows that a little shift may cause a large difference in Pseudo-Wasserstein metric. And so it is possible that the convolution of μ with $\frac{1}{\epsilon^d}\varphi(\frac{\cdot}{\epsilon})$ (φ is a bump function and ϵ is a small positive value) is far away from μ in view of the Pseudo-Wasserstein metric.

CHAPTER 4

Aggregation Equations with Singular Drifts

4.1 Introduction

In this chapter, we consider the following equation

$$u_t = \Delta u^m - \nabla \cdot (u \nabla \mathcal{K}_s u) \text{ in } \mathbb{R}^d \times [0, \infty), \tag{4.1.1}$$

with nonnegative initial data $u(x, 0) = u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, where the degeneracy arises due to the range of m, $m > 1$. The non-local drift is given by the Riesz kernel

$$\mathcal{K}_s u = cK_s * u \quad \text{ where } s \in \left(0, \frac{d}{2}\right), \, K_s(z) = |z|^{-d+2s}, c > 0. \tag{4.1.2}$$

When $d \geq 3$, for a suitable choice of $c = c(d, s) > 0$, the convolution is governed by a fractional diffusion process: $\mathcal{K}_s u = (-\Delta)^{-s} u$ ([66]). This chapter is from my paper [74].

The model arises from the macroscopic description of cell motility due to cell adhesion and chemotaxis phenomena, see [12, 22, 69]. In the context of biological aggregation, u describes the population density and the degenerate diffusion models the local repulsion taking over-crowding effects into consideration. This effect can also be found in many physical applications, including fluids in porous medium ([44, 70]). The homogeneous singular kernel models long-range attractive interactions between cells, with smaller s representing stronger aggregation at near-distances and therefore more singular. For larger s, we consider stronger force at long-distances. The competition between the diffusion and the non-local aggregation is one of the core subjects in the study of aggregation models.

To find the balance of the two competing effects, we use a scaling argument, also see [11,23].

Define

$$u_r(x,t) := r^d u(rx, r^{d(m-1)+2}t), \tag{4.1.3}$$

and then formally $(-\Delta)^{-s}u_r = r^{d-2s}(-\Delta)^{-s}u$. It is straightforward to check

$$\partial_t u_r = \Delta u_r^m - r^{2d-dm-2s}\nabla\cdot(u_r\nabla\cdot\mathcal{K}_s u_r).$$

So $m = 2-2s/d$ leads to a compensation between the diffusion and the aggregation. The range $m > 2-2s/d$ where the diffusion dominates over the aggregation is often referred to as the *subcritical regime*. The range $m < 2-2s/d$ is called *supercritical*.

When $s = 1$, $\mathcal{K}_1$ represents the *Newtonian potential* and (4.1.1) is the well-known degenerate Patlak-Keller-Segel equation. In the corresponding subcritical regime, the well-posedness, boundedness and continuity regularity properties of solutions have been established in [5, 10, 29]. When $m = 2-2/d$, it has been shown in [11, 39] that the mass of the initial data plays an important role. More precisely, if the initial mass is larger than one critical value, solutions can blow up in finite time and otherwise they always stay regular. In the supercritical regime, finite time blow up is again possible, see [5, 67].

In this chapter, we consider the natural extension of the Newtonian potential: $\mathcal{K}_s = (-\Delta)^{-s}$ with $s \in (0, \frac{d}{2})$ (see (4.1.7) for details). For this kernel, to the best of our knowledge, only stationary solutions have been analyzed before. [19] studied the existence of stationary solutions in the fair competition regime: $m = 2-2s/d$. It was shown in [22] that stationary solutions are radially symmetric decreasing with compact supports and have certain regularity properties in most of the subcritical regime. Recently in [21, 24], the equations with repulsive-attractive kernel, of the form $K(z) = \frac{|z|^a}{a} - \frac{|z|^b}{b}$ with $2 \geq a, b > -d$, are studied. [21] analyzed the asymptotic and the singular limits as $m \to \infty$, and [24] proved the boundedness of solutions when the repulsive potential has a stronger singularity.

Our goal is to initiate investigating the dynamic equation (4.1.1) in the subcritical regime, starting with its well-posedness and regularity properties. Many questions stay open as we discuss below.

4.1.1 Summary of Results

Throughout the chapter, we assume

$$\begin{aligned}&(1)\, d \geq 3,\ m > 2 - 2s/d,\\ &(2)\, u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \text{ and } u_0 \text{ is nonnegative.}\end{aligned} \tag{4.1.4}$$

Next we give the notion of weak solutions to (4.1.1) which is similar to those in [6, 14].

Definition 4.1.1. *Let $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be nonnegative and $T \in (0, \infty]$. We say that a nonnegative function $u : \mathbb{R}^d \times [0, T] \to [0, \infty)$ is a weak solution to* (4.1.1) *in time $[0, T]$ with initial data u_0 if*

$$\begin{aligned}u \in C([0,T], L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times [0,T]), \quad u^m \in L^2(0, T, \dot{H}^1(\mathbb{R}^d)),\\ \text{and} \quad u\nabla\mathcal{K}_s u \in L^1(\mathbb{R}^d \times [0,T])\end{aligned} \tag{4.1.5}$$

and for all test function $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T))$,

$$\iint_{\mathbb{R}^d\times[0,T]} u\phi_t dxdt = \int_{\mathbb{R}^d} u_0(x)\phi(0,x)dx + \iint_{\mathbb{R}^d\times[0,T]} (\nabla u^m + u\nabla\mathcal{K}_s u)\nabla\phi\, dxdt. \tag{4.1.6}$$

Theorem 4.1.1. *[Existence and Boundedness] Suppose* (4.1.4)*, and either $s > \frac{1}{2}$ or $m < 2$ hold. Then there exists a nonnegative weak solution u to* (4.1.1) *with mass preserved and u is uniformly bounded for all $t \in [0, \infty)$. The bound only depends on s, m, d, and $\|u_0\|_1 + \|u_0\|_\infty$.*

This can be seen as a variate result as compared to [5, 43, 68] where Keller-Segel systems or equations are considered. We approach the problem by two approximations: regularization of the gradient of the kernel and elimination of the degeneracy, see (4.2.3). The existence of smooth solutions for the approximated problems is standard [6].

With the approximations in mind, the first goal is to obtain a priori uniform in time L^∞-bound of solutions. We use an iteration method which can be found in [52], and show a sequence of differential inequalities. The key is to bound the attracting term by the degenerate diffusion. By scaling, the condition $m > 2 - 2s/d$ is critical, the use of which will be highlighted.

If $1/2 < s \leq 1$, the uniform bound is obtained separately when $m < 2$ and $m \geq 2$, and only for the former range of m if $s \leq 1/2$. Here $s = \frac{1}{2}$ is a borderline, because $|\nabla K_s|$ is only locally integrable when $s > \frac{1}{2}$. In the proof, we will apply Sobolev inequalities and properties of fractional Laplacian. It is essential that each estimate needs to be consistent with the scaling and this turns out to be a useful hint for us, for example the choice of exponents in inequality (4.2.20). When $s > 1$, interpolation inequalities for fractional differentiation are no longer helpful. To go around the technical difficulty, we adopt a different argument by studying the singular convolution integrals in three different ways according to the steps of the iteration, see the proof of Theorem 4.2.4.

As for the remaining range $m \geq 2$ and $s \leq 1/2$, we conjecture the same a priori bound. While likely a technical issue, challenges for $m > 2$ arise as well in [22], where stationary solutions are shown to be in $W^{1,\infty}(\mathbb{R}^d)$ only when $m \leq 2$.

With aforementioned a priori bound, a compactness argument yields the existence solution to (4.1.1), see Theorems 4.3.1- 4.3.3. One hard part is to justify the singular interaction term when $s \leq 1/2$. Due to the loss of the integrability of $|\nabla K_s|$, $\nabla K_s * u$ is not well-defined for $u \in L^1 \cap L^\infty$. To overcome this difficulty, we observe the following a priori estimate under the condition $m < 2$,

$$\nabla u \in L^2_{loc}([0, \infty), L^2(\mathbb{R}^d)).$$

Using this, we can make sense of $\nabla(-\Delta)^{-s}u$ in the space $L^2_{loc}([0, \infty), L^2(\mathbb{R}^d))$, see Lemma 4.3.2.

Next let us state the uniqueness result.

Theorem 4.1.2. *[Uniqueness] When $s \geq 1$, there is a unique weak solution to* (4.1.1) *with initial data u_0.*

In [5,6], the uniqueness problem was solved when $s = 1$. We will take their approach and prove for $s \in (1, \frac{d}{2})$.

As for the regularity property, with the help of [50], we have the following theorem.

Theorem 4.1.3. *[Hölder Regularity] Suppose $s \in (\frac{1}{2}, \frac{d}{2})$. Let $u(\cdot, t)$ be a weak solution to (4.1.1) with initial data u_0. Then for any $\tau > 0$, u is Hölder continuous in $\mathbb{R}^d \times (\tau, \infty)$.*

A lot of open questions remain to be investigated in the subcritical regime: existence result for $s < 1/2$ and $m > 2$, uniqueness for $s < 1$ and Hölder regularity for $s \leq 1/2$.

Let us comment that our results and proofs adapt to more general kernels K_s such that $|K_s(x,y,t)|$, $|\nabla_x K_s(x,y,t)|$, $|D_x^2 K_s(x,y,t)|$ share the same singularity as $|x-y|^{-d+2s}$, $|x-y|^{-d-1+2s}$, $|x-y|^{-d-2+2s}$ respectively near $x = y$. Some modifications are needed if we only assume $|K_s(x,y,t)|$, $|\nabla_x K_s(x,y,t)|$, $|D_x^2 K_s(x,y,t)|$ to be bounded away from $x = y$.

4.1.2 The Singular Kernel

We use the notation $-(-\Delta)^r$ with $r \in (0,1]$ for fractional Laplacian operator which is defined on the Schwartz class of functions on $\mathbb{R}^d$ by Fourier multiplier with symbol $-|\xi|^{2r}$, see chapter V [66]. Alternatively, $-(-\Delta)^r$ can also be realized as the following singular integral in the sense of Cauchy principal value, see [53].

$$-(-\Delta)^r u(x) = \lim_{R \to 0^+} \frac{2^{2r}\Gamma(\frac{d+2r}{2})}{\pi^{d/2}|\Gamma(-r)|} \int_{\mathbb{R}^d \setminus B_R(x)} \frac{u(x+y) - u(x)}{|y|^{d+2r}} dy.$$

We denote the constant in front of the singular integral as $c_{d,r}$. The domain of the operator can be extended naturally to the Sobolev space $W^{2r,2}(\mathbb{R}^d)$.

We write $|\nabla|^{2r} := (-\Delta)^r$.

We define the following bilinear form associated to the space $W^{r,2}(\mathbb{R}^d)$:

$$\mathcal{B}_r(v,w) = c_{d,r} \int \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{d+2r}} dx dy$$

for $v, w \in W^{r,2}(\mathbb{R}^d)$. Then

$$\mathcal{B}_r(v,w) = \langle (-\Delta)^r v, w \rangle_{L^2} := \int_{\mathbb{R}^d} w (-\Delta)^r v \, dx.$$

Using Parseval's identity and definitions, we have for $0 < r_1 < r$

$$\langle (-\Delta)^r v, w \rangle_{L^2} = \langle |\nabla|^{r-r_1} v, |\nabla|^{r_1} w \rangle_{L^2}.$$

For details, we refer readers to [53] and Section 3 [13].

Proposition 4.1.4 (Proposition 3.2 [13]). *For every $v, w \in W^{1,2}(\mathbb{R}^d)$, we have*

$$\mathcal{B}_r(v, w) = C \int \frac{\nabla v(x) \cdot \nabla w(y)}{|x-y|^{d-2+2r}} dxdy.$$

The inverse operator of fractional Laplacian is denoted by $-(-\Delta)^{-s}$ which can be realized as the Riesz potential

$$(-\Delta)^{-s} u(x) := \int_{\mathbb{R}^d} K_s(x, y) u(y) dy; \tag{4.1.7}$$

$$\text{and } \; K_s(x, y) := \frac{2^{2s}\Gamma(\frac{d-2s}{2})}{\pi^{d/2}\Gamma(s)} |x-y|^{-d+2s}.$$

Here $s \in (0, \frac{d}{2})$ and u is a function integrable enough for (4.1.7) to make sense. We refer readers to [14, 53, 62] for more details.

When $s > \frac{1}{2}$, $\nabla \mathcal{K}_s u$ is well defined for $u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. When $s < \frac{1}{2}$, if we further assume that u is γ-Hölder continuous with $\gamma \geq 1 - 2s$, $\nabla \mathcal{K}_s u$ can be defined via a Cauchy principal value

$$\nabla \mathcal{K}_s u(x) := \int_{\mathbb{R}^d} \nabla_x K_s(x, y)(u(y) - u(x)) dy.$$

4.2 A Priori Estimates

In this section several a priori estimates ($L_t^\infty L_x^p$ and $L_t^\infty L_x^\infty$ bounds) are obtained.

Let us regularize the equation (4.1.1). Instead of modifying $\mathcal{K}_s$, we consider the following approximation

$$V_{s,\epsilon}(x) := \zeta_\epsilon(x) \nabla_x K_s(x, 0) \tag{4.2.1}$$

where $\epsilon > 0$ is a small parameter and ζ_ϵ is a smooth, radially symmetric, nonnegative function that

$$\begin{aligned} &\zeta_\epsilon = 0 \text{ for } |x| \leq \epsilon \text{ and } |x| \geq 2/\epsilon, \quad \zeta_\epsilon = 1 \text{ for } |x| \in [2\epsilon, 1/\epsilon], \\ &|\nabla \zeta_\epsilon| \lesssim 1/\epsilon \text{ for } |x| \leq 2\epsilon, \quad |\nabla \zeta_\epsilon| \lesssim \epsilon \text{ for } |x| \geq 1/\epsilon. \end{aligned} \tag{4.2.2}$$

It is not hard to see

(1) $V_{s,\epsilon}$ is a smooth vector field and $V_{s,\epsilon}(x) = c(-d+2s)|x|^{-d-2+2s}x$ for $|x| \in [2\epsilon, 1/\epsilon]$;

(2) $|\nabla \cdot V_{s,\epsilon}(x)| \le C|x|^{-d-2+2s}$ holds for some $C > 0$ only depending on d, s and for all x.

Consider the following problem

$$\begin{cases} \dfrac{\partial}{\partial t} u_\epsilon = \epsilon \Delta u_\epsilon + \Delta u_\epsilon^m - \nabla \cdot (u_\epsilon V_{s,\epsilon} * u_\epsilon) = 0 & \text{in } \mathbb{R}^d \times [0,\infty), \\ u_\epsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^d \end{cases} \tag{4.2.3}$$

which is uniformly parabolic with smooth compactly supported interaction kernel. By Theorem 4.2 [6], there exists a unique solution u_ϵ to (4.2.3) which is nonnegative and smooth.

In the following subsequent theorems, we are going to prove that u_ϵ are uniformly bounded for all time independent of ϵ. As mentioned before, we will treat the following five cases separately: $\{m < 2, 1/2 < s \le 1\}$, $\{m \ge 2, 1/2 < s \le 1\}$, $\{m < 2, s \le 1/2\}$, $\{m < 2, 1 < s < d/2\}$ and $\{m \ge 2, 1 < s < d/2\}$.

Theorem 4.2.1. *Suppose* (4.1.4), $s \in (\frac{1}{2}, 1]$ *and let* $u := u_\epsilon$ *be a solution to* (4.2.3). *Then* u *is uniformly bounded for all time and the bound only depends on* d, s, m *and* $\|u_0\|_{L^1} + \|u_0\|_{L^\infty}$.

Proof. Define a sequence $\{n_k, k \in \mathbb{N}^+\} \subset \mathbb{R}^+$ by

$$n_0 = 1, \quad n_{k+1} := 2n_k + 1 - m \text{ for all } k \ge 0. \tag{4.2.4}$$

We find $n_k = 2^k(2-m) - 1 + m$. It follows from $m < 2$ that $n_k \to \infty$ as $k \to \infty$.

Fix any $k \ge 1$ and we know $n_k \ge n_1 = 3 - m$. For simplicity of notation, let us write

$$n = n_k, \quad l = n_{k-1} = \frac{m+n-1}{2} < n.$$

Now, without loss of generality, suppose that the total mass of u_0 is 1 and so is the total mass of $u(\cdot, t)$ by the equation. Since u is smooth, we multiply u^{n-1} on both sides of (4.2.3) and find

$$\begin{aligned} \partial_t \int_{\mathbb{R}^d} u^n dx &\le -n \int_{\mathbb{R}^d} \nabla u^m \nabla u^{n-1} dx + n \int_{\mathbb{R}^d} (u V_{s,\epsilon} * u) \cdot \nabla u^{n-1} dx \\ &\le -C_m \int_{\mathbb{R}^d} \left| \nabla u^{\frac{n+m-1}{2}} \right|^2 dx + (n-1) \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u^n dx. \end{aligned} \tag{4.2.5}$$

By property (2) of $V_{s,\epsilon}$, we obtain

$$\begin{aligned} X &:= \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u^n dx = \int_{\mathbb{R}^d} (-\nabla \cdot V_{s,\epsilon}) * u \, u^n dx \\ &\leq C \iint_{\mathbb{R}^{2d}} \frac{(u(x)-u(y))\,(u^n(x)-u^n(y))}{|x-y|^{d+2-2s}} dxdy \end{aligned}$$

Since u is nonnegative

$$(u(x)-u(y))\,(u^n(x)-u^n(y)) \leq \left(u^l(x)-u^l(y)\right)\left(u^{n+1-l}(x)-u^{n+1-l}(y)\right),$$

and thus

$$\begin{aligned} X &\leq C \iint_{\mathbb{R}^{2d}} \frac{\left(u^l(x)-u^l(y)\right)\left(u^{n+1-l}(x)-u^{n+1-l}(y)\right)}{|x-y|^{d+2-2s}} dxdy \\ &= C \int_{\mathbb{R}^d} \nabla(-\Delta)^{-s} u^l \nabla u^{n+1-l} dx \quad \text{(by Proposition 4.1.4)} \\ &\leq C \int_{\mathbb{R}^d} \left|(-\Delta)^{1-s} u^l\right| u^{n+1-l} dx \\ &\leq C \left\| |\nabla|^{2-2s} u^l \right\|_2 \left\| u^{n+1-l} \right\|_2 \quad \text{(by Hölder's inequality)} \\ &= C \left\| |\nabla|^{2-2s} u^l \right\|_2 \left\| u^l \right\|_{2\frac{n+1-l}{l}}^{\frac{n+1-l}{l}}. \end{aligned}$$

By Gagliardo-Nirenberg interpolation inequality

$$\begin{aligned} &\left\| |\nabla|^{2-2s} u^l \right\|_2 \lesssim \left\| \nabla u^l \right\|_2^{\alpha} \left\| u^l \right\|_1^{1-\alpha}, \\ &\left\| u^l \right\|_{2\frac{n+1-l}{l}} \lesssim \left\| \nabla u^l \right\|_2^{\beta} \left\| u^l \right\|_1^{1-\beta} \end{aligned} \tag{4.2.6}$$

with $\alpha(n), \beta(n)$ satisfying

$$\frac{1}{2} = \frac{2-2s}{d} + \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + 1 - \alpha, \quad \frac{1}{2}\frac{l}{n+1-l} = \left(\frac{1}{2} - \frac{1}{d}\right)\beta + 1 - \beta.$$

It can be checked that $\alpha > 2-2s$ if and only if $s > \frac{1}{2}$. The conditions of Theorem A.1.1 are fulfilled.

Letting $\theta(n) := \alpha + \frac{n+1-l}{l}\beta$ yields

$$X \leq \left\| \nabla u^l \right\|_2^{\theta} \left\| u^l \right\|_1^{1+\frac{n+1-l}{l}-\theta}.$$

It follows from (4.2.6) that

$$\left(\frac{1}{2} + \frac{1}{d}\right)\theta(n) = \frac{2-2s}{d} + \frac{n+1-l}{l} = \frac{2-2s}{d} + \frac{4-2m}{n-1+m} + 1.$$

Since $m < 2$, $\{\theta(n)\}$ is decreasing as $n \to \infty$ and the limit equals $\left(\frac{2-2s}{d}+1\right) \Big/ \left(\frac{1}{2}+\frac{1}{d}\right)$ which is less than 2. Very importantly when $n = 3-m$, $\theta(3-m) < 2$ is equivalent to

$$\left(\frac{2-2s}{d}+3-m\right) \Big/ \left(\frac{1}{2}+\frac{1}{d}\right) < 2 \iff m > 2 - \frac{2s}{d}.$$

So in all

$$\inf_{n \geq 3-m} 2-\theta(n) > 0 \text{ and } \inf_{n \geq 3-m} \theta(n) > 0. \tag{4.2.7}$$

Now by Hölder's inequality, for any small $\delta > 0$

$$X \leq \frac{\delta}{n} \left\|\nabla u^l\right\|_2^2 + C_\delta n^{c_n} \left\|u^l\right\|_1^{\theta'} \tag{4.2.8}$$

where

$$\theta' = \theta'(n) := 2 + \frac{2(2-m)}{l(2-\theta(n))} \leq 2 + Cn^{-1} \quad \text{ and } \quad c_n := \frac{\theta(n)}{2-\theta(n)}.$$

According to (4.2.7), $\{c_n\}$ are uniformly bounded for all $n \geq 3-m$. By Gagliardo-Nirenberg inequality

$$\left\|u^l\right\|_{\frac{n}{l}} \lesssim \left\|\nabla u^l\right\|_2^\gamma \left\|u^l\right\|_1^{1-\gamma} \text{ where } \gamma = \left(\frac{n-l}{n}\right) \Big/ \left(\frac{1}{2}+\frac{1}{d}\right).$$

Direct calculation shows $\frac{\gamma n}{l} < 2$. Next by Young's inequality

$$\int_{\mathbb{R}^d} u^n dx \lesssim \left\|\nabla u^l\right\|_2^{\frac{\gamma n}{l}} \left\|u^l\right\|_1^{\frac{(1-\gamma)n}{l}} \leq \delta \left\|\nabla u^l\right\|_2^2 + C_\delta \left\|u^l\right\|_1^{\gamma'} \tag{4.2.9}$$

where

$$\gamma' = \gamma'(n) := 2 - \frac{2(m-1)}{2l-\gamma n} \leq 2 - Cn^{-1} \text{ and (not hard to check) } \gamma' > 0.$$

Finally by (4.2.5), (4.2.8) and (4.2.9), we obtain for all $n \geq 3-m$

$$\partial_t \int_{\mathbb{R}^d} u^n dx + c\int_{\mathbb{R}^d} u^n dx \leq Cn \left(\int_{\mathbb{R}^d} u^l dx\right)^{\gamma'} + Cn^{c_n+1} \left(\int_{\mathbb{R}^d} u^l dx\right)^{\theta'}$$

where c, C are independent of n.

Recall (4.2.4). Since for some universal C

$$n_k \sim_m 2^k,\ \theta', \gamma' \leq 2 + Cn^{-1},\ c_n \leq C,$$

writing $A_k = \int_{\mathbb{R}^d} u^{n_k} dx$ yields

$$\frac{d}{dt} A_{k+1} + cA_{k+1} \leq C^k + C^k A_k^{2+C2^{-k}} \quad \text{for all } k \geq 0.$$

Finally applying Lemma 1.2.2 concludes the theorem.

□

Theorem 4.2.2. *Theorem 4.2.1 holds in the regime $m \geq 2$ and $s \in (\frac{1}{2}, 1]$.*

Proof. Denote $u_1 = \max\{u - 1, 0\}$, $\tilde{u} = \min\{u, 1\}$ and so $u = u_1 + \tilde{u}$. For some $n \geq 2$, let us multiply u_1^{n-1} on both sides of (4.2.3). We get

$$\partial_t \int_{\mathbb{R}^d} u_1^n dx = n \int_{\mathbb{R}^d} u_1^{n-1} u_t dx \leq -mn \int_{\mathbb{R}^d} u^{m-1} \nabla u \nabla u_1^{n-1} dx + n \underbrace{\int_{\mathbb{R}^d} (V_{s,\epsilon} * u)\, u \nabla u_1^{n-1} dx}_{X:=}.$$

Because

$$\begin{aligned}\int_{\mathbb{R}^d} u^{m-1} \nabla u \nabla u_1^{n-1} dx &= \int_{\mathbb{R}^d} (u_1 + 1)^{m-1} \nabla u_1 \nabla u_1^{n-1} dx \\ &\geq \frac{C_m}{n} \int_{\mathbb{R}^d} \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 + \left| \nabla u_1^{\frac{n}{2}} \right|^2 dx\end{aligned}$$

for some $C_m > 0$ bounded from below for all $m \geq 2$, we obtain

$$\partial_t \int_{\mathbb{R}^d} u_1^n dx \leq -C_m \int_{\mathbb{R}^d} \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 + \left| \nabla u_1^{\frac{n}{2}} \right|^2 dx + nX. \tag{4.2.10}$$

Let us now estimate X:

$$\begin{aligned} X &= \int_{\mathbb{R}^d} V_{s,\epsilon} * u_1 u \nabla u_1^{n-1} dx + \int_{\mathbb{R}^d} V_{s,\epsilon} * \tilde{u}\; u \nabla u_1^{n-1} dx \\ &\lesssim \int_{\mathbb{R}^d} V_{s,\epsilon} * u_1 \nabla u_1^n dx + \int_{\mathbb{R}^d} V_{s,\epsilon} * u_1 \nabla u_1^{n-1} dx + \int_{\mathbb{R}^d} V_{s,\epsilon} * \tilde{u}\; u \nabla u_1^{n-1} dx \\ &=: Y_n + Y_{n-1} + X_1. \end{aligned} \tag{4.2.11}$$

We will first consider X_1. By the fact that

$$s > \frac{1}{2},\ \tilde{u} \leq 1,\ \tilde{u} \in L^1 \text{ and } |V_{s,\epsilon}(x)| \lesssim |x|^{-d-1+2s},$$

we have

$$\left|V_{s,\epsilon}*\tilde{u}\right|(x) \lesssim \int_{\mathbb{R}^d} |x-y|^{-d-1+2s}\tilde{u}(y)dy \leq C\int_{\mathbb{R}^d}\tilde{u}dy + \int_{|x-y|\leq 1}|x-y|^{-d-1+2s}dy \leq C.$$

Hence for any small $\delta>0$

$$\begin{aligned}
X_1 = C\int_{\mathbb{R}^d} u\left|\nabla u_1^{n-1}\right|dx &\lesssim \int_{\mathbb{R}^d} u\, u_1^{\frac{n}{2}-1}\left|\nabla u_1^{\frac{n}{2}}\right|dx \\
&\leq C_\delta n\int_{\mathbb{R}^d}\left(u_1^n+u_1^{n-2}\right)dx + \frac{\delta}{n}\|\nabla u_1^{\frac{n}{2}}\|_2^2 \\
&\leq C_\delta n\int_{\mathbb{R}^d} u_1^n dx + C_\delta n + \frac{\delta}{n}\|\nabla u_1^{\frac{n}{2}}\|_2^2. \qquad (4.2.12)
\end{aligned}$$

In the last inequality (4.2.12), we applied

$$\int_{\mathbb{R}^d} u_1^{n-2}dx \leq \int_{u_1\geq 1} u_1^n dx + \int_{1\leq u\leq 2} 1dx \leq \left\|u_1^{\frac{n}{2}}\right\|_2^2 + 1.$$

Next by Gagliardo-Nirenberg and Young's inequalities

$$C_\delta\left\|u_1^{\frac{n}{2}}\right\|_2^2 \leq C_\delta C_\alpha\left\|\nabla u_1^{\frac{n}{2}}\right\|_2^{2\alpha}\left\|u_1^{\frac{n}{2}}\right\|_1^{2(1-\alpha)} \leq C_\delta' n^d\left\|u_1^{\frac{n}{2}}\right\|_1^2 + \frac{\delta}{n^2}\left\|\nabla u_1^{\frac{n}{2}}\right\|_2^2 \qquad (4.2.13)$$

where we picked

$$\alpha = \frac{1}{2}\Big/\left(\frac{1}{2}+\frac{1}{d}\right).$$

So by (4.2.12), for some universal small $\delta>0$

$$X_1 \leq C_\delta n + C_\delta' n^{d+1}\left\|u_1^{\frac{n}{2}}\right\|_1^2 + \frac{\delta}{n}\left\|\nabla u_1^{\frac{n}{2}}\right\|_2^2. \qquad (4.2.14)$$

For Y_l with $l=n-1,n$, as proved before (in Theorem 4.2.1)

$$Y_l \lesssim \iint_{\mathbb{R}^{2d}} \frac{\left(u_1^{\frac{l+1}{2}}(x)-u_1^{\frac{l+1}{2}}(y)\right)^2}{|x-y|^{d+2-2s}}dxdy \lesssim \int_{\mathbb{R}^d}\nabla(-\Delta)^{-s}u_1^{\frac{l+1}{2}}\nabla u_1^{\frac{l+1}{2}}dx.$$

By Fourier transformation and Hölder's inequality,

$$\begin{aligned}
Y_l &\lesssim \int_{\mathbb{R}^d}|\xi|^{2-2s}\left|\widehat{u_1^{\frac{l+1}{2}}}\right|^2 d\xi \lesssim \left(\int_{\mathbb{R}^d}|\xi|^2\left|\widehat{u_1^{\frac{l+1}{2}}}\right|^2 d\xi\right)^{1-s}\left(\int_{\mathbb{R}^d}\left|\widehat{u_1^{\frac{l+1}{2}}}\right|^2 d\xi\right)^s \\
&\lesssim \left(\int_{\mathbb{R}^d}\left|\nabla u_1^{\frac{l+1}{2}}\right|^2 dx\right)^{1-s}\left(\int_{\mathbb{R}^d} u_1^{l+1}dx\right)^s \\
&\leq C_\delta n^{\frac{1-s}{s}}\left\|u_1^{\frac{l+1}{2}}\right\|_2^2 + \frac{\delta}{n}\left\|\nabla u_1^{\frac{l+1}{2}}\right\|_2^2 \\
&\leq C_\delta n\left\|u_1^{\frac{l+1}{2}}\right\|_2^2 + \frac{\delta}{n}\left\|\nabla u_1^{\frac{l+1}{2}}\right\|_2^2 \quad (\text{ since } s>\frac{1}{2}\ ). \qquad (4.2.15)
\end{aligned}$$

When $l = n-1$, by (4.2.15) and (4.2.13), we get for some C only depending on δ

$$Y_{n-1} \le Cn^{d+1} \left\| u_1^{\frac{n}{2}} \right\|_1^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2.$$

When $l = n$, as done previously

$$Y_n \le Cn^{d+1} \left\| u_1^{\frac{n+1}{2}} \right\|_1^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{n+1}{2}} \right\|_2^2.$$

By Gagliardo-Nirenberg,

$$\left\| u_1^{\frac{n+1}{2}} \right\|_1^2 = \left\| u_1^{\frac{n}{2}} \right\|_{\frac{n+1}{n}}^{2\frac{n+1}{n}} \le C \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^{2\beta_1} \left\| u_1^{\frac{n}{2}} \right\|_1^{2\beta_2}$$

where

$$\beta_1 = \beta_1(n) = \frac{1}{n} \Big/ \left(\frac{1}{2} + \frac{1}{d} \right), \quad \beta_2 = \beta_2(n) = \frac{n+1}{n} - \beta_1.$$

By Young's inequalities

$$\begin{aligned} C \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^{2\beta_1} \left\| u_1^{\frac{n}{2}} \right\|_1^{2\beta_2} &\le C\Big(\frac{\epsilon^p \| \nabla u_1^{\frac{n}{2}} \|_2^{2\beta_1 p}}{p} + \frac{\| u_1^{\frac{n}{2}} \|_1^{2\beta_2 q}}{\epsilon^q q}\Big) \\ &\le C_\delta n^{c_n} \left\| u_1^{\frac{n}{2}} \right\|_1^{\gamma_n} + \frac{\delta}{n^{d+2}} \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2 \end{aligned}$$

where we pick

$$p = \frac{1}{\beta_1} \sim n, \quad q = \frac{p}{p-1}, \quad C\epsilon^p / p = \frac{\delta}{n^{d+2}}.$$

Thus $\epsilon^p \sim \frac{\delta}{n^{d+1}}$. Now since $\frac{-q}{p} = -\frac{1}{p-1} = -\frac{\beta_1}{1-\beta_1}$, we find $\epsilon^{-q} = C_\delta n^{\frac{\beta_1(d+1)}{1-\beta_1}}$,

$$\gamma_n = \frac{2\beta_2}{1-\beta_1} = 2 + \frac{1}{n} \frac{2}{(1-\beta_1)} \quad \text{and} \quad c_n = \frac{\beta_1(d+1)}{1-\beta_1}.$$

It is not hard to check that for all $n \ge 2$, $\beta_1(n) \le \beta_1(2) < 1$. And so c_n is uniformly bounded for all $n \ge 2$. Thus we proved that for any small $\delta > 0$

$$Y_n \le C_\delta n^{c_n} \left\| u_1^{\frac{n}{2}} \right\|_1^{\gamma_n} + \frac{\delta}{n} \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{n+1}{2}} \right\|_2^2.$$

Combining with (4.2.11) and (4.2.14), for some c $(= c_n + d + 1) > 0$ uniformly bounded for all $n \ge 2$

$$nX \le C_\delta n^2 + C_\delta n^c \left(\left\| u_1^{\frac{n}{2}} \right\|_1^2 + \left\| u_1^{\frac{n}{2}} \right\|_1^{\gamma_n} \right) + \delta \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2 + \delta \left\| \nabla u_1^{\frac{n+1}{2}} \right\|_2^2.$$

Therefore by (4.2.10)

$$\frac{d}{dt}\|u_1^n\|_1 + \left(\left\|\nabla u_1^{\frac{n}{2}}\right\|_2^2 + \left\|\nabla u_1^{\frac{n+1}{2}}\right\|_2^2\right) \lesssim n^2 + n^c\left(\left\|u_1^{\frac{n}{2}}\right\|_1^2 + \left\|u_1^{\frac{n}{2}}\right\|_1^{\gamma_n}\right). \tag{4.2.16}$$

Again by Galiardo-Nirenberg inequality and Young's inequality

$$\left\|u_1^{\frac{n}{2}}\right\|_2 \lesssim \left\|\nabla u_1^{\frac{n}{2}}\right\|_2^{\theta}\left\|u_1^{\frac{n}{2}}\right\|_1^{1-\theta} + \left\|u_1^{\frac{n}{2}}\right\|_1 \lesssim \left\|\nabla u_1^{\frac{n}{2}}\right\|_2 + \left\|u_1^{\frac{n}{2}}\right\|_1$$

where $\theta = \frac{1}{2}/\left(\frac{1}{2}+\frac{1}{d}\right)$. So for some universal $C, c > 0$

$$\left\|\nabla u_1^{\frac{n}{2}}\right\|_2^2 \geq C\left\|u_1^{\frac{n}{2}}\right\|_2^2 - c\left\|u_1^{\frac{n}{2}}\right\|_1^2. \tag{4.2.17}$$

By (4.2.16), (4.2.17), we have

$$\frac{d}{dt}\|u_1^n\|_1 + \|u_1^n\|_1 \lesssim n^2 + n^c\left\|u_1^{\frac{n}{2}}\right\|_1^{\gamma_n}.$$

Recall here $\gamma_n \leq 2 + \frac{C}{n}$.

Now letting $n = 2^k$ with $k \in \mathbb{N}^+$ and $A_k = \int_{\mathbb{R}^d} u_1^{n_k} dx$ gives

$$\frac{d}{dt}A_{k+1} + cA_{k+1} \leq C^k + C^k A_k^{2+C2^{-k}}. \tag{4.2.18}$$

By Lemma 1.2.2 and (4.2.18), $u_1(x,t)$ is uniformly bounded for all $t \geq 0$ and so is $u(x,t)$. □

Theorem 4.2.3. *Theorem 4.2.1 holds in the regime: $m \in (2-2s/d, 2)$ and $s \in (0, \frac{1}{2}]$.*

Proof. Recall (4.2.4), and for any $k \geq 1$ let $n = n_k \geq 3-m$, $l = n_{k-1} = \frac{m+n-1}{2}$. Multiplying u^{n-1} on both sides of (4.2.3), we obtain

$$\partial_t \int_{\mathbb{R}^d} u^n dx \leq -C_m \int_{\mathbb{R}^d} |\nabla u^l|^2 dx + Cn\int_{\mathbb{R}^d} V_{s,\epsilon} * u\nabla u^n dx. \tag{4.2.19}$$

For the interaction term:

$$\begin{aligned}
&\int_{\mathbb{R}^d} V_{s,\epsilon} * u\nabla u^n dx = \int_{\mathbb{R}^d} (-\nabla\cdot V_{s,\epsilon}) * u\ u^n dx \\
\lesssim &\int_{\mathbb{R}^d} \nabla(-\Delta)^{-s} u\nabla u^n dx \quad \text{(by property (2) of } V_{s,\epsilon} \text{ and Proposition 4.1.4)} \\
\lesssim &\int_{\mathbb{R}^d} \nabla(-\Delta)^{-s} u^l \nabla u^{n+1-l} dx \\
\lesssim &\int_{\mathbb{R}^d} \left||\nabla|^{1-2s} u^l\right| \left||\nabla| u^{n+1-l}\right| dx \quad \text{(by Fourier analysis).}
\end{aligned}$$

By Young's inequality, for any $p > 1, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the above

$$\begin{aligned}
&\lesssim n^{p/q} \left\| |\nabla|^{1-2s} u^l \right\|_p^p + \frac{1}{n} \left\| |\nabla| u^{n+1-l} \right\|_q^q \\
&:= n^{p/q} X_1^p + \frac{1}{n} X_2.
\end{aligned}$$

To choose the p, q, we use the scaling by considering $u_r = r^d u(rx, t)$. Then

$$\left\| |\nabla|^{1-2s} u_r^l \right\|_p^p = \left(r^{1-2s} r^{dl} \right)^p \left\| |\nabla|^{1-2s} u^l \right\|_p^p ;$$

$$\left\| |\nabla| u_r^{n+1-l} \right\|_q^q = \left(r r^{d(n+1-l)} \right)^q \left\| |\nabla| u^{n+1-l} \right\|_q^q .$$

To match the scaling, we want $(1-2s+dl)p = (1+d(n+1-l))q$ in the case when $m = 2 - \frac{2s}{d}$. Using $\frac{1}{p} + \frac{1}{q} = 1$, we find

$$p(n) = \frac{2 + d(m+n-1)}{1 + d(m+l-2)}, \; q(n) = \frac{2 + d(m+n-1)}{1 + d(n+1-l)}. \tag{4.2.20}$$

These are the values we pick for p, q. When $n = 3 - m, l = 1$, we obtain

$$p(3-m) = \frac{2+2d}{1+d(m-1)}, \; q(3-m) = \frac{2+2d}{1+d(3-m)}. \tag{4.2.21}$$

While as $n \to \infty$, $p(n)$ is monotonically decreasing, $q(n)$ is monotonically increasing and

$$p(n) \to 2, \; q(n) \to 2.$$

Also it is not hard to see that

$$p(n) > 1, \; q(n) > 1, \; 1 \leq \frac{p(n)}{q(n)} \leq \frac{1 + d(3-m)}{1 + d(m-1)} =: c_1(d, m),$$

$$p(n) - 2 = \frac{2d(2-m)}{1 + d(m+l-2)} \sim \frac{1}{n}, \tag{4.2.22}$$

$$2 - q(n) = \frac{2d(2-m)}{1 + d(n+1-l)} \sim \frac{1}{n}. \tag{4.2.23}$$

By Lemma A.1.6 and Young's inequality, for any $\delta \in (0, 1)$

$$\begin{aligned}
X_2 &\leq C \left\| \nabla u^{n+1-l} \right\|_q^q = C' \int_{\mathbb{R}^d} u^{(n+1-2l)q} \left| \nabla u^l \right|^q dx \\
&\leq C_\delta n \left\| u^{(n+1-2l)q} \right\|_{\frac{2}{2-q}}^{\frac{2}{2-q}} + \delta \left\| \left| \nabla u^l \right|^q \right\|_{\frac{2}{q}}^{\frac{2}{q}} \\
&= C_\delta n \left\| u^l \right\|_2^2 + \delta \left\| \nabla u^l \right\|_2^2 .
\end{aligned}$$

In the last inequality, we used (4.2.23). Now by Gagliardo-Nirenberg interpolation inequality

$$C_\delta \left\|u^l\right\|_2^2 \leq C_\delta \left\|\nabla u^l\right\|_2^{2\beta} \left\|u^l\right\|_1^{2(1-\beta)} \leq \frac{\delta}{n} \left\|\nabla u^l\right\|_2^2 + C_\delta n^{c_0} \left\|u^l\right\|_1^2$$

where

$$\beta = \frac{1}{2}/(\frac{1}{2} + \frac{1}{d}),\ c_0 = \frac{\beta}{1-\beta} = \frac{d}{2}.$$

Therefore

$$X_2 \leq C_\delta n^{c_0+1} \left\|u^l\right\|_1^2 + 2\delta \left\|\nabla u^l\right\|_2^2. \tag{4.2.24}$$

As for X_1, again by Gagliardo-Nirenberg interpolation inequality

$$X_1 = \left\||\nabla|^{1-2s} u^l\right\|_p \lesssim \left\|\nabla u^l\right\|_2^\alpha \left\|u^l\right\|_1^{1-\alpha}$$

where we need to put

$$\alpha = \alpha(n) = \left(\frac{1-2s}{d} - \frac{1}{p} + 1\right) \Big/ \left(\frac{1}{d} + \frac{1}{2}\right).$$

It is not hard to check that $s < \alpha(n) < 1$ uniformly for all $n \geq 3 - m$. Moreover, we claim that

$$\sup_{n \geq 3-m} \alpha(n) p(n) < 2.$$

By monotonicity of $\alpha(n)p(n)$ in n, we only need to check when $n = (3-m)$. By (4.2.21) and direct calculations,

$$\alpha(3-m)\, p(3-m) < 2 \iff m > 2 - \frac{2s}{d}.$$

With this, we obtain

$$X_1^p \lesssim \frac{\delta}{n^{c_1+1}} \|\nabla u^l\|_2^2 + C_\delta n^{c_2} \|u^l\|_1^{2+\gamma} \tag{4.2.25}$$

where c_2 is a constant that

$$c_2 \geq (c_1 + 1) \frac{\alpha(1-\alpha)p^2}{2 - \alpha p}$$

and by (4.2.22)

$$\gamma = \gamma(n) = \frac{2(p-2)}{2 - \alpha p} \sim \frac{1}{n}. \tag{4.2.26}$$

Putting together (4.2.24) and (4.2.25) shows

$$\begin{aligned} n\int V_{s,\epsilon} * u\nabla u^n dx &\leq n^{c_1+1}X_1^p + X_2 \\ &\leq C_\delta n^{c_2+c_1+1}\|u^l\|_1^{2+\gamma} + C_\delta n^{c_0+1}\left\|u^l\right\|_1^2 + 3\delta\left\|\nabla u^l\right\|_2^2. \end{aligned}$$

Picking δ small enough, (4.2.19) shows for $c^* = \max\{c_2+c_1+1, c_0+1\}$

$$\partial_t \int_{\mathbb{R}^d} u^n dx + \int_{\mathbb{R}^d} |\nabla u^l|^2 dx \leq C_\delta n^{c^*}\left\|u^l\right\|_1^{2+\gamma}. \tag{4.2.27}$$

As done in (4.2.9), for some $\gamma' \in (0,2)$

$$\|u^n\|_1 \lesssim \left\|\nabla u^l\right\|_2^2 + \left\|u^l\right\|_1^{\gamma'} \lesssim \left\|\nabla u^l\right\|_2^2 + \left\|u^l\right\|_1^{2+\gamma} + 1.$$

To conclude, we find out that

$$\frac{d}{dt}\|u^n\|_1 + \|u^n\|_1 \leq C + Cn^{c^*}\left\|u^l\right\|_1^{2+\gamma}$$

where $C, c^* > 0$ only depends on s, d, m.

Finally as in Theorem 4.2.1, since $n = n_k, l = n_{k-1}$ in (4.2.4), we proved the desired differential inequalities for all k. By considering $A_k = \int_{\mathbb{R}^d} u^{n_k} dx$, we conclude the proof after applying Lemma 1.2.2.

□

Theorem 4.2.4. *Theorem 4.2.1 holds in the regime:* $m \in (2-2s/d, 2)$ *and* $s \in (1, \frac{d}{2})$*.*

Proof. For $n \geq 3-m$, denote $l = \frac{n+m-1}{2} \geq 1$. We multiply u^{n-1} on both sides of (4.2.3) and obtain

$$\begin{aligned} \partial_t \int_{\mathbb{R}^d} u^n dx &\leq -mn\int_{\mathbb{R}^d} u^{m-1}\nabla u\nabla u^{n-1}dx + n\int_{\mathbb{R}^d}\left(V_{s,\epsilon} * u\right)u\nabla u^{n-1}dx \\ &\leq -C_m\int_{\mathbb{R}^d}|\nabla u^l|^2 dx - (n-1)\underbrace{\int_{\mathbb{R}^d}\left(\nabla\cdot V_{s,\epsilon} * u\right)u^n dx}_{X:=}. \end{aligned} \tag{4.2.28}$$

Let $\chi(x) = \chi_{|x|\leq 1}(x)$ be an indicator function. Taking $A_1 := \chi\nabla\cdot V_{s,\epsilon}$ and $A_2 := (1-\chi)\nabla\cdot V_{s,\epsilon}$ yield

1. A_1 is compactly supported and $|A_1|(z) \leq |z|^{-d-2+2s}$,

2. $|A_1|$ bounded in $L^{\frac{d}{d+2-2s'}}(\mathbb{R}^d)$ for all $1 < s' < s$,

3. A_2 is bounded.

Fix one s' such that

$$s' \in (1, s) \quad \text{and} \quad m > 2 - \frac{2s'}{d}.$$

We have

$$X \leq \int_{\mathbb{R}^d} |A_1| * u\, u^n dx + C \int_{\mathbb{R}^{2d}} |A_2|_\infty u(y) u^n(x) dx dy =: X_1 + X_2.$$

By Young's convolution inequality

$$X_1 \leq \|u^n\|_p \|u\|_q \tag{4.2.29}$$

with

$$p,\, q \geq 1 \quad \text{satisfying } \frac{1}{p} + \frac{1}{q} = 1 + \frac{2s'-2}{d}. \tag{4.2.30}$$

Claim: for any $\delta > 0$, there exist $C(\delta), c(\delta) > 0$ such that for all $n \geq 3 - m, l = \frac{n+m-1}{2}$

$$X_1 \leq Cn^c + Cn^c \left\|u^l\right\|_1^{2+\frac{c}{n}} + \frac{\delta}{n} \left\|\nabla u^l\right\|_2^2. \tag{4.2.31}$$

We will discuss the proof below according to different values of n.

(i) Suppose $n \leq \frac{d}{2s'-2}$. We can write

$$\|u^n\|_p = \left\|u^l\right\|_{\frac{np}{l}}^{\frac{n}{l}}, \quad \|u\|_q = \left\|u^l\right\|_{\frac{q}{l}}^{\frac{1}{l}}. \tag{4.2.32}$$

By Gagliardo-Nirenberg inequality and Young's inequality,

$$\left\|u^l\right\|_{\frac{np}{l}} \leq C \left\|\nabla u^l\right\|_2^\alpha \left\|u^l\right\|_1^{1-\alpha}, \quad \left\|u^l\right\|_{\frac{q}{l}} \leq C \left\|\nabla u^l\right\|_2^\beta \left\|u^l\right\|_1^{1-\beta} \tag{4.2.33}$$

where α, β are given by

$$\frac{l}{np} + (\frac{1}{2} + \frac{1}{d})\alpha = 1, \quad \frac{l}{q} + (\frac{1}{2} + \frac{1}{d})\beta = 1. \tag{4.2.34}$$

Now we take

$$p = \frac{n+1}{n} / \left(1 + \frac{2s'-2}{d}\right),\ q = (n+1) / \left(1 + \frac{2s'-2}{d}\right).$$

Since $n \leq \frac{d}{2s'-2}$, (4.2.30) is fulfilled. With this choice of p, q, by (4.2.34) we have

$$\alpha = \beta = \left(\frac{1}{2} + \frac{1}{d}\right)^{-1} \left(1 - \frac{l}{n+1}\left(1 + \frac{2s'-2}{d}\right)\right) > 0$$

uniformly in $n \geq 3-m$ by $n \leq \frac{d}{2s'-2}$. By definitions of n, l, we compute

$$\begin{aligned} 2 - \alpha\frac{n+1}{l} &= (\frac{1}{2} + \frac{1}{d})^{-1}(2 - \frac{n+1}{l} + \frac{2s'}{d}) \\ &\geq (\frac{1}{2} + \frac{1}{d})^{-1}(m - 2 - \frac{2s'}{d}) \end{aligned}$$

with equality holds when $n = 3-m$. Thus we get

$$\sup_{n \geq 3-m} \alpha \frac{n+1}{l(n)} < 2 \tag{4.2.35}$$

and this is exactly equivalent to $m > 2 - \frac{2s'}{d}$. In particular, we have $\sup_{n\geq 3-m} \alpha < 1$ due to $n+1 \geq 2l$.

From (4.2.29), (4.2.32) and (4.2.33), it follows that

$$X_1 \leq C \left\|\nabla u^l\right\|_2^{\alpha\frac{n+1}{l}} \left\|u^l\right\|_1^{(1-\alpha)\frac{n+1}{l}}.$$

By Young's inequality,

$$X_1 \leq \frac{\epsilon^{p_1} \left\|\nabla u^l\right\|_2^{\alpha\frac{n+1}{l}p_1}}{p_1} + \frac{C^{q_1}\left\|u^l\right\|_1^{(1-\alpha)\frac{n+1}{l}q_1}}{\epsilon^{q_1} q_1} \text{ for any } \epsilon > 0,$$

where we select

$$p_1(n) = \frac{2}{\alpha\frac{n+1}{l}}, \quad q_1(n) = \frac{p_1}{p_1 - 1}.$$

By (4.2.35), we obtain $\inf_{n\geq 3-m} p_1(n) > 1$ and thus q_1 is uniformly bounded. Pick $\epsilon = (\delta/n)^{1/p_1}$. Finally because q_1 is bounded and $\frac{n+1}{l} \leq 2 + \frac{c}{n}$ for some universal c, we conclude with (4.2.31).

(ii) If $l \leq \frac{d}{2s'-2} \leq n$, use $q = \frac{d}{2s'-2}$, $p = 1$ in (4.2.29) and calculate α, β accordingly by (4.2.34). In this situation, we get

$$1 > \frac{l}{np}, \frac{l}{q} > \frac{1}{2},$$

and thus it is immediately to check that $\alpha, \beta \in (0,1)$. (4.2.29) and (4.2.33) yield

$$X_1 \leq C \left\|\nabla u^l\right\|_2^{\alpha\frac{n}{l}+\beta\frac{1}{l}} \left\|u^l\right\|_1^{(1-\alpha)\frac{n}{l}+(1-\beta)\frac{1}{l}}. \tag{4.2.36}$$

Direction computation shows

$$\sup_{n\geq 3-m} \alpha\frac{n}{l(n)} + \beta\frac{1}{l(n)} < 2 \quad \text{and this is equivalent to} \quad m > 2 - \frac{2s'}{d}.$$

Because

$$\alpha\frac{n}{l} + \beta\frac{1}{l} + (1-\alpha)\frac{n}{l} + (1-\beta)\frac{1}{l} = \frac{n+1}{l} \sim 2 + \frac{c}{n}$$

holds for some universal $c > 0$, we obtain (4.2.31) by (4.2.36) and Hölder's inequality.

(iii) Lastly suppose $n > l(n) \geq \frac{d}{2s'-2}$. We take $p = 1$, $q = \frac{d}{2s'-2}$ in (4.2.29). Since $\|u\|_1$ is bounded, the set $\{u > 1\}$ is of finite measure. By the assumption, we get $q \leq l$. By Jensen's inequality

$$\int_{\mathbb{R}^d} u^q dx \leq \int_{u<1} u dx + \int_{u>1} u^q dx \leq C + C\|u\|_l^q.$$

Thus

$$X_1 = \|u^n\|_1 \|u\|_q \leq C\|u^n\|_1 (1 + \left\|u^l\right\|_1^{\frac{1}{l}}). \tag{4.2.37}$$

By Gagliardo-Nirenberg,

$$\|u^n\|_1^{\frac{l}{n}} = \left\|u^l\right\|_{\frac{n}{l}} \leq C\left\|\nabla u^l\right\|_2^{\alpha} \left\|u^l\right\|_1^{1-\alpha}$$

where α is given by $\frac{l}{n} + (\frac{1}{2} + \frac{1}{d})\alpha = 1$. From this we get

$$\begin{aligned} X_1 &\leq C\left\|\nabla u^l\right\|_2^{\alpha\frac{n}{l}} \left(\left\|u^l\right\|_1^{(1-\alpha)\frac{n}{l}} + \left\|u^l\right\|_1^{(1-\alpha)(\frac{n}{l}+\frac{1}{l})}\right) \\ &\leq C\left\|\nabla u^l\right\|_2^{\alpha\frac{n}{l}} \left(1 + \left\|u^l\right\|_1^{(1-\alpha)(\frac{n}{l}+\frac{1}{l})}\right) \\ &\leq Cn^c + \frac{\delta}{n}\left\|\nabla u^l\right\|_2^2 + C\left\|\nabla u^l\right\|_2^{\alpha\frac{n}{l}} \left\|u^l\right\|_1^{(1-\alpha)(\frac{n}{l}+\frac{1}{l})}. \end{aligned} \tag{4.2.38}$$

In the last line we used the mean inequality and

$$\alpha\frac{n}{l}=\frac{n-l}{l}/\left(\frac{1}{2}+\frac{1}{d}\right)<\frac{d}{d+2}<2.$$

Next compute

$$2-\frac{n}{l}\alpha=(\frac{1}{2}+\frac{1}{d})^{-1}(2+\frac{2}{d}-\frac{n}{l})\geq(\frac{1}{2}+\frac{1}{d})^{-1}\frac{2}{d},$$

and we find

$$\sup_{n\geq 3-m}\frac{n}{l(n)}\alpha<2. \tag{4.2.39}$$

As before after applying Hölder's inequality in (4.2.38), we can prove (4.2.31). (4.2.39) shows that the constants in (4.2.31) can be chosen independent of n.

In all we finished the proof of the claim and proved (4.2.31). As for X_2, note that $X_2\leq C\left\|u^n\right\|_1$, so it can be handled similarly as the estimate (4.2.37).

Putting together the estimates (4.2.28), (4.2.31) and taking δ to be small, we obtain for all $n\geq 3-m$

$$\partial_t\left\|u^n\right\|_1+c\left\|\nabla u^l\right\|_2^2\leq C_\delta n^c+C_\delta n^c\left\|u^l\right\|_1^{2+\frac{c}{n}} \tag{4.2.40}$$

for some $C,c>0$ independent of n. Using (4.2.9) and (4.2.40), we get

$$\frac{d}{dt}A_{k+1}+cA_{k+1}\leq C^k+C^kA_k^{2+C2^{-k}}$$

with $A_k=\|u^{n_k}\|_1$ and $n_k=2^k(2-m)-1+m$. Finally the proof follows as before.

□

Theorem 4.2.5. *Theorem 4.2.1 holds in the regime: $m\geq 2$ and $s\in(1,\frac{d}{2})$.*

Proof. For any $n>1$, we multiply u_1^{n-1} on both sides of (4.2.3) where $u_1=(u-1)_+$. We have

$$\begin{aligned}\partial_t\int_{\mathbb{R}^d}u_1^n dx=&n\int_{\mathbb{R}^d}u_1^{n-1}u_t dx\\ &\leq -mn\int_{\mathbb{R}^d}u^{m-1}\nabla u\nabla u_1^{n-1}dx+n\underbrace{\int_{\mathbb{R}^d}\left(V_{s,\epsilon}*u\right)u\nabla u_1^{n-1}dx}_{Y:=}\end{aligned}$$

Since $m \geq 2$,

$$n \int_{\mathbb{R}^d} u^{m-1} \nabla u \nabla u_1^{n-1} dx \geq n \int_{\mathbb{R}^d} (1+u_1) \nabla u_1 \nabla u_1^{n-1} dx \geq C \int_{\mathbb{R}^d} \left| \nabla u_1^{\frac{n}{2}} \right|^2 + \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 dx,$$

we obtain

$$\partial_t \int_{\mathbb{R}^d} u_1^n dx \leq -C_m \int_{\mathbb{R}^d} \left| \nabla u_1^{\frac{n}{2}} \right|^2 + \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 dx + nY. \tag{4.2.41}$$

Recall the notation $\tilde{u} = u - u_1$. For Y, we have

$$\begin{aligned}
Y &= \frac{n-1}{n} \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u_1^n dx + \int_{\mathbb{R}^d} V_{s,\epsilon} * u \; \nabla u_1^{n-1} dx \\
&\lesssim \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u \, u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u \, u_1^{n-1} dx \\
&\lesssim \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * (u_1 + \tilde{u}) \, u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u \, (u_1^n + 1_{\{u_1<1\}}) dx \\
&\lesssim \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u_1 \, u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * \tilde{u} \, u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * 1_{\{u_1<1\}} \, u \, dx.
\end{aligned}$$

Because $|\nabla \cdot V_{s,\epsilon}| \lesssim |x|^{-d-2+2s}$, $s > 1$ and

$$|\tilde{u}|, \, |1_{\{u_1<1\}}| \leq 1; \quad \tilde{u}(\cdot,t), \, 1_{\{u_1<1\}}(\cdot,t) \in L^1(\mathbb{R}^d),$$

we have for some universal constant C

$$|\nabla \cdot V_{s,\epsilon}| * \tilde{u} + |\nabla \cdot V_{s,\epsilon}| * 1_{\{u_1<1\}} \leq C.$$

Also due to (4.2.42) and $u(\cdot,t) \in L^1(\mathbb{R}^d)$, we deduce

$$Y \lesssim \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u_1 \, u_1^n dx + \int_{\mathbb{R}^d} u_1^n dx + 1.$$

Next fix one $\tilde{s} \in (1, s)$. By Young's convolution inequality,

$$Y \lesssim \|u_1^n\|_p \|u_1\|_q + \|u_1^n\|_1 + 1 \tag{4.2.42}$$

where p, q satisfy

$$p, \, q \geq 1, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{2\tilde{s} - 2}{d}.$$

Now fix one $\tilde{m} \in (2 - \frac{2\tilde{s}}{d}, 2)$ and set $l = \frac{n+\tilde{m}-1}{2}$. Note that

$$\left\|\nabla u_1^l\right\|_2^2 \lesssim \left\| \nabla u_1^{\frac{n}{2}} \right\|_2^2 + \left\| \nabla u_1^{\frac{n+1}{2}} \right\|_2^2.$$

It then follows from (4.2.41) and (4.2.42) that

$$\partial_t \|u^n\|_1 + c\left\|\nabla u^l\right\|_2^2 \le Cn\|u_1^n\|_p\|u_1\|_q + Cn\|u_1^n\|_1 + Cn.$$

Then we only need to show (4.2.40) with m, s replaced by $\tilde{m}, \tilde{s}$, and after that the proof follows the same as before.

To show (4.2.40), actually $\|u_1^n\|_1$ can be treated the same as in (4.2.9). For $\|u_1^n\|_p\|u_1\|_q$, we go to (4.2.29) and follow the proof of the Claim below it.

$\square$

4.3 Existence of Solutions

In this section, we show the existence of weak solutions to (4.1.1) in the subcritical regime with $s \in (0, \frac{d}{2})$. Let u_ϵ be a solution to (4.2.3). By Theorems 4.2.1-4.2.5,

$$\sup_{\epsilon\in(0,1),t\ge 0} \|u_\epsilon(\cdot,t)\|_1 + \|u_\epsilon(\cdot,t)\|_\infty < \infty. \tag{4.3.1}$$

First consider the case when $s > \frac{1}{2}$. Notice $|V_{s,\epsilon}|$ is locally integrable near the origin, and so

$$|V_{s,\epsilon} * u_\epsilon(x)| \le C\int_{|x-y|\le 1} |V_{s,\epsilon}(x-y)|dy + C\int_{|x-y|>1} u_\epsilon(y)dy < \infty \tag{4.3.2}$$

uniformly in ϵ. We have the following theorem.

Theorem 4.3.1. *Assume* (4.1.4) *and* $s \in (\frac{1}{2}, \frac{d}{2})$. *Then there exists a weak solution* u *to* (4.1.1) *with initial data* u_0 *and* u *preserves the mass.*

This existence result can be established by a compactness argument by taking $\epsilon \to 0$ in (4.2.3). With the help of (4.3.1), (4.3.2), the proof (which we skip) is parallel to the one in [5], see their Theorems 1,2,7.

Let us focus on the situation $s \le \frac{1}{2}$. We need the following a priori estimate.

Lemma 4.3.2. *Assume (4.1.4), $s \in (0, \frac{1}{2}]$ and let u_ϵ be the solution to (4.2.3). Then for any $T > 0$, there exists a constant C_T independent of ϵ such that*

$$\|\nabla u_\epsilon\|_{L^2(\mathbb{R}^d\times[0,T])} + \|V_{s,\epsilon} * u_\epsilon\|_{L^2(\mathbb{R}^d\times[0,T])} \le C_T. \tag{4.3.3}$$

Proof. Recall (4.2.27) and we take $n = 3 - m$. After integrating in time, we find

$$\int_{\mathbb{R}^d} u_\epsilon^{3-m} dx(T) + \iint_{\mathbb{R}^d\times[0,T]} |\nabla u_\epsilon|^2 dxdt \le \int_{\mathbb{R}^d} u_\epsilon^{3-m} dx(0) + C|3-m|^c \int_0^T \|u_\epsilon\|_1^{2+\gamma}\, dt.$$

Since $\|u_\epsilon\|_{3-m}(t), \|u_\epsilon\|_1(t)$ are uniformly bounded in time,

$$\|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])} \le C + CT. \tag{4.3.4}$$

Denote $\rho = |x|$ and recall (4.2.1), we have

$$V_{s,\epsilon}(x) := c\,\zeta_\epsilon(\rho)\nabla_x\, \rho^{-d+2s}.$$

We can find $g(\rho) : (0,\infty) \to \mathbb{R}$ such that

$$g'(\rho) = c(-d+2s)\zeta_\epsilon(\rho)\rho^{-d-1+2s} \quad \text{and} \quad g(1) = c. \tag{4.3.5}$$

By (4.2.2), we have

$$|g(\rho)| \le C\rho^{-d+2s}, \quad |g'(\rho)| \le C\rho^{-d-1+2s}$$

for some $C > 0$ only depend on d, s. Actually for $\rho \in [2\epsilon, 1/\epsilon]$, we know $g(\rho) = c\rho^{-d+2s}$.

Let $\varphi : [0,\infty) \to [0,1]$ be a smooth bump function that $\varphi(\rho) = 1$ for $\rho \le 1$ and $\varphi(\rho) = 0$ for $\rho \ge 2$. Let us decompose g into two parts and write

$$g = g_s + g_b := \varphi g + (1-\varphi)g.$$

Seeing from (4.3.5), for some universal constant $C = C(d,s) > 0$

$$\|g_s\|_1 \le c\int_\epsilon^2 \rho^{-d+2s}\rho^{d-1} d\rho \le C, \tag{4.3.6}$$

$$\begin{aligned}\|\nabla g_b\|_2^2 &\le C\int_{|x|\ge 1} |(\nabla\varphi)\, g|^2(x) + |\nabla g|^2(x) dx \\ &\le C\int_1^2 (\rho^{-d+2s})^2\rho^{d-1} d\rho + C\int_1^\infty (\rho^{-d-1+2s})^2 \rho^{d-1} d\rho \\ &\le C \quad (\text{ since } d \ge 3, s \le 1).\end{aligned} \tag{4.3.7}$$

It is not hard to see

$$\|V_{s,\epsilon} * u_\epsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])} \le 2\,\|g_s(|x|) * \nabla u_\epsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])} + 2\,\|\nabla g_b(|x|) * u_\epsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])}\,.$$
$$=: 2X_1 + 2X_2.$$

Using Proposition A.1.5, (4.3.4) and (4.3.6) give

$$X_1 = \int_0^T \|g_s * \nabla u_\epsilon\|_2^2\,dt \le \int_0^T \|g_s\|_1^2\,\|\nabla u_\epsilon\|_2^2\;dt$$
$$= \|g_s\|_1^2\,\|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])} < \infty.$$

For X_2, by (4.3.7) we obtain

$$X_2 \le \iint_{\mathbb{R}^d\times[0,T])} \int_{\mathbb{R}} |\nabla g_b|^2(x-y)u_\epsilon^2(y)dydxdt \le C \iint_{\mathbb{R}\times[0,T]} u_\epsilon^2(y)dydt < \infty.$$

In all, we have proved

$$\|V_{s,\epsilon} * u_\epsilon\|_{L^2(\mathbb{R}^d\times[0,T])} \le C$$

where C only depends on d, s, T and $\|u_\epsilon\|_1 + \|u_\epsilon\|_\infty$.

□

Theorem 4.3.3. *Assume* (4.1.4), $s \in (0, \frac{1}{2}]$*. Then there exists a weak solution to* (4.1.1) *with initial data* u_0*.*

Proof. For any small $\epsilon > 0$, let u_ϵ be a solution to (4.2.3). Let us show the following tightness of $\{u_\epsilon(\cdot, t)\}_\epsilon$ in $L^1(\mathbb{R}^d)$: for any $T > 0$

$$\lim_{R\to\infty} \int_{B_{2R}^c} u_\epsilon(x,t)dx \to 0 \text{ uniformly in } \epsilon \in (0,1) \text{ and } t \in [0,T]. \tag{4.3.8}$$

Take a function $\varphi = \varphi_{N,R} \in C_0^\infty(\mathbb{R}^d)$ such that for some $N >> 1$

$$\varphi = 0 \quad \text{ in } |x| \le R,$$
$$\varphi = 1 \quad \text{ in } 2R \le |x| \le NR,$$
$$0 \le \varphi \le 1,\ |\nabla\varphi| \lesssim R^{-1}, |\Delta\varphi| \lesssim R^{-1} \quad \text{ in } \mathbb{R}^d.$$

By the equation (4.2.3), for any $t \in (0,T]$

$$\int_{\mathbb{R}^d} u_\epsilon \varphi \, dx(t) = \int_{\mathbb{R}^d} u_\epsilon \varphi \, dx(0) + \underbrace{\int_0^t \int_{\mathbb{R}^d} (\epsilon u_\epsilon + u_\epsilon^m) \Delta \varphi \, dxdt}_{Y_1 :=} + \underbrace{\int_0^t \int_{\mathbb{R}^d} (u_\epsilon V_{s,\epsilon} * u_\epsilon) \nabla \varphi \, dxdt}_{Y_2 :=} .$$

By (4.3.1) and the condition $|\Delta \varphi| \lesssim R^{-1}$, Y_1 converges to 0 as $R \to \infty$ uniformly in ϵ and $t \leq T$. Next by Hölder's inequality and Lemma 4.3.2,

$$Y_2 \leq \|V_{s,\epsilon} * u_\epsilon\|_2 \, \|u_\epsilon \nabla \varphi\|_2 \leq C_T R^{-1}.$$

Combining the assumption that $u_0 \in L^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u_\epsilon(x,t) \varphi(x) dx \to 0 \text{ as } R \to 0$$

uniformly in ϵ, N. Finally letting $N \to \infty$, we proved (4.3.8).

Next by (4.3.3) in Lemma 4.3.2, $\|\nabla u_\epsilon\|_{L^2(\mathbb{R}^d \times [0,T])} \leq C_T$. This, as well as (4.3.1) and (4.3.8), implies that $\{u_\epsilon\}_\epsilon$ is precompact in $L^1(0,T,L^1(\mathbb{R}^d))$. The proof follows from the work of [5,6]. Thus by passing $\epsilon \to 0$ along subsequences, we have $u_\epsilon \to u$ in $L^1(0,T,L^1(\mathbb{R}^d))$. Again due to (4.3.3), we can have

$$u \in L^2(0,T,\dot{H}^1(\mathbb{R}^d)).$$

Therefore $\nabla(-\Delta)^{-s}u$ is well-defined which is a bounded function in $L^2(\mathbb{R}^d \times [0,T])$. Now we need to show the weak convergence of $V_{s,\epsilon} * u_\epsilon$ to $\nabla(-\Delta)^{-s}u$.

Let $\xi \in C_0^\infty(\mathbb{R}^d \times [0,T], \mathbb{R}^d)$ be a test function. We have

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times [0,T]} \left(V_{s,\epsilon} * u_\epsilon - \nabla(-\Delta)^{-s}u\right) \xi \, dxdt \\
&= \iint_{\mathbb{R}^d \times [0,T]} V_{s,\epsilon} * \xi \, u_\epsilon - \nabla \cdot (-\Delta)^{-s} \xi \, u \, dxdt \\
&\leq C \underbrace{\iint_{\mathbb{R}^d \times [0,T]} \left|V_{s,\epsilon} * \xi \; - \nabla \cdot (-\Delta)^{-s}\xi\right| dxdt}_{X :=} + C \iint_{\mathbb{R}^d \times [0,T]} \left|\nabla \cdot (-\Delta)^{-s}\xi\right| |u_\epsilon - u| \, dxdt.
\end{aligned} \tag{4.3.9}$$

Here we used that u_ϵ, u are uniformly bounded. Note that $u_\epsilon \to u$ in $L^1(\mathbb{R}^d \times [0,T])$, and hence to show the above integral converges to 0 as $\epsilon \to 0$, we only need to show $X \to 0$.

Suppose $\xi = 0$ in $B^c_{R_\xi} \times [0,T]$ for some $R_\xi \in (0, 4/\epsilon)$ and hence by (4.2.1), (4.2.2)

$$
\begin{aligned}
X &\leq C \iiint_{\mathbb{R}^{2d}\times[0,T]} |(\zeta_\epsilon(x-y)-1)\nabla_x K_s(x,y)|\,|\xi(y,t)-\xi(x,t)|\,dydxdt \\
&\leq C Lip(\xi) \iiint_{|x-y|\leq 2\epsilon} |x-y|^{-d-1+2s}|x-y| \left(\chi_{|x|\leq R_\xi} + \chi_{|y|\leq R_\xi}\right) dxdydt \\
&\leq 2C\, Lip(\xi)\, T \iint_{|x|\leq R_\xi, |z|\leq 2\epsilon} |z|^{-d+2s} dzdx \\
&\leq C(d,s)\, Lip(\xi)\, T\, R^d_\xi\, \epsilon^{2s}
\end{aligned}
$$

which converges to 0 as $\epsilon \to 0$. Thus $V_{s,\epsilon} * u_\epsilon \to \nabla(-\Delta)^{-s}u$ weakly in distribution.

Again by (4.3.3) and interpolation,

$$\|\nabla(-\Delta)^{-s}u_\epsilon\|_{L^2(\mathbb{R}^d\times[0,T])} \leq C_T$$

So actually we have

$$V_{s,\epsilon} * u_\epsilon \to \nabla(-\Delta)^{-s}u \text{ weakly in } L^2(\mathbb{R}^d \times [0,T])$$

which gives

$$u_\epsilon V_{s,\epsilon} * u_\epsilon \to u\nabla(-\Delta)^{-s}u \text{ weakly in } L^1(\mathbb{R}^d \times [0,T]).$$

We proved the existence of weak solutions.

From the equation and (4.3.8), we deduce the mass preservation of u: for all $t > 0$. Finally the property $u \in C([0,T], L^1(\mathbb{R}^d))$ follows from [5,6].

$\square$

4.4 Uniqueness and Hölder Regularity

This section is concerned with the uniqueness and the continuity properties of (4.1.1) in the subcritical regime for some s.

Theorem 4.4.1. *Assume* (4.1.4) *and* $s \in (1, \frac{d}{2})$. *Then there is a unique weak solution to* (4.1.1) *with initial data* u_0.

Proof. Fix any $T > 0$, let u_1, u_2 be two weak solutions in $\mathbb{R}^d \times [0, T]$ with the same initial data. By definition they satisfy (4.1.5).

We will follow the approach of [5,6] and estimate the difference of u_1, u_2 in $\dot{H}^{-1}$. For each $t > 0$, define $\phi(\cdot, t)$ through

$$\Delta\phi(x,t) = u_1(x,t) - u_2(x,t) \quad \text{and} \quad \lim_{|x|\to\infty} \phi(x,t) = 0.$$

By the equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla\phi|^2 dx = \int_{\mathbb{R}^d} (\nabla u_1^m - \nabla u_2^m)\nabla\phi\, dx - \int_{\mathbb{R}^d} (u_1 - u_2)(\nabla\mathcal{K}_s u_1)\nabla\phi\, dx \\ - \int_{\mathbb{R}^d} u_2(\nabla\mathcal{K}_s(u_1 - u_2))\nabla\phi\, dx =: X_1 + X_2 + X_3.\end{aligned}$$

Direct computations yields

$$X_1 = -\int_{\mathbb{R}^d} (u_1^m - u_2^m)(u_1 - u_2) \le 0.$$

Note that $|D^2K_s(z)| \sim |z|^{-d-2+2s}$ and $d + 2 - 2s \in (2, d)$. Therefore, denoting

$$A_1(z) := \chi_{|z|\ge 1} D^2K_s(z), \quad A_2(z) := \chi_{|z|<1} D^2K_s(z),$$

we have $A_1(z)$ is bounded and $A_2(z) \in L^1$. Since u_1 is uniformly bounded for $t \in [0, T]$,

$$\begin{aligned}\left|D^2K_s * u_1\right|(x) &\le C\int_{\mathbb{R}^d} |A_1(x-y)|u_1(y)dy + C\int_{\mathbb{R}^d} |A_2(x-y)|u_1(y)dy \\ &\lesssim \int_{\mathbb{R}^d} u_1(y)dy + \int_{\mathbb{R}^d} |A_2(x-y)|dy \le C.\end{aligned}$$

We get

$$X_2 = -\int_{\mathbb{R}^d} \Delta\phi\, \nabla\mathcal{K}_s u_1 \nabla\phi\, dx = \int_{\mathbb{R}^d} \nabla\phi\, D^2\mathcal{K}_s u_1 \nabla\phi\, dx \le C\int_{\mathbb{R}^d} |D^2\mathcal{K}_s u_1||\nabla\phi|^2 dx \le C\|\nabla\phi\|_2^2.$$

As for X_3, by Young's convolution inequality,

$$\begin{aligned}X_3 &= \int_{\mathbb{R}^d} u_2(D^2K_s * \nabla\phi)\nabla\phi\, dx \\ &= \int_{\mathbb{R}^d} u_2(A_1(z) * \nabla\phi)\nabla\phi\, dx + \int_{\mathbb{R}^d} u_2(A_2(z) * \nabla\phi)\nabla\phi\, dx \\ &\le C\left\|A_1 * \nabla\phi\right\|_2 \left\|\nabla\phi\right\|_2 + C\left\|A_2\right\|_1 \left\|\nabla\phi\right\|_2^2 \\ &\le C\left\|\nabla\phi\right\|_2^2 \quad (\text{ since } A_1 \text{ is bounded}).\end{aligned}$$

Set $\eta(t) = \|\nabla\phi\|_2^2$ and we find

$$\frac{d}{dt}\eta(t) \leq C\eta(t).$$

Also we have $\eta(0) = 0$ due to $u_1(x,0) - u_2(x,0) = 0$. By Gronwall's inequality $\eta(t) = 0$ and we find $u_1(\cdot,t) = u_2(\cdot,t)$ in $\dot{H}^{-1}$ for all $t \in [0,T]$. Since T is arbitrary, we conclude the proof of the theorem.

$\square$

Now consider the regularity problems with $s > 1/2$. Let u be a solution to (4.1.1) and denote

$$V(x,t) := \nabla\mathcal{K}_s u(x,t).$$

Then we can rewrite the equation as

$$u_t = \Delta u^m + \nabla\cdot(Vu). \tag{4.4.1}$$

By Theorems 4.2.1- 4.2.5, in the subcritical regime, u is uniformly bounded in $L^\infty(\mathbb{R}^d \times [0,\infty))$ and $\|u(\cdot,t)\|_1 = \|u_0\|_1 < \infty$. Thus

$$\begin{aligned}
|V(x,t)| = |\nabla\mathcal{K}_s u(x,t)| &\lesssim \int_{\mathbb{R}^d} |x-y|^{-d-1+2s} u(y,t)dy \\
&\lesssim \int_{|x-y|\leq 1} |y|^{-d-1+2s}dy + \int_{|x-y|\geq 1} u(y,t)dy \\
&\leq C
\end{aligned} \tag{4.4.2}$$

which only depending on $d, s, \|u_0\|_1 + \|u\|_\infty$. Following from the proof of Theorem 4.1 [50] which studied (4.4.1), we deduce that weak solutions to (4.1.1) are Hölder continuous for all $t > 0$ and we proved Theorem 4.1.3.

APPENDIX A

Appendix

A.1 Some Inequalities

A.1.1 Embedding Inequalities

Take $p \geq 1$ and consider the Banach spaces

$$V^p(\Omega_T) := L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$$

and

$$V_0^p(\Omega_T) := L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)),$$

both equipped with the norm $v \in V^p(\Omega_T)$,

$$\|v\|_{V^p(\Omega_T)} := \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{p,\Omega} + \|\nabla v\|_{p,\Omega_T}.$$

Now we introduce some embedding inequalities (Refer Chapter I in [36]).

Theorem A.1.1. *(Gagliardo-Nirenberg embedding inequality) Let $v \in W_0^{1,p}(\Omega)$, $p \geq 1$. For every fixed number $s \geq 1$ there exists a constant C depending only upon d, p and s such that*

$$\|v\|_{q,\Omega} \leq C\|\nabla v\|_{p,\Omega}^{\alpha}\|v\|_{s,\Omega}^{1-\alpha},$$

where $\alpha \in [0,1]$, $p, q \geq 1$, are linked by

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{d} - \frac{1}{p} + \frac{1}{s}\right)^{-1},$$

and their admissible range is

$$\begin{cases} q \in [s, \infty], \ \alpha \in [0, \frac{p}{p+s(p-1)}], & \textit{if } d = 1, \\ q \in [s, \frac{dp}{d-p}], \ \alpha \in [0, 1], & \textit{if } 1 \le p < d, \ s \le \frac{dp}{d-p}, \\ q \in [\frac{dp}{d-p}, s], \ \alpha \in [0, 1], & \textit{if } 1 \le p < d, \ s \ge \frac{dp}{d-p}, \\ q \in [s, \infty), \ \alpha \in [0, \frac{dp}{dp+s(p-d)}), & \textit{if } 1 < d \le p. \end{cases}$$

Theorem A.1.2. *(Sobolev embedding theorem) Let $p > 1$. There exists a constant C depending only upon d, p such that for every $v \in L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$,*

$$\iint_{\Omega_T} |v(x,t)|^q \, dxdt \le C^q \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v(x,t)|^p \, dx \right)^{p/d} \left(\iint_{\Omega_T} |\nabla v(x,t)|^p \, dxdt \right)$$

where $q = \frac{p(d+p)}{d}$.

By Hölder's inequality, we have

Corollary A.1.3. *Let v be as the above. Then*

$$\|v\|_{q,\Omega_T} \le C \|v\|_{V^p(\Omega_T)} \ \textit{and}$$
$$\|v\|_{p,\Omega_T}^p \le C \, |\{|v| > 0\}|^{\frac{p}{d+p}} \, \|v\|_{V^p(\Omega_T)}^p.$$

Proposition A.1.4. *There exists a constant C depending only upon d and p such that for every $v \in V_0^p(\Omega_T)$,*

$$\|v\|_{q,r;\Omega_T} \le C \|v\|_{V^p(\Omega_T)}$$

where the numbers $q, r \ge 1$ are linked by

$$\frac{1}{r} + \frac{d}{pq} = \frac{d}{p^2},$$

and their admissible range is

$$\begin{cases} q \in (p, \infty), \ r \in (p^2, \infty); & \textit{for } d = 1, \\ q \in (p, \frac{dp}{d-p}), \ r \in (p, \infty); & \textit{for } 1 < p < d, \\ q \in (p, \infty), \ r \in (\frac{p^2}{d}, \infty); & \textit{for } 1 < d \le p. \end{cases}$$

Proof. Let $v \in V_0^p(\Omega_T)$ and let $r \geq 1$ to be chosen. From Theorem A.1.1 with $s = p$ follows that

$$\begin{aligned}&\left(\int_0^T \|v(\cdot,\tau)\|_{q,\Omega}^r \, d\tau\right)^{1/r} \\ &\leq C \left(\int_0^T \|\nabla v(\cdot,\tau)\|_p^{\alpha r} \, d\tau\right)^{1/r} \operatorname*{ess\,sup}_{0\leq r\leq T} \|v(\cdot,\tau)\|_{p,\Omega_T}^{1-\alpha}.\end{aligned}$$

Choose α such that $\alpha r = p$. $\square$

Proposition A.1.5. *[Young's convolution inequality] For all $p, q, r \in [1, \infty]$ satisfying $1 + 1/q = 1/p + 1/r$, we have for all functions $f \in L^p(\mathbb{R}^d), g \in L^r(\mathbb{R}^d)$*

$$\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}.$$

The following lemma is useful which can be proved by using Calderón-Zygmund inequality, see Theorem 4.3.3 [41].

Lemma A.1.6. *There exists a constant $C > 0$ such that for all $1 < p < \infty$ and $u \in W^{1,p}(\mathbb{R}^d)$*

$$\left\| |\nabla| u \right\|_p \leq C \max\{p,\, (p-1)^{-1}\} \left\| \nabla u \right\|_p.$$

A.1.2 Homogeneous Sobolev Space

We refer readers to [4].

Definition A.1.1. *Let $s \in \mathbb{R}$. The homogeneous Sobolev space is the space of tempered distributions f over $\mathbb{R}^d$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^d)$ and satisfies*

$$\|f\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi < \infty.$$

Proposition A.1.7. *If $|s| < \frac{d}{2}$, $\dot{H}^s$ can be considered as the dual space of $\dot{H}^{-s}$ through the following bilinear functional: for any $f \in \dot{H}^s, g \in \dot{H}^{-s}$, $(f, g) \to \int_{\mathbb{R}^d} f(x)g(x)dx$. Furthermore if $s = 1$, $\dot{H}^1$ is the subset of tempered distributions with locally integrable Fourier transforms and such that $|\nabla f| \in L^2(\mathbb{R}^d)$.*

A.1.3 Brunn-Minkowski Theorem

Lemma A.1.8. *Let $d \geq 1$ and let A and B be two nonempty compact subsets of $\mathbb{R}^d$. Then the following inequality holds:*

$$vol\{A+B\}^{1/d} \geq vol\{A\}^{1/d} + vol\{B\}^{1/d},$$

where $A+B$ denotes the Minkowski sum:

$$A+B := \{\, a+b \in \mathbb{R}^d \,|\, a \in A,\, b \in B \,\}.$$

A.2 Additional Computations for Chapter 1

A.2.1 Proof of Lemma 1.2.2

Suppose $B_k(0), B_0(t)$ are bounded by M, then $A_k(0) \leq M^{n_k}$. Solving the differential inequality gives that for all $t \geq 0$

$$A_k(t) \leq e^{-C_0 t} \int_0^t e^{C_0 s} \left({C_1}^{n_k} + {C_1}^{k} A_{k-1}^{2+C_1 n_k^{-1}}(s) \right) ds + M^{n_k}.$$

If $A_{k-1}(t)$ are uniformly bounded by M_{k-1} for all t, we can choose a constant C_2 depending only on (C_0, C_1, M) such that

$$A_k(t) \leq CC_1^{n_k} + CC_1^k M_{k-1}^{2+C_1 n_k^{-1}} + M^{n_k} \leq C_2^{n_k} + {C_2}^k M_{k-1}^{2+C_1 n_k^{-1}}. \tag{A.2.1}$$

We claim that it can be proved by induction that

$$A_k(t) \leq C_3^{c_k} \text{ for some constants } C_3(C_0, C_1, M), c_k(C_1, k). \tag{A.2.2}$$

Here $\{c_k\}$ is defined inductively by

$$c_0 = 1, \quad c_k := (2 + \frac{C_1}{n_k})c_{k-1} + k + 1. \tag{A.2.3}$$

By a slight abuse of notation, we will write C's as constants which only depend on C_1, C_0, M, a (independent of k) and they may vary from one expression to the other.

To see the claim, by induction taking $M_{k-1} = C_3^{c_{k-1}}$ in (A.2.1), we only need

$$C_2^{n_k} + C_2^k C_3^{c_{k-1}(2+C_1 n_k^{-1})} \le C_3^{c_k}.$$

And it is not hard to see by definition, $n_k \lesssim k + c_{k-1}(2 + C_1 n_k^{-1})$. So if choosing C_3 large enough, we only need

$$C_3^{k+1+c_{k-1}(2+C_1 n_k^{-1})} \le C_3^{c_k}$$

which is exactly (A.2.3). We proved the claim.

By (A.2.3) and simple calculations,

$$c_k = \left(\sum_{j=1}^{k} j\ b_{j,k}\right) + k + 1 \quad \text{where } b_{j,k} := \Pi_{i=j}^{k}(2 + \frac{C_1}{n_i}).$$

Notice $n_k = 2^k(a+1) - a$, there is a constant $C_4(a, C_1)$ that $\frac{C_1}{n_k} \le C_4 2^{-k}$ for all $k \ge 0$. So

$$b_{j,k} \le 2^{k-j+1} \Pi_{i=j}^{k}(1 + C_4 2^{-k-1}).$$

Then we apply the fact that given $x_n \ge 0$ and $\sum_n x_n \le C_4$, we have for some other constant $C > 0$

$$\Pi_n(1 + x_n) \le C + C \sum_n x_n.$$

We find out

$$b_{j,k} \le 2^{k-j+1} C(1 + C_4) \lesssim 2^{k-j} \quad \text{and}$$

$$c_k \le C 2^k \sum_{j=1}^{k} \frac{j}{2^j} + k + 1 \lesssim 2^k \lesssim n_k.$$

So $c_k \le C n_k$. By (A.2.2), we proved that

$$A_k^{(n_k^{-1})}(t) \le C_2^C \quad \text{uniformly for all } k \in \mathbb{N}^0 \text{ and } t \ge 0.$$

A.3 Additional Computations for Chapter 2

A.3.1 Proof of Lemma 2.4.3

The idea in Lemma 9 [15] is to compute

$$\overline{\lim}_{r\to 0}\left(\fint_{B_r} f(x) - f(0) dx\right).$$

Suppose locally near the origin

$$f(x) = \inf_{|\nu|=1} v\left(x + \varphi(x)\nu\right).$$

Choosing an appropriate system of coordinates, we can have

$$f(0) = v(\varphi(0)e_n);$$

$$\nabla\varphi(0) = \alpha e_1 + \beta e_n.$$

We will evaluate w by above by choosing $\nu(x) = \nu_*/|\nu_*|$ where

$$\nu_* = e_n + \frac{\beta x_1 - \alpha x_n}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)} \left(\Sigma_{i=2}^{d-1} x_i\, e_i\right)$$

with γ such that

$$(1+\gamma)^2 = (1+\beta)^2 + \alpha^2.$$

With this choice of ν, we define $y = x + \varphi(x)\nu(x)$. Next we can write

$$y = x + Y_* + o(|x|^2)$$

such that $Y_* - \varphi(0)e_n$ is a first-order term that can be thought as a rotation combined with an expansion. And

$$y(0) = \varphi(0)e_n, \qquad \left|\frac{D(Y_* - \varphi(0)e_n)}{Dx}\right| \leq \sigma\|\nabla\varphi\|_\infty.$$

Explicit formulas can be found in [15]. Then

$$\begin{aligned}\fint_{B_r} f(x) - f(0)dx &\leq \fint_{B_r} v(y(x)) - v(y(0))dx \\ &\leq \fint_{B_r} v(y(x)) - v(Y_*(x))dx + \fint_{B_r} v(Y_*(x)) - v(y(0))dx.\end{aligned}$$

By the condition on φ and the computations done in Lemma 9 [15], the first term is non-negative.

If v is smooth, the second term converges to $\left|\frac{DY_*}{Dx}\right|_{x=0} \Delta v(y(0))$ as $r \to 0$ which is bounded by

$$\sigma\|\nabla\varphi\|_\infty \max\{\Delta v(y(0)), 0\} = \sigma\|\nabla\varphi\|_\infty \max\{\Delta v(\varphi(0)e_n), 0\}.$$

Thus we finished the proof.

A.3.2 Proof of Lemma 2.4.4

Let us suppose $x = 0$ and only compute $\partial_1 f = \partial_{x_1} f$. If $\nabla v(y) = 0$, it is not hard to see

$$\partial_1 f(0) = \partial_1 v(y) = 0.$$

Otherwise suppose $\nabla v(y) \neq 0$, then $y \in \partial B(0, \varphi(0))$ and v obtains its minimum over $B(0, \varphi(0))$ at point y. Let us assume

$$y = (y_1, y_2, 0, ..., 0) \quad \text{ and thus } |y_1|^2 + |y_2|^2 = (\varphi(0))^2.$$

For smooth v, it is not hard to see that

$$\nabla v(y) = -ky \quad \text{with } k = \frac{|\nabla v|}{\varphi(0)}.$$

Near point y

$$v(x) - v(y) = -ky_1(x_1 - y_1) - ky_2(x_2 - y_2) + o(|x - y|).$$

To estimate $w((\delta, 0, ..., 0))$, consider the leading terms:

$$A(\delta) := -ky_1(x_1 - y_1) - ky_2(x_2 - y_2) = -ky_1(x_1 - \delta) - ky_2 x_2 + ky_1^2 + ky_2^2 - ky_1\delta.$$

By a standard argument, under the constrain

$$|x_1 - \delta|^2 + |x_2|^2 + |x_3|^2 + ... + |x_n|^2 \leq \varphi(\delta, 0...0)^2,$$

$A(\delta)$ achieves its minimum at

$$x_1 = y_1\varphi(\delta, 0...0)/(y_1^2 + y_2^2)^{\frac{1}{2}} + \delta, \; x_2 = y_2\varphi(\delta, 0...0)/(y_1^2 + y_2^2)^{\frac{1}{2}}$$

with value

$$-k\varphi(\delta, 0...0)(y_1^2 + y_2^2)^{\frac{1}{2}} + ky_1^2 + ky_2^2 - ky_1\delta = -k\varphi(\delta, 0...0)\varphi(0) + k\varphi(0)^2 - ky_1\delta.$$

Thus

$$\partial_1 f(0) = \lim_{\delta \to 0} A(\delta)/\delta = -k\varphi(0)\,\partial_1\varphi(0) - ky_1.$$

So we find

$$\partial_1 f(0) - \partial_1 v(y) = -k\varphi(0)\,\partial_1\varphi(0) = -|\nabla v|\,\partial_1\varphi(0).$$

This leads to the conclusion.

A.3.3 Proof of Lemma 2.5.6

Let $g = \hat{\nabla}_p v$ which then solves

$$g_t = (m-1)g\Delta v + 2\nabla v \cdot \nabla g + (m-1)v\Delta g + \nabla g \cdot (\vec{b}(x+X) - \vec{b}(X)) + (m-1)g\nabla \cdot \vec{b}$$
$$+\nabla v \cdot \hat{\nabla}_p \vec{b}(x+X) + (m-1)v\nabla \cdot \hat{\nabla}_p \vec{b}.$$

By the condition $(A_k)(C_k)$, as before

$$|\nabla v \cdot \hat{\nabla}_p \vec{b}(x+X)| + |(m-1)v\nabla \cdot \hat{\nabla}_p \vec{b}| \le \sigma L \delta J^k.$$

Now apply Harnack's inequality in $(B_{\frac{7}{8}} \times [-3r, 3r]) \cap \{v \ge \frac{1}{2}\epsilon\}$. As done in Proposition 2.2 in [18], if we restrict to a smaller region $(B_{\frac{3}{4}} \times (-2r, 2r)) \cap \{v \ge \epsilon\}$ for r small enough (depending on ϵ), there exist C, C' (depending on L, r, ϵ) such that

$$\hat{\nabla}_p v(x,t) \ge C\hat{\nabla}_p v(\mu, -2r) - C'\delta J^k.$$

By (D_k), we have $\hat{\nabla}_p v(\mu, -2r) \ge J^k$. Thus we can select δ small enough such that for some $C > 0$

$$\hat{\nabla}_p v(x,t) \ge C\hat{\nabla}_p v(\mu, -2r) \text{ in } (B_{\frac{3}{4}} \times (-2r, 2r)) \cap \{u \ge \epsilon\}. \tag{A.3.1}$$

To show the assertion, we need to show

$$\frac{v((x,t)+\gamma p) - v(x,t)}{\gamma} \ge \tau v(x,t) = \tau\epsilon$$

which holds by the definition of τ and (A.3.1). Lastly we can take $r \le \frac{\epsilon}{2L}$ and thus by (A_k) and $0 \in \Gamma_0$, we have $v \le \epsilon$ in Q_{2r}.

A.3.4 Proof of Lemma 2.5.7

Let $\alpha \in (-2r, 2r)$. Let f be a non-negative C^1 function defined in $B_{\frac{1}{2}}$ such that

$$f = 0 \text{ in } B_{\frac{1}{4}};\ |\nabla f| \le \epsilon,\ |\Delta f| \le 10\epsilon \text{ and } f = \epsilon \text{ if } |x| = \frac{1}{2}.$$

Define

$$\omega(x,t) = v(x,t) + \tau\gamma(v(x,t) + \epsilon(t+\alpha) - f(x))^+.$$

Then we claim that ω is a subsolution in $\Sigma := (B_{\frac{1}{2}} \times (-2r, -\alpha)) \cap \{v \leq \epsilon\}$ if ϵ is small enough independent of $r < \frac{1}{3}$. Let us follow [18] and only point out the differences coming from the drift. We denote the following two operators as

$$\widetilde{\mathcal{L}}\omega := \omega_t - (m-1)\,\omega\,\Delta\omega - |\nabla\omega|^2,$$
$$\mathcal{L}_2\omega := \omega_t - (m-1)\,\omega\,\Delta\omega - |\nabla\omega|^2 - \nabla\omega \cdot (\vec{b}(x+X) - \vec{b}(X)) - (m-1)\,\omega\,\nabla\cdot\vec{b}(x+X).$$

Let $g(s) := \tau\gamma s^+$ and thus $g' = \tau\gamma\,\chi_{\{s>0\}}$. Following the computations in Lemma 3.1 in [18] we have

$$\begin{aligned}
&\omega_t = (1+g')v_t + \epsilon g',\\
&\nabla\omega = \nabla v + g'\nabla(v-f),\\
&\Delta\omega = (1+g')\Delta v - g'\Delta f + g''|\nabla(v-f)|^2,\\
&\widetilde{\mathcal{L}}\omega \leq (1+g')\widetilde{\mathcal{L}}v - \left(\frac{1}{L^2} - C\epsilon\right) g' \quad \text{with } C \text{ only depending on } L \text{ and } \sigma.
\end{aligned}$$

Since $\mathcal{L}_2\omega - \widetilde{\mathcal{L}}\omega = -\nabla\omega\cdot(\vec{b}(x+X) - \vec{b}(X)) - (m-1)\omega\nabla\cdot\vec{b}$, then

$$\begin{aligned}
\mathcal{L}_2\omega &\leq (1+g')\widetilde{\mathcal{L}}v - \left(\frac{1}{L^2} - C\epsilon\right) g' - \nabla\omega\cdot(\vec{b}(x+X) - \vec{b}(X)) - (m-1)\omega\nabla\cdot\vec{b}\\
&= (1+g')\mathcal{L}_2 v + (1+g')(\nabla v\cdot(\vec{b}(x+X) - \vec{b}(X)) + (m-1)v\nabla\cdot\vec{b})\\
&\quad - (\nabla v + g'\nabla(v-f))(\vec{b}(x+X) - \vec{b}(X)) - (m-1)(v+g)\nabla\cdot\vec{b} - \left(\frac{1}{L^2} - C\epsilon\right) g'\\
&= g'\nabla f\cdot(\vec{b}(x+X) - \vec{b}(X)) + (m-1)g\nabla\cdot\vec{b} - \left(\frac{1}{L^2} - C\epsilon\right) g'.
\end{aligned}$$

By (D_k), we have $\|\vec{b}\|_\infty \leq \sigma, \|\nabla\vec{b}\|_\infty \leq \sigma\delta J^k$. And since we assumed $\delta \leq \epsilon$ and $J < 1$, $\|\nabla\vec{b}\|_\infty \leq \sigma\epsilon$. Also since $g(s) \leq \epsilon g'(s)$ for $s \leq \epsilon$, we have for $(x,t) \in Q_{1/2}$

$$|g'\nabla f\cdot(\vec{b}(x+X) - \vec{b}(X)) + (m-1)g\nabla\cdot\vec{b}| \leq \sigma\epsilon^2 g'.$$

Thus $\mathcal{L}_2\, y \leq 0$ if ϵ is small enough.

The rest of the proof follows from the proof of Proposition 2.3 [18], where we compare w and ω in Σ to conclude that

$$w(x,-\alpha) \geq (1+\tau\gamma)v(x,-\alpha) \text{ in } B_{\frac{1}{4}} \cap \{v \leq \epsilon\} \tag{A.3.2}$$

for all $\alpha \in (-2r, 2r)$.

A.3.5 Proof of Lemma 2.5.8

Based on $(A_k) - (B_k)$ and the elliptic regularity estimate, one can argue as in Lemma 3.2 of [18] to conclude that

$$vD_{ij}v \geq -C_4, \quad \text{for all } i,j = 1,...,d \text{ in } Q_{2r}, \tag{A.3.3}$$

where C_4 depends only on L, universal constants and the Lipschitz constant of $\Gamma(v)$. We will use this fact in the computation below.

For a given vector $\bar{\eta} \in \mathcal{S}^{d-1}$, define h such that

$$h(x,t) := (1+\tau\gamma)v(x+(t+2r)\phi\bar{\eta}, t), \quad y := x+(t+2r)\phi\bar{\eta}.$$

Note that $|y-x| \leq \kappa\tau\gamma$.

Next Lemma 2.5.7 implies that $w \geq h$ on the parabolic boundary of

$$\Sigma := (B_{\frac{1}{4}} \times (-2r, 2r)) \cap \{v \leq \epsilon\}.$$

We claim that $\mathcal{L}_2 h \leq 0$ in Σ. Write $\tau' := \tau\gamma$. We have

$$\begin{aligned}
h_t &= (1+\tau')(v_t + v_{\bar{\eta}}\phi), \\
\nabla h &= (1+\tau')\left(\nabla v + v_{\bar{\eta}}(t+2r)\nabla\phi\right), \\
\Delta h &= (1+\tau')\left(\Delta v + 2(t+2r)\nabla v_{\bar{\eta}} \cdot \nabla\phi + v_{\bar{\eta}\bar{\eta}}(t+2r)^2|\nabla\phi|^2 + v_{\bar{\eta}}(t+2r)\Delta\phi\right),
\end{aligned}$$

From (A.3.3) and the computations in Proposition 2.4

$$\widetilde{\mathcal{L}}h \leq (1+\tau')\widetilde{\mathcal{L}}v(y,t) - \tau'\left(\frac{1}{L} - C\kappa\right)$$

where C depends only on m, L, C_4, σ. Thus

$$\begin{aligned}
\mathcal{L}_2 h &\leq (1+\tau')\widetilde{\mathcal{L}}\, v(y,t) - \tau'\left(\frac{1}{L} - C\kappa\right) - \nabla h \cdot \left(\vec{b}(x+X) - \vec{b}(X)\right) - (m-1)h\nabla\cdot\vec{b}(x+X) \\
&= (1+\tau')\mathcal{L}_2 v(y,t) - \tau'\left(\frac{1}{L} - C\kappa\right) - (1+\tau')\nabla v \cdot \left(\vec{b}(x+X) - \vec{b}(y+X)\right) \\
&\quad - (m-1)(1+\tau')v(y,t)\nabla\cdot\left(\vec{b}(x+X) - \vec{b}(y+X)\right) \\
&\quad - (1+\tau')v_{\bar\eta}(t+2r)\nabla\phi)\cdot\left(\vec{b}(x+X) - \vec{b}(X)\right) \\
&\leq -\tau'\left(\frac{1}{L} - C\kappa\right) + (1+\tau')|\nabla v| \left\|\nabla\vec{b}\right\|_\infty |x-y| + (m-1)(1+\tau')v\left\|D^2\vec{b}\right\|_\infty |x-y| \\
&\quad + (1+\tau')\left|v_{\bar\eta}\right|(t+2r)\left|\nabla\phi\right| \left\|\nabla\vec{b}\right\|_\infty |x|.
\end{aligned}$$

Now apply (C_k) and since $\delta \leq \epsilon$, we have $\left\|\nabla\vec{b}\right\|_\infty \leq \sigma\epsilon, \|D^2 V\|_\infty \leq \sigma\epsilon^2$. Since $|\nabla\phi| \leq \kappa\tau'$, we obtain

$$\begin{aligned}
\mathcal{L}_2 h &\leq -\tau'\left(\frac{1}{L} - C\kappa\right) - \sigma L\epsilon\kappa\tau' - \sigma L\epsilon r\kappa\tau' - \sigma L\epsilon^2\kappa\tau' \\
&\leq -\tau'\left(\frac{1}{L} - C\kappa - \sigma L\kappa\right) \leq 0 \quad \text{ in } \Sigma,
\end{aligned}$$

if κ is small enough. Take $\bar\eta = \mu$. By comparison principle we can conclude that

$$w \geq (1+\tau\gamma)v(x+(t+2r)\phi(x)\mu, t) \geq v(x+(t+2r)\phi(x)\mu, t) \text{ in } Q_{2r}.$$

As a corollary of Proposition 2.5.10, the $C^{1,\alpha}$ regularity of Γ (Theorem 2.5.1) follows from the relation $\theta_k = \theta_{k-1} + S(\pi/2 - \theta_{k-1})$ given in above Proposition. We refer to Theorem 1 in [18] for the proof.

A.3.6 Proof of Lemma 2.6.2

Fix one non-negative $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$. Denote

$$U_0 := \{\phi > 0\} \cap \{\psi > 0\}.$$

For any $\epsilon > 0$, take finitely many space time balls $U_i, i = 1, ..., n$ such that

1. for each $i \geq 1$, $|U_i| \leq \epsilon^d$ and U_i is in the ϵ-neighbourhood of $\Gamma(\psi)$,

2. $\{U_i\}_{i=1,\ldots,n}$ is an open cover of $\Gamma(\psi)\cap\{\phi>0\}$.

Since $\Gamma(\psi)$ is of dimension $d-1$, we can assume

$$n \lesssim 1/\epsilon^{d-1}. \tag{A.3.4}$$

Take a partition of unity $\{\rho_i, i=0,...,n\}$ which is subordinate to the open cover $\{U_i\}_{i\geq 0}$. Then for $i \geq 1$,

$$|\nabla \rho_i| + |\partial_t \rho_i| \lesssim 1/\epsilon. \tag{A.3.5}$$

By the assumption, ψ is a supersolution in the interior of its positive set. And since ϵ can be arbitrarily small, to show (2.6.5) we only need to show

$$I_\epsilon := \sum_{i=1}^{n(\epsilon)} \left(\int_0^T \int_{\mathbb{R}^d} \psi\,(\phi\rho_i)_t - (\nabla\psi^m + \psi\,\vec{b})\nabla(\phi\rho_i)\;dxdt - \int_{\mathbb{R}^d} \psi(0,x)\phi(0,x)\rho_i dx \right) \to 0$$

as $\epsilon \to 0$.

By property 1 of U_i and the regularity assumption on ψ, in all $U_i, i \geq 1$ we have

$$\psi \leq C\epsilon^{\frac{1}{\alpha}}, \quad |\nabla\psi^m| \leq C\psi^{m-\alpha}|\nabla\psi^\alpha| \leq C\epsilon^{\frac{m-\alpha}{\alpha}}.$$

Now from (A.3.4), (A.3.5) and $\alpha < m$, it follows that

$$\begin{aligned} |I_\epsilon| &\leq C\epsilon^{-d+1}\left(\iint_{U_i} \frac{1}{\epsilon}(\psi + |\nabla\psi^m|)\;dxdt + \int_{U_i\cap\{t=0\}} \psi(0,x)dx \right) \\ &\leq C(\epsilon^{\frac{1}{\alpha}} + \epsilon^{\frac{m-\alpha}{\alpha}} + \epsilon) \end{aligned}$$

which indeed converges to 0 as $\epsilon \to 0$.

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