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Higher Categories and Topological Quantum Field Theories

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Higher Categories and Topological Quantum Field Theories

To my parents, my wife, and my sister,
who have been the most important sources of my happiness in life.

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Abstract

Higher Categories and Topological Quantum Field Theories

by

Xingshan Cui

We give a construction of Turaev-Viro type $(3+1)$-TQFT out of a G-crossed braided spherical fusion category for G a finite group. The resulting invariant of 4-manifolds generalizes several known invariants in literature such as the Crane-Yetter invariant and Yetter's invariant from homotopy 2-types. Some concrete examples will be provided to show the calculations. If the category is concentrated only at the sector indexed by the trivial group element, a co-cycle in $H^4(G, U(1))$ can be introduced to produce another invariant, which reduces to the twisted Dijkgraaf-Witten theory in a special case. It can be shown that with a G-crossed braided spherical fusion category, one can construct a monoidal 2-category with certain extra structure, but these structures do not satisfy all the axioms of a spherical 2-category given by M. Mackaay. Although not proven, it is believed that our invariant is strictly different from other known invariants. It remains to see if the invariant has the power to detect any smooth structures.

Contents

Chapter 1

Introduction

The notion of a topological quantum field theory (TQFT) was invented by E. Witten based on path integrals in physics [1]. M. Atiyah gave a mathematical formulation of TQFTs [2]. Since then there has been a vast study of TQFTs both from the physics side and mathematics side. Roughly speaking, a $(d+1)$-TQFT associates to every d-manifold a Hilbert space and to every $(d+1)$-manifold a vector in the Hilbert space corresponding to the boundary of the $(d+1)$-manifold. These assignments should satisfy certain compatible properties as specified by the axioms. Each $(d+1)$-TQFT produces a scalar invariant, called the partition function, for smooth closed $(d+1)$-manifolds. This is particularly useful in topology as it may have the power to detect smooth structures on a given manifold.

The study of TQFTs is closely related to higher category theories [3][4]. In general, an n-TQFT is to be described by the data of an n-category. On one hand, strict n-categories are well-defined for any n [5] [6], but this is not enough for the purpose of constructing TQFTs since many important higher categories are not strict and can not be strictified either [7]. On the other hand, weak n-categories are only rigorously defined for small n (like $n = 1, 2, 3, 4$), and it is still controversial what should be the right notion of weak n-categories for higher n, although there have been many efforts in this direction [8] [9] [10]. In the following, by n-categories we mean weak n-categories. Special n-categories can be constructed from k-categories with certain extra structures for $k < n$. For instance, a monoidal 1-category is a 2-category and a braided monoidal 1-category is a 3-category. When constructing TQFTs, especially $(2+1)$- and $(3+1)$-TQFTs, one usually utilize these special n-categories to bypass the barrier of the poor definition of general n-categories. We give a brief overview of some categorical constructions of $(2+1)$- and $(3+1)$-TQFTs below.

There has been a significant amount of achievements in (2+1)-TQFTs. N. Reshetikhin and V. Turaev constructed an invariant of 3-manifolds (Reshetikhin-Turaev invariant) using modular tensor categories, which is believed to be the mathematical realization of Witten's TQFT from non-abelian Chern-Simon theories [11]. V. Turaev and O. Viro gave a state-sum invariant of 3-manifolds (Turaev-Viro invariant) with the data from a ribbon fusion category [12]. Later this invariant was generalized by J. Barrett and B. Westbury so that it takes a spherical fusion category as input [13]. The Reshetikhin-Turaev invariant with the center of a spherical fusion category $\mathcal{C}$ as input is shown to equal the Turaev-Viro invariant with $\mathcal{C}$ as input [14] [15] [16]. Apart from these categorical constructions of TQFTs, another approach is by using certain Hopf algebras, among which the Kuperberg invariant [17] and the Hennings invariant [18] [19] are non-semisimple generalizations of the Turaev-Viro invariant and the Reshetikhin-Turaev invariant, respectively. A special case of the Kuperberg invariant reduces to the Dijkgraaf-Witten theory [20]. The study of $(2+1)$-TQFTs has had great effects on quantum groups, $3d$ topology and knot theories. For example, the Turaev-Viro invariant can distinguish certain manifolds which are homotopy equivalent.

Going one dimensional higher. The theory of $(3+1)$-TQFTs, however, is not understood as well as its counter-part in $(2+1)$ dimension. The Dijkgraaf-Witten invariant

[20] in $(3+1)$ dimension, as well as in other dimensions, measures the number of group morphisms from the fundamental group of a 4-manifold to a given finite group. L. Crane and I. Frankel constructed a 4-manifold invariant out of some algebraic structure, called Hopf categories, and they also proposed a construction of a family of Hopf categories [21]. In [22], L. Crane and D. Yetter gave a state-sum invariant of 4-manifolds (Crane-Yetter invariant) using a semi-simple sub-quotient of the category of representations of $U_q(sl_2)$ for q a certain principal root of unity. L. Crane, L. Kauffman and D. Yetter generalized the construction for a ribbon fusion category [23].[1] The Walker-Wang model in [24] is a Hamiltonian realization in $3d$ of the Crane-Yetter invariant. The modular Crane-Yetter invariant, which is obtained from a modular tensor category, turns out to be a function of the Euler characteristics and the signature [25], and thus is a classical invariant. From a different direction, D. Yetter gave a construction of $(3+1)$-TQFT from homotopy 2-types [26], which is equivalent to a crossed module or a categorical group. Along a similar line, A. Kapustin [27] and M. Mackaay [28] obtained 4-manifold invariants from 2-groups with some additional structures. More recently, R. Kashaev produced a $(3+1)$-TQFT out of a cyclic group $\mathbb{Z}_N$ [29]. Although simple and elementary, it is not clear if Kashaev's invariant is related with those from categorical constructions.

Now we focus a little more on categorical state-sum constructions of TQFTs. In (2+1) dimension, the typical state-sum model (Turaev-Viro invariant) involves a spherical fusion (1-)category. Thus in $(3+1)$ dimension, one would expect there to be a 'spherical fusion 2-category' and all known categorical constructions of $4d$-invariants should be a special case. Modoidal 2-categories are defined in [30], and monoidal 2-categories with duals are given in [31]. In [32], M. Mackaay proposed a definition of spherical 2-categories by introducing certain extra structures based on monoidal 2-categories with

[1] In [23], it was called a semi-simple tortile category. The generalized invariant is still called in the Crane-Yetter invariant.

duals. However, as explained later, his definition is too restrictive and excluded many interesting examples. Therefore, the question of what should be a 'good' definition of spherical fusion 2-categories is still open.

Here we will not try to define spherical 2-categories. Rather, we consider a special class, but still more general than those studied in literature along this line, of 'properly defined spherical 2-categories'. The main result of this thesis is a state-sum construction of 4-manifold invariants out of a G-crossed braided spherical fusion category (G-BSFC) $\mathcal{C}_G^\times$ for G a finite group. G-crossed braided (spherical) fusion categories were studied in [33][34][35][36] and also have applications in condensed matter physics [37]. Roughly speaking, a G-BSFC is a G-graded spherical fusion category with a G-action and a G-crossed braiding which satisfy certain compatible conditions. Note that a G-BSFC is in general not a braided tensor category. Our construction is a state-sum model and a natural generalization of the Tuaev-Viro invariant from $(2+1)$ to $(3+1)$. Although in this thesis, we only describe the invariants for closed 4-manifolds, it is straight forward to extend it to 3-manifolds and produce a $(3+1)$-TQFT.

It can be shown that out of a G-BSFC one can construct a monoidal 2-category with some extra structures (See Section 2.3). One might claim that these extra structures make it into a 'spherical fusion 2-category'. However, they violate some conditions in the definition of a spherical 2-category given in [32]. This suggests that the definition of spherical 2-categories might need to be revised. Also it means that our result is not included in [32].

The invariant from a G-BSFC generalizes most known categorical invariants in literature. If G is the trivial group, then $\mathcal{C}_G^\times$ is just a ribbon fusion category and the invariant we get is just the Crane-Yetter invariant. If $\mathcal{C}_G^\times$ is obtained from a crossed module (See Section 3.3.2), then we get the Yetter's invariant from homotopy 2-types, a special case of which is the untwisted Dijkgraaf-Witten invariant. Our invariant also includes some cases of the invariant in [28][27]. Moreover, if the G-grading is concentrated on the trivial sector, namely, only the sector indexed by the trivial group element is non-trivial, then a 4-cocycle in $H^4(G, U(1))$ can be introduced into the construction to produce another invariant, which reduces to the twisted Dijkgraaf-Witten invariant as a special case. It remains to see if the invariant from G-BSFCs are strictly stronger than those invariants and if it has the power to detect smooth structures.

Lastly, G-BSFCs are not rare. In [35][38][39], it was proved that equivalent classes of G-BSFCs are in one-to-one correspondence, by equivariatization and de-equivariatization, with equivalent classes of spherical braided categories containing $Rep(G)$ as a subcategory. Also, given a group morphism from G to the group of automorphisms of a unitary braided fusion category $\mathcal{C}$, if certain obstructions vanish, then $\mathcal{C}$ can be extended to a unitary G-crossed braided fusion category, which is also a G-BSFC, with $\mathcal{C}$ as the sector indexed by the trivial group element [33].

The structure of the thesis is organized as follows. In Chapter 2, we give a review of category theories. We start from monoidal categories and then introduce different structures including dual, braiding, G-action, etc. (Section 2.1). G-BSFCs are discussed in Section 2.2. In Section 2.3, it is shown that with a G-BSFC one can construct a monoidal 2-category with duals and certain extra structures. In Chapter 3, we define the invariant of 4-manifolds out of a G-BSFC. Section 3.1 provides three equivalent definitions of the invariant, which are defined on triangulated 4-manifolds. In Section 3.2, a formula of the invariant is expressed in terms of the data of a G-BSFC such as the F-symbols, R-symbols, etc. In Section 3.3, several examples are discussed. In particular, it shows that some of known invariants arise naturally as special cases of our invariant. Section 3.4 gives a variation of the invariant. Chapter 4 contains a complete proof that the invariant defined above is indeed an ‘invariant’. Finally, in Chapter 5 we point out some open questions/directions for future study.

Chapter 2

Higher Categories

In this chapter, we give a brief review of categories with extra structures such as tensor product, dual, braiding, etc. The ultimate goal of this chapter is to introduce G-crossed braided spherical fusion categories (G-BSFC) in Section 2.2, which are the input to the (3+1) topological quantum field theory to be defined in Section 3.1. All of these categories are special examples of higher categories, e.g., 2-, 3-, 4-categories. But we will not go into details on general higher categories, except in Section 2.3 the concept of monoidal 2-categories is sketched, and we show that with any G-BSFC one can construct a monoidal 2-category with duals and certain addition structures, but these structures do not make it into a spherical 2-category under the definition in [32]. We assume the readers are familiar with the concepts of abelian categories, functors and natural transformations. There are a number of excellent references such as [40] [41] [42] [43], etc.

A few notations are in order. If $\mathcal{C}$ is a category, denote the set of objects by $\mathcal{C}^0$ and the set of morphisms by $\mathcal{C}^1$. Capital letters, A, B, X, Y, are often used for objects and little letters, f, g, h, for morphisms. For $A, B \in \mathcal{C}^0$, $\mathrm{Hom}(A, B)$ is the set of morphisms from A to B. Compositions of morphisms are to be read from right to left except in Section 2.3. The identity functor is denoted by $Id_{\mathcal{C}}$. The identity morphism from A to

A is denoted by I_A or I when it is clear from the context what the sub index should be.

2.1 Ribbon Fusion Categories

In this section, we give a review of monoidal category theories, which are the preliminaries to the concept of G-crossed braided spherical fusion categories. In the following subsections, we will talk about some basic ingredients such as tensor product, dual, braiding, fusion, etc.

2.1.1 Monoidal Categories

A bi-functor $\otimes$ on $\mathcal{C}$ is a functor $\otimes : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ such that $\otimes(ff', gg') = F(f,g)F(f',g')$. A bi-functor is usually thought of as a binary operation and $\otimes(A,B), \otimes(f,g)$ are written as $A \otimes B, f \otimes g$, respectively. Given $A \in \mathcal{C}^0$, define $A \otimes \cdot : \mathcal{C} \longrightarrow \mathcal{C}$ as a functor by $(A\otimes\cdot)(B) := A\otimes B$ and $(A\otimes\cdot)(f) := Id_A \otimes f$. The functor $\cdot\otimes A$ is defined analogously.

A monoidal category is a tuple $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ is a bi-functor, $\mathbf{1} \in \mathcal{C}^0$ is an object, and α, l, r are natural isomorphisms $\otimes\circ(\otimes\times Id_{\mathcal{C}}) \longrightarrow \otimes\circ(Id_{\mathcal{C}}\times\otimes)$, $\mathbf{1}\otimes\cdot \longrightarrow Id_{\mathcal{C}}$, $\cdot\otimes\mathbf{1} \longrightarrow Id_{\mathcal{C}}$, respectively. Namely, for $A, B, C \in \mathcal{C}^0$, there are natural isomorphisms $\alpha_{A,B,C} : (A\otimes B)\otimes C \xrightarrow{\simeq} A\otimes(B\otimes C)$, $l_A : \mathbf{1}\otimes A \xrightarrow{\simeq} A$, $r_A : A\otimes\mathbf{1} \xrightarrow{\simeq} A$. These isomorphisms need to satisfy the Pentagon Axiom and Triangle Axiom:

1. Pentagon Axiom:

$$\begin{tikzcd}[column sep=small]
& ((A\otimes B)\otimes C)\otimes D \arrow[dl, "\alpha_{A,B,C}\otimes I_D"'] \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "\alpha_{A,B\otimes C,D}"'] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha_{A,B,C\otimes D}"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "I_A\otimes\alpha_{B,C,D}"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd} \tag{2.1}$$

2. Triangle Axiom:

$$\begin{tikzcd}
(A\otimes\mathbf{1})\otimes B \arrow[r, "\alpha_{A,\mathbf{1},B}"] \arrow[d, "r_A\otimes I_B"'] & A\otimes(\mathbf{1}\otimes B) \arrow[dl, "I_A\otimes l_B"] \\
A\otimes B &
\end{tikzcd} \tag{2.2}$$

The bi-functor $\otimes$ is called the tensor product functor, and $A\otimes B$, $f\otimes g$ are called the tensor product of A and B, the tensor product of f and g, respectively.

Thus a monoidal category can be thought of as a generalization (or more precisely, a categorification) of a monoid such that the product and unit are given by the tensor product functor $\otimes$ and the object $\mathbf{1}$, respectively, and that the axioms for associativity and unit hold only up to isomorphisms. In a monoid (or a group), different orders of multiplying a sequence of elements result in the same answer due to the axioms of associativity and unit. In a monoidal category, since associativity only holds up to isomorphism, we can only expect different orders of taking tensor product of a sequence of objects result in isomorphic objects.

Given a sequence of objects $\mathcal{A} = (A_1, \cdots, A_n)$, a representation X of $\mathcal{A}$ is an expression obtained by first inserting $\mathbf{1}$'s in the sequence $A_1\cdots A_n$ and then putting in parentheses so that it gives a well-defined order of taking tensor product of the objects. Denote by $\otimes(X)$ the object resulting from taking tensor product in X. For instance, $X_1 = (A_1A_2)A_3$ and $X_2 = (A_1\mathbf{1})(A_2A_3)$ are both representations of (A_1, A_2, A_3). It is an elementary fact that one representation can be converted into any other one by finitely many local moves:

$$\begin{aligned} \cdots(A_1A_2)A_3\cdots &\quad\longleftrightarrow\quad \cdots A_1(A_2A_3)\cdots \\ \cdots(\mathbf{1}A)\cdots &\quad\longleftrightarrow\quad \cdots A\cdots \\ \cdots(A\mathbf{1})\cdots &\quad\longleftrightarrow\quad \cdots A\cdots \end{aligned}$$

If X_1, X_2 are two representations which differ by one of the above local moves, then there is an isomorphism $\otimes(X_1) \longrightarrow \otimes(X_2)$ given by one of α, l, r tensoring with some identity morphisms. Thus if X_1, X_2 are any two representations, there exists an isomorphism $\otimes(X_1) \longrightarrow \otimes(X_2)$ which is a composition of isomorphisms described above. In the Pentagon Axiom, there are two sequences of local moves converting $((A_1A_2)A_3)A_4$ into $A_1(A_2(A_3A_4))$, and correspondingly there are two isomorphisms $((A_1\otimes A_2)\otimes A_3)\otimes A_4 \longrightarrow A_1\otimes(A_2\otimes(A_3\otimes A_4))$, one going along the left and bottom side of the pentagon, the other going along the right side. The Pentagon Axiom requires that these two isomorphisms are equal. Similarly, in Triangle Axiom, the two isomorphisms $(A\otimes\mathbf{1})\otimes B \longrightarrow A\otimes B$ are also required to be equal. A little surprisingly, these two axioms actually imply that these isomorphisms are always equal, which is given by the MacLane Coherence Theorem.

Theorem 2.1.1 (MacLane Coherence Theorem) *[44] If X_1, X_2 are any two representations of the same sequence of objects in a monoidal category $\mathcal{C}$, then there is a unique isomorphism between $\otimes(X_1)$ and $\otimes(X_2)$ consisting of compositions of α, l, r's.*

We call the unique isomorphism in Theorem 2.1.1 a *canonical* isomorphism. A monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ is sometimes written as $\mathcal{C}$ for simplicity.

A monoidal category $\mathcal{C}$ is defined to be *strict* if all the α, l, r's are identity maps, namely, the canonical isomorphism between different representations are actually the identity. Another theorem of Mac Lane shows that every monoidal category is monoidal equivalent to a strict monoidal category. The concept of 'monoidal equivalence ' is defined below.

Let $\mathcal{C}_1, \mathcal{C}_2$ be two monoidal categories, and we use the same symbols $\{\otimes, \mathbf{1}, \alpha, l, r\}$ for $\mathcal{C}_1, \mathcal{C}_2$. A monoidal functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a triple (F, Φ, ψ), where $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a functor, Φ is a natural isomorphism $F \circ (\cdot \otimes \cdot) \longrightarrow F(\cdot) \times F(\cdot)$, i.e., Φ is a family of isomorphisms $\{\Phi_{A,B} : F(A \otimes B) \longrightarrow F(A) \otimes F(B) \mid A, B \in \mathcal{C}^0\}$, and $\psi : F(\mathbf{1}) \xrightarrow{\simeq} \mathbf{1}$ is an isomorphism such that the following diagrams hold:

$$\begin{tikzcd}
F((A_1 \otimes A_2) \otimes A_3) \arrow[r, "\Phi_{A_1\otimes A_2, A_3}"] \arrow[d, "F(\alpha_{A_1,A_2,A_3})"'] & F(A_1 \otimes A_2) \otimes F(A_3) \arrow[r, "\Phi_{A_1,A_2}\otimes I"] & (F(A_1) \otimes F(A_2)) \otimes F(A_3) \arrow[d, "\alpha_{F(A_1),F(A_2),F(A_3)}"] \\
F(A_1 \otimes (A_2 \otimes A_3)) \arrow[r, "\Phi_{A_1, A_2\otimes A_3}"'] & F(A_1) \otimes F(A_2 \otimes A_3) \arrow[r, "I\otimes\Phi_{A_2,A_3}"'] & F(A_1) \otimes (F(A_2) \otimes F(A_3))
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
F(\mathbf{1} \otimes A) \arrow[r, "\Phi_{\mathbf{1},A}"] \arrow[d, "F(l_A)"'] & F(\mathbf{1}) \otimes F(A) \arrow[r, "\psi\otimes I"] & \mathbf{1} \otimes F(A) \arrow[d, "l_{F(A)}"] \\
F(A) \arrow[rr, "I"'] & & F(A)
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
F(A \otimes \mathbf{1}) \arrow[r, "\Phi_{A,\mathbf{1}}"] \arrow[d, "F(r_A)"'] & F(A) \otimes F(\mathbf{1}) \arrow[r, "I\otimes\psi"] & F(A) \otimes \mathbf{1} \arrow[d, "r_{F(A)}"] \\
F(A) \arrow[rr, "I"'] & & F(A)
\end{tikzcd} \tag{2.5}$$

Roughly speaking, a monoidal functor is a functor which preserves the monoidal structure.

Definition 2.1.2 *Two monoidal categories $\mathcal{C}_1, \mathcal{C}_2$ are monoidal equivalent if there there are monoidal functors $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2,\ G : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$, such that $F \circ G$ is equivalent to $Id_{\mathcal{C}_2}$ and $G \circ F$ is equivalent to $Id_{\mathcal{C}_1}$.*

Theorem 2.1.3 *[44] Every monoidal category $\mathcal{C}$ is monoidal equivalent to a strict monoidal* *category $\mathcal{C}^{str}$.*

Due to Theorem 2.1.3, we can always assume monoidal categories are strict.

Example 2.1.4 *1. $\mathcal{V}ect$: The objects are finite dimensional vector spaces over $\mathbb{C}$ and morphisms are linear maps between vector spaces. The tensor product functor $\otimes$ is the usual tensor product, and the unit $\mathbf{1}$ is $\mathbb{C}$.*

2. Given a finite group G, one can form three monoidal categories.

- *$\mathcal{V}ect_G$: The objects are G-graded vector spaces over $\mathbb{C}$ and morphisms are G-graded linear maps.If $V = \bigoplus_{g\in G} V_g, W = \bigoplus_{g\in G} W_g$, then $V \bigotimes W = \bigoplus_{g\in G} (V \otimes W)_g$, where $(V \otimes W)_g = \bigoplus_{g=g_1g_2} V_{g_1} \otimes W_{g_2}$. The unit $\mathbf{1} = \bigoplus_{g\in G} \mathbf{1}_g$, where $\mathbf{1}_g = \delta_{g,e}\mathbb{C}$.*
- *$\underline{G}$: The objects are members of G. For $g, h \in G$, $\mathrm{Hom}(g, h) = \{I_g\}$ if $g = h$, and $\emptyset$ otherwise. The tensor product and unit are the multiplication and unit in G, respectively.*
- *$Rep(G)$: Objects are left modules (representations) of G and morphisms are G-module morphisms.*

3. $Rep(\mathcal{H})$: the category of representations of a bi-algebra $\mathcal{H}$. The previous example $Rep(G)$ is a special case corresponding to $\mathcal{H} = \mathbb{C}[G]$, the group algebra.

4. $Aut_{\otimes}(\mathcal{C})$: The objects are monoidal equivalences of a monoidal category $\mathcal{C}$, and the morphisms are natural isomorphisms. The tensor product is the composition of equivalences and the unit is $Id_{\mathcal{C}}$.

In the next few subsections, we are going to introduce some extra structures on a monoidal category. In the following, let $\mathcal{C}$ be always a monoidal category. Whether or not assuming $\mathcal{C}$ is strict depends on personal flavors. If one assumes $\mathcal{C}$ is strict, many diagrams and equations can be simplified since the α, l, r's are identity maps. By Theorem 2.1.3, there is no loss of generality for such an assumption. Thus we will assume $\mathcal{C}$ is strict. But one should keep in mind that all the extra structures can be equally defined in a non-strict category and all commuting diagrams still make sense by inserting some canonical isomorphisms involving α, l, r's at appropriate locations. For instance, if $b_A : \mathbf{1} \longrightarrow A \otimes A^*$, $d_A : A^* \otimes A \longrightarrow \mathbf{1}$ are two morphisms, then the following diagram in a strict category should be replaced by Diagram 2.6 in a general category instead.

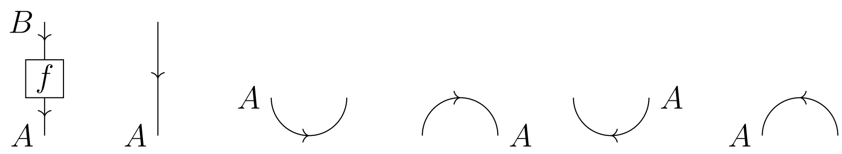
$$A \xrightarrow{b_A \otimes I} A \otimes A^* \otimes A \xrightarrow{I \otimes d_A} A$$

$$A \xrightarrow{l_A^{-1}} \mathbf{1} \otimes A \xrightarrow{b_A \otimes I} (A \otimes A^*) \otimes A \xrightarrow{\alpha_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{I \otimes d_A} A \otimes \mathbf{1} \xrightarrow{r_A} A \tag{2.6}$$

2.1.2 Rigid Monoidal Categories

Given an object A, a right dual of A is an object A^* together with two morphisms $b_A : \mathbf{1} \longrightarrow A \otimes A^*$, $d_A : A^* \otimes A \longrightarrow \mathbf{1}$ such that $(I_A \otimes d_A) \circ (b_A \otimes I_A) = I_A$ and $(d_A \otimes I_{A^*}) \circ (I_{A^*} \otimes b_A) = I_{A^*}$. The left hand side of the two identities above can be written as Diagram 2.7 and 2.8, respectively, if there is no assumption of $\mathcal{C}$ being strict.

$$A \xrightarrow{l_A^{-1}} \mathbf{1} \otimes A \xrightarrow{b_A \otimes I} (A \otimes A^*) \otimes A \xrightarrow{\alpha_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{I \otimes d_A} A \otimes \mathbf{1} \xrightarrow{r_A} A \tag{2.7}$$



$a)\ f$ $b)\ I_A$ $c)\ b_A$ $d)\ d_A$ $e)\ b'_A$ $f)\ d'_A$

Figure 2.1: Pictures of some morphisms

$$A^* \xrightarrow{r_{A^*}^{-1}} A^* \otimes \mathbf{1} \xrightarrow{I \otimes b_A} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha_{A^*,A,A^*}^{-1}} (A^* \otimes A) \otimes A^* \xrightarrow{d_A \otimes I} \mathbf{1} \otimes A^* \xrightarrow{l_{A^*}} A^* \tag{2.8}$$

Similarly, a left dual of A is an object ${}^*A$ with $b'_A : \mathbf{1} \longrightarrow {}^*A \otimes A$, $d'_A : A \otimes {}^*A \longrightarrow \mathbf{1}$ such that $(d'_A \otimes I_A) \circ (I_A \otimes b'_A) = I_A$ and $(I_{^*A} \otimes d'_A) \circ (b'_A \otimes I_{^*A}) = I_{^*A}$.

The conditions defining left and right duals will be easier to memorize with the tools of pictorial calculus. Here we only give a minimal illustration of using pictorial calculus. In Section 2.1.4, more details will be filled in. For a more rigorous treatment of it, see [45] [14].

A morphism $f : A \longrightarrow B$ is represented by Figure 2.1 $a)$. If f is the identity map, we also represent it by Figure 2.1 $b)$. The b_A, d_A, b'_A and d'_A are represented by Figure 2.1 $c), d), e)$ and $f)$, respectively.

Here are some rules for the representation of morphisms with pictures. Pictures consist of arcs and boxes. Arcs are directed as specified by an arrow, and the segments of the arc near its ends are required to be vertically aligned. There is an object attached to each arc. Each end of the arc represents an object. If the segment near one end is directed downwards, then the object the end represents is the same as the one attached to the arc; otherwise, it is the the right dual or the left dual. At this point it is not clear which dual it should be unless we specify it. Later on when more structures are introduced, e.g.

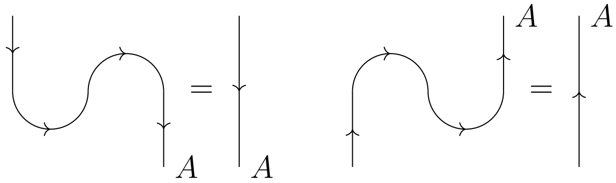


Figure 2.2: Identities of a right dual

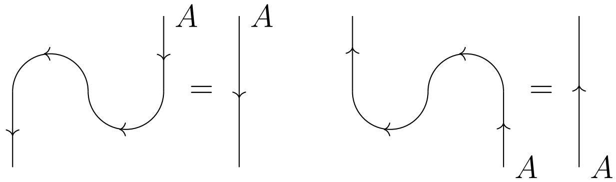


Figure 2.3: Identities of a left dual

pivotal structure, the left dual will be the same as the right dual and thus the ambiguity will be gone. Composition of two morphisms corresponds to stacking one on top of the other; tensor products corresponds to juxtaposition. With these conventions, the two identities defining a right dual are illustrated by pictures in Figure 2.2, and the identities defining a left dual in Figure 2.3.

The following lemma shows that if a right dual exists, then it is unique up to a unique isomorphism. In the same way one can also show the left dual is also unique.

Lemma 2.1.5 *[41] If $(A^*, b_A, d_A), (\tilde{A}^*, \tilde{b}_A, \tilde{d}_A)$ are both right duals of A, then there exists a unique isomorphism $\phi : A^* \longrightarrow \tilde{A}^*$, such that $\tilde{b}_A = (I \otimes \phi) \circ b_A$ and $\tilde{d}_A = d_A \circ (\phi^{-1} \otimes I)$.*

Definition 2.1.6 *A rigid monoidal category is a monoidal category such that there exists a right dual and a left dual for each object.*

In a rigid monoidal category, one can define the right (resp. left) dual of a morphism. For $f \in \mathrm{Hom}(A, B)$, the right dual $f^* \in \mathrm{Hom}(B^* \otimes A^*)$ is defined as $(d_B \otimes I)(I \otimes f \otimes I) \circ (I \otimes b_A)$, or pictorially as Figure 2.4. Similarly, the left dual ${}^*f \in \mathrm{Hom}({}^*B, {}^*A)$ is defined as $(I \otimes d'_B) \circ (I \otimes f \otimes I) \circ (b'_A \otimes I)$, or pictorially as Figure 2.4.

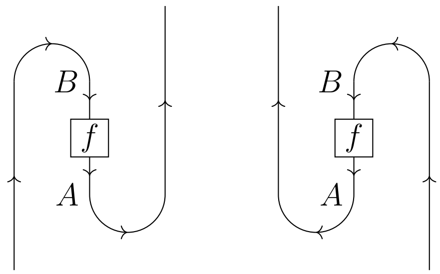


Figure 2.4: f^* (Left) and $^*f$ (Right)

Proposition 2.1.7 *The assignment $A \longmapsto A^*$, $f \longmapsto f^*$ defines a monoidal functor $(\cdot)^* : \mathcal{C} \longrightarrow \mathcal{C}^{op,rev}$. Similarly, $^*(\cdot) : \mathcal{C} \longrightarrow \mathcal{C}^{op,rev}$ sending A to $^*A$ and f to $^*f$ is also a monoidal functor.*

Since $(\mathcal{C}^{op,rev})^{op,rev} = \mathcal{C}$, $(\cdot)^{**}$ and $^{**}(\cdot)$ are both monoidal functors from $\mathcal{C}$ to itself.

2.1.3 Fusion Categories

An abelian category is $\mathbb{C}$-linear if the morphism spaces are vector spaces over $\mathbb{C}$ and compositions of morphisms are $\mathbb{C}$-bilinear. A functor from a $\mathbb{C}$-linear abelian category to a $\mathbb{C}$-linear abelian category is $\mathbb{C}$-linear if it is a $\mathbb{C}$-linear map restricted on morphism spaces. All functors on between $\mathbb{C}$-linear abelian categories are assumed to be $\mathbb{C}$-linear.

An object A in a $\mathbb{C}$-linear abelian category is *simple* if $\mathrm{Hom}(A, A) \simeq \mathbb{C}$. It can be shown that if A, B are simple objects that are not isomorphic, then $\mathrm{Hom}(A, B) = 0$. An abelian category is *semi-simple* if every object is isomorphic to a direct sum of simple objects.

If $\mathcal{C}$ is semi-simple, denote by $L(\mathcal{C})$ the set of isomorphism classes of simple objects.

Definition 2.1.8 *A fusion category is a semi-simple $\mathbb{C}$-linear rigid monoidal category $\mathcal{C}$ such that $\mathbf{1}$ is simple and $L(\mathcal{C})$ is finite.*

2.1.4 Pivotal and Spherical Categories

Let $\mathcal{C}$ be a rigid monoidal category. A pivotal structure on $\mathcal{C}$ is a natural isomorphism $\delta : Id_{\mathcal{C}} \longrightarrow (\cdot)^{**}$, namely, for each object A, there is an isomorphism $\delta_A : A \longrightarrow A^{**}$, such that

1. $\delta_{A\otimes B} = \delta_A \otimes \delta_B$;
2. $\delta_A^* = \delta_{A^*}^{-1}$;
3. $\delta_{\mathbf{1}} = I_{\mathbf{1}}$.

Definition 2.1.9 *A pivotal category is a rigid monoidal category with a pivotal structure.*

In a pivotal category $\mathcal{C}$, the left dual can be defined in terms of the right dual. Specifically, given an object A and its right dual A^* with b_A, d_A, one can define $b'_A := (I \otimes \delta_A^{-1}) \circ b_{A^*}, d'_A = d_{A^*} \circ (\delta_A \otimes I)$. It is straight forward to check that (A^*, b'_A, d'_A) satisfies the conditions of a left dual. Thus as an object, the left dual and the right dual are the same, and in the following we will simply call the dual of an object.

If $\mathcal{C}$ is a $\mathbb{C}$-linear pivotal category with $\mathbf{1}$ simple, we can define the concept of left trace and right trace of a morphism. Since $\mathrm{Hom}(\mathbf{1}, \mathbf{1}) \simeq \mathbb{C}$, one can fix an isomorphism sending $I_{\mathbf{1}}$ to $1 \in \mathbb{C}$. Thus every morphism in $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$ can be identified with a scalar in $\mathbb{C}$.

For $f \in \mathrm{Hom}(A, A)$, define left trace $\mathrm{Tr}^l(f) := d_A \circ (I \otimes f) \circ b'_A$ and the right trace $\mathrm{Tr}^r(f) := d'_A \circ (f \otimes I) \circ b_A$. Pictorially, these traces are illustrated in Figure 2.5.

The following properties on the left traces explains the notion of a 'trace '. The same properties also hold for right traces.

Proposition 2.1.10 *1.* $\mathrm{Tr}^l(\cdot)$ *is a linear functional on* $\mathrm{Hom}(A, A)$*.*

2. $\mathrm{Tr}^l(f \circ g) = \mathrm{Tr}^l(g \circ f)$ *for* $f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(A, A)$*.*

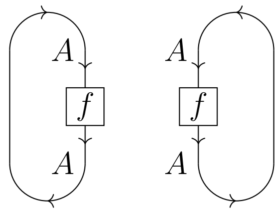


Figure 2.5: $\mathrm{Tr}^l(f)$ (Left) and $\mathrm{Tr}^r(f)$ (Right)

3. $\mathrm{Tr}^l(f \otimes g) = \mathrm{Tr}^l(f)\,\mathrm{Tr}^l(g)$ *for* $f \in \mathrm{Hom}(A, A), g \in \mathrm{Hom}(B, B)$.

4. $\mathrm{Tr}^l(I_{\mathbf{1}}) = 1$.

Definition 2.1.11 1. *A* $\mathbb{C}$*-linear pivotal category with* $\mathbf{1}$ *simple is spherical if* $\mathrm{Tr}^l = \mathrm{Tr}^r$.

2. *A spherical category* $\mathcal{C}$ *is strict if it is strict as a monoidal category and the pivotal structure is the identity natural isomorphism, namely,* $Id_{\mathcal{C}} = (\cdot)^{**}$.

Thus in a spherical category, we simple use the notion 'the trace of f ', and the notation $\mathrm{Tr}(f)$ for either the left trace or the right trace. By [46], every spherical category is equivalent to a strict spherical category. Therefore, we will assume spherical categories are strict.

In a spherical category, one can use the trace operator to define a bilinear pairing. Explicitly, define $\langle\ ,\ \rangle : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, A) \longrightarrow \mathbb{C}$ by $\langle f, g\rangle = \mathrm{Tr}(f \circ g)$.

Definition 2.1.12 *A spherical category is non-degenerate if the pairing* $\langle\ ,\ \rangle$ *is non-degenerate for all* A, B.

Proposition 2.1.13 *Spherical fusion categories are non-degenerate.*

Definition 2.1.14 1. *The quantum dimension* d_A *of an object* A *is* $d_A = \mathrm{Tr}(I_A)$.

2. *If $\mathcal{C}$ is a spherical fusion category, the total quantum dimension D of $\mathcal{C}$ is the positive square root of $D^2 := \sum_{a\in L(\mathcal{C})} d_a^2$. The notion 'positive square root ' makes sense by Lemma 2.1.15 below.*

Lemma 2.1.15 *[42] If $\mathcal{C}$ is a spherical fusion category, then $d_A = d_{A^*}$ and d_A is real.*

Note that we have used the same symbol d_A for the morphism $d_A : A^* \otimes A \longrightarrow \mathbf{1}$ and the quantum dimension of A, but in most cases it will be clear which quantity we mean by d_A, and otherwise we will point it out explicitly.

If $\mathcal{C}$ is a spherical fusion category, then for $a, b \in L(\mathcal{C})$, we have

$$a \otimes b = \bigoplus_{c\in L(\mathcal{C})} N_{ab}^c \; c. \tag{2.9}$$

The N_{ab}^c's are non-negative integers, called fusion rules. A category is multiplicity free if N_{ab}^c is either 0 or 1. Denote by $\bar{a}$ the dual a^*. The following properties hold.

$$N_{\mathbf{1}a}^b = N_{a\mathbf{1}}^b = \delta_{a,b},\;\; N_{ab}^c = N_{\bar{c}a}^{\bar{b}} = N_{b\bar{c}}^{\bar{a}} = N_{\bar{b}\bar{a}}^{\bar{c}}$$

At the end of this subsection, we give a brief introduction on pictorial calculus in a (strict) spherical category. See [14] for more detailed discussions.

We first define pictures in the plane $\mathbb{R}^2$ with the standard orientation. A picture in $\mathbb{R}^2$ is a union of rectangles and directed arcs (open or closed) restricted in $\mathbb{R} \times [0, 1]$ subjected to the following conditions:

1. All rectangles are disjoint and all arcs are disjoint. The interior of any arc is disjoint with any rectangle. The end points of an arc are either on $\mathbb{R} \times \{0, 1\}$ or on the bottom or the top edge of a rectangle. Arcs intersect with $\mathbb{R} \times \{0, 1\}$ or a rectangle transversally.

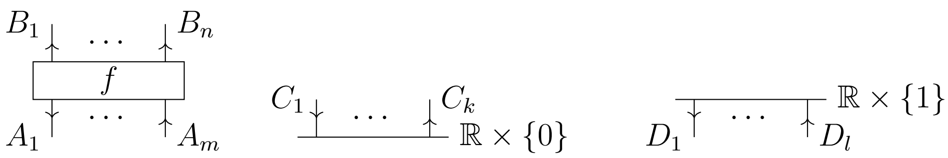


Figure 2.6: A local picture near a rectangle (Left), $\mathbb{R} \times \{0\}$ (Middle), $\mathbb{R} \times \{1\}$ (Right)

2. Each arc is labelled with an object. Each rectangle is labelled by a morphism $f : A \longrightarrow B$ where the description of f is as follows. Let the picture around the rectangle be as shown in Figure 2.6, then $A = A_1^{\epsilon_1} \otimes \cdots \otimes A_m^{\epsilon_m}$ and $B = B_1^{\beta_1} \otimes \cdots \otimes B_n^{\beta_n}$, where $\epsilon_i = 1$ if the arc labelled by A_i is directed downwards near the rectangle and $\epsilon = -1$ if it is directed upwards, β_j is defined in the same way depending on the direction of the arc labelled by B_j, and for an object C, C^{+1} is C and C^{-1} is C^*.

Given a picture $\mathcal{R}$, let $\mathcal{R}$ be as in Figure 2.6 (Middle) and Figure 2.6 (Right) near the intersections of $\mathcal{R}$ with $\mathbb{R} \times \{0\}$ and with $\mathbb{R} \times \{1\}$, respectively. Then define $S(\mathcal{R}) = C_1^{\epsilon_1} \otimes \cdots \otimes C_k^{\epsilon_k}$ and $T(\mathcal{R}) = D_1^{\beta_1} \otimes \cdots \otimes D_l^{\beta_l}$, where ϵ_i and β_j are defined in the same way as above. A picture $\mathcal{R}$ can be interpreted as a morphism from $S(\mathcal{R})$ to $T(\mathcal{R})$ by the following rules.

Interpret the pictures in Figure 2.1 as the morphism below them. If $\mathcal{R}_1, \mathcal{R}_2$ are disjoint and put side by side, then $\mathcal{R}_1 \sqcup \mathcal{R}_2$ is the tensor product of $\mathcal{R}_1$ and $\mathcal{R}_2$. If $\mathcal{R}_1, \mathcal{R}_2$ are two pictures such as $T(\mathcal{R}_1) = S(\mathcal{R}_2)$, then we can glue $\mathcal{R}_2$ on top of $\mathcal{R}_1$ along their common end points and compress the resulting graph to within $\mathbb{R} \times [0, 1]$. The morphism represented by the glued picture is the composition of $\mathcal{R}_1$ followed by $\mathcal{R}_2$, i.e., $\mathcal{R}_2 \circ \mathcal{R}_1$.

By the above rules, any picture can be interpreted as a morphism which is a composition of morphisms, each of which is a tensor product of some morphisms from Figure 2.1. By [14], the morphism thus obtained is well-defined, and *if there is an isotopy mapping one picture to another picture and preserving the labels of arcs, then the two pictures* *represent the same morphism.*

Proposition 2.1.16 *[14] If $\mathcal{C}$ is a spherical category, then*

1. *If the direction of an arc is reversed and the label attached to that arc is changed to its dual, then the new picture represents the same morphism as the old picture.*

2. *Let $S^2 = \mathbb{R}^2 \sqcup \{\infty\}$ be the oriented sphere and naturally embed $\mathbb{R}^2$ into S^2. If $\mathcal{R}_1, \mathcal{R}_2$ are pictures in $\mathbb{R}^2$ that are isotopic in S^2, then they represent the same morphism.*

2.1.5 Braided Monoidal (Tensor) Categories

A braided monoidal (tensor) category is a monoidal category $\mathcal{C}$, together with a natural isomorphism $c : \otimes \longrightarrow \otimes \circ Swap$, where $Swap$ is the functor which swaps the two components of $\mathcal{C} \times \mathcal{C}$, such that Diagram 2.10 and 2.11 commute. For $A, B \in \mathcal{C}^0$, denote by $c_{A,B}$ the natural isomorphism, namely,

$$c_{A,B} : A \otimes B \longrightarrow B \otimes A.$$

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[dl, "c_{A,B} \otimes I"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"'] & & A \otimes (B \otimes C) \arrow[d, "c_{A,B \otimes C}"] \\
B \otimes (A \otimes C) \arrow[dr, "I \otimes c_{A,C}"'] & & (B \otimes C) \otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B \otimes (C \otimes A) &
\end{tikzcd} \tag{2.10}$$

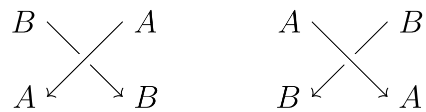


Figure 2.7: $c_{A,B}$ (Left) and $c^{-1}_{A,B}$ (Right)

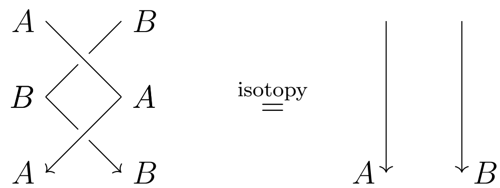


Figure 2.8: $c^{-1}_{A,B} \circ c_{A,B} = I_{A\otimes B}$

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[dl, "c^{-1}_{B,A}\otimes I"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"'] & & A\otimes (B\otimes C) \arrow[d, "c^{-1}_{B\otimes C,A}"] \\
B\otimes (A\otimes C) \arrow[dr, "I\otimes c^{-1}_{C,A}"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B\otimes (C\otimes A) &
\end{tikzcd} \tag{2.11}$$

The morphisms $c_{A,B}$ and $c^{-1}_{A,B}$ are represented by the pictures in Figure 2.7 (Left) and (Right), respectively.

Thus $I_{A\otimes B} = c^{-1}_{A,B} \circ c_{A,B}$ corresponds to the picture of the LHS of Figure 2.8, which is isotopic to the RHS representing the identity morphism.

With this convention, Diagram 2.10 and 2.11 can be represented as equalities of pictures shown in Figure 2.9 (Left) and (Right), respectively.

A spherical braided category is called a *ribbon* category. The picture calculus can be extend from spherical categories to ribbon categories by adding the convention that the braiding and its inverse shown in Figure 2.7. Then two pictures represent the same morphism in a ribbon category if and only if they are related by a regular isotopy.

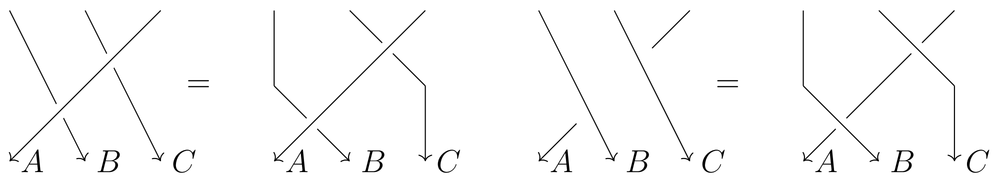


Figure 2.9: Equalities of the braiding

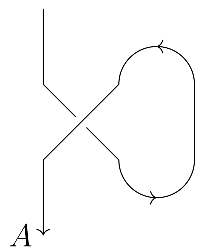


Figure 2.10: The twist θ_A

Let $\mathcal{C}$ be a ribbon category, and define $\theta_A : A \longrightarrow A$, called the twist, to be the isomorphism represented by the picture in Figure 2.10. Note that this picture is not regular isotopic to a vertical line representing the identity map in the plane.

The following proposition is a summary on properties of the twist.

Proposition 2.1.17 *The twist $\theta_{(\cdot)}$ defined for each object is a natural isomorphism $Id_{\mathcal{C}} \longrightarrow Id_{\mathcal{C}}$ that satisfies the following properties.*

1. $\theta_{A\otimes B} = (\theta_A \otimes \theta_B) \circ c_{B,A} \circ c_{A,B}$;

2. $\theta_{A^*} = (\theta_A)^*$;

3. $\theta_{\mathbf{1}} = I$.

2.2 G-crossed Braided Spherical Fusion Categories (G-BSFC)

G-BSFCs are the main ingredients we wish to introduce. In [47], they are called G-equivariant fusion categories. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a tensor category and G be a finite group. Recall in Section 2.1.1 that $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is the monoidal category where objects are auto tensor equivalences of $\mathcal{C}$, morphisms are natural transformations, and tensor product of two tensor equivalences are compositions of functors. Also recall that $\underline{G}$ is the monoidal category where the objects are elements of G, the only morphism between two elements is either the identity map if they are the same or none otherwise, and tensor product is given by group multiplication.

Definition 2.2.1 *A G-action on $\mathcal{C}$ is a monoidal functor $(F, \delta, \epsilon) : \underline{G} \longrightarrow Aut_{\otimes}(\mathcal{C})$.*

The above definition is a highly concise way to describe a G-action. To be more clear about it, we unpack the definition as follows. For $g \in G, X \in \mathcal{C}^0, f \in \mathcal{C}^1$, $F(g)(X), F(g)(f)$ are often written as ${}^gX, {}^gf$, respectively, to indicate F is an action.

For each object $g \in \underline{G}^0$, i.e., $g \in G$, $F(g)$ is an auto monoidal equivalence, and thus it comes with natural isomorphisms:

$$\gamma_{g;X,Y} : {}^g(X \otimes Y) \longrightarrow {}^gX \otimes {}^gY, \text{ and } \sigma_g : {}^g\mathbf{1} \longrightarrow \mathbf{1}$$

These natural isomorphisms satisfy the axioms in Equations 2.3 2.4 2.5.

Since (F, δ, ϵ) is a monoidal functor, there is a natural isomorphism $\delta_{g,h} : F(gh) \longrightarrow F(g) \circ F(h)$ and a natural isomorphism $\epsilon : F(e) \longrightarrow Id_{\mathcal{C}}$. Or more specifically, for each $X \in \mathcal{C}^0$, there is an isomorphism $\delta_{g,h;X} : {}^{gh}X \longrightarrow {}^g({}^hX)$ and $\epsilon_X : {}^eX \longrightarrow X$, such that the following diagrams commute:

$$\begin{tikzcd}
{}^{gh}X \arrow[r, "\delta_{g,h;X}"] \arrow[d, "{}^{gh}f"'] & {}^{g}({}^{h}X) \arrow[d, "{}^{g}({}^{h}f)"] \\
{}^{gh}Y \arrow[r, "\delta_{g,h;Y}"'] & {}^{g}({}^{h}Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^{e}X \arrow[r, "\epsilon_X"] \arrow[d, "{}^{e}f"'] & X \arrow[d, "f"] \\
{}^{e}Y \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}
\tag{2.12}$$

$$\begin{tikzcd}
{}^{gh}(X \otimes Y) \arrow[r, "\delta_{g,h;X\otimes Y}"] \arrow[d, "\gamma_{gh;X,Y}"'] & {}^{g}({}^{h}(X \otimes Y)) \arrow[r, "{}^{g}(\gamma_{h;X,Y})"] & {}^{g}({}^{h}X \otimes {}^{h}Y) \arrow[d, "\gamma_{g;{}^{h}X,{}^{h}Y}"] \\
{}^{gh}X \otimes {}^{gh}Y \arrow[rr, "\delta_{g,h;X} \otimes \delta_{g,h;Y}"'] & & {}^{g}({}^{h}X) \otimes {}^{g}({}^{h}Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^{e}(X \otimes Y) \arrow[r, "\epsilon_{X\otimes Y}"] \arrow[d, "\gamma_{e;X,Y}"'] & X \otimes Y \\
{}^{e}X \otimes {}^{e}Y \arrow[ur, "\epsilon_X \otimes \epsilon_Y"'] &
\end{tikzcd}
\tag{2.13}$$

$$\begin{tikzcd}
{}^{ghk}X \arrow[r, "\delta_{gh,k;X}"] \arrow[d, "\delta_{g,hk;X}"'] & {}^{gh}({}^{k}X) \arrow[d, "\delta_{g,h;{}^{k}X}"] \\
{}^{g}({}^{hk}X) \arrow[r, "{}^{g}(\delta_{h,k;X})"'] & {}^{g}({}^{h}({}^{k}X))
\end{tikzcd}
\tag{2.14}$$

$$\begin{tikzcd}
{}^{ge}X \arrow[r, "\delta_{g,e;X}"] \arrow[dr, "I_{{}^{g}X}"'] & {}^{g}({}^{e}X) \arrow[d, "{}^{g}(\epsilon_X)"] \\
& {}^{g}X
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^{eg}X \arrow[r, "\delta_{e,g;X}"] \arrow[dr, "I_{{}^{g}X}"'] & {}^{e}({}^{g}X) \arrow[d, "\epsilon_{{}^{g}X}"] \\
& {}^{g}X
\end{tikzcd}
\tag{2.15}$$

Definition 2.2.2 *A G-action on a strict monoidal category is strict if all the isomorphisms $\gamma_{g;X,Y}, \sigma_g, \delta_{g,h;X}, \epsilon_X$ are identity maps.*

A G-crossed braided spherical fusion category is a G-graded spherical fusion category $\mathcal{D} = \bigoplus_{g\in G} \mathcal{C}_g$ together with a G-action and a family of natural isomorphisms $\{c_{X,Y} : X \otimes Y \longrightarrow {}^{g}Y \otimes X \mid X \in \mathcal{C}_g, Y \in \mathcal{D}\}$ such that the following conditions are satisfied.

1. Each $\mathcal{C}_g$ is a full subcategory, and it is called the g-sector. There is only zero morphism between two objects from different sectors.

2. $\mathcal{C}_g \otimes \mathcal{C}_{g'} \subset \mathcal{C}_{gg'}$.

3. ${}^{g}(\mathcal{C}_{g'}) \subset \mathcal{C}_{gg'g^{-1}}$.

4. The $c_{X,Y}$'s, called crossed braiding, are self consistent, namely, for $X \in \mathcal{C}_g, Y \in \mathcal{C}_h, Z \in \mathcal{C}_k$,

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[dl, "c_{X,Y} \otimes I"'] \arrow[dr, "\alpha_{X,Y,Z}"] & \\
({}^{g}Y \otimes X) \otimes Z \arrow[d, "\alpha_{{}^{g}Y,X,Z}"'] & & X \otimes (Y \otimes Z) \arrow[d, "c_{X,Y\otimes Z}"] \\
{}^{g}Y \otimes (X \otimes Z) \arrow[d, "I \otimes c_{X,Z}"'] & & {}^{g}(Y \otimes Z) \otimes X \arrow[d, "\gamma_{g;Y,Z} \otimes I"] \\
{}^{g}Y \otimes ({}^{g}Z \otimes X) & & ({}^{g}Y \otimes {}^{g}Z) \otimes X \arrow[ll, "\alpha_{{}^{g}Y,{}^{g}Z,X}"]
\end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd}
 & ({}^{h}({}^{k}X) \otimes Y) \otimes Z \arrow[dl, "c^{-1}_{Y,{}^{k}X} \otimes I"'] \arrow[dr, "(\delta^{-1}_{h,k;X} \otimes I) \otimes I"] & \\
(Y \otimes {}^{k}X) \otimes Z \arrow[d, "\alpha_{Y,{}^{k}X,Z}"'] & & ({}^{hk}X \otimes Y) \otimes Z \arrow[d, "\alpha_{{}^{hk}X,Y,Z}"] \\
Y \otimes ({}^{k}X \otimes Z) \arrow[d, "I \otimes c^{-1}_{Z,X}"'] & & {}^{hk}X \otimes (Y \otimes Z) \arrow[d, "c^{-1}_{Y\otimes Z,X}"] \\
Y \otimes (Z \otimes X) & & (Y \otimes Z) \otimes X \arrow[ll, "\alpha_{Y,Z,X}"]
\end{tikzcd} \tag{2.17}$$

5. The G-action is consistent with the crossed braiding, namely, for $X \in \mathcal{C}_g, Y \in \mathcal{D}$, the diagram commutes:

$$\begin{tikzcd}
{}^{g'}(X \otimes Y) \arrow[rrr, "\gamma_{g';X,Y}"] \arrow[d, "{}^{g'}(c_{X,Y})"'] & & & {}^{g'}X \otimes {}^{g'}Y \arrow[d, "c_{{}^{g'}X,{}^{g'}Y}"] \\
{}^{g'}({}^{g}Y \otimes X) \arrow[r, "\gamma_{g';{}^{g}Y,X}"'] & {}^{g'}({}^{g}Y) \otimes {}^{g'}X \arrow[r, "\delta^{-1}_{g',g;Y} \otimes I"'] & {}^{g'g}Y \otimes {}^{g'}X \arrow[r, "\delta_{g'g\bar{g'},g';Y} \otimes I"'] & {}^{g'g\bar{g'}}({}^{g'}Y) \otimes {}^{g'}X
\end{tikzcd} \tag{2.18}$$

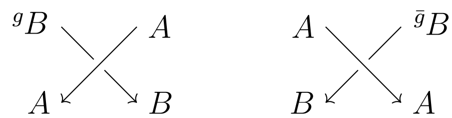


Figure 2.11: $c_{A,B}$ (Left) and $c^{-1}_{A,B}$ (Right) for $A \in \mathcal{C}_g$

Note that if G is trivial, then the definitions above are the same as those of spherical braided fusion categories, so a $\{e\}$-crossed braided spherical fusion category is simply a ribbon fusion category.

Every G-BSFC is equivalent to a strict one as indicated the following theorem. Roughly speaking, an equivalence of G-BSFCs is an equivalence as a monoidal functor and it preserves all additional structures, e.g., crossed braiding, G-action, etc. For the rigorous definition, see [48].

Theorem 2.2.3 *[48] Let $\mathcal{D}$ be a G-BSFC, then there exist a strict G-BSFC $\mathcal{D}'$ with a strict G-action and an equivalence $F : \mathcal{D} \longrightarrow \mathcal{D}'$ of G-crossed braided categories.*

Thus there is no loss of generality to restrict the study on strict G-BSFCs with strict G-actions, which is the assumption for the rest of the paper. Namely, one can assume that the $\alpha_{X,Y,Z}, l_X, r_X, \gamma_{g;X,Y}, \sigma_g, \delta_{g,h;X}, \epsilon_X$ are identities.

Following [47], we can use the same picture for a regular braiding to represent the G-crossed braiding as shown in Figure 2.11.

Similar to braided spherical fusion categories, here we can also define the twist $\theta_A : A \longrightarrow {}^{g}A$ for $g \in \mathcal{C}_A$ by the same picture:

The following proposition is parallel to the properties of the regular twist.

Proposition 2.2.4 *[47] In a G-crossed braided spherical fusion category, the twist $\theta_{(\cdot)}$ satisfies the following properties:*

1. $\theta_{A\otimes B} = (\theta_{{}^{gh\bar{g}}A} \otimes \theta_{{}^{g}B}) \circ c_{{}^{g}B,A} \circ c_{A,B},\ A \in \mathcal{C}_g, B \in \mathcal{C}_h;$

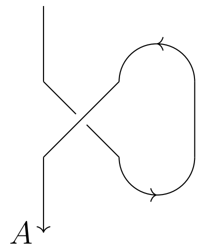


Figure 2.12: The twist θ_A

2. $\theta_{A^*} = (\theta_A)^*$;

3. $\theta_{\mathbf{1}} = I$;

4. ${}^h(\theta_A) = \theta_{{}^hA}$.

2.3 Monoidal 2-categories with Duals

In this section we show that a monoidal 2-category with certain extra structures can be constructed from a G-crossed braided spherical fusion category (G-BSFC) introduced in the previous section. The readers can skip this section if they are only interested in the $(3+1)$-TQFT out of a G-BSFC. Here we do not intend to list the full definition of monoidal 2-categories as it would be of too much distraction. Rather we only give the core structures and refer the readers to [30][31][32] for detailed discussions.

A (strict) 2-category has 0-morphisms (objects), 1-morphisms, and 2-morphisms. For any two objects A, B, there is a set of 1-morphisms $\mathrm{Hom}^1(A, B)$ and for any two 1-morphisms f, g with same the domain and codomain, there is a set of 2-morphisms $\mathrm{Hom}^2(f, g)$. Usually, 0-morphisms will be denoted by capital letters A, B, C, etc., 1-morphisms by lowercase letters f, g, h, etc., and 2-morphisms by Greek letters α, β, γ, etc. We use regular arrows for 1-morphisms and double arrows for 2-morphisms, namely, $f : A \longrightarrow B$, $\alpha : f \Longrightarrow g$. Both 1- and 2-morphisms can be composed. *Only in this* *section, we will use the convention that composition of morphisms are to be read from left to right.* Explicitly,

- Composition of 1-morphisms:

$$\begin{aligned} \circ \ : \mathrm{Hom}^1(A,B) \times \mathrm{Hom}^1(B,C) &\longrightarrow \mathrm{Hom}^1(A,C) \\ (f,g) \qquad\qquad &\longmapsto \quad f \circ g \end{aligned}$$

- 'Vertical' composition of 2-morphisms: for $f, g, h \in \mathrm{Hom}^1(A,B)$,

$$\begin{aligned} * \ : \mathrm{Hom}^2(f,g) \times \mathrm{Hom}^2(g,h) &\longrightarrow \mathrm{Hom}^2(f,h) \\ (\alpha,\beta) \qquad\qquad &\longmapsto \quad \alpha * \beta \end{aligned}$$

- 'Horizontal' composition of 2-morphisms: for $f, g \in \mathrm{Hom}^1(A,B)$, $f', g' \in \mathrm{Hom}^1(B,C)$,

$$\begin{aligned} \circ \ : \mathrm{Hom}^2(f,g) \times \mathrm{Hom}^2(f,g') &\longrightarrow \mathrm{Hom}^2(f \circ f', g \circ g') \\ (\alpha,\alpha') \qquad\qquad &\longmapsto \quad \alpha \circ \alpha' \end{aligned}$$

These compositions satisfy various associativity axioms. (In the non-strict case, the associativity axioms need to be replaced by certain Pentagon Identity similar to that in the definition of monoidal categories.) In fact, monoidal categories are in one-to-one correspondence with 2-categories with only one object. Explicitly, if $\mathcal{C}$ is a monoidal category, define a 2-category $\tilde{\mathcal{C}}$ with a single object as follows:

Denote the object by $*$ (or anything else). Then 1-morphisms between $*$ and $*$ are objects of $\mathcal{C}$ and 2-morphisms of $\tilde{\mathcal{C}}$ are 1-morphisms of $\mathcal{C}$. Composition of 1-morphisms and horizontal composition of 2-morphisms in $\tilde{\mathcal{C}}$ are given by tensor product of objects and tensor product of morphisms in $\mathcal{C}$, respectively. Vertical composition is just the regular composition of morphisms in $\mathcal{C}$.

Conversely, given $\tilde{\mathcal{C}}$, a 2-category with one object $*$, by reversing the rules in the above paragraph one can define a monoidal category.

Thus, a 2-category is in some sense a generalization of monoidal categories. And similar to the case of monoidal categories, the coherence theorem for 2-categories shows that every 2-category is equivalent to a strict 2-category in an appropriate sense.

A (semi-strict) monoidal 2-category with duals is a (strict) 2-category $\mathcal{C}$ with the following structures:

- For $A, B \in \mathcal{C}^0$, there is an object $A \otimes B \in \mathcal{C}^0$;
- For $f \in \mathrm{Hom}^1(A, B), C \in \mathcal{C}^0$, there are 1-morphisms $f \otimes C \in \mathrm{Hom}^1(A \otimes C, B \otimes C)$ and $C \otimes f \in \mathrm{Hom}^1(C \otimes A, C \otimes B)$;
- For $\alpha \in \mathrm{Hom}^2(f, g), C \in \mathcal{C}^0$, there are 2-morphisms $\alpha \otimes C \in \mathrm{Hom}^2(f \otimes C, g \otimes C)$ and $C \otimes \alpha \in \mathrm{Hom}^2(C \otimes f, C \otimes g)$;
- For $f \in \mathrm{Hom}^1(A, B), g \in \mathrm{Hom}^1(C, D)$, there is a 2-morphism, called the tensorator, $\otimes_{f,g} : (A \otimes g) \circ (f \otimes D) \Longrightarrow (f \otimes C) \circ (B \otimes g)$;
- There is a unit object $\mathbf{1} \in \mathcal{C}^0$;
- For $f \in \mathrm{Hom}^1(A, B)$, there is a 1-morphism $f^* \in \mathrm{Hom}^1(B, A)$, a 2-morphism $i_f : I_A \Longrightarrow f \circ f^*$, and a 2-morphism $e_f : f^* \circ f \Longrightarrow I_B$ called the dual, the unit, and the counit of f, respectively;
- For $A \in \mathcal{C}^0$, there is an object $A^* \in \mathcal{C}^0$, a 1-morphism $i_A : \mathbf{1} \longrightarrow A \otimes A^*$ and a 1-morphism $e_A : A^* \otimes A \longrightarrow \mathbf{1}$ called the dual, the unit and the counit of A,

respectively, and a 2-morphism $T_A : (i_A \otimes A) \circ (A \otimes e_A) \Longrightarrow I_A$ called the triangulator of A.

Again these structures need satisfy certain compatible axioms. See [31].

Now we show that from a G-BSFC we can construct a monoidal 2-category with duals and certain extra structures as well. Let $\mathcal{C}_G^\times$ be a G-BSFC and we denote the morphism space $\mathrm{Hom}(A, B)$ by $\mathrm{Hom}_{\mathcal{C}_G^\times}(A, B)$ to avoid confusion. Define a 2-category $\tilde{\mathcal{C}}$ as follows.

- 0-morphisms $\tilde{\mathcal{C}}^0 := G$.

- 1-morphisms $\mathrm{Hom}^1(g_1, g_2) := \mathcal{C}^0_{\bar{g}_1 g_2}$ for $g_1, g_2 \in G$; namely, the 1-morphisms from g_1 to g_2 are objects in $\mathcal{C}_{\bar{g}_1 g_2}$, where $\bar{g}_1$ is the inverse of g_1.

- 2-morphisms $\mathrm{Hom}^2(A, B) := \mathrm{Hom}_{\mathcal{C}_G^\times}(A, B)$ for $A, B \in \mathrm{Hom}^1(g_1, g_2) = \mathcal{C}^0_{\bar{g}_1 g_2}$; namely, the 2-morphisms from A to B are the regular morphisms from A to B in $\mathcal{C}_G^\times$. Note that here and only in this example, the notations for each level of morphisms are different from those specified at the beginning of the section. Namely, in $\tilde{\mathcal{C}}$, we will use $g_1, g_1', \cdots,$ for objects, capital letters $A, B, \cdots$, for 1-morphisms, and Greek letters $\alpha, \beta, \cdots$, for 2-morphisms.

- For $A \in \mathrm{Hom}^1(g_1, g_2), B \in \mathrm{Hom}^1(g_2, g_3)$, the composition $A \circ B := A \otimes B \in \mathrm{Hom}^1(g_1, g_3)$.

- For $\alpha \in \mathrm{Hom}^2(A, B), \beta \in \mathrm{Hom}^2(B, C)$, the vertical composition $\alpha * \beta := \alpha \circ \beta \in \mathrm{Hom}^2(A, C)$; namely, vertical composition in $\tilde{\mathcal{C}}$ corresponds to regular composition in $\mathcal{C}_G^\times$.

- For $\alpha \in \mathrm{Hom}^2(A, B), \alpha' \in \mathrm{Hom}^2(A', B')$, where $A, B \in \mathrm{Hom}^1(g_1, g_2)$, $A', B' \in \mathrm{Hom}^1(g_2, g_3)$, the horizontal composition $\alpha \circ \alpha' := \alpha \otimes \alpha' \in \mathrm{Hom}^2(A \otimes A', B \otimes B')$.

One can check this defines a 2-category. Now we introduce the monoidal structure and duals on $\tilde{\mathcal{C}}$. To avoid confusion, the tensor product in $\tilde{\mathcal{C}}$ will be denoted by the symbol $\boxtimes$ instead of $\otimes$, while in $\mathcal{C}_G^\times$ still by $\otimes$. We list below explicit constructions corresponding to the items in the definition of monoidal 2-categories with duals.

- $g_1 \boxtimes g_2 := g_1 g_2$.
- For $A \in \mathrm{Hom}^1(g_1, g_2), g \in G$, $A \boxtimes g := {}^{\bar{g}}A \in \mathrm{Hom}^1(g_1 g, g_2 g)$ and $g \boxtimes A := A \in \mathrm{Hom}^1(gg_1, gg_2)$. Note that as a set, $\mathrm{Hom}^1(g_1, g_2) = \mathrm{Hom}^1(gg_1, gg_2)$.
- For $\alpha \in \mathrm{Hom}^2(A, B), g \in G$, $\alpha \boxtimes g := {}^{\bar{g}}\alpha \in \mathrm{Hom}^2(A \boxtimes g, B \boxtimes g) = \mathrm{Hom}^2({}^{\bar{g}}A, {}^{\bar{g}}B)$ and $g \boxtimes \alpha := \alpha \in \mathrm{Hom}^2(g \boxtimes A, g \boxtimes B) = \mathrm{Hom}^2(A, B)$.
- For $A \in \mathrm{Hom}^1(g_1, g_2)$, $A' \in \mathrm{Hom}^1(g_1', g_2')$, note that $(g_1 \boxtimes A') \circ (A \boxtimes g_2') = A' \otimes {}^{\overline{g_2'}}A$ and $(A \boxtimes g_1') \circ (g_2 \boxtimes A') = {}^{\overline{g_1'}}A \otimes A'$. Define the tensorator $\boxtimes_{A,A'} := c_{A', {}^{\overline{g_2'}}A}$.
- The unit $\mathbf{1} := e$ is the unit of G.
- For $A \in \mathrm{Hom}^1(g_1, g_2)$, A^* is the same as the dual of A when viewed as an object of $\mathcal{C}_G^\times$; The i_A and e_A are defined as b_A and d_A, respectively.
- For $g \in G$, $g^* := \bar{g}$, and i_g, e_g are defined as the unit $\mathbf{1}$ of $\mathcal{C}_G^\times$, and T_g is the identity map $I_{\mathbf{1}}$.

It is direct, though a little tedious, to check $\tilde{\mathcal{C}}$ is indeed a monoidal 2-category with duals.

Below we show that actually there are more structures in $\tilde{\mathcal{C}}$. However, these structures do not make it into a 'spherical 2-category ' defined in [32].

Given $A \in \mathrm{Hom}^1(g_1, g_2)$, define ${}^{\#}A := {}^{g_2}A$, $A^{\#} := {}^{g_1}A \in \mathrm{Hom}(\bar{g}_2, \bar{g}_1)$ as follows [32].

$$\bar{g}_2 \xrightarrow{=} \bar{g}_1 \boxtimes g_1 \boxtimes \bar{g}_2 \xrightarrow{\bar{g}_1 \boxtimes A \boxtimes \bar{g}_2} \bar{g}_1 \boxtimes g_2 \boxtimes \bar{g}_2 \xrightarrow{=} \bar{g}_1$$

$$\bar{g}_2 \xrightarrow{=} \bar{g}_2 \boxtimes g_1 \boxtimes \bar{g}_1 \xrightarrow{\bar{g}_2 \boxtimes A \boxtimes \bar{g}_1} \bar{g}_2 \boxtimes g_2 \boxtimes \bar{g}_1 \xrightarrow{=} \bar{g}_1$$

Therefore, ${}^{\#}A := {}^{g_2}A,\ A^{\#} := {}^{g_1}A$. Similarly, for $\alpha \in \mathrm{Hom}^2(A, B), A, B \in \mathrm{Hom}^1(g_1, g_2)$, define ${}^{\#}\alpha := {}^{g_2}\alpha \in \mathrm{Hom}^2({}^{\#}A, {}^{\#}B),\ \alpha^{\#} := {}^{g_1}\alpha \in \mathrm{Hom}^1(A^{\#}, B^{\#})$. Note that there is the isomorphism:

$$\theta^{-1}_{A^{\#}} :\ {}^{\#}A \longrightarrow A^{\#}$$

According to Definition 2.4 in [32], the isomorphisms $\theta^{-1}_{A^{\#}}$ make $\tilde{\mathcal{C}}$ a pivotal 2-category. According to Definition 2.5 in [32], for any $g \in G,\ A \in \mathrm{Hom}^1(g, g) = \mathcal{C}_e$, the left trace $\mathrm{Tr}_L(A) := g \boxtimes A = A$ and the right trace $\mathrm{Tr}_R(A) := A \boxtimes g = {}^{\bar{g}}A$. According to Definition 2.8 in [32], for the category to be spherical, there should be a 2-isomorphism $\sigma_A : \mathrm{Tr}_L(A) \Longrightarrow \mathrm{Tr}_R(A)$ which satisfies certain conditions as specified there. However, in a G-BSFC, this implies that for any $A \in \mathcal{C}_e, g \in G$, there is an isomorphism $\sigma_A : A \longrightarrow {}^{\bar{g}}A$, which in general does not exist.

As a summary, out of a G-BSFC we have constructed a pivotal 2-category, which in general violates Definition 2.8 in [32], and thus is not necessarily a spherical 2-category.

Chapter 3

Invariants of 4-manifolds

In this chapter, we introduce the key construction of the 4-manifold invariants out of a G-BSFC. The invariant, called partition function, is a state-sum model and a natural generalization of the Tuaev-Viro invariant. Although we will only describe the invariants for oriented closed 4-manifolds, it is straight forward to extend the construction to 3-manifolds and to 4-manifolds with boundaries, and thus produce a $(3+1)$-TQFT. In Section 3.1, three equivalent definitions of the partition function are given based on ordered triangulations. To show these indeed define an invariant of 4-manifolds, one needs to prove the partition function is independent of the choice of ordered triangulations on a given manifold. Considering that the proof is quite technical and lengthy, we defer it to Chapter 4. In Section 3.3, several examples of the partition function are given showing that our invariant generalizes several known categorical invariants in literature such as the Crane-Yetter invariant and Yetter's invariant from homotopy 2-types. We also give a concrete example of a non-trivial G-BSFC, which is obtained by the de-equivariatization process. Finally in Section 3.4, we study a variation of the invariant, namely, when the a G-BSFC is concentrated on the sector indexed by the trivial group element, then a 4-cohomology class in $H^4(G, U(1))$ can be introduced into the model. The resulting invariant reduces to the twisted Dijkgraaf-Witten invariant as a special case.

3.1 Partition Function

An ordered triangulation of a manifold is a triangulation with an ordering of its vertices by $0, 1, 2, \cdots$. Let $\mathcal{T}$ be an ordered triangulation of a closed oriented 4-manifold M, and for $k = 0, 1, 2, 3, 4$, let $\mathcal{T}^k$ be the set of k-simplicies of the triangulation. The restriction of the ordering on each k-simplex $\sigma \in \mathcal{T}^k$ induces a relative ordering of the vertices of σ. Under this relative ordering, we write σ as $(0\ 1\ \cdots\ k-1)$. For each $\sigma \in \mathcal{T}^4$, we define the sign, $\epsilon(\sigma)$, of σ to be $+$ if the orientation on σ induced from that of M coincides with that coming from the ordering of its vertices; $\epsilon(\sigma)$ is defined to be $-$ otherwise.

Let $\mathcal{D} = \mathcal{C}_G^\times = \bigoplus_{g \in G} \mathcal{C}_g$ be a G-BSFC, and let $L(\mathcal{D})$ be the set of isomorphism classes of simple objects.

Definition 3.1.1 *A $\mathcal{D}$-coloring of $\mathcal{T}$ is a pair of maps $F = (g, f)$, $g : \mathcal{T}^1 \to G$, $f : \mathcal{T}^2 \to L(\mathcal{D})$, such that for any simplex $\beta = (012) \in \mathcal{T}^2$ with the induced ordering on its vertices,*

$$f(012) \in \mathcal{C}_{g(02)^{-1}g(01)g(12)}.$$

Given a $\mathcal{D}$-coloring $F = (g, f)$, for each $\beta = (012) \in \mathcal{T}^2$, we arbitrarily choose a representative in the class $f(012)$ and denote it by f_{012} or just 012 when no confusion arises. For an edge (ij), we also write $g(ij)$ as g_{ij} or ij, $g(ij)^{-1}$ as $\bar{g}_{ij}$ or $\overline{ij}$.

Remark 3.1.2 *Since the representative for each triangle color is chosen arbitrarily, it is possible that even if two triangles are assigned the same color, their representatives could still be different. Our approach is thus more general than the common one, where a representative for each isomorphism class is fixed at the beginning and these representatives* *are assigned as triangle colors. The approach here is necessary and will become clear in Section 4.2 where we prove the partition function does not depend on the ordering of the triangulation.*

Assume now a representative for each triangle has been chosen. For any $\tau = (0123) \in \mathcal{T}^3$ with the induced ordering on its vertices, consider the boundary map:

$$\partial(0123) = (123) - (023) + (013) - (012).$$

We assign to (0123) two vector spaces:

$$V_F^+(0123) := \mathrm{Hom}(f_{023} \otimes {}^{\bar{g}_{23}}f_{012}, f_{013} \otimes f_{123}),$$

$$V_F^-(0123) := \mathrm{Hom}(f_{013} \otimes f_{123}, f_{023} \otimes {}^{\bar{g}_{23}}f_{012}).$$

Note that by the properties of a coloring, the objects $f_{023} \otimes {}^{\bar{g}_{23}}f_{012}$ and $f_{013} \otimes f_{123}$ are both in the sector $\mathcal{C}_{\bar{g}_{03}g_{01}g_{12}g_{23}}$. This is necessary since otherwise the spaces $V_F^+(0123)$ and $V_F^-(0123)$ would always be trivial.

Recall that for any two objects $X, Y \in \mathcal{D}$, the pairing on $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X)$ is defined as follows:

$$\begin{aligned} \langle\ ,\ \rangle : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X) &\longrightarrow \mathbb{C} \\ (\phi, \psi) &\mapsto \mathrm{Tr}(\phi\psi) \end{aligned}$$

The pairing $\langle\ ,\ \rangle$ is non-degenerate, and thus there is a natural isomorphism between $V_F^+(0123)$ and $V_F^-(0123)^*$, $V_F^-(0123)$ and $V_F^+(0123)^*$ induced by the pairing.

Finally for any 4-simplex $\sigma = (01234) \in \mathcal{T}^4$ with the induced ordering on its vertices, the partition function $Z_F^{\epsilon(\sigma)}(\sigma)$ is defined as follows. Note that the boundary map is given by:

$$\partial(01234) = (1234) - (0234) + (0134) - (0124) + (0123).$$

If $\epsilon(\sigma) = +$, we first define the functional $\tilde{Z}_F^+(01234) : V_F^+(0234) \otimes V_F^+(0124) \otimes V_F^-(1234) \otimes V_F^-(0134) \otimes V_F^-(0123) \to \mathbb{C}$ by the picture shown in Figure 3.1 (Left). The meaning of the picture is interpreted in the following way. Given an element $\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$, each ϕ_i is to be inserted in the corresponding box, and the resulting graph represents a morphism in $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$, which is a scalar multiple of $Id_{\mathbf{1}}$. Then $\tilde{Z}_F^+(01234)(\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4)$ is defined to be that scalar. Note that the label $\bar{g}_{34}$ to the left top corner of the box containing $V_F^-(0123)$ in the picture means the action of $\bar{g}_{34}$ on the space $V_F^-(0123)$. It is direct to figure out the object labels on each edge of the picture and check that all the compositions of morphisms make sense.

Using the non-degenerate pairing $\langle \; , \; \rangle$, this defines a linear map $Z_F^+(01234)$:

$$Z_F^+(01234) : V_F^+(0234) \otimes V_F^+(0124) \longrightarrow V_F^+(1234) \otimes V_F^+(0134) \otimes V_F^+(0123),$$

such that $\langle Z_F^+(01234)(\phi_0 \otimes \phi_1), \phi_2 \otimes \phi_3 \otimes \phi_4 \rangle = \tilde{Z}_F^+(01234)(\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4)$.

Note that the domain of $Z_F^+(01234)$ corresponds to the negative boundaries, and the codomain corresponds to the positive boundaries. Thus we can think of $Z_F^+(01234)$ as a map from negative boundaries to positive boundaries.

Similarly, if $\epsilon(\sigma) = -$, consider instead the functional $\tilde{Z}_F^-(01234) : V_F^-(0234) \otimes V_F^-(0124) \otimes V_F^+(1234) \otimes V_F^+(0134) \otimes V_F^+(0123) \to \mathbb{C}$ by the picture shown in Figure 3.1 (Right), which is obtained by reflecting the picture in Figure 3.1 (Left) along a horizon line. In the same way we get a linear map:

$$Z_F^-(01234) : V_F^+(1234) \otimes V_F^+(0134) \otimes V_F^+(0123) \longrightarrow V_F^+(0234) \otimes V_F^+(0124),$$

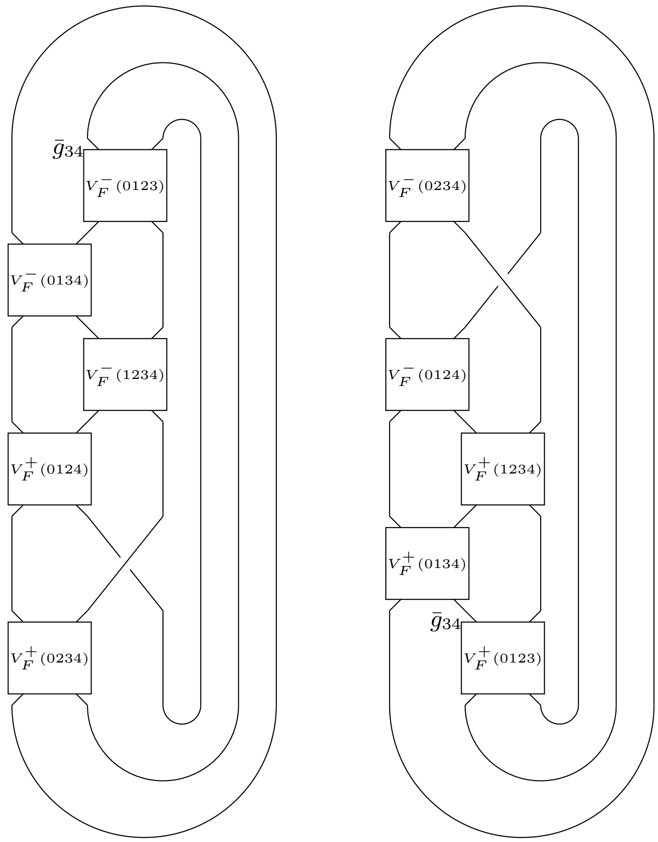


Figure 3.1: (Left): $\tilde{Z}_F^+(01234)$; (Right): $\tilde{Z}_F^-(01234)$

by $\langle \phi_0 \otimes \phi_1, Z_F^-(01234)(\phi_2 \otimes \phi_3 \otimes \phi_4)\rangle = \tilde{Z}_F^-(01234)(\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4)$. Noting that

$$\partial(-01234) = -(1234) + (0234) - (0134) + (0124) - (0123),$$

we can also think of $Z_F^-(01234)$ as a map from negative boundaries to positive boundaries.

Each 3-simplex τ is the intersection of exactly two 4-simplices σ_1 and σ_2. If τ is a negative boundary in σ_1, it must be a positive boundary in σ_2. Thus, $V_F^+(\tau)$ appears exactly once in the domain and codomain for some 4-simplex. Thus we have

$$\bigotimes_{\sigma \in \mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma) : \bigotimes_{\tau \in \mathcal{T}^3} V_F^+(\tau) \longrightarrow \bigotimes_{\tau \in \mathcal{T}^3} V_F^+(\tau).$$

Definition 3.1.3 *Given a G-BSFC $\mathcal{D}$ and an ordered triangulation $\mathcal{T}$ of a closed oriented 4-manifold M, the partition function $Z_{\mathcal{D}}(M;\mathcal{T})$ of the pair $(M,\mathcal{T})$ is defined by:*

$$Z_{\mathcal{D}}(M;\mathcal{T}) = \sum_{F=(g,f)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod_{\beta\in\mathcal{T}^2} d_{f(\beta)})\,\mathrm{Tr}(\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}}, \tag{3.1}$$

where F runs through all $\mathcal{D}$-colorings of $\mathcal{T}$.

Theorem 3.1.4 (Main Theorem) *The partition function $Z_{\mathcal{D}}(M;\mathcal{T})$ is an invariant of closed oriented 4-manifolds.*

To prove the Main Theorem, we need to show that $Z_{\mathcal{D}}(M;\mathcal{T})$ is independent on:

1. the choice of representatives for each triangle color,

2. the ordering of vertices of a triangulation,

3. the choice of a triangulation.

These will be proved in Section 4.1, Section 4.2, and Section 4.4, respectively.

Below we derive two other formulas for the partition function, which will be used in the invariance proof.

For any two objects X, Y, the non-degenerate pairing $\langle\ ,\ \rangle_{X,Y}$ induces canonical isomorphisms $\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(Y,X)^*$ and $\mathrm{Hom}(Y,X) \simeq \mathrm{Hom}(X,Y)^*$. Define $\langle\ ,\ \rangle^*_{X,Y}$ to be the dual map of $\langle\ ,\ \rangle_{X,Y}$:

$$\langle\ ,\ \rangle^*_{X,Y} : \mathbb{C} \longrightarrow \mathrm{Hom}(X,Y)^* \otimes \mathrm{Hom}(Y,X)^* \simeq \mathrm{Hom}(Y,X) \otimes \mathrm{Hom}(X,Y),$$

and let $\phi_{X,Y} = \langle\ ,\ \rangle^*_{X,Y}(1)$.

If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in I}$ are a basis of $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$, respectively, such that $\langle v_i, w_j\rangle = d_i\delta_{i,j}$ for some non-zero numbers d_i, then it is direct to check that $\phi_{X,Y} =$

$\sum_{i \in I} d_i^{-1} w_i \otimes v_i$.

Now assume an coloring $F = (g, f)$ and a representative for each triangle color have been chosen. For each 3-simplex $\tau = (0123) \in \mathcal{T}^3$, there is thus the element

$$\phi_{F;\tau} := \phi_{f_{023} \otimes \bar{g}_{23} f_{012}, f_{013} \otimes f_{123}} \in V_F^-(0123) \otimes V_F^+(0123).$$

Let $V_F = \bigotimes_{\tau \in \mathcal{T}^3} V_F^-(\tau) \otimes V_F^+(\tau)$.

From the definition of $\tilde{Z}_F^{\epsilon(\sigma)}(\sigma)$ for each $\sigma \in \mathcal{T}^4$, again it is not hard to see that if $\tau \in \mathcal{T}^3$ is the intersection of two 4-simplices σ_1, σ_2, then $V_F^+(\tau) \otimes V_F^-(\tau)$ appears as a component in the domain of $\tilde{Z}_F^{\epsilon(\sigma_1)}(\sigma_1) \otimes \tilde{Z}_F^{\epsilon(\sigma_2)}(\sigma_2)$. And therefore, the map $\bigotimes_{\sigma \in \mathcal{T}^4} \tilde{Z}_F^{\epsilon(\sigma)}(\sigma)$ is a functional on V_F.

Proposition 3.1.5 *Let $\phi_F = \bigotimes_{\tau \in \mathcal{T}^3} \phi_{F;\tau} \in V_F$, then $Z_{\mathcal{D}}(M; \mathcal{T})$ is given by the following formula:*

$$Z_{\mathcal{D}}(M; \mathcal{T}) = \sum_{F=(g,f)} \frac{(D^2/|G|)^{|\mathcal{T}^0|} (\prod_{\alpha \in \mathcal{T}^2} d_{f(\alpha)})}{(D^2)^{|\mathcal{T}^1|}} (\bigotimes_{\sigma \in \mathcal{T}^4} \tilde{Z}_F^{\epsilon(\sigma)}(\sigma))(\phi_F) \tag{3.2}$$

We will give a third formulation of the partition function as a state sum model. For simplicity, we assume that the category $\mathcal{D}$ is multiplicity free. The more general case can be considered in a similar way.

Let $\tilde{L}(\mathcal{D})$ be an arbitrary complete set of representatives, one for each isomorphism of simple objects. Recall for each triple (a, b, c) of simple objects such that $N_{ab}^c = 1$, namely, (a, b, c) is admissible, we have chosen a basis element $B_c^{ab} \in \mathrm{Hom}(c, a \otimes b)$, and a basis element $B_{ab}^c \in \mathrm{Hom}(a \otimes b, c)$, such that for any $c, c' \in \tilde{L}(\mathcal{D})$, $B_{ab}^{c'} B_c^{ab} = \delta_{c,c'} Id_c$, and that $\sum_{c \in \tilde{L}(\mathcal{D})} B_c^{ab} B_{ab}^c = Id_{a \otimes b}$. These identities are also illustrated in Figure 3.2.

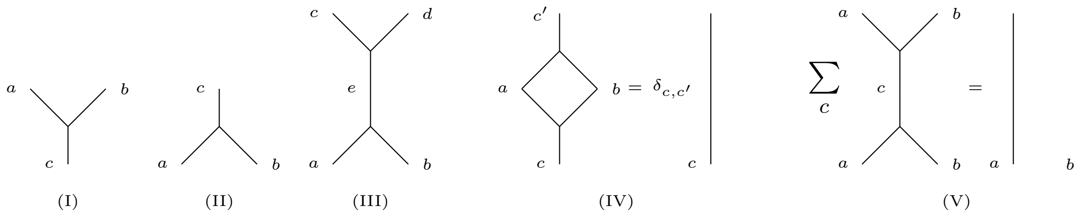


Figure 3.2: Some identities

For simple objects (a, b, c, d), let $B^e_{ab,cd} = B^{cd}_e B^e_{ab}$, thus

$$\{B^e_{ab,cd} : e \in \tilde{L}(\mathcal{D}), (a, b, e), (c, d, e) \text{ admissible}\}$$

is a basis of $\mathrm{Hom}(a \otimes b, c \otimes d)$, and moreover, we have $\langle B^e_{ab,cd}, B^{e'}_{cd,ab}\rangle = \delta_{e,e'} d_e$.

Definition 3.1.6 *An extended $\mathcal{D}$-coloring on the ordered triangulation $\mathcal{T}$ is a triple $\hat{F} = (g, f, t)$, $g : \mathcal{T}^1 \to G$, $f : \mathcal{T}^2 \to L(\mathcal{D})$, $t : \mathcal{T}^3 \to L(\mathcal{D})$, such that,*

1. *for any $\beta = (012) \in \mathcal{T}^2$, $f(012) \in \mathcal{C}_{\bar{g}_{02}g_{01}g_{12}}$;*

2. *for any $\tau = (0123) \in \mathcal{T}^3$, $t(0123) \in \mathcal{C}_{\bar{g}_{03}g_{01}g_{12}g_{23}}$.*

As before, we choose arbitrarily a representative for each $f(012)$ and $t(0123)$, and denote these representatives as f_{012}, t_{0123} or $012, 0123$. Then, for each $\sigma \in \mathcal{T}^4$, the $25j$-symbol $\hat{Z}^{\epsilon(\sigma)}_{\hat{F}}(\sigma)$ is defined to be the evaluation of the graph in Figure 3.3 (Left) if $\epsilon = +$, and Figure 3.3 (Right) if $\epsilon = -$.

Proposition 3.1.7 *The partition function has the state sum model:*

$$Z_{\mathcal{D}}(M; \mathcal{T}) = \sum_{\hat{F}=(g,f,t)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod_{\sigma\in\mathcal{T}^4} \hat{Z}^{\epsilon(\sigma)}_{\hat{F}}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod_{\tau\in\mathcal{T}^3} d_{t(\tau)})} \tag{3.3}$$

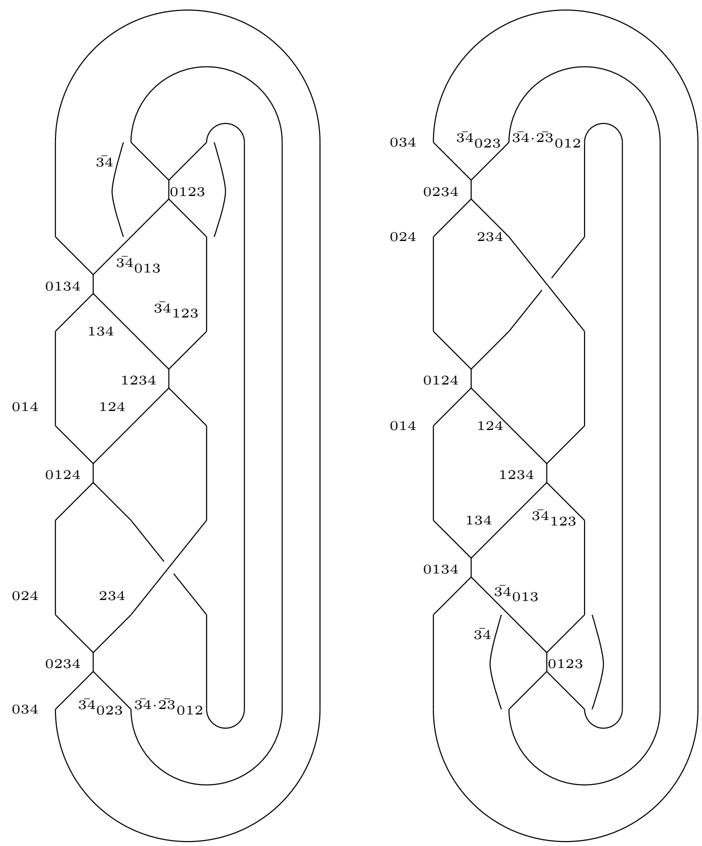


Figure 3.3: (Left): $\hat{Z}^{+}_{\hat{F}}(01234)$; (Right): $\hat{Z}^{+}_{\hat{F}}(01234)$

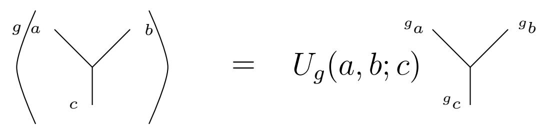


Figure 3.4: Definition of U-symbol

3.2 A Formula for the Partition Function

Here we give an explicit formula for the $25j$-symbol $\hat{Z}_{\hat{F}}^{\epsilon(\sigma)}(\sigma)$ in Figure 3.3 in terms of the data of a G-BSFC $\mathcal{D} = \mathcal{C}_G^\times$ such as the F-symbol, R-symbol, etc. This formula will be convenient for computer calculations. For simplicity, we assume $\mathcal{D}$ is multiplicity free. The formula for the general case can be obtained in a similar way. Let us first define the relevant data in $\mathcal{D}$.

Let $L(\mathcal{D})$ be a complete set of representatives of simple objects. Recall that given $a, b, c \in L(\mathcal{D})$ such that (a, b, c) is admissible, namely, $N_{ab}^c = 1$, the morphism spaces $\mathrm{Hom}(c, a \otimes b)$ and $\mathrm{Hom}(a \otimes b, c)$ are both one-dimensional, and we have chosen a basis element B_c^{ab} in $\mathrm{Hom}(c, a \otimes b)$ and B_{ab}^c in $\mathrm{Hom}(a \otimes b, c)$. See Figure 3.2 for their pictorial representations and related identities. For the rest of the section, all the summation variables will run through $L(\mathcal{D})$ unless otherwise noted.

- **U-symbol**: for any $g \in G, a, b, c \in L(()\mathcal{D})$ such that (a, b, c) is admissible, $g(B_c^{ab})$ is a scalar multiple of $B_{{}^g c}^{{}^g a {}^g b}$. Denote this scalar by $U_g(a, b; c)$. See Figure 3.4 for the pictorial definition.

- **η-symbol**: for $g, h \in G, a \in L(\mathcal{D})$, $\eta_a(g, h)$ is defined to be the isomorphism from ${}^{gh}a$ to ${}^g({}^h a)$. [1]

- **F-symbol**: for any $a, b, c, d \in L(\mathcal{D})$ such that $\mathrm{Hom}(d, a \otimes b \otimes c) \neq 0$, there are two

[1] In Section 2.2, $\eta_a(g, h)$ is denoted by $\delta_{g,h;a}$.

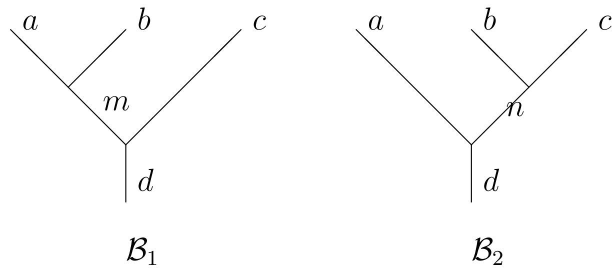


Figure 3.5: Two Bases $\mathcal{B}_1$ and $\mathcal{B}_2$

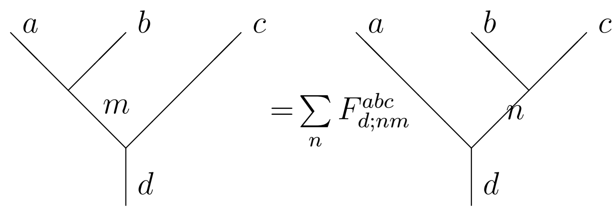


Figure 3.6: Definition of the F-symbol

set of bases:

$$\mathcal{B}_1 = \{(B^{ab}_m \otimes I) \circ B^{mc}_d \mid m \in L(\mathcal{D}), (a, b, m), (m, c, d) \text{ admissible}\}$$

$$\mathcal{B}_2 = \{(I \otimes B^{bc}_n) \circ B^{an}_d \mid n \in L(\mathcal{D}), (b, c, n), (a, n, d) \text{ admissible}\}$$

These two bases are represented pictorially in Figure . The matrix relating the two bases are called F-symbol or F-matrix. Explicitly, the F-symbols and F^{-1}-symbols are defined as shown in Figure 3.6 and 3.7, respectively.

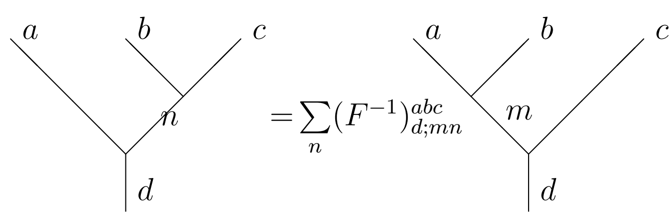


Figure 3.7: Definition of the F^{-1}-symbol

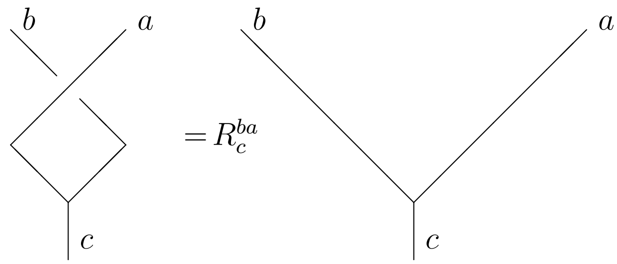


Figure 3.8: Definition of the R-symbol R_c^{ba}

- R**-symbol**: the R-symbol is defined as a scalar shown in Figure 3.8.

With the data defined above, the $25j$-symbol $\hat{Z}_{\hat{F}}^{\epsilon(\sigma)}(\sigma)$ in Figure 3.3 can be expressed as a concrete formula. Explicitly, let $\sigma = (01234)$ assume an extended coloring $\hat{F}$ has been assigned to σ, then $\hat{Z}_{\hat{F}}^{+}(\sigma)$ and $\hat{Z}_{\hat{F}}^{+}(\sigma)$ are given by Equation 3.4 and 3.5, respectively.

$$\begin{aligned}\hat{Z}_{\hat{F}}^{+}(\sigma) = \sum_{d,a} & F_{d;a,0234}^{024,234,{}^{\overline{34}\cdot\overline{23}}012}\ \eta_{012}^{-1}(\overline{34},\overline{23})\ R_a^{{}^{\overline{24}}012,234}\ (F^{-1})_{d;0124,a}^{024,{}^{\overline{24}}012,234} \\ & F_{d;1234,0124}^{014,124,234}\ (F^{-1})_{d;0134,1234}^{014,134,{}^{\overline{34}}123}\ F_{d;{}^{\overline{34}}0123,0134}^{034,{}^{\overline{34}}013,{}^{\overline{34}}123} \\ & U_{\overline{34}}(023,{}^{\overline{23}}012;0123)\ U_{\overline{34}}^{-1}(013,123;0123)\ (F^{-1})_{d;0234,{}^{\overline{34}}0123}^{034,{}^{\overline{34}}023,{}^{\overline{34}\cdot\overline{23}}012}\ d_d \end{aligned} \tag{3.4}$$

$$\begin{aligned}\hat{Z}_{\hat{F}}^{-}(\sigma) = \sum_{d,a} & (F^{-1})_{d;0234,a}^{024,234,{}^{\overline{34}\cdot\overline{23}}012}\ \eta_{012}(\overline{34},\overline{23})\ (R_a^{{}^{\overline{24}}012,234})^{-1}\ F_{d;a,0124}^{024,{}^{\overline{24}}012,234} \\ & (F^{-1})_{d;0124,1234}^{014,124,234}\ F_{d;1234,0134}^{014,134,{}^{\overline{34}}123}\ (F^{-1})_{d;0134,{}^{\overline{34}}0123}^{034,{}^{\overline{34}}013,{}^{\overline{34}}123} \\ & U_{\overline{34}}^{-1}(023,{}^{\overline{23}}012;0123)\ U_{\overline{34}}(013,123;0123)\ F_{d;{}^{\overline{34}}0123,0234}^{034,{}^{\overline{34}}023,{}^{\overline{34}\cdot\overline{23}}012}\ d_d \end{aligned} \tag{3.5}$$

3.3 Examples

In this section, we give several examples of the partition function defined in Section 3.1. Some of the known invariants in literature emerge as special cases of our construction.

3.3.1 Crane-Yetter invariant

The Crane-Yetter invariant was introduced in [22], where the authors gave a state sum construction of 4-manifold invariants with the modular tensor category $Rep(U_q(sl_2))$ as input, where $Rep(U_q(sl_2))$ is the category of representations of $U_q(sl_2)$ with q some principal $4r$-th root of unity. Later on the construction was generalized so that it takes any semi-simple ribbon category as input [23] [2]. This generalized state sum invariant is still called the Crane-Yetter invariant denoted by $CY_{\mathcal{C}}(M)$.

For a G-BSFC $\mathcal{D} = \mathcal{C}_G^{\times} = \bigoplus_{g\in G} \mathcal{C}_g$, if $G = \{e\}$ is the trivial group, then $\mathcal{D}$ has only one sector $\mathcal{C} = \mathcal{C}_e$ and it is a ribbon fusion category. In this case, the only color on a 1-simplex is the unit e, so we can just assume there is no color at all on 1-simplices. The colors on each 2-, 3-simplex run through a complete set of representatives in $\mathcal{C}$. For each colored 4-simplex, the partition function is given by Figure 3.3 where all the relevant group elements are trivial, and this is equal to that defined in [23]. It is then trivial to see that the resulting partition function $Z_{\mathcal{C}^{\times}_{\{e\}}}(M)$ is the exactly the Crane-Yetter invariant $CY_{\mathcal{C}}(M)$.

3.3.2 Yetter's invariant from homotopy 2-types

A (strict) categorical group is a rigid tensor category $\mathcal{G}$ such that $(\otimes, \mathbf{1}, (\cdot)^*)$ satisfies the axioms of groups strictly and that every morphism is invertible. Note that here the morphism spaces are not required to be a vector space. Namely, in a categorical

[2] In the paper, such a category was called semi-simple tortile category.

group, all the structural isomorphisms α, l, r are identity maps and $A^* \otimes A = A \otimes A^* = \mathbf{1}$, $f \otimes (f^{-1})^* = I$, $(f^{-1})^* \otimes f = I$ for any object A and morphism f.

There is a one-to-one correspondence between categorical groups and crossed modules which are defined below.

Let G, H be two finite groups. A crossed module is a quad (H, G, ρ, a), where $\rho : H \longrightarrow G$ is a group morphism, and $a : G \times H \longrightarrow H$ is a group action of G on H, such that the following two conditions are satisfied.

1. ρ commutes with the G-action, i.e., for any $g \in G, h \in H$, $\rho(a(g,h)) = {}^g\rho(h)$, where the right hand side is the conjugation action.

2. The action a extends the conjugation action of H on itself, i.e., for any $h, h' \in H$, $a(\rho(h'), h) = {}^{h'}h$.

We will usually write the action $a(g,h) = {}^gh$.

Given a categorical group $\mathcal{G}$, let $G = \mathcal{G}^0$ and $H = \sqcup_{g \in G} \mathrm{Hom}(\mathbf{1}, g)$. It is clear that G is a group with the product $\otimes$, the unit $e = \mathbf{1}$, and the inverse $\overline{(\cdot)} = (\cdot)^*$. Define $\rho : H \longrightarrow G$ by $\rho(h) = t(h)$, and define the G-action by $a(g,h) = I_g \otimes h \otimes I_{\bar{g}}$. It is an exercise to check that H is also a group, and (H, G, ρ, a) is a crossed module, which we denote by $\mathrm{Mod}(\mathcal{G})$.

For the converse direction, given a crossed module (H, G, ρ, a), define the categorical group $\mathrm{Cat}(H, G, \rho, a)$ as follows.

1. Objects are elements of G. For $g_1, g_2 \in G$, $\mathrm{Hom}(g_1, g_2) := \rho^{-1}(\bar{g}_1 g_2) \subset H$. The composition of morphisms are the product in H.

2. For $h \in \mathrm{Hom}(g_1, g_2), h' \in \mathrm{Hom}(g_1', g_2')$, $g_1 \otimes g_1' := g_1 g_1'$, and $h \otimes h' = {}^{\overline{g_1'}}h h'$. The unit $\mathbf{1}$ is the unit in G. The α, l, r's are identity maps.

3. For $g \in G$, the dual $g^* := \bar{g}$, and the b_g, d_g's are identities.

Again, it can be checked that $\mathrm{Cat}(H, G, \rho, a)$ is a categorical group.

Proposition 3.3.1 *If (H, G, ρ, a) is a crossed module and $\mathcal{G}$ is a categorical group, then $Mod(Cat(H, G, \rho, a)) = (H, G, \rho, a))$ and $Cat(Mod(\mathcal{G})) = \mathcal{G}$.*

Proof: Direct verification. ■

From a crossed module (H, G, ρ, a) we can construct a G-BSFC denoted by $\mathcal{D} = \mathcal{D}(H, G, \rho, a) = \bigoplus_{g \in G} \mathcal{C}_g$. As a category, $\mathcal{D}(H, G, \rho, a)$ is the same as $\mathcal{V}ect_H$, the H-graded category. We identify the simple objects in $\mathcal{D}$ with elements of H. For each $g \in G$, the g-sector $\mathcal{C}_g$ is spanned by all simple objects in $\rho^{-1}(g)$. This defines a G-grading on $\mathcal{D}$ due to the fact that ρ is a group morphism. The G-action on simple objects of $\mathcal{D}$ is defined to be the action a on H. It can be checked that this is a well-defined action, and Condition (1) above guarantees that the G-action g sends the g'-sector to the ${}^g g'$-sector.

Let $h \in \mathcal{C}_g, h' \in \mathcal{C}_{g'}$, namely, $\rho(h) = g, \rho(h') = g'$, then $h \otimes h' = hh' \in \mathcal{C}_{gg'}$, and ${}^g h' \otimes h = {}^{\rho(h)} h' \otimes h = {}^h h' \otimes h = hh' \in \mathcal{C}_{gg'}$, where the second equality is due to Condition (2). We define the G-crossed braiding by the identity map, namely,

$$c_{h,h'} := Id : h \otimes h' \longrightarrow {}^g h' \otimes h$$

We now describe the partition function associated with $\mathcal{D}(H, G, \rho, a)$. Let $\hat{F} = (g, f, t)$ be an extended coloring. Then for each 2-simplex (012), $f_{012} \in \mathcal{C}_{\bar{g}_{02} g_{01} g_{12}}$. For any 3-simplex (0123), if the space $V^{\pm}(0123)$ is to be non-zero, we need to have

$$f_{034} {}^{\bar{g}_{34}} f_{012} = f_{013} f_{123} = t_{0123} \tag{3.6}$$

Thus the color on a 3-simplex is uniquely determined by those on its boundary faces. Given an extended coloring such that the condition from Equation 3.6 is satisfied for every 3-simplex, it is direct to see that for each 4-simplex σ, $\hat{Z}^{\pm}(\sigma) = 1$ by Figure 3.3.

Definition 3.3.2 *Given a crossed module (H, G, ρ, a) and an ordered triangulation $\mathcal{T}$ of M, an admissible coloring is a map $F = (g, f)$, $g : \mathcal{T}^1 \longrightarrow G$, $f : \mathcal{T}^2 \longrightarrow H$, such that,*

1. *for any 2-simplex (012), $\rho(f_{012}) = \bar{g}_{02}g_{01}g_{12}$;*
2. *for any 3-simplex (0123), ${f_{034}}^{\bar{g}_{34}}f_{012} = f_{013}f_{123}$.*

An admissible coloring is a $\mathcal{C}(H, G, \rho, a)$-color in the sense of [26]. The following property shows actually the partition function associated with $\mathcal{D}(H, G, \rho, a)$ is exactly the Yetter's invariant $Y_{\mathcal{C}(H,G,\rho,a)}$ associated with $\mathrm{Cat}(H, G, \rho, a)$ in [26].

Proposition 3.3.3 *Let $\mathcal{D} = \mathcal{D}(H, G, \rho, a)$ be the G-BSFC obtained from a crossed module, then*

$$Z_{\mathcal{D}}(M; \mathcal{T}) = \frac{|H|^{|\mathcal{T}^0|-|\mathcal{T}^1|}}{|G|^{|\mathcal{T}^0|}}\#(H, G, \rho, a) = Y_{\mathcal{C}(H,G,\rho,a)}(M), \tag{3.7}$$

where $\#(H, G, \rho, a)$ is the number of admissible colorings.

Proof: In $\mathcal{D}$, it is clear that the quantum dimension of each simple object is 1, and the total dimension square $D^2 = |H|$. Then the first equality follows from Equation 3.1.3. The second equality follows from [26]. ■

If $H = \{e\}$ is the trivial group, there is a unique group morphism ρ_0 from H to G, and a unique action (the trivial action) a_0 of G on H. Then according to Definition 3.3.2, an *admissible* coloring is simply a map $g : \mathcal{T}^1 \longrightarrow G$, such that for each 2-simplex (012), $\bar{g}_{02}g_{01}g_{12} = e$. In this case our partition function is reduced to the untwisted Dijkgraaf-Witten invariant $DW_G(M)$ [20].

Proposition 3.3.4 *Let $\mathcal{D}_0 = \mathcal{D}(\{e\}, G, \rho_0, a_0)$, then*

$$Z_{\mathcal{D}_0}(M; \mathcal{T}) = \frac{1}{|G|^{|\mathcal{T}^0|}}\#(\{e\}, G, \rho_0, a_0) = \frac{|\operatorname{Hom}(\pi_1(M), G)|}{|\operatorname{Hom}(\pi_0(M), G)|} = DW_G(M), \tag{3.8}$$

where $\pi_0(M)$ is the set of connected components of M and $\mathrm{Hom}(\pi_0(M), G)$ *is the set of maps from $\pi_0(M)$ to G.*

Proof: The first equality is by Proposition 3.3.3.

Note that both of the two sides of the second equality is multiplicative with respect to disjoint union of connected components. Thus it suffices to prove the equality for a connected manifold M, namely, $|\pi_0(M)| = 1$.

Choose a maximal spanning tree K, which is a sub complex of $\mathcal{T}^1$ with $|\mathcal{T}^0| - 1$ edges. Then it is easy to see that there is a $|G|^{|\mathcal{T}^0|-1}$ to one correspondence between the set of *admissible* colorings and $\mathrm{Hom}(\pi_1(M), G)$. ■

3.3.3 Trivial G-grading with trivial G-action

Given a ribbon fusion category $\mathcal{C}$ and a finite group G, we can construct a G-BSFC $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$, where $\mathcal{C}_g = \mathcal{C}$ if $g = e$, and $\mathcal{C}_g = 0$ otherwise, and G acts on $\mathcal{D}$ by identity. We consider the partition function associated to $\mathcal{D}$.

Since the nontrivial part of $\mathcal{D}$ is constrained in the trivial sector, the coloring g on 1-simplices needs to satisfy the Dijkgraaf-Witten coloring, namely, $\bar{g}_{02} g_{01} g_{12} = e$ for each 2-simplex (012). The colorings on 2- and 3-simplices are independent of those on 1-simplices, and they are the same as the colorings of Crane-Yetter model. Moreover, the partition function corresponding to each colored 4-simplex is the same as that of Crane-Yetter since the group action here is trivial.

Proposition 3.3.5 *If $\mathcal{D}$ has a trivial G-grading and a trivial G-action where the trivial sector of $\mathcal{D}$ is $\mathcal{C}$, then $Z_{\mathcal{D}}(M; \mathcal{T}) = CY_{\mathcal{C}}(M) DW_G(M)$.*

Proof: By Equation 3.3 and the argument above,

$$
\begin{aligned}
Z_{\mathcal{D}}(M;\mathcal{T}) &= \sum_{\hat{F}=(g,f,t)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod\limits_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod\limits_{\sigma\in\mathcal{T}^4} \hat{Z}_{\hat{F}}^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod\limits_{\tau\in\mathcal{T}^3} d_{t(\tau)})} \\
&= \sum_g \frac{1}{|G|^{|\mathcal{T}^0|}} \sum_{f,t} \frac{(D^2)^{|\mathcal{T}^0|}(\prod\limits_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod\limits_{\sigma\in\mathcal{T}^4} \hat{Z}^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod\limits_{\tau\in\mathcal{T}^3} d_{t(\tau)})} \\
&= \sum_g \frac{1}{|G|^{|\mathcal{T}^0|}} CY_{\mathcal{C}}(M) \\
&= DW_G(M) CY_{\mathcal{C}}(M),
\end{aligned}
$$

where $\hat{Z}^{\epsilon(\sigma)}(\sigma)$ is the partition function of σ for which we have hidden the dependence on the coloring.

■

In Section 3.4, we will give some variations of the construction of the the partition function, in which the twisted Dijkgraaf-Witten invariant will appear as a special case.

3.4 Variation: Trivial G-grading

In this section, we study a variation of the partition function.

Let $\mathcal{C}$ be a ribbon fusion category, and let $\mathcal{D} = \bigoplus\limits_{g\in G} \mathcal{C}_g$ be a G-BSFC with a trivial G-grading, namely, $\mathcal{C}_e = \mathcal{C}$, and $\mathcal{C}_g = 0$ for $g \neq e$. Thus $\mathcal{D}$ is a ribbon fusion category with a G-action. In this case, we show that we can introduce a 4-cocycle $\omega \in H^4(G, U(1))$ in the construction of the partition function, and this results in a new invariant.

Since the G-grading is trivial, the colors g on 1-simplices must satisfy the condition $\bar{g}_{02}g_{01}g_{12} = e$ for each 2-simplex (012). The colors on 2- and 3-simplices run through a complete set of representatives in $\mathcal{C}$. For each fixed coloring, we will introduce an ω

factor to the partition function. Explicitly, using the notations in 3.1, the new invariant can be defined as follows:

Definition 3.4.1 *Given a G-BSFC $\mathcal{D}$ with a trivial G-grading, a 4-cocycle $\omega \in H^4(G, U(1))$, and an ordered triangulation $\mathcal{T}$ of a closed oriented 4-manifold M, the partition function $Z_{\mathcal{D},\omega}(M;\mathcal{T})$ of the pair $(M, \mathcal{T})$ is defined by:*

$$Z_{\mathcal{D},\omega}(M;\mathcal{T}) = \sum_{F=(g,f)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod_{\beta\in\mathcal{T}^2} d_{f(\beta)}) \operatorname{Tr}(\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}} \omega(g_{01}, g_{12}, g_{23}, g_{34}), \quad (3.9)$$

or written in state sum model by:

$$Z_{\mathcal{D},\omega}(M;\mathcal{T}) = \sum_{\hat{F}=(g,f,t)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod_{\sigma\in\mathcal{T}^4} \hat{Z}_{\hat{F}}^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod_{\tau\in\mathcal{T}^3} d_{t(\tau)})} \omega(g_{01}, g_{12}, g_{23}, g_{34}). \quad (3.10)$$

The invariance proof of $Z_{\mathcal{D},\omega}(M;\mathcal{T})$ can be processed in a similar way as that of $Z_{\mathcal{D}}(M;\mathcal{T})$ with slight modifications, so we will omit the details. For instance, to show it is invariant under the 3-3 pachner move, Equation 4.26 is replaced by Equation 3.11:

$$\begin{aligned}
&\sum_{\substack{I_2,\\ I_3}} \{ \frac{d_{024}}{d_{0124}d_{0234}d_{0245}} \hat{Z}^+(01234)\hat{Z}^+(01245)\hat{Z}^+(02345) \\
&\quad \cdot \omega(01,12,23,34)\omega(01,12,24,45)\omega(02,23,34,45)\} \\
=&\sum_{\substack{I_2',\\ I_3'}} \{ \frac{d_{135}}{d_{0135}d_{1235}d_{1345}} \hat{Z}^+(01235)\hat{Z}^+(01345)\hat{Z}^+(12345) \\
&\quad \cdot \omega(01,12,23,35)\omega(01,13,34,45)\omega(12,23,34,45)\} \qquad (3.11)
\end{aligned}$$

In Equation 3.11, the product of the three ω factors on the left hand side is equal to the product of the three factors on the right hand side, and this is precisely because ω is a 4-cocycle and the colors on 1-simplices satisfy $\bar{g}_{02}g_{01}g_{12}$ for each 2-simplex (012). Cancelling the ω factors on both sides, we get back to Equation 4.26.

Proposition 3.4.2 *If $\mathcal{D}$ is a G-BSFC with a trivial G-grading and a trivial G-action, and $\omega \in H^4(G, U(1))$, then*

$$Z_{\mathcal{D},\omega}(M;\mathcal{T}) = DW_G^{\omega}(M)CY_{\mathcal{C}}(M), \tag{3.12}$$

where $DW_G^{\omega}(M)$ is the twisted Dijkgraaf-Witten invariant [20]. In particular, If $\mathcal{D} = \mathcal{V}ect$, then $Z_{\mathcal{D},\omega} = DW_G^{\omega}(M)$.

Proof: A similar argument as that in Section 3.3.3 shows that $Z_{\mathcal{D},\omega}$ can be written as follows:

$$\begin{aligned}
Z_{\mathcal{D},\omega}(M;\mathcal{T}) &= \sum_{\hat{F}=(g,f,t)} \frac{(D^2/|G|)^{|\mathcal{T}^0|}(\prod_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod_{\sigma\in\mathcal{T}^4} \hat{Z}_{\hat{F}}^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod_{\tau\in\mathcal{T}^3} d_{t(\tau)})}\omega(g_{01}, g_{12}, g_{23}, g_{34}) \\
&= \sum_{g} \frac{1}{|G|^{|\mathcal{T}^0|}}\omega(g_{01}, g_{12}, g_{23}, g_{34}) \sum_{f,t} \frac{(D^2)^{|\mathcal{T}^0|}(\prod_{\alpha\in\mathcal{T}^2} d_{f(\alpha)})(\prod_{\sigma\in\mathcal{T}^4} \hat{Z}^{\epsilon(\sigma)}(\sigma))}{(D^2)^{|\mathcal{T}^1|}(\prod_{\tau\in\mathcal{T}^3} d_{t(\tau)})} \\
&= \sum_{g} \frac{1}{|G|^{|\mathcal{T}^0|}}\omega(g_{01}, g_{12}, g_{23}, g_{34})CY_{\mathcal{C}}(M) \\
&= DW_G^{\omega}(M)CY_{\mathcal{C}}(M).
\end{aligned}$$

■

Chapter 4

Proof of Invariance

In this chapter we will prove in turn that $Z_{\mathcal{D}}(M;\mathcal{T})$ is independent on the choice of representatives for each triangle color (Section 4.1), the ordering of vertices of a triangulation (Section 4.2), and the choice of a triangulation 4.4.

4.1 Invariance under Choice of Representatives

To define the partition function $Z_{\mathcal{D}}(M;\mathcal{T})$, for each coloring $F=(g,f)$, we chose arbitrarily a representative f_β for the triangle color $f(\beta)$, $\beta \in \mathcal{T}^2$. In the following, we show that $\mathrm{Tr}(\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma))$ is independent on the choice of representatives, and thus $Z_{\mathcal{D}}(M;\mathcal{T})$ is also independent on the choice of representatives by Definition 3.1.3.

Let $\phi_A : A \xrightarrow{\sim} A'$, $\phi_B : B \xrightarrow{\sim} B'$ be isomorphisms in $\mathcal{D}$. Denote by $T_{\phi_A}^{\phi_B}$ the following linear isomorphism:

$$\begin{aligned} T_{\phi_A}^{\phi_B} : \mathrm{Hom}(A,B) &\xrightarrow{\sim} \mathrm{Hom}(A',B') \\ \phi_A^B &\longmapsto \phi_B \circ \phi_A^B \circ \phi_A. \end{aligned}$$

That is, the diagram in Diagram 4.1 commutes.

$$\begin{array}{ccc} B & \xrightarrow{\phi_B} & B' \\ {\scriptstyle \phi_A^B}\uparrow & & \uparrow {\scriptstyle T_{\phi_A}^{\phi_B}(\phi_A^B)} \\ A & \xrightarrow{\phi_A} & A' \end{array} \tag{4.1}$$

The isomorphism $T_{\phi_A}^{\phi_B}$ satisfies functorial properties, namely,

1. If $\phi_C : C \xrightarrow{\sim} C'$, then $T_{\phi_B}^{\phi_C}(\phi_B^C) \circ T_{\phi_A}^{\phi_B}(\phi_A^B) = T_{\phi_A}^{\phi_C}(\phi_B^C \circ \phi_A^B)$.

2. $T_{\phi_A}^{\phi_A}(Id_A) = Id_{A'}$.

Lemma 4.1.1 *The isomorphism $T_{\phi_A}^{\phi_B}$ preserves the pairing, i.e.,*

$$\langle \phi_B^A, \phi_A^B \rangle = \langle T_{\phi_B}^{\phi_A}(\phi_B^A), T_{\phi_A}^{\phi_B}(\phi_A^B) \rangle.$$

For the rest of the section, we fix a coloring $F = (g, f)$. For each 2-simplex $(ijk) \in \mathcal{T}^2$, assume we have arbitrarily chosen two representatives of $f(ijk)$ denoted by ijk and ijk', respectively. To distinguish these two choices, we attach an apostrophe to all quantities related to the second the choice, e.g., $V_F^+(0123)'$, $Z_F^-(01234)'$, etc.

For each $\beta = (ijk) \in \mathcal{T}^2$, choose any isomorphism $\phi_{ijk} : ijk \longrightarrow ijk'$. If $\tau = (0123) \in \mathcal{T}^3$, let $\phi_{-\tau}$, $\phi_{+\tau}$ be the isomorphisms defined below.

$$\phi_{-\tau} := \phi_{023} \otimes {}^{\overline{23}}(\phi_{012}) : 023 \otimes {}^{\overline{23}}012 \xrightarrow{\sim} 023' \otimes {}^{\overline{23}}012'$$

$$\phi_{+\tau} := \phi_{013} \otimes \phi_{123} : 013 \otimes 123 \xrightarrow{\sim} 013' \otimes 123'$$

Then we have the isomorphisms $T_{\phi_{-\tau}}^{\phi_{+\tau}} : V_F^+(\tau) \longrightarrow V_F^+(\tau)'$, $T_{\phi_{+\tau}}^{\phi_{-\tau}} : V_F^-(\tau) \longrightarrow V_F^-(\tau)'$. See Diagram 4.2. When it is clear from the context, we will drop the subscript/superscript, and simply write $T^{\phi_{+\tau}}_{\phi_{-\tau}}$ as T.

$$\begin{tikzcd} 013 \otimes 123 \arrow[r, "\phi_{+\tau}"] & 013' \otimes 123' \\ 023 \otimes {}^{\overline{23}}012 \arrow[u, "\phi"] \arrow[r, "\phi_{-\tau}"] & 023' \otimes {}^{\overline{23}}012' \arrow[u, "T^{\phi_{+\tau}}_{\phi_{-\tau}}(\phi)"'] \end{tikzcd} \tag{4.2}$$

Lemma 4.1.2 *For any 4-simplex $\sigma = (01234) \in \mathcal{T}^4$, Diagram 4.3 and 4.4 both commute.*

$$\begin{tikzcd} V_F^+(0234) \otimes V_F^+(0124) \otimes V_F^-(1234) \otimes V_F^-(0134) \otimes V_F^-(0123) \arrow[r, "\tilde{Z}_F^+(01234)"] \arrow[d, "T\otimes T\otimes T\otimes T\otimes T"] & \mathbb{C} \\ V_F^+(0234)' \otimes V_F^+(0124)' \otimes V_F^-(1234)' \otimes V_F^-(0134)' \otimes V_F^-(0123)' \arrow[ur, "\tilde{Z}_F^+(01234)"'] & \end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd} V_F^-(0234) \otimes V_F^-(0124) \otimes V_F^+(1234) \otimes V_F^+(0134) \otimes V_F^+(0123) \arrow[r, "\tilde{Z}_F^-(01234)"] \arrow[d, "T\otimes T\otimes T\otimes T\otimes T"] & \mathbb{C} \\ V_F^-(0234)' \otimes V_F^-(0124)' \otimes V_F^+(1234)' \otimes V_F^+(0134)' \otimes V_F^+(0123)' \arrow[ur, "\tilde{Z}_F^-(01234)'"'] & \end{tikzcd} \tag{4.4}$$

Proof: We only prove the case for Diagram 4.3. The other case can be proved in the same way. Let $\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$ be in the domain of $\tilde{Z}_F^+(01234)$. Consider the following diagram, where it is not hard to see that the the two maps on the sides of the horizontal arrows represent identical maps.

$$
\begin{tikzcd}
034 \otimes {}^{\overline{34}}023 \otimes {}^{\overline{34}.\overline{23}}012 \arrow[r, "\phi_{-0234}\otimes{}^{\overline{34}.\overline{23}}(\phi_{012})", "\phi_{034}\otimes{}^{\overline{34}}(\phi_{-0123})"'] & 034' \otimes {}^{\overline{34}}023' \otimes {}^{\overline{34}.\overline{23}}012' \\
034 \otimes {}^{\overline{34}}013 \otimes {}^{\overline{34}}123 \arrow[u, "Id\otimes{}^{\overline{34}}\phi_4"] \arrow[r, "\phi_{034}\otimes{}^{\overline{34}}(\phi_{+0123})", "\phi_{-0134}\otimes{}^{\overline{34}}(\phi_{123})"'] & 034' \otimes {}^{\overline{34}}013' \otimes {}^{\overline{34}}123' \arrow[u, "Id\otimes{}^{\overline{34}}T(\phi_4)"'] \\
014 \otimes 134 \otimes {}^{\overline{34}}123 \arrow[u, "\phi_3\otimes Id"] \arrow[r, "\phi_{+0134}\otimes{}^{\overline{34}}(\phi_{123})", "\phi_{014}\otimes\phi_{-1234}"'] & 014' \otimes 134' \otimes {}^{\overline{34}}123' \arrow[u, "T(\phi_3)\otimes Id"'] \\
014 \otimes 124 \otimes 234 \arrow[u, "Id\otimes\phi_2"] \arrow[r, "\phi_{014}\otimes\phi_{+1234}", "\phi_{+0124}\otimes\phi_{234}"'] & 014' \otimes 124' \otimes 234' \arrow[u, "Id\otimes T(\phi_2)"'] \\
024 \otimes {}^{\overline{24}}012 \otimes 234 \arrow[u, "\phi_1\otimes Id"] \arrow[r, "\phi_{-0124}\otimes\phi_{234}", "\phi_{024}\otimes{}^{\overline{24}.23.34}({}^{\overline{34}.\overline{23}}(\phi_{012}))\otimes\phi_{234}"'] & 024' \otimes {}^{\overline{24}}012' \otimes 234' \arrow[u, "T(\phi_1)\otimes Id"'] \\
024 \otimes 234 \otimes {}^{\overline{34}.\overline{23}}012 \arrow[u, "Id\otimes c_{234,{}^{\overline{34}.\overline{23}}012}"] \arrow[r, "\phi_{024}\otimes\phi_{234}\otimes{}^{\overline{34}.\overline{23}}(\phi_{012})", "\phi_{+0234}\otimes{}^{\overline{34}.\overline{23}}(\phi_{012})"'] & 024' \otimes 234' \otimes {}^{\overline{34}.\overline{23}}012' \arrow[u, "Id\otimes c_{234',{}^{\overline{34}.\overline{23}}012'}"'] \\
034 \otimes {}^{\overline{34}}023 \otimes {}^{\overline{34}.\overline{23}}012 \arrow[u, "\phi_0\otimes Id"] \arrow[r, "\phi_{-0234}\otimes{}^{\overline{34}.\overline{23}}(\phi_{012})"'] & 034' \otimes {}^{\overline{34}}023' \otimes {}^{\overline{34}.\overline{23}}012' \arrow[u, "T(\phi_0)\otimes Id"']
\end{tikzcd}
\tag{4.5}
$$

The second square diagram commutes by the naturality of the G-crossed braiding. All other square diagrams commute by Diagram 4.2. Thus, the whole diagram commutes.

Denote the composition of the vertical maps on the left (resp. right) side by L (resp. R), then $\tilde{Z}_F^+(01234)(\phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4) = \mathrm{Tr}(L)$, $\tilde{Z}_F^+(01234)'(T(\phi_0) \otimes T(\phi_1) \otimes T(\phi_2) \otimes T(\phi_3) \otimes T(\phi_4)) = \mathrm{Tr}(R)$. Since L, R are conjugate by the above diagram, thus $\mathrm{Tr}(L) = \mathrm{Tr}(R)$ and we have $\tilde{Z}_F^+(01234) = \tilde{Z}_F^+(01234)'(T^{\otimes 5})$. ■

Proposition 4.1.3 *For any $\sigma = (01234) \in \mathcal{T}^4$, $Z_F^{\epsilon(\sigma)}(\sigma)$ commutes with T. More pre-*

cisely, the following diagrams commute.

$$
\begin{tikzcd}
V_F^+(0234)\otimes V_F^+(0124) \arrow[r, "Z_F^+(01234)"] \arrow[d, "T\otimes T"] & V_F^+(1234)\otimes V_F^+(0134)\otimes V_F^+(0123) \arrow[d, "T\otimes T\otimes T"] \\
V_F^+(0234)'\otimes V_F^+(0124)' \arrow[r, "Z_F^+(01234)'"'] & V_F^+(1234)'\otimes V_F^+(0134)'\otimes V_F^+(0123)'
\end{tikzcd}
\tag{4.6}
$$

$$
\begin{tikzcd}
V_F^+(0234)\otimes V_F^+(0124) \arrow[d, "T\otimes T"] & V_F^+(1234)\otimes V_F^+(0134)\otimes V_F^+(0123) \arrow[l, "Z_F^-(01234)"'] \arrow[d, "T\otimes T\otimes T"] \\
V_F^+(0234)'\otimes V_F^+(0124)' & V_F^+(1234)'\otimes V_F^+(0134)'\otimes V_F^+(0123)' \arrow[l, "Z_F^-(01234)'"]
\end{tikzcd}
\tag{4.7}
$$

Proof: Again we only prove the case of Diagram 4.6.

Let $\phi_0\otimes\phi_1 \in V_F^+(0234)\otimes V_F^+(0124)$, $\phi_2\otimes\phi_3\otimes\phi_4 \in V_F^+(1234)\otimes V_F^+(0134)\otimes V_F^+(0123)$, *then*

$$
\begin{aligned}
&\langle (T\otimes T\otimes T)\circ Z_F^+(01234)(\phi_0\otimes\phi_1), (T\otimes T\otimes T)(\phi_2\otimes\phi_3\otimes\phi_4)\rangle \\
=&\langle Z_F^+(01234)(\phi_0\otimes\phi_1), \phi_2\otimes\phi_3\otimes\phi_4\rangle \\
=&\tilde{Z}_F^+(01234)(\phi_0\otimes\phi_1\otimes\phi_2\otimes\phi_3\otimes\phi_4)
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&\langle Z_F^+(01234)'\circ (T\otimes T)(\phi_0\otimes\phi_1), (T\otimes T\otimes T)(\phi_2\otimes\phi_3\otimes\phi_4)\rangle \\
=&\tilde{Z}_F^+(01234)'(T(\phi_0)\otimes T(\phi_1)\otimes T(\phi_2)\otimes T(\phi_3)\otimes T(\phi_4)) \\
=&\tilde{Z}_F^+(01234)'\circ (T\otimes T\otimes T\otimes T\otimes T)(\phi_0\otimes\phi_1\otimes\phi_2\otimes\phi_3\otimes\phi_4) \\
=&\tilde{Z}_F^+(01234)(\phi_0\otimes\phi_1\otimes\phi_2\otimes\phi_3\otimes\phi_4)
\end{aligned}
\tag{4.9}
$$

The first "= " in Equation 4.8 is by Lemma 4.1.1, and the third "= " in Equation 4.9 is by Lemma 4.1.2.

Since the equalities in Equation 4.8 and 4.9 hold for any $\phi_i, i = 0,1,2,3,4.$ *this* *implies Diagram 4.6 commutes.* ■

Theorem 4.1.4 *Given a coloring $F = (g, f)$, we have*

$$\mathrm{Tr}(\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma)) = \mathrm{Tr}(\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma)').$$

As a consequence, the partition function $Z_{\mathcal{D}}(M;\mathcal{T})$ is independent on the choice of representatives for each triangle color.

Proof: It suffices to prove Diagram 4.10 commutes, which follows from Proposition 4.1.3 since each $V_F^+(\tau)$ is acted on by exactly one $Z_F^{\epsilon(\sigma)}(\sigma)$.

$$\begin{tikzcd} \bigotimes_{\tau\in\mathcal{T}^3} V_F^+(\tau) \arrow[r, "\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma)"] \arrow[d, "\bigotimes_{\tau\in\mathcal{T}^3} T"] & \bigoplus_{\tau\in\mathcal{T}^3} V_F^+(\tau) \arrow[d, "\bigotimes_{\tau\in\mathcal{T}^3} T"] \\ \bigotimes_{\tau\in\mathcal{T}^3} V_F^+(\tau)' \arrow[r, "\bigotimes_{\sigma\in\mathcal{T}^4} Z_F^{\epsilon(\sigma)}(\sigma)'"'] & \bigotimes_{\tau\in\mathcal{T}^3} V_F^+(\tau)' \end{tikzcd} \tag{4.10}$$

■

4.2 Invariance under change of ordering

When defining the partition function, we require that the triangulation be equipped with an ordering on its vertices. In this section, we show that the partition function is actually independent of the ordering, and we will use Equation 3.2 as the definition of the partition function.

Let S_n be the permutation group on the set $\{0, 1, \cdots, n-1\}$ with the standard generators $s_i \in \mathrm{S}_n, i = 0, 1, \cdots, n-2$ swapping i and $i+1$. Let F_n be the free group on n generators $\tilde{s}_0, \cdots, \tilde{s}_{n-1}$, and denote by $p_n : F_{n-1} \longrightarrow \mathrm{S}_n$ the quotient map sending $\tilde{s}_i$

to s_i. If $\tilde{s}$ and $\tilde{s}'$ are any two elements in F_{n-1} such that $p_n(\tilde{s}) = p_n(\tilde{s}') = s \in \mathrm{S}_n$, then we say $\tilde{s}$ and $\tilde{s}'$ represent the same permutation s and $\tilde{s}$ is a representative of s.

For each $s \in \mathrm{S}_n$ and any non-empty subset $T \subset \{0, 1, \cdots, n-1\}$, if we order the elements in T and in $s(T)$, respectively, by $0, 1, \cdots, |T|-1$ according to their relative order, then we get a permutation s_T, called the restriction of s on T, on $\{0, 1, \cdots, |T|-1\}$, namely, $s_T(i) = j$ if s maps the i-th greatest element in T to the j-th greatest element in $s(T)$. When no confusion arises, we simply write s_T as s. For example, if $s = s_0 \in \mathrm{S}_3$, then $s_{\{0,1\}} = s_0 \in \mathrm{S}_2$, $s_{\{0,2\}} = Id \in \mathrm{S}_2$. It is clear that s_T is a group morphism in the sense that $s'_{s(T)} s_T = (s's)_T$.

The rough idea of proof of invariance under reordering goes as follows. Let $\mathcal{T}$ be a fixed ordered triangulation with vertices ordered by $0, 1, \cdots, N-1$, where $N = |\mathcal{T}^0|$. Let $\mathcal{T}'$ be any ordered triangulation obtained from $\mathcal{T}$ by reordering its vertices. Apparently, each reordering of vertices corresponds to an element of S_N. To make it clear, if $s \in \mathrm{S}_N$, then we obtain $\mathcal{T}'$ from $\mathcal{T}$ by replacing the label i of a vertex by $s(i)$. See Figure 4.1 for the reordering of a 2-simplex. **That is, we are thinking that the 'physical' unordered triangulation is fixed all the time; only the labels on vertices are permuting.** For a k-simplex $\sigma = (a_0 a_1 \cdots a_{k-1}) \in \mathcal{T}^k$, denote by σ_s the corresponding k-simplex $(\sigma(a_0)\sigma(a_1) \cdots \sigma(a_{k-1})) \in (\mathcal{T}')^k$. Note that σ and σ_s are the same as an unordered simplex. Given a coloring $F = (g, f)$ on $\mathcal{T}$, we wish to define a corresponding coloring $F' = (g', f')$ on $\mathcal{T}'$ so that,

1. for each 2-simplex $\beta \in \mathcal{T}^2$, $d_\beta = d_{\beta_s}$;

2. for each 4-simplex σ, there exists an isomorphism $\Phi^s_\sigma : V_F(\partial\sigma) \longrightarrow V_{F'}(\partial\sigma_s)$ such that $\tilde{Z}_F(\sigma) = \tilde{Z}_{F'}(\sigma_s) \circ \Phi^s_\sigma$;

3. $\Phi := \bigotimes_{\sigma \in \mathcal{T}^4} \Phi^s_\sigma : V_F(\mathcal{T}) \longrightarrow V_{F'}(\mathcal{T}')$ maps ϕ_F to $\phi_{F'}$.

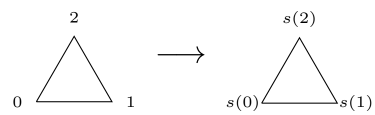
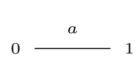
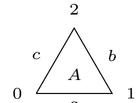


Figure 4.1: Reordering by a permutation s

If the above conditions are satisfied, then it is clear that $Z_{\mathcal{D}}(M;\mathcal{T}) = Z_{\mathcal{D}}(M;\mathcal{T}')$. The first condition is satisfied trivially as will be seen later. The Φ^s_σ clearly depends on the reordering restricted to the 4-simplex σ. It is natural to try to construct $\Phi^{s_i}_\sigma$ for each reordering $s_i \in \mathrm{S}_5$ and assume $\Phi^{ss'}_\sigma = \Phi^s_{\sigma_{s'}}\Phi^{s'}_\sigma$. Namely, we wish Φ^s_σ acts like a group action. Unfortunately, this turns out to be false. For instance, $\Phi^{s_i}_{\sigma_{s_i}}\Phi^{s_i}_\sigma$ is not equal to the identity; it is, however, equal to the identity up to certain canonical isomorphisms. Thus in the following, we will actually first define an action $\Phi^{\tilde{s}}_\sigma$ for $\tilde{s} \in F_4$ and show that that if $\tilde{s}$ and $\tilde{s}'$ represent the same permutation in S_5, then there is a canonical way to identity them. To define the action $\Phi^{\tilde{s}}_\sigma$, we will need to consider in turn actions of F_i on colors of i-simplices for $i = 1, 2, 3, 4$.

First we fix some notations. In general letters s, s' are used to represent elements of S_n, and $\tilde{s}, \tilde{s}'$ elements of F_{n-1}. We will still think of the generators $\tilde{s}_i$ of F_{n-1} as swapping i with $i+1$, although this is actually the role of $p_n(\tilde{s}_i)$. By a colored k-simplex, $k = 1, 2, 3, 4$, we mean an ordered k-simplex with a coloring on its sub 1-simplices and sub 2-simplices (if any). Let $\mathcal{S}^k$ be the set of all colored k-simplices, where two k-simplices are identified up to relative ordering of their vertices. For instance,

$\mathcal{S}^1 = \{$  $: \ a \in G\}$

$\mathcal{S}^2 = \{$  $: a, b, c \in G, A$ is simple in $\mathcal{C}_{\bar{c}ab}\}$

.

For $k = 2, 3, 4$, associated to each colored k-simplex σ is an array $\sigma[0], \cdots, \sigma[k]$, where $\sigma[i]$ is the colored $(k-1)$-simplex corresponding to the face $\partial_i\sigma$. We combine the indices if there are multiple of them, e.g., $\sigma[i][j] = \sigma[i,j]$. We use $C(\sigma)$ to denote the group element color on σ if $k=1$, and the object color if $k=2$. For $k=3,4$, a colored k-simplex σ is uniquely determined by the arrays $\sigma[i]$'s. For $k=1,2$, a colored k-simplex σ is uniquely determined by the arrays $\sigma[i]$'s and their colors.

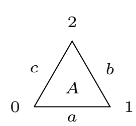
In the following, we consider in order the effect of the reordering of a k-simplex on its colorings for $k=1,2,3,4$. This involves studying the action of F_k (resp. S_{k+1}) on $\mathcal{S}^k$.

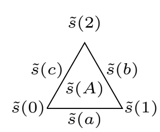
Firstly, define the action of F_1 on $\mathcal{S}^1$ by

$$\tilde{s}_0\left({}_{0}\overset{a}{\text{------}}{}_{1}\right) = {}_{1}\overset{\bar{a}}{\text{------}}{}_{0}\,.$$

This action factors through S_2 since $\tilde{s}_0^2$ acts as the identity. Thus when the vertices of a colored 1-simplex are reordered by s_0, the color on the 1-simplex is changed to its inverse. By abusing of notations, we also write $s_0(a) = \bar{a}, a \in G$.

Secondly, consider the action of F_2 on $\mathcal{S}^2$. For each $\tilde{s} \in F_2$, define

$\tilde{s}($ 

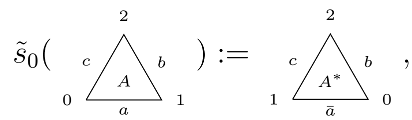
$) :=$ 

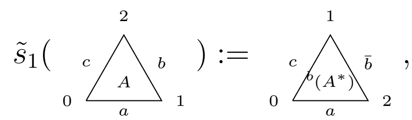
,

where $\tilde{s}(i) := p_3(\tilde{s})(i), i = 0,1,2,$ is the reordering, $\tilde{s}(a)$ is the action of $p(\tilde{s})_{\{0,1\}}$ on ${}_{0}\overset{a}{\text{------}}{}_{1}$, and $\tilde{s}(b), \tilde{s}(c)$ are defined analogously, and

$$\tilde{s}(A) = \begin{cases} A^* & \tilde{s} = \tilde{s}_0 \\ {}^{b}(A^*) & \tilde{s} = \tilde{s}_1 \end{cases}$$

Explicitly, the generators act as





or written in another way, $\tilde{s}_0(b,c,a;A) = (c,b,\bar{a};A^*)$, $\tilde{s}_1(b,c,a;A) = (\bar{b},a,c;{}^b(A^*))$, where we identify a colored 2-simplex σ with the quad $(C(\sigma[0]),C(\sigma[1]),C(\sigma[2]);C[\sigma])$.

It is easy to check that $\tilde{s}_i$ sends colored 2-simplices to colored 2-simplices. The action of F_2 does not in general factor through S_3 since, for instance, $\tilde{s}_1^2(b,c,a;A) = (b,c,a;{}^{\bar{b}}(({}^bA^*)^*))$, but ${}^{\bar{b}}(({}^bA^*)^*)$ is not always equal to A. However, since ${}^{\bar{b}}(({}^bA^*)^*)$ is canonically isomorphic to A, this does give us some hints on relating the action of $\tilde{s}$ and $\tilde{s}'$ which represent the same permutation.

Let us recall some structures in $\mathcal{D}$.

$$\delta_A : A \xrightarrow{\sim} A^{**}$$

$$\theta_A : A \xrightarrow{\sim} {}^{g'}A \qquad\qquad A \in \mathcal{C}_{g'}$$

$$\phi_{A,g} : {}^g(A^*) \xrightarrow{\sim} ({}^gA)^* \qquad\qquad \forall g \in G$$

Definition 4.2.1 *1. If A, B are two objects of $\mathcal{D}$, a morphism $f \in \mathrm{Hom}(A,B)$ is called a canonical isomorphism if it is a composition of morphisms which are tensor products of δ, θ, ϕ, I and their inverses.*

2. A linear map $\Phi : \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A',B')$ is called a canonical isomorphism if $\Phi(\psi) = g \circ \psi \circ f$ for some canonical isomorphisms $f \in \mathrm{Hom}(A',A), g \in \mathrm{Hom}(B,B')$.

Canonical isomorphisms will be denoted by *can*. It is clear that composition, tensor products, inverse, dual, G-action of canonical isomorphisms are still canonical isomorphisms. And a canonical isomorphism from A to B is unique, if it exists.

It is straight forward to check the following equalities.

$$\tilde{s}_0^2(b, c, a; A) = (b, c, a; A^{**}) \tag{4.11}$$

$$\tilde{s}_1^2(b, c, a; A) = (b, c, a; {}^{\bar{b}}(({}^{b}A^*)^*)) \tag{4.12}$$

$$\tilde{s}_0\tilde{s}_1\tilde{s}_0(b, c, a; A) = (\bar{a}, \bar{c}, \bar{b}; ({}^{c}(A^{**}))^*) \tag{4.13}$$

$$\tilde{s}_1\tilde{s}_0\tilde{s}_1(b, c, a; A) = (\bar{a}, \bar{c}, \bar{b}; {}^{a}(({}^{b}A^*)^{**})) \tag{4.14}$$

Lemma 4.2.2 *Let $\tilde{s}, \tilde{s}' \in F_2$ be such that $p_3(\tilde{s}) = p_3(\tilde{s}') = s \in S_3$, then for any colored 2-simplex σ,*

1. *$\tilde{s}(\sigma)[s(i)] = s_{\hat{i}}(\sigma[i])$ for $i = 0, 1, 2$, where $s_{\hat{i}}$ is the restriction of s on the edge $\hat{i}$. Namely, if $\tilde{s}$ is a reordering of σ, then the effect of $\tilde{s}$ on any sub 1-simplex is equal to the action of the restriction of s on that simplex.*

2. *There is a canonical isomorphism from $C(\tilde{s}(\sigma))$ to $C(\tilde{s}'(\sigma))$.*

Proof: It can be checked that if the first part of the statement is true for $\tilde{s}$ and $\tilde{s}''$, then it is true for their product $\tilde{s}\tilde{s}''$. Thus it suffices to verify it on the generators $\tilde{s}_0$ and $\tilde{s}_1$, which is straight forward.

To prove the second part, note that by the identities in Equations 4.11 − 4.14, there is a canonical isomorphism $C(\sigma) \longrightarrow C(\tilde{s}_0^2(\sigma))$, $C(\sigma) \longrightarrow C(\tilde{s}_1^2(\sigma))$, $C(\tilde{s}_0\tilde{s}_1\tilde{s}_0(\sigma)) \longrightarrow C(\tilde{s}_1\tilde{s}_0\tilde{s}_1(\sigma))$. The general case follows from the naturality of these isomorphisms. ∎

Lemma 4.2.2 can be applied to the following situation. Assume σ' and σ'' are two colored 2-simplices in $\mathcal{T}$ with a common colored 1-simplex α. Let $s \in \mathrm{S}_4$ be an reordering of $\sigma \cup \sigma'$, and $\tilde{s}'$, $\tilde{s}''$ be a representative of $s_{\sigma'}$, $s_{\sigma''}$, respectively. Then the effect of $\tilde{s}'$ and $\tilde{s}''$ on α coincides, both equal to the action of s_α.

Also by Lemma 4.2.2, if $\tilde{s}$ and $\tilde{s}'$ represent the same element in S_3 and σ is a colored 2-simplex, then the effect of $\tilde{s}$ and $\tilde{s}'$ on $C(\sigma)$ is related by a canonical isomorphism, namely, there is a canonical isomorphism *can* from $C(\tilde{s}(\sigma))$ to $C(\tilde{s}'(\sigma))$.

Thirdly, we consider the reordering of 3-simplices. Again for the same reason as before, we will need to study the action of F_3 on colored 3-simplices. If $s \in \mathrm{S}_4$ is a reordering of a colored 3-simplex σ and $\tilde{s}$ is a representative of s, it is natural to require the effect of $\tilde{s}$ on a sub 2-simplex α to be the action of a representative of s_α. There is no canonical choice of representatives. We start with generators $\tilde{s}_i$ and choose the shortest word as the representative of $(s_i)_\alpha$. Explicitly, $\tilde{s}_i(\sigma)$ is defined as in Equation 4.15, where the $\tilde{s}_i$'s on the right hand side are considered as elements of F_2 and we identify σ with the quad $(\sigma[0], \sigma[1], \sigma[2], \sigma[3])$.

$$\tilde{s}_i(\sigma) = \tilde{s}_i(\sigma[0], \sigma[1], \sigma[2], \sigma[3]) = \begin{cases} (\sigma[1], \sigma[0], \tilde{s}_0(\sigma[2]), \tilde{s}_0(\sigma[3])) & i = 0 \\ (\tilde{s}_0(\sigma[0]), \sigma[2], \sigma[1], \tilde{s}_1(\sigma[3])) & i = 1 \\ (\tilde{s}_1(\sigma[0]), \tilde{s}_1(\sigma[1]), \sigma[3], \sigma[2]) & i = 2 \end{cases} \tag{4.15}$$

The following lemma shows the action defined indeed satisfies the natural requirement mentioned above.

Lemma 4.2.3 *Let $\tilde{s} \in F_3, s \in S_4$ be such that $p_4(\tilde{s}) = s$, and let σ be any colored 3-simplex, then*

$$\tilde{s}(\sigma)[s(i)] = L_{\tilde{s}}^{s(i)}(\sigma[i]),\ i = 0, 1, 2, 3, \tag{4.16}$$

where $L^i_{(-)} : F_3 \longrightarrow F_2$ is a map such that $p_3(L_{\tilde{s}}^{s(i)}) = s_{\hat{i}}$. In particular, if $p_4(\tilde{s}) = p_4(\tilde{s}')$, then $p_3(L^i_{\tilde{s}}) = p_3(L^i_{\tilde{s}'})$.

Proof: Again the statement holds for the generators, and it holds for the product of two elements if it does for each of them. ■

Lemma 4.2.3 has the following consequence. Assume a 2-simplex β is the intersection of two 3-simplices σ' and σ'', $\beta = \sigma'[i] = \sigma''[j]$, and $s \in \mathrm{S}_5$ is a reordering of $\sigma' \cup \sigma''$. Restricting to each 3-simplex, we get an reordering s' on σ' and s'' on σ''. Choose an arbitrary representative $\tilde{s}'$ and $\tilde{s}''$ for s' and s'', respectively. Then $L_{\tilde{s}'}^{s'(i)}$ and $L_{\tilde{s}''}^{s''(j)}$ represent the same permutation, namely, s_β, and thus there is a canonical isomorphism from $\tilde{s}'(\sigma')[s'(i)]$ to $\tilde{s}''(\sigma'')[s''(j)]$.

Recall that for a colored 3-simplex σ, there are associated Hilbert spaces $V^{\pm}(\sigma)$, where we have dropped the subscript symbol F since σ is assumed to be colored. Specifically,

$$V^{+}(\sigma) = \mathrm{Hom}(C(\sigma[1]) \otimes {}^{\overline{C(\sigma[0,0])}}C(\sigma[3]), C(\sigma[2]) \otimes C(\sigma[0]))$$

$$V^{-}(\sigma) = \mathrm{Hom}(C(\sigma[2]) \otimes C(\sigma[0]), C(\sigma[1]) \otimes {}^{\overline{C(\sigma[0,0])}}C(\sigma[3]))$$

If σ' and σ are two colored 3-simplices which differ by a reordering, we show below that their associated Hilbert spaces are related by canonical isomorphisms.

For each $\tilde{s}_i \in F_3$, we will define $V(\tilde{s}_i)$ as a linear isomorphism from $V^{\pm}(\sigma)$ from $V^{\mp}(s_i(\sigma))$, and extend the action to $V(\tilde{s})$ for any $\tilde{s} \in F_3$. Assume $\sigma = (0123)$ and denote the colors of sub 1- and 2- simplices of σ by the simplices themselves, e.g., $12, 23, 123$, etc. Then by Equation 4.15, one can write out $\tilde{s}(\sigma)$ explicitly. For instance, $\sigma[3] = (12, 02, 01; 012)$, $\tilde{s}_0(\sigma)[3] = \tilde{s}_0(\sigma[3]) = (02, 12, \overline{01}; 012^*)$. It is direct to check that

$$V^{-}(\tilde{s}_0(\sigma)) = \mathrm{Hom}(013^* \otimes 023, 123 \otimes {}^{\overline{23}}012^*).$$

Similarly, once can find the expression for $V^{\pm}(\tilde{s}_i(\sigma))$. Then the action $V(\tilde{s}_i)$ is defined in Figure 4.2, 4.3 and 4.4.

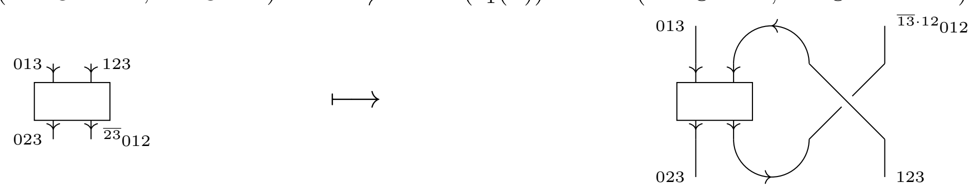
If $p_4(\tilde{s}) = p_4(\tilde{s}') = s$ and σ is a colored 3-simplex, then by Lemma 4.2.2 and Lemma 4.2.3, there is a canonical isomorphism *can*from $C(\tilde{s}(\sigma)[i])$ to $C(\tilde{s}'(\sigma)[i])$. If $g \in G$, then

$$V^+(\sigma) = \mathrm{Hom}(023 \otimes {}^{\overline{23}}012, 013 \otimes 123) \xrightarrow{V(\tilde{s}_0)} V^-(\tilde{s}_0(\sigma)) = \mathrm{Hom}(013^* \otimes 023, 123 \otimes {}^{\overline{23}}012^*)$$

$$V^-(\sigma) = \mathrm{Hom}(013 \otimes 123, 023 \otimes {}^{\overline{23}}012) \xrightarrow{V(\tilde{s}_0)} V^+(\tilde{s}_0(\sigma)) = \mathrm{Hom}(123 \otimes {}^{\overline{23}}012^*, 013^* \otimes 023)$$

Figure 4.2: Definition of $V(\tilde{s}_0)$

$$V^+(\sigma) = \mathrm{Hom}(023 \otimes {}^{\overline{23}}012, 013 \otimes 123) \xrightarrow{V(\tilde{s}_1)} V^-(\tilde{s}_1(\sigma)) = \mathrm{Hom}(023 \otimes 123^*, 013 \otimes {}^{\overline{13}\cdot 12}012^*)$$



$$V^-(\sigma) = \mathrm{Hom}(013 \otimes 123, 023 \otimes {}^{\overline{23}}012) \xrightarrow{V(\tilde{s}_1)} V^+(\tilde{s}_1(\sigma)) = \mathrm{Hom}(013 \otimes {}^{\overline{13}\cdot 12}012^*, 023 \otimes 123^*)$$

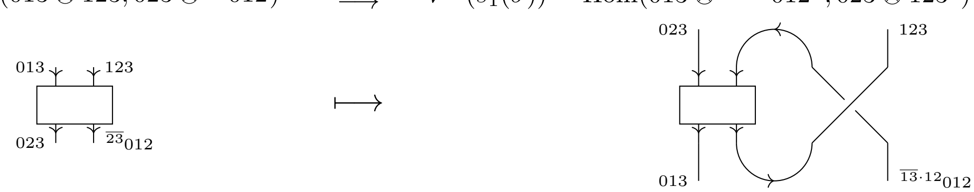


Figure 4.3: Definition of $V(\tilde{s}_1)$

$$V^+(\sigma) = \mathrm{Hom}(023 \otimes {}^{\overline{23}}012, 013 \otimes 123) \xrightarrow{V(\tilde{s}_2)} V^-(\tilde{s}_2(\sigma)) = \mathrm{Hom}(012 \otimes {}^{23}123^*, {}^{23}023^* \otimes {}^{23}013)$$

$$V^-(\sigma) = \mathrm{Hom}(013 \otimes 123, 023 \otimes {}^{\overline{23}}012) \xrightarrow{V(\tilde{s}_2)} V^+(\tilde{s}_2(\sigma)) = \mathrm{Hom}({}^{23}023^* \otimes {}^{23}013, 012 \otimes {}^{23}123^*)$$

Figure 4.4: Definition of $V(\tilde{s}_2)$

the g-action on any canonical isomorphisms are still canonical isomorphisms. Thus by composing and pre-composing canonical isomorphisms, we have an isomorphism can : $V^{\pm}(\tilde{s}(\sigma)) \longrightarrow V^{\pm}(\tilde{s}'(\sigma))$.

Lemma 4.2.4 *For any colored 3-simplex σ, and $\tilde{s}, \tilde{s}' \in F_3$ such that $p_4(\tilde{s}) = p_4(\tilde{s}') = s \in S_4$, the following diagram commutes:*

$$\begin{tikzcd} V^{\pm}(\sigma) \arrow[r, "V(\tilde{s})"] \arrow[dr, "V(\tilde{s}')"'] & V^{\pm sgn(s)}(\tilde{s}(\sigma)) \arrow[d, "can"] \\ & V^{\pm sgn(s)}(\tilde{s}'(\sigma)) \end{tikzcd} \tag{4.17}$$

where $sgn(s) = \pm$ is the sign the s and we use the convention that $++ = -- = +, +- = -+ = -$.

Proof: It suffices to prove the statement for the pairs $(\tilde{s}, \tilde{s}') = (\tilde{s}_i^2, e), i = 0, 1, 2$ and $(\tilde{s}_0\tilde{s}_1\tilde{s}_0, \tilde{s}_1\tilde{s}_0\tilde{s}_1)$, $(\tilde{s}_1\tilde{s}_2\tilde{s}_1, \tilde{s}_2\tilde{s}_1\tilde{s}_2)$, $(\tilde{s}_0\tilde{s}_2, \tilde{s}_2\tilde{s}_0)$. These can be checked by direct pictorial calculus. ■

Lemma 4.2.5 *For any $\tilde{s}, \tilde{s'} \in F_3$ such that $p_4(\tilde{s}) = p_4(\tilde{s}')$, σ a colored 3-simplex,*

1. $V(\tilde{s}) \otimes V(\tilde{s}) : V^+(\sigma) \otimes V^-(\sigma) \longrightarrow V^{sgn(s)}(\tilde{s}(\sigma)) \otimes V^{-sgn(s)}(\tilde{s}(\sigma))$ *is pairing preserving.*

2. $can \otimes can : V^+(\tilde{s}(\sigma)) \otimes V^-(\tilde{s}(\sigma)) \longrightarrow V^+(\tilde{s}'(\sigma)) \otimes V^-(\tilde{s}'(\sigma))$ *is pairing preserving.*

Finally we consider the reordering of 4-simplices. For a colored 4-simplex σ, denote by $V^{\pm}(\partial\sigma)$ the tensor product of the Hilbert spaces associated with its boundary:

$$V^{\pm}(\partial\sigma) = V^{\pm}(\sigma[1]) \otimes V^{\pm}(\sigma[3]) \otimes V^{\mp}(\sigma[0]) \otimes V^{\mp}(\sigma[2]) \otimes V^{\mp}(\sigma[4])$$

The generators of F_4 act on $\mathcal{S}^4$ as follows:

$$\tilde{s}_i(\sigma) = \begin{cases} (\sigma[1], \sigma[0], \tilde{s}_0(\sigma[2]), \tilde{s}_0(\sigma[3]), \tilde{s}_0(\sigma[4])) & i = 0 \\ (\tilde{s}_0(\sigma[0]), \sigma[2], \sigma[1], \tilde{s}_1(\sigma[3]), \tilde{s}_1(\sigma[4])) & i = 1 \\ (\tilde{s}_1(\sigma[0]), \tilde{s}_1(\sigma[1]), \sigma[3], \sigma[2], \tilde{s}_2(\sigma[4])) & i = 2 \\ (\tilde{s}_2(\sigma[0]), \tilde{s}_2(\sigma[1]), \tilde{s}_2(\sigma[2]), \sigma[4], \sigma[3]) & i = 3 \end{cases} \tag{4.18}$$

We need to check that RHS of Equation 4.18 in each case is indeed a valid colored 4-simplex. If σ is a colored 4-simplex, then for $0 \leq i \leq 4, 0 \leq j \leq 3$, we have

$$\sigma[i,j] = \begin{cases} \sigma[j, i-1] & i > j \\ \sigma[j+1, i] & i \leq j \end{cases} \tag{4.19}$$

The following lemma asserts that the action of F_4 preserves this property.

Lemma 4.2.6 *If $\tilde{s} \in F_4$, and σ is a colored 4-simplex, then $\tilde{s}(\sigma)$ satisfies Equation 4.19.*

Proof: Direct verifications on the generators. ■

Similar to Lemma 4.2.3, we have the following lemma.

Lemma 4.2.7 *If $p_5(\tilde{s}) = s \in S_5$ and $\sigma \in \mathcal{S}^4$ is a colored 4-simplex, then*

$$\tilde{s}(\sigma)[s(i)] = H_{\tilde{s}}^{s(i)}(\sigma[i]),\ i = 0, 1, 2, 3, 4, \tag{4.20}$$

where $H_{(-)}^i : F_4 \longrightarrow F_3$ is a map such that $p_4(H_{\tilde{s}}^{s(i)}) = s_{\hat{i}}$.

By Lemma 4.2.7,

$$V(H_{\tilde{s}}^{s(i)}) : V^{\pm}(\sigma[i]) \xrightarrow{\simeq} V^{sgn(s)\pm}(\tilde{s}(\sigma)[s(i)]),$$

Thus $V(\tilde{s}) := \bigotimes_{i=0}^{4} V(H_{\tilde{s}}^{s(i)}) : V^{\pm}(\partial\sigma) \xrightarrow{\simeq} V^{sgn(s)\pm}(\partial\tilde{s}(\sigma))$ is an isomorphism.

Lemma 4.2.8 *For each colored 4-simplex $\sigma \in \mathcal{S}^4$ and $\tilde{s} \in F_4$ such that $p_5(\tilde{s}) = s \in S_5$, the following diagram commutes, where and for the rest of the section we drop the subscript F for the $\tilde{Z}_F^{\pm}$.*

$$\begin{tikzcd} V^{\pm}(\partial\sigma) \arrow[r, "V(\tilde{s})"] \arrow[dr, "\tilde{Z}^{\pm}(\sigma)"'] & V^{\pm sgn(s)}(\partial\tilde{s}(\sigma)) \arrow[d, "\tilde{Z}^{\pm sgn(s)}(\tilde{s}(\sigma))"] \\ & \mathbb{C} \end{tikzcd} \tag{4.21}$$

Proof: It suffices to prove for the case $\tilde{s} = \tilde{s}_i$, which again is done by pictorial calculus. ■

Now we are ready to prove the main theorem of the section on the independence of the partition function on the ordering of vertices.

Theorem 4.2.9 *Let $\mathcal{T}, \mathcal{T}'$ be two ordered triangulations of M which differ by a reordering $s \in S_N$, then for each coloring $F = (g, f)$ of $\mathcal{T}$, there is a corresponding coloring $F' = (g', f')$ and an isomorphism $\Phi : V_F(\mathcal{T}) \longrightarrow V_{F'}(\mathcal{T}')$, such that $d_{f(\alpha)} = d_{f'(\alpha_s)}$ for* *any 2-simplex $\alpha \in \mathcal{T}^2$, $\Phi(\phi_F) = \phi_{F'}$, and that the following diagram commutes.*

$$\begin{tikzcd} V_F(\mathcal{T}) \arrow[r, "\Phi"] \arrow[dr, "\bigotimes_{\sigma\in\mathcal{T}^4} Z^{\epsilon(\sigma)}(\sigma)"'] & V_{F'}(\mathcal{T}') \arrow[d, "\bigotimes_{\sigma\in(\mathcal{T}')^4} Z^{\epsilon(\sigma)}(\sigma)"] \\ & \mathbb{C} \end{tikzcd} \tag{4.22}$$

In particular, this implies that $Z_{\mathcal{D}}(M;\mathcal{T}) = Z_{\mathcal{D}}(M;\mathcal{T}')$.

Proof: Fix a coloring $F = (g, f)$ on $\mathcal{T}$. For each simplex $\sigma \in \mathcal{T}$, denote by σ' the corresponding simplex σ_s in $\mathcal{T}'$.

For each colored 1-simplex α, identify α' with $s_\alpha(\alpha)$. That is, color α' with $s_\alpha(g_\alpha)$.

For each colored 2-simplex $\beta = (\beta[0], \beta[1], \beta[2]; C(\beta))$, arbitrarily choose a representative $\tilde{s}_\beta$ for s_β, and identify β' with $\tilde{s}_\beta(\beta)$, namely, transport the colors on $\tilde{s}_\beta(\beta)$ to β' according to their relative ordering. Lemma 4.2.2 guarantees the coloring on common edges of different 2-simplices are consistent, which is equal to the coloring already given above. Denote the resulting coloring on $\mathcal{T}'$ by $F' = (g', f')$. The map $F \mapsto F'$ gives a one-to-one correspondence between the set of colorings on $\mathcal{T}$ and the set of colorings on $\mathcal{T}'$.

Since the dual functor and the G-action preserve the quantum dimension of objects, it is clear from the definition of the F_2-action on $\mathcal{S}^2$ that $d_{f(\alpha)} = d_{f'(\alpha')}$.

For each colored 4-simplex σ, arbitrarily choose a representative $\tilde{s}_\sigma$ for s_σ, and let $\tilde{s}_\sigma(\sigma) = \tilde{\sigma}$. By Lemma 4.2.8, the following diagram commutes:

$$\begin{tikzcd} V^{\epsilon(\sigma)}(\partial\sigma) \arrow[r, "V(\tilde{s}_\sigma)"] \arrow[dr, "\tilde{Z}^{\epsilon(\sigma)}(\sigma)"'] & V^{\epsilon(\tilde{\sigma})}(\partial\tilde{\sigma}) \arrow[d, "\tilde{Z}^{\epsilon(\tilde{\sigma})}(\tilde{\sigma})"] \\ & \mathbb{C} \end{tikzcd} \tag{4.23}$$

Note that here we cannot identify $\tilde{\sigma}$ with σ' directly since the color on a sub 2-simplex of $\tilde{\sigma}$ may not be the same as that on the corresponding sub 2-simplex of σ'. However, it is true that the colors are both determined by the action of some elements in F_2 whose image in S_3 is the restriction of s on that 2-simplex. Thus there is a canonical isomorphism connecting these two colors and hence there is a canonical isomorphism from $V^{\pm}(\tilde{\sigma}[i])$ to $V^{\pm}_{F'}(\sigma'[i])$. And the following diagram commutes:

$$\begin{tikzcd} V^{\epsilon(\sigma)}(\partial\sigma) \arrow[r, "V(\tilde{s}_\sigma)"] \arrow[dr, "\tilde{Z}^{\epsilon(\sigma)}(\sigma)"'] & V^{\epsilon(\tilde{\sigma})}(\partial\tilde{\sigma}) \arrow[r, "can"] \arrow[d, "\tilde{Z}^{\epsilon(\tilde{\sigma})}(\tilde{\sigma})"] & V^{\epsilon(\sigma')}_{F'}(\partial\sigma') \arrow[dl, "\tilde{Z}^{\epsilon(\sigma')}(\sigma')"] \\ & \mathbb{C} & \end{tikzcd} \tag{4.24}$$

Define $\Phi : V_F(\mathcal{T}) \longrightarrow V_{F'}(\mathcal{T}')$ to be the tensor product over all 4-simplices of the isomorphism on horizontal line in Diagram 4.24, i.e., $\Phi = \bigotimes_{\sigma\in\mathcal{T}^4} can \circ V(\tilde{s}_\sigma)$.

This implies the diagram in the theorem commutes.

Next we show $\Phi(\phi_F) = \phi_{F'}$.

Let τ be the common face of two 4-simplices σ_1, σ_2, and assume $\tau = \sigma_1[i] = \sigma_2[j]$. Without loss of generality, assume $V^+_F(\tau)$ appears as a component in the domain of $\tilde{Z}^{\epsilon(\sigma_1)}(\sigma_1)$, and thus $V^-_F(\tau)$ must be a component in the domain of $\tilde{Z}^{\epsilon(\sigma_2)}(\sigma_2)$. Only for the rest of the proof, let $s_1 = s_{\sigma_1}, s_2 = s_{\sigma_2}, \tilde{s}_1 = \tilde{s}_{\sigma_1}, \tilde{s}_2 = \tilde{s}_{\sigma_2}$, and let $\tilde{\sigma}_1 = \tilde{s}_{\sigma_1}(\sigma_1)$ and $\tilde{\sigma}_2 = \tilde{s}_{\sigma_2}(\sigma_2)$.

By Lemma 4.2.7, we have

$$\tilde{\sigma}_1[s_1(i)] = H^{s_1(i)}_{\tilde{s}_1}(\tau),\ \tilde{\sigma}_2[s_2(j)] = H^{s_2(j)}_{\tilde{s}_2}(\tau),$$

and $p_4(H^{s_1(i)}_{\tilde{s}_1}) = (s_1)_{\hat{i}} = s_\tau = p_4(H^{s_2(j)}_{\tilde{s}_2})$.

Let $H_1 = H^{s_1(i)}_{\tilde{s}_1}, H_2 = H^{s_2(j)}_{\tilde{s}_2}$. Note that $H_1(\tau), H_2(\tau)$ and τ' have the same underlying 3-simplex.

The map Φ restricted to the component $V_F^+(\tau) \otimes V_F^-(\tau)$ is the map Φ':

$$\begin{tikzcd} V_F^+(\tau) \otimes V_F^-(\tau) \arrow[r, "V(H_1)\otimes V(H_2)"] & V^{sgn(s_\tau)}(H_1(\tau)) \otimes V^{-sgn(s_\tau)}(H_2(\tau)) \arrow[d, "can\otimes can"] \\ & V_{F'}^{sgn(s_\tau)}(\tau') \otimes V_{F'}^{-sgn(s_\tau)}(\tau') \end{tikzcd}$$

To show $\Phi'(\phi_{F,\tau}) = \phi_{F',\tau'}$, it suffices to show Φ' preserves the pairing. Since the canonical isomorphisms preserves the pairing, it suffices then to show $(I \otimes can) \circ (V(H_1) \otimes V(H_2))$ in the the following diagram does.

$$\begin{tikzcd} V_F^+(\tau) \otimes V_F^-(\tau) \arrow[r, "V(H_1)\otimes V(H_2)"] \arrow[dr, "V(H_1)\otimes V(H_1)"'] & V^{sgn(s_\tau)}(H_1(\tau)) \otimes V^{-sgn(s_\tau)}(H_2(\tau)) \arrow[d, "I\otimes can"] \\ & V^{sgn(s_\tau)}(H_1(\tau)) \otimes V^{-sgn(s_\tau)}(H_1(\tau)) \end{tikzcd}$$

By Lemma 4.2.4, the above diagram commutes, and by Lemma 4.2.5, $V(H_1) \otimes V(H_1)$ preserves the pairing.

$\blacksquare$

4.3 Pachner Moves

Let $M = M^n$ be any closed manifold of dimension n. A triangulation $\mathcal{T}$ of M means a simplicial complex $\mathcal{T}$ such that the underline topological space $\mathcal{T}^{top}$ is homeomorphic to M. Pachner moves are set of local operations on triangulations which convert a triangulation of M to any other triangulation of the same manifold.

Let σ_n be any n-simplex. Its boundary, denoted by $\partial(\sigma_n)$, consists of $n+1$ $(n-1)$-simplices. Let $\partial(\sigma_n) = I \sqcup J$ be a bi-partition of $\partial(\sigma_n)$ into (disjoint) sets I and J such that $|I| = k, |J| = n+1-k$. We say two triangulations $\mathcal{T}, \mathcal{T}'$ are related by a pachner move of type $(k, n+1-k)$ if $I \subset \mathcal{T}$ and $\mathcal{T}' = (\mathcal{T} \setminus I) \cup J$ for some σ_k, I, J. Apparently,

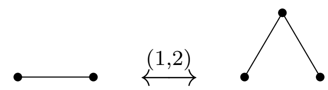


Figure 4.5: Pachner move $(1, 2)$; $n = 1$

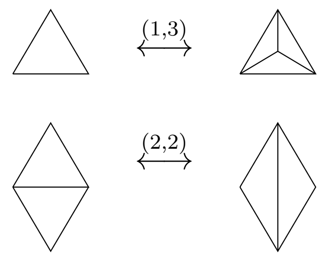


Figure 4.6: Pachner moves $(1, 3)$, $(2, 2)$; $n = 2$

pachner moves of types $(k, n + 1 - k)$ and $(n + 1 - k, k)$ are inverse operations if the partitions are taken to be (I, J) and (J, I) respectively. If two triangulations are related by a pachner move, then their underlying spaces are homeomorphic.

Theorem 4.3.1 *[49] Let $M = M^n$ a closed manifold and $\mathcal{T}, \mathcal{T}'$ be two triangulations of M, then there is a sequence of triangulations of M, $\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \cdots, \mathcal{T}_k = \mathcal{T}'$, such that any two neighboring triangulations are related by a pachner move of type $(k, n + 2 - k)$, $1 \leq k \leq \frac{n}{2} + 1$, or its inverse.*

Some examples of pachner moves are listed as follows.

If $dim(M) = n = 1$, we have pachner moves of type $(1, 2)$ and $(2, 1)$ which are inverse to each other. See Figure 4.5.

If $n = 2$, we have pachner moves of type $(1, 3)$, $(2, 2)$, and $(3, 1)$. See Figure 4.6.

We are more interested in the case $n = 4$, where we have pachner moves of types $(3, 3), (2, 4), (1, 5)$ and their inverses. Given a 5-simplex σ_5, we order its vertices by $0, 1, 2, 3, 4$, and denote the face which does not contain the vertex i by $(0 \cdots \hat{i} \cdots 4)$. For each type of pachner moves, we pick a specific partition I, J, and call it the *typical*

	I_k			I'_k		
$k=4$	02345	01245	01234	12345	01345	01235
$k=3$	0124	0234	0245	0135	1235	1345
$k=2$	024			135		

Table 4.1: Comparison of $\mathcal{T}$ and $\mathcal{T}'$ for 3-3 move

pachner move. The *typical* pachner moves for $n=4$ are as follows:

$$(02345)(01245)(01234) \quad \xleftrightarrow{(3,3)} \quad (12345)(01345)(01235)$$

$$(02345)(01245)(01234)(12345) \quad \xleftrightarrow{(2,4)} \quad (01345)(01235)$$

$$(02345)(01245)(01234)(12345)(01345) \quad \xleftrightarrow{(1,5)} \quad (01235)$$

4.4 Invariance under Pachner Moves

We prove that the partition function $Z_{\mathcal{D}}(M;\mathcal{T})$ defined in Section 3.1 is invariant under pachner moves. Since in Section 4.2 it has been shown that $Z_{\mathcal{D}}(M;\mathcal{T})$ does not depend on the ordering of the vertices, we only need to consider *typical* pachner moves of types 3-3, 2-4, and 1-5. Let $\mathcal{T},\mathcal{T}'$ be two ordered triangulations whose vertices are ordered in such a way that they only differ by a *typical* pachner move. In each of the cases, let $I_k = \mathcal{T}^k \setminus (\mathcal{T}')^k$ be the set of k-simplices which belong to $\mathcal{T}$ but not $\mathcal{T}'$, and similarly let $I'_k = (\mathcal{T}')^k \setminus \mathcal{T}^k$. Tables 4.1, 4.2, and 4.3 list the differences between $\mathcal{T}$ and $\mathcal{T}'$ corresponding to each *typical* pachner move 3-3, 2-4, and 1-5, respectively. Without loss of generality, assume the 4-simplices 02345, 01245, 01234 in $\mathcal{T}$ are positive, while 12345, 01345, 01235 in $\mathcal{T}'$ are positive. Thus if any of 12345, 01345, 01235 are in $\mathcal{T}$, they must be negative.

By a complete set of representatives we mean a set of objects which contains exactly

	I_k			I'_k	
$k=4$	02345	01245	01234	01345	01235
	12345				
$k=3$	0124	0234	0245	0135	
	2345	1245	1234		
$k=2$	024				
	245	234	124		
$k=1$	24				

Table 4.2: Comparison of $\mathcal{T}$ and $\mathcal{T}'$ for 2-4 move

	I_k						I'_k
$k=4$	02345	01245	01234	12345			01235
	01345						
$k=3$	0124	0234	0245	2345	1245	1234	
	0134	0145	1345	0345			
$k=2$	024	245	234	124			
	034	014	134	045	145	345	
$k=1$	24						
	04	14	34	45			
$k=0$	4						

Table 4.3: Comparison of $\mathcal{T}$ and $\mathcal{T}'$ for 1-5 move

one representative for each isomorphism class of simple objects. For the rest of the section, a summation variable concerning simple objects in the category is always assumed to run through an arbitrary complete set of representatives, unless otherwise stated. Recall that an extended coloring is a map $\hat{F} = (g, f, t)$, which assigns a group element to each 1-simplex, an isomorphism class of simple objects to each 2-, and 3-simplex, such that these assignments satisfy some restrictions, see Definition 3.1.6. Then to define the partition function, a representative is arbitrarily chosen for each 2- and 3-simplex with an extended coloring. The partition function is a sum over all extended colorings, see Equation 3.3. In light of the observations above, we can rephrase the definition of the partition function as follows.

We let each 1-simplex run through the set of all elements in G, and let each 2-, 3-simplex run through an arbitrary complete set of representatives (i.e., each 2-, 3-simplex has its own complete set). Then an extended coloring can be viewed as a choice of value for 1-, 2-, and 3-simplices, so that the resulting configuration satisfies the restriction in Definition 3.1.6. If a configuration is not an extended coloring, we set its contribution to zero. As before, we denote the color of a simplex by the simplex itself. Then Equation 3.3 can be rewritten as Equation 4.25, where we have omitted the dependence of $\hat{Z}^{\epsilon(\sigma)}(\sigma)$ $\hat{F}$.

$$Z_{\mathcal{D}}(M;\mathcal{T}) = \sum_{\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3} \frac{(D^2/|G|)^{|\mathcal{T}^0|}}{(D^2)^{|\mathcal{T}^1|}} \frac{\prod\limits_{\alpha\in\mathcal{T}^2} d_\alpha}{\prod\limits_{\tau\in\mathcal{T}^3} d_\tau} \prod_{\sigma\in\mathcal{T}^4} \hat{Z}^{\epsilon(\sigma)}(\sigma) \tag{4.25}$$

By Equation 4.25, to prove $Z_{\mathcal{D}}(M;\mathcal{T}) = Z_{\mathcal{D}}(M;\mathcal{T}')$, it suffices to show that Equations 4.26, 4.27, and 4.28 hold, corresponding to the *typical* pachner move 3-3, 2-4, 1-5, respectively, where all simplices which do not present among the summation are assumed to have a fixed coloring on them.

$$\sum_{\substack{I_2,\\ I_3}} \frac{d_{024}}{d_{0124}d_{0234}d_{0245}} \hat{Z}^+(01234)\hat{Z}^+(01245)\hat{Z}^+(02345)$$
$$= \sum_{\substack{I_2',\\ I_3'}} \frac{d_{135}}{d_{0135}d_{1235}d_{1345}} \hat{Z}^+(01235)\hat{Z}^+(01345)\hat{Z}^+(12345) \tag{4.26}$$

$$\sum_{\substack{I_1,\\ I_2,\\ I_3}} \frac{1}{D^2} \frac{\prod\limits_{\alpha\in I_2} d_\alpha}{\prod\limits_{\tau\in I_3} d_\tau} \hat{Z}^+(01234)\hat{Z}^+(01245)\hat{Z}^+(02345)\hat{Z}^-(12345)$$
$$= \sum_{I_3'} \frac{1}{d_{0135}} \hat{Z}^+(01235)\hat{Z}^+(01345) \tag{4.27}$$

$$\sum_{I_0,I_1,I_2,I_3} \frac{(D^2/|G|)^{|I_0|}}{(D^2)^{|I_1|}} \frac{\prod\limits_{\alpha\in I_2} d_\alpha}{\prod\limits_{\tau\in I_3} d_\tau} \hat{Z}^+(01234)\hat{Z}^+(01245)\hat{Z}^+(02345)\hat{Z}^-(12345)\hat{Z}^-(01345)$$
$$= \hat{Z}^+(01235) \tag{4.28}$$

The above three equations are to be proved in Section 4.4.1, 4.4.2, and 4.4.3, respectively. In the proof, the following simple lemmas will be heavily used. For the section of invariance proof under pachner moves, we take one more convention that all diagrams of morphisms in figures are assumed to have their top and bottom identified, namely, the diagrams in figures are actually the trace of the morphisms drawn.

Lemma 4.4.1 (Merging Formula) *Let A, B be two objects of $\mathcal{D}$, and $\{e_{A,B,i} : i \in I\}, \{e_{B,A,j} : j \in I\}$ be a basis of* $\mathrm{Hom}(A, B)$ *and* $\mathrm{Hom}(B, A)$*, respectively, such that $\langle e_{A,B,i}, e_{B,A,j}\rangle = \alpha_i \delta_{i,j}$, then for $F \in \mathrm{Hom}(A, B), G \in \mathrm{Hom}(B, A)$, we have $\langle G, F\rangle =$*

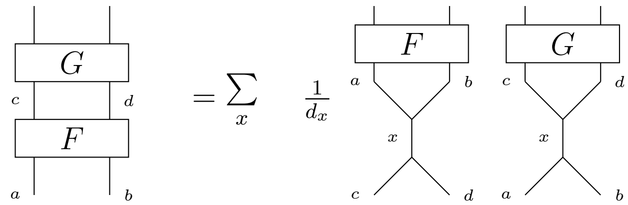


Figure 4.7: Trace Identity (1)

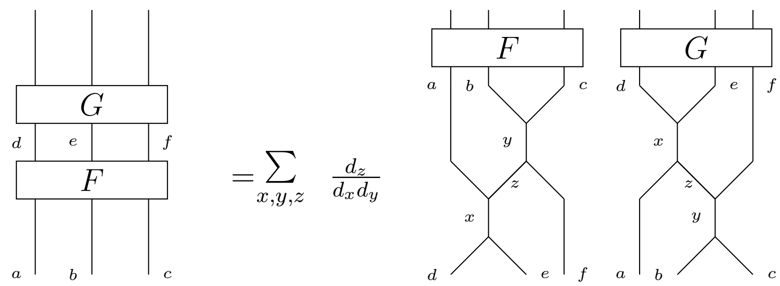


Figure 4.8: Trace Identity (2)

$\sum_{i \in I} \frac{1}{\alpha_i} \langle G, e_{A,B,i} \rangle \langle F, e_{B,A,i} \rangle$. *Specifically, the following special examples will be used in the invariance proof.*

1. *Let* $F \in \mathrm{Hom}(a \otimes b, c \otimes d)$, $G \in \mathrm{Hom}(c \otimes d, a \otimes b)$, *then*

$$\langle G, F \rangle = \sum_x \frac{1}{d_x} \langle F, B^x_{cd,ab} \rangle \langle G, B^x_{ab,cd} \rangle.$$

2. *Let* $F \in \mathrm{Hom}(a \otimes b \otimes c, d \otimes e \otimes f)$, $G \in \mathrm{Hom}(d \otimes e \otimes f, a \otimes b \otimes c)$, *then*

$$\langle G, F \rangle = \sum_{x,y,z} \frac{d_z}{d_x d_y} \langle F, (Id \otimes B^y_{zf,bc}) \circ (B^x_{de,az} \otimes Id) \rangle \langle G, (B^x_{az,de} \otimes Id) \circ (Id \otimes B^y_{bc,zf}) \rangle.$$

The identities in Part 1 and Part 2 are also illustrated in Figure 4.7 and Figure 4.8, respectively.

Lemma 4.4.2 *Let* $g, g' \in G$ *and* a, b, c, d, e *be simple objects of* $\mathcal{D}$ *such that* $a \otimes b \in \mathcal{C}_{gg'}$, *then the equality is Figure 4.9 holds.*

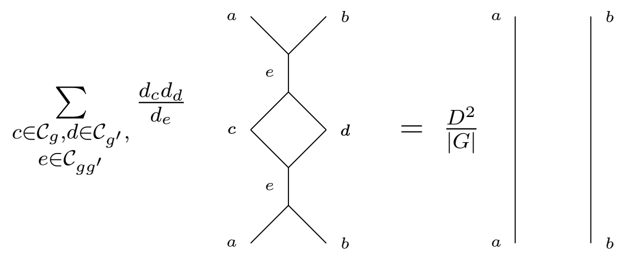


Figure 4.9: Dimension Identity

Proof:

$$
\begin{aligned}
\text{LHS} &= \sum_{c,d,e} \frac{d_c d_d}{d_e} N_{cd}^{e} B_{ab,ab}^{e} & &= \sum_{e\in\mathcal{C}_{gg'}} \sum_{c\in\mathcal{C}_g} \frac{d_c}{d_e} B_{ab,ab}^{e} \sum_{d\in\mathcal{C}_{g'}} d_{d^*} N_{e^*c}^{d^*} \\
&= \sum_{e\in\mathcal{C}_{gg'}} \sum_{c\in\mathcal{C}_g} \frac{d_c^2 d_{e^*}}{d_e} B_{ab,ab}^{e} & &= \sum_{e\in\mathcal{C}_{gg'}} \frac{D^2}{|G|} B_{ab,ab}^{e} \\
&= \frac{D^2}{|G|} Id_{a\otimes b}
\end{aligned}
$$

■

4.4.1 Invariance under 3-3 move

In this subsection and the next two subsections, picture calculus is used heavily. For the partition functions defined in Figure 3.3, we will omit the labels on 2-simplices, and only keep those on 3-simplices. Given a colored 3-simplex (0123), we will use the the short notation illustrated in Figure 4.10. From this notation, it is direct to work out the color of each edge in the figures representing partition functions. The idea of proof is to express each side of the equation as a weighted sum of diagrams, combine these diagrams into one using the *Merging Formula*, and show that the resulting diagrams for two side of the equation are homotopic and thus represent the same morphism.

Proposition 4.4.3 *Let $\mathcal{T}, \mathcal{T}'$ be two ordered triangulations which differ by a typical* 3-3

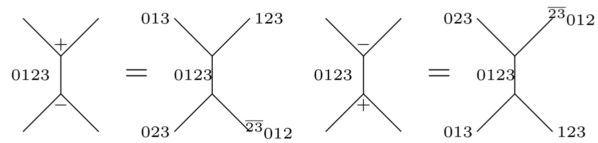


Figure 4.10: Short notations for basis elements

pachner move, see Table 4.1 for their comparison. Then the identity in Equation 4.26 holds, assuming all simplices which do not present among the summation variables have been assigned a fixed coloring.

Proof: The proof is best illustrated using picture calculus and Lemma 4.4.1. Note that the top and bottom of each diagram (see below) are identified, so within each diagram morphisms are cyclically ordered, namely, one can move a morphism on the bottom to the top and vice versa. Also, the summation terms are written below the diagrams. It is direct to check that the final diagrams in both LHS and RHS represent the same morphism, which justifies the equation.

Below we give a brief explanation of each "= " sign.

LHS:

① By definition.

② Apply $\overline{45}$ to the first diagram. Move the top two morphisms involving $0245, 0234$ in the third diagram to the bottom.

③ Apply Lemma 4.4.1 to the second and third diagram.

④ Isotope the second diagram.

⑤ Apply Lemma 4.4.1.

RHS:

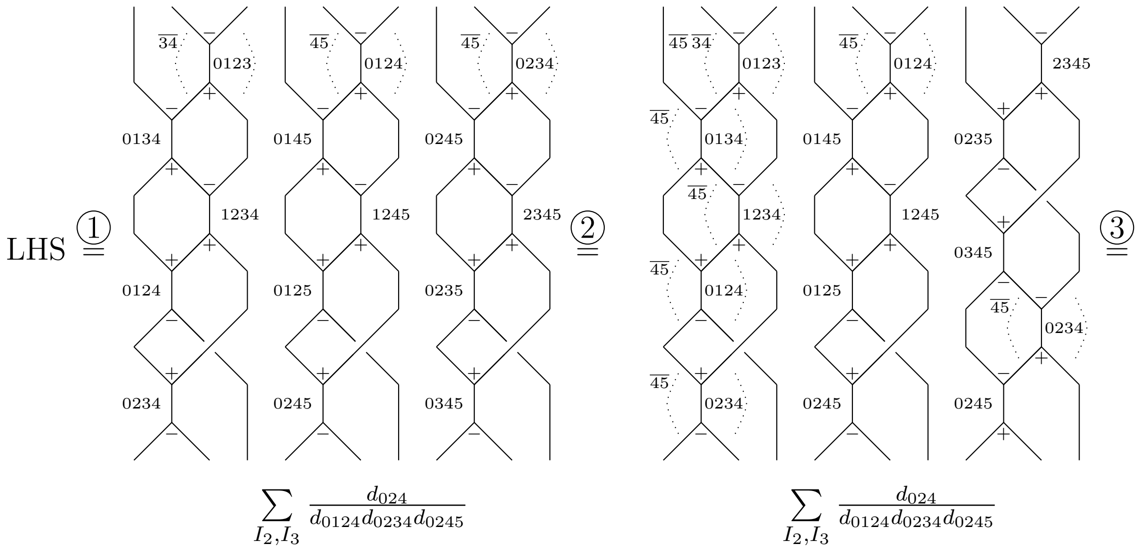
① By definition.

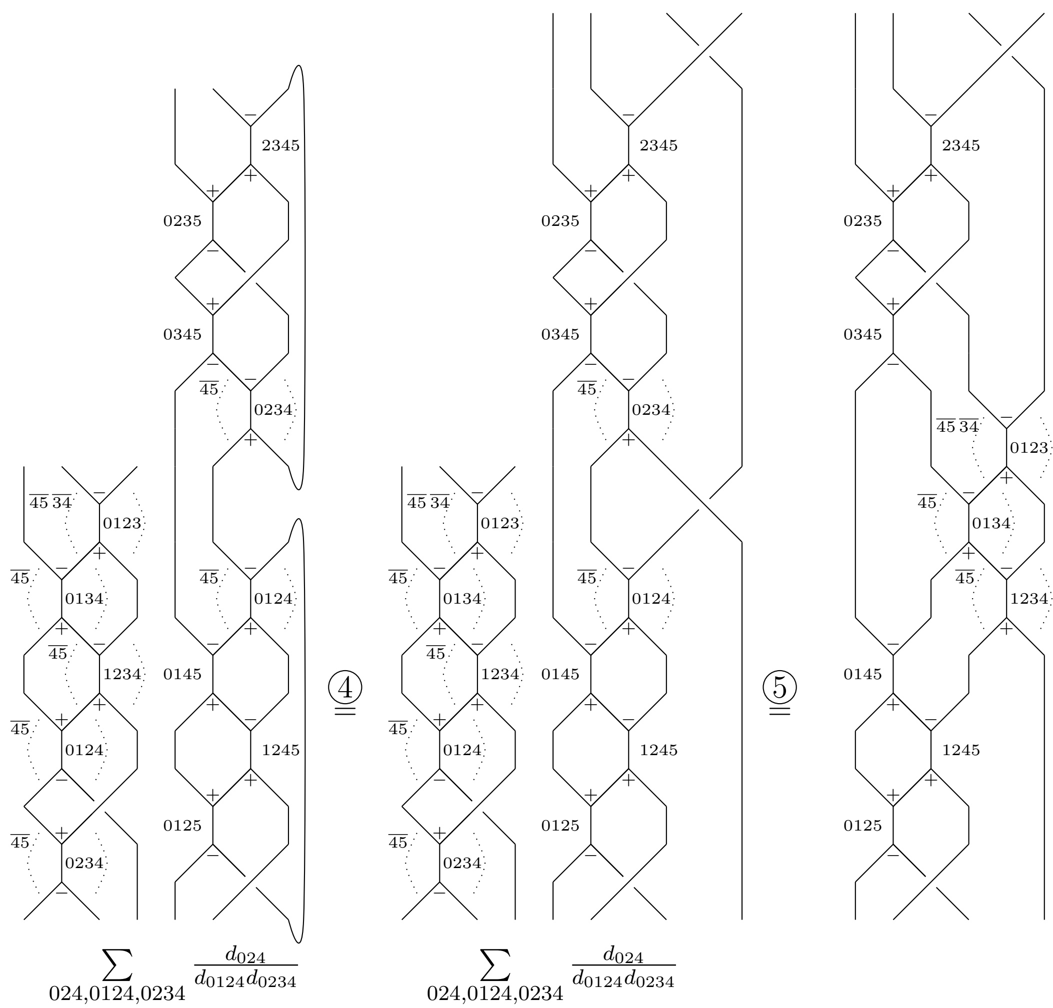
② Apply Lemma 4.4.1 to the first and third diagram.

③ Apply Lemma 4.4.1 to the two diagrams.

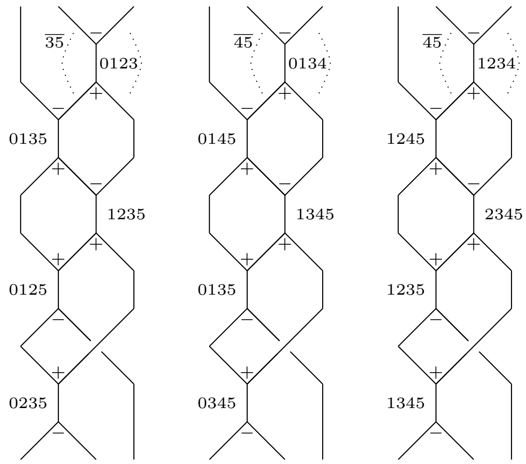
④ Isotope the diagram. Note that the group element acting on 0123 changes from $\overline{35}$ to $\overline{45}\,\overline{34}$ due to braiding.

■


LHS ①=
②=
③=
34 0123 0134 1234 0124 0234
45 0124 0145 1245 0125 0245
45 0234 0245 2345 0235 0345
45 34 0123 45 0134 45 1234 45 0124 45 0234
45 0124 0145 1245 0125 0245
2345 0235 0345 45 0234 0245
$\sum_{I_2,I_3} \frac{d_{024}}{d_{0124}d_{0234}d_{0245}}$
$\sum_{I_2,I_3} \frac{d_{024}}{d_{0124}d_{0234}d_{0245}}$

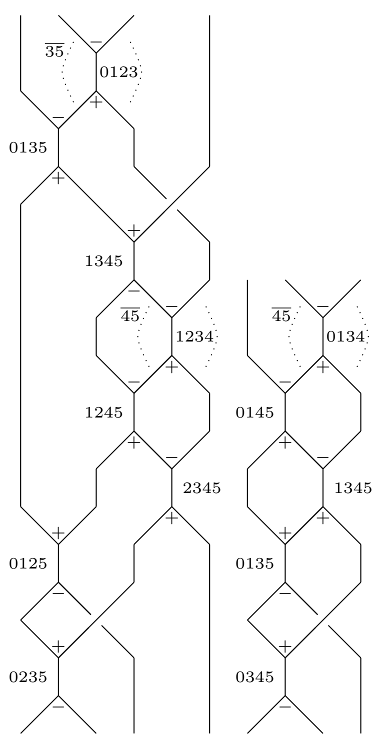

2345 0235 0345 45 0234
45 34 0123 45 0134 45 1234 45 0124 45 0234
45 0124 0145 1245 0125
④=
45 34 0123 45 0134 45 1234 45 0124 45 0234
2345 0235 0345 45 0234 45 0124 0145 1245 0125
⑤=
2345 0235 0345 45 34 0123 45 0134 45 1234 0145 1245 0125
$\sum_{024,0124,0234} \frac{d_{024}}{d_{0124}d_{0234}}$
$\sum_{024,0124,0234} \frac{d_{024}}{d_{0124}d_{0234}}$

RHS $\overset{①}{=}$

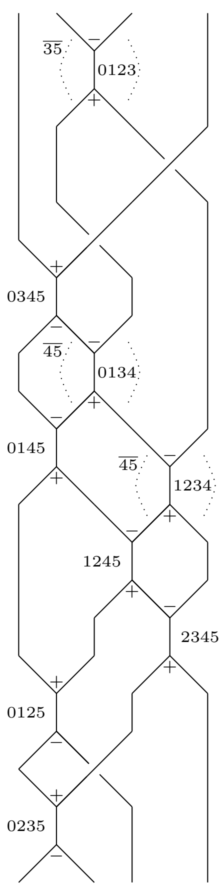


$\overset{②}{=}$

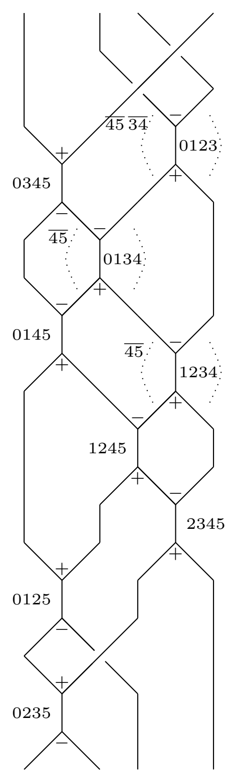
$$\sum_{I_2',I_3'} \frac{d_{135}}{d_{0135}d_{1235}d_{1345}}$$



$\overset{③}{=}$



$\overset{④}{=}$



$$\sum_{135,0135,1345} \frac{d_{135}}{d_{0135}d_{1345}}$$

4.4.2 Invariance under 2-4 move

Proposition 4.4.4 *Let $\mathcal{T}, \mathcal{T}'$ be two ordered triangulations which differ by a typical* 2-4 *pachner move, see Table 4.2 for their comparison. Then the identity in Equation 4.27 holds, assuming all simplices which do not present among the summation variables have been assigned a fixed coloring.*

Proof: LHS:

① By definition.

② Use the result of the LHS in the proof of Equation 4.26.

③ Apply Lemma 4.4.1 to the two diagrams.

④ Move the morphism involving 2345 and the braiding above it from the top to the bottom.

⑤ Isotope the diagram.

⑥ Apply Lemma 4.4.2. Note that after summing over the variables $245, 234, 2345$, the factor remained is $\frac{1}{|G|}$, which is cancelled with the variable 24.

RHS:

① By definition.

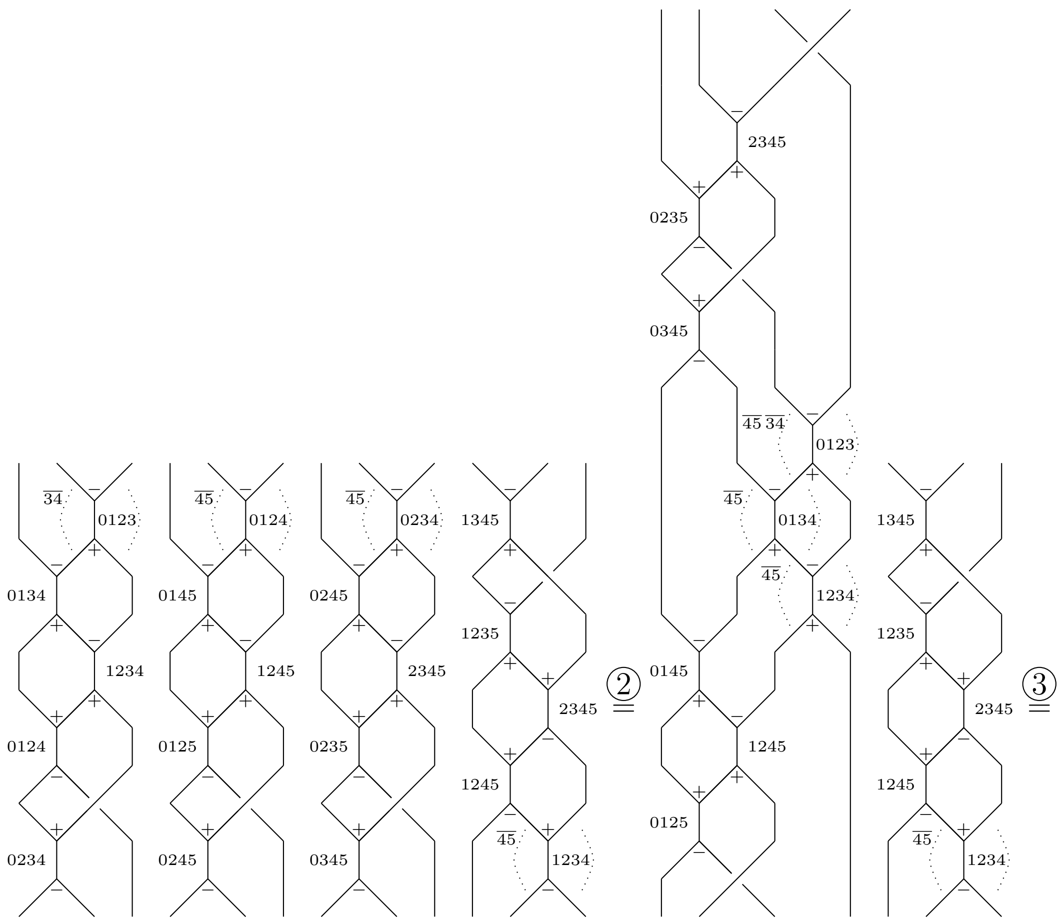
② Move the morphism involving 0123 in the first diagram from the top to the bottom.

③ Apply Lemma 4.4.1 to the two diagrams.

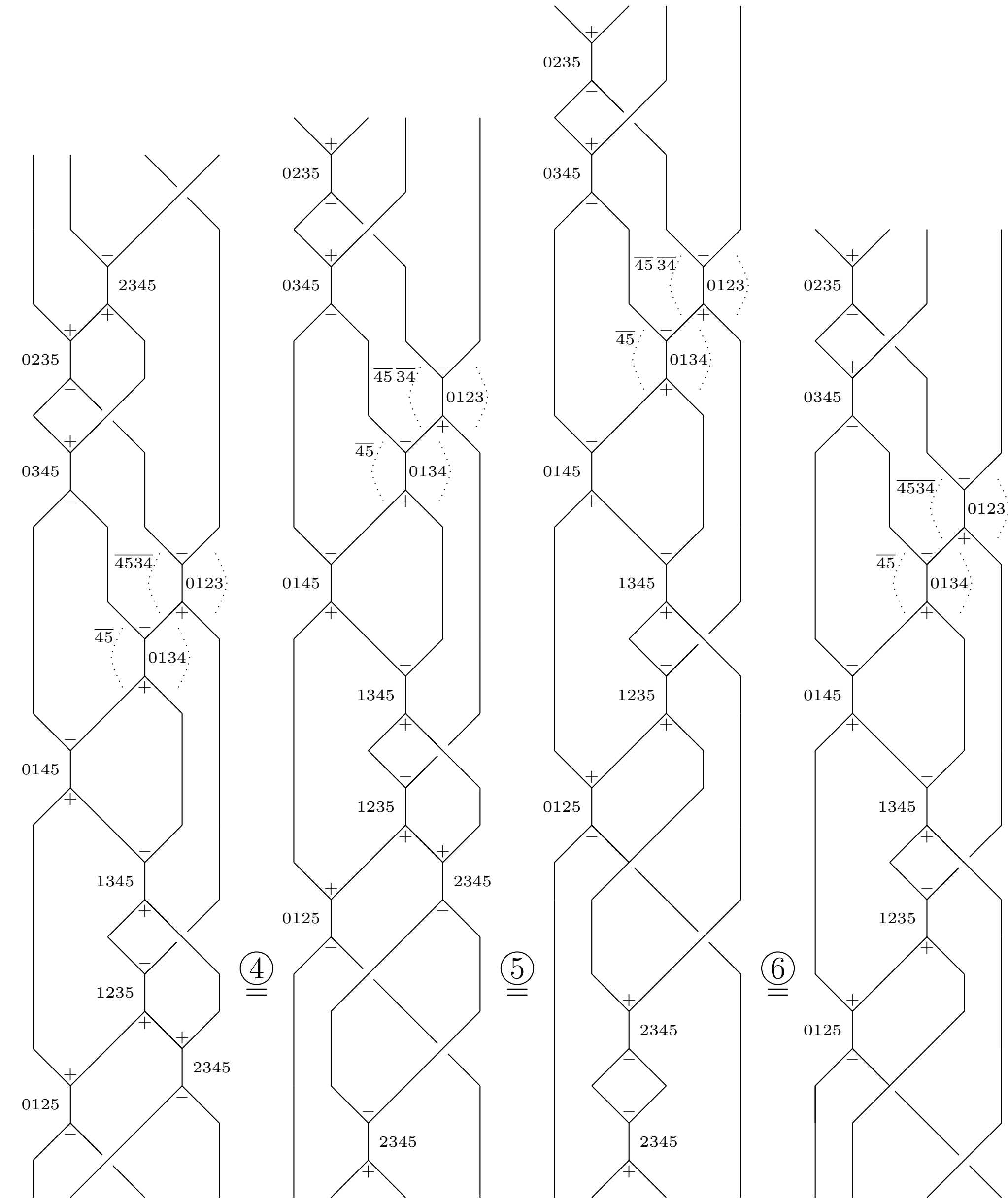
④ Isotope the diagram.

⑤ Isotope the diagram.

LHS $\overset{①}{=}$

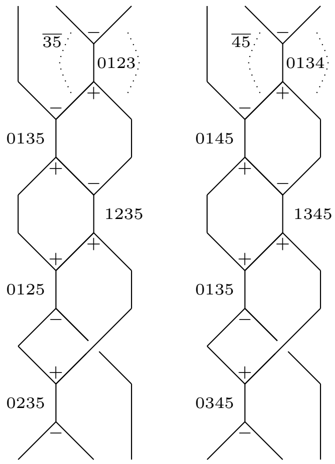


$$\sum_{I_1,I_2,I_3} \frac{1}{D^2} \prod_{\alpha\in I_2} d_\alpha \prod_{\tau\in I_3} d_\tau^{-1} \qquad \sum_{\substack{24\\245,234,124\\2345,1245,1234}} \frac{1}{D^2}\frac{d_{245}d_{234}d_{124}}{d_{2345}d_{1245}d_{1234}}$$



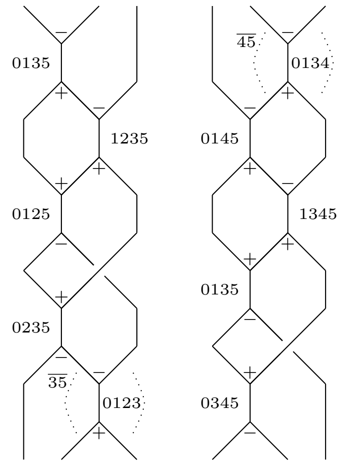
$$\sum_{\substack{24\\245,234\\2345}} \frac{1}{D^2}\frac{d_{245}d_{234}}{d_{2345}} \qquad \sum_{\substack{24\\245,234\\2345}} \frac{1}{D^2}\frac{d_{245}d_{234}}{d_{2345}} \qquad \sum_{\substack{24\\245,234\\2345}} \frac{1}{D^2}\frac{d_{245}d_{234}}{d_{2345}}$$

RHS $\overset{①}{=}$



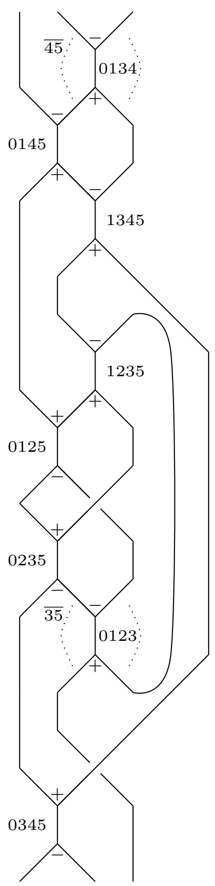
$\sum_{0135} \frac{1}{d_{0135}}$

$\overset{②}{=}$

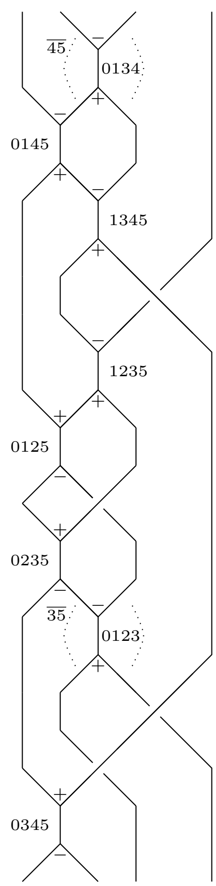


$\sum_{0135} \frac{1}{d_{0135}}$

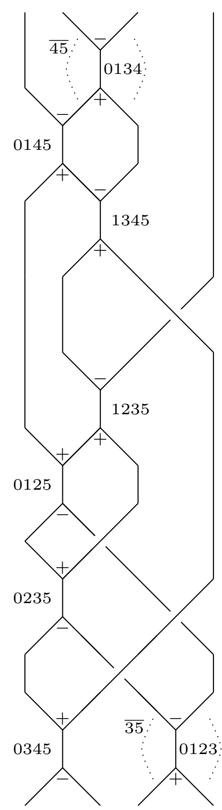
$\overset{③}{=}$



$\overset{④}{=}$



$\overset{⑤}{=}$



■

4.4.3 Invariance under 1-5 move

Proposition 4.4.5 *Let $\mathcal{T}, \mathcal{T}'$ be two ordered triangulations which differ by a typical* 1-5 *pachner move, see Table 4.3 for their comparison. Then the identity in Equation 4.28 holds, assuming all simplices which do not present among the summation variables have been assigned a fixed coloring.*

Proof:

LHS:

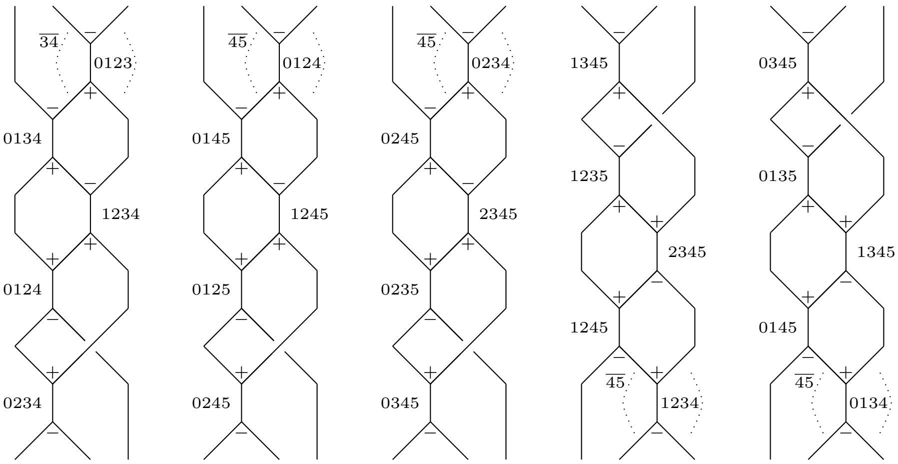
① By definition.

② Use the result of the LHS in the proof of Equation 4.27.

③ Apply Lemma 4.4.1 to the two diagrams.

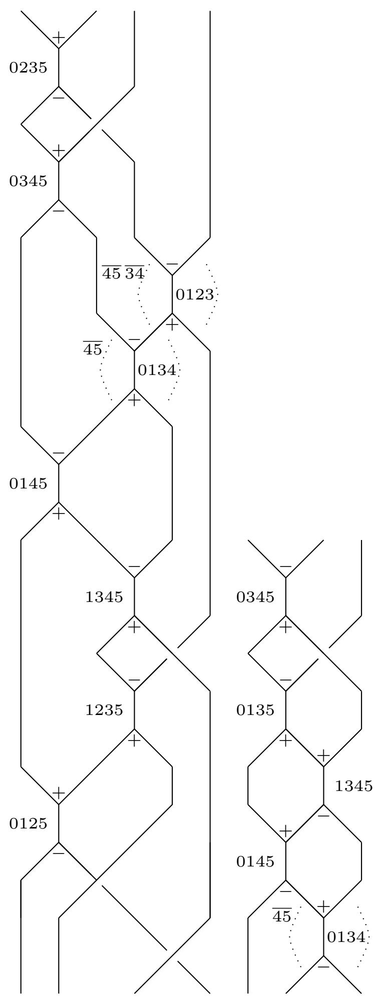
④ Apply Lemma 4.4.2.

⑤ The strand labeled by 345 can be isotoped to an isolated loop, and thus it contributes a factor d_{345}. Note that the resulting diagram is equal to the RHS of Equation 4.28 after a cyclic shift of morphisms.

LHS $\overset{①}{=}$  $\overset{②}{=}$

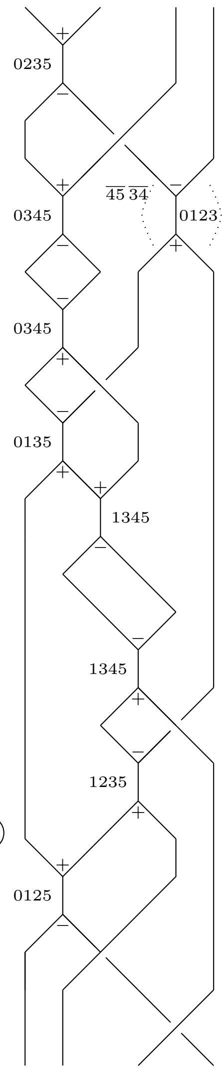
$$\sum_{I_1,I_2,I_3} \frac{D^2}{|G|} \frac{1}{(D^2)^{|I_1|}} \prod_{\alpha\in I_2} d_\alpha \prod_{\tau\in I_3} d_\tau^{-1}$$

■



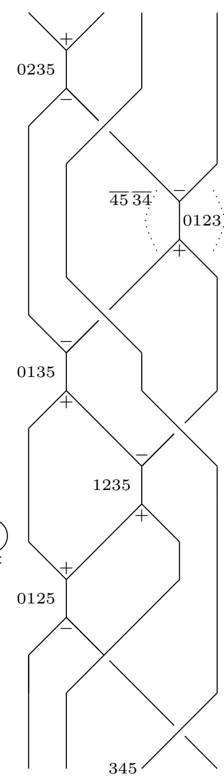
$$\sum_{\substack{04,14,34,45\\ J_2:=\{034,014,134,\\ 045,145,345\}\\ J_3:=\{0134,0145,\\ 1345,0345\}}} \frac{D^2}{|G|}\frac{1}{(D^2)^4}\frac{\prod\limits_{\alpha\in J_2} d_\alpha}{\prod\limits_{\tau\in J_3} d_\tau}$$

③ =



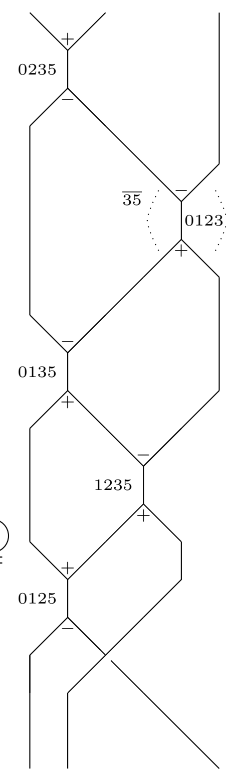
$$\sum_{\substack{04,14,34,45\\ J_2':=\{034,134,\\ 045,145,345\}\\ J_3':=\{1345,0345\}}} \frac{D^2}{|G|}\frac{1}{(D^2)^4}\frac{\prod\limits_{\alpha\in J_2'} d_\alpha}{\prod\limits_{\tau\in J_3'} d_\tau}$$

④ =



$$\sum_{\substack{04,14,\\ 34,45\\ 345}} \Big(\frac{D^2}{|G|}\Big)^3 \frac{d_{345}}{(D^2)^4}$$

⑤ =



$$\sum_{\substack{04,14,\\ 34,45\\ 345}} \Big(\frac{D^2}{|G|}\Big)^3 \frac{d_{345}^2}{(D^2)^4} = 1$$

Chapter 5

Future Directions and Open Questions

We have constructed a 4-manifold invariant out of a G-crossed braided spherical fusion category and this invariant is a generalization of several known invariants in literature. Here we point out a few directions/questions for future study.

1. What is the power of the invariants? Can they distinguish any smooth structures? In particular, it is worth checking if the invariants are strictly better than the previous known ones, e.g. the Crane-Yetter invariant.

The Crane-Yetter invariant from a modular category is a classical invariant, which can be expressed in terms of the signature and the Euler characteristic [25]. It would be interesting to also give the invariant from a G-BSFC an interpretation in terms of the intrinsic properties of the manifolds. We believe that our invariant is related to the homotopy 3-types with the exact relation to be studied in the future.

2. It is appealing to generalize the invariant in the following three directions.

In [50] and [51], a refined version of the Crane-Yetter invariant was introduced, namely, the invariant was associated with a pair (M, ω), where M is a closed oriented 4-manifold and $\omega \in H_2(M, \mathbb{Z}_2)$, and the original Crane-Yetter invariant is a normalized sum of the refined invariant over all $\omega \in H_2(M, \mathbb{Z}_2)$. Moreover, the refined invariant gives a state sum formula of the second Stiefel-Whitney class and the Pontrjagin squares of second cohomology classes. It is interesting to see if our invariant also has a similar refinement.

Secondly, the extension of a braided category to a G-crossed braided category depends on the vanishing condition of some obstruction in $O_4 \in H^4(G, U(1))$. We conjecture that even if this obstruction does not vanish, one can still get a 4-manifold invariant from some structures beyond G-crossed braided categories.

Lastly but more importantly, we are interested in defining a G-BSFC where G is allowed to be an infinite group, such as $U(1)$, and in modifying the state sum model so that the partition function converges. It is expected that the resulting invariant would be much stronger than the current one, and might be related to Donaldson/Seiberg-Witten invariants.

3. By [35][38][39], equivalent classes G-BSFCs are in one-to-one correspondence, by equivariatization and de-equivariatization, with equivalent classes of ribbon fusion categories containing $Rep(G)$ as a subcategory. Ribbon fusion categories are the input to the Crane-Yetter invariant. Thus we wonder whether there are any relations between the invariant from a G-BSFC and Crane-Yetter invariant from the corresponding ribbon fusion category containing $Rep(G)$.

4. In $(2+1)$ dimension, a non-semisimple generalization of the Turaev-Viro invariant is the Kuperberg invariant [17], which is defined based on Hopf algebras. Our invariant is a $(3+1)$ analog of the Turaev-Viro invariant. Thus it is interesting to see what the

$(3+1)$ analog of the Kuperberg invariant is. We speculate that the base to the $(3+1)$ Kuperberg invariant is some special class of Hopf algebras. In [29], a 4-manifold invariant is produced from the cyclic group $\mathbb{Z}_N$. We think this should be a special case of the $(3+1)$ Kuperberg invariant where the Hopf algebra $\mathbb{C}[\mathbb{Z}_N]$ is a particular case in the relevant class of Hopf algebras.

5. On the physics side, the Walker-Wang model [24] is a Hamiltonian realization of the Crane-Yetter invariant. What is the Hamiltonian realization of the invariant from a G-BSFC?

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