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CREATING ACADEMIC COMMUNITY FOR FIRST-GENERATION COLLEGE STUDENTS A Graduate Student Instructor Guidebook

Thinking like an Expert: Scaffolding Mathematical Concepts in Physics Courses

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Most college educators aspire to develop expertise among students. However, the manners in which experts learn and organize their knowledge are often not emphasized with novice learners. In this study, we implemented a supplementary curriculum of advanced materials in an introductory modern physics course to examine whether novice learners can benefit from an early exposure to expert modes of knowledge organization. Students were given weekly web-based exercises employing multiple learning components which were intended to introduce and promote familiarity with the mathematical concepts that experts use to conceptually organize the theory of quantum mechanics. The data results obtained from student surveys and weekly write-ups revealed a bi-modal response to the supplementary curriculum: more than half the students reported significant learning benefits, while the remaining students indicated little effect. Those who benefited from the curriculum reported an increased interest in quantum mechanics, strengthened conceptual understanding of the theory, and improved knowledge and proficiency with high-level mathematical concepts. Moreover, the study revealed students engaging in learning transfer and metacognitive thinking as a result of the exercises, and it highlighted the effectiveness of using multiple learning formats to improve student comprehension and achievement. Given the frequently low achievement of student learning outcomes in quantum mechanics, the positive results of this study warrant further investigation into the benefit of training novice learners to think like experts.

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Abstract

Most college educators aspire to develop expertise among students. However, the manners in which experts learn and organize their knowledge are often not emphasized with novice learners. In this study, we implemented a supplementary curriculum of advanced materials in an introductory modern physics course to examine whether novice learners can benefit from an early exposure to expert modes of knowledge organization. Students were given weekly web-based exercises employing multiple learning components which were intended to introduce and promote familiarity with the mathematical concepts that experts use to conceptually organize the theory of quantum mechanics. The data results obtained from student surveys and weekly write-ups revealed a bi-modal response to the supplementary curriculum: more than half the students reported significant learning benefits, while the remaining students indicated little effect. Those who benefited from the curriculum reported an increased interest in quantum mechanics, strengthened conceptual understanding of the theory, and improved knowledge and proficiency with high-level mathematical concepts. Moreover, the study revealed students engaging in learning transfer and metacognitive thinking as a result of the exercises, and it highlighted the effectiveness of using multiple learning formats to improve student comprehension and achievement. Given the frequently low achievement of student learning outcomes in quantum mechanics, the positive results of this study warrant further investigation into the benefit of training novice learners to think like experts.

Introduction

Most college educators aspire to develop expertise among students; however, in light of pressing preparatory knowledge needs in introductory and gateway coursework, we might not immediately prioritize building expert knowledge in novice scholars. Recent research into cognition, highlighted in the National Research Council's *How People Learn* (2000), has yielded exciting findings that offer new opportunities to improve teaching and help students learn most effectively, based on the principles and structures of expertise [1]. Some findings with particularly important implications for education reform come from an understanding of how experts learn and organize their knowledge. Studies have revealed that "(experts') knowledge is not simply a list of facts and formulas that are relevant to their domain; instead, their knowledge is organized around core concepts or "big ideas" that guide their thinking about their domains". This principle is true in diverse fields from physics to history and indicates a common foundation on which expert knowledge is built [2]. This claim begs the question: Could introducing expert ways of organizing knowledge early in a learning process benefit novice scholars?

If so, this suggests that curricula should emphasize organizational structures that lead to conceptual understanding. In other words, an early introduction to advanced concepts encourages students' ability to develop familiarity with expert modes of thinking.

In this article, we offer a preliminary exploration into the question of whether novice learners can benefit from introductory course curricula emphasizing the organizational structures characteristic of expert knowledge. Our findings are based on a case study conducted with students in an introductory modern physics course intended to provide students with a first introduction to quantum mechanics. The course was restructured to foster student familiarity with the mathematical organizing structure of quantum mechanics via a new supplementary web-based curriculum. The new curriculum included exercises with a variety of instructional formats, allowing a comparison of the relative merit of different components. Based on data collected during the semester, this intervention produced a bi-modal response: more than half the students reported significant learning benefits from the restructured curriculum, while the remaining students saw little gain. Not surprisingly, the learning benefits were closely correlated with the students' overall engagement with the supplementary materials. Interestingly, many of the students who found the exercises beneficial reported that the knowledge gained from the exercises helped them in other courses as well. Initial findings also indicate improved student achievement, particularly in applying mathematical approaches to abstract theories.

While the results of this study are somewhat preliminary and mixed, the positive learning outcomes indicate a need for further research into the benefit of organizing course curricula to reflect expert modes of thinking. This case study can give the reader a sense for how this kind of intervention can work and benefit students. Moreover, we believe that the curriculum design model is not limited to introductory modern physics courses, but could potentially benefit other disciplines as well since the components of expert knowledge are quite universal. Our hope is that this investigation may prove useful in enabling educators to develop better teaching methods that can help students learn most effectively.

The remainder of this article is organized into the following sections: (i) motivation and background, (ii) curriculum design and implementation, (iii) assessment methodology and student demographics, (iv) data results and analysis and (v) discussion. In conclusion, we offer recommendations on best practices for implementing a curriculum of core conceptual materials in an introductory course.

Motivation and Background

Quantum Mechanics is an ideal field for investigating an early introduction to expert modes of thinking for numerous reasons. To begin, traditional instructional practices in quantum physics support the need for an investigation of varied teaching methods. Also, the subject matter of quantum mechanics is particularly well suited for

research on expert knowledge structures. The non-intuitive concepts of quantum theory and its close connection with mathematics place great importance on the subject's theoretical organization for effective learning and comprehension. Not to mention, an introductory course in quantum physics provides a gateway to the physics major, providing a sample of motivated students keen on gaining deeper knowledge of their subject area. Of course, the strongest motivation for any research on improving instructional methods in quantum physics is found by looking at the current state of education: a recent study involving extensive testing and interviews revealed that a significant fraction of students who complete an advanced undergraduate quantum mechanics course are still not proficient in basic skills [3]. Certainly, there remains ample room for improvement.

Other education researchers have noted and investigated the importance of core organizing concepts in physics. A well-known early example comes from the Feynman Lectures of the 1960's. In his lectures, Richard Feynman sought to instruct lower division physics' students the core conceptual principles of quantum mechanics in a traditional lecture format. Although Feynman himself questioned the effectiveness of his lectures on student learning, the books he wrote based on them continue to be a favorite of physics' students the world over [4]. A more recent example is found in the research of Peter Hewson, who investigated using 'change' as a core organizing concept in an introductory Newtonian physics course for non-majors. Interviewing two students, he found that this emphasis was helpful for one student, but not for the other [5]. In the field of classical mechanics, Ruth Chabay developed a curriculum and accompanying text book organized around a limited number of core conservation principles that remains in use today [6]. Specific to quantum mechanics, Ileana Greca developed a curriculum organized around the key idea of 'quantum state' and reported that 65% of the students completing the course showed a reasonable understanding of the basics of quantum mechanics [7]. This is a rather effective outcome when compared with overall achievement levels in quantum mechanics.

A more common approach in teaching quantum mechanics, however, is to delay the introduction of core organizing principles. Following this approach, introductory course curricula are typically designed to present students with a broad survey of topics at a simplified level of analysis. While this may provide a good overview of natural phenomena and introduce important ideas, the analysis is shallow and lacks the mathematical structure of a unified conceptual perspective of quantum theory. Even at the advanced undergraduate level, many popular quantum mechanics' texts, such as David Griffiths' *Introduction to Quantum Mechanics* (2005), tend to dedicate a disproportionate amount of time to lengthy mathematical calculations and applications relative to the underlying conceptual organizing principles [8]. Often, it is not until graduate level study that the core mathematical organizing principles of quantum mechanics are emphasized as, for example, in the commonly used graduate text *Modern Quantum Mechanics* (1994) by J.J. Sakurai [9]. At this level, the tendency towards broad

generalization of results often leads to a highly abstract and formal mathematical presentation, which is an unfamiliar format and requires high-order skills to interpret.

Since the traditional method of instruction in quantum mechanics tends to neglect the connection between mathematical structure and core organizational concepts, it may inhibit students' ability to organize knowledge meaningfully. Thus, this 'delay' may contribute to student learning difficulties. For example, researchers at Oregon State University found that physics students often struggled with their initial upper-division courses. The decline in achievement was attributed to an inundation of formidable mathematical techniques, combined with challenging conceptual ideas. Lacking sufficient preparation, students became overwhelmed and mentally saturated by the challenging mathematics [10]. What is more, it seems likely that such an experience could impede students' ability to identify and synthesize core concepts, a crucial step towards developing expert knowledge [11].

A quantum physics education starting with an early exposure to expert knowledge structures may provide an effective alternative to conventional instructional practices. There may be a variety of factors to consider. In physics, core concepts that guide expert modes of thinking are commonly organized via mathematical frameworks [12]. Moreover, as physics has progressed, the reliance on mathematical organizational structures to provide an understanding of natural phenomena has steadily increased. In quantum mechanics, the mathematics of linear algebra not only provide a consistent framework for describing quantum states, but also forms the basis for developing a unified conceptual understanding of the theory. Unlike classical physics, where students can draw on personal experience and illustrative models to guide their thinking, comprehension and intuition in quantum mechanics stems almost exclusively from an understanding of its conceptual organizing principles [13]. Consequently, if we desire that students not only perform quantum mechanics exercises, but also to contextualize the theory, then it is important to build competency with the mathematical organizing structure as early as possible.

Another anticipated benefit of teaching students core conceptual constructs in introductory courses is the allowance of additional time and opportunity for students to absorb and revisit difficult concepts. Core concepts in quantum mechanics are initially unfamiliar and non-intuitive and are presented with a level of abstraction far exceeding that of previous courses. As such, early familiarization with these concepts may improve the likelihood that they will be effectively absorbed and retained by students. This notion is supported by cognition research, which has revealed that the integration of complex subject matter takes time and implies that learning should not be rushed [14]. It is also supported by previous physics education researchers, who have advocated that quantum mechanical concepts be given sufficient time and repetition so that students may properly comprehend them [10, 15, 16, 17]. The supplemental introductory course environment

may provide a positive setting for students to initiate the cognitive processes necessary for knowledge integration.

At last, we are brought back to the challenges associated with traditional instructional methods in quantum mechanics, and how earlier instruction on expert modes of knowledge organization may be beneficial. The close relationship between conceptual organization and mathematics naturally affords students the opportunity to gain familiarity and practice with the mathematical techniques of quantum mechanics. With early exposure to these ideas at a simplified level and in a low-pressure, exploratory environment, students may become better prepared for the challenge of transitioning from lower to upper division course work. Such a sentiment is common among physics researchers, who have advocated for earlier and better mathematical preparation as an important component for improving student performance in quantum mechanics [11, 17, 18]. Moreover, as students are able to explore the connection between conceptual organization and mathematics from the beginning, they are likely to be better equipped for developing a unified conceptual perspective of the theory.

Curriculum Design and Implementation

The main objective in this research project was to develop and assess a supplementary curriculum aimed at helping novice learners gain an initial familiarity with expert modes of knowledge organization in quantum mechanics. A set of weekly exercises were developed and administered to students via the school's online project collaboration and courseware system, UCM CROPS. The weekly web-based exercises were supplementary to the traditional course lecture and discussion format, which remained mostly unchanged. The intent was to enhance student learning with the new materials, while avoiding overly burdensome additional workloads for the instructor and students alike. As such, the exercises were intended to take about an hour's time to complete and roughly three-quarters of the students surveyed reported being able to do so. An important objective was to create a relaxed learning environment for students to explore ideas and engage in reflective thinking, as low stakes assignments have been shown to provide important opportunities for students to practice core concepts [19]. Towards this end, participation was incentivized by positive rather than negative reinforcement (i.e. grading was based on participation, not 'correctness'). In an attempt to maximize student engagement and learning, the exercises incorporated a variety of instructional formats, which also provided an opportunity to assess the relative merit of different learning components. The principal learning outcome from these exercises was for students to gain an initial familiarity with the basic ideas underlying expert knowledge and organizational structures. This underlies a broader goal of developing students' ability to think about the subject matter from a unified and coherent perspective and to improve students' preparation for comprehending these ideas in more advanced form in future courses.

The resulting exercises introduced key elements of the linear algebraic organizational framework of quantum mechanics without the added pressure of difficult computations and intricate applications. They begin by introducing the notion of quantum state and proceed to describe the linear algebraic structure of quantum states, including base states and superposition states, finally leading up to the paradigm of eigenstates of linear operators and their eigenvalues in relation to measurable observables (see Appendix I for a complete description of the weekly exercises and Appendix II for an example exercise). A consistent effort was made to connect these ideas to specific physical phenomena and towards this end examples of the hydrogen atom and the infinite square well (i.e. particle in a box) were frequently used. Key ideas from quantum mechanics, including intrinsic probability, wavefunction collapse, and the uncertainty principle, were introduced in relation to the mathematical organizing structure in order to promote a unified conceptual perspective of quantum mechanics. The content and format of the exercises was motivated in large part by *The Feynman Lectures on Physics Vol. III* and *A Quantum Mechanics Primer* by Daniel Gillespie [20]. There were a total of eleven exercises given over the course of the semester.

Simultaneous with introducing the conceptual organizing structure of quantum mechanics, the exercises also developed students' familiarity and practice with important mathematical techniques used in quantum mechanics. Introducing the linear algebraic structure of quantum states provided a natural opportunity for introducing the key ideas of abstract vectors and function spaces. Following the approach of *The Feynman Lectures*, the exercises sought to tap into students' previous knowledge of ordinary Euclidean vectors by introducing quantum state vectors and their properties as an extension of familiar mathematics [21]. Dirac notation was introduced as a new notation for state vectors and the students studied and practiced manipulating the state vectors to learn about base states, superposition states, inner products and the statistical nature of quantum measurements. Fourier analysis was presented as a further generalization of the vector concept, allowing students to apply the ideas and properties of vectors to trigonometric functions. Finally, the ideas of linear transformations and eigenvectors were established, drawing on the analogy of vector rotations in Euclidean space. Again, the mathematical techniques were consistently applied to the description and interpretation of quantum phenomena, underscoring the direct connection between mathematics and conceptual organization in quantum mechanics.

As mentioned above, the exercises incorporated a variety of instructional formats. The students' primary activity was to read a weekly text article written to introduce the main concepts. In addition, students were frequently directed to external websites to use Java applet simulations (freely available on the web) related to the material so that they could interactively explore the concepts from a visual perspective¹. In response to the

¹ One short educational video was also assigned, but it proved difficult to find videos on the web that matched the content of the exercises. Nevertheless, videos could provide an additional source of engaging material.

weekly exercise, students were asked to submit a short written reflection (100-200 words) for course credit. On occasion, this was replaced with a mathematical problem for the students to solve. The written reflections were often prompted with specific questions to provide guidance, though the students were encouraged to use their reflections to freely share their thoughts and questions about the material. In response to the students' submitted reflections, the author of the exercises posted a weekly write-up to an online classroom discussion board, providing a mix of general feedback and specific responses to select questions from students.

To incentivize student participation, the supplementary curriculum exercises comprised 10% of the course grade. To receive credit, the students were required to electronically submit their reflection/math problem prior to a weekly submission deadline. Submitted work was not graded for content. Rather, full credit was given for completed assignments submitted on time. Electronic submission deadlines proved to be an effective means for motivating participation, collecting work and assigning credit.

Assessment Methodology and Student Demographics

To assess the effectiveness of the supplementary exercises, the students were given two anonymous surveys, one at mid-semester (Survey 1) and one at the end of the semester (Survey 2). Written reflections also provided insight into student attitudes and opinions regarding the weekly exercises. Taken together, this data allows a fair assessment of the effectiveness of the supplementary exercises and their impact on student learning.

The two surveys consisted of about ten 'best match' questions plus a few short answer questions at the end. Apart from several questions on student demographics, the survey questions were chosen to assess the overall effectiveness of the exercises in accomplishing the main research objectives of this study (see Appendix III for the complete surveys). To assure decisiveness on the best match questions, students were given an even number of response options. These response options were grouped into two categories, 'favorable' and 'unfavorable', to facilitate the data analysis². Both surveys were voluntary and anonymous, and they were conducted during the course laboratory (Survey 1) and discussion sections (Survey 2), respectively. In each instance, all the students in attendance chose to complete the survey, amounting to a sample size of seventeen students for Survey 1 and ten for Survey 2. The total course enrollment was twenty, but data from minors under the age of eighteen years was omitted, eliminating the survey responses of one student. The lower number of respondents for the second survey is due to the fact that the laboratories were mandatory, while the discussion sections were optional. Chronologically, Survey 1 was given at mid-semester when approximately half of the supplementary exercises had been completed, while Survey 2

² In this context, 'favorable' corresponds to a positive response or agreement with the statement and 'unfavorable' to a negative response or disagreement with the statement.

was given at the end of the semester after all the exercises had been completed. This allowed us to examine changes in student opinions over the course of the semester.

The surveys included some basic demographic questions about the student population, which revealed a fairly typical student group for an introductory modern physics course. They were predominantly physics majors, rounded out by a few math and engineering majors, and were evenly distributed between sophomore and junior level standing. Their course preparation was also fairly standard, with at least three quarters of the students being concurrently enrolled or having previously completed all of the following common prerequisite courses for introductory modern physics: a lower division introductory physics sequence, multi-variable calculus, linear algebra and differential equations, and probability and statistics.

Before proceeding to a detailed analysis of the data, we first mention some strengths and weaknesses associated with the data collection. First, the small sample size that is typical for this course renders the statistics less robust than for courses with higher enrollment. Moreover, the uneven number of student responses from Survey 1 to Survey 2 can further skew the results within this sample. This is especially true if the students are somehow self-selected from the larger group in a way that may be correlated with learning/educational experience, which is a possibility here since Survey 2 was administered during a voluntary discussion section. In this research, however, this last limitation is partially overcome by supplementing the survey data with data collected from the students' weekly written reflections, which the students' frequently used as an outlet for providing feedback and expressing their opinions about the exercises. In particular, a majority of the students used the final written reflection to offer their thoughts, opinions and suggestions about the supplementary curriculum as a whole. The candid responses in the written reflections provide a deeper insight into individual student thought processes than is possible from the survey questions alone, thereby contributing a personal perspective on student learning that greatly enriches the study results.

Data Results and Analysis

The complete data results of the best match portion of Surveys 1 and 2 are given in Figures 1 and 2, while the short answer survey questions are listed in Figure 3. Additionally, Figure 4 shows how student responses to repeat questions varied from Survey 1 to Survey 2. The exercises were commonly referred to as 'ERAs', an acronym for educational research activities. In the following data analysis, we combine the written reflection responses with the aggregate survey results to provide the fullest picture of the successes and shortcomings of the supplementary curriculum in obtaining its core objectives.

To begin the analysis, we present an overview of general results and highlight the broad trends that consistently appeared throughout the data. An overall sense of the

supplementary curriculum’s impact on student achievement can be gained from reviewing student responses to best match questions concerning how the exercises influenced their interest and comprehension of quantum mechanics (Survey 1, questions 2 & 3 and Survey 2, questions 3 & 4). In Survey 1, 65% of the students reported an increased interest in quantum mechanics and 71% reported improved understanding of the subject as a result of the exercises. Survey 2 affirmed this trend, albeit somewhat weaker, with 60% of the students reporting an increased interest and 50% reporting improved understanding, respectively. Furthermore, after all exercises had been completed, 50% of the students felt that the supplementary exercises had helped to improve their preparedness for future courses and would recommend the exercises for future students taking the modern physics course (Survey 2, questions 8 & 9). Thus, according to the students, the supplementary curriculum was beneficial for at least half the class. Nevertheless, the data also revealed a high degree of polarization in student opinions regarding the supplementary exercises. For the majority of students who found the exercises useful, the survey responses were resoundingly positive, while for another sizeable group of students, the responses were uniformly negative. This bi-modal trend proved consistent throughout the study, and is keyed to other engagement and success factors.

	Question³	Favorable	Unfavorable
1	How often do you complete the ERAs? ⁴	17 (100%)	0 (0%)
2	Have these activities been useful in improving your understanding of QM?	12 (71%)	5 (29%)
3	Have these activities increased your interest in QM?	11 (65%)	6 (35%)
4	Have you found the web resources – interactive applets and videos – useful in increasing your understanding and/or interest in QM?	14 (82%)	3 (18%)
5	Do you feel that writing the reflections about these activities enhances your understanding of the concepts?	10 (59%)	7 (41%)
6	Have you discussed the material in the ERAs with your classmates?	2 (12%)	15 (88%)

³ The question numbering differed on the original surveys since early demographic questions have been removed in this figure. See Appendices III and IV for original question sequence.

⁴ Completion of the ERAs is considered ‘favorable’.

7	How often do you read the feedback posted on the UCM CROPS student lounge? ⁵	5 (31%)	11 (69%)
8	Do you find the feedback on your submissions useful? ⁶	9 (75%)	3 (25%)
9	Have these activities helped to improve your understanding of how the theory of QM is organized?	7 (41%)	10 (59%)

Figure 1 Student responses to best match questions from Survey 1.

	Question	Favorable	Unfavorable
1	How often did you complete the ERAs?	9 (90%)	1 (10%)
2	If the ERAs were not worth course credit, would you still complete them?	4 (40%)	6 (60%)
3	Now that the course is nearly over, do you feel that the ERAs have been useful in improving your understanding of QM?	5 (50%)	5 (50%)
4	Have these activities increased your interest in QM?	6 (60%)	4 (40%)
5	How comfortable are you with the ideas of base states, functions as vectors and Fourier analysis in QM?	7 (70%)	3 (30%)
6	How well are you able to relate the ERAs to the rest of the course material?	6 (60%)	4 (40%)
7	Have these activities helped to improve your understanding of how the theory of QM is organized?	6 (60%)	4 (40%)
8	Do you feel these activities have helped improve your preparedness for future quantum mechanics courses?	5 (50%)	5 (50%)

⁵ Data from one student, who marked multiple responses to this question, has been excluded.

⁶ Five students who selected 'N/A' in response to this question are excluded.

9	Would you recommend these exercises for future students in this course?	5 (50%)	5 (50%)
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Figure 2 Student responses to best match questions from Survey 2.

Long Answer Survey Questions	
Survey 1	
1	These activities are designed to introduce mathematical concepts. Is the emphasis on relating new concepts to previously familiar mathematics within a physics context effective? If not, please explain. If so, how so?
2	How well are you able to relate these exercises to the rest of the course material? If not, please explain. If so, how so?
Survey 2	
1	What did you like the most and the least about the ERAs? Do you have any recommendations for improving the ERAs?
2	What do you imagine when you think of a quantum mechanical electron?
3	In your opinion, why is there uncertainty in quantum mechanics?

Figure 3 Long answer questions from Survey 1 and Survey 2.

Question	Mid- Semester Survey		End of Semester Survey	
	Favorable	Unfavorable	Favorable	Unfavorable
How often did you complete the ERAs?	17 (100%)	0 (0%)	9 (90%)	1 (10%)
Do you feel that the ERAs have been useful in improving your understanding of QM?	12 (71%)	5 (29%)	5 (50%)	5 (50%)
Have these activities increased your	11 (65%)	6 (35%)	6 (60%)	4 (40%)

interest in QM?				
Have these activities helped to improve your understanding of how the theory of QM is organized?	7 (41%)	10 (59%)	6 (60%)	4 (40%)

Figure 4 Comparison of student responses to repeated questions in Surveys 1 and 2.

In the following, we present the remaining data results and provide analysis to assess the effectiveness of the supplementary curriculum, focusing primarily on the following principal research questions:

1. Did the exercises enable students to gain an initial familiarity with expert modes of knowledge organization in quantum mechanics, namely the linear algebraic description, and how did this impact student learning and interest in the subject?
2. Did the introduction and practice with advanced mathematical techniques help students develop skills and thought processes to enhance their conceptual understanding of quantum mechanics and increase their mathematical preparation for future courses?
3. Which of the various instructional formats (or combinations thereof) in the supplementary curriculum were most effective in engaging students and supporting their learning?

The first two questions address the effectiveness of the supplementary curriculum in achieving its major learning objectives for the students, namely improving conceptual understanding based on expert modes of knowledge organization and developing greater familiarity with advanced mathematical techniques. To assess the effectiveness of this first objective, the students were asked whether they felt the exercises had improved their understanding of how quantum mechanics is organized. Initially in Survey 1, only 41% of the students reported that the exercises had helped improve their understanding of how quantum mechanics is organized (Survey 1, question 9). However, this percentage increased to 60% in Survey 2 following completion of all the supplementary curriculum exercises (Survey 2, question 7). As another means of assessment, we sought to survey students' ability to extend or transfer their knowledge to new contexts, a common trait of experts and an indicator of effective learning [22]. Students were asked how well they were able to relate the supplementary exercises to the rest of the course material (long answer question 1 in Survey 1 and question 6 in Survey 2). Initially, as indicated by the long answer responses from the mid-semester survey, most of the students were unable to relate the supplementary exercises with the rest of the coursework. A common sentiment was that the supplementary exercises were ahead of the other coursework and did not match up, as conveyed by one student's response: "...these activities tend to be fairly far ahead of the course progression so we might discuss something in the activity and not see it till weeks later in class." However, by the end of the course, 60% of the students replied

that they were able to relate the supplementary curriculum to the rest of the coursework somewhat or very well. In fact, the students seemed pleased later in the semester when the exercises and course lessons began to clearly align. As one student commented in a reflection response to exercise 8, "I thought that this ERA was really nice in how it connected things we are currently going over in class right now, which is the Schrodinger equation, and things that we have been talking about in our previous ERA's, such as the Fourier series." All in all, the data reveals that after a slow start, a majority of the students by the end reported an improved understanding of how quantum theory is organized and felt they were able to relate this knowledge to the various topics introduced in the standard course lessons.

Other survey questions served to assess the effectiveness of the supplementary exercises in helping students gain familiarity with advanced mathematical concepts. In Survey 1, the students were asked to comment on the effectiveness of the strategy of introducing new mathematical concepts by relating them to familiar ones within a physics context (Survey 1, long answer question 1). Overwhelmingly, the students affirmed the effectiveness of this strategy; only less than a quarter reported it as ineffective. As one student put it, "Yes, when relating a hard new topic (quantum) to previously known material always makes it easier to grasp." At the end of the course in Survey 2, the students were asked to rate their level of comfort with the core mathematical ideas of base states, functions as vectors and Fourier analysis that were a focus of the exercises (Survey 2, question 5). In response, 70% of the students replied that they were somewhat to very comfortable with these ideas, showing that by the end of the supplementary curriculum, a majority of the students felt at least somewhat comfortable with the mathematical knowledge that the exercises sought to teach.

The student reflection responses further revealed the effectiveness of teaching based on prior knowledge and revealed that the mathematical knowledge gained was useful in other courses as well. The following comments are taken from student reflections following exercise 6, in which function spaces and Fourier series were introduced. As one student wrote,

"I think this is a cool way of thinking of Fourier series. By making use of our knowledge of classical vectors, we all have a chance to understand this series in a better, more concise and fun way ... Personally, I am experiencing some trouble understanding Fourier series, as I need it for another class that I am taking. However, after reading this paper, I have started to understand it better, which I hope will result in better performance in my other class."

Numerous reflection responses captured students' emerging awareness of the new mathematical techniques. As one student commented,

"When I learned about vectors in Calculus, I never dealt with trigonometric functions ... I also thought that it was interesting that we could find the

orthonormality of two trigonometric functions when they are vectors. The interesting part was how we found the orthonormality, which we did simply by using dot products. All of this was more simple than I thought it would be because all we did was use our knowledge of vectors from beforehand to do all of this.”

Still more students remarked in their reflections that familiarity with mathematics in the supplementary exercises proved helpful in other courses:

“Being in a differential equations/linear algebra class concurrently with this one is quite interesting because of the expansion of knowledge surrounding vectors. I find this new perception of vectors to be quite revealing for me” and “In fact I just learned the orthogonality, referred as Fourier’s trick, in my electrodynamics class; that class only briefly went over this concept. This activity enhanced my understanding to a higher degree ...”

In addition, the responses revealed students making progress in understanding how mathematical concepts relate to and provide structure for the theory of quantum mechanics. As one student stated, “I think it’s pretty neat that we can treat sine and cosine functions like basis vectors . . . My mental image of the base states for the H-atom [*hydrogen atom*] and how vectors behave in infinite dimensions is getting clearer as more and more of these ERAs are put out.” As the data shows, numerous students gained an understanding of the new mathematical techniques introduced in the supplementary curriculum and this understanding helped them not only in the introductory modern physics course, but in other courses as well.

The remaining survey questions polled student opinion on the effectiveness of the various instructional formats utilized in the supplementary exercises, including the written text articles, Java applets, students’ written reflections and feedback on the reflections. In Survey 1, 82% of the students replied that they found the web applets useful in increasing interest and understanding in quantum mechanics, while 59% of the students indicated that the written reflections enhanced their understanding of the concepts (Survey 1, questions 4 and 5). Survey 1 also indicated that only 31% of the students regularly read the posted feedback response to their reflections (Survey 1, question 7). However, of the 71% of students who had read some feedback, 75% found it useful (Survey 1, question 8). In Survey 2, the students were asked to rank the various instructional formats from most to least effective. The results are shown in Figure 5. The text articles and java applets shared marks for most effective, while the written reflections were ranked as the least effective. None of the students selected the feedback as most effective, although 56% of the students ranked them as the second most effective component of the exercises.

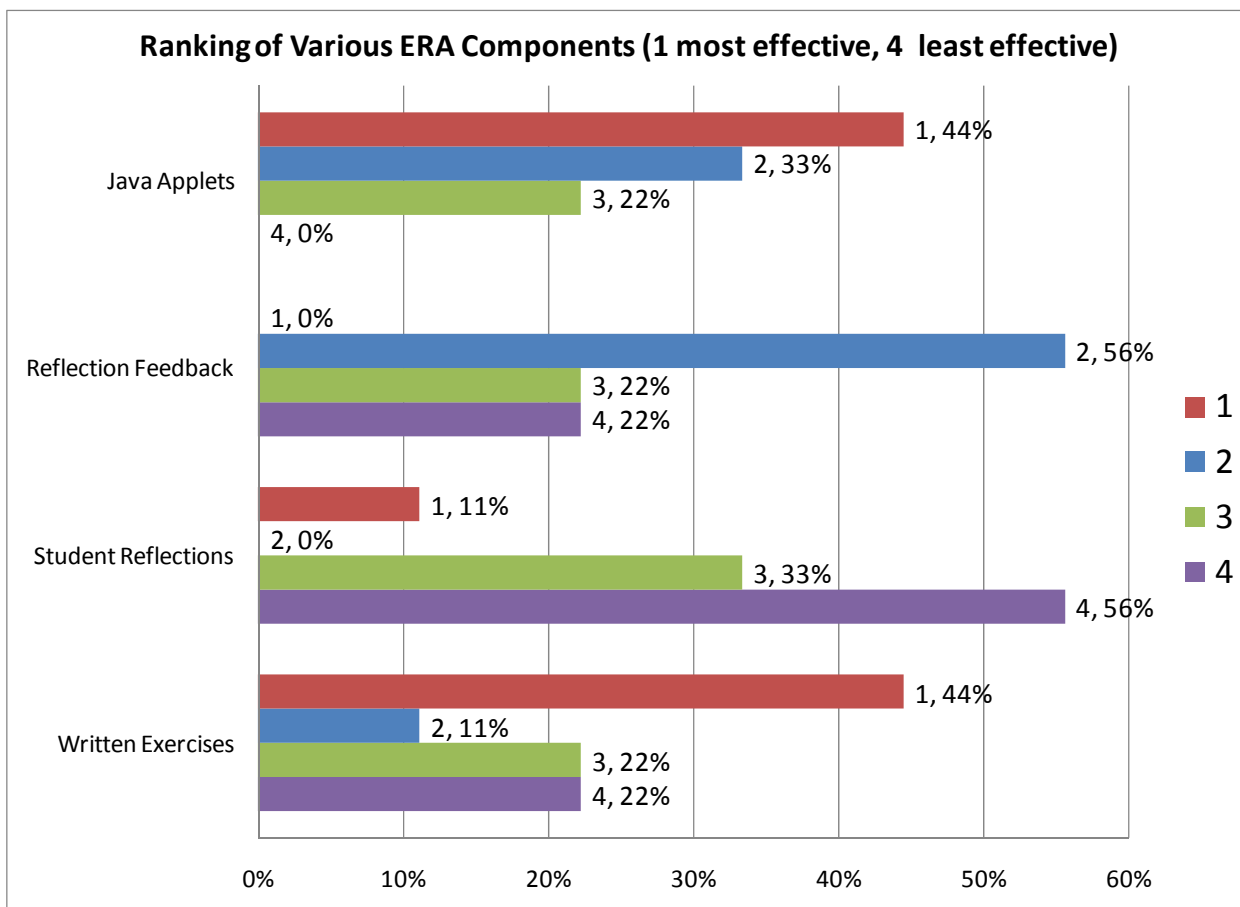


Figure 5 Student ranking of supplementary exercise formats in order of effectiveness, 1 being most effect and 4 being least effective.

Written reflections also provided valuable information about the various learning formats in the supplementary curriculum. A majority of the students found the Java web applets especially useful. Following exercise 10, which contained a java applet on the wave/particle duality, numerous students wrote favorably of the applets and how they aided their learning. Their reflection comments included: “This activity was probably my favorite so far. I really enjoyed playing with the applet and making as many waves as I can and many other things. This really did help me understand what wave packets are more, and see how they change depending on the length of the box, and all the other variables. ... I just found this ERA to be much more understandable then reading a bunch of words and trying to image it”, and “With the aid of the applet, I understood the concepts behind the uncertainty principle much more effectively because I was able to play around with it and see the modifications in the different graphs”. Moreover, the applets were useful in helping students improve reading comprehension, as is evidenced by the following student response to exercise 7, which contained an applet on Fourier series: “I would like to comment on the applet. This visual aid was quite helpful in understanding the example explained [*in the written text article*] as well as explaining last

week's ERA even better. I found the example to also be a great help in understanding how the Fourier series explains quantum mechanics." Additionally, positive experience with the applets led some students to seek out more examples on the web. One student remarked "I found a lot of cool, useful, and helpful applets from the website you provided that can help me visualize more mathematical and physical concepts."

While the written reflections were not as popular as the applets, there is evidence that they were useful in promoting metacognition among the students. Metacognition is the ability to monitor one's current level of understanding and decide when it is not adequate; it is a common trait among experts in wide-ranging fields of knowledge. When asked about their favored activities, students described metacognition outcomes (Survey 2, long answer question 1): "The reflections were helpful in making me think about what actually made me confused or knowledgeable about; they made me highlight my problem areas." In addition, students often used the reflections to summarize their knowledge. The following reflection from exercise 6 reveals a student elaborating on new knowledge about vectors:

"This activity challenged my understanding of concept on vectors. Forget the concept of cross products, I wonder what the requirement for something to be a unit vector is; as in the recent activities, we are making "everything" into vectors. I conclude that first unit vectors must be linearly independent at first. Then there should be a definition of dot-product—forget cross-products. The definition of dot-product should be an operation that performs an operation on two commutative parameters and yield a scale number as its return value. Most crucially, based on that definition of dot-product, the dot-product of two same unit vectors must yield one, and yields zero for different unit vectors. I think this should be a complete vector system—correct me if there are other requirements. Any set of elements that satisfies the requirements above can expand a vector system. Once they can be used as vectors, their concepts and calculations can be simplified as vector is a quite developed system."

It was a common occurrence among some students to include questions about the new material in their written reflections, for which they could anticipate a response in the following feedback. In this manner, the supplementary curriculum provided the students with access to another person knowledgeable in the subject matter.

As a final indicator of the overall effectiveness of the supplementary curriculum, we display Figure 6 summarizing the final course grade distribution versus the number of exercises completed per student. The data shows that grade levels in the course were strongly correlated with participation in the supplementary exercises. In particular, of the nine students who received a grade of A- or better in the course, eight of them either completed every supplementary exercise or only missed one.

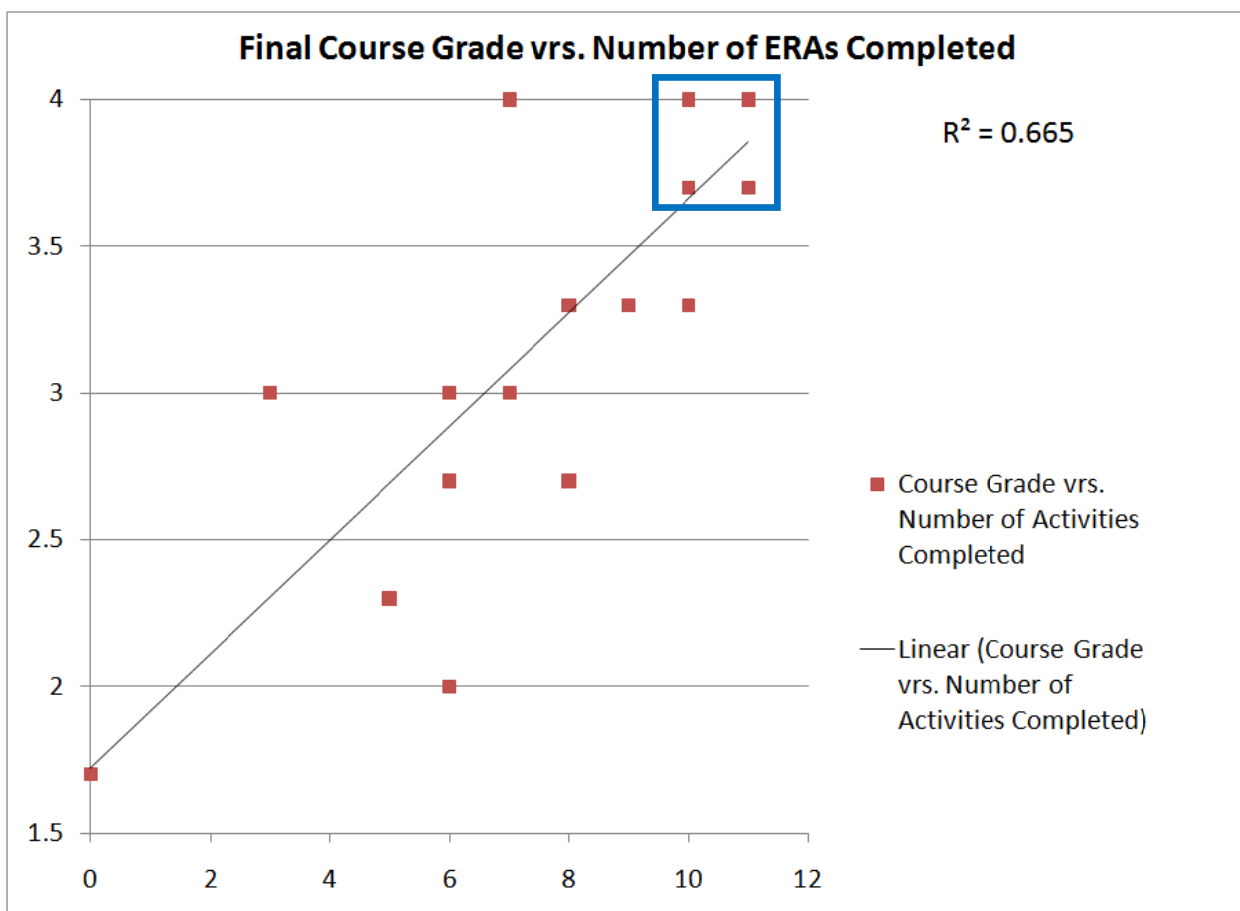


Figure 6 Correlation between final course grade (expressed as standard 4.0 scale grade points on the vertical axis) and number of ERAs completed during the semester (horizontal axis). A best fit linear regression line is also shown. *The highlighted box actually contains eight data points due to the following redundancies: (11, 4.0) two, (10, 4.0) two and (10, 3.7) three.*

Discussion

As is evident from the data, the supplementary curriculum proved that it is possible for beginning students to benefit from exposure to expert modes of knowledge organization, although this did not occur for everyone. It is interesting to note that it took until Survey 2 before a majority of the students reported that the supplementary curriculum had improved their understanding of how quantum mechanics is organized, despite the fact that a higher percentage of students responded favorably to the exercises as a whole in Survey 1 than Survey 2. This delayed outcome was anticipated since the supplementary exercises sought to develop the mathematical organizing structure in advance of direct application, resulting in the early exercises being fairly ahead of the

other course lessons. Consequently, this data result indicates that the overall design of the supplementary curriculum worked as intended.

There were many positive effects revealed from this study. However, it is equally important to consider how the exercises fell short for some students and how this might be improved. Several students related their criticisms in the long answer portion of the surveys. In Survey 1, writing in the section for additional comments, students stated that “some of the mathematics was not explained well (Dirac notation)” and “I don’t like how you skip steps in some of the math. It makes it hard to follow.” In Survey 2, addressing the question of what they liked and disliked about the exercises and the recommendations they would suggest for improvement (Survey 2, question 1), students wrote “Make them shorter, give clearer explanation” and “The very first one was the only one I could follow, quantum requires discussion, not material.” This suggests that the effectiveness of the exercises might be improved for some students by providing better explanation of the mathematical portions and by incorporating more student discussion into the model. In fact, with more time for development and the advantage of hindsight and student feedback, more exercises could be added and others modified for later use. As concerns the degree of student to student interaction, Survey 1 revealed that few students discussed the exercises with their classmates (Survey 1, question 6). An original intent was to use a web-based discussion board as a forum to facilitate discussion amongst the students, but instead this ended up being a communication portal for the curriculum administrator to provide feedback to students’ on their written reflections. Nevertheless, it could be worthwhile to reconsider additional mechanisms for encouraging inter-student discussion to improve the effectiveness and scope of the supplementary curriculum.

Despite these shortcomings, there were many tangible gains for a majority of the students that resulted from the supplementary curriculum and revealed a variety of effective teaching strategies. It was interesting to learn that, for some students, the mathematical training had a positive impact on other courses. This not only reveals the effectiveness of the mathematical instruction in the exercises, but also highlights the importance of these techniques for the students’ continued success in their field of study. Moreover, it illustrates that the supplementary exercises provided learning experiences that lead to transfer, the ability to extend what has been learned in one context to new contexts. In relating the concepts from the supplementary curriculum with the other course materials, the students also engaged in a form of transfer. Perhaps the low stakes environment and inquiry-based atmosphere fostered by the supplementary curriculum was useful in encouraging students to creatively explore and apply the new ideas in a variety of contexts.

The study also revealed the learning benefits of employing a diversity of instructional formats. While individual students expressed preferences for writing reflections or solving math problems, it was evident that the inclusion of both formats provided some appreciated variety. In fact, it may be beneficial to allow students some

choice between these options, as was done following exercise 7. Other students mentioned that the written exercises were a useful supplement to the book and an additional aid to learning quantum mechanics. Statements included: "Outline with steps and reasoning was beneficial in that it was a good reference and often a substitute for certain areas in the book" and "What I like the most about ERAs is that it is an extra aide in learning quantum mechanics". Moreover, the written reflections often revealed students pensively assessing their knowledge and testing their comprehension, engaging in metacognition. In summary, the data results seem to indicate that more than any single instructional format, a combination of formats is best suited for maximizing student engagement and achievement in the learning process.

Finally, we end our discussion by sharing select student comments on the supplementary exercises, written during the final reflection following exercise 11. The responses were mostly positive and provide further insight into student opinion on the supplementary curriculum and how it impacted their education. The full list of student reflection responses following the final exercise is given in Appendix IV.

"I have been taking math 24 (the one with linear algebra) this semester, so, after noticing some of the similarities between everything being talked about, it is satisfying to see that they are in fact as closely related as they would seem. . . . These ERA's have been useful throughout the semester in demystifying some of the ideas in quantum mechanics. I would hear/read people talking about things like Hilbert spaces and Fourier Transforms, but never knew what they actually were. Giving these ideas some kind of basis and definition makes everything seem a bit less complicated."

"For me, this ERA simply reinforced the fact that mathematics is very useful in many branches of physics, including quantum physics. It surprises me how certain quantities in quantum physics can be thought of as vectors, and how they can be transformed, just like the regular vectors we are used to in our math classes. That is, I thought vectors, and other topics from math could only be applied to classical physics, but that is not the case. This allows us to understand and manipulate physical quantities more efficiently. In my opinion, the ERAs assigned during this course were very helpful because they describe physical phenomenon in ways our text does not, i.e. the ERAs are straighter forward, clearer, and give better examples than the book. I think the material in this class would be harder to understand if not for the ERAs assigned."

"For this Final ERA I would like to say a few things about the assignments over the course of the semester. In the Beginning I kept up with the articles very well, they may not have been straight forward, but they were all within my realm of understanding. But as the Semester went on the Articles got more and more complex. And not understanding something at the beginning of the article, made it very hard to read the rest. As such toward the end of the semester I could do

little but run my lines over the words hoping to understand them. But usually to no avail. The early concepts are perfectly understandable, but toward the end... I found that if I was to understand I would need someone to explain it to me. After a certain point the ERA's require aid, eventually the concepts get so complicated, that an article does not suffice."

Conclusion

As the data reveal, an early introduction to expert modes of thinking can lead to improved understanding and increased interest in subject matter for a majority of students. In quantum mechanics in particular, the emphasis on the mathematical organizing structure of the theory provided multiple benefits for many students, helping them to develop an improved understanding of core organizing concepts and providing familiarity with mathematical techniques applicable to additional courses. The data showed evidence of students making a connection between the unity of mathematics and physical concepts. Also, the diversity of instructional formats included in the supplementary exercises helped to increase student engagement with the material. The Java applets were particularly useful in this respect. It may be that the benefits of these exercises could be extended to an even greater proportion of students by further expanding the instructional formats employed. For instance, the data results suggest that for some of the students who saw no apparent benefit, the outcome might be improved by introducing more inter-student interaction and discussion into the model.

Given the tangible benefits seen from this study, it is important to assess the costs and benefits of implementing such a supplementary curriculum and to outline best practices and areas for improvement learned from this research. The initial time commitment required to develop the supplementary materials is lengthy and likely impractical for instructors already busy with a full workload. Factoring out curriculum development time, administration of this supplemental instruction for a small course would require perhaps an additional four hours per week, which would entail reading and responding to students submitted work as well as making small changes/refinements to the exercises. As such, the eleven completed exercises from this research project provide a good starting point for a standard supplementary curriculum to modern physics and could be improved and expanded on to fill out a complete semester of materials. In the initial development of the course materials, student opinion can be gainfully used to determine the effectiveness of various exercise components and to help identify areas for improvement. Also, it may be desirable and beneficial for a competent instructional assistant to administer the supplementary curriculum instead of the main course instructor, as this provides the students with an additional knowledgeable person with whom to consult and would likely benefit all parties involved.

Based on the data results and observations from this research, certain best practices can be given for implementing such a supplementary curriculum. As this project reveals, it is important to motivate student participation by providing course credit for the supplementary exercises. However, it seems equally important that credit be allotted for participation rather than graded for correctness. This enables students to engage in discovery-based learning in a safe environment without fear of negative repercussions for incorrect thinking. As such, students are more likely to provide candid reflections and honest responses to the material, as was evidenced in the student responses from this study. Furthermore, as this research reveals, it is important to provide students with a diversity of educational formats to encourage interaction with the course material. Such an approach seems to maximize the reach of course materials to student learners with different needs. Moreover, this study suggests that multiple instructional formats aid in promoting knowledge transfer and metacognitive thinking, both important aspects of expert knowledge development. It is always good to be creative in exploring new ways to engage students with the subject matter.

Given the initial successful outcomes of this research, it would be interesting to conduct a longitudinal study on how students fair in subsequent quantum mechanics courses following an introductory course with supplementary instruction on expert modes of knowledge organization. Such a model may be beneficial in diverse fields of study, as research shows that experts in all fields of study exhibit common thought processes that vary greatly from those of novice learners. This type of supplementary curriculum appears to be beneficial in helping students bridge this gap.

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Appendix I – Description of the Supplementary Curriculum Exercises

Exercise	Title	Description
1	Quantum Phenomena	Examples of quantum phenomena, focusing on the wave-particle duality, including a web video on electron double slit experiments and a description of Otto Wiener's 1890 experiment using photographic plates to measure standing light waves.
2	Quantum States	Introduction of quantum states, providing examples from the hydrogen atom and the polarization states of a photon.
3	Mathematical Concepts	Description of quantum states as vectors. Describes state vector properties by analogy to Cartesian vectors in Euclidean space. Introduces ideas of base states associated with a definite value of a measurable property, superposition states and the dot (inner) product. Introduces Dirac notation as a new way of notating vectors and vector operations.
4	Mathematical Concepts II: QM Base States, Superposition States, Probability Amplitudes and Measurements	More practice with ideas from previous activity, including normalization, orthonormality, superposition states and projection onto base states. Physical interpretation of state vectors in quantum mechanics, including probability amplitudes, probability of measuring physical attributes and uncertainty in quantum mechanics. Application to H-atom energy eigenstates.
5	Quantum Behavior: Collapse of the Quantum State and Measuring the Energy of a 'Particle in a Box'	Examples of superposition states and probability calculations for the energy states of the hydrogen atom. Discussion about measurement and the collapse of the wavefunction. Link to Java applet allowing students to create their own superposition states for a particle in a box and to make measurements to observe wavefunction collapse.
6	Fourier Series: Expanding Our Notion of What It Means to Be a Vector	Introduction to Fourier series, drawing on analogy with vectors in Euclidean space.

7	More Fourier Series: An Example and an Interactive Applet	Detailed example of the Fourier series representation of a square wave. A discussion of the significance of Fourier analysis for interpreting position and momentum in quantum mechanics. Link to a java applet allowing students to construct their own Fourier series for a variety of periodic functions.
8	Wavefunctions of a 'Quantum Particle in a Box'	Introduction of Schrodinger equation, analogy with Newton's 2nd law. Discussion of relationship between Schrodinger equation, wavefunctions and energy base states. Application to a particle in a box, including a discussion of energy quantization and (base) states of definite energy. Relationship of Fourier series to superpositions of base states of the particle in a box. A brief introduction to time-evolution in quantum mechanics and the concept of stationary states.
9	Position and Momentum	Discussion of position and momentum in the context of the particle in a box example. Relation of Fourier harmonics to momentum and momentum base states, leading to an introduction to Fourier analysis of non-periodic functions. Discussion of the Heisenberg uncertainty principle.
10	Wavepackets	Activity based around a java applet that allows students to manipulate wavepackets by adjusting their Fourier components. Applet allows students to investigate the transition from discrete to continuous Fourier analysis. Continues discussion of position, momentum and the uncertainty principle. Reveals how uncertainty in quantum mechanics is an intrinsic property of the mathematical theory.
11	Observables, Operators and Basis States	Introduces linear transformations, eigenvectors and eigenvalues using the rotation matrix in Euclidean space as an example. Introduces the paradigm by which each quantum mechanical observable is related to a linear operator, whose eigenvectors represent a complete basis of quantum states and whose eigenvalues represent the only possible values which a measurement of the observable may yield. The Schrodinger equation for a particle in a box is given as an example of an eigenvector equation in quantum mechanics. The concepts of uncertainty, compatibility, measurement and quantum numbers are discussed in the

		context of the eigenvector paradigm, followed by a brief discussion of conservation rules in quantum mechanics.
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Appendix II – Sample Supplementary Curriculum Exercise

Activity 10 – Essential Quantum Concepts:

Wavepackets

Introduction

Today's activity will continue to investigate the connection between position and momentum in quantum mechanics. Rather than give you a reading intensive exercise, I will ask you to explore a really nice Java applet on wave packets. In my opinion, wave packets are one of the best things to study if you want to improve your understanding of position, momentum and the position-momentum uncertainty principle in quantum mechanics. Wave packets also provide deep insight into the connection between waves and particles in quantum mechanics. I think this applet really presents a nice graphical picture of all these ideas. Let me know if you agree!

Wave Packet Applet

Instructions for Opening Applet:

You can access the Java applet for this activity from the following website:

<http://www.educyclopedia.be/electronics/javafourier.htm>

Once this webpage opens, you will see a long list of Fourier analysis related applets. Select the applet that says "**Fourier – Making Waves**" (you may need to scroll down the page a bit to find it). After clicking on the link, the Java applet should immediately load. The applet consists of three tabs. The first tab allows you to make Fourier series and the second one is a game related to Fourier series. We will focus on the third tab entitled "**Discrete to Continuous.**"

Instructions for Using Applet:

Background:

Wave packets are wavefunctions that represent a particle traveling through space. They are as close as we can get to the classical idea of a particle trajectory in quantum mechanics. For simplicity, this applet is one dimensional (along the x-axis) and only shows the wave packet at a fixed moment in time (it doesn't move). A real particle would be described by a wave packet that moves through three dimensional space, with the

position of the particle given by the (average) location of the wave packet and its momentum related to the (average) wavelength of the wave packet.

Position, Momentum and the Superposition Principle

As you can see from the applet, wave packets are also constructed by adding together sine and cosine functions (Fourier analysis). Each of these functions contains information about the particle's position and momentum. Consider a single sine wave with wavelength λ . The particle's position is related to the value of the sine function squared and the momentum is related to the wavelength through the de Broglie relationship⁷ (see Figure 1).

De Broglie relation: $p = \hbar k = \hbar \frac{2\pi}{\lambda} = \frac{h}{\lambda}$

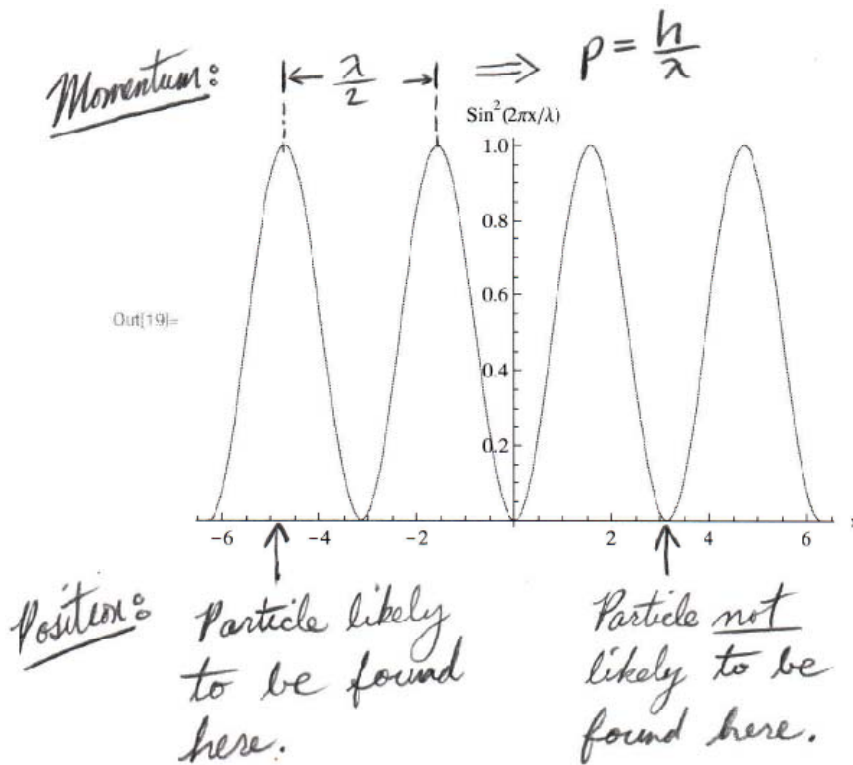


Figure 1 Relation of particle position and momentum to a sine wave.

⁷ Actually, the true momentum base states are complex exponentials $e^{ikx} = \cos(kx) + i \sin(kx)$, but this is a subtle point that you can ignore for now.

Of course, we can see from the above graph that a single sine wave is a poor representation of a *particle*. The likely position is spread out all over the place!⁸ In fact, a single sine function is indicative of *wave properties* in quantum mechanics since a sinusoidal wave can interfere nicely with other similar waves. If we wish to represent a *particle*, we'd like the wavefunction to be localized in one region of space. Luckily, we can achieve this by using the superposition principle to add together sine and cosine waves of different wavelength (i.e. momentum). Amazingly, as we have seen with our study of Fourier analysis, the addition of sinusoidal waves with different frequencies results in regions of constructive and destructive interference that can create a localized wavefunction (i.e. a wave packet).

Fourier Analysis: From Discrete to Continuous

Look at the default graphics in the **Discrete to Continuous** tab of the applet. The top graph gives the relative amplitude (i.e. component) of the sine waves in the Fourier sum as a function of their wave number k . The middle graph plots *all* of these individual sine waves simultaneously in an attempt to show you how they interfere (this is a bit confusing at first). The bottom graph shows the *sum* of all the sine functions plotted in the middle graph (i.e. the wave packet or wavefunction).

First, notice that since the default setting is for a Fourier series with a '**Spacing between Fourier components**' equal to π (i.e. $k_1 = \pi$), the wave packet repeats itself with a periodic length of $L = \frac{2\pi}{k_1} = \frac{2\pi}{\pi} = 2$, as shown by the yellow line on the bottom graph. Try changing the spacing to 2π and you will now see the 'wave packets' get closer together, now repeating with a period of $L = 1$. Try reducing this spacing and witness that the repetition length L of the wave packets increases according to the same formula.

This shows us why we cannot represent a wave packet by a Fourier series: the wave packet for a sum of sinusoidal functions is always a repeating periodic function! If we wish to represent a *particle*, we need a wave packet that exists in only *one* region of space! Fortunately, we see that as we use more and more wave numbers in the sum, the repetition length increases (the separate packets get further apart). If we decide to use a continuous range of wave numbers in the sum, then we get just a single wave packet that never repeats! We can do this with the applet by setting the spacing between Fourier components equal to zero. This, of course, is the Fourier transform, the transition from a discrete number to an infinite number of terms in our superposition of sine functions! (Notice that when we set the k spacing to zero we see the following consequences of transitioning to the Fourier transform: the equation above the lower graph changes to an integral, in the upper graph the fundamental frequency $k_1 = 0$ and the amplitude A

⁸ Since the momentum is precisely known, we are completely uncertain about the position – the uncertainty principle in action!

becomes a function of the continuous variable k and the middle graph says “can’t show an infinite number of components”).

Position, Momentum and the Heisenberg Uncertainty Principle

Now we would like to understand what wave packets can teach us about the position-momentum uncertainty principle. First, set the applet to the Fourier transform setting (i.e. set the k spacing equal to zero). Observe that the wave packet contains a superposition of momentum states with a Gaussian distribution centered at the “**Wave packet center**” value. The average value of the momentum is given by the center value of this Gaussian distribution and the uncertainty in the momentum is given by the standard deviation of this Gaussian distribution. Look at the wave packet and notice that it seems to have an average wavelength; this is the wavelength corresponding to the average momentum. Try changing the center wavelength to a larger k value, which corresponds to a greater average momentum. Observe that the wavelength of the wave packet gets shorter, indicating an increase in the particle’s momentum. Click on the “**x-space envelope**” to see the location and spatial extent of the particle (remember that particle position is determined by the square of the wave packet). The standard deviation of this ‘spatial envelope’ describes our uncertainty in the particle’s position. Click on the “**Width Indicators**” box and a graphic makes the uncertainties in momentum and position apparent. Indeed, we see that a wave packet is similar to the classical idea of a particle trajectory: both the particle’s position and momentum are relatively well defined. Of course, the key difference is that a quantum mechanical wave packet always has some uncertainty in both the position and momentum.

Now, let’s see what happens when we change the uncertainties. We can do this in the applet by changing the “**Wave packet width**” sliders. Give it a try. Notice that when you change the width of one uncertainty, the other one also changes, but in the opposite direction. The uncertainties in position and momentum are not independent! This is the uncertainty principle in action! If we want to confine the wave packet to a smaller area (i.e. decreasing its position uncertainty), then we are required to use a wider range of wave numbers in the superposition of momentum states to form that wave packet. But a wider range of wave numbers implies increased uncertainty in the momentum! Now try going the other direction; if we want a sharp spike in the wave number distribution (i.e. decreased momentum uncertainty), then the spatial extent of the wave function must increase! It can be proven from Fourier analysis that the product of the position and momentum uncertainties can *never be less than* $\hbar/2$. The bottom line: The uncertainty principle is a consequence of the wave particle duality and is enforced by Fourier analysis. According to quantum mechanics, uncertainty does not arise from a failure of our instruments to measure without disturbance. Rather, uncertainty it is a fundamental part of nature that is built into and inseparable from the mathematics of quantum mechanics.

Weekly Submission

I can tell by your responses that the last ERA was rather challenging. I've tried to address this in the feedback for ERA#9 and I also hope this ERA has helped improve your understanding. For this ERA, please submit a written reflection and let me know if this has helped you make better sense of things. As always, I welcome any questions/comments.

Appendix III – Complete Student Surveys

Survey 1:

Educational Research Activities Mid-Semester Survey – Physics 10

This is an anonymous survey designed to get your feedback on the Educational Research Activities you have been doing this semester. The results will be used to assess the effectiveness of these activities in supporting your learning. Survey completion is completely voluntary. Whether you decide to complete the survey or not, it will in no way affect your grade in the course.

Instructions: Unless otherwise indicated, please circle the response that best matches your answer to the following questions.

Your major (**write-in**):

Year: **freshman / sophomore / junior / senior**

Are you 18 or older? **Y / N**

- 1) Have you previously taken or are currently enrolled in the following math or physics courses:
 - a. 3rd Semester calculus (multi-variable calculus) **Y / N**
 - b. Linear algebra **Y / N**
 - c. Probability and statistics **Y / N**
 - d. Differential equations **Y / N**
 - e. Quantum Mechanics (Phys 137) **Y / N**
- 2) How often do you complete the Educational Research Activities (ERAs)?
never / rarely / often / always
- 3) How long does it take you to complete an ERA on average?
less than 1 hour / about 1 hour / more than 1 hour
- 4) Have these activities been useful in improving your understanding of quantum mechanics?
not useful / barely useful / somewhat useful / very useful
- 5) Have these activities increased your interest in quantum mechanics? **Y / N**
- 6) Have you found the web resources – interactive applets and videos – useful in increasing your understanding and/or interest in quantum mechanics? **Y / N**
- 7) Do you feel that writing reflections about these activities enhances your understanding of the concepts? **Y / N**

- 8) Have you discussed the material in the ERAs with your classmates? **Y / N**
- 9) How often do you read the feedback posted on the UCM CROPS Student Lounge?
never / rarely / frequently / always
- 10) Do you find the feedback on your submissions useful?
***not applicable* / not useful / barely useful / somewhat useful / very useful**
- 11) Have these activities helped to improve your understanding of how the theory of quantum mechanics is organized?
no improvement / slight improvement / moderate improvement / good improvement
- 12) These activities are designed to introduce mathematical concepts. Is the emphasis on relating new concepts to previously familiar mathematics within a physics context effective?
If not, please explain. If so, how so?
- 13) How well are you able to relate these exercises to the rest of the course material?
If not, please explain. If so, how so?
- 14) Any additional comments?

Survey 2:

Educational Research Activities End of Semester Survey – Physics 10

This is an anonymous survey designed to get your feedback on the Educational Research Activities you have been doing this semester. The results will be used to assess the effectiveness of these activities in supporting your learning. You do not have to complete this survey if you do not want to and whether you decide to complete the survey or not, it will in no way affect your grade in the course.

Instructions: Unless otherwise indicated, please circle the response that best matches your answer to the following questions.

Your major (**write-in**):

Year: **freshman / sophomore / junior / senior**

Gender: **male / female**

Are you 18 or older? **Y / N**

- 1) How often did you complete the Educational Research Activities (ERAs)?
never / rarely / often / always
- 2) If the ERAs were not worth course credit, would you still complete them? **Y / N**
- 3) Now that the course is nearly over, do you feel that the ERAs have been useful in improving your understanding of quantum mechanics?
not useful / barely useful / somewhat useful / very useful
- 4) Have these activities increased your interest in quantum mechanics? **Y / N**
- 5) How comfortable are you with the ideas of base states, functions as vectors and Fourier analysis in quantum mechanics?
not comfortable / barely comfortable / somewhat comfortable / very comfortable
- 6) How well are you now able to relate the ERAs to the rest of the course material?
Not able to relate / barely able to relate / somewhat able to relate / able to relate very well
- 7) Have these activities helped to improve your understanding of how the theory of quantum mechanics is organized?
no / barely / somewhat / yes

- 8) Do you feel like these activities have helped to improve your preparedness for future quantum mechanics courses? **Y / N**
- 9) Please rank the following ERA components in order of their effectiveness, marking 1 (most effective) through 4 (least effective) :
- [] Reading the written articles (ERAs)

 - [] Interactive Java Applets

 - [] Writing your reflections

 - [] Receiving feedback on your reflections
- 10) Would you recommend these exercises for future Physics 10 students? **Y / N**
- 11) Would you be interested in participating in a longitudinal study, which tracks the effectiveness of these ERAs in helping you to understand and succeed in the upper division quantum mechanics course? (It would be an anonymous study, just like this one)
Y / N / Not Applicable
- 12) What did you like the most and the least about the ERAs? Do you have any recommendations for improving the ERAs?
- 13) What do you imagine when you think of a quantum mechanical electron?
- 14) In your opinion, why is there uncertainty in quantum mechanics?
- 15) Additional comments?

Appendix IV – Complete List of Student Comments from the Final Reflection⁹

“Overall I often found the activities a bit confusing, and while I’d often find myself asking reasonable questions while reading, you’d usually answer most of them in the next few sentences or paragraphs. While this is definitely a good thing it often caused me to read the papers with more of a might I be able to say about this in a reflection, then how does this work.”

“One idea that this ERA reminded me of is the concept of observables. I think this is one of the concepts presented in the ERAs that will stick with me the most due to its very fundamental nature. Observables represent values we can measure in quantum mechanics such as energy, position, and momentum. Every observable can be represented by a set of unique base state vectors.”

“In terms of the ERAs in general, I found them most useful when the concepts being explained to us corresponded to concepts we were concurrently being taught in class. For example, the ERA pertaining to black body radiation coincided perfectly with our class lessons. As a result, I felt comfortable answering the ERA and had some knowledge solidified as well. This raised a question in my mind about the nature of the ERAs. Why the ERAs teaching us different information from our class lessons as opposed to reinforcing what we were learning in class? I’m not trying to say that they weren’t helpful, just curious as to the reasoning behind this decision.”

“For this Final ERA I would like to say a few things about the assignments over the course of the semester. In the Beginning I kept up with the articles very well, they may not have been straight forward, but they were all within my realm of understanding. But as the Semester went on the Articles got more and more complex. And not understanding something at the beginning of the article, made it very hard to read the rest. As such toward the end of the semester I could do little but run my lines over the words hoping to understand them. But usually to no avail. The early concepts are perfectly understandable, but toward the end... I found that if I was to understand I would need someone to explain it to me. After a certain point the ERA’s require aid, eventually the concepts get so complicated, that an article does not suffice.”

“I have been taking math 24 (the one with linear algebra) this semester, so, after noticing some of the similarities between everything being talked about, it is satisfying to see that they are in fact as closely related as they would seem. . . . These ERA's have been useful throughout the semester in demystifying some of the ideas in quantum mechanics. I would hear/read people talking about things like Hilbert spaces and Fourier Transforms, but never knew what they actually were. Giving these ideas some kind of basis and definition makes everything seem a bit less complicated.”

⁹ Some information particular to the content of exercise 11 has been excluded here to focus on the comments relating to the students’ overall opinion of the supplementary exercises.

"This ERA taught me a lot more fundamental knowledge of the "real" quantum mechanics before I leave this class. And I also noticed that quantum mechanics is based on a lot of advanced mathematical concepts ... I learned the fundamental and mathematical sides of QM from these ERA's rather than the lectures. I'm sure they'll be very helpful for my future study on QM."

"QM is such an complex and interesting subject because of the math that is involved and how the real life implications of the math turn out to be counter intuitive. The ERA's that you've made have been a real help in learning about the math involved in quantum mechanics. Although I already took linear algebra, I've never applied it to problems involving quantum mechanics. These ERA'S were perfect starting points for introducing a mathematical treatment of quantum mechanics."

"For me, this ERA simply reinforced the fact that mathematics is very useful in many branches of physics, including quantum physics. It surprises me how certain quantities in quantum physics can be thought of as vectors, and how they can be transformed, just like the regular vectors we are used to in our math classes. That is, I thought vectors, and other topics from math could only be applied to classical physics, but that is not the case. This allows us to understand and manipulate physical quantities more efficiently. In my opinion, the ERAs assigned during this course were very helpful because they describe physical phenomenon in ways our text does not, i.e. the ERAs are straighter forward, clearer, and give better examples than the book. I think the material in this class would be harder to understand if not for the ERAs assigned."

"Although I did not understand all of the material, I found it interesting to read about. Just like in the previous ERA's, the information I learned helped me understand quantum mechanics a little better, and I found it interesting even though this is not my specific area of study."

"I enjoyed the ERA's and I would recommend continuing the program for the future. I felt the ERA's taught me more about quantum mechanics than many of the initial lectures did."

"The activities this semester were definitely useful, as I was actually learning something rather than doing something 'useless.' However, the activities were not always clear, although this may be from the content of QM itself and not the structure of the ERAs (either/or or both). Your responses have been helpful, although I found it hard to keep up with each one each week. As for discussing QM with other students, the structure of the whole class and our class standings don't encourage much of it (in my case, I am a sophomore that has only taken introductory mechanics and EM, so I'm in the position of learning than actually being able to offer something to discuss). Or maybe the majority of us are more shy than not. Overall, the ERAs have been mildly helpful."