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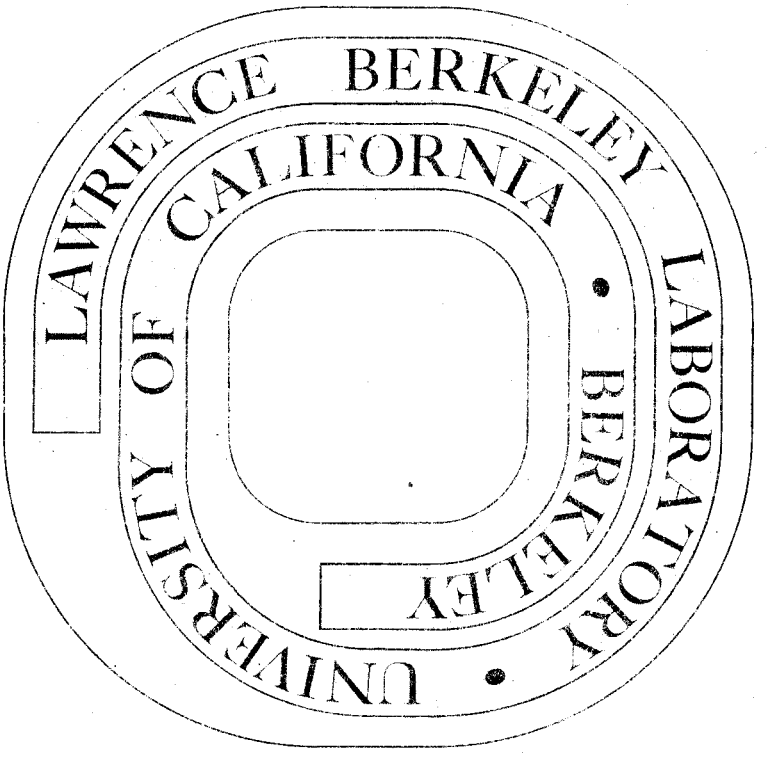

DISCLAIMER

IS IT POSSIBLE TO GENERATE A PHYSICAL ρ-MESON WITH NO CDD POLES?*

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July 6, 1971

ABSTRACT

The possibility of generating a physical ρ meson in ππ p wave by the N/D method with no CDD pole is reinvestigated. A general model with some dual properties is introduced to represent high-energy behavior of the input potential, so the arbitrary cutoff is eliminated. This model requires that the Pomeranchuk intercept be less than one, and shows that the high-energy behavior of the potential is dominated by the Pomeranchuk trajectory. This model also gives a repulsive potential in the high energy region, and suggests the necessity of CDD poles to generate a ρ meson with correct mass. Also in the absence of CDD poles, it is shown that near-by singularities mainly control the width of the output particle and distant singularities mainly determine its mass.

I. INTRODUCTION

The problem of generating the I = 1, ππ p-wave resonance (ρ meson) has been investigated many times by a variety of methods since the original study by Chew and Mandelstam.[1] Many investigations[2-11] have been based on the assumed absence of CDD poles,[12] but with the failure of such attempts to generate a narrow-width ρ meson, the importance of inelastic channels as revealed by multi-channel calculations,[13,14] and the progress made in understanding the connection between CDD poles and inelastic resonances,[15-18] the speculation grows that CDD poles are necessary to generate a narrow width ρ meson.

Previous analysis of the no-CDD-pole problem nevertheless suffers from certain defects. The most common defect is the failure to recognize the importance of inelasticity and the effect of distant singularities even if CDD poles are assumed absent.[19] A second point to keep in mind concerning such calculations is that both near-by and distant "input" singularities lie in the unphysical region and are usually model dependent. A third point is that some of the previous investigations involve numerical calculations, which leave uncertain the possibility of improving the output result. Therefore even though the collection of devices heretofore employed to generate a narrow width ρ meson with no CDD pole has failed, all possibilities may still not have been exhausted. In this article we will review the problem of the ρ as a ππ composite in a qualitative but analytic way based on the N/D method, in order to investigate the possibilities of improving the output width of the ρ meson and to see whether or not we can generate a ρ meson at its physical mass with no CDD poles.

We accept the fact that both near-by and distant singularities are model dependent, and we begin by representing both with arbitrary

parameters. The elementary ρ-meson exchange model is used as a standard to compare with other models of near-by singularities. By requiring a ρ meson of correct width and mass to be generated in the $\pi\pi$ p-wave amplitude, we obtain necessary conditions which must be satisfied by near-by and distant singularities. This analysis is performed in Sec. 3. The condition for near-by singularities is that the input potential attraction at low energy, which is assumed to be dominated by near-by singularities, must be substantially weaker than the attraction from the elementary ρ-meson exchange potential. (The elementary ρ-meson exchange potential generates a ρ meson with a width too large by a factor ≈ 3, when distant singularities are adjusted to give correct ρ mass, 750 MeV.) Through the analysis of Sec. 3, it is shown that the mass of the output resonance is mainly controlled by distant singularities, whereas the output width is mainly determined by near-by singularities.

In Sec. 4 we consider models that represent near-by singularities by cross-channel Regge poles. The original study of Regge-pole exchange potential in Ref. (20) was discussed in the language of the new form of the strip approximation.[6] In our formulation of the Regge-pole exchange potential, which only represents near-by singularities, reference to the strip approximation can be avoided. The essential problem of such a model is how to modify the cross-channel Regge-pole terms in order to let them have correct boundaries of double spectral functions. In principle this can be done by the Chew-Jones representation,[6] but it is very difficult to perform an analytic calculation suitable for a qualitative discussion. We only consider ρ and Pomeranchuk trajectories, assuming that the $J = 1$ state, i.e., ρ meson, is dominant in the Regge ρ-exchange potential, while the $J = 0$

state, i.e., the ghost with a proper ghost killing factor, dominates the Pomeranchuk exchange potential. An approximation can be made based on these assumptions, and we can restore the correct boundary of the Regge-pole terms through the Froissart-Gribov projection.[21,22] This result can be written down analytically, and the contributions of the Regge ρ-exchange potential and Pomeranchuk exchange potential at the $4\mu^2$ threshold are compared with that of elementary ρ-exchange potentials.

Our analysis in Sec. 4 implies that Regge-pole exchange potentials for the $\pi\pi$ p wave depend very much on the t dependence of Regge residues and on the slope of Regge trajectories. In the case of a ρ-Regge-pole exchange potential with the residue function and trajectory given by the $\pi\pi \to \pi\pi$ Veneziano model,[23-26] the attraction of the potential at low energy remains almost the same as that of the elementary ρ-meson exchange potential, so introduction of the Regge ρ-exchange potential will not help to reduce the output ρ width. The Pomeranchuk exchange potential for the $\pi\pi$ p wave, with a "normal" residue function and trajectory slope (as, for example, in the model of Wong[27]), is attractive rather than repulsive in the low energy, so the output ρ width will be increased further by including this potential. This result coincides qualitatively with that of Ref. (28). But if the slope of the Pomeranchuk trajectory becomes small or the residue decreases abnormally fast away from the forward direction, the Pomeranchuk exchange potential for the $\pi\pi$ p-wave can become repulsive in the low-energy region and may reduce the output width.

In Sec. 5 we introduce a model, which bases on some general dual properties extrapolated from the $\pi\pi$ Veneziano model[23-26] for the ρ-f_0 trajectory, and then generalize this model to include a

Pomeranchuk trajectory. Our result shows that the high-energy behavior of the input potential, which is controlled by distant singularities, is dominated by the Pomeranchuk trajectory. Therefore the attempts[29,30] to obtain a potential from the $\pi\pi$ Veneziano model with no Pomeranchuk trajectory are not enough to conclude the necessity of CDD poles to generate a ρ meson with correct mass.

The model for distant singularities consists of the following three assumptions;

(1) A definite isospin $\pi\pi$ elastic amplitude is built up from a linear combination of three functions A_{st}, A_{su}, and A_{tu}, where A_{st} contains only s- and t-channel singularities but no u-channel singularities, and A_{su} and A_{tu} have corresponding properties. The function A_{st} has leading Regge asymptotic behavior at the limit s (or t) $\to \pm\infty$, t (or s) fixed but damps out faster than these leading Regge asymptotic behaviors at the limit s (or t) $\to \pm\infty$, u fixed. The functions A_{su} and A_{st} have corresponding asymptotic behaviors.

(2) The Regge asymptotic behavior of A_{st} at the limit s (or t) $\to +\infty$, with t (or s) fixed is generated by s-(or t-) channel singularities. The functions A_{su} and A_{tu} have corresponding properties.

(3) A dispersion relation with a finite number of subtractions is satisfied by the reduced partial-wave amplitudes.

We show in Sec. 5 that this model leads to the conclusion that the Pomeranchuk trajectory intercept must be less than one; otherwise the three assumptions become inconsistent. Further, we show that the Pomeranchuk contribution to distant singularities causes a repulsive

high-energy behavior, whereas the analysis of Sec. 3 requires an overall attractive potential in the region above the low-energy resonances to generate a physical ρ meson. This suggests the necessity of CDD poles in this model.

In the Appendix a detailed formulation is given of the N/D integral equation with no CDD poles.

II. FORMULATION OF THE PROBLEM

In this section we will write the N/D integral equation for the $\pi\pi$ p wave and also define for later usage the elementary ρ-meson exchange potential. Certain detailed aspects of the N/D integral equation are discussed in the Appendix.

The $\ell = 1$, $\pi\pi$-partial wave amplitude $A_1(s)$ is defined as

$$A_1(s) = \frac{1}{32\pi} \oint_{-1}^{1} dz_s \, P_1(z_s) \, A^{I=1}(s,z_s), \tag{2.1}$$

where the end points of the contour of the above integration on the z_s plane are fixed at $z_s = \pm 1$, but the contour may deviate from the real z_s axis if t or u singularities of $A^{I=1}(s,z_s)$ cross the physical z_s interval as s is continued away from the s-channel physical region. A reduced partial wave amplitude $B_1(s)$ is defined as[1]

$$B_1(s) = \frac{A_1(s)}{q_s^2}, \qquad q_s^2 = \frac{s - 4\mu^2}{4}.$$

We assume that $B_1(s)$ satisfies a partial-wave dispersion relation with no subtraction:

$$B_1(s) = \frac{1}{2\pi i} \int_{\text{L.H.C.}} ds' \, \frac{\text{Disc.}[B_1(s')]}{s' - s} + \frac{1}{2\pi i} \int_{4\mu^2}^{\infty} ds' \times \frac{\text{Disc.}[B_1(s')]}{s' - s}, \tag{2.1'}$$

where the contour $\int_{\text{L.H.C.}} ds'$ is taken around the left-hand singularities generated by the coincidence of t and u singularities with

the end points $z_s = +1$ in Eq. (2.1) fixed. The potential $V_1(s)$ is defined as

$$V_1(s) = \frac{1}{2\pi i} \int_{\text{L.H.C.}} ds' \, \frac{\text{Disc.}[B_1(s')]}{s' - s}. \tag{2.2}$$

We write the partial-wave unitarity relation through the R-function method[1]:

$$\left.\begin{aligned} \text{Im}\left[\frac{1}{B_1(s)}\right] &= -\rho_1(s) \, R_1(s), \\ \rho_1(s) &= \frac{s - 4\mu^2}{4} \left(\frac{s - 4\mu^2}{s}\right)^{\frac{1}{2}}, \\ R_1(s) &= \frac{\text{Im } B_1(s)}{\rho_1(s)|B_1(s)|^2} = \frac{\sigma^{\text{total}}_{\ell=1}(s)}{\sigma^{\text{elastic}}_{\ell=1}(s)} \end{aligned}\right\}. \tag{2.3}$$

In the absence of CDD poles, the N/D integral equation from the R-function[1] method and that from Frye-Warnock's method[31] are equivalent. We choose the R-function method, because its form is simpler.

We assume the absence of CDD poles, that is, our solution is assumed to be uniquely determined by a knowledge of $V_1(s)$ and $R_1(s)$. This assumption can be formulated in the following precise statements[19] through the N/D method,[1]

(a) The decomposition $B_1(s) = [N_1(s)]/[D_1(s)]$ can be made in a way such that $N_1(s)$ contains only the left-hand cut and $D_1(s)$ contains only the cuts from $4\mu^2$ to $+\infty$, and the zeros of $D_1(s)$ are in one-to-one correspondence with the poles of the amplitude $B_1(s)$.

(b) The function $N_1(s)$ satisfies a dispersion relation with no subtraction, and $D_1(s)$ satisfies a dispersion relation with one subtraction. No poles are present in either $N_1(s)$ or $D_1(s)$.

(c) The N/D integral equation constructed by Uretsky's method[32,33] is Fredholm.

From the above statements (a) - (c), the N/D integral equation can be constructed as

$$\left.\begin{aligned} D_1(s) &= 1 - \frac{s}{\pi}\int_{4\mu^2}^{\infty} ds' \ \frac{\rho_1(s')\, R_1(s')\, N_1(s')}{s'(s'-s)} \ , \\ h_1^N(s) &= N_1(s)\left(\frac{\rho_1(s)\, R_1(s)}{s}\right)^{\frac{1}{2}} , \\ h_1^N(s) &= V_1(s)\, D_1(s)\left(\frac{\rho_1(s)\, R_1(s)}{s}\right)^{\frac{1}{2}} + \int_{4\mu^2}^{\infty} ds'\, K_1(s;s') \\ &\qquad \times h_1^N(s') \ , \\ K_1(s;s') &= \frac{1}{\pi}\left(\frac{\rho_1(s)\, R_1(s)}{s}\cdot\frac{\rho_1(s')\, R_1(s')}{s'}\right)^{\frac{1}{2}} \\ &\qquad \cdot \frac{s'\, V_1(s') - s\, V_1(s)}{s'-s} \ , \end{aligned}\right\} \tag{2.4}$$

where we have normalized the subtraction constant of the dispersion relation for $D_1(s)$ to one, since both $D_1(s)$ and $N_1(s)$ are proportional to that subtraction constant and the quotient $N_1(s)/D_1(s)$ is independent of it. The condition to make the integral equation for $h_1^N(s)$ Fredholm is

$$\left.\begin{aligned} &\int_{4\mu^2}^{\infty} ds \int_{4\mu^2}^{\infty} ds' \left\{ \frac{\rho_1(s) R_1(s) \rho_1(s') R_1(s')}{ss'} \cdot \left| \frac{s' V_1(s') - s V_1(s)}{s' - s} \right|^2 \right\} < +\infty \\ &\text{and} \\ &\int_{4\mu^2}^{\infty} ds \ \frac{\rho_1(s)\, R_1(s)}{s} \cdot |V_1(s)|^2 < +\infty . \end{aligned}\right\} \tag{2.5}$$

The condition of Eq. (2.5) is satisfied for the input $R_1(s)$ and $V_1(s)$ if they are constrained by

$$\left.\begin{aligned} &\lim_{s\to+\infty} sV_1(s) < +\infty \\ &\lim_{s\to+\infty} \frac{R_1(s)}{s^{1-\epsilon}} = 0 \qquad \text{for some positive } \epsilon . \end{aligned}\right\} \tag{2.6}$$

In this article we consider only the input information which satisfies Eq. (2.6).

As discussed above, the left-hand singularities in Eq. (1) are generated when the cross channel singularities in $A^{I=1}(s,z_s)$ in the z_s plane encounter the end points $z_s = \pm 1$. We assume that the low-energy behavior of the potential $V_1(s)$ defined by Eq. (2.2) is dominated by near-by singularities. Near-by singularities may be evaluated by approximating the amplitude $A^{I=1}(s,z_s)$ by some explicit function and partial wave projecting on the s-channel p-wave amplitude as in Eq. (2.1) to find the discontinuity across the left-hand cut.

The elementary ρ-meson exchange potential is derived by approximating the s-channel isospin one $\pi\pi$ elastic amplitude by one t-channel and one u-channel ρ-meson term,

$$A_\rho^{I=1}(s,z_s) = \frac{1}{2}\left\{\frac{(g_{\pi\pi\rho})^2\, T^\alpha(t)\, P_1(z_t)}{\mathrm{Re}\, m_\rho^2 - t} + \frac{(g_{\pi\pi\rho})^2\, T^\alpha(u)\, P_1(z_u)}{\mathrm{Re}\, m_\rho^2 - u}\right\}, \tag{2.7}$$

where

$$z_t = 1 + \frac{2s}{t - 4\mu^2},$$

$$z_u = 1 + \frac{2t}{u - 4\mu^2},$$

$$z_s = 1 + \frac{2t}{s - 4\mu^2}.$$

The factor $1/2$ is the isospin crossing matrix element,[1] and the function $T^\alpha(x)$ is the threshold function. We define $T^\alpha(x)$ as

$$T^\alpha(x) = q_x^2/q^2_{x=\mathrm{Re}m_\rho^2} \qquad \text{for } x \text{ near } 4\mu^2$$

and

$$\lim_{x\to\pm\infty} T^\alpha(x) = \pm 1.$$

The $\pi\pi\rho$ coupling constant $g^2_{\pi\pi\rho}$ is defined by

$$\frac{(g_{\pi\pi\rho})^2}{32\pi} = \frac{|\mathrm{Im}\, m_\rho^2|\cdot[\mathrm{Re}\, m_\rho^2]^{\frac{1}{2}}}{2[\mathrm{Re}\, m_\rho^2 - 4\mu^2]^{\frac{1}{2}}}. \tag{2.8}$$

By partial-wave projecting the expression for $A_\rho^{I=1}(s,z_s)$ in Eq. (2.7) we can see that there are left-hand cuts starting from $s = 4\mu^2 - \mathrm{Re}\, m_\rho^2$, but no right-hand singularities. Therefore the elementary ρ-meson exchange potential $V_1^{e\ell\cdot\rho}(s)$ can be defined directly as

$$V_1^{e\ell\cdot\rho}(s) = \frac{1}{q_s^2}\cdot\frac{1}{32\pi}\int_{-1}^{1} dz_s\, A_\rho^{I=1}(s,z_s). \tag{2.9}$$

The potential can be evaluated explicitly, and it varies slowly through the low-energy region from numerical calculation. Its threshold value and asymptotic behavior are

$$V_1^{e\ell\cdot\rho}(4\mu^2) = \frac{(g_{\pi\pi\rho})^2}{32\pi}\cdot\frac{4}{3}\cdot\frac{1}{\mathrm{Re}\, m_\rho^2}\cdot\frac{1}{\mathrm{Re}\, m_\rho^2 - 4\mu^2}\cdot\left(1 + \frac{4\mu^2}{\mathrm{Re}\, m_\rho^2}\right), \tag{2.10}$$

and

$$\lim_{s\to+\infty} sV_1^{e\ell\cdot\rho}(s) = \frac{(g_{\pi\pi\rho})^2}{32\pi}\cdot\frac{20}{\mathrm{Re}\, m_\rho^2 - 4\mu^2}\cdot\left\{\ell n\, 2 + \ell n\left(1 - \frac{4\mu^2}{\mathrm{Re}\, m_\rho^2}\right)\right\}.$$

We note that there are alternative methods for defining the elementary ρ-meson exchange potential, but the behavior at low energy is always roughly equivalent to that of our potential.

The above-defined elementary ρ-meson exchange potential will serve as the standard in this article when we compare various other models for representing near-by singularities, especially the Regge-pole exchange potential in Sec. 4.

III. ANALYSIS OF THE RELATIVE ROLES OF NEAR-BY AND DISTANT SINGULARITIES

We have stated that both near-by and distant singularities are model dependent. In this section we will approximately represent both types of singularities through arbitrary parameters, and investigate the relative role of near-by and distant singularities in the solution. We shall find necessary conditions which the parameters must satisfy in order to generate a ρ meson with physical mass and width.

We assume

$$V_1(s) = \begin{cases} a_1 & \text{for } 4\mu^2 < s < s_c \\ \dfrac{b_1}{s} & \text{for } s_c < s \end{cases} , \tag{3.1}$$

$$b_1 = a_1 s_c .$$

The motivation for such an assumption is that the elementary ρ-meson exchange potential in Sec. 2 causes a relatively slow variation in the low-energy region, and the asymptotic behavior b_1/s is consistent with the asymptotic behavior of the reduced partial-wave amplitude given by Regge theory. In addition, the parameter b_1 can be chosen very large or very small to represent small deviations from an exact $1/s$ asymptotic behavior. By the assumption of Eq. (3.1), we can approximate the kernel $K_1(s; s')$ of Eq. (2.4) as

$$\frac{\pi K_1(s; s')}{\left[\dfrac{\rho_1(s) R_1(s) \rho_1(s') R_1(s')}{ss'}\right]^{\frac{1}{2}}} \approx \begin{cases} a_1 & \text{for } 4\mu^2 < s < s_c, \quad 4\mu^2 < s' < s_c , \\ \dfrac{b_1}{s'} & \text{for } 4\mu^2 < s < s_c, \quad s_c < s' , \\ \dfrac{b_1}{s} & \text{for } s_c < s, \quad 4\mu^2 < s' < s_c . \\ 0 & \text{for } s_c < s, \quad s_c < s' . \end{cases} \tag{3.2}$$

With the kernel approximated by Eq. (3.2), the integral equation for $h_1^N(s)$, i.e., Eq. (2.4), can be solved explicitly to give

$$\operatorname{Re} D_1(s) = 1 - \frac{s}{\pi} \cdot \left(a_1 + \frac{a_1 x}{\pi} + \frac{b_1 Y}{\pi} \right) \cdot I_1(s; s_c) - \frac{s}{\pi} \cdot \left(b_1 + \frac{b_1 x}{\pi} + \frac{b_1 Y}{\pi} \right) \cdot I_2(s; s_c), \tag{3.3}$$

$$\operatorname{Im} D_1(s) = -s\, p_1^2(s) \cdot \left(a_1 + \frac{a_1 x}{\pi} + \frac{b_1 Y}{\pi} \right) \quad \text{for } 4\mu^2 < s < s_c,$$

where

$$p_1^2(s) = \frac{\rho_1(s) R_1(s)}{s} ,$$

$$I_1(s; s_c) = P \int_{4\mu^2}^{s_c} ds' \, \frac{p_1^2(s')}{s' - s} ,$$

$$I_2(s; s_c) = \int_{s_c}^{\infty} ds' \, \frac{p_1^2(s')}{s'(s' - s)} ,$$

and

$$X = \frac{\pi E_1(b_1^2 E_2 + \pi a_1)}{\pi(\pi - a_1 E_1) - b_1^2 E_1 E_2} ,$$

$$Y = \frac{\pi^2 b_1 E_2}{\pi(\pi - a_1 E_1) - b_1^2 E_1 E_2} ,$$

$$E_1 = \int_{4\mu^2}^{s_c} ds\, p_1^2(s) ,$$

$$E_2 = \int_{s_c}^{\infty} ds\, \frac{p_1^2(s)}{s^2} .$$

We assume

$$R_1(s) = \begin{cases} 1 & \text{for } 4\mu^2 < s < s_d , \\ \frac{1}{s_e - s_d}\cdot\{(R_\infty - 1)s + (s_e - R_\infty s_d)\} & \text{for } s_d < s < s_e , \\ R_\infty & \text{for } s_e < s. \end{cases} \qquad (3.4)$$

This assumption is consistent with experimental data.[34] We must note that the form assumed in Eq. (3.4) does not prohibit a possible logarithmic dependence of the energy s at high energy, but we approximate it by a constant. From experimental evidence about the $\pi\pi$ p-wave phase shift and the inelastic factor,[34] we put

$$s_d = 70\,\mu^2, \qquad s_e = 100\,\mu^2, \qquad R_\infty = 5. \qquad (3.5)$$

We only need to consider the case $s_e < s_c$, since if $s_c < s_e$, distant singularities turn out to be too weak to generate a ρ meson at 750 MeV. We put

$$\mathrm{Re}\, m_\rho^2 = 30\,\mu^2,$$

and require

$$\mathrm{Re}\, D_1(30\,\mu^2) = 0, \qquad (3.6)$$

which corresponds to a requirement that a ρ-meson pole be generated at about 750 MeV. The output width Γ_{out} is given by

$$\Gamma_{out} = -\frac{1}{30\,\mu^2}\cdot\frac{\mathrm{Im}\, D_1(30\,\mu^2)}{\left[\frac{d}{ds}\,\mathrm{Re}\, D_1(s)\right]_{s=30\mu^2}} . \qquad (3.7)$$

In Eqs. (3.6) and (3.7) the right-hand sides are both functions of a_1 and s_c (or b_1). Table I shows combinations of a_1 and s_c that satisfy Eq. (3.6), i.e., that give the correct ρ-meson mass. The corresponding values of the output width are also shown.

From the results in Table I we observe that in order to generate a ρ meson with mass 750 MeV and width 120 MeV, a_1 must be much weaker than $V_1^{el\cdot p}(4\mu^2) \approx 5.1 \times 10^{-3}/\mu^2$. Furthermore, in the region of a_1 where the output width is not far from 120 MeV, b_1 varies slowly even though the changes in a_1 are large. These observations imply the following three qualitative properties;

(1) The mass of the output ρ meson is mainly controlled by the distant singularities.

(2) The width of the output ρ meson is mainly controlled by the near-by singularities.

(3) If we assume that a physical ρ-meson can be generated through the present model, i.e., an N/D equation with no CDD poles, the attraction of the potential at low energy, which is caused by near-by singularities, should be much weaker than that of the elementary ρ-meson exchange potential.

In the following sections some models to represent near-by and distant singularities are considered.

IV. REGGE-POLE EXCHANGE POTENTIALS[20]

In this section, a model to represent near-by singularities by cross-channel Regge poles is considered. Such a model is called the Regge pole exchange potential.[20] We consider only the ρ and Pomeranchuk exchange potentials in this section. In Ref. 20, Regge-pole exchange potentials are formulated in the language of the new form of the strip approximation,[6,35] and they represent both distant and near-by singularities. But the idea of Regge-pole exchange potentials for the representation of near-by singularities can be formulated without referring to the strip approximation. The essence of this model is to approximate the $\pi\pi$ elastic amplitude by several t- and u-channel Regge poles, and then to project them onto the s-channel p-wave amplitude to find their contribution to the discontinuity across the left-hand cut. The difficulty arises due to the fact that an ordinary Regge pole term does not have the correct boundary for its double spectral functions, and a naive s-channel partial-wave projection of such a term will pick up some unphysical near-by left-hand singularities. In principle this difficulty can be resolved by modifying the ordinary Regge pole term through the Chew-Jones representation[6] in order to restore the correct boundary of double spectral functions. Such a process will involve extensive numerical calculations,[28,36-38] and is not appropriate for our qualitative discussion. Instead we make the following approximation to simplify the calculation.

We first consider a t-channel ρ-Regge pole, which can be written as

$$A_\rho(t,z_t) = -\frac{G_\rho(t)}{\sin \pi \alpha_\rho(t)} \cdot \frac{1}{2} \cdot \{P_{\alpha_\rho(t)}(-z_t) - P_{\alpha_\rho(t)}(z_t)\}. \tag{4.1}$$

We may approximate $A_\rho(t,z_t)$ by the lowest nonvanishing partial wave amplitude, t-channel p wave in this case, and write it as

$$A_\rho(t,z_t) \approx 16\pi \times 3\, A_1^\rho(t)\, P_1(z_t), \tag{4.2}$$

where $A_1^\rho(t)$ is calculated from Eq. (4.1) through Froissart-Gribov projection.[21,22] We may argue alternatively that since ρ-meson contribution is dominant over other higher spin resonances on the ρ trajectory, we may approximate $A_\rho(t,z_t)$ as in Eq. (4.2).

To accomplish this projection we begin by rewriting Eq. (4.1) into a definite signature form

$$A_\rho^-(t,z_t) = -\frac{G_\rho(t)}{\sin \pi\alpha_\rho(t)} \cdot P_{\alpha_\rho(t)}(-z_t).$$

The discontinuity of the above expression with respect to z_t is

$$D_{R(z_t)}^-(t,z_t) = \begin{cases} G_\rho(t)\, P_{\alpha_\rho(t)}(z_t) & \text{for } z_t > 1, \\ 0 & \text{for } z_t \leqslant 1. \end{cases} \tag{4.3}$$

The physical discontinuity should vanish in the region

$$1 \leqslant z_t \leqslant z_t(t,s=4\mu^2),$$

in order to give the correct boundary of double spectral functions. The Froissart-Gribov projection implies

$$A_\ell^\rho(t) = \frac{1}{16\pi^2} \cdot \int_{z_t(t,4\mu^2)}^{\infty} dz'\, Q_\ell(z')\, D_{R(z_t)}(t,z'),$$

so the correct boundary is restored for the partial wave amplitude. Substituting Eq. (4.3) into the right-hand side for the case $\ell = 1$, we have

$$A_1^\rho(t) = \frac{1}{16\pi^2} \cdot G_\rho(t) \cdot \int_{z_t(t,4\mu^2)}^{\infty} dz'\, Q_1(z')\, P_{\alpha_\rho(t)}(z').$$

This can be evaluated as[39]

$$A_1^\rho(t) = \frac{1}{16\pi^2} \cdot G_\rho(t) \cdot \frac{F_\rho(t)}{\{1 - \alpha_\rho(t)\}\cdot\{2 + \alpha_\rho(t)\}},$$

$$F_\rho(t) = [(x^2 - 1)\cdot\{P_{\alpha_\rho(t)}(x)\cdot Q_1'(x) - Q_1(x)\cdot P_{\alpha_\rho(t)}'(x)\}]_{x=z_t(t,4\mu^2)}^{x=\infty}. \tag{4.4}$$

Since

$$z_t(t,4\mu^2) = 1 + \frac{8\mu^2}{t - 4\mu^2} \xrightarrow[t\to 4\mu^2+0]{} +\infty,$$

by the behavior[39]

$$Q_\ell(x) \xrightarrow[x\to+\infty]{} \frac{(\pi)^{\frac{1}{2}}}{2^{\ell+1}} \cdot \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} \cdot (x)^{-\ell-1},$$

$$P_\nu(x) \xrightarrow[x\to+\infty]{} \frac{2^\nu}{(\pi)^{\frac{1}{2}}} \cdot \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} \cdot (x)^\nu,$$

we have

$$F_\rho(t = 4\mu^2 + \Delta) = (2)^{\alpha_\rho(t)-2} \cdot \frac{\Gamma(2)\,\Gamma\left(\alpha_\rho(t) + \frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\alpha_\rho(t) + 1\right)} \cdot \{\alpha_\rho(t) + 2\}$$

$$\cdot \left(\frac{8\mu^2}{\Delta}\right)^{\alpha_\rho(t)-1} .$$

Defining a reduced residue $\tilde{G}_\rho(t)$ as

$$\tilde{G}_\rho(t) = \left(\frac{2\mu^2}{q_t^2}\right)^{\alpha_\rho(t)} \cdot G_\rho(t) = \left(\frac{8\mu^2}{\Delta}\right)^{\alpha_\rho(t)} \cdot G_\rho(t),$$

we may rewrite Eq. (4.4) as

$$A_1^\rho(t) = \frac{\tilde{G}_\rho(t)}{16\pi^2} \cdot \frac{4}{3} \cdot \frac{1}{(\pi)^{\frac{1}{2}}} \cdot (2)^{\alpha_\rho(t)-2} \cdot \frac{\Gamma\left(\alpha_\rho(t) + \frac{1}{2}\right)}{\Gamma\left(\alpha_\rho(t) + 1\right)} \cdot \frac{t - 4\mu^2}{8\mu^2}$$

$$\cdot \frac{1}{1 - \alpha_\rho(t)} . \qquad (4.5)$$

Substituting Eq. (4.5) into Eq. (4.2), we have

$$A_\rho(t,z_t) = \frac{4}{(\pi)^{3/2}} \cdot \tilde{G}_\rho(t) \cdot \frac{\Gamma\left(\alpha_\rho(t) + \frac{1}{2}\right)}{\Gamma\left(\alpha_\rho(t) + 1\right)} \cdot (2)^{\alpha_\rho(t)-2}$$

$$\cdot \left(\frac{t - 4\mu^2}{8\mu^2}\right) \cdot \frac{P_1(z_t)}{1 - \alpha_\rho(t)} . \qquad (4.6)$$

We note that in Eq. (4.6) the right-hand side has no s singularities, and it has t singularities only for $t > 4\mu^2$. Therefore the Regge ρ-exchange potential $V_1^{R\cdot\rho}(s)$ may be defined as

$$V_1^{R\cdot\rho}(s) = \frac{1}{q_s^2} \cdot \frac{1}{32\pi} \int_{-1}^{1} dz_s\, P_1(z_s) \times \frac{1}{2}\{A_\rho(t,z_t) + A_\rho(u,z_u)\}. \qquad (4.7)$$

By evaluating Eq. (4.7) explicitly with the substitution of Eq. (4.6), its threshold value is

$$V_1^{R\cdot\rho}(4\mu^2) = \frac{1}{16\pi^2} \cdot \frac{\beta_\rho(0)}{1 - \alpha_\rho(0)} \cdot \left\{\frac{\beta_\rho'(0)}{\beta_\rho(0)} + \frac{\alpha_\rho'(0)}{1 - \alpha_\rho(0)} + \frac{1}{4\mu^2}\right\} , \qquad (4.8)$$

$$\beta_\rho(t) = (2)^{\alpha_\rho(t)-2} \cdot \frac{4}{3(\pi)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\alpha_\rho(t) + \frac{1}{2}\right)}{\Gamma\left(\alpha_\rho(t) + 1\right)} \cdot \tilde{G}_\rho(t) .$$

We approximate $\tilde{G}_\rho(t)$ by comparing the ρ-Regge pole term of Eq. (4.1) in the limit $s \to +\infty$, t fixed, with the corresponding limit of the "standard" Veneziano model for the $\pi\pi$ elastic scattering amplitude (with ρ-f_0 trajectory only but no Pomeranchuk trajectory).[23-26] We find

$$\tilde{G}_\rho(t) = \frac{\beta(\pi)^{3/2}}{\Gamma\left(\alpha_\rho(t)\right)} \cdot \frac{\Gamma\left(\alpha_\rho(t) + 1\right)}{\Gamma\left(\alpha_\rho(t) + \frac{1}{2}\right)} \cdot (2\mu^2 a)^{\alpha_\rho(t)} , \qquad (4.9)$$

where

$$\alpha_\rho(t) = at + b ,$$

$$a = \frac{1}{2\ \mathrm{Re}\ m_\rho^2} , \qquad (4.10)$$

$$b = \frac{1}{2} ,$$

$$\beta = \frac{(g_{\pi\pi\rho})^2}{\mathrm{Re}\ m_\rho^2 - 4\mu^2} .$$

Substituting Eqs. (4.9) and (4.10) into Eq. (4.8), we obtain

$$V_1^{R\cdot\rho}(4\mu^2) = \frac{(g_{\pi\pi\rho})^2}{32(\pi)^{3/2}} \cdot \frac{4}{3} \cdot \left(\frac{2\mu^2}{\text{Re } m_\rho^2}\right)^{\frac{1}{2}} \cdot \frac{1}{\text{Re } m_\rho^2 - 4\mu^2}$$

$$\cdot \left\{ \frac{2 - \frac{1}{2}\,\ell n \frac{\text{Re } m_\rho^2}{2\mu^2}}{\text{Re } m_\rho^2} + \frac{1}{4\mu^2} \right\} . \qquad (4.11)$$

Comparing $V_1^{R\cdot\rho}(4\mu^2)$ with $V_1^{e\ell\cdot\rho}(4\mu^2)$ of Eq. (2.10), and taking

$$\text{Re } m_\rho^2 = 30\,\mu^2,$$

we have

$$V_1^{R\cdot\rho}(4\mu^2) \approx 1.05\, V_1^{e\ell\cdot\rho}(4\mu^2) .$$

This result implies that even if we replace the elementary ρ-meson exchange potential by the Regge ρ-exchange potential, the width of the output ρ-meson does not improve. This conclusion agrees with the result of Ref. 28 and Ref. 37. We note that our result is qualitatively different from that of Ref. 20, since the contribution of a Regge-pole exchange potential to an s-channel partial wave amplitude depends critically on whether the angular momentum of the s-channel partial wave is even or odd. The conclusion of Ref. 20 is only applicable to the potential for an even angular momentum partial wave. In our model, if we calculate the ρ-exchange potential for the $\pi\pi$ s wave, we also get

$$V_{\ell=0}^{R\cdot\rho}(4\mu^2) \approx 0.65\, V_{\ell=0}^{e\ell\cdot\rho}(4\mu^2) .$$

The Pomeranchuk exchange potential can be derived in a way similar to that of the Regge ρ-exchange potential. This time we assume that the $J = 0$ ghost with a proper ghost killing factor is dominant, and approximate the t-channel Pomeranchuk term

$$A_P(t,z_t) = -\frac{G_P(t)}{\sin \pi\, \alpha_P(t)} \cdot P_{\alpha_P(t)}(-z_t) \qquad (4.12)$$

by the expression

$$A_P(t,z_t) \approx 16\, \pi\, A_0^P(t) .$$

By a calculation similar to that for the ρ-exchange potential, we have

$$A_P(t,z_t) = -\frac{\beta_P(t)}{\alpha_P(t)} \cdot \frac{1}{\pi}$$

$$\beta_P(t) = (2)^{\alpha_P(t)-1} \cdot \frac{\Gamma\left(\alpha_P(t) + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\alpha_P(t) + 1\right)} \cdot \tilde{G}_P(t), \qquad (4.13)$$

$$\tilde{G}_P(t) = \left(\frac{8\mu^2}{t - 4\mu^2}\right)^{\alpha_P(t)} \cdot G_P(t) .$$

The Pomeranchuk exchange potential is defined as

$$V_1^P(s) = \frac{1}{q_s^2} \cdot \frac{1}{32\,\pi} \cdot \frac{1}{3} \int_{-1}^{1} dz_s\, P_1(z_s) \cdot \{A_P(t,z_t) - A_P(u,z_u)\}. \qquad (4.14)$$

Substituting Eq. (4.13) into Eq. (4.14) and evaluating at $s = 4\mu^2$, we have

$$V_1^P(4\mu^2) = -\frac{1}{32\pi^2}\cdot\frac{2}{9}\cdot\frac{\beta_P(0)}{\alpha_P(0)}\cdot\left\{\frac{\beta_P'(0)}{\beta_P(0)}-\frac{\alpha_P'(0)}{\alpha_P(0)}\right\} . \qquad (4.15)$$

We can approximate the residue $G_P(t)$ by comparing with the Regge asymptotic behavior of Wong's model,[27] which is a dual resonance model for the $\pi\pi$ elastic amplitude with a Pomeranchuk trajectory. We then have

$$\beta_P(t) = (2)^{\alpha_P(t)}\cdot\frac{\pi\,\beta_x}{\Gamma(\alpha_P(t))}\cdot(2\mu^2 a)^{\alpha_P(t)} , \qquad (4.16)$$

$$\alpha_P(t) = a_P t + b_P, \qquad a_P = a = \frac{1}{2\ \mathrm{Re}\ m_\rho^2} ,$$

$$b_P = \frac{1}{2} ,$$

$$\frac{\beta_x}{\beta} = 0.56 .$$

Substituting Eq. (4.16) into Eq. (4.15) and comparing with $V_1^{R\cdot\rho}(4\mu^2)$, we have

$$\frac{V_1^P(4\mu^2)}{V_1^{R\cdot\rho}(4\mu^2)} \approx 0.8 \times 10^{-2} \ll 1 .$$

This result implies that the Pomeranchuk exchange potential for the $\pi\pi$ p wave with residue function and trajectory given by Eq. (4.16) is weakly attractive at the threshold; therefore, the inclusion of such a potential will not improve the width of the output ρ meson. This result seems to agree qualitatively with Ref. 28.

We note that in Eq. (4.15) if

$$\frac{\beta_P'(0)}{\beta_P(0)} \gg \frac{\alpha_P'(0)}{\alpha_P(0)} ,$$

i.e., if either the residue decreases, as $|t|$ increases, much faster than in the model of Eq. (4.16) or the slope of the Pomeranchuk trajectory at $t = 0$ is much smaller than that of Eq. (4.16), we can get a repulsive potential at low energy and may reduce the output ρ-meson width.

Again we note that the conclusion of Ref. 20 about the Pomeranchuk exchange potential only applies to the even angular momentum partial wave potential.

V. A MODEL FOR DISTANT SINGULARITIES

From the argument in Sec. 3 that distant singularities mainly control the mass of the output pole and near-by singularities mainly determine the width of the output pole, we can meaningfully study various models to represent distant singularities, even though the nature of the near-by singularities may remain uncertain.

In this section we first present a model with some of its assumptions extrapolated from the Veneziano model for the $\pi\pi$ elastic amplitude with only the ρ-f_0 trajectory,[23-26] and subsequently generalize it to a model with the Pomeranchuk trajectory. We will see that the application of this model to an amplitude with only exchange-degenerated ρ and f_0 trajectories, but no Pomeranchuk trajectory (like the $\pi\pi$ Veneziano model), will lead to a high-energy behavior of the input potential, which is too weak to generate a ρ meson with the correct mass. This result, of course, agrees with the attempts[29,30] to extrapolate a potential for use in N/D calculation from the $\pi\pi$ Veneziano model with only the ρ-f_0 trajectory. By applying our model to an amplitude with both Pomeranchuk trajectory and exchange degenerated ρ-f_0 trajectories, we will see that the high-energy behavior of the potential, which is dominated by distant singularities, is determined by the Pomeranchuk trajectory. Therefore it is not meaningful to question whether CDD poles are necessary or not for generating a ρ-meson with the correct mass if the Pomeranchuk trajectory is not contained.

We find, in the model with both the Pomeranchuk and the exchange degenerated ρ-f_0 trajectory, that the Pomeranchuk intercept must be less than one for our model to be consistent. The high-energy behavior of the potential given by this model is repulsive, whereas

we have shown in Sec. 2 that an attractive behavior is required to generate a ρ meson with the correct mass. This will imply the necessity of inclusion of CDD poles in this model.

The model consists of the following three assumptions;

(1) A definite isospin $\pi\pi$ elastic-scattering amplitude is built up from a linear combination of three functions A_{st}, A_{su}, and A_{tu}, where A_{st} contains only s- and t-channel singularities but no u-channel singularities. The functions A_{su} and A_{tu} have corresponding properties. The function A_{st} has the leading Regge asymptotic behavior (they can be that of Pomeranchuk trajectory, or ρ and f_0 trajectories, depending on what kinds of Regge trajectories we contain in this model) in the limit s (or t) $\to \pm\infty$ with t (or s) fixed, but it damps out faster in the limit s (or t) $\to \pm\infty$ with u fixed. The functions A_{su} and A_{tu} have corresponding asymptotic behaviors with the limits changed properly.

(2) The Regge asymptotic behavior of A_{st} in the limit s (or t) $\to +\infty$ with t (or s) fixed is generated by s-(or t-) channel singularities. The functions A_{su} and A_{tu} have corresponding properties.

(3) A dispersion relation with a finite number of subtractions is satisfied by the reduced partial-wave amplitude $B_\ell(s)$. (The reduced $\pi\pi$ p-wave amplitude satisfies a dispersion relation with no subtraction.)

Conditions (1) - (3) can be stated more explicitly as follows: A definite isospin s-channel amplitude has a Regge asymptotic behavior

$$A_s^I \xrightarrow[\substack{s\to+\infty \\ t \text{ fixed}}]{} \frac{\beta(t)}{\sin \pi\alpha(t)} \cdot \{s^{\alpha(t)} + \tau(-s)^{\alpha(t)}\} ,$$

where τ is the signature. Condition (1) means that the term $s^{\alpha(t)}$ comes from the asymptotic behavior of A_{tu}, and the term $(-s)^{\alpha(t)}$ comes from that of A_{st}. Condition (2) further implies that the term $(-s)^{\alpha(t)}$ is generated by s-channel singularities. From condition (3) we can define a potential $V_\ell^I(s)$ as

$$V_\ell^I(s) = \frac{1}{2\pi i} \int_{\text{L.H.C.}} ds' \frac{\text{Disc.}[B_\ell^I(s')]}{s' - s} ;$$

where we did not write the possible subtractions explicitly. We see that $V_\ell^I(s)$ does not contain any s-channel singularities. Condition (2) then implies that the term $(-s)^{\alpha(t)}$ does not contribute to the high-energy behavior of $V_\ell^I(s)$, but the term $s^{\alpha(t)}$ does.

We next consider the "standard" $\pi\pi$ Veneziano model,[23-26] which contains only the exchange degenerated ρ-f_0 trajectory but no Pomeranchuk trajectory. Condition (3) is only true for the isospin one partial wave amplitude[40,41] and the resolution of this difficulty will be discussed later. Here we only consider the isospin one partial-wave amplitudes. In the limit $s \to +\infty$, t fixed, the t-channel isospin zero and two amplitudes have the Regge asymptotic behaviors

$$A_t^{I=0} \xrightarrow[\substack{s\to+\infty \\ t \text{ fixed}}]{} -\frac{3}{2} \cdot \frac{\beta(t)}{\sin \pi\alpha(t)} \cdot \{s^{\alpha(t)} + (-s)^{\alpha(t)}\} ,$$

$$(f_0 \text{ trajectory}) ,$$

$$A_t^{I=1} \xrightarrow[\substack{s\to+\infty \\ t \text{ fixed}}]{} \frac{\beta(t)}{\sin \pi\alpha(t)} \cdot \{s^{\alpha(t)} - (-s)^{\alpha(t)}\}, \quad (\rho \text{ trajectory}).$$

By introducing proper isospin crossing matrix elements,[1] the contribution of these two trajectories to the s-channel isospin one amplitude is

$$A_s^{I=1} = \frac{1}{2} A_t^{I=1} + \frac{1}{3} A_t^{I=0} \xrightarrow[\substack{s\to+\infty \\ t \text{ fixed}}]{} -\frac{\beta(t)}{\sin \pi\alpha(t)} \cdot (-s)^{\alpha(t)} .$$

At the limit $s \to +\infty$ with u fixed, we obtain

$$A_s^{I=1} \xrightarrow[\substack{s\to+\infty \\ u \text{ fixed}}]{} -\frac{\beta(u)}{\sin \pi\alpha(u)} \cdot (-s)^{\alpha(u)} .$$

We see that in the $\pi\pi$ Veneziano model, the terms $s^{\alpha(t)}$ and $s^{\alpha(u)}$ are absent in the asymptotic behavior of the $I = 1$ s-channel amplitude. If the terms $s^{\alpha(t)}$ and $s^{\alpha(u)}$ are present, and $\alpha(t)$ and $\alpha(u)$ are linear trajectories with sufficiently rapid damping residue $\beta(t)$ and $\beta(u)$, they will give the potential $V_{\ell=1}^{I=1}(s)$ an asymptotic behavior $1/s^{2-b}\ \ell n\ s$ (b is the intercept of the ρ-f_0 trajectory) from conditions (1) - (3). The absence of the terms $s^{\alpha(t)}$ and $s^{\alpha(u)}$ then implies

$$\lim_{s\to+\infty} \left\{ V_{\ell=1}^{I=1}(s) \Big/ \frac{1}{s^{2-b}\ \ell n\ s} \right\} = 0. \qquad (5.1)$$

Comparing this result with the analysis of Sec. 3, it is apparent that such a potential cannot generate a ρ-meson with the correct mass.[29,30]

Hereafter we consider a model with both the Pomeranchuk trajectory and the exchange degenerated ρ-f_0 trajectory, which is similar to that of the $\pi\pi$ Veneziano model. Since there is no $I = 1$ exchange-degenerated partner for the Pomeranchuk trajectory, the terms $s^{\alpha_P(t)}$ and $s^{\alpha_P(u)}$ will appear in the asymptotic behavior of $A_s^{I=1}$, which contribute to the high-energy behavior of $V_\ell^{I=1}(s)$ with odd ℓ

according to conditions (1) - (3). Therefore the high-energy behavior of the potential is controlled by the Pomeranchuk trajectory.

We must note that in the $\pi\pi$ Veneziano model the $I = 1$ partial-wave amplitudes satisfy condition (3),[40] but the $I = 0$ and $I = 2$ partial wave amplitudes do not satisfy condition (3).[41] The difference between even and odd isospin amplitudes is due to the presence and the absence of the function corresponding to A_{tu}. But this property of the $\pi\pi$ Veneziano model, i.e., the violation of condition (3) if the function corresponding to A_{tu} is present, does not seem to be essential. We may, for example, assume that a proper unitarization of the Veneziano model, or the addition of proper secondary terms[42] will resolve this difficulty. In the following discussion we always assume condition (3) is satisfied, since we do not need any explicit representation of the amplitudes as in the $\pi\pi$ Veneziano model.

The Regge asymptotic behavior of the Pomeranchuk trajectory for the $I = 1$ s-channel amplitude can be written as

$$A_s^{I=1} \xrightarrow[\substack{s\to+\infty \\ t \text{ fixed}}]{} -\frac{1}{3}\cdot\frac{\beta_P(t)}{\sin\pi\alpha_P(t)}\cdot\{s^{\alpha_P(t)} + (-s)^{\alpha_P(t)}\}$$

$$\xrightarrow[\substack{s\to+\infty \\ u \text{ fixed}}]{} \frac{1}{3}\cdot\frac{\beta_P(u)}{\sin\pi\alpha_P(u)}\cdot\{s^{\alpha_P(u)} + (-s)^{\alpha_P(u)}\},$$

where the factor $1/3$ is the isospin crossing matrix element, and

$$\beta_P(0) > 0.$$

From the previous discussion about condition (1) - (3), we see that the high-energy behavior of $V_{\ell=1}^{I=1}(s)$ is given by

$$V_1(s) \xrightarrow[s\to+\infty]{} \frac{1}{32\pi}\cdot\frac{1}{q_s^2}\cdot\int_{-1}^{1} dz_s\, P_1(z_s)$$

$$\cdot\frac{1}{3}\left\{-\beta_P(t)\cdot\frac{(s)^{\alpha_P(t)}}{\sin\pi\alpha_P(t)} + \beta_P(u)\cdot\frac{(s)^{\alpha_P(u)}}{\sin\pi\alpha_P(u)}\right\}. \qquad (5.2)$$

The right-hand side of Eq. (5.2) is indefinite if

$$\alpha_P(0) = 1.$$

Therefore we must assume

$$\alpha_P(0) = 1 - \epsilon \qquad (\epsilon \text{ positive})$$

in order to make condition (3) consistent with conditions (1) and (2).

We assume that the Pomeranchuk trajectory is linear, then in Eq. (5.2) the t-channel Pomeranchuk trajectory mainly contributes to the forward direction and the u-channel one to the backward direction. Therefore we have

$$\lim_{s\to+\infty} sV_1(s) < 0.$$

This implies that the high-energy behavior of the potential from this model is repulsive, whereas, in Sec. 2, we see that

$$b_1 = \lim_{s\to+\infty} sV_1(s) \approx 0.7 > 0$$

is necessary to generate a ρ meson with mass 750 MeV. Therefore we cannot generate a physical ρ meson in this model.

We must note that the b_1/s behavior of the potential in the high-energy region is only an effective approximation. For example, a potential with sufficiently large attraction in the intermediate energy region and a repulsive high-energy tail can still generate a ρ meson with the correct mass. Such a strong attraction in the intermediate energy can not be considered by the models with no CDD poles. Therefore we may conclude that CDD poles are necessary to generate a physical ρ-meson in the present model to represent distant singularities.

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APPENDIX

In Sec. 2 the three condtions (a), (b), and (c) are quoted as being equivalent to the absence of CDD poles. We will discuss their necessity in some detail in this Appendix. The reduced partial wave amplitude $B_1(s)$ is assumed to satisfy a partial-wave dispersion relation with no subtraction

$$B_1(s) = \frac{A_1(s)}{q_s^{\,2}} ,$$

$$B_1(s) = \frac{1}{2\pi i}\int_{L.H.C.} ds' \frac{\text{Disc.}[B_1(s')]}{s' - s} + \frac{1}{2\pi i}\int_{4\mu^2}^{\infty} ds' \frac{\text{Disc.}[B_1(s')]}{s' - s} , \qquad (A.1)$$

$$V_1(s) = \frac{1}{2\pi i}\int_{L.H.C.} ds' \frac{\text{Disc.}[B_1(s')]}{s' - s} .$$

We only consider the case

$$\lim_{s\to+\infty} sV_1(s) < +\infty.,$$

which is consistent with linear Regge trajectories. The partial-wave unitarity relation in terms of the R-function[1] is

$$\text{Im}\left[\frac{1}{B_1(s)}\right] = -\rho_1(s)\, R_1(s) ,$$

$$\rho_1(s) = \frac{s - 4\mu^2}{4}\left[\frac{s - 4\mu^2}{s}\right]^{\frac{1}{2}} , \qquad (A.2)$$

$$R_1(s) = \frac{\sigma^{\text{total}}_{\ell=1}}{\sigma^{\text{elastic}}_{\ell=1}(s)} = \frac{\text{Im}\, B_1(s)}{\rho_1(s)|B_1(s)|^2} .$$

The absence of CDD poles in the formulation of N/D method is interpreted to mean that the solution of the N/D integral equation is uniquely determined by the input information $R_1(s)$ and $V_1(s)$. This requirement of uniqueness of the solution will be satisfied if the following three conditions (a), (b), and (c) are assumed;

(a) The decomposition $B_1(s) = \frac{N_1(s)}{D_1(s)}$ can be made in a way such that $N_1(s)$ contains only the left-hand cuts and $D_1(s)$ contains only the cuts from $4\mu^2$ to $+\infty$. The zeros of $D_1(s)$ are in one-to-one correspondence with the poles of the amplitude $B_1(s)$.

(b) The function $N_1(s)$ satisfies a dispersion relation with no subtractions and $D_1(s)$ satisfies a dispersion relation with one subtraction. No poles are present in either $D_1(s)$ or $N_1(s)$.

(c) The N/D integral equation constructed using Uretsky[36] and Mandelstam's[37] method is Fredholm.

Condition (c) is necessary from the analysis of Refs. 16 - 18. In order to see the necessity of condition (b), we first construct the N/D integral equation which satisfies conditions (a), (b), and (c), and show that its solution is uniquely determined by the input information $R_1(s)$ and $V_1(s)$; then we show that the solution is no longer unique if condition (b) is violated; i.e., arbitrary constants, which cannot be determined from $R_1(s)$ and $V_1(s)$, will be contained in the solution. In the following discussion we do not use Levinson's theorem,[43] which is violated if Regge trajectories rise indefinitely.[44,45]

A function $C_1(s)$ is defined as

$$C_1(s) = N_1(s) - V_1(s)\, D_1(s).$$

From condition (a) we have

$$\text{Im } D_1(s) = \begin{cases} -\rho_1(s)\, R_1(s)\, N_1(s) & \text{for } s \geq 4\mu^2, \\ 0 & \text{for } s < 4\mu^2, \end{cases}$$

and

$$\text{Im } C_1(s) = \begin{cases} \rho_1(s)\, R_1(s)\, V_1(s)\, N_1(s) & \text{for } s \geq 4\mu^2, \\ 0 & \text{for } s < 4\mu^2 . \end{cases}$$

From the definition of $V_1(s)$ and condition (b), we see that $C_1(s)$ also satisfies a dispersion relation with no subtraction,

$$C_1(s) = N_1(s) - V_1(s)\, D_1(s) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' \, \frac{\rho_1(s')R_1(s')V_1(s')N_1(s')}{s' - s}. \tag{A.3}$$

The once-subtracted dispersion relation for $D_1(s)$ is

$$D_1(s) = D_1(0) - \frac{s}{\pi}\int_{4\mu^2}^{\infty} ds' \, \frac{\rho_1(s')\, R_1(s')\, N_1(s')}{s'(s' - s)}, \tag{A.4}$$

where we have chosen the subtraction point at $s = 0$. The subtraction point is, of course, not significant. If we insist on choosing an arbitrary subtraction point, for example, at $s = s_1$, we need only replace $D_1(0)$ in Eq. (A.4) by

$$\lambda(s_1) = D_1(s_1) - \frac{s_1}{\pi}\int_{4\mu^2}^{\infty} ds' \, \rho_1(s')\, R_1(s')\, N_1(s') \cdot \frac{1}{s'(s' - s)} ,$$

and the following discussion still applies. Since it is not an essential complication, we only consider Eq. (A.4) here. Substituting Eq. (A.4) into Eq. (A.3), we have

$$N_1(s) = D_1(0)\, V_1(s) + \int_{4\mu^2}^{\infty} ds' \, K_1(s; s')\, N_1(s'), \tag{A.5}$$

$$K_1(s; s') = \frac{1}{\pi} \cdot \rho_1(s')\, R_1(s') \cdot \frac{s'\, V_1(s') - sV_1(s)}{s'(s' - s)} .$$

The kernel $K_1(s; s')$ is independent of $D_1(0)$, and is determined completely by the input information $R_1(s)$ and $V_1(s)$. The Fredholm condition (c) can be satisfied by properly choosing $R_1(s)$ and $V_1(s)$. In particular, the conditions

$$\lim_{s \to +\infty} sV_1(s) < +\infty ,$$

$$|R_1(s)| \leq \text{constant} \quad \text{for any } s$$

will guarantee that the integral equation Eq. (A.6) is Fredholm. The resolvent of the kernel $K_1(s; s')$, to be denoted by $H_1(s; s')$, satisfies the integral equation

$$H_1(s; s') = K_1(s; s') + \int_{4\mu^2}^{\infty} ds'' \, K_1(s; s'')\, H_1(s'', s').$$

Since $K_1(s; s')$ is independent of the subtraction constant $D_1(0)$, so is $H_1(s; s')$. From the property of a Fredholm integral equation, the solution of Eq. (A.5) can be written as

$$N_1(s) = D_1(0) \cdot \tilde{N}_1(s),$$

$$\tilde{N}_1(s) = V_1(s) + \int_{4\mu^2}^{\infty} ds' \, H_1(s; s') \, V_1(s'), \tag{A.7}$$

where $\tilde{N}_1(s)$ is independent of $D_1(0)$. By substituting Eq. (A.6) into Eq. (4.4), we obtain

$$D_1(s) = D_1(0) \cdot \tilde{D}_1(s),$$

$$\tilde{D}_1(s) = 1 - \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \, \frac{\rho_1(s') \, R_1(s') \, \tilde{N}_1(s')}{s'(s' - s)} .$$

Again the function $\tilde{D}_1(s)$ is independent of $D_1(0)$. Therefore the quotient

$$B_1(s) = \frac{N_1(s)}{D_1(s)} = \frac{\tilde{N}_1(s)}{\tilde{D}_1(s)}$$

is independent of $D_1(0)$, and is uniquely determined by the input information $R_1(s)$ and $V_1(s)$.

Next we show that if condition (b) is not satisfied, the solution will contain arbitrary constants which cannot be determined by the input information $R_1(s)$ and $V_1(s)$. We consider only the case that $N_1(s)$ satisfies a dispersion relation with one subtraction and $D_1(s)$ satisfies a dispersion relation with two subtractions. Other numbers of subtractions are trivial generalizations of this case. We write $N_1(s)$, $D_1(s)$, and $C_1(s)$ in this case as $N_1^{(1)}(s)$, $D_1^{(1)}(s)$, and $C_1^{(1)}(s)$. From the definition of $C_1^{(1)}(s)$,

$$C_1^{(1)}(s) = N_1^{(1)}(s) - V_1(s) \, D_1^{(1)}(s),$$

and the restriction

$$\lim_{s \to +\infty} sV_1(s) < +\infty,$$

$C_1^{(1)}(s)$ also satisfies a dispersion relation with one subtraction,

$$C_1^{(1)}(s) = N_1^{(1)}(s) - V_1(s) \, D_1^{(1)}(s)$$

$$= \lambda_0 + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \, \frac{\rho_1(s') \, R_1(s') \, V_1(s') \, N_1^{(1)}(s')}{s'(s' - s)} ,$$

where (A.7)

$$\lambda_0 = C_1^{(1)}(0),$$

where we have chosen the subtraction point at $s = 0$ and the generalization to an arbitrary subtraction point is a trivial process as discussed following Eq. (A.4). The dispersion relation for $D_1^{(1)}(s)$ can be written as

$$D_1^{(1)}(s) = \lambda_1 + s\lambda_2 - \frac{s^2}{\pi} \int_{4\mu^2}^{\infty} ds' \, \frac{\rho_1(s') \, R_1(s') \, N_1^{(1)}(s')}{s'^2(s' - s)} ,$$

where

$$\lambda_1 = D_1^{(1)}(0) \tag{A.8}$$

$$\lambda_2 = \left[\frac{d}{ds} D_1^{(1)}(s) \right]_{s=0} .$$

Substituting Eq. (A.8) into Eq. (A.7), we have the N/D integral equation for $N_1^{(1)}(s)$,

$$h(s) = \frac{1}{s} \cdot N_1^{(1)}(s)$$

$$h(s) = \frac{\lambda_0}{s} + \lambda_1 \cdot \frac{V_1(s)}{s} + \lambda_2 V_1(s) + \int_{4\mu^2}^{\infty} ds' \, K_1(s; s')h(s') \,, \tag{A.9}$$

$$K_1(s; s') = \frac{1}{\pi} \cdot \rho_1(s') R_1(s') \cdot \frac{s'V_1(s') - sV_1(s)}{s'(s' - s)} \,.$$

We note that the kernel $K_1(s; s')$ of Eq. (A.9) is the same as that of Eq. (A.5), so the solution of Eq. (A.9) can be written as

$$N_1^{(1)}(s) = \lambda_0 F_0(s) + \lambda_1 F_1(s) + \lambda_2 F_2(s) \,,$$

where

$$F_0(s) = s \int_{4\mu^2}^{\infty} ds' \, \frac{1}{s'} \cdot H_1(s; s') \,,$$

$$F_1(s) = s \int_{4\mu^2}^{\infty} ds' \, H_1(s; s') \cdot \frac{V_1(s')}{s'} \,, \tag{A.10}$$

$$F_2(s) = s \int_{4\mu^2}^{\infty} ds' \, H_1(s; s') \, V_1(s'),$$

where the functions $F_i(s)$'s are independent of λ_0, λ_1, and λ_2. Substituting Eq. (A.10) into Eq. (A.8), we obtain

$$D_1^{(1)}(s) = \lambda_0 G_0(s) + \lambda_1 G_1(s) + \lambda_2 G_2(s) \,,$$

where $G_0(s)$, $G_1(s)$, and $G_2(s)$ are independent of the λ's. The quotient

$$B_1(s) = \frac{N_1^{(1)}(s)}{D_1^{(1)}(s)} = \frac{F_0(s) + \gamma_1 F_1(s) + \gamma_2 F_2(s)}{G_0(s) + \gamma_1 G_1(s) + \gamma_2 G_2(s)} \,,$$

where

$$\gamma_1 = \frac{\lambda_1}{\lambda_0} \,,$$

$$\gamma_2 = \frac{\lambda_2}{\lambda_0} \,.$$

This result shows explicitly that $B_1(s)$ contains two parameters γ_1 and γ_2, which cannot be determined from the input information $V_1(s)$ and $R_1(s)$.

FOOTNOTES AND REFERENCES

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Table I. Variation of parameters in the potential $V_1(s)$ and the corresponding output width.

a_1	s_c	$b_1 = a_1 s_c$	Γ_{output}
$5.1 \times 10^{-3}/\mu^2$	$165\ \mu^2$	0.84	380 MeV
$4.0 \times 10^{-3}/\mu^2$	$199\ \mu^2$	0.80	300
$3.0 \times 10^{-3}/\mu^2$	$251\ \mu^2$	0.75	240
$2.1 \times 10^{-3}/\mu^2$	$348\ \mu^2$	0.73	190
$1.5 \times 10^{-3}/\mu^2$	$467\ \mu^2$	0.70	165
$1.2 \times 10^{-3}/\mu^2$	$576\ \mu^2$	0.69	148
$1.0 \times 10^{-3}/\mu^2$	$688\ \mu^2$	0.69	138
$0.8 \times 10^{-3}/\mu^2$	$854\ \mu^2$	0.684	127
$0.7 \times 10^{-3}/\mu^2$	$974\ \mu^2$	0.681	121
$0.6 \times 10^{-3}/\mu^2$	$1132\ \mu^2$	0.679	115

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